Chapter 14

Curves and Surfaces

Some three-dimensional geometry is needed for understanding functions of two variables.

The main subject of this chapter is surfaces in three-dimensional space. In preparation for this, we begin with a study of some special curves in the plane—the conic sections. In the last two sections, we will do some calculus with curves in space. Applications of calculus to surfaces are given in Chapters 15 and 16.

14.1 The Conic Sections

All the curves described by quadratic equations in two variables can be obtained by cutting a cone with planes.

The ellipse, hyperbola, parabola, and circle are called *conic sections* because they can all be obtained by slicing a cone with a plane (see Fig. 14.1.1). The

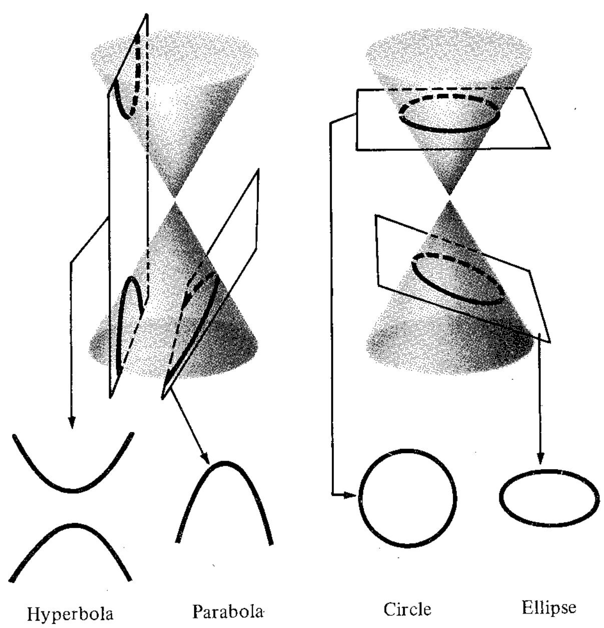


Figure 14.1.1. Conic sections are obtained by slicing a cone with a plane; which conic section is obtained depends on the direction of the slicing plane.

theory of these curves, developed by Apollonius of Perga (262–200 B.C.), is a masterwork of Greek geometry. We will return to the three-dimensional origin of the conics in Section 14.4, after we have studied some analytic geometry in space. For now, we will treat these curves, beginning with the ellipse, purely as objects in the plane.

Definition of Ellipse

An *ellipse* is the set of points in the plane for which the sum of the distances from two fixed points is constant. These two points are called the *foci* (plural of *focus*).

An ellipse can be drawn with the aid of a string tacked at the foci, as shown in Fig. 14.1.2.

To find an equation for the ellipse, we locate the foci on the x axis at the points $F' = (-c, 0)$ and $F = (c, 0)$. Let $2a > 0$ be the sum of the distances from a point on the ellipse to the foci. Since the distance between the foci is $2c$, and the length of a side of a triangle is less than the sum of the lengths of the other sides, we must have $2c < 2a$; i.e., $c < a$. Referring to Fig. 14.1.3, we

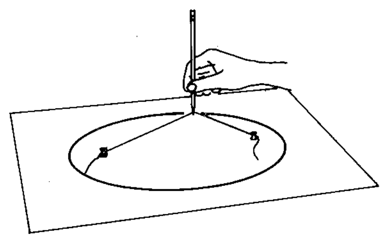


Figure 14.1.2. Mechanical construction of an ellipse.

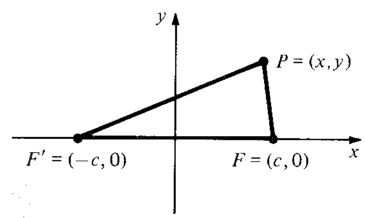


Figure 14.1.3. P is on the ellipse when $|FP| + |F'P| = 2a$.

see that a point $P = (x, y)$ is on the ellipse precisely when

$$|FP| + |F'P| = 2a.$$

That is,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Transposing $\sqrt{(x-c)^2 + y^2}$, squaring, simplifying, and squaring again yields

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Let $a^2 - c^2 = b^2$ (remember that $a > c > 0$ and so $a^2 - c^2 > 0$). Then, after division by a^2b^2, the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the *equation of an ellipse in standard form*.

Since $b^2 = a^2 - c^2 < a^2$, we have $b < a$. If we had put the foci on the y axis, we would have obtained an equation of the same form with $b > a$; the length of the "string" would now be $2b$ rather than $2a$. (See Fig. 14.1.4.) In either case, the length of the long axis of the ellipse is called the *major axis*, and the length of the short axis is the *minor axis*.

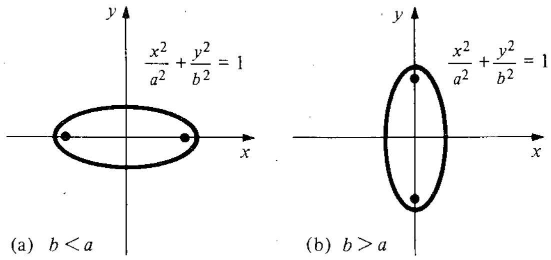


Figure 14.1.4. The appearance of an ellipse in the two cases $b < a$ and $b > a$.

Example 1 Sketch the graph of $4x^2 + 9y^2 = 36$. Where are the foci? What are the major and minor axes?

Solution Dividing both sides of the equation by 36, we obtain the standard form

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Hence $a = 3$, $b = 2$, and $c = \sqrt{a^2 - b^2} = \sqrt{5}$. The foci are $(\pm\sqrt{5}, 0)$, the y intercepts are $(0, \pm 2)$, and the x intercepts are $(\pm 3, 0)$. The major axis is 6 and the minor axis is 4. The graph is shown in Fig. 14.1.5.

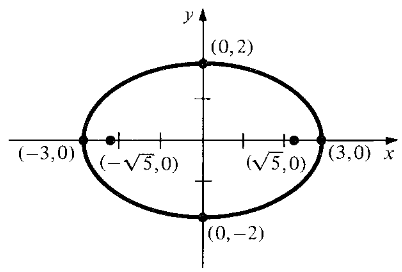


Figure 14.1.5. The graph of $4x^2 + 9y^2 = 36$.

Example 2 Sketch the graph of $9x^2 + y^2 = 81$. Where are the foci?

Solution Dividing by 81, we obtain the standard form $x^2/3^2 + y^2/9^2 = 1$. The graph is sketched in Fig. 14.1.6. The foci are at $(0, \pm 6\sqrt{2})$. ▲

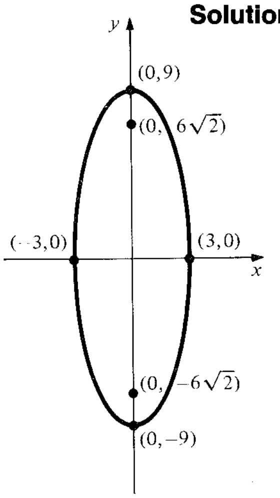


Figure 14.1.6. The ellipse $9x^2 + y^2 = 81$.

Ellipse

Equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (standard form).

Foci: $(\pm c, 0)$ where $c = \sqrt{a^2 - b^2}$ if $a > b$.
$(0, \pm c)$ where $c = \sqrt{b^2 - a^2}$ if $a < b$.
If $a = b$, the ellipse is a circle.

x intercepts: $(a, 0)$ and $(-a, 0)$.

y intercepts: $(0, b)$ and $(0, -b)$.

If P is any point on the ellipse, the sum of its distances from the foci is $2a$ if $b < a$ or $2b$ if $b > a$.

The second type of conic section, to which we now turn, is the hyperbola.

Definition of Hyperbola

A hyperbola is the set of points in the plane for which the *difference* of the distances from two fixed points is constant. These two points are called the *foci*.

To draw a hyperbola requires a mechanical device more elaborate than the one for the ellipse (see Fig. 14.1.7); however, we can obtain the equation in the same way as we did for the ellipse. Again let the foci be placed at $F' = (-c, 0)$

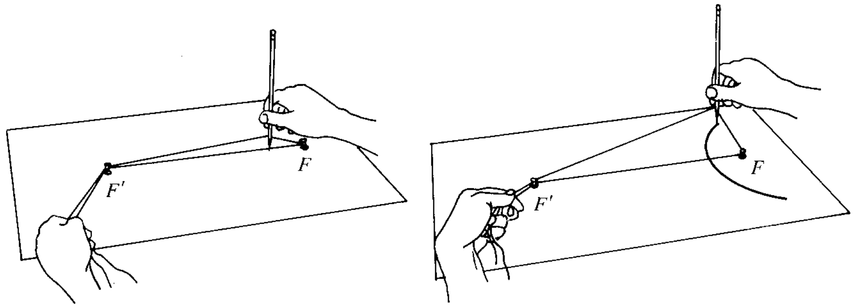


Figure 14.1.7. Mechanical construction of a hyperbola.

and $F = (c, 0)$, and let the difference in question be $2a$, $a > 0$. Since the difference of the distances from the two foci is $2a$ and we must have $|F'P| < |FP| + |F'F|$, it follows that $|F'P| - |FP| < |F'F|$, and so $2a < 2c$. Thus we must have $a < c$ (see Fig. 14.1.8). The point $P = (x, y)$ lies on the

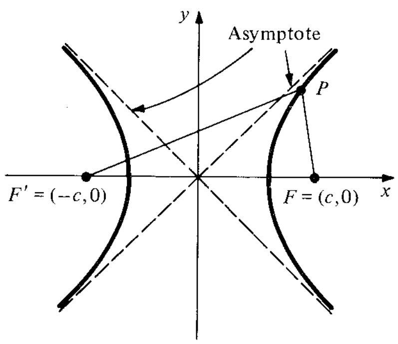


Figure 14.1.8. P is on the hyperbola when $|F'P| - |FP| = \pm 2a$.

hyperbola exactly when

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

After some calculations (squaring, simplifying and squaring again), we get

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

If we let $c^2 - a^2 = b^2$ (since $a < c$), we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is the equation of a hyperbola in standard form.

For x large in magnitude, the hyperbola approaches the two lines $y = \pm(b/a)x$, which are called the *asymptotes* of the hyperbola. To see this, for x and y positive, we first solve for y in the equation of the hyperbola, obtaining $y = (b/a)\sqrt{x^2 - a^2}$. Subtracting this from the linear function $(b/a)x$, we find that the vertical distance from the hyperbola to the line $y = (b/a)x$ is given by

$$d = \frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right).$$

To study the behavior of this expression as x becomes large, we multiply by $(x + \sqrt{x^2 - a^2})/(x + \sqrt{x^2 - a^2})$ and simplify to obtain $ab/(x + \sqrt{x^2 - a^2})$. As x becomes larger and larger, the denominator increases as well, so the quantity d approaches zero. Thus the hyperbola comes closer and closer to the line. The other quadrants are treated similarly. (See Fig. 14.1.9.)

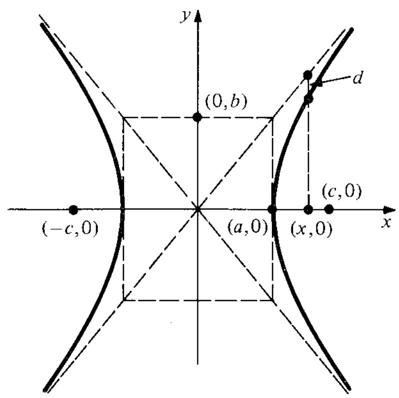


Figure 14.1.9. The vertical distance d from the hyperbola to its asymptote $y = (b/a)x$ is

$$\frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right) = \frac{ab}{x + \sqrt{x^2 - a^2}}.$$

Example 3 Sketch the curve $25x^2 - 16y^2 = 400$.

Solution Dividing by 400, we get the standard form $x^2/16 - y^2/25 = 1$, so $a = 4$ and $b = 5$. The asymptotes are $y = \pm\frac{5}{4}x$, and the curve intersects the x axis at $(\pm 4, 0)$ (see Fig. 14.1.10). ▲

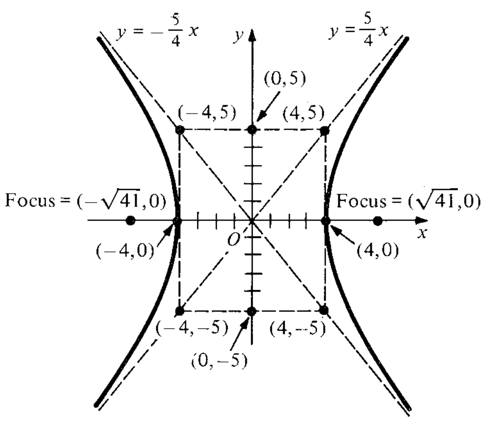


Figure 14.1.10. The hyperbola $25x^2 - 16y^2 = 400$.

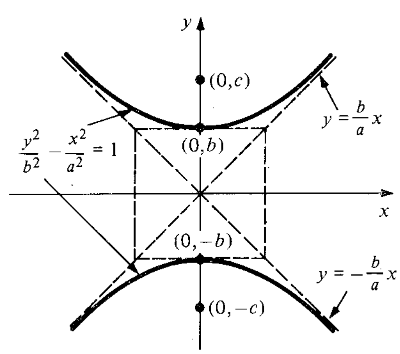


Figure 14.1.11. A hyperbola with foci on the y axis.

If the foci are located on the y axis, the equation of the hyperbola takes the second standard form $y^2/b^2 - x^2/a^2 = 1$ (see Fig. 14.1.11).

Notice that if we draw the rectangle with $(\pm a, 0)$ and $(0, \pm b)$ at the midpoints of its sides, then the asymptotes are the lines through opposite corners, as shown in Figs. 14.1.10 and 14.1.11.

Example 4 Sketch the graph of $4y^2 - x^2 = 4$.

Solution Dividing by 4, we get $y^2 - x^2/2^2 = 1$, which is in the second standard form with $a = 2$ and $b = 1$. The hyperbola and its asymptotes are sketched in Fig. 14.1.12. ▲

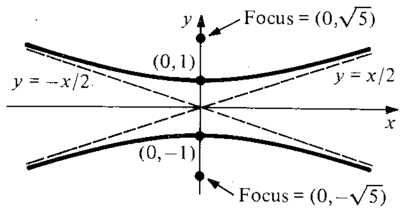


Figure 14.1.12. The hyperbola $4y^2 - x^2 = 4$.

Hyperbola

Case 1: Foci on x axis	**Case 2:** Foci on y axis
Equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$
Foci: $(\pm c, 0)$, $c = \sqrt{a^2 + b^2}$	$(0, \pm c)$, $c = \sqrt{a^2 + b^2}$
x intercepts: $(\pm a, 0)$	none
y intercepts: none	$(0, \pm b)$
Asymptotes: $y = \pm \dfrac{b}{a} x$	$y = \pm \dfrac{b}{a} x$

If P is any point on the hyperbola, the difference between its distances from the two foci is $2a$ in case 1 and $2b$ in case 2.

We are already familiar with the circle and parabola from Section R.5. The circle is a special case of an ellipse in which $a = b$; that is, the foci coincide. The parabola can be thought of as a limiting case of the ellipse or hyperbola, in which one of the foci has moved to infinity. It can also be described as follows:

Definition of Parabola

A parabola is the set of points in the plane for which the distances from a fixed point, the *focus*, and a fixed line, the *directrix*, are equal.

Placing the focus at $(0, c)$ and the directrix at the line $y = -c$ leads, as above, to an equation relating x and y. Here we have (see Fig. 14.1.13) $|PF| = |PG|$.

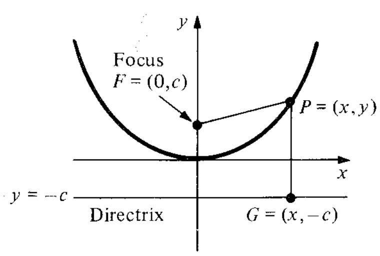


Figure 14.1.13. P is on the parabola when $|PF| = |PG|$.

That is, $\sqrt{x^2 + (y - c)^2} = |y + c|$, so $x^2 + (y - c)^2 = (y + c)^2$, which gives

$$x^2 - 4cy = 0,$$

$$y = \frac{x^2}{4c}$$

which is the form of a parabola as given in Section R.5.

If we place the focus at $(c, 0)$ on the axis and use $x = -c$ as the directrix, we get the "horizontal" parabola $x = y^2/4c$.

Parabola

Case 1: Focus on y axis		**Case 2:** Focus on x axis	
Equation: $y = ax^2$	$\left(a = \frac{1}{4c}\right)$	$x = by^2$	$\left(b = \frac{1}{4c}\right)$
Focus: $(0, c)$		$(c, 0)$	
Directrix: $y = -c$		$x = -c$	

If P is any point on the parabola, its distances from the focus and directrix are equal.

Example 5 (a) Find the equation of the parabola with focus $(0, 2)$ and directrix $y = -2$.
(b) Find the focus and directrix of the parabola $x = 10y^2$.

Solution (a) Here $c = 2$, so $a = 1/4c = 1/8$, and so the parabola is $y = x^2/8$.
(b) Here $b = 10 = 1/4c$, so $c = 1/40$. Thus the focus is $(1/40, 0)$ and the directrix is the line $x = -1/40$. ▲

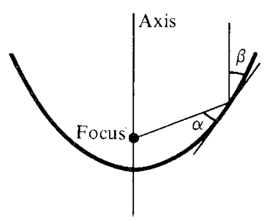


Figure 14.1.14. The angles α and β are equal.

The conic sections appear in a number of physical problems, two of which will be mentioned here; we will see additional ones in later sections. The first application we discuss is to parabolic mirrors. The parabola has the property that the angles α and β shown in Fig. 14.1.14 are equal. This fact, called the *reflecting property* of the parabola, was demonstrated in Review Exercise 86 of Chapter 1. Since the angles of incidence and reflection are equal for a beam of light, this implies that a parallel beam of light impinging on a parabolic mirror will converge at the focus. This is the basis of parabolic telescopes (visual and radio) as well as solar-energy collectors. Similarly, a searchlight will produce a parallel beam of light if a light source is placed at the focus of a parabolic mirror.

Example 6 A parabolic mirror for a searchlight is to be constructed with width 1 meter and depth 0.2 meter. Where should the light source be placed?

Solution We set up the parabola on the coordinate axes as shown in Fig. 14.1.15. The equation of the parabola is $y = ax^2$. Since $y = 0.2$ when $x = 0.5$, we get

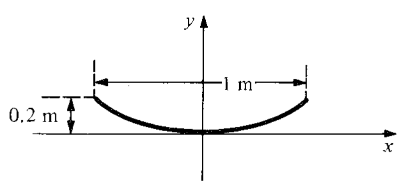


Figure 14.1.15. Find the focus of the searchlight.

$a = 0.2/0.25 = 0.8$. The focus is at $(0, c)$, where $a = 1/4c$, so $c = 1/4a = 0.3125$. Thus the light source should be placed on the axis, 0.3125 meters from the mirror. ▲

In courses in mechanics, it is shown that bodies revolving about the sun (planets, asteroids, and comets) do so in elliptic, parabolic, or hyperbolic orbits with the sun at one focus. We shall see part of the derivation of this fact in Section 14.7. Most planetary orbits are nearly circular. To measure the departure from circularity, the *eccentricity* is introduced. It is defined by

$$e = \frac{c}{a},$$

where a, b, and c are defined as on p. 697, with $a > b$. Thus a and b are the semi-major and semi-minor axes and c is the distance of a focus from the center; $c = \sqrt{a^2 - b^2}$. An ellipse is circular when $e = 0$, and as e approaches 1, the ellipse grows longer and narrower.

Example 7 The eccentricity of Mercury's orbit is 0.21. How wide is its orbit compared to its length?

Solution Since $e = 0.21$, $c = 0.21a$, so $c^2 = a^2 - b^2$, and therefore $b^2 = a^2 - c^2 = a^2(1 - (0.21)^2) = 0.9559a^2$. Hence $b \approx 0.9777a$, so the orbit is 0.9777 times as wide as it is long. ▲

Exercises for Section 14.1

1. Sketch the graph of $x^2 + 9y^2 = 36$. Where are the foci?
2. Sketch the graph of $x^2 + \frac{1}{9}y^2 = 1$. Where are the foci?
3. Sketch the graphs of $x^2 + 4y^2 = 4$, $x^2 + y^2 = 4$, and $4x^2 + 4y^2 = 4$ on the same set of axes.
4. Sketch the graphs of $x^2 + 9y^2 = 9$, $9x^2 + y^2 = 9$, and $9x^2 + 9y^2 = 9$ on the same set of axes.
5. Sketch the graph of $y^2 - x^2 = 2$, showing its asymptotes and foci.
6. Sketch the graph of $3x^2 = 2 + y^2$, showing asymptotes and foci.
7. Sketch the graphs $x^2 + 4y^2 = 4$ and $x^2 - 4y^2 = 4$ on the same set of axes.
8. Sketch the graphs of $x^2 - y^2 = 4$ and $x^2 + y^2 = 4$ on the same set of axes.

Find the equation of the parabolas in Exercises 9 and 10 with the given focus and directrix.

9. Focus $(0, 4)$, directrix $y = -4$
10. Focus $(0, 3)$, directrix $y = -3$.

Find the focus and directrix of the parabolas in Exercises 11–14.

11. $y = x^2$
12. $y = 5x^2$
13. $x = y^2$
14. $x = 4y^2$

Find the equations of the curves described in Exercises 15–20.

15. The circle with center $(0, 0)$ and radius 5.
16. The ellipse consisting of those points whose distances from $(-2, 0)$ and $(2, 0)$ sum to 8.
17. A parabola with vertex at $(0, 0)$ and passing through $(2, 1)$.
18. The circle centered at $(0, 0)$ and passing through $(1, 1)$.
19. The hyperbola with foci at $(0, 2)$ and $(0, -2)$ and passing though $(0, 1)$.
20. The ellipse with x intercept $(1, 0)$ and foci $(0, -2)$ and $(0, 2)$.

21. A parabolic mirror to be used in a searchlight has width 0.8 meters and depth 0.3 meters. Where should the light source be placed?
22. A parabolic disk 10 meters in diameter and 5 meters deep is to be used as a radio telescope. Where should the receiver be placed?
23. The eccentricity of Pluto's orbit is 0.25. What is the ratio of the length to width of this orbit?
24. A comet has an orbit 20 times as long as it is wide. What is the eccentricity of the orbit?

★25. Prove the reflecting property of the ellipse: light originating at one focus converges at the other (*Hint*: Use implicit differentiation.)

★26. A planet travels around its sun on the polar path $r = 1/(2 + \cos\theta)$, the sun at the origin.
   (a) Verify that the path is an ellipse by changing to (x, y) coordinates.
   (b) Compute the *perihelion* distance (minimum distance from the sun to the planet).

14.2 Translation and Rotation of Axes

Whatever their position or orientation, conics are still described by quadratic equations.

In Section R.5 we studied the shifted parabola: if we move the origin to (p, q), $y = ax^2$ becomes $(y - q) = a(x - p)^2$. We can do the same for the other conic sections:

Shifted Conic Sections

Shifted ellipse: $\dfrac{(x-p)^2}{a^2} + \dfrac{(y-q)^2}{b^2} = 1$ (shifted circle if $a = b$).

Shifted hyperbola: $\dfrac{(x-p)^2}{a^2} - \dfrac{(y-q)^2}{b^2} = 1$ (horizontal);

$\dfrac{(y-q)^2}{b^2} - \dfrac{(x-p)^2}{a^2} = 1$ (vertical).

Shifted parabola: $y - q = a(x - p)^2$ (vertical);

$x - p = b(y - q)^2$ (horizontal).

Example 1 Graph the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and shifted ellipse $\dfrac{(x-5)^2}{9} + \dfrac{(y-4)^2}{4} = 1$ on the same xy axes.

Solution The graph of $x^2/9 + y^2/4 = 1$ may be found in Fig. 14.1.5. If (x, y) is any point on this graph, then the point $(x + 5, y + 4)$ satisfies the equation $(x-5)^2/9 + (y-4)^2/4 = 1$; thus the graph of $(x-5)^2/9 + (y-4)^2/4 = 1$ is obtained by shifting the original ellipse 5 units to the right and 4 units upward. (See Fig. 14.2.1.) ▲

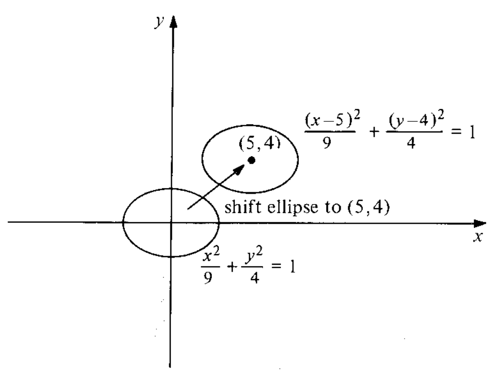


Figure 14.2.1. The graph $(x-5)^2/9 + (y-4)^2/4 = 1$ is an ellipse centered at $(5, 4)$.

Although we referred to the second graph in Example 1 as a "shifted ellipse," it is really just an ellipse, since it satisfies the geometric definition given in Section 14.1. (Can you locate the foci?) To emphasize this, we may introduce new "shifted variables," $X = x - 5$ and $Y = y - 4$, for which the equation becomes $X^2/9 + Y^2/4 = 1$. If we superimpose X and Y axes on our graph as in Fig. 14.2.2, the "shifted" ellipse is now centered at the origin of our new coordinate system. We refer to this process as *translation of axes*.

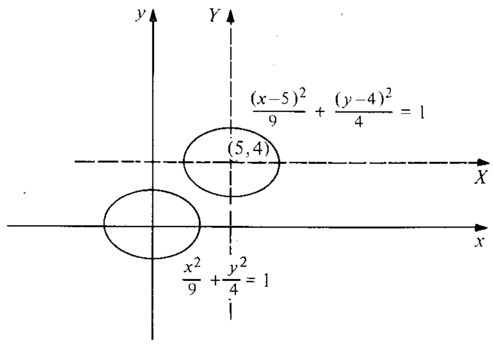


Figure 14.2.2. The ellipse $(x-5)^2/9 + (y-4)^2/4 = 1$ is centered at the origin in a shifted coordinate system.

The importance of translation of axes is that it is possible to bring any equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

into the simpler form

$$AX^2 + CY^2 + G = 0 \tag{2}$$

of a conic by letting $X = x - a$ and $Y = y - b$ for suitable choices of constants a and b. Thus, (1) always describes a shifted conic. The way to find the quantities a and b by which the axes are to be shifted is by completing the square, as was done for circles and parabolas in Chapter R. (Notice that in equation (1) there is no xy term. We shall deal with such terms by means of rotation of axes in the second half of this section.)

Example 2 Sketch the graph of $x^2 - 4y^2 - 2x + 16y = 19$.

Solution We complete the square twice:

$$x^2 - 2x = (x-1)^2 - 1,$$

$$-4y^2 + 16y = -4(y^2 - 4y) = -4\left[(y-2)^2 - 4\right].$$

Thus

$$0 = x^2 - 4y^2 - 2x + 16y - 19 = (x-1)^2 - 1 - 4\left[(y-2)^2 - 4\right] - 19$$
$$= (x-1)^2 - 4(y-2)^2 - 4.$$

Hence our equation is

$$\frac{(x-1)^2}{4} - (y-2)^2 = 1$$

which is the hyperbola $x^2/4 - y^2 = 1$ shifted over to $(1,2)$. (See Fig. 14.2.3.)

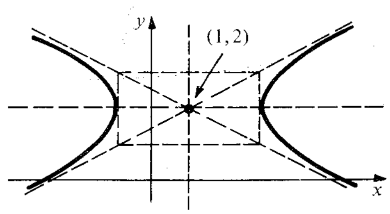


Figure 14.2.3. The hyperbola $x^2 - 4y^2 - 2x + 16y = 19$.

An alternative procedure is to write

$$x^2 - 4y^2 - 2x + 16y - 19 = (x-a)^2 - 4(y-b)^2 + G.$$

Expanding and simplifying, we get

$$-2x + 16y - 19 = -2ax + a^2 + 8by - 4b^2 + G.$$

We find a, b, and G by comparing both sides, which gives $a = 1$, $b = 2$, and $-19 = a^2 - 4b^2 + G$, or $G = -19 - 1 + 16 = -4$. This gives the same answer as above. ▲

Example 3 Sketch the curve $y^2 + x + 3y - 8 = 0$.

Solution Completing the square, we get $y^2 + 3y = (y + \frac{3}{2})^2 - \frac{9}{4}$, so that $y^2 + x + 3y - 8 = 0$ becomes $(y + \frac{3}{2})^2 + x - \frac{41}{4} = 0$; that is, $x - \frac{41}{4} = -(y + \frac{3}{2})^2$. This is a shifted parabola opening to the left, as in Fig. 14.2.4. ▲

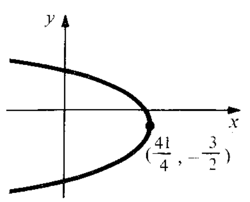


Figure 14.2.4. The parabola $y^2 + x + 3y - 8 = 0$.

We next turn our attention to rotation of axes. The geometric definitions of the ellipse, hyperbola, and parabola given in Section 14.1 do not depend on how these figures are shifted or oriented with respect to the coordinate axes. In the preceding examples we saw how the equations are changed when the coordinate axes are shifted; now we examine how they are changed when the axes are rotated.

In Figure 14.2.5 we have drawn a new set of XY axes which have been rotated by an angle α relative to the old xy axes. The corresponding unit vectors along the axes are denoted $\mathbf{i}, \mathbf{j}$ and $\mathbf{I}, \mathbf{J}$, as shown in the figure.

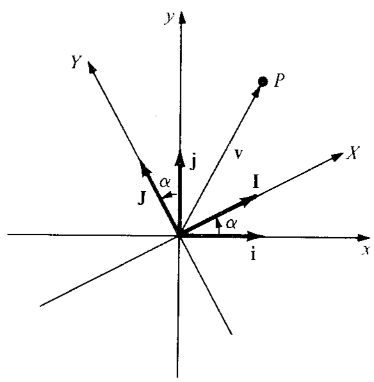


Figure 14.2.5. The XY coordinate system is obtained by rotating the xy system through an angle α.

To understand how to change coordinates from the xy to XY systems, we will use vector methods. Note that as vectors in the plane,

$$\begin{aligned} \mathbf{I} &= \mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha \\ \mathbf{J} &= -\mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha. \end{aligned} \tag{3}$$

Observe that either a direct examination of Fig. 14.2.5 or the fact that $\mathbf{J} = \mathbf{k} \times \mathbf{I}$ can be used to derive the formula for $\mathbf{J}$.

Now consider a point P in the plane and the vector $\mathbf{v}$ from O to P. The coordinates of P relative to the two systems are denoted (x, y) and (X, Y), respectively, and satisfy

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} = X\mathbf{I} + Y\mathbf{J}. \tag{4}$$

Substituting (3) into (4), we get

$$x\mathbf{i} + y\mathbf{j} = X(\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha) + Y(-\mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha).$$

Comparing coefficients of $\mathbf{i}$ and $\mathbf{j}$ on both sides gives

$$\begin{aligned} x &= X\cos\alpha - Y\sin\alpha \\ y &= X\sin\alpha + Y\cos\alpha. \end{aligned} \tag{5}$$

To solve these equations for X, Y in terms of x, y, we notice that the roles of (X, Y) and (x, y) are reversed if we change α to $-\alpha$. In other words, the xy

axes are obtained from the XY axes by a rotation through an angle $-\alpha$. Thus we can interchange (x, y) and (X, Y) in (5) if we switch the sign of α:

$$\begin{aligned} X &= x\cos\alpha + y\sin\alpha \\ Y &= -x\sin\alpha + y\cos\alpha. \end{aligned} \tag{6}$$

This conclusion can be verified by substituting (6) into (5) or (5) into (6).

Example 4 Write down the change of coordinates corresponding to a rotation of 30°.

Solution We have $\cos 30° = \sqrt{3}/2$ and $\sin 30° = 1/2$, so (5) and (6) become

$$x = \frac{\sqrt{3}}{2}X - \frac{1}{2}Y,$$

$$y = \frac{1}{2}X + \frac{\sqrt{3}}{2}Y$$

and $X = \dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}y,$

$$Y = -\frac{1}{2}x + \frac{\sqrt{3}}{2}y. \blacktriangle$$

Now suppose we have a rotated conic, such as the ellipse shown in Fig. 14.2.6. In the XY coordinate system, such a conic has the form given by (1):

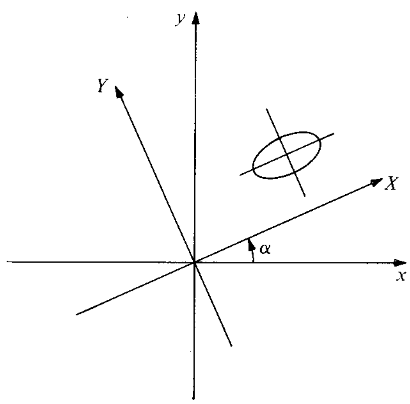


Figure 14.2.6. The conic is aligned with the rotated coordinate system (X, Y) but is rotated relative to the (x, y) coordinate system.

$$\bar{A}X^2 + \bar{C}Y^2 + \bar{D}X + \bar{E}Y + \bar{F} = 0. \tag{7}$$

Substituting (6) into (7) gives

$$\bar{A}(x\cos\alpha + y\sin\alpha)^2 + \bar{C}(-x\sin\alpha + y\cos\alpha)^2$$
$$+ \bar{D}(x\cos\alpha + y\sin\alpha) + \bar{E}(-x\sin\alpha + y\cos\alpha) + \bar{F} = 0.$$

Expanding, we find

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{8}$$

where

$$\left.\begin{aligned} A &= \bar{A}\cos^2\alpha + \bar{C}\sin^2\alpha, \\ B &= (\bar{A} - \bar{C})\cdot 2\cos\alpha\sin\alpha = (\bar{A} - \bar{C})\sin 2\alpha, \\ C &= \bar{A}\sin^2\alpha + \bar{C}\cos^2\alpha, \\ D &= \bar{D}\cos\alpha - \bar{E}\sin\alpha, \\ E &= \bar{D}\sin\alpha + \bar{E}\cos\alpha, \\ F &= \bar{F}. \end{aligned}\right\} \tag{9}$$

Notice the introduction of the xy term in (8). If we are *given* an equation of the form (8), we may determine the type of conic it is and the rotation angle α by finding the rotated form (7). To accomplish this, we notice from (9) that

$$\begin{aligned} A - C &= \bar{A}(\cos^2\alpha - \sin^2\alpha) - \bar{C}(\cos^2\alpha - \sin^2\alpha) \\ &= (\bar{A} - \bar{C})\cos 2\alpha. \end{aligned}$$

Therefore,

$$B = (\bar{A} - \bar{C})\sin 2\alpha = (A - C)\tan 2\alpha.$$

Thus,

$$\tan 2\alpha = \frac{B}{A - C} \tag{10}$$

($\alpha = 45°$ if $A = C$). Equation (10) enables one to solve for α given equation (8).

Equation (8) will describe an ellipse only when (7) does, i.e., when $\bar{A}$ and $\bar{C}$ have the same sign, or $\bar{A}\bar{C} > 0$. To recognize this condition directly from (8), we use (9) to obtain

$$\begin{aligned} AC &= (\bar{A}\cos^2\alpha + \bar{C}\sin^2\alpha)(\bar{A}\sin^2\alpha + \bar{C}\cos^2\alpha) \\ &= (\bar{A}^2 + \bar{C}^2)\cos^2\alpha\sin^2\alpha + \bar{A}\bar{C}(\cos^4\alpha + \sin^4\alpha). \end{aligned}$$

However, $B = (\bar{A} - \bar{C})2\cos\alpha\sin\alpha$, so $\frac{1}{4}B^2 = (\bar{A}^2 + \bar{C}^2 - 2\bar{A}\bar{C})(\cos^2\alpha\sin^2\alpha)$, and thus

$$\begin{aligned} AC - \tfrac{1}{4}B^2 &= \bar{A}\bar{C}(\cos^4\alpha + \sin^4\alpha + 2\cos^2\alpha\sin^2\alpha) \\ &= \bar{A}\bar{C}(\cos^2\alpha + \sin^2\alpha)^2 = \bar{A}\bar{C}. \end{aligned}$$

Thus (8) is an ellipse if $AC - \frac{1}{4}B^2 > 0$; i.e., $B^2 - 4AC < 0$. The other conics are identified in a similar way, as described in the following box.

Rotation of Axes

The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

(with A, B, and C not all zero) is a conic; it is

an ellipse if $B^2 - 4AC < 0$;
a hyperbola if $B^2 - 4AC > 0$;
a parabola if $B^2 - 4AC = 0$.

To graph this conic, proceed as follows:

1. Find α from $\alpha = \frac{1}{2}\tan^{-1}\left[\dfrac{B}{A - C}\right]$.
2. Let $x = X\cos\alpha - Y\sin\alpha$, $y = X\sin\alpha + Y\cos\alpha$, and substitute into the given equation. You will get an equation of the form
$$\bar{A}X^2 + \bar{C}Y^2 + \bar{D}X + \bar{E}Y + \bar{F} = 0.$$
3. This is a shifted conic in XY coordinates which may be plotted by completing the square (as in Examples 2 and 3).
4. Place your conic in the XY coordinates in the xy plane by rotating the axes through an angle α, as in Figure 14.2.6.

Example 5 What type of conic is given by $x^2 + 3y^2 - 2\sqrt{3}\,xy + 2\sqrt{3}\,x + 2y = 0$?

Solution This is a rotated conic because it has an xy term. Here $A = 1$, $B = -2\sqrt{3}$, $C = 3$, $D = 2\sqrt{3}$, and $E = 2$. To find the type, we compute the quantity $B^2 - 4AC = 4\cdot 3 - 4\cdot 3 = 0$, so this is a parabola. ▲

Example 6 Sketch the graph of the conic in Example 5.

Solution We follow the four steps in the preceding box:

$$1.\ \alpha = \tfrac{1}{2}\tan^{-1}\left[B/(A-C)\right] = \tfrac{1}{2}\tan^{-1}\left[-2\sqrt{3}/(1-3)\right] = \tfrac{1}{2}\tan^{-1}\sqrt{3} = \pi/6.$$

Thus $\alpha = \pi/6$ or $30°$.

2. As in Example 4, we have

$$x = \frac{\sqrt{3}}{2}X - \frac{1}{2}Y \quad\text{and}\quad y = \frac{1}{2}X + \frac{\sqrt{3}}{2}Y.$$

Substituting into $x^2 + 3y^2 - 2\sqrt{3}\,xy + 2\sqrt{3}\,x + 2y = 0$, we get

$$\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right)^2 + 3\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right)^2$$
$$-2\sqrt{3}\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right)\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right)$$
$$+2\sqrt{3}\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right) + 2\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right) = 0.$$

Expanding, we get

$$\left(\frac{3}{4}X^2 + \frac{1}{4}Y^2 - \frac{\sqrt{3}}{2}XY\right) + \left(\frac{3}{4}X^2 + \frac{9}{4}Y^2 + \frac{3\sqrt{3}}{2}XY\right)$$
$$-2\sqrt{3}\left(\frac{\sqrt{3}}{4}X^2 + \frac{1}{2}XY - \frac{\sqrt{3}}{4}Y^2\right) + 3X - \sqrt{3}\,Y + X + \sqrt{3}\,Y = 0$$

which simplifies to $4Y^2 + 4X = 0$ or $X = -Y^2$.

3. The conic $X = -Y^2$ is a parabola opening to the left in XY coordinates.

4. We plot the graph in Fig. 14.2.7. ▲

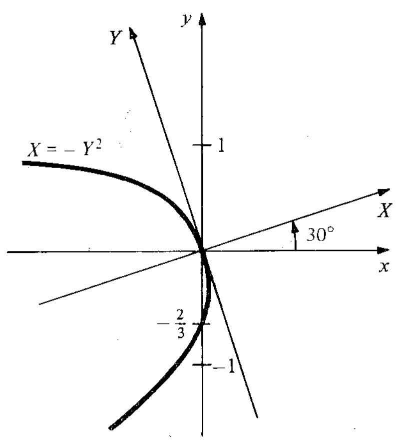


Figure 14.2.7. The graph of $x^2 + 3y^2 - 2\sqrt{3}\,xy + 2\sqrt{3}\,x + 2y = 0$.

Example 7 Sketch the graph of $3x^2 + 3y^2 - 10xy + 18\sqrt{2}\,x - 14\sqrt{2}\,y + 38 = 0$.

Solution Let us first determine the type of conic. Here $B^2 - 4AC = 100 - 4 \cdot 3 \cdot 3 = 100 - 36 = 64 > 0$, so it is a hyperbola.

1. $\alpha = \frac{1}{2}\tan^{-1}[B/(A-C)] = \frac{1}{2}\tan^{-1}\infty = \pi/4$ or $45°$.
2. $x = (1/\sqrt{2})(X - Y)$, $y = (1/\sqrt{2})(X + Y)$; substituting and simplifying, one arrives at

$$X^2 - 4Y^2 - 2X + 16Y - 19 = 0.$$

3. This is the hyperbola in Fig. 14.2.3.
4. See Fig. 14.2.8. ▲

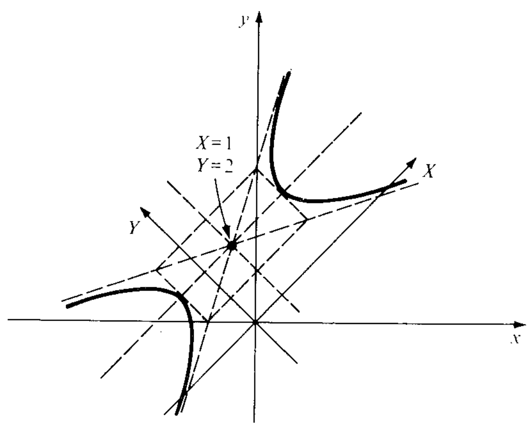


Figure 14.2.8. The graph of $3x^2 + 3y^2 - 10xy + 18\sqrt{2}\,x - 14\sqrt{2}\,y + 38 = 0$.

Exercises for Section 14.2

In Exercises 1–4, graph the conics and shifted conics on the same xy axes.

1. $y = -x^2$, $y - 2 = -(x+1)^2$.
2. $x^2 - y^2 = 1$, $(x-2)^2 - (y+3)^2 = 1$.
3. $x^2 + y^2 = 4$, $(x+3)^2 + (y-8)^2 = 4$.
4. $x^2/9 + y^2/16 = 1$, $(x-1)^2/9 + (y-2)^2/16 = 1$.

Identify the equations in Exercises 5–10 as shifted conic sections and sketch their graphs.

5. $x^2 + y^2 - 2x = 0$
6. $x^2 + 4y^2 - 8y = 0$
7. $2x^2 + 4y^2 - 6y = 8$
8. $x^2 + 2x + y^2 - 2y = 2$
9. $x^2 + 2x - y^2 - 2y = 1$
10. $3x^2 - 6x + y - 7 = 0$

In Exercises 11–14, write down the transformation of coordinates corresponding to a rotation through the given angle.

11. $60°$
12. $\pi/4$
13. $15°$
14. $2\pi/3$

In Exercises 15–18, determine the type of conic.

15. $xy = 2$.
16. $x^2 + xy + y^2 = 4$.
17. $\frac{19}{4}x^2 + \frac{43}{12}y^2 + \frac{7\sqrt{3}}{6}xy = 48$.
18. $3x^2 + 3y^2 - 2xy - \frac{6}{\sqrt{2}}x - \frac{6}{\sqrt{2}}y = 8$.

Sketch the graphs of the conics in Exercises 19–22.

19. The conic in Exercise 15.
20. The conic in Exercise 16.
21. The conic in Exercise 17.
22. The conic in Exercise 18.

Find the equations of the curves described in Exercises 23–28.

23. The circle with center $(2,3)$ and radius 5.
24. The ellipse consisting of those points whose distances from $(0,0)$ and $(2,0)$ sum to 8.
25. The parabola with vertex at $(1,0)$ and passing through $(0,1)$ and $(2,1)$.
26. The circle passing through $(0,0)$, $(1,\frac{1}{2})$, and $(2,0)$.
27. The hyperbola with foci at $(0,-1)$ and $(0,3)$ and passing through $(0,2)$.
28. The ellipse with x intercept $(1,0)$ and foci $(0,0)$ and $(0,2)$.

29. Find the equation of the conic in Exercise 8 rotated through $\pi/3$ radians.
30. Find the equation of the conic in Exercise 9 rotated through $45°$.
31. Show that $A + C$ is unchanged under a rotation or translation of axes.
32. Show that $D^2 + E^2$ is unchanged under a rotation of axis, but not under translation.

★33. Show that if $B^2 - 4AC < 0$, the area of the ellipse $Ax^2 + Bxy + Cy^2 = 1$ is $2\pi/\sqrt{4AC - B^2}$.

14.3 Functions, Graphs, and Level Surfaces

The graph of a function of two variables is a surface in space.

The daily weather map of North America shows the temperatures of various locations at a fixed time. If we let x be the longitude and y the latitude of a point, the temperature T at that point may be thought of as a function of the *pair* (x, y). Weather maps often contain curves through points with the same temperature. These curves, called *isotherms*, help us to visualize the temperature function; for instance, in Fig. 14.3.1 they help us to locate a hot spot in the southwestern U.S. and a cold spot in Canada.

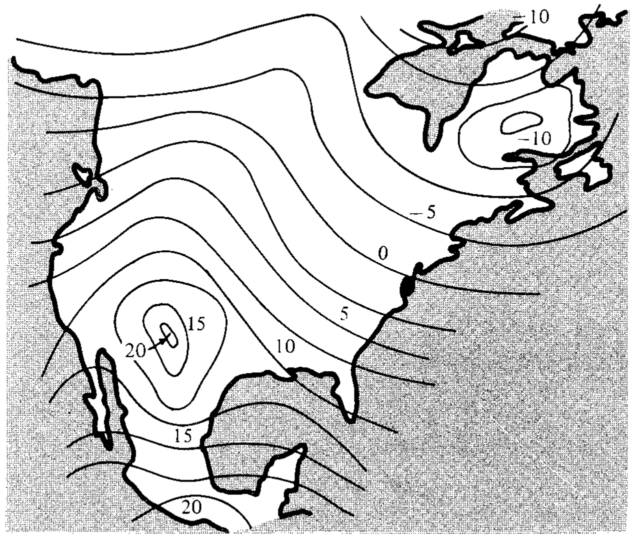


Figure 14.3.1. Isotherms are lines of constant temperature (in degrees Celsius).

Functions of two variables arise in many other contexts as well. For instance, in topography the height h of the land depends on the two coordinates that give the location. The reaction rate σ of two chemicals A and B depends on their concentrations a and b. The altitude α of the sun in the sky on June 21 depends on the latitude l and the number of hours t after midnight.

Many quantities depend on more than two variables. For instance, the temperature can be regarded as a function of the time t as well as of x and y to give a function of three variables. (Try to imagine visualizing this function by watching the isotherms move and wiggle as the day progresses.) The rate of a reaction involving 10 chemicals is a function of 10 variables.

In this book we limit our attention to functions of two and three variables. Readers who have mastered this material can construct for themselves, or find in a more advanced work,[1] the generalizations of the concepts presented here to functions of four and more variables.

[1] See, for example, J. Marsden and A. Tromba, *Vector Calculus*, Second Edition, W. H. Freeman and Co., 1980.

The mathematical development of functions of several variables begins with some definitions.

Functions of Two Variables

A *function of two variables* is a rule which assigns a number $f(x, y)$ to each point (x, y) of a domain in the xy plane.

Example 1 Describe the domain of $f(x, y) = x/(x^2 + y^2)$. Evaluate $f(1,0)$ and $f(1,1)$.

Solution As given, this function is defined as long as $x^2 + y^2 \neq 0$, that is, as long as $(x, y) \neq (0,0)$. We have

$$f(1,0) = \frac{1}{1^2 + 0^2} = 1 \quad \text{and} \quad f(1,1) = \frac{1}{1^2 + 1^2} = \frac{1}{2}. \blacktriangle$$

The Graph of a Function

The *graph* of a function $f(x, y)$ of two variables consists of all points (x, y, z) in space such that (x, y) is in the domain of the function and $z = f(x, y)$.

Some particularly simple graphs can be drawn on the basis of our work in earlier chapters.

Example 2 Sketch the graph of (a) $f(x, y) = x - y + 2$ and (b) $f(x, y) = 3x$.

Solution (a) We recognize $z = x - y + 2$ (that is, $x - y - z + 2 = 0$) as the equation of a plane. Its normal is $(1, -1, -1)$ and it meets the axes at $(-2,0,0)$, $(0,2,0)$, $(0,0,2)$. From this information we sketch its graph in Fig. 14.3.2.
(b) The graph of $f(x, y) = 3x$ is the plane $z = 3x$. It contains the y axis and is shown in Fig. 14.3.3. ▲

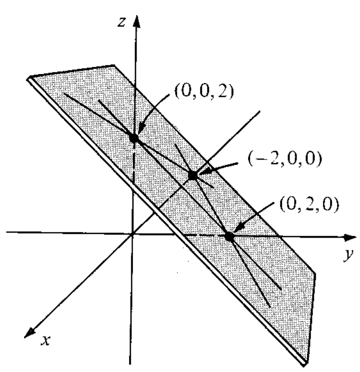


Figure 14.3.2. The graph of $z = x - y + 2$ is a plane.

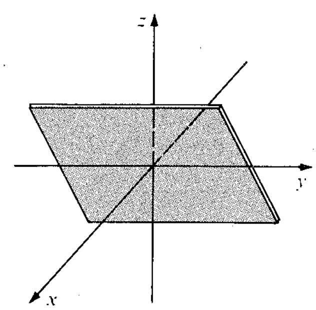


Figure 14.3.3. The graph of $z = 3x$.

Using level curves instead of graphs makes it possible to visualize a function of two variables by a two-dimensional rather than a three-dimensional picture.

Level Curves

Let f be a function of two variables and let c be a constant. The set of all (x, y) in the plane such that $f(x, y) = c$ is called a *level curve* of f (with value c).

Isotherms are just the level curves of a temperature function, and a contour plot of a mountain consists of representative level curves of the height function.

Example 3 Sketch the level curves with values $-1, 0, 1$ for $f(x, y) = x - y + 2$.

Solution The level curve with value -1 is obtained by setting $f(x, y) = -1$; that is,

$$x - y + 2 = -1, \qquad \text{that is,} \qquad x - y + 3 = 0,$$

which is a straight line in the plane (see Fig. 14.3.4). The level curve with value zero is the line

$$x - y + 2 = 0,$$

and the curve with value 1 is the line

$$x - y + 2 = 1, \qquad \text{that is,} \qquad x - y + 1 = 0. \blacktriangle$$

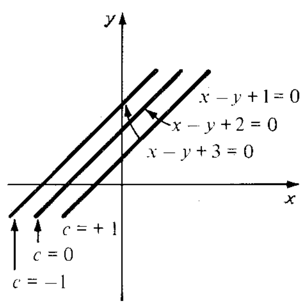


Figure 14.3.4. Three level curves of the function $f(x, y) = x - y + 2$.

Example 4 How is the intersection of the plane $z = c$ with the graph of $f(x, y)$ related to the level curves of f? Sketch.

Solution The intersection of the plane $z = c$ and the graph of f consists of the points (x, y, c) in space such that $f(x, y) = c$. This set has the same shape as the level curve with value c, but it is moved from the xy plane up to the plane $z = c$. (See Fig. 14.3.5.) ▲

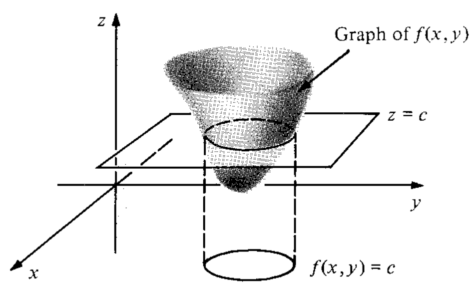


Figure 14.3.5. The level curve of $f(x, y)$ with value c is obtained by finding the intersection of the graph of f with the plane $z = c$ and moving it down to the (x, y) plane.

We turn now to functions of three variables.

Functions of Three Variables

A *function of three variables* is a rule which assigns a number $f(x, y, z)$ to each point (x, y, z) of a domain in (x, y, z) space.

The graph of a function $w = f(x, y, z)$ of three variables would have to lie in four-dimensional space, so we cannot visualize it; but the concept of level curve has a natural extension.

Level Surfaces

Let f be a function of three variables and let c be a constant. The set of all points (x, y, z) in space such that $f(x, y, z) = c$ is called a *level surface* of f (with value c).

Example 5 (a) Let $f(x, y, z) = x - y + z + 2$. Sketch the level surfaces with values 1, 2, 3.
(b) Sketch the level surface of $f(x, y, z) = x^2 + y^2 + z^2 - 8$ with value 1.

Solution (a) In each case we set $f(x, y, z) = c$:

$$c = 1: \quad x - y + z + 2 = 1 \qquad \text{(that is, } x - y + z + 1 = 0\text{)},$$
$$c = 2: \quad x - y + z + 2 = 2 \qquad \text{(that is, } x - y + z = 0\text{)},$$
$$c = 3: \quad x - y + z + 2 = 3 \qquad \text{(that is, } x - y + z - 1 = 0\text{)}.$$

These surfaces are parallel planes and are sketched in Fig. 14.3.6.

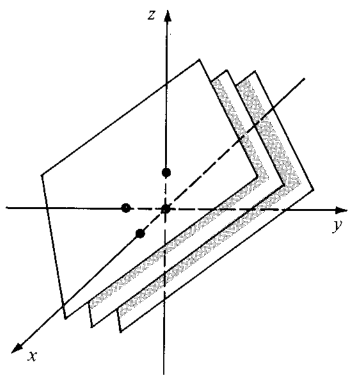


Figure 14.3.6. Three level surfaces of the function $f(x, y, z) = x - y + z + 2$.

(b) The surface $x^2 + y^2 + z^2 - 8 = 1$ (that is, $x^2 + y^2 + z^2 = 9$) is the set of points (x, y, z) whose distance from the origin is $\sqrt{9} = 3$; it is a sphere with radius 3 and center at the origin. (See Fig. 14.3.7.) ▲

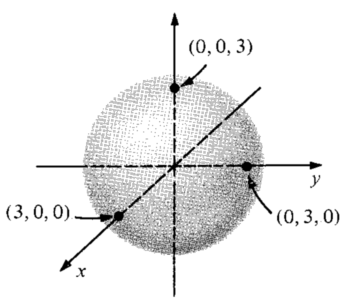


Figure 14.3.7. The level surface of $x^2 + y^2 + z^2 - 8$ with value 1 is a sphere of radius 3.

Plotting surfaces in space is usually more difficult than plotting curves in the plane. It is rare that plotting a few points on a surface will give us enough information to sketch the surface. Instead we often plot several *curves* on the surface and then interpolate between the curves. This technique, called the *method of sections*, is useful for plotting surfaces in space, whether they be graphs of functions of two variables or level surfaces of functions of three variables. The idea behind the method of sections is to obtain a picture of the surface in space by looking at its slices by planes parallel to one of the coordinate planes. For instance, for a graph $z = f(x, y)$ the section $z = c$ is illustrated in Fig. 14.3.8.

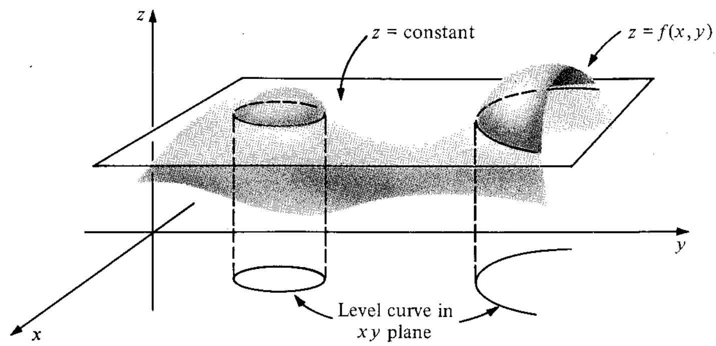


Figure 14.3.8. The section $z = c$ of the graph $z = f(x, y)$.

Example 6 Sketch the surfaces in xyz space given by (a) $z = -y^2$ and (b) $x^2 + y^2 = 25$.

Solution (a) Since x is missing, all sections $x =$ constant look the same; they are copies of the parabola $z = -y^2$. Thus we draw the parabola $z = -y^2$ in the yz plane and extend it parallel to the x axis as shown in Fig. 14.3.9. The surface is called a *parabolic cylinder*.

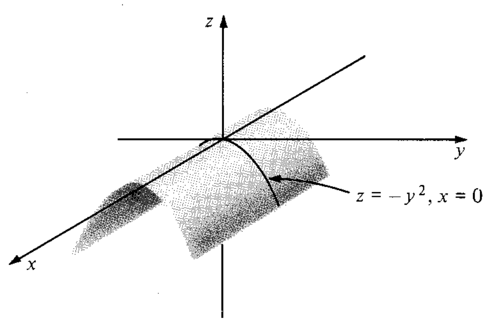


Figure 14.3.9. The graph $z = -y^2$ is a parabolic cylinder.

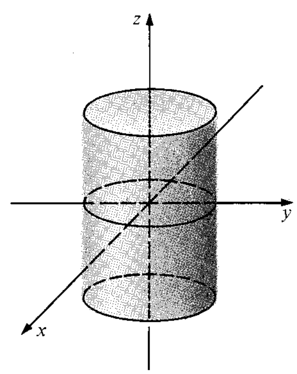


Figure 14.3.10. The graph $x^2 + y^2 = 25$ is a right circular cylinder.

(b) The variable z does not occur in the equation, so the surface is a cylinder parallel to the z axis. Its cross section is the plane curve $x^2 + y^2 = 25$, which is a circle of radius 5, so the surface is a right circular cylinder, as shown in Fig. 14.3.10. ▲

Example 7 (a) Sketch the graph of $f(x, y) = x^2 + y^2$ (this graph is called a *paraboloid of revolution*). (b) Sketch the surface $z = x^2 + y^2 - 4x - 6y + 13$. [*Hint:* Complete the square.]

Solution (a) If we set $z =$ constant, we get $x^2 + y^2 = c$, a circle. Taking $c = 1^2, 2^2, 3^2, 4^2$, we get circles of radius 1, 2, 3, and 4. These are placed on the planes $z = 1^2 = 1$, $z = 2^2 = 4$, $z = 3^2 = 9$, and $z = 4^2 = 16$ to give the graph shown in Fig. 14.3.11.

If we set $x = 0$, we obtain the parabola $z = y^2$; if we set $y = 0$, we obtain the parabola $z = x^2$. The graph is symmetric about the z axis since z depends only on $r = \sqrt{x^2 + y^2}$.

(b) Completing the square, we write $z = x^2 + y^2 - 4x - 6y + 13$ as

$$\begin{aligned} z &= x^2 - 4x + \quad y^2 - 6y \quad + 13 \\ &= x^2 - 4x + 4 + y^2 - 6y + 9 + 13 - 13 \\ &= (x-2)^2 + (y-3)^2. \end{aligned}$$

The level surface for value c is thus the circle $(x-2)^2 + (y-3)^2 = c$ with center $(2,3)$ and radius $\sqrt{c}$. Comparing this result with (a) we find that the surface is again a paraboloid of revolution, with its axis shifted to the line $(x, y) = (2,3)$. (See Fig. 14.3.12.) ▲

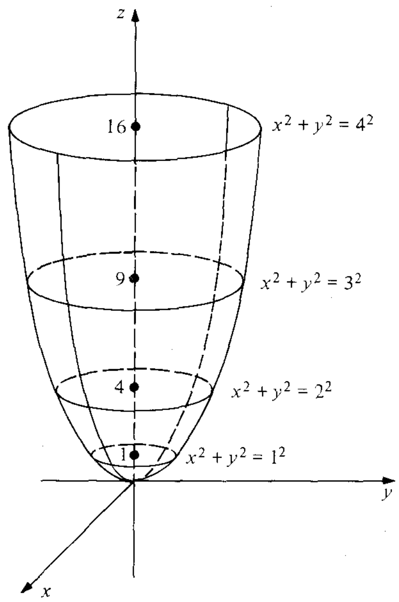


Figure 14.3.11. The sections of the graph $z = x^2 + y^2$ by planes $z = c$ are circles.

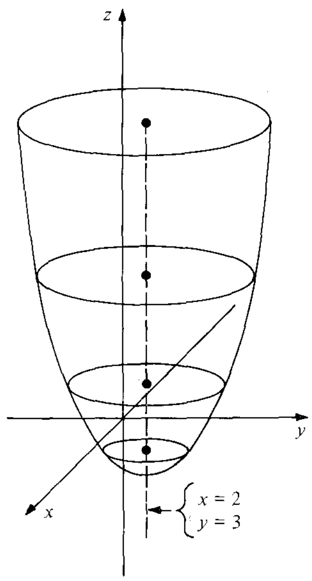


Figure 14.3.12. The graph $z = x^2 + y^2 - 4x - 6y + 13$ is a shifted paraboloid of revolution.

Plotting Surfaces: Methods of Sections

1. Note any symmetries of the graph.
2. See if any variables x, y, or z are missing from the equation. If so, the surface is a "cylinder" parallel to the axis of the missing variable, and its cross section is the curve in the other variables (see Example 6).
3. If the surface is a graph $z = f(x, y)$, find the level curves $f(x, y) = c$ for various convenient values of c and draw these curves on the planes $z = c$. Smoothly join these curves with a surface in space. Draw the curves obtained by setting $x = 0$ and $y = 0$ or other convenient values to help clarify the picture.
4. If the surface has the form $F(x, y, z) = c$, then either:
   (a) Solve for one of the variables in terms of the other two and use step 2 if it is convenient to do so.
   (b) Set x equal to various constant values to obtain curves in y and z; draw these curves on the corresponding $x =$ constant planes. Repeat with $y =$ constant or $z =$ constant or both. Fill in the curves obtained with a surface.

In the next section, we will use our knowledge of conic sections to plot the graphs of more complicated quadratic functions.

The computer can help us graph surfaces that may be tedious or impossible to plot by hand. The computer draws the graph either by drawing sections perpendicular to the x and y axes or by sections perpendicular to the z axis—that is, level curves lifted to the graph. When this is done for the function $z = x^2 + y^2$ (Example 7), Figs. 14.3.13(a) and 14.3.13(b) result. (The pointed tips appear because a rectangular domain has been chosen for the function.)[2]

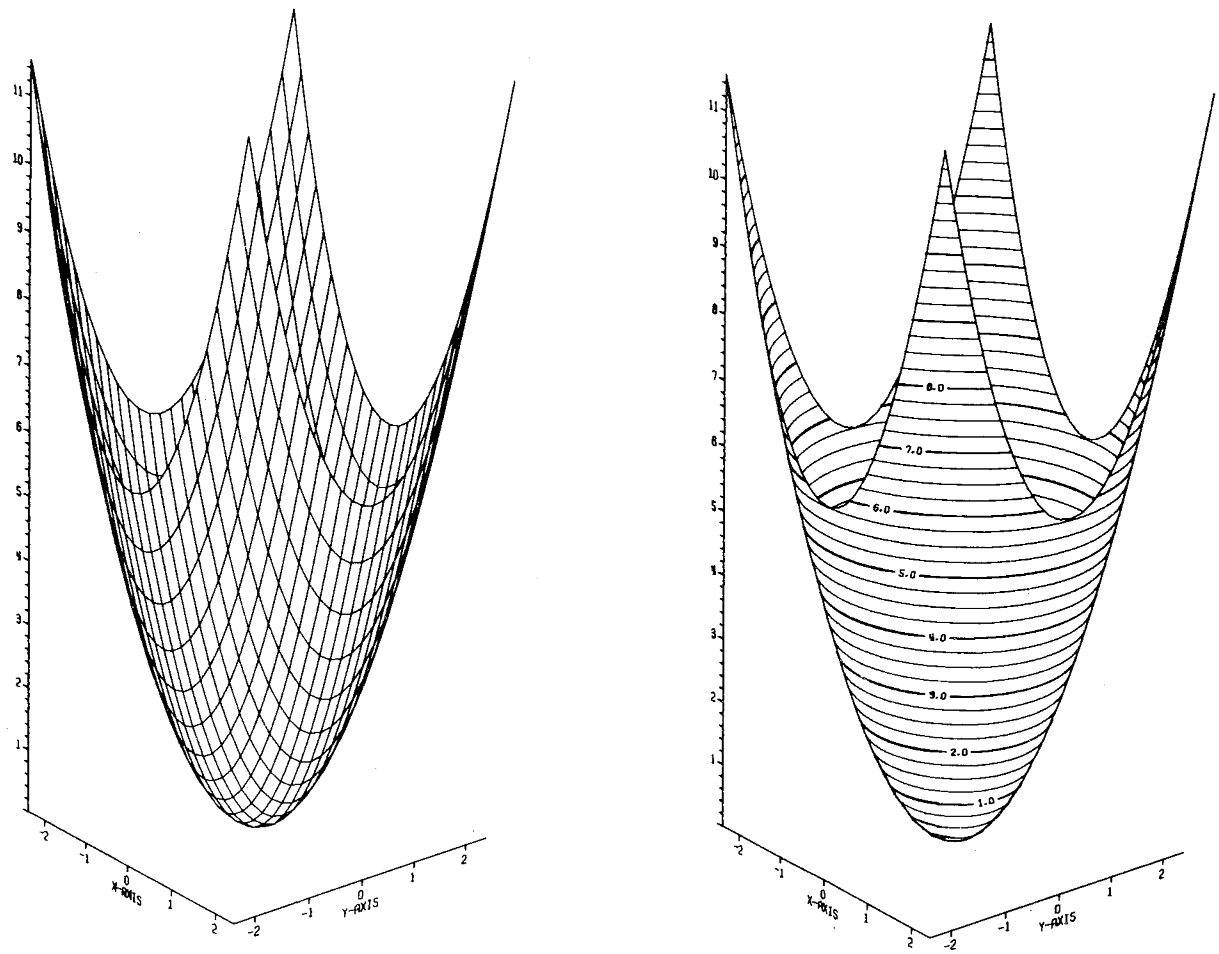


Figure 14.3.13. The graph of $z = x^2 + y^2$ drawn by computer in two ways.

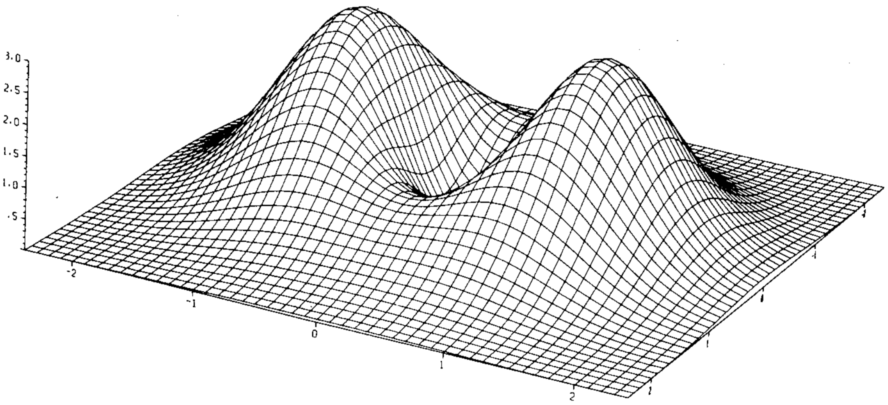
The computer-generated graph in Fig. 14.3.14 shows the function

$$z = (x^2 + 3y^2)e^{1-(x^2+y^2)}.$$

Fig. 14.3.15 shows the level curves of this function in the xy plane, viewed first from an angle and then from above. Study these pictures to help develop your powers of three-dimensional visualization; attempt to reconstruct the graph in your mind by looking at the level curves.

[2] The authors are indebted to Jerry Kazdan for preparing most of the computer-generated graphs in this book.

(a)



(b)

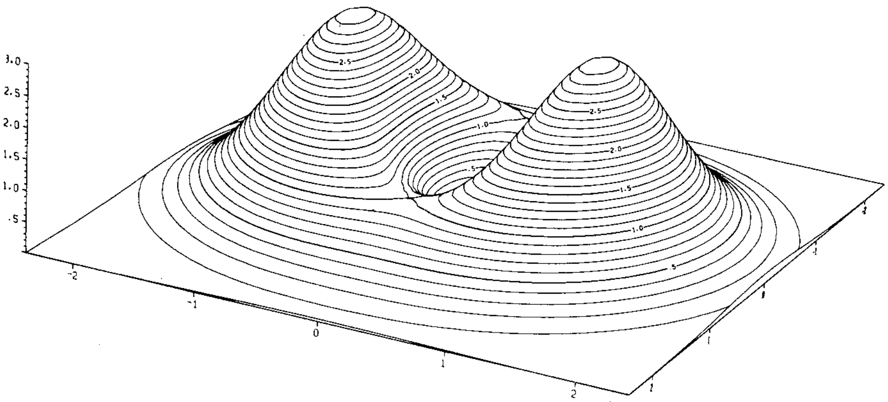
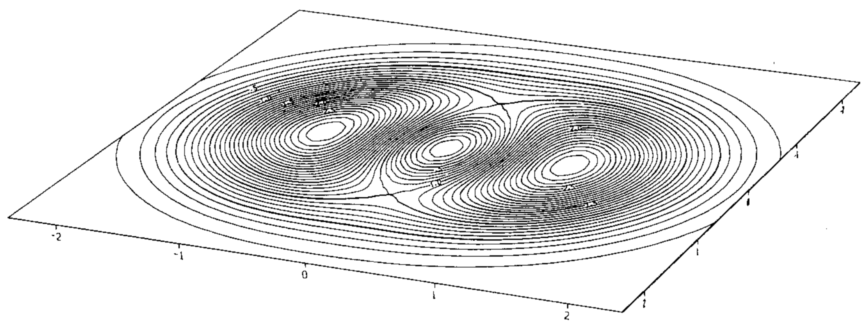


Figure 14.3.14. Computer-generated graphs of $z = (x^2 + 3y^2)e^{1-(x^2+y^2)}$.

(a)



(b)

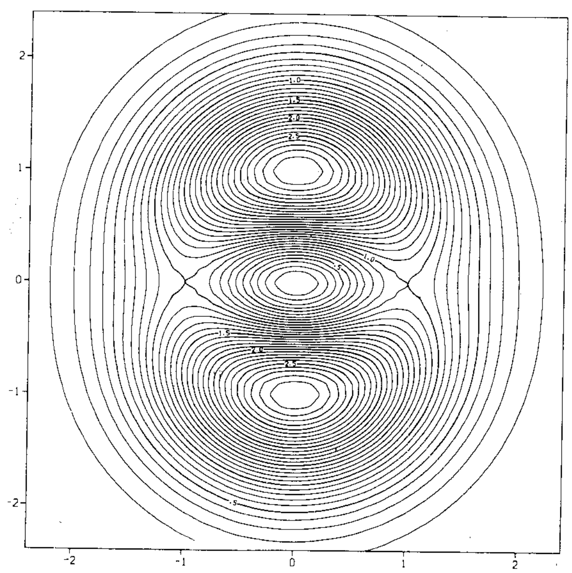


Figure 14.3.15. Level curves for the function $z = (x^2 + 3y^2)e^{1-(x^2+y^2)}$.

Exercises for Section 14.3

In Exercises 1–8, describe the domain of each of the given functions and evaluate the function at the indicated points.

1. $f(x, y) = \dfrac{y}{x}$; $(1,0)$, $(1,1)$.
2. $f(x, y) = \dfrac{x+y}{x-y}$; $(1,-1)$, $(1,0.9)$.
3. $f(x, y) = \dfrac{x+y}{x^2+y^2-1}$; $(1,1)$, $(-1,1)$.
4. $f(x, y) = \dfrac{2xy}{x^2+y^2}$; $(1,1)$, $(-1,1)$.
5. $f(x, y, z) = \dfrac{2x+y-z}{x^2+y^2+z^2-1}$; $(1,1,1)$, $(0,0,2)$.
6. $f(x, y, z) = \dfrac{z}{x^2-4y^2-1}$; $(1,0.5,1)$, $(0.1,0.5,1)$.
7. $f(x, y) = \dfrac{2x-\sin y}{1+\cos x}$; $(0,\pi/4)$, $(\pi/4,\pi)$.
8. $f(x, y) = \dfrac{e^x-e^y}{1+\sin x}$; $(0,1)$; $(\pi/4,-1)$.

Sketch the graphs of the functions in Exercises 9–12.

9. $f(x, y) = 1 - x - y$
10. $f(x, y) = -1 - x - y$
11. $z = x - y$
12. $z = x + 2$

Sketch the level curves for the indicated functions and values in Exercises 13–18.

13. f in Exercise 9, values $1, -1$.
14. f in Exercise 10, values $1, -1$.
15. f in Exercise 3, values $-2, -1, 1, 2$ and describe the level curve for the general value.
16. f in Exercise 4, values $-2, -1, 1, 2$ and describe the level curve for the general value.
17. f in Exercise 2, values $-2, -1, 1, 2$ and describe the level curve for the general value.
18. $f(x, y) = 3^{-1/(x^2+y^2)}$, value $1/e$.

Sketch the level surfaces in Exercises 19–22.

19. $x + y - 2z = 8$
20. $3x - 2y - z = 4$
21. $x^2 + y^2 + z^2 = 4$
22. $x^2 + y^2 - z = 4$

Draw the level curves $f(x, y) = c$—first in the xy plane and then lifted to the graph in space—for the functions and values in Exercises 23–26.

23. $f(x, y) = x^2 + 2y^2$; $c = 0, 1, 2$.
24. $f(x, y) = x^2 - y^2$; $c = -1, 0, 1$.
25. $f(x, y) = x - y^2$; $c = -2, 0, 2$.
26. $f(x, y) = y - x^2$; $c = -1, 0, 1$.

Sketch the surface in space defined by each of the equations in Exercises 27–40.

27. $z = x^2 + 2$
28. $z = |y|$
29. $z^2 + x^2 = 4$
30. $x^2 + y = 2$
31. $z = (x-1)^2 + y^2$
32. $x = -8z^2 + z$
33. $z = x^2 + y^2 - 2x + 8$.
34. $z = 3x^2 + 3y^2 - 6x + 12y + 15$.
35. $z = \sqrt{x^2 + y^2}$
36. $z = \max(|x|, |y|)$. [*Note:* $\max(|x|, |y|)$ is the maximum of $|x|$ and $|y|$.]
37. $z = \sin x$ (the "washboard").
38. $z = 1/(1 + y^2)$.
39. $4x^2 + y^2 + 9z^2 = 1$.
40. $x^2 + 4y^2 + 16z^2 = 1$.
41. Let $f(x, y) = e^{-1/(x^2+y^2)}$; $f(0,0) = 0$.
    (a) Sketch the level curve $f(x, y) = c$ for $c = 0.001$, $c = 0.01$, $c = 0.5$, and $c = 0.9$.
    (b) What happens if c is less than zero or greater than 1?
    (c) Sketch the cross section of the graph in the vertical plane $y = 0$ (that is, the intersection of the graph with the xz plane).
    (d) Argue that this cross section looks the same in any vertical plane through the origin.
    (e) Describe the graph in words and sketch it.
42. The formula
$$z = \frac{2x}{(y - y^{-1})^2 + (2x)^2}$$
appears in the study of steady state motions of a mechanical system with viscous damping subjected to a harmonic external force. The *average power input* by the external force is proportional to the variable z (with proportionality constant $k > 0$). The variable y is the ratio of input frequency to natural frequency. The variable x measures the viscous damping constant.
    (a) Plot z versus y for $x = 0.2, 0.5, 2.0$ on the same axes. Use the range of values $0 < y \leqslant 2.0$.
    (b) The average power input is a maximum when $y = 1$, that is, when the input and natural frequencies are the same. Verify this both graphically and algebraically.
43. The potential difference E between electrolyte solutions separated by a membrane is given by
$$E = \frac{RT}{F}\,\frac{x-y}{x+y}\ln z.$$
(The symbols R, T, F are the universal gas constant, absolute temperature, and Faraday unit, respectively—these are constants. The symbols x and y are the mobilities of Na^+ and Cl^- respectively. The symbol z is c_1/c_2, where c_1 and c_2 are the mean salt (NaCl) concentrations on each side of the membrane.) Assume hereafter that $RT/F = 25$.
    (a) Write the level surface $E = -12$ in the form $z = f(x, y)$.
    (b) In practice, $y = 3x/2$. Plot E versus z in this case.

★44. Describe the behavior, as c varies, of the level curve $f(x, y) = c$ for each of these functions:
    (a) $f(x, y) = x^2 + y^2 + 1$;
    (b) $f(x, y) = 1 - x^2 - y^2$;
    (c) $f(x, y) = x^2 + xy$;
    (d) $f(x, y) = x^3 - x$.

14.4 Quadric Surfaces

Quadric surfaces are defined by quadratic equations in x, y, and z.

The methods of Section 14.3, together with our knowledge of conics, enable us to graph a number of interesting surfaces defined by quadratic equations.

Example 1 Sketch the graph of

$$z = f(x, y) = x^2 - y^2 \qquad \text{(a \textit{hyperbolic paraboloid})}.$$

Solution To visualize this surface, we first draw the level curves $x^2 - y^2 = c$ for $c = 0, \pm 1, \pm 4$. For $c = 0$ we have $y^2 = x^2$ (that is, $y = \pm x$), so this level set consists of two straight lines through the origin. For $c = 1$ the level curve is $x^2 - y^2 = 1$, which is a hyperbola that passes vertically through the x axis at the points $(\pm 1, 0)$ (see Fig. 14.4.1). Similarly, for $c = 4$ the level curve is $x^2/4 - y^2/4 = 1$, the hyperbola passing vertically through the x axis at $(\pm 2, 0)$. For $c = -1$ we obtain the hyperbola $x^2 - y^2 = -1$ passing horizontally through the y axis at $(0, \pm 1)$, and for $c = -4$ the hyperbola through $(0, \pm 2)$ is obtained. These level curves are shown in Fig. 14.4.1. To aid us in visualizing the graph of f, we will also compute two sections. First, set $x = 0$ to obtain $z = -y^2$, a parabola opening downward. Second, setting $y = 0$ gives the parabola $z = x^2$ opening upward.

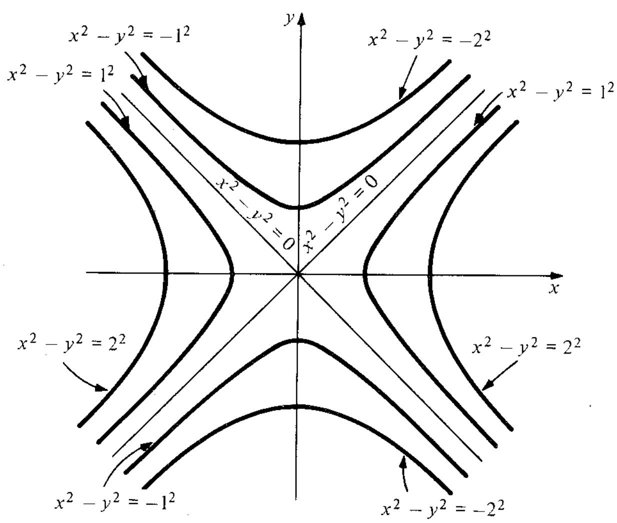


Figure 14.4.1. Some level curves of $f(x) = x^2 - y^2$.

The graph may now be visualized if we lift the level curves to the appropriate heights and smooth out the resulting surface. The placement of the lifted curves is aided by the use of the parabolic sections. This procedure generates the saddle-shaped surface indicated in Fig. 14.4.2. The graph is unchanged under reflection in the yz plane and in the xz plane. When accurately plotted by a computer, this graph has the appearance of Fig. 14.4.3; the level curves are shown in Fig. 14.4.4. (The graph has been rotated by 90° about the z axis.) ▲

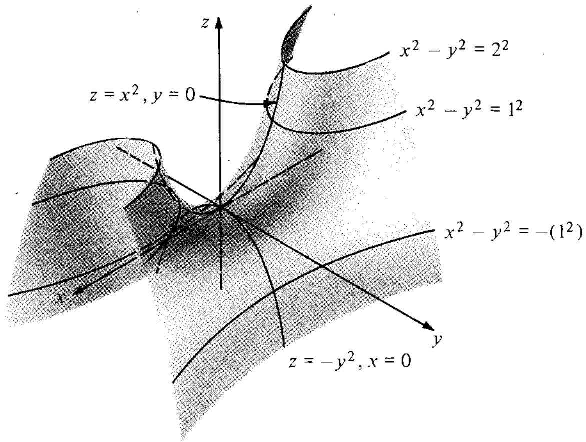
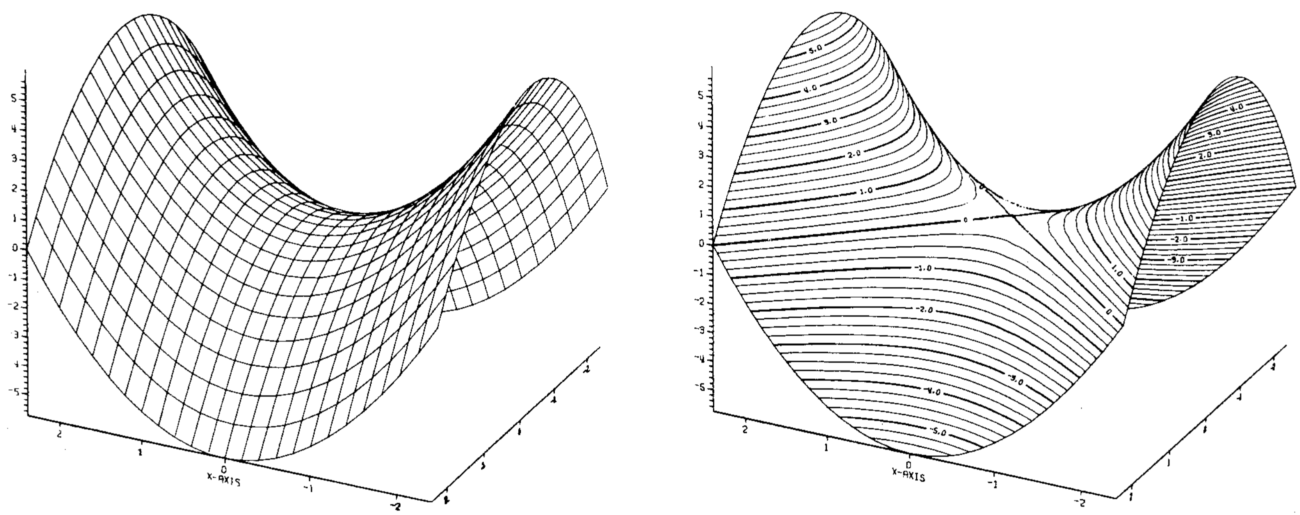


Figure 14.4.2. The graph $z = x^2 - y^2$ is a hyperbolic paraboloid, or "saddle."



(a) (b)

Figure 14.4.3. Computer-generated graph of $z = x^2 - y^2$.

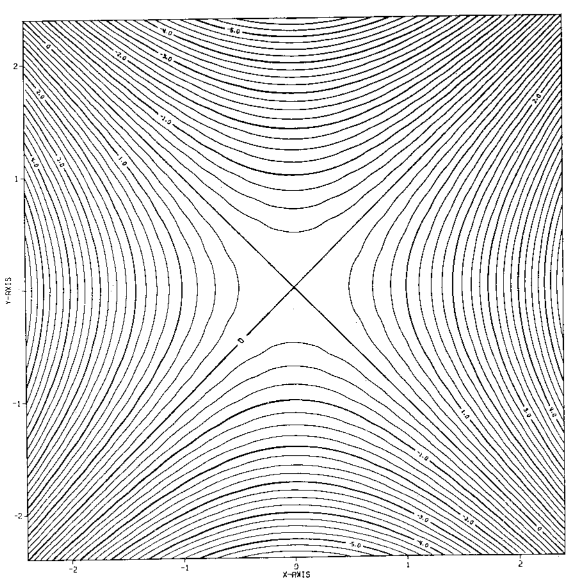


Figure 14.4.4. Level curves of $z = x^2 - y^2$ drawn by computer.

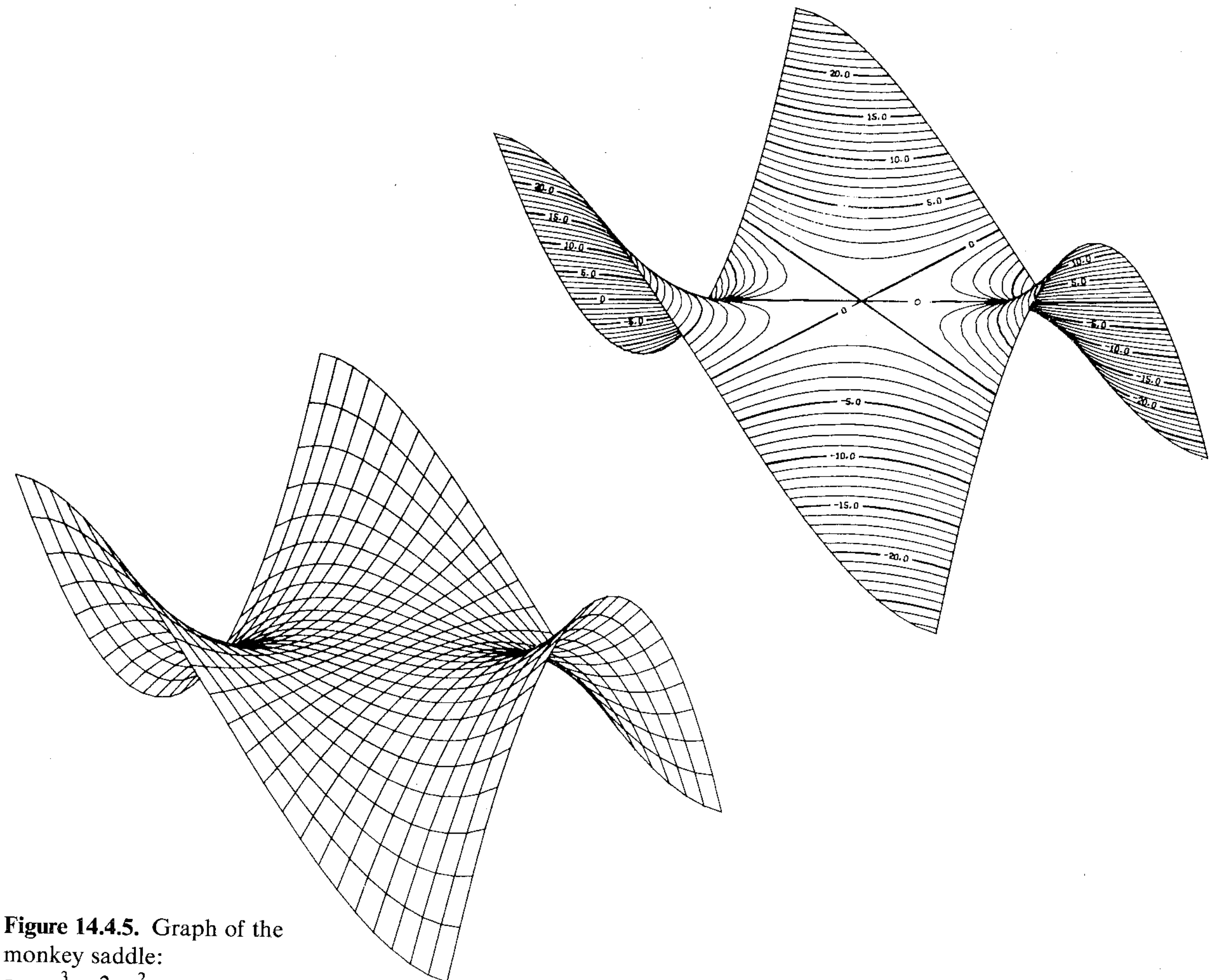


Figure 14.4.5. Graph of the monkey saddle: $z = x^3 - 3xy^2$.

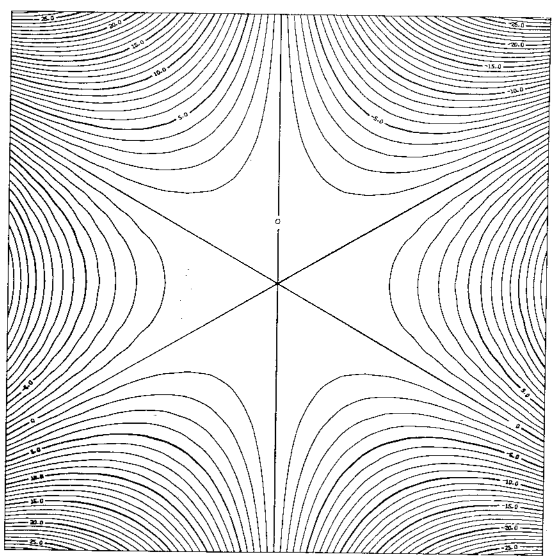


Figure 14.4.6. Level curves for the monkey saddle.

The origin is called a *saddle point* for the function $z = x^2 - y^2$ because of the appearance of the graph. We will return to the study of saddle points in Chapter 16, but it is worth noting another kind of saddle here. Figure 14.4.5 on the preceding page shows the graph of $z = x^3 - 3xy^2$, again plotted by a computer using sections and level curves. The origin now is called a *monkey saddle*, since there are two places for the legs and one for the tail. Figure 14.4.6 shows the contour lines in the plane. Figure 14.4.7 shows the four-legged or *dog saddle*: $z = 4x^3y - 4xy^3$.

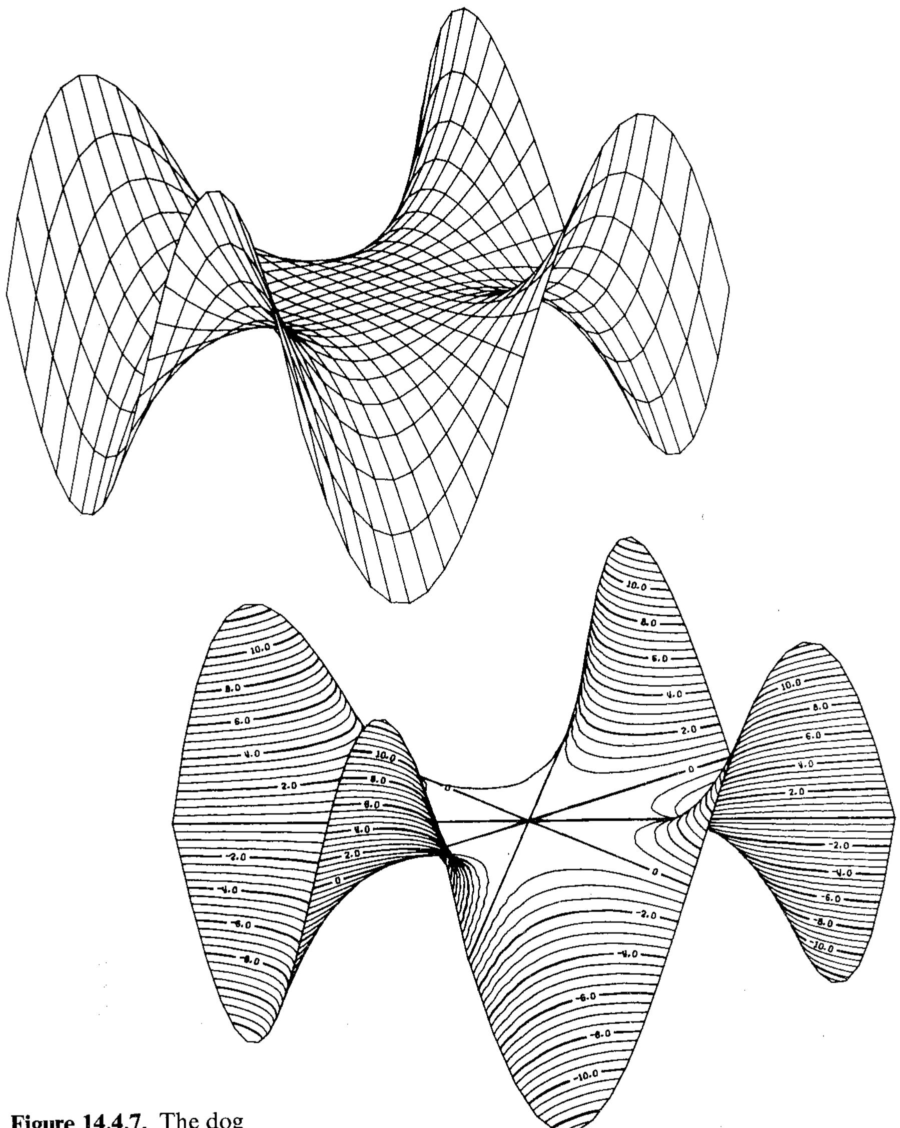


Figure 14.4.7. The dog saddle: $z = 4x^3y - 4xy^3$.

A *quadric surface* is a three-dimensional figure defined by a quadratic equation in three variables:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + m = 0.$$

The quadric surfaces are the three-dimensional versions of the conic sections, studied in Section 14.1, which were defined by quadratic equations in two variables.

Example 2 Particular conic sections can degenerate to points or lines. Similarly, some quadric surfaces can degenerate to points, lines, or planes. Match the sample equations to the appropriate descriptions.

(a) $x^2 + 3y^2 + z^2 = 0$	(1) No points at all
(b) $z^2 = 0$	(2) A single point
(c) $x^2 + y^2 = 0$	(3) A line
(d) $x^2 + y^2 + z^2 + 1 = 0$	(4) One plane
(e) $x^2 - y^2 = 0$	(5) Two planes

Solution Equation (a) matches (2) since only $(0,0,0)$ satisfies the equation; (b) matches (4) since this is the plane $z = 0$; (c) matches (3) since this is the z axis, where $x = 0$ and $y = 0$; (d) matches (1) since a non-negative number added to 1 can never be zero; (e) matches (5) since the equation $x^2 - y^2 = 0$ is equivalent to the two equations $x + y = 0$ or $x - y = 0$, which define two planes. ▲

If one variable is missing from an equation, we only have to find a curve in one plane and then extend it parallel to the axis of the missing variable. This procedure produces a generalized *cylinder*, either *elliptic*, *parabolic*, or *hyperbolic*.

Example 3 Sketch the surface $z = y^2 + 1$.

Solution The intersection of this surface with a plane x = constant is a parabola of the form $z = y^2 + 1$. The surface, a *parabolic cylinder*, is sketched in Fig. 14.4.8. (See also Example 6, Section 14.3). ▲

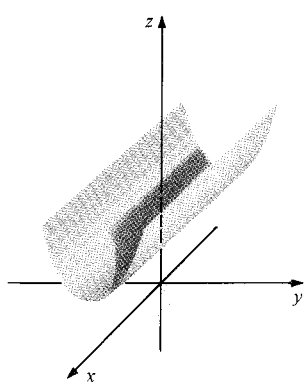


Figure 14.4.8. The surface $z = y^2 + 1$ is a parabolic cylinder.

Example 4 The surface defined by an equation of the form $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$ is called a *hyperboloid of two sheets*. Sketch the surface $x^2 + 4y^2 - z^2 = -4$.

Solution The section by the plane $z = c$ has the equation $x^2 + 4y^2 = c^2 - 4$. This is an ellipse when $|c| > 2$, a point when $c = \pm 2$, and is empty when $|c| < 2$. The section with the xz plane is the hyperbola $x^2 - z^2 = -4$, and the section with the yz plane is the hyperbola $4y^2 - z^2 = -4$. The surface is symmetric with respect to each of the coordinate planes. A sketch is given in Fig. 14.4.9. ▲

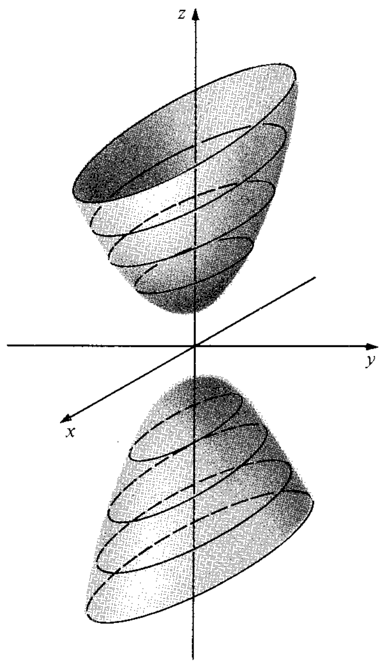


Figure 14.4.9. The surface $x^2 + 4y^2 - z^2 = -4$ is a hyperboloid of two sheets (shown with some of its sections by planes of the form z = constant).

Example 5 The surface defined by an equation of the form $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is called an *ellipsoid*. Sketch the surface $x^2/9 + y^2/16 + z^2 = 1$.

Solution First, let z be constant. Then we get $x^2/9 + y^2/16 = 1 - z^2$. This is an ellipse centered at the origin if $-1 < z < 1$. If $z = 1$, we just get a point $x = 0, y = 0$. Likewise, $(0, 0, -1)$ is on the surface. If $|z| > 1$ there are no (x, y) satisfying the equation.

Setting x = constant or y = constant, we also get ellipses. We must have $|x| \leqslant 3$ and, likewise, $|y| \leqslant 4$. The surface, shaped like a stepped-on football, is easiest to draw if the intersections with the three coordinate planes are drawn first. (See Fig. 14.4.10.) ▲

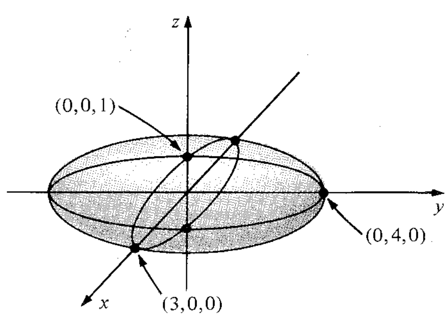


Figure 14.4.10. The surface $(x^2/9) + (y^2/16) + z^2 = 1$ is an ellipsoid.

Example 6 The surface defined by an equation of the form $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ is called a *hyperboloid of one sheet*. Sketch the surface $x^2 + y^2 - z^2 = 4$.

Solution If z is a constant, then $x^2 + y^2 = 4 + z^2$ is a circle. Thus, in any plane parallel to the xy plane, we get a circle. Our job of drawing the surface is simplified if we note right away that the surface is rotationally invariant about the z axis (since z depends only on $r^2 = x^2 + y^2$). Thus we can draw the curve traced by the surface in the yz plane (or xz plane) and revolve it about the z axis. Setting $x = 0$, we get $y^2 - z^2 = 4$, a hyperbola. Hence we get the surface shown in Fig. 14.4.11, a one-sheeted hyperboloid. Since this surface is symmetric about the z axis, it is also called a hyperboloid of revolution. ▲

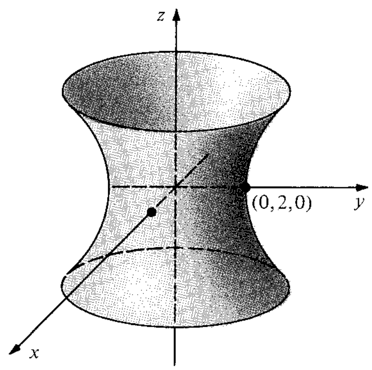


Figure 14.4.11. The surface $x^2 + y^2 - z^2 = 4$ is a one-sheeted hyperboloid of revolution.

The hyperboloid of one sheet has the property that it is *ruled*: that is, the surface is composed of straight lines (see Review Exercise 76). It is therefore easy to make with string models and is useful in architecture. (See Fig. 14.4.12.)

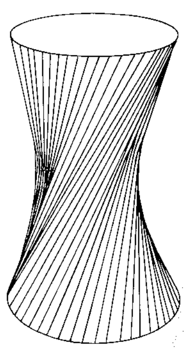


Figure 14.4.12. One can make a hyperboloid with a wire frame and string.

Example 7 Consider the equation $x^2 + y^2 - z^2 = 0$.

(a) What are the horizontal cross sections for $z = \pm 1, \pm 2, \pm 3$?

(b) What are the vertical cross sections for $x = 0$ or $y = 0$? (Sketch and describe.)

(c) Show that this surface is a cone by showing that any straight line through the origin making a 45° angle with the z axis lies in the surface.

(d) Sketch this surface.

Solution (a) Rewriting the equation as $x^2 + y^2 = z^2$ shows that the horizontal cross sections are circles centered around the z axis with radius $|z|$. Therefore, for $z = \pm 1$, ± 2, and ± 3, the cross sections are circles of radius 1, 2, and 3.

(b) When $x = 0$, the equation is $y^2 - z^2 = 0$ or $y^2 = z^2$ or $y = \pm z$, whose graph is two straight lines. When $y = 0$, the equation is $x^2 - z^2 = 0$ or $x = \pm z$, again giving two straight lines.

(c) Any point on a straight line through the origin making a 45° angle with the z axis satisfies $|z|/\sqrt{x^2 + y^2 + z^2} = \cos 45° = 1/\sqrt{2}$. Squaring gives $1/2 = z^2/(x^2 + y^2 + z^2)$, or $x^2 + y^2 + z^2 = 2z^2$, or $x^2 + y^2 - z^2 = 0$, which is the original equation.

(d) Draw a line as described in part (c) and rotate it around the z axis (see Figure 14.4.13). ▲

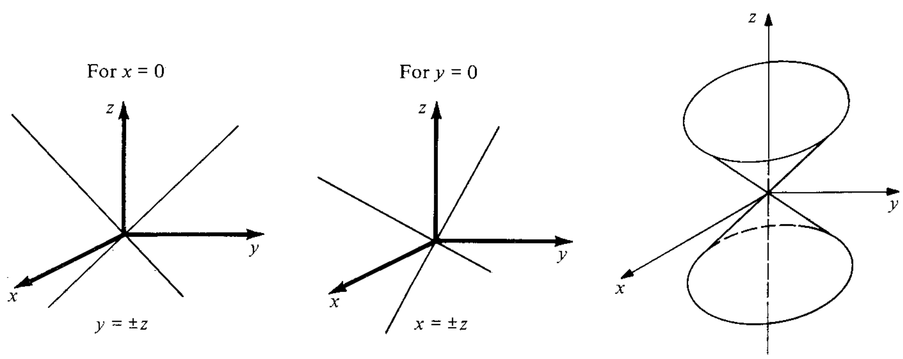


Figure 14.4.13. The cone $x^2 + y^2 - z^2 = 0$.

We now discuss how the conic sections, as introduced in the first section of this chapter, can actually be obtained by slicing a cone.

Example 8 Show that the intersection of the cone $x^2 + y^2 = z^2$ and the plane $y = 1$ is a hyperbola (see Figure 14.4.14).

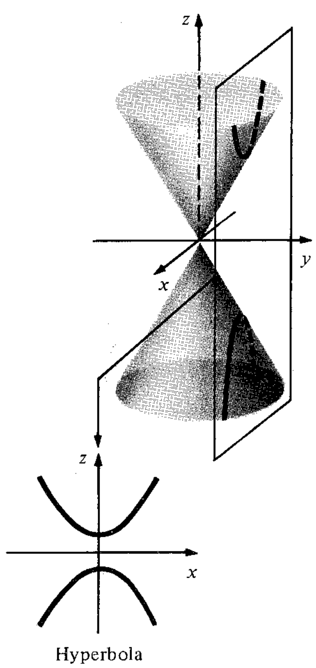


Figure 14.4.14. The intersection of this vertical plane and the cone is a hyperbola.

Solution The intersection of these two surfaces consists of all points (x, y, z) such that $x^2 + y^2 = z^2$ and $y = 1$. We can use x and z as coordinates to describe points in the plane. Thus, eliminating y, we get $x^2 + 1 = z^2$ or $z^2 - x^2 = 1$. From Section 14.1, we recognize this as a hyperbola with foci at $x = 0$, $z = \pm\sqrt{2}$ in the plane $y = 1$, with the branches opening vertically as in the figure. ▲

Example 9 Show that the intersection of the cone $x^2 + y^2 = z^2$ and the plane $z = y - 1$ is a parabola (see Figure 14.4.15).

Solution We introduce rectangular coordinates on the plane as follows. A normal vector to the plane is $\mathbf{n} = (0, 1, -1)$, and so a vector $\mathbf{w} = (a, b, c)$ is parallel to the plane if $0 = \mathbf{n} \cdot \mathbf{w} = b - c$. Two such vectors that are orthogonal and of unit length are

$$\mathbf{u} = \mathbf{i} \quad \text{and} \quad \mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k}).$$

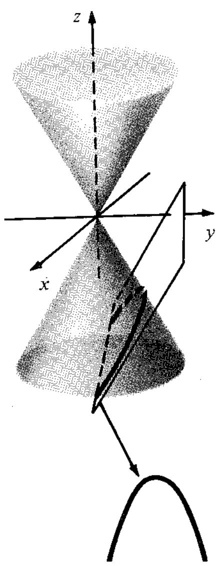


Figure 14.4.15. The intersection of the plane tilted at 45° and the cone is a parabola.

Pick a point on the plane, say $P_0 = (0, 0, -1)$, and write points $P = (x, y, z)$ in the plane in terms of coordinates (ξ, η) by writing

$$\overrightarrow{P_0P} = \xi\mathbf{u} + \eta\mathbf{v}$$

(see Fig. 14.4.16). In terms of (x, y, z), this reads

$$x = \xi, \quad y = \frac{\eta}{\sqrt{2}}, \quad \text{and} \quad z = \frac{\eta}{\sqrt{2}} - 1.$$

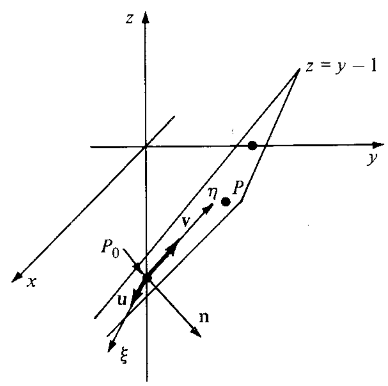


Figure 14.4.16. Coordinates (ξ, η) in the plane $z = y - 1$.

Substitution into $x^2 + y^2 = z^2$ gives

$$\xi^2 + \frac{\eta^2}{2} = \left(\frac{\eta}{\sqrt{2}} - 1\right)^2 = \frac{\eta^2}{2} - \sqrt{2}\eta + 1,$$

or

$$\xi^2 = -\sqrt{2}\,\eta + 1,$$

or

$$\eta = -\frac{1}{\sqrt{2}}\xi^2 + \frac{1}{\sqrt{2}}.$$

This, indeed, is a parabola opening downwards in the $\xi\eta$ plane. ▲

Other sections of the cone can be analyzed in a similar way, and one can prove that a conic will always result (see Exercise 27).

Exercises for Section 14.4

Sketch the surfaces in three-dimensional space defined by each of the equations in Exercises 1–16.

1. $y^2 + z^2 = 1$
2. $x^2 + y^2 = 0$
3. $9x^2 + 4z^2 = 36$
4. $4x^2 + y^2 = 2$
5. $z^2 - 8y^2 = 0$
6. $x^2 - z^2 = 1$
7. $8x^2 + 3z^2 = 0$
8. $x^2 = 4z^2 + 9$
9. $x^2 + y^2 + \frac{z^2}{4} = 1$
10. $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$
11. $x = 2z^2 - y^2$
12. $y = 4x^2 - z^2$
13. $x^2 + 9y^2 - z^2 = 1$
14. $x^2 + y^2 + 4z^2 = 1$
15. $16z^2 = 4x^2 + y^2 + 16$
16. $z^2 + 4y^2 = x^2 + 4$
17. This problem concerns the *hyperbolic paraboloid.* (A surface of this kind was studied in Example 1.) A standard form for the equation is $z = ax^2 - by^2$, with a and b both positive or both negative.
    (a) Sketch the graph of $z = x^2 - 2y^2$.
    (b) Show that $z = xy$ also determines a hyperbolic paraboloid. Sketch some of its level curves.
18. This exercise concerns the *elliptic paraboloid*:
    (a) Sketch the graph of $z = 2x^2 + y^2$.
    (b) Sketch the surface given by
    $$x = -3y^2 - 2z^2.$$
    (c) Consider the equation $z = ax^2 + by^2$, where a and b are both positive or both negative. Describe the horizontal cross sections where z = constant. Describe the sections obtained in the planes $x = 0$ and $y = 0$. What is the section obtained in the vertical plane $x = c$? (The special case in which $a = b$ is a *paraboloid of revolution* as in Example 7(a), Section 14.3.)
19. Sketch the cone $z^2 = 3x^2 + 3y^2$.
20. Sketch the cone $(z-1)^2 = x^2 + y^2$.
21. Sketch the cone $z^2 = x^2 + 2y^2$.
22. Sketch the cone $z^2 = x^2/4 + y^2/9$.
23. Show that the intersection of the cone $x^2 + y^2 = z^2$ and the plane $z = 1$ is a circle.
24. Show that the intersection of the cone $x^2 + y^2 = z^2$ and the plane $2z = y + 1$ is an ellipse.
25. This problem concerns the *elliptic cone*. Consider the equation $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$.
    (a) Describe the horizontal cross sections z = constant.
    (b) Describe the vertical cross sections $x = 0$ and $y = 0$.
    (c) Show that this surface has the property that if it contains the point (x_0, y_0, z_0), then it contains the whole line through $(0,0,0)$ and (x_0, y_0, z_0).

★26. The quadric surfaces may be shifted and rotated in space just as the conic sections may be shifted in the plane. These transformations will produce more complicated cases of the general quadratic equation in three variables. Complete squares to bring the following to one of the standard forms (shifted) and sketch the resulting surfaces:
    (a) $4x^2 + y^2 + 4z^2 + 8x - 4y - 8z + 8 = 0$;
    (b) $2x^2 + 3y^2 - 4z^2 + 4x + 9y - 8z + 10 = 0$.

★27. Show that the intersection of the cone $x^2 + y^2 = z^2$ and any plane is a conic section as follows. Let $\mathbf{u}$ and $\mathbf{v}$ be two orthonormal vectors and P_0 a point. Consider the plane described by points P such that $\overrightarrow{P_0P} = \xi\mathbf{u} + \eta\mathbf{v}$, which introduces rectangular coordinates (ξ, η) in the plane. Substitute an expression for (x, y, z) in terms of (ξ, η) into $x^2 + y^2 = z^2$ and show that the result is a conic section in the $\xi\eta$ plane.

14.5 Cylindrical and Spherical Coordinates

There are two ways to generalize polar coordinates to space.

In Sections 5.1, 5.6, and 10.5, we saw the usefulness of polar coordinates in the plane. In space there are two different coordinate systems analogous to polar coordinates, called cylindrical and spherical coordinates.

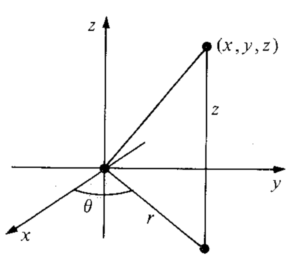


Figure 14.5.1. The cylindrical coordinates of the point (x, y, z).

The *cylindrical coordinates* of a point (x, y, z) in space are the numbers (r, θ, z), where r and θ are the polar coordinates of (x, y); that is,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad z = z.$$

See Fig. 14.5.1. As with polar coordinates, we can solve for r and θ in terms of x and y: squaring and adding gives

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2, \qquad \text{so} \quad r = \pm\sqrt{x^2 + y^2}\,.$$

Dividing gives

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

As in polar coordinates, it is sometimes convenient to allow negative r; thus (r,θ) and $(-r,\theta+\pi)$ represent the same point. Also, we recall that (r,θ) and $(r,\theta+2\pi)$ represent the same point. Sometimes we specify $r \geqslant 0$ (with $r=0$ corresponding to the z-axis) and a definite range for θ. If we choose θ between π and $-\pi$ and choose $\tan^{-1}u$ between $-\pi/2$ and $\pi/2$, then the solution of $y/x = \tan\theta$ is $\theta = \tan^{-1}(y/x)$ if $x > 0$, and $\theta = \tan^{-1}(y/x) + \pi$ if $x < 0$ ($\theta = \pi/2$ if $x = 0$ and $y > 0$, and $\theta = -\pi/2$ if $x = 0$ and $y < 0$).

Cylindrical Coordinates

If the cartesian coordinates of a point in space are (x, y, z), then the cylindrical coordinates of the point are (r, θ, z), where

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z;$$

or, if we choose $r \geqslant 0$ and $-\pi < \theta \leqslant \pi$,

$$r = \sqrt{x^2+y^2}\,,$$

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geqslant 0, \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0. \end{cases}$$

Example 1 (a) Find the cylindrical coordinates of $(6,6,8)$. Plot.

(b) If a point has cylindrical coordinates $(3, -\pi/6, -4)$, what are its cartesian coordinates? Plot.

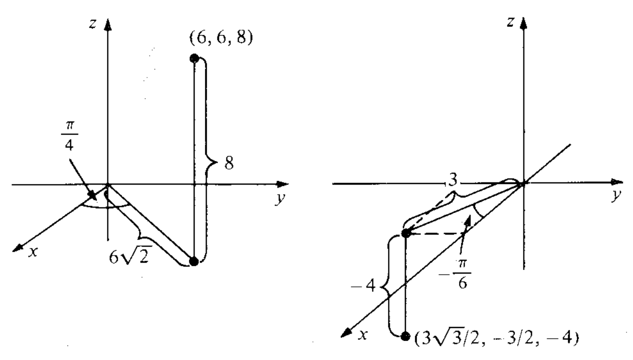
(c) Let a point have cartesian coordinates $(2,-3,6)$. Find its cylindrical coordinates and plot.

(d) Let a point have cylindrical coordinates $(2, 3\pi/4, 1)$. Find its cartesian coordinates and plot.

Solution (a) Here $r = \sqrt{6^2+6^2} = 6\sqrt{2}$ and $\theta = \tan^{-1}(\frac{6}{6}) = \tan^{-1}(1) = \pi/4$. Thus the cylindrical coordinates are $(6\sqrt{2}, \pi/4, 8)$. See Fig. 14.5.2(a).

(b) $x = r\cos\theta = 3\cos(-\pi/6) = 3\sqrt{3}/2$, and $y = r\sin\theta = 3\sin(-\pi/6) = -3/2$. Thus the cartesian coordinates are $(3\sqrt{3}/2, -3/2, -4)$. See Fig. 14.5.2(b).

(c) $r = \sqrt{x^2+y^2} = \sqrt{2^2+(-3)^2} = \sqrt{13}$; $\theta = \tan^{-1}(-\frac{3}{2}) \approx -0.983 \approx -56.31°$; $z = 6$. See Fig. 14.5.3(a).



(a) (b)

Figure 14.5.2. Comparing the cylindrical and cartesian coordinates of two points.

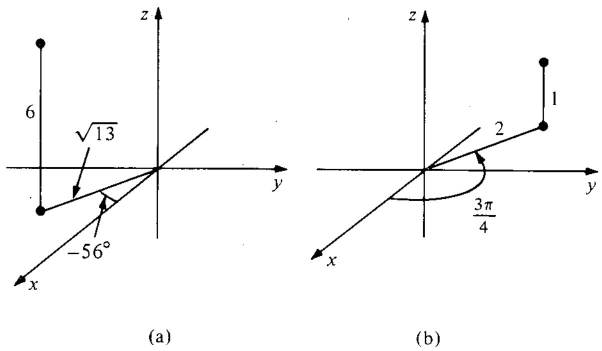


Figure 14.5.3. Two points in cylindrical coordinates.

(d) $x = r\cos\theta = 2\cos(3\pi/4) = 2\cdot(-\sqrt{2}/2) = -\sqrt{2}$;

$y = r\sin\theta = 2\sin(3\pi/4) = 2\cdot(\sqrt{2}/2) = \sqrt{2}$; $z = 1$.

See Fig. 14.5.3(b). ▲

Many surfaces are easier to describe in cylindrical than in cartesian coordinates, just as many curves are easier to work with using polar rather than cartesian coordinates.

Example 2 Plot the two surfaces described in cylindrical coordinates by (a) $r = 3$ and (b) $r = \cos 2\theta$.

Solution (a) Note that r is the distance from the given point to the z axis. Therefore the points with $r = 3$ lie on a cylinder of radius 3 centered on the z axis. See Fig. 14.5.4.

(b) The curve $r = \cos 2\theta$ in the xy plane is a four-petaled rose (see Example 1, Section 5.6). Thus in cylindrical coordinates we obtain a vertical cylinder with the four-leafed rose as a base, as shown in Fig. 14.5.5. ▲

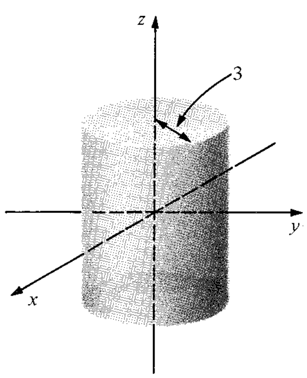
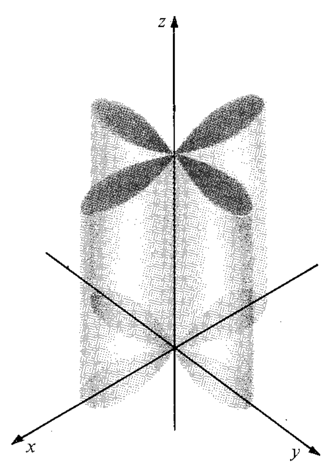


Figure 14.5.4. The cylinder has a very simple equation in cylindrical coordinates.

Figure 14.5.5. The surface $r = \cos 2\theta$ is a cylinder with a four-petaled rose as its base.



Example 3 Describe the geometric meaning of replacing (r,θ,z) by $(r,\theta+\pi,-z)$.

Solution Increasing θ by π is a rotation through 180° about the z axis. Switching z to $-z$ reflects in the xy plane (see Fig. 14.5.6). Combining the two operations results in reflection through the origin. ▲

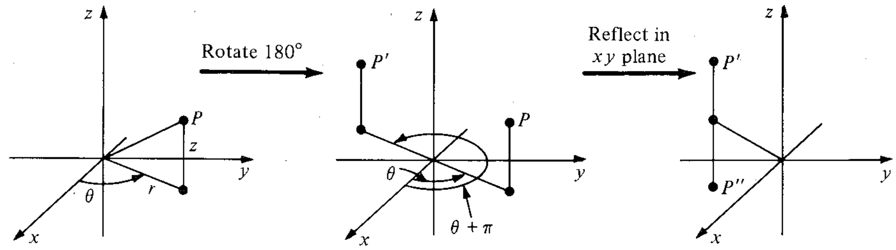


Figure 14.5.6. The effect of replacing (r,θ,z) by $(r,\theta+\pi,-z)$ is to replace P by $-P$.

Example 4 Show that the surface $r=f(z)$ is a surface of revolution.

Solution If we set $y=0$ and take $x \geqslant 0$, then $r=x$ and $r=f(z)$ becomes $x=f(z)$; the remaining points satisfying $r=f(z)$ are then obtained by revolving the graph $x=f(z)$ about the z axis; note that $r=c$, $z=d$ is a circle centered on the z axis. Thus we get a surface of revolution with symmetry about the z axis. ▲

Cylindrical coordinates are best adapted to problems which have cylindrical symmetry—that is, a symmetry about the z axis. Similarly, for problems with spherical symmetry—that is, symmetry with respect to all rotations about the origin in space—the *spherical coordinate system* is useful.

The *spherical coordinates* of a point (x,y,z) in space are the numbers (ρ,θ,ϕ) defined as follows (see Fig. 14.5.7).

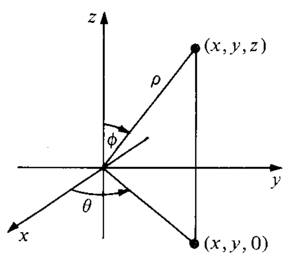


Figure 14.5.7. Spherical coordinates.

$\rho=$ distance from (x,y,z) to the origin;
$\theta=$ cylindrical coordinate θ (angle from the positive x axis to the point (x,y));
$\phi=$ the angle (in $[0,\pi]$) from the positive z axis to the line from origin to (x,y,z).

To express the cartesian coordinates in terms of spherical coordinates, we first observe that the cylindrical coordinate $r=\sqrt{x^2+y^2}$ is equal to $\rho\sin\phi$ and that $z=\rho\cos\phi$ (see Fig. 14.5.7). Therefore

$$x=r\cos\theta=\rho\sin\phi\cos\theta,\quad y=r\sin\theta=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi.$$

We may solve these equations for ρ, θ, and ϕ. The results are given in the following box.

Spherical Coordinates

If the cartesian coordinates of a point in space are (x,y,z), then the spherical coordinates of the point are (ρ,θ,ϕ), where

$$x=\rho\sin\phi\cos\theta,$$
$$y=\rho\sin\phi\sin\theta,$$
$$z=\rho\cos\phi,$$

or, if we choose $\rho \geqslant 0$, $-\pi<\theta\leqslant\pi$ and $0\leqslant\phi\leqslant\pi$,

$$\rho=\sqrt{x^2+y^2+z^2}\,,$$

$$\theta=\begin{cases}\tan^{-1}(y/x) & \text{if } x\geqslant 0,\\ \tan^{-1}(y/x)+\pi & \text{if } x<0,\end{cases}$$

$$\phi=\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}\,.$$

Notice that the spherical coordinates θ and ϕ are similar to the geographic coordinates of longitude and latitude if we take the earth's axis to be the z axis. There are differences, though: the geographical longitude is $|\theta|$ and is called east or west longitude according to whether θ is positive or negative; the geographical latitude is $|\pi/2-\phi|$ and is called north or south latitude according to whether $\pi/2-\phi$ is positive or negative.

Example 5 (a) Find the spherical coordinates of $(1,-1,1)$ and plot.
(b) Find the cartesian coordinates of $(3,\pi/6,\pi/4)$ and plot.
(c) Let a point have cartesian coordinates $(2,-3,6)$. Find its spherical coordinates and plot.
(d) Let a point have spherical coordinates $(1,-\pi/2,\pi/4)$. Find its cartesian coordinates and plot.

Solution (a)
$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{1^2+(-1)^2+1^2}=\sqrt{3}\,,$$
$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{-1}{1}\right)=-\frac{\pi}{4}\,,$$
$$\phi=\cos^{-1}\left(\frac{z}{\rho}\right)=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 0.955\approx 54.74°.$$

See Fig. 14.5.8(a).

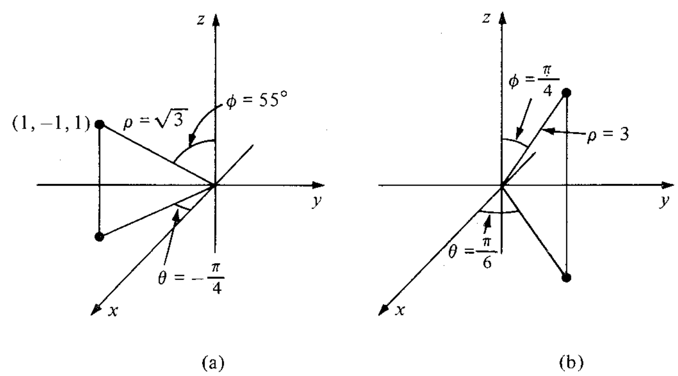


Figure 14.5.8. Finding the spherical coordinates of the point $(1,-1,1)$ and the cartesian coordinates of $(3,\pi/6,\pi/4)$.

(b)
$$x=\rho\sin\phi\cos\theta=3\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)=3\left(\frac{1}{\sqrt{2}}\right)\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2\sqrt{2}}\,,$$
$$y=\rho\sin\phi\sin\theta=3\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)=3\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)=\frac{3}{2\sqrt{2}}\,,$$
$$z=\rho\cos\phi=3\cos\left(\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}\,.$$

See Fig. 14.5.8(b).

(c)
$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{2^2+(-3)^2+6^2}=\sqrt{49}=7,$$
$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{-3}{2}\right)\approx-0.983\approx-56.31°$$
$$\phi=\cos^{-1}\left(\frac{z}{\rho}\right)=\cos^{-1}\left(\frac{6}{7}\right)\approx 0.541\approx 31.0°.$$

See Fig. 14.5.9(a) (the point is the same as in Example 1(d)).

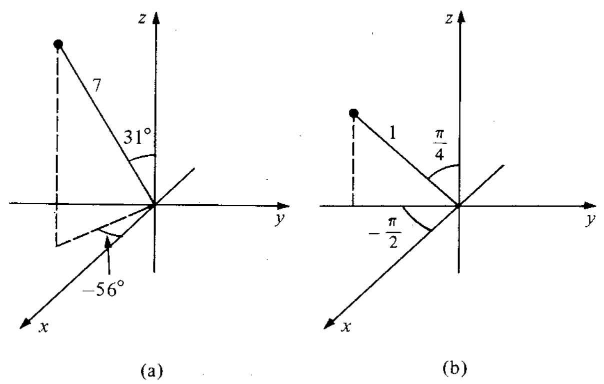


Figure 14.5.9. Two points in spherical coordinates.

(d) $x = \rho \sin\phi \cos\theta = 1 \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{-\pi}{2}\right) = \left(\frac{\sqrt{2}}{2}\right) \cdot 0 = 0,$

$$y = \rho \sin\phi \sin\theta = 1 \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{-\pi}{2}\right) = \left(\frac{\sqrt{2}}{2}\right)(-1) = -\frac{\sqrt{2}}{2},$$

$$z = \rho \cos\phi = 1 \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

See Fig. 14.5.9(b). ▲

Example 6 Find the equation in spherical coordinates of $x^2 + y^2 - z^2 = 4$ (a hyperboloid of revolution).

Solution To take advantage of the relationship $x^2 + y^2 + z^2 = \rho^2$, write

$$x^2 + y^2 - z^2 = (x^2 + y^2 + z^2) - 2z^2 = \rho^2 - 2\rho^2\cos^2\phi,$$

since $z = \rho\cos\phi$. Also, we can note that

$$\rho^2 - 2\rho^2\cos^2\phi = \rho^2(1 - 2\cos^2\phi) = -\rho^2\cos 2\phi.$$

Thus the surface is

$$\rho^2\cos 2\phi + 4 = 0. ▲$$

Example 7 (a) Describe the surface given in spherical coordinates by $\rho = 3$. (b) Describe the geometric meaning of replacing (ρ, θ, ϕ) by $(\rho, \theta + \pi, \phi)$.

Solution (a) In spherical coordinates, ρ is the distance from the point (x, y, z) to the origin. Thus $\rho = 3$ consists of all points a distance 3 from the origin—that is, a sphere of radius 3 centered at the origin. (b) Increasing θ by π has the effect of rotating about the z axis through an angle of 180°. ▲

Example 8 Show that the surface $\rho = f(\phi)$ is a surface of revolution.

Solution The equation $\rho = f(\phi)$ does not involve θ and hence is independent of rotations about the z axis; thus it is a surface of revolution. If we set $y = 0$, then $\rho = \sqrt{x^2 + z^2}$ and $\phi = \cos^{-1}(z/\sqrt{x^2 + z^2})$. Thus the surface $\rho = f(\phi)$ is obtained by revolving the curve in the xz plane given by

$$\sqrt{x^2 + z^2} = f\left(\cos^{-1}\left(\frac{z}{\sqrt{x^2 + z^2}}\right)\right),$$

about the z axis. ▲

Exercises for Section 14.5

In Exercises 1–6, convert from cartesian to cylindrical coordinates and plot:

1. $(1, -1, 0)$
2. $(\sqrt{2}, 1, 1)$
3. $(3, -2, 1)$
4. $(0, 6, -2)$
5. $(6, 0, -2)$
6. $(-1, 1, 1)$

In Exercises 7–12, convert from cylindrical to cartesian coordinates and plot.

7. $(1, \pi/2, 0)$
8. $(3, 45°, 8)$
9. $(-1, \pi/6, 4)$
10. $(2, 0, 1)$
11. $(0, \pi/18, 6)$
12. $(2, -\pi/4, 3)$
13. Sketch the surface described in cylindrical coordinates by $r = 1 + 2\cos\theta$.
14. Sketch the surface given in cylindrical coordinates by $r = 1 + \cos\theta$.

In Exercises 15–18, describe the geometric meaning of the stated replacement.

15. (r, θ, z) by $(r, \theta, -z)$
16. (r, θ, z) by $(2r, \theta, z)$
17. (r, θ, z) by $(2r, \theta, -z)$
18. (r, θ, z) by $(2r, \theta + \pi, z)$
19. Describe the surfaces $r =$ constant, $\theta =$ constant, and $z =$ constant in cylindrical coordinates.
20. Describe the surface given in cylindrical coordinates by $z = \theta$.

In Exercises 21–26, convert from cartesian to spherical coordinates and plot.

21. $(0, 1, 1)$
22. $(1, 0, 1)$
23. $(-2, 1, -3)$
24. $(1, 2, 3)$
25. $(-3, -2, -4)$
26. $(1, 1, 1)$

In Exercises 27–32, convert from spherical to cartesian coordinates and plot.

27. $(3, \pi/3, \pi)$
28. $(2, -\pi/6, \pi/3)$
29. $(3, 2\pi, 0)$
30. $(1, \pi/6, \pi/3)$
31. $(8, -\pi/3, \pi)$
32. $(1, \pi/2, \pi/2)$
33. Express the surface $xz = 1$ in spherical coordinates.
34. Express the surface $z = x^2 + y^2$ in spherical coordinates.
35. Describe the surface given in spherical coordinates by $\theta = \pi/4$.
36. Describe the surface given in spherical coordinates by $\rho = \phi$.
37. Describe the geometric meaning of replacing (ρ, θ, ϕ) by $(2\rho, \theta, \phi)$.
38. Describe the geometric meaning of replacing (ρ, θ, ϕ) by $(\rho, \theta, \phi + \pi/2)$ in spherical coordinates.
39. Describe the curve given in spherical coordinates by $\rho = 1$, $\phi = \pi/2$.
40. Describe the curve given in spherical coordinates by $\rho = 1$, $\theta = 0$.

In Exercises 41–46, convert each of the points from cartesian to cylindrical and spherical coordinates and plot.

41. $(0, 3, 4)$
42. $(-\sqrt{2}, 1, 0)$
43. $(0, 0, 0)$
44. $(-1, 0, 1)$
45. $(-2\sqrt{3}, -2, 3)$
46. $(-1, 1, 0)$

In Exercises 47–52, the points are given in cylindrical coordinates. Convert to cartesian and spherical coordinates:

47. $(1, \pi/4, 1)$
48. $(3, \pi/6, -4)$
49. $(0, \pi/4, 1)$
50. $(2, -\pi/2, 1)$
51. $(-2, -\pi/2, 1)$
52. $(1, -\pi/6, 2)$

In Exercises 53–58, the points are given in spherical coordinates. Convert to cartesian and cylindrical coordinates and plot.

53. $(1, \pi/2, \pi)$
54. $(2, -\pi/2, \pi/6)$
55. $(0, \pi/8, \pi/35)$
56. $(2, -\pi/2, -\pi)$
57. $(-1, \pi, \pi/6)$
58. $(-1, -\pi/4, \pi/2)$
59. Express the surface $z = x^2 - y^2$ (a hyperbolic paraboloid) in (a) cylindrical and (b) spherical coordinates.
60. Express the plane $z = x$ in (a) cylindrical and (b) spherical coordinates.
61. Show that in spherical coordinates:
    (a) ρ is the length of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;
    (b) $\phi = \cos^{-1}(\mathbf{v} \cdot \mathbf{k}/\|\mathbf{v}\|)$, where $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$;
    (c) $\theta = \cos^{-1}(\mathbf{u} \cdot \mathbf{i}/\|\mathbf{u}\|)$, where $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.
62. Two surfaces are described in spherical coordinates by the equations $\rho = f(\theta, \phi)$ and $\rho = -2f(\theta, \phi)$, where $f(\theta, \phi)$ is a function of two variables. How is the second surface obtained geometrically from the first?
63. A circular membrane in space lies over the region $x^2 + y^2 \leqslant a^2$. The maximum deflection z of the membrane is b. Assume that (x, y, z) is a point on the deflected membrane. Show that the corresponding point (r, θ, z) in cylindrical coordinates satisfies the conditions
    $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $|z| \leqslant b$.
64. A tank in the shape of a right circular cylinder of radius 10 feet and height 16 feet is half filled and lying on its side. Describe the air space inside the tank by suitably chosen cylindrical coordinates.
65. A vibrometer is to be designed which withstands the heating effects of its spherical enclosure of diameter d, which is buried to a depth $d/3$ in the earth, the upper portion being heated by the sun. Heat conduction analysis requires a description of the buried portion of the enclosure, in spherical coordinates. Find it.
66. An oil filter cartridge is a porous right circular cylinder inside which oil diffuses from the axis to the outer curved surface. Describe the cartridge in cylindrical coordinates, if the diameter of the filter is 4.5″, the height is 5.6″ and the center of the cartridge is drilled (all the way through) from the top to admit a $\frac{5}{8}$″ diameter bolt.

★67. Describe the surface given in spherical coordinates by $\rho = \cos 2\theta$.

14.6 Curves in Space

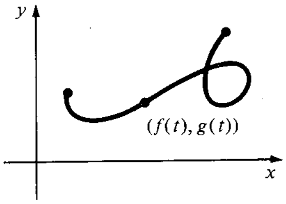


Figure 14.6.1. A parametric curve in the plane.

Tangents and velocities of curves in space can be computed by vector methods.

We continue our study of three-dimensional geometry by considering curves in space. We can consider tangents to these curves by using calculus, since only the calculus of functions of one variable and a knowledge of vectors are required. (To determine tangent planes to surfaces, we will need the calculus of functions of several variables.)

Recall from Section 2.4 that a parametric curve in the plane consists of a pair of functions $(x, y) = (f(t), g(t))$. As t ranges through some interval (on which f and g are defined), the point (x, y) traces out a curve in the plane; see Fig. 14.6.1.

Example 1 What curve is traced out by $(\sin t, 2\cos t)$, $0 \leqslant t \leqslant 2\pi$?

Solution Since $x = \sin t$ and $y/2 = \cos t$, (x, y) satisfies $x^2 + y^2/4 = 1$, so the curve traced out is an ellipse. As t goes from zero to 2π, the moving point goes once around the ellipse, starting and ending at P (Fig. 14.6.2). ▲

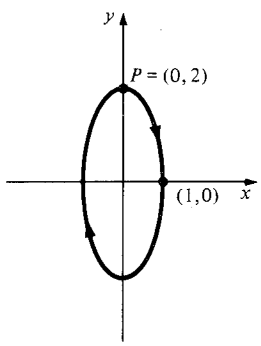


Figure 14.6.2. The ellipse traced out by $(\sin t, 2\cos t)$.

The step from two to three dimensions is accomplished by adding one more function; i.e., we state the following definition: A *parametric curve in space* consists of three functions $(x, y, z) = (f(t), g(t), h(t))$ defined for t in some interval on which f, g, and h are defined.

The curve we "see" is the path traced out by the point (x, y, z) as t varies, just as for curves in the plane.

Example 2 (a) Sketch the parametric curve $(x, y, z) = (3t + 2, 8t - 1, t)$. (b) Describe the curve $x = 3t^3 + 2$, $y = t^3 - 8$, $z = 4t^3 + 3$.

Solution (a) If we write $P = (x, y, z)$, then

$$P = (2, -1, 0) + t(3, 8, 1)$$

which is a straight line through $(2, -1, 0)$ in the direction $(3, 8, 1)$ (see Section 13.3). To sketch it, we pick the points obtained by setting $t = 0$ and $t = 1$, that is, $(2, -1, 0)$ and $(5, 7, 1)$; see Fig. 14.6.3 on the next page.

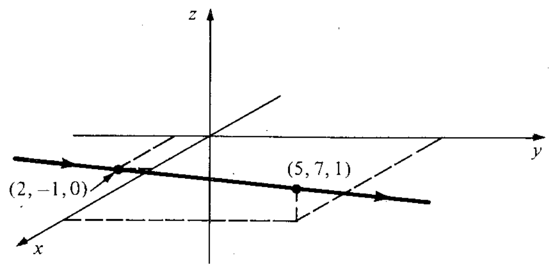


Figure 14.6.3. The parametric curve $(3t+2, 8t-1, t)$ is a straight line.

(b) We find

$$\begin{aligned}(x, y, z) &= (3t^3+2, t^3-8, 4t^3+3)\\ &= (2, -8, 3) + t^3(3, 1, 4);\end{aligned}$$

so the curve is a straight line through $(2, -8, 3)$ in the direction $(3, 1, 4)$. ▲

Example 3 (a) Sketch the curve given by $x = \cos t$, $y = \sin t$, $z = t$, where $-\infty < t < \infty$.
(b) Sketch the curve $(\cos t, 2\sin t, 2t)$.

Solution (a) As t varies, the point (x, y) traces out a circle in the plane. Thus (x, y, z) is a path which circles around the z axis, but at value t, its height above the xy plane is $z = t$. Thus we get the helix shown in Fig. 14.6.4. (It is called a *right circular helix*, since it lies on the right circular cylinder $x^2 + y^2 = 1$.)

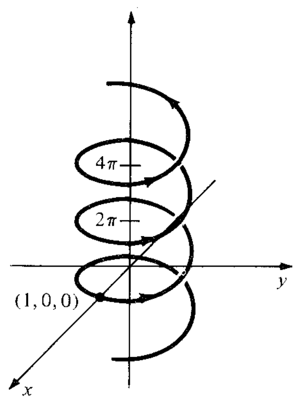


Figure 14.6.4. The curve $(\cos t, \sin t, t)$ is a helix.

In Fig. 14.6.4, the z axis has been drawn with a different scale than the x and y axes so that more coils of the helix can be shown. It is often useful to do something like this when displaying sketches of curves or graphs. You should be careful, however, not to give a false impression—label the axes to show the scale when necessary.

(b) Since $x = \cos t$ and $y/2 = \sin t$, the point $(x, y, 0)$ satisfies $x^2 + y^2/4 = 1$, so the curve lies over this ellipse in the xy plane. As t increases from zero to 2π, the projection in the xy plane goes once around the same ellipse as in Example 1 (Fig. 14.6.2), only now it starts at $(1, 0, 0)$ at $t = 0$ and proceeds counterclockwise since x behaves like $\cos t$ and y like $2\sin t$. Meanwhile, z increases steadily with t according to the formula $z = 2t$. The net result is a helix winding around the z axis, much like that of part (a), but no longer circular. It now lies on a cylinder of elliptical cross section (see Fig. 14.6.5). ▲

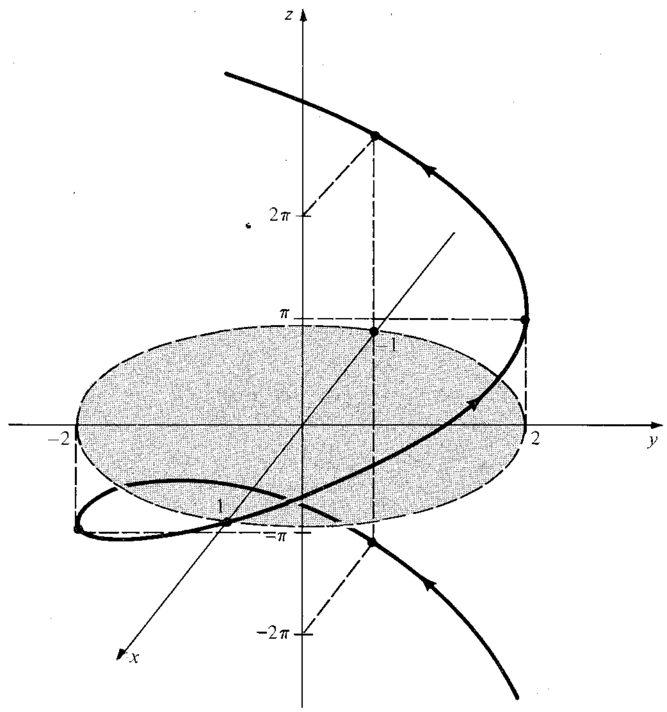


Figure 14.6.5. The curve $(\cos t, 2\sin t, 2t)$ is an elliptical helix.

Example 4 Sketch the curve $(t, 2t, \cos t)$.

Solution If we ignore z temporarily, we note that $(t, 2t)$ describes the line $y = 2x$ in the xy plane. As t varies, $(t, 2t)$ moves along this line. Thus $(t, 2t, \cos t)$ moves along a curve over this line with the z component oscillating as $\cos t$. Thus we get the curve shown in Fig. 14.6.6. ▲

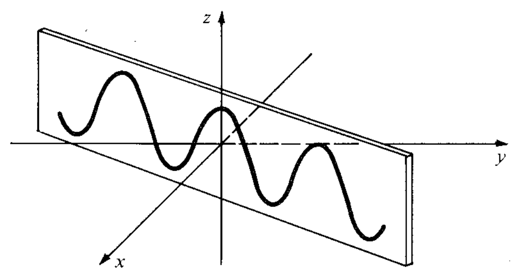


Figure 14.6.6. The curve $(t, 2t, \cos t)$ lies in the plane $y = 2x$.

In doing calculus with parametric curves, it is useful to identify the point $P = (x, y, z) = (f(t), g(t), h(t))$ with the vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

This vector is a *function* of t, according to the following definition.

Vector Functions

A *vector function* of one variable is a rule $\boldsymbol{\sigma}$ which associates a vector $\mathbf{r} = \boldsymbol{\sigma}(t)$ in space (or the plane) to each real number t in some domain.

If $\boldsymbol{\sigma}$ is a vector function and t is in its domain, we can express $\boldsymbol{\sigma}(t)$ in terms of the standard basis vectors, **i**, **j**, and **k**. The coefficients will themselves depend upon t, so we may write

$$\boldsymbol{\sigma}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where f, g, and h are scalar (real-valued) functions with the same domain as $\boldsymbol{\sigma}$. Notice that the functions $f(t)$, $g(t)$, $h(t)$ define a parametric curve such that the displacement vector from the origin to $(f(t), g(t), h(t))$ is just $\boldsymbol{\sigma}(t)$. The functions f, g, and h are called the *component functions* of the vector function $\boldsymbol{\sigma}(t)$. To summarize, we may say that parametric curves, vector functions, and triples of scalar functions are mathematically equivalent objects; we simply visualize them differently. For instance, the wind velocity at a fixed place on earth, or the cardiac vector (see Fig. 13.2.14), may be visualized as a vector depending on time.

Example 5 Let **u**, **v**, and **w** be three vectors such that **v** and **w** are perpendicular and have the same length r, and let[3] $\boldsymbol{\sigma}(t) = \mathbf{u} + \mathbf{v}\cos t + \mathbf{w}\sin t$.

(a) Describe the motion of the tip of $\boldsymbol{\sigma}(t)$ if the tail of $\boldsymbol{\sigma}(t)$ is fixed at the origin. (That is, describe the parametric curve corresponding to $\boldsymbol{\sigma}(t)$.)

(b) Find the component functions of $\boldsymbol{\sigma}(t)$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} - \mathbf{k}$, and $\mathbf{w} = \mathbf{j} + \mathbf{k}$.

Solution (a) We observe first that the vector $\mathbf{v}\cos t + \mathbf{w}\sin t$ always lies in the plane spanned by **v** and **w** and that the square of its length is

$$\begin{aligned}(\mathbf{v}\cos t + \mathbf{w}\sin t)\cdot(\mathbf{v}\cos t + \mathbf{w}\sin t)\\ &= \mathbf{v}\cdot\mathbf{v}\cos^2 t + 2\mathbf{v}\cdot\mathbf{w}\sin t\cos t + \mathbf{w}\cdot\mathbf{w}\sin^2 t\\ &= r^2\cos^2 t + r^2\sin^2 t = r^2(\cos^2 t + \sin^2 t) = r^2,\end{aligned}$$

so the tip of the vector $\mathbf{v}\cos t + \mathbf{w}\sin t$ moves in a circle of radius r if its tail is fixed. Adding **u** to $\mathbf{v}\cos t + \mathbf{w}\sin t$ to get $\boldsymbol{\sigma}(t)$, we find that the tip of $\boldsymbol{\sigma}(t)$ moves in a circle of radius r whose center is at the tip of **u**. (See Fig. 14.6.7.)

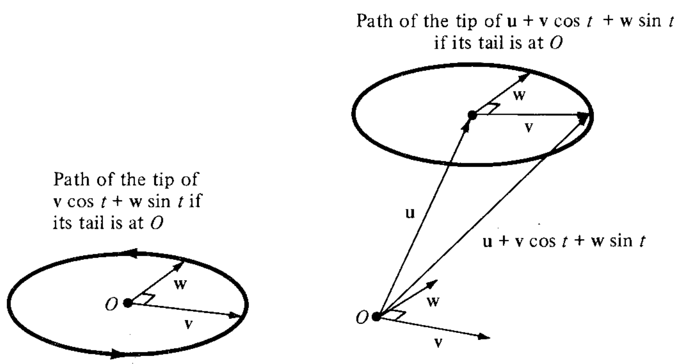


Figure 14.6.7. The tip of $\mathbf{u} + \mathbf{v}\cos t + \mathbf{w}\sin t$ moves in a circle of radius r with center at the tip of **u** and in a plane parallel to that spanned by **v** and **w**.

(b) We have

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathbf{u} + \mathbf{v}\cos t + \mathbf{w}\sin t = 2\mathbf{i} + \mathbf{j} + (\mathbf{j} - \mathbf{k})\cos t + (\mathbf{j} + \mathbf{k})\sin t\\ &= 2\mathbf{i} + (1 + \cos t + \sin t)\mathbf{j} + (-\cos t + \sin t)\mathbf{k},\end{aligned}$$

so the component functions are 2, $1 + \cos t + \sin t$, and $-\cos t + \sin t$. ▲

[3] Formulas involving vector functions are sometimes clearer to write and read if scalars are placed to the right of vectors. Any expression of the form $\mathbf{v}f(t)$ is to be interpreted as $f(t)\mathbf{v}$.

We now wish to define the rate of change, or *derivative*, of a vector function $\boldsymbol{\sigma}(t)$ with respect to t. If $\boldsymbol{\sigma}(t)$ is the displacement from a fixed origin to a moving point, this derivative will represent the velocity of the point. To see how the derivative should be defined, we examine the case of uniform rectilinear motion.

Example 6 Let $\boldsymbol{\sigma}(t) = \mathbf{u} + t\mathbf{v}$, so that $\boldsymbol{\sigma}(t)$ is the displacement from the origin to a point moving uniformly with velocity vector $\mathbf{v}$. Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$.

(a) Find the component functions of $\boldsymbol{\sigma}(t)$.
(b) Show that the components of the velocity vector are obtained by differentiating the component functions of $\boldsymbol{\sigma}(t)$.

Solution (a) We have

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathbf{u} + t\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} + t(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\\ &= (a + lt)\mathbf{i} + (b + mt)\mathbf{j} + (c + nt)\mathbf{k},\end{aligned}$$

so the component functions are $a + lt$, $b + mt$, and $c + nt$.
(b) The derivatives of the component functions of $\boldsymbol{\sigma}(t)$ are the constants l, m, and n; these are precisely the components of the velocity vector $\mathbf{v}$. ▲

Example 7 Let $\boldsymbol{\sigma}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ be a vector function in the plane. Show that the tangent line at time t_0 to the parametric curve corresponding to $\boldsymbol{\sigma}(t)$ (with the tail of $\boldsymbol{\sigma}(t)$ fixed at zero) has the direction of the vector $f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j}$.

Solution Recall from Section 2.4 that if $(f(t), g(t))$ is a parametrized curve in the plane, then the slope of its tangent line at $(f(t_0), g(t_0))$ is $g'(t_0)/f'(t_0)$. A line in the direction of $f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j}$ has slope $g'(t_0)/f'(t_0)$, so it is in the same direction as the tangent line. ▲

Guided by Examples 6 and 7, we make the following definition.

Derivative of a Vector Function

Let $\boldsymbol{\sigma}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function. If the coordinate functions f, g, and h are all differentiable at t_0, then we say that $\boldsymbol{\sigma}$ is differentiable at t_0, and we define the *derivative* $\boldsymbol{\sigma}'(t_0)$ to be the vector $f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$:

$$\boldsymbol{\sigma}'(t_0) = f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}.$$

The derivative of $\boldsymbol{\sigma}$ is a function of the value of t at which the derivative is evaluated. Thus $\boldsymbol{\sigma}'(t)$ is a new vector function, and we may consider the second derivative $\boldsymbol{\sigma}''(t)$, as well as higher derivatives.

We will sometimes use Leibniz notation for derivatives of vector functions: if $\mathbf{r} = \boldsymbol{\sigma}(t)$, we will write $d\mathbf{r}/dt$ for $\boldsymbol{\sigma}'(t)$ and $d^2\mathbf{r}/dt^2$ for $\boldsymbol{\sigma}''(t)$.

The derivative of a vector function can also be expressed as a limit of difference quotients. If $\mathbf{r} = \boldsymbol{\sigma}(t)$, we write $\Delta\mathbf{r} = \boldsymbol{\sigma}(t + \Delta t) - \boldsymbol{\sigma}(t)$. Then $\Delta\mathbf{r}/\Delta t$ (i.e., the scalar $1/\Delta t$ times the vector $\Delta\mathbf{r}$) is a vector which approaches $\boldsymbol{\sigma}'(t)$ as $\Delta t \to 0$. (See Fig. 14.6.8 on the next page and Exercise 52.)

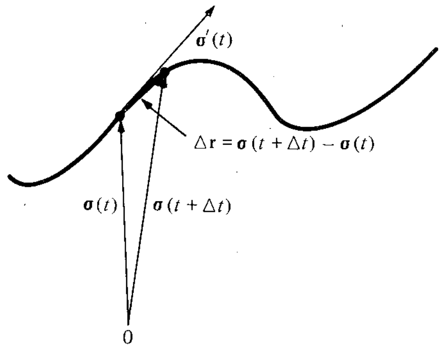


Figure 14.6.8. As $\Delta t \to 0$, the quotient $[\sigma(t+\Delta t) - \sigma(t)]/\Delta t$ approaches $\sigma'(t)$; i.e., $\Delta \mathbf{r}/\Delta t \to d\mathbf{r}/dt$.

Example 8 Let $\sigma(t)$ be the vector function of Example 5, with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as in part (b) of that example. Find $\sigma'(t)$ and $\sigma''(t)$.

Solution In terms of components,

$$\sigma(t) = 2\mathbf{i} + (1 + \cos t + \sin t)\mathbf{j} + (-\cos t + \sin t)\mathbf{k}.$$

Differentiating the components, we have

$$\sigma'(t) = (-\sin t + \cos t)\mathbf{j} + (\sin t + \cos t)\mathbf{k}$$

and

$$\sigma''(t) = (-\cos t - \sin t)\mathbf{j} + (\cos t - \sin t)\mathbf{k}. \blacktriangle$$

The differentiation of vector functions is facilitated by algebraic rules which follow from the corresponding rules for scalar functions. We list the rules in the following box.

Differentiation Rules for Vector Functions

To differentiate a vector function $\sigma(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, differentiate it component by component: $\sigma'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$. Let $\sigma(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ be vector functions and let $p(t)$ and $q(t)$ be scalar functions.

Sum Rule: $\frac{d}{dt}[\sigma_1(t) + \sigma_2(t)] = \sigma_1'(t) + \sigma_2'(t).$

Scalar Multiplication Rule: $\frac{d}{dt}[p(t)\sigma(t)] = p'(t)\sigma(t) + p(t)\sigma'(t).$

Dot Product Rule: $\frac{d}{dt}[\sigma_1(t) \cdot \sigma_2(t)] = \sigma_1'(t) \cdot \sigma_2(t) + \sigma_1(t) \cdot \sigma_2'(t).$

Cross Product Rule: $\frac{d}{dt}[\sigma_1(t) \times \sigma_2(t)] = \sigma_1'(t) \times \sigma_2(t) + \sigma_1(t) \times \sigma_2'(t).$

Chain Rule: $\frac{d}{dt}[\sigma(q(t))] = q'(t)\sigma'(q(t)).$

For example, to prove the dot product rule, let $\sigma_1(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}$ and $\sigma_2(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}$. Hence,

$$\sigma_1(t) \cdot \sigma_2(t) = f_1(t)f_2(t) + g_1(t)g_2(t) + h_1(t)h_2(t),$$

so by the sum and product rules for real-valued functions, we have

$$\frac{d}{dt}[\sigma_1(t) \cdot \sigma_2(t)] = [f_1'(t)f_2(t) + f_1(t)f_2'(t)] + [g_1'(t)g_2(t) + g_1(t)g_2'(t)] + [h_1'(t)h_2(t) + h_1(t)h_2'(t)].$$

Regrouping terms, we can rewrite this as

$$\begin{aligned}&\left[f_1'(t)f_2(t) + g_1'(t)g_2(t) + h_1'(t)h_2(t)\right]\\&\quad + \left[f_1(t)f_2'(t) + g_1(t)g_2'(t) + h_1(t)h_2'(t)\right]\\&\quad = \left[f_1'(t)\mathbf{i} + g_1'(t)\mathbf{j} + h_1'(t)\mathbf{k}\right]\cdot\left[f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}\right]\\&\qquad + \left[f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}\right]\cdot\left[f_2'(t)\mathbf{i} + g_2'(t)\mathbf{j} + h_2'(t)\mathbf{k}\right]\\&\quad = \boldsymbol{\sigma}_1'(t)\cdot\boldsymbol{\sigma}_2(t) + \boldsymbol{\sigma}_1(t)\cdot\boldsymbol{\sigma}_2'(t),\end{aligned}$$

so the dot product rule is proved. The other rules are proved in a similar way (Exercises 53–56).

Example 9 Show that if $\boldsymbol{\sigma}(t)$ is a vector function such that $\|\boldsymbol{\sigma}(t)\|$ is constant, then $\boldsymbol{\sigma}'(t)$ is perpendicular to $\boldsymbol{\sigma}(t)$ for all t.

Solution Since $\|\boldsymbol{\sigma}(t)\|$ is constant, so is its square $\|\boldsymbol{\sigma}(t)\|^2 = \boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t)$. The derivative of this constant is zero, so by the dot product rule we have

$$0 = \frac{d}{dt}\left[\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}(t)\right] = \boldsymbol{\sigma}'(t)\cdot\boldsymbol{\sigma}(t) + \boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}'(t) = 2\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}'(t);$$

so $\boldsymbol{\sigma}(t)\cdot\boldsymbol{\sigma}'(t) = 0$; that is, $\boldsymbol{\sigma}'(t)$ is perpendicular to $\boldsymbol{\sigma}(t)$. ▲

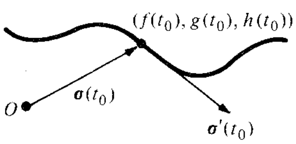


Figure 14.6.9. The velocity vector of a parametric curve is the derivative of the vector $\boldsymbol{\sigma}(t)$ from the origin to the curve.

Let $(f(t), g(t), h(t))$ be a parametric curve. If f, g, and h are differentiable at t_0, the vector $f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ is called the *velocity vector* of the curve at t_0. Notice that if $\boldsymbol{\sigma}(t)$ is the vector function corresponding to the curve $(f(t), g(t), h(t))$, then the velocity vector at t_0 is just $\boldsymbol{\sigma}'(t_0)$ (see Fig. 14.6.9). We often write $\mathbf{v}$ for the velocity vector—that is, $\mathbf{v} = \boldsymbol{\sigma}'(t)$. In Leibniz notation, if $\mathbf{r} = \boldsymbol{\sigma}(t)$, we have $\mathbf{v} = d\mathbf{r}/dt$.

Several other quantities of interest may be defined in terms of the velocity vector. If $\mathbf{v} = \boldsymbol{\sigma}'(t_0)$ is the velocity of a curve at t_0, then the length $v = \|\mathbf{v}\| = \|\boldsymbol{\sigma}'(t_0)\|$ is called the *speed* along the curve at t_0, and the line through $\boldsymbol{\sigma}(t_0)$ in the direction of $\boldsymbol{\sigma}'(t_0)$ (assuming $\boldsymbol{\sigma}'(t_0) \neq 0$) is called the *tangent line* to the curve (see Example 7). Thus the tangent line is given by $\mathbf{r} = \boldsymbol{\sigma}(t_0) + t\boldsymbol{\sigma}'(t_0)$.

For a curve describing uniform rectilinear motion, the velocity vector is constant (see Example 6). In general, the velocity vector is a vector function $\mathbf{v} = \boldsymbol{\sigma}'(t)$ which depends on t. The derivative $\mathbf{a} = d\mathbf{v}/dt = \boldsymbol{\sigma}''(t)$ is called the *acceleration vector* of the curve. Notice that if the curve is $(f(t), g(t), h(t))$, then the acceleration vector is

$$\mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}.$$

The terms *velocity*, *speed*, and *acceleration* come from physics, where parametric curves represent the motion of particles. These topics will be discussed in the next section.

Example 10 A particle moves in a helical path along the curve $(\cos t, \sin t, t)$. (a) Find its velocity and acceleration vectors. (b) Find its speed. (c) Find the tangent line at $t_0 = \pi/4$.

Solution (a) Differentiating the components, we have $\mathbf{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$, and $\mathbf{a} = d\mathbf{v}/dt = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$. Notice that the acceleration vector points directly from $(\cos t, \sin t, t)$ to the z axis and is perpendicular to the axis as well as to the velocity vector (see Fig. 14.6.10).

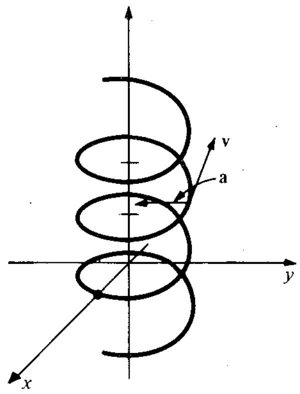


Figure 14.6.10. The velocity and acceleration of a particle moving on a helix.

(b) The velocity vector is $\mathbf{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$, so the speed is

$$v = \|\mathbf{v}\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}$$
$$= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}\,.$$

(c) The tangent line is

$$\mathbf{r} = \boldsymbol{\sigma}(t_0) + t\boldsymbol{\sigma}'(t_0) = (\cos t_0)\mathbf{i} + (\sin t_0)\mathbf{j} + t_0\mathbf{k} + t\big[(-\sin t_0)\mathbf{i} + (\cos t_0)\mathbf{j} + \mathbf{k}\big].$$

At $t_0 = \pi/4$, we get

$$\mathbf{r} = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) + \frac{\pi}{4}\mathbf{k} + t\left[-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \mathbf{k}\right]$$
$$= \frac{1-t}{\sqrt{2}}\mathbf{i} + \frac{1+t}{\sqrt{2}}\mathbf{j} + \left(\frac{\pi}{4} + t\right)\mathbf{k}. \ \blacktriangle$$

Example 11 A particle moves in such a way that its acceleration is constantly equal to $-\mathbf{k}$. If the position when $t = 0$ is $(0,0,1)$ and the velocity at $t = 0$ is $\mathbf{i} + \mathbf{j}$, when and where does the particle fall below the plane $z = 0$? Describe the path travelled by the particle.

Solution Let $(f(t), g(t), h(t))$ be the parametric curve traced out by the particle, so that the velocity vector is $\boldsymbol{\sigma}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$. The acceleration $\boldsymbol{\sigma}''(t)$ is equal to $-\mathbf{k}$, so we must have $f''(t) = 0$, $g''(t) = 0$, and $h''(t) = -1$. It follows that $f'(t)$ and $g'(t)$ are constant functions, and $h'(t)$ is a linear function with slope -1. Since $\boldsymbol{\sigma}'(0) = \mathbf{i} + \mathbf{j}$, we must have $\boldsymbol{\sigma}'(t) = \mathbf{i} + \mathbf{j} - t\mathbf{k}$. Integrating again and using the initial position $(0,0,1)$, we find that $(f(t), g(t), h(t)) = (t, t, 1 - \frac{1}{2}t^2)$. The particle drops below the plane $z = 0$ when $1 - \frac{1}{2}t^2 = 0$; that is, $t = \sqrt{2}$. At that time, the position is $(\sqrt{2}, \sqrt{2}, 0)$. The path travelled by the particle is a parabola in the plane $y = x$. (See Fig. 14.6.11.) $\blacktriangle$

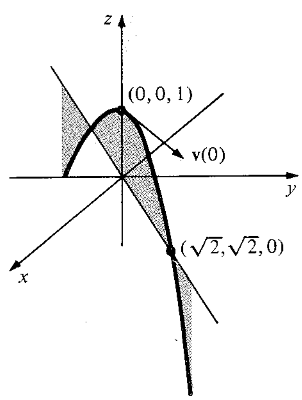


Figure 14.6.11. The path of the parabola with initial position $(0, 0, 1)$, initial velocity $\mathbf{i} + \mathbf{j}$, and constant acceleration $-\mathbf{k}$ is a parabola in the plane $y = x$.

Exercises for Section 14.6

Sketch the curves in Exercises 1–10.

1. $x = \sin t, y = 4\cos t, 0 \leqslant t \leqslant 2\pi$.
2. $x = 2\sin t, y = 4\cos t, 0 \leqslant t \leqslant 2\pi$.
3. $x = 2t - 1; y = t + 2; z = t$.
4. $x = -t; y = 2t; z = 1/t; 1 \leqslant t \leqslant 3$.
5. $x = -t; y = t; z = t^2; 0 \leqslant t \leqslant 3$.
6. $(t, -t, t^2); 0 \leqslant t \leqslant 2$.
7. $(4\cos t, 2\sin t, t); 0 \leqslant t \leqslant 2\pi$.
8. $x = \cos t; y = \sin t; z = t/2\pi; -2\pi \leqslant t \leqslant 2\pi$.
9. $(t, 1/t, t); 1 \leqslant t \leqslant 3$.
10. $(\cosh t, \sinh t, t); -1 \leqslant t \leqslant 1$.

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three vectors such that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular and have the same length r. In Exercises 11 and 12, (a) describe the motion of the tip of the vector $\boldsymbol{\sigma}(t)$ and (b) find the components of $\boldsymbol{\sigma}(t)$ if $\mathbf{u} = \mathbf{i} - \mathbf{j}$, $\mathbf{v} = 2(\mathbf{j} + \mathbf{k})$, $\mathbf{w} = 2(\mathbf{j} - \mathbf{k})$.

11. $\boldsymbol{\sigma}(t) = \mathbf{u} + 2\mathbf{v}\cos t + 4\mathbf{w}\sin t$.
12. $\boldsymbol{\sigma}(t) = \mathbf{u} + 3\mathbf{v}\cos t - 5\mathbf{w}\sin t$.

13. Let $\boldsymbol{\sigma}(t) = 3\cos t\mathbf{i} - 8\sin t\mathbf{j} + e^t\mathbf{k}$. Find $\boldsymbol{\sigma}'(t)$ and $\boldsymbol{\sigma}''(t)$.
14. (a) Give the "natural" domain for this vector function:
$$\boldsymbol{\sigma}(t) = \frac{1}{t}\mathbf{i} + \frac{1}{t-1}\mathbf{j} + \frac{1}{t-2}\mathbf{k}.$$
(b) Find $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$.

In Exercises 15–20, let $\boldsymbol{\sigma}_1(t) = e^t\mathbf{i} + (\sin t)\mathbf{j} + t^3\mathbf{k}$ and $\boldsymbol{\sigma}_2(t) = e^{-t}\mathbf{i} + (\csc t)\mathbf{j} - 2t^3\mathbf{k}$. Find each of the stated derivatives in two different ways:

15. $\dfrac{d}{dt}[\boldsymbol{\sigma}_1(t) + \boldsymbol{\sigma}_2(t)]$
16. $\dfrac{d}{dt}[\boldsymbol{\sigma}_1(t) \cdot \boldsymbol{\sigma}_2(t)]$
17. $\dfrac{d}{dt}[\boldsymbol{\sigma}_1(t) \times \boldsymbol{\sigma}_2(t)]$
18. $\dfrac{d}{dt}\{\boldsymbol{\sigma}_1(t) \cdot [2\boldsymbol{\sigma}_2(t) + \boldsymbol{\sigma}_1(t)]\}$
19. $\dfrac{d}{dt}e^t\boldsymbol{\sigma}_1(t)$
20. $\dfrac{d}{dt}[\boldsymbol{\sigma}_1(t^2)]$

21. Show that if the acceleration of an object is always perpendicular to the velocity, then the speed of the object is constant. [*Hint:* See Example 9.]
22. Show that, at a local maximum or minimum of $\|\boldsymbol{\sigma}(t)\|$, $\boldsymbol{\sigma}'(t)$ is perpendicular to $\boldsymbol{\sigma}(t)$.

Compute (a) the velocity vector, (b) the acceleration vector, and (c) the speed for each of the curves in Exercises 23–32.

23. The curve in Exercise 1.
24. The curve in Exercise 2.
25. The curve in Exercise 3.
26. The curve in Exercise 4.
27. The curve in Exercise 5.
28. The curve in Exercise 6.
29. The curve in Exercise 7.
30. The curve in Exercise 8.
31. The curve in Exercise 9.
32. The curve in Exercise 10.

For each of the curves in Exercises 33–38, determine the velocity and acceleration vectors for all t and the equation for the tangent line at the specified value of t.

33. $(6t, 3t^2, t^3)$; $t = 0$.
34. $(\sin 3t, \cos 3t, 2t^{3/2})$; $t = 1$.
35. $(\cos^2 t, 3t - t^3, t)$; $t = 0$.
36. $(t\sin t, t\cos t, \sqrt{3}\,t)$; $t = 0$.
37. $(\sqrt{2}\,t, e^t, e^{-t})$; $t = 0$.
38. $(2\cos t, 3\sin t, t)$; $t = \pi$.

39. Suppose that a particle follows the path $(e^t, e^{-t}, \cos t)$ until it flies off on a tangent at $t = 1$. Where is it at $t = 2$?
40. If the particle in Exercise 39 flies off the path at $t = 0$ instead of $t = 1$, where is it at $t = 2$?
41. Describe and sketch the curves specified by the following data:
(a) $\boldsymbol{\sigma}'(t) = (1, 0, 1)$; $\boldsymbol{\sigma}(0) = (0, 0, 0)$,
(b) $\boldsymbol{\sigma}'(t) = (-1, 1, 1)$; $\boldsymbol{\sigma}(0) = (1, 2, 3)$,
(c) $\boldsymbol{\sigma}'(t) = (-1, 1, 1)$; $\boldsymbol{\sigma}(0) = (0, 0, 0)$.
42. Suppose that a curve $\boldsymbol{\sigma}(t)$ has the velocity vector $\boldsymbol{\sigma}'(t) = (a, b, \sin t)$, where a and b are constants. Sketch the curve if $a = -1$, $b = 2$, and assuming $\boldsymbol{\sigma}(0) = (0, 0, 1)$.
43. Suppose that a curve has the velocity vector $\mathbf{v} = \boldsymbol{\sigma}'(t) = (\sin t, -\cos t, d)$, where d is a constant. (a) Describe the curve. (b) Sketch the curve if you know that $\boldsymbol{\sigma}(0) = \mathbf{i}$. (c) What if in addition, $d = 0$?
44. Suppose that $\boldsymbol{\sigma}(t)$ is a vector function such that $\boldsymbol{\sigma}'(t) = -\boldsymbol{\sigma}(t)$. Show that $\boldsymbol{\sigma}(t) = e^{-t}\boldsymbol{\sigma}(0)$. (*Hint:* See Chapter 8.) What is the behavior of $\boldsymbol{\sigma}(t)$ as $t \to \infty$?
45. (a) Let $\boldsymbol{\sigma}_1(t)$ and $\boldsymbol{\sigma}_2(t)$ satisfy the differential equation $\boldsymbol{\sigma}''(t) = -\boldsymbol{\sigma}(t)$. Show that for any constants A_1 and A_2, $A_1\boldsymbol{\sigma}_1(t) + A_2\boldsymbol{\sigma}_2(t)$ satisfies the equation as well.
(b) Find as many solutions of $\boldsymbol{\sigma}''(t) = -\boldsymbol{\sigma}(t)$ as you can.
46. Suppose that $\boldsymbol{\sigma}(t)$ satisfies the differential equation $\boldsymbol{\sigma}''(t) + \omega^2\boldsymbol{\sigma}(t) = \mathbf{0}$. Describe and sketch the curve if $\boldsymbol{\sigma}(0) = (0, 0, 1)$ and $\boldsymbol{\sigma}'(0) = (0, \omega, \omega)$.
47. (a) Sketch the following curves. On each curve, indicate the points obtained when $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.
(i) $x = -t; y = 2t; z = 3t; 0 \leqslant t \leqslant 1$.
(ii) $x = -t^2; y = 2t^2; z = 3t^2; -1 \leqslant t \leqslant 1$.
(iii) $x = -\sin(\pi t/2); y = 2\sin(\pi t/2)$; $z = 3\sin(\pi t/2); 0 \leqslant t \leqslant 1$.
(b) Show that the set of points in space covered by each of these curves is the same. Discuss differences between the curves thought of as functions of t. (How fast do you move along the curve as t changes? How many times is each point covered?)

48. For each curve in Exercise 47, find the velocity vector $\mathbf{v}$ and the speed v as functions of t. Compute $\int_a^b v\,dt$, where $[a,b]$ is the defining interval for t in each case. What should this number represent? Explain the difference between the result for part (ii) and that for (i) and (iii).

49. Let θ and ϕ be fixed angles, and consider the following two curves:
(a) $x = \sin\phi\cos t,$
$y = \sin\phi\sin t, \quad 0 \leqslant t \leqslant 2\pi$
$z = \cos\phi;$
(b) $x = \sin t\cos\theta,$
$y = \sin t\sin\theta, \quad 0 \leqslant t \leqslant 2\pi$
$z = \cos t.$
Show that each curve is a circle lying on the sphere of radius 1 centered at the origin. Find the center and radius of each circle. Sketch the curves for $\phi = 45°$ and for $\theta = 45°$.

50. Suppose that $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant \phi \leqslant 2\pi$, and let
$$\sigma(\theta,\phi) = ((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi).$$
(Note that this is a vector function of two variables.)
(a) Describe each of the following curves:
  (i) $\sigma(\theta,0);\quad 0 \leqslant \theta \leqslant 2\pi;$
  (ii) $\sigma(\theta,\pi);\quad 0 \leqslant \theta \leqslant 2\pi;$
  (iii) $\sigma(\theta,\pi/2);\ 0 \leqslant \theta \leqslant 2\pi;$
  (iv) $\sigma(0,\phi);\quad 0 \leqslant \phi \leqslant 2\pi;$
  (v) $\sigma(\pi/2,\phi);\ 0 \leqslant \phi \leqslant 2\pi;$
  (vi) $\sigma(\pi/4,\phi);\ 0 \leqslant \phi \leqslant 2\pi.$
(b) Show that the point $\sigma(\theta,\phi)$ lies on the circle of radius $2+\cos\phi$ parallel to the xy plane and centered at $(0,0,\sin\phi)$.
(c) Show that $\sigma(\theta,\phi)$ lies on the doughnut-shaped surface (a *torus*) shown in Fig. 14.6.12.
(d) Describe and sketch the curve $((2+\cos t)\cos t, (2+\cos t)\sin t, \sin t)$.

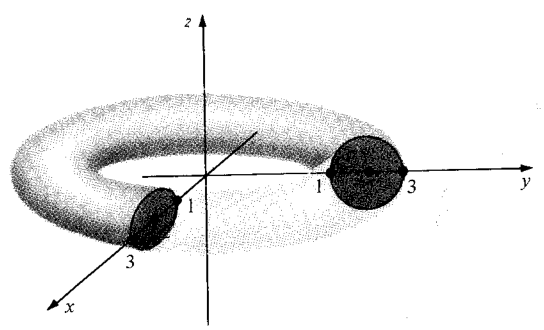


Figure 14.6.12. The points $\sigma(\theta,\phi)$ (Exercise 50) lie on this surface.

51. Suppose that $P_0 = (x_0, y_0, 0)$ is a point on the unit circle in the xy plane. Describe the set of points lying directly above or below P_0 on the right circular helix of Example 3. What is the vertical distance between coils of the helix?

★52. If $\sigma(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is a vector function, we may define $\lim_{t\to t_0}\sigma(t)$ componentwise; that is,
$$\lim_{t\to t_0}\sigma(t) = \left[\lim_{t\to t_0} f(t)\right]\mathbf{i} + \left[\lim_{t\to t_0} g(t)\right]\mathbf{j} + \left[\lim_{t\to t_0} h(t)\right]\mathbf{k}$$
if the three limits on the right-hand side all exist. Using this definition, show that
$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}[\sigma(t_0+\Delta t) - \sigma(t_0)] = \sigma'(t_0).$$

Prove the rules in Exercises 53–56 for vector functions.
★53. The sum rule.
★54. The scalar multiplication rule.
★55. The cross product rule.
★56. The chain rule.

★57. Let $\mathbf{r} = \sigma(t)$ be a parametric curve.
(a) Suppose there is a unit vector $\mathbf{u}$ (constant) such that $\sigma(t)\cdot\mathbf{u} = 0$ for all values of t. What can you say about the curve $\sigma(t)$?
(b) What can you say if $\sigma(t)\cdot\mathbf{u} = c$ for some constant c?
(c) What can you say if $\sigma(t)\cdot\mathbf{u} = b\|\sigma(t)\|$ for some constant b with $0 < b < 1$?

★58. Consider the curve given by
$$x = r\cos\omega t,\quad y = r\sin\omega t,\quad \text{and}\quad z = ct,$$
where r, ω, and c are positive constants and $-\infty < t < \infty$.
(a) What path is traced out by (x,y) in the plane?
(b) The curve in space lies on what cylinder?
(c) For what t_0 does the curve trace out one coil of the helix as t goes through the interval $0 \leqslant t \leqslant t_0$?
(d) What is the vertical distance between coils?
(e) The curve is a right-circular helix. Sketch it.

14.7 The Geometry and Physics of Space Curves

Particles moving in space according to physical laws can trace out geometrically interesting curves.

This section is concerned with applications of calculus: arc length, Newton's second law, and some geometry of space curves.

In Section 10.4, we found the arc length formula

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

for a parametric curve in the plane. A similar formula, with one term added, applies to curves in space.

Arc Length

Let $(x, y, z) = (f(t), g(t), h(t))$ be a parametric curve in space. The *length* of the curve, for t in the interval $[a, b]$, is defined to be

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

or

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \tag{1}$$

Example 1 Find the length of the helix $(\cos t, \sin t, t)$ for $0 \leqslant t \leqslant \pi$.

Example Here $f'(t) = -\sin t$, $g'(t) = \cos t$, and $h'(t) = 1$, so the integrand in the arc length formula (1) is $\sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, a constant. Thus the length is simply

$$L = \int_0^\pi \sqrt{2}\, dt = \pi\sqrt{2}\,. \blacktriangle$$

Notice that the integrand in the arc length formula is precisely the speed $\|\boldsymbol{\sigma}'(t)\|$ of a particle moving along the parametric curve. Thus the arc length, which can be written as $L = \int_a^b \|\boldsymbol{\sigma}'(t)\|\, dt$, is the integral of speed with respect to time and represents the total distance travelled by the particle between time a and time b.

Example 2 Find the arc length of $(\cos t, \sin t, t^2)$, $0 \leqslant t \leqslant \pi$.

Solution The curve $\boldsymbol{\sigma}(t) = (\cos t, \sin t, t^2)$ has velocity vector $\mathbf{v} = (-\sin t, \cos t, 2t)$. Since $\|\mathbf{v}\| = \sqrt{1 + 4t^2} = 2\sqrt{t^2 + (\frac{1}{2})^2}$, the arc length is

$$L = \int_0^\pi 2\sqrt{t^2 + (\tfrac{1}{2})^2}\, dt.$$

This integral may be evaluated using the formula (43) from the table of integrals:

$$\int \sqrt{x^2+a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\ln\left(x+\sqrt{x^2+a^2}\right)\right] + C.$$

Thus

$$\begin{aligned} L &= 2\cdot\tfrac{1}{2}\left[t\sqrt{t^2+\left(\tfrac{1}{2}\right)^2} + \left(\tfrac{1}{2}\right)^2\ln\left(t+\sqrt{t^2+\left(\tfrac{1}{2}\right)^2}\right)\right]\Bigg|_{t=0}^{\pi} \\ &= \pi\sqrt{\pi^2+\tfrac{1}{4}} + \tfrac{1}{4}\ln\left(\pi+\sqrt{\pi^2+\tfrac{1}{4}}\right) - \tfrac{1}{4}\ln\left(\sqrt{\tfrac{1}{4}}\right) \\ &= \frac{\pi}{2}\sqrt{1+4\pi^2} + \tfrac{1}{4}\ln\left(2\pi+\sqrt{1+4\pi^2}\right) \\ &\approx 10.63. \blacktriangle \end{aligned}$$

Example 3 Find the arc length of (e^t, t, e^t), $0 \leqslant t \leqslant 1$. [*Hint:* Use $u = \sqrt{1+2e^{2t}}$ to evaluate the integral.]

Solution $\sigma(t) = (e^t, t, e^t)$, so $\mathbf{v} = (e^t, 1, e^t)$, and $\|\mathbf{v}\| = \sqrt{1+2e^{2t}}$; so $L = \int_0^1 \sqrt{1+2e^{2t}}\,dt$. To evaluate this integral, set $u = \sqrt{1+2e^{2t}}$, which leads to

$$\begin{aligned} \int \sqrt{1+2e^{2t}}\,dt &= \int u\frac{u\,du}{u^2-1} \\ &= \int\left[1+\frac{1}{2}\left(\frac{1}{u-1}\right) - \frac{1}{2}\left(\frac{1}{u+1}\right)\right]du \quad \text{(partial fractions)} \\ &= u + \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1) + C \\ &= \sqrt{1+2e^{2t}} + \frac{1}{2}\ln\frac{\sqrt{1+2e^{2t}}-1}{\sqrt{1+2e^{2t}}+1} + C. \end{aligned}$$

(This result may be checked by differentiation.) To find L, we evaluate the last expression at $t = 0$ and $t = 1$ and subtract, to obtain

$$L = \sqrt{1+2e^2} + \frac{1}{2}\ln\frac{\sqrt{1+2e^2}-1}{\sqrt{1+2e^2}+1} - \sqrt{3} - \frac{1}{2}\ln\frac{\sqrt{3}-1}{\sqrt{3}+1} \approx 2.64. \blacktriangle$$

We turn next to the study of curves followed by physical particles subject to forces.

If a particle of mass m moves in space, the total force $\mathbf{F}$ acting on it at any time is a vector which is related to the acceleration by *Newton's second law* (see Section 8.1): $\mathbf{F} = m\mathbf{a}$.

In many situations, the force is a given function of position $\mathbf{r}$ (the "force law"), and the problem of interest is to find the vector function $\mathbf{r} = \sigma(t)$ describing a particle's motion, given the initial position and velocity. Thus, Newton's second law becomes a differential equation for $\sigma(t)$, and techniques of differential equations can be used to solve it (as we solved the spring equation in Section 8.1). For example, a planet moving around the sun (considered to be located at the origin) obeys to a high degree of accuracy *Newton's law of gravitation*:

$$\mathbf{F} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} = -\frac{GmM}{r^3}\mathbf{r},$$

This integral may be evaluated using the formula (43) from the table of integrals:

$$\int \sqrt{x^2+a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\ln\left(x+\sqrt{x^2+a^2}\right)\right] + C.$$

Thus

$$\begin{aligned} L &= 2\cdot\tfrac{1}{2}\left[t\sqrt{t^2+\left(\tfrac{1}{2}\right)^2} + \left(\tfrac{1}{2}\right)^2\ln\left(t+\sqrt{t^2+\left(\tfrac{1}{2}\right)^2}\right)\right]\Bigg|_{t=0}^{\pi} \\ &= \pi\sqrt{\pi^2+\tfrac{1}{4}} + \tfrac{1}{4}\ln\left(\pi+\sqrt{\pi^2+\tfrac{1}{4}}\right) - \tfrac{1}{4}\ln\left(\sqrt{\tfrac{1}{4}}\right) \\ &= \frac{\pi}{2}\sqrt{1+4\pi^2} + \tfrac{1}{4}\ln\left(2\pi+\sqrt{1+4\pi^2}\right) \\ &\approx 10.63. \ \blacktriangle \end{aligned}$$

Example 3 Find the arc length of (e^t, t, e^t), $0 \leqslant t \leqslant 1$. [*Hint:* Use $u = \sqrt{1+2e^{2t}}$ to evaluate the integral.]

Solution $\boldsymbol{\sigma}(t) = (e^t, t, e^t)$, so $\mathbf{v} = (e^t, 1, e^t)$, and $\|\mathbf{v}\| = \sqrt{1+2e^{2t}}$; so $L = \int_0^1\sqrt{1+2e^{2t}}\,dt$. To evaluate this integral, set $u = \sqrt{1+2e^{2t}}$, which leads to

$$\begin{aligned} \int\sqrt{1+2e^{2t}}\,dt &= \int u\frac{u\,du}{u^2-1} \\ &= \int\left[1+\frac{1}{2}\left(\frac{1}{u-1}\right)-\frac{1}{2}\left(\frac{1}{u+1}\right)\right]du \quad \text{(partial fractions)} \\ &= u + \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1) + C \\ &= \sqrt{1+2e^{2t}} + \frac{1}{2}\ln\frac{\sqrt{1+2e^{2t}}-1}{\sqrt{1+2e^{2t}}+1} + C. \end{aligned}$$

(This result may be checked by differentiation.) To find L, we evaluate the last expression at $t=0$ and $t=1$ and subtract, to obtain

$$L = \sqrt{1+2e^2} + \frac{1}{2}\ln\frac{\sqrt{1+2e^2}-1}{\sqrt{1+2e^2}+1} - \sqrt{3} - \frac{1}{2}\ln\frac{\sqrt{3}-1}{\sqrt{3}+1} \approx 2.64. \ \blacktriangle$$

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If a particle of mass m moves in space, the total force $\mathbf{F}$ acting on it at any time is a vector which is related to the acceleration by *Newton's second law* (see Section 8.1): $\mathbf{F} = m\mathbf{a}$.

In many situations, the force is a given function of position $\mathbf{r}$ (the "force law"), and the problem of interest is to find the vector function $\mathbf{r} = \boldsymbol{\sigma}(t)$ describing a particle's motion, given the initial position and velocity. Thus, Newton's second law becomes a differential equation for $\boldsymbol{\sigma}(t)$, and techniques of differential equations can be used to solve it (as we solved the spring equation in Section 8.1). For example, a planet moving around the sun (considered to be located at the origin) obeys to a high degree of accuracy *Newton's law of gravitation*:

$$\mathbf{F} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} = -\frac{GmM}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the vector pointing from the sun to the planet at time t, M is the mass of the sun, m that of the planet, $r = \|\mathbf{r}\|$, and G is the gravitational constant ($G = 6.67 \times 10^{-11}$ newton meter^2 per kilogram^2). The differential equation arising from this force law is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^3}\mathbf{r}.$$

Rather than solving this equation here, we shall content ourselves with understanding its consequences for the case of circular motion.

Example 4 A particle of mass m is moving in the xy plane at constant speed v in a circular path of radius r_0. Find the acceleration of the particle and the force acting on it.

Solution Let $\mathbf{r}$ be the vector from the center of the circle to the particle at time t. Motion of the type described is given by

$$\mathbf{r} = r_0\cos\left(\frac{tv}{r_0}\right)\mathbf{i} + r_0\sin\left(\frac{tv}{r_0}\right)\mathbf{j}.$$

Differentiating twice, we see that

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\frac{v^2}{r_0}\cos\left(\frac{tv}{r_0}\right)\mathbf{i} - \frac{v^2}{r_0}\sin\left(\frac{tv}{r_0}\right)\mathbf{j} = -\frac{v^2}{r_0^2}\mathbf{r}.$$

The force acting on the particle is $\mathbf{F} = m\mathbf{a} = -(mv^2/r_0^2)\mathbf{r}$. ▲

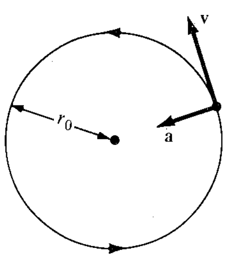

Figure 14.7.1. The acceleration vector of a particle in uniform circular motion points to the center of the circle.

Example 4 shows that in uniform circular motion, the acceleration vector points in a direction opposite to $\mathbf{r}$—that is, it is directed toward the center of the circle (see Fig. 14.7.1). This acceleration, multiplied by the mass of the particle, is called the *centripetal force*. Note that even though the speed is constant, the *direction* of the velocity vector is continually changing, which is why there is an acceleration. By Newton's law, some force must cause the acceleration which keeps the particle moving in its circular path. In whirling a rock at the end of a string, you must constantly be pulling on the string. If you stop that force by releasing the string, the rock will fly off in a straight line tangent to the circle. The force needed to keep a planet or satellite bound into an elliptical or circular orbit is supplied by gravity. The force needed to keep a car going around a curve may be supplied by the friction of the tires against the road or by direct pressure if the road is banked (see Exercise 12).

Suppose that a satellite is moving with a speed v around a planet with mass M in a circular orbit of radius r_0. Then the force computed in Example 4 must equal that in Newton's law:

$$-\frac{v^2}{r_0^2}\mathbf{r} = -\frac{GM}{r_0^3}\mathbf{r}.$$

The lengths of the vectors on both sides of this equation must be equal. Hence

$$v^2 = \frac{GM}{r_0}. \tag{2}$$

If T is the period of one revolution, then $2\pi r_0/T = v$ (distance/time = speed); substituting this value for v in equation (2) and solving for T^2, we obtain the rule:

$$T^2 = r_0^3\frac{(2\pi)^2}{GM}. \tag{3}$$

The square of the period is proportional to the cube of the radius. This law is one of the famous three which were discovered empirically by Kepler before Newton's laws were formulated; it enables one to compute the period of a satellite given the radius of its orbit or to determine the radius of the orbit if the period is prescribed. If both the radius and period are known, (3) can be used to determine GM, and hence if G is known, M can be computed.

Example 5 Suppose we want to have a satellite in circular orbit about the earth in such a way that it stays fixed in the sky over one point on the equator. What should be the radius of such an orbit? (The mass of the earth is 5.98×10^{24} kilograms.)

Solution The period of the satellite should be 1 day, so $T = 60 \times 60 \times 24 = 86{,}400$ seconds. By formula (3), the radius of the orbit should satisfy

$$r_0^3 = \frac{T^2GM}{(2\pi)^2} = \frac{(86{,}400)^2 \times (6.67 \times 10^{-11}) \times (5.98 \times 10^{24})}{(2\pi)^2}$$
$$= 7.54 \times 10^{22} \text{ meters}^3,$$

so $r_0 = 4.23 \times 10^7$ meters $= 42{,}300$ kilometers
$= 26{,}200$ miles. ▲

Example 6 Let $\mathbf{r} = \boldsymbol{\sigma}(t)$ be the vector from a fixed point to the position of an object, $\mathbf{v}$ the velocity, and $\mathbf{a}$ the acceleration. Suppose that $\mathbf{F}$ is the force acting at time t.

(a) Prove that $(d/dt)(m\mathbf{r} \times \mathbf{v}) = \boldsymbol{\sigma} \times \mathbf{F}$, (that is, "*rate of change of angular momentum = torque*"). What can you conclude if $\mathbf{F}$ is parallel to $\mathbf{r}$? Is this the case in planetary motion?

(b) Prove that a planet moving about the sun does so in a fixed plane. (This is another of Kepler's laws.)

Solution (a) We use the rules of differentiation for vector functions:

$$\frac{d}{dt}(m\mathbf{r} \times \mathbf{v}) = m\frac{d\mathbf{r}}{dt} \times \mathbf{v} + m\mathbf{r} \times \frac{d\mathbf{v}}{dt} = m\mathbf{v} \times \mathbf{v} + m\mathbf{r} \times \mathbf{a}$$
$$= \mathbf{0} + \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \mathbf{F}.$$

If $\mathbf{F}$ is parallel to $\mathbf{r}$, then this last cross product is $\mathbf{0}$. Thus $m\mathbf{r} \times \mathbf{v}$ must be a constant vector. It represents the angular momentum, a quantity which measures the tendency of a spinning body to keep spinning. The magnitude of $m\mathbf{r} \times \mathbf{v}$ measures the amount of angular momentum, and the direction is along the axis of spin. If the derivative above is zero, it means that angular momentum is conserved; both its magnitude and its direction are preserved. This is the case for our model of planetary motion in which the sun is regarded as fixed and the gravitational force

$$\mathbf{F} = -\frac{GmM}{\|r\|^3}\mathbf{r}$$

is parallel to the vector $\mathbf{r}$ from the sun to the planet. (The actual situation is a bit more complicated than this: in fact, both the sun and the planet move around their common center of mass. However, the mass M of the sun is so much greater than the mass m of the planet that this center of mass is very close to the center of the sun, and our approximation is quite good. Things would be more complicated, for example, in a double-star system where the masses were more nearly the same and the center of gravity somewhere between. What is conserved is the total angular momentum of the whole system, taking both stars into account.)

(b) Let $\mathbf{l} = m\mathbf{r} \times \mathbf{v}$ be the angular momentum vector. Then certainly $\mathbf{r} \cdot \mathbf{l} = 0$. We argued above that $d\mathbf{l}/dt = \mathbf{0}$, so $\mathbf{l}$ is a constant vector. Since $\mathbf{r}$ satisfies $\mathbf{r} \cdot \mathbf{l} = 0$, the planet stays in the plane through the sun with normal vector $\mathbf{l}$. ▲

Our third and final application in this section is to the geometry of space curves.

Differential geometry is the branch of mathematics in which calculus is used to study the geometry of curves, surfaces, and higher dimensional objects. When we studied the arc length of curves, we were already doing differential geometry—now we will go further and introduce the important idea of *curvature*.

The curvature of a curve in the plane or in space is a measure of the rate at which the direction of motion along the curve is changing. A curve with curvature zero is just a straight line. We can define the curvature as the rate of change of the velocity vector, if the length of this vector happens to be 1; otherwise the change in length of the velocity vector confuses the issue. We therefore make the following definitions.

Parametrization by Arc Length

Let $\mathbf{r} = \boldsymbol{\sigma}(t)$ be a parametric curve.

1. The curve is called *regular* if $\mathbf{v} = \boldsymbol{\sigma}'(t)$ is not equal to $\mathbf{0}$ for any t.
2. If the curve is regular, the vector $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\| = \boldsymbol{\sigma}'(t)/\|\boldsymbol{\sigma}'(t)\|$ is called the *unit tangent vector* to the curve.
3. If the length of $\boldsymbol{\sigma}'(t)$ is constant and equal to 1 (in which case $\mathbf{T} = \mathbf{v}$), the curve is said to be *parametrized by arc length*.

Example 7 Suppose that the curve $\mathbf{r} = \boldsymbol{\sigma}(t)$ is parametrized by arc length. Show that the length of the curve between $t = a$ and $t = b$ is simply $b - a$.

Solution The integrand in the arc length formula (1) is constant and equal to 1 if the curve is parametrized by arc length. Thus

$$L = \int_a^b 1\,dt = b - a. \ \blacktriangle$$

If a curve $\mathbf{r} = \boldsymbol{\sigma}(t)$, as it is presented to us, is regular but not parametrized by arc length, we can introduce a new independent variable so that the new curve *is* parametrized by arc length. In fact, we can choose a value a in the domain of the curve and define $s = p(t)$ to be the arc length $\int_a^t \|\boldsymbol{\sigma}'(u)\|\,du$ of the curve between a and t. We have $ds/dt = \|\boldsymbol{\sigma}'(t)\| > 0$ since the curve is regular, so the inverse function $t = q(s)$ exists (see Section 5.3). Now look at the new curve $\mathbf{r} = \boldsymbol{\sigma}_1(s) = \boldsymbol{\sigma}(q(s))$, which goes through the same points in space as the original curve but at a different speed. In fact, the new speed is

$$\begin{aligned}\|\boldsymbol{\sigma}_1'(s)\| &= \|q'(s)\boldsymbol{\sigma}'(q(s))\| \quad \text{(chain rule)}\\ &= q'(s)\|\boldsymbol{\sigma}'(q(s))\| \quad (q'(s) \text{ is positive})\\ &= \frac{1}{p'(q(s))}\|\boldsymbol{\sigma}'(q(s))\|.\end{aligned}$$

However, by the fundamental theorem of calculus and the definition of p, $p'(t) = \|\boldsymbol{\sigma}'(t)\|$, so $\|\boldsymbol{\sigma}_1'(s)\| = 1$, and so the new curve is parametrized by arc length.

Example 8 Find the arc length parametrization for the helix $(\cos t, \sin t, t)$.

Solution We have $\|\mathbf{v}\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. Taking $a = 0$, we have $s = p(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$, so $s = \sqrt{2}\, t$, and $t = s/\sqrt{2}$; the curve in arc length parametrization is therefore $(\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})$. ▲

The arc length parametrization is mostly useful for theoretical purposes, since the integral in its definition is often impossible to evaluate. Still, the *existence* of this parametrization makes the definitions which follow much simpler.

Whenever a curve is parametrized by arc length, we will denote this parameter by s. Notice that in this case $\mathbf{T} = \mathbf{v} = d\mathbf{r}/ds$. Now we can define the curvature of a curve.

Curvature

Let $\mathbf{T}$ be the unit tangent vector of a curve parametrized by arc length. The scalar $k = \|d\mathbf{T}/ds\|$ is called the *curvature* of the curve. If $k \neq 0$, the unit vector $\mathbf{N} = (d\mathbf{T}/ds)/\|d\mathbf{T}/ds\|$ is called the *principal normal vector* to the curve.

Let us show that the principal normal vector is perpendicular to the unit tangent vector. Since $\mathbf{T}$ has constant length, we know, by Example 9 of the previous section, that $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$. Since $\mathbf{N}$ has the same direction as $d\mathbf{T}/ds$, it is perpendicular to $\mathbf{T}$ as well.

Example 9 Compute the curvature and principal normal vector of the helix in Example 8.

Solution We have $\mathbf{T} = -(1/\sqrt{2})\sin(s/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\cos(s/\sqrt{2})\mathbf{j} + (1/\sqrt{2})\mathbf{k}$, so $d\mathbf{T}/ds = -(1/2)\cos(s/\sqrt{2})\mathbf{i} - (1/2)\sin(s/\sqrt{2})\mathbf{j}$; the curvature is

$$\sqrt{\frac{1}{4}\cos^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{4}\sin^2\left(\frac{s}{\sqrt{2}}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2},$$

and the principal normal vector is $-\cos(s/\sqrt{2})\mathbf{i} - \sin(s/\sqrt{2})\mathbf{j}$. ▲

If a curve is not parametrized by arc length, it is possible to compute the curvature and principal normal vector directly by the following formulas:

$$k = \frac{\|\mathbf{v} \times \mathbf{v}'\|}{\|\mathbf{v}\|^3}, \tag{4}$$

$$\mathbf{N}(t) = \frac{(\mathbf{v}\cdot\mathbf{v})\mathbf{v}' - (\mathbf{v}'\cdot\mathbf{v})\mathbf{v}}{\|(\mathbf{v}\cdot\mathbf{v})\mathbf{v}' - (\mathbf{v}'\cdot\mathbf{v})\mathbf{v}\|}. \tag{5}$$

We now prove formula (4). The curvature is defined as $\|d\mathbf{T}/ds\|$. We must use the chain rule to express this in terms of the original parametrization t. First of all, we have $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\| = \mathbf{v}/(\mathbf{v}\cdot\mathbf{v})^{1/2}$, so

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{1}{(ds/dt)}\frac{d\mathbf{T}}{dt}.$$

Now $ds/dt = \|\mathbf{v}\|$ and so

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= \frac{d}{dt}\left(\frac{\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{1/2}}\right)\\ &= \left(\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})^{-1/2}\right)\mathbf{v} + (\mathbf{v}\cdot\mathbf{v})^{-1/2}\frac{d\mathbf{v}}{dt}\\ &= -\frac{1}{2}(\mathbf{v}\cdot\mathbf{v})^{-3/2}2\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)\mathbf{v} + (\mathbf{v}\cdot\mathbf{v})^{-1/2}\frac{d\mathbf{v}}{dt}\\ &= (\mathbf{v}\cdot\mathbf{v})^{-3/2}\left[-\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)\mathbf{v} + (\mathbf{v}\cdot\mathbf{v})\frac{d\mathbf{v}}{dt}\right],\end{aligned}$$

so

$$\begin{aligned}\left\|\frac{d\mathbf{T}}{dt}\right\|^2 &= (\mathbf{v}\cdot\mathbf{v})^{-3}\left[-\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)\mathbf{v} + (\mathbf{v}\cdot\mathbf{v})\frac{d\mathbf{v}}{dt}\right]\cdot\left[-\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)\mathbf{v} + (\mathbf{v}\cdot\mathbf{v})\frac{d\mathbf{v}}{dt}\right]\\ &= (\mathbf{v}\cdot\mathbf{v})^{-3}\left[\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)^2(\mathbf{v}\cdot\mathbf{v}) - 2\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)^2(\mathbf{v}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v})^2\left(\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt}\right)\right]\\ &= (\mathbf{v}\cdot\mathbf{v})^{-2}\left[(\mathbf{v}\cdot\mathbf{v})\left(\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt}\right) - \left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)^2\right] = \|\mathbf{v}\|^{-4}\left\|\mathbf{v}\times\frac{d\mathbf{v}}{dt}\right\|^2.\end{aligned}$$

(See Exercise 38a, Section 13.5.) Thus

$$k = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\|^{-2}\left\|\mathbf{v}\times\frac{d\mathbf{v}}{dt}\right\| = \frac{\|\mathbf{v}\times d\mathbf{v}/dt\|}{\|\mathbf{v}\|^3},$$

which is formula (4).

In Exercise 20, the reader is asked to derive (5) using similar methods.

Example 10 Find the curvature of the exponential spiral $(e^{-t}\cos t, e^{-t}\sin t, 0)$ (Fig. 14.7.2). What happens as $t\to\infty$?

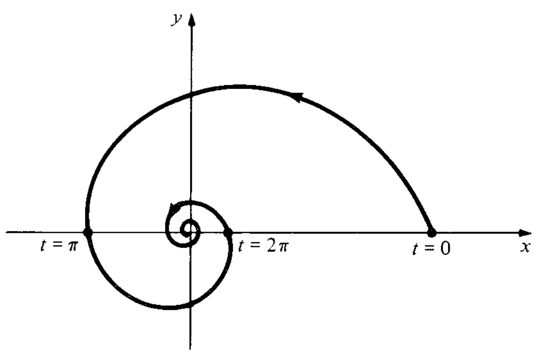

Figure 14.7.2. Graph of the exponential spiral in the (x, y) plane.

Solution We have

$$\mathbf{v} = (-e^{-t}\cos t - e^{-t}\sin t)\mathbf{i} + (-e^{-t}\sin t + e^{-t}\cos t)\mathbf{j}$$

and

$$\begin{aligned}\mathbf{v}' &= (e^{-t}\cos t + e^{-t}\sin t + e^{-t}\sin t - e^{-t}\cos t)\mathbf{i}\\ &\quad + (e^{-t}\sin t - e^{-t}\cos t - e^{-t}\cos t - e^{-t}\sin t)\mathbf{j}\\ &= 2e^{-t}(\sin t\mathbf{i} - \cos t\mathbf{j}).\end{aligned}$$

Then

$$\begin{aligned}\mathbf{v}\times\mathbf{v}' &= 2e^{-2t}\begin{vmatrix} -\cos t - \sin t & -\sin t + \cos t\\ \sin t & -\cos t\end{vmatrix}\mathbf{k}\\ &= 2e^{-2t}(\cos^2 t + \cos t\sin t + \sin^2 t - \cos t\sin t)\mathbf{k} = 2e^{-2t}\mathbf{k},\end{aligned}$$

and so $\|\mathbf{v} \times \mathbf{v}'\| = 2e^{-2t}$. Since

$$\|\mathbf{v}\| = \{[e^{-t}(\cos t + \sin t)]^2 + [e^{-t}(\cos t - \sin t)]^2\}^{1/2},$$

$$\|\mathbf{v}\|^3 = e^{-3t}\left[(\cos t + \sin t)^2 + (\cos t - \sin t)^2\right]^{3/2} = e^{-3t}2^{3/2}.$$

By formula (4),

$$k = \frac{2e^{-2t}}{e^{-3t}2^{3/2}} = \frac{e^t}{\sqrt{2}}.$$

As $t \to \infty$, the curvature approaches infinity as the spiral wraps more and more tightly about the origin. ▲

Exercises for Section 14.7

Find the arc length of the given curve on the specified interval in Exercises 1–6.

1. $(2\cos t, 2\sin t, t)$; $0 \leqslant t \leqslant 2\pi$.
2. $(1, 3t^2, t^3)$; $0 \leqslant t \leqslant 1$.
3. $(\sin 3t, \cos 3t, 2t^{3/2})$; $0 \leqslant t \leqslant 1$.
4. $(t + 1, \frac{2\sqrt{2}}{3} t^{3/2} + 7, \frac{1}{2} t^2)$ for $1 \leqslant t \leqslant 2$.
5. (t, t, t^2) for $1 \leqslant t \leqslant 2$.
6. $(t, t\sin t, t\cos t)$; $0 \leqslant t \leqslant \pi$.
7. A body of mass 2 kilograms moves in a circular path on a circle of radius 3 meters, making one revolution every 5 seconds. Find the centripetal force acting on the body.
8. Find the centripetal force acting on a body of mass 4 kilograms, moving on a circle of radius 10 meters with a frequency of 2 revolutions per second.
9. A satellite is in a circular orbit 500 miles above the surface of the earth. What is the period of the orbit? (See Example 5; 1 mile = 1.609 kilometer; the radius of the earth is 6370 km).
10. What is the gravitational acceleration on the satellite in Exercise 9? The centripetal acceleration?
11. For a falling body near the surface of the earth, the force of gravity can be approximated very well as a constant downward force with magnitude $F = GmM/R^2$, where G is the gravitational constant, M the mass of the earth, m the mass of the body, and R the radius of the earth (6.37×10^6 meters).
   (a) Show that this approximation means that any body falling freely (neglecting air resistance) near the surface of the earth experiences a constant acceleration of $g = 9.8$ meters per second per second. Note that this acceleration is independent of m: any two bodies fall at the same rate.
   (b) Show that the flight path of a projectile or a baseball is a parabola (see Example 11 in Section 14.6).
12. (a) Suppose that a car is going around a circular curve of radius r at speed v. It will then exert an outward horizontal force on the roadway due to the centripetal acceleration and a vertical force due to gravity. At what angle θ should the roadway be banked so that the total force tends to press the car directly into (perpendicular to) the roadway? (See Fig. 14.7.3.) How does the bank angle depend on r and on the speed v?

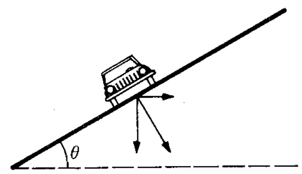

Figure 14.7.3. For what value of θ does the total force press directly into the roadway?

   (b) Discuss how you might treat the design problem in part (a) for a curve that is not part of a circle. Design an elliptical racetrack with major axis 800 meters, minor axis 500 meters, and speed 160 kilometers per hour.
13. A particle with charge q moving with velocity vector $\mathbf{v}$ through a magnetic field is acted on by the force $\mathbf{F} = (q/c)\mathbf{v} \times \mathbf{B}$, where c is the speed of light and $\mathbf{B}$ is a vector describing the magnitude and direction of the magnetic field. Suppose that:
   (1) The particle has mass m and is following a path $\mathbf{r} = \boldsymbol{\sigma}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
   (2) $\boldsymbol{\sigma}(0) = \mathbf{i}$; $\boldsymbol{\sigma}'(0) = a\mathbf{j} + c\mathbf{k}$.
   (3) The magnetic field is constant and uniform given by a vector $\mathbf{B} = b\mathbf{k}$.
   (a) Use the equation $\mathbf{F} = (q/c)\mathbf{v} \times \mathbf{B}$ to write differential equations relating the components of $\mathbf{a} = \boldsymbol{\sigma}''(t)$ and $\mathbf{v} = \boldsymbol{\sigma}'(t)$.

(b) Solve these equations to obtain the components of $\boldsymbol{\sigma}(t)$. [*Hint:* Integrate the equations for d^2x/dt^2 once, use item (2) in the list above to determine the constant of integration, and substitute the resulting expression for dx/dt into the equation for d^2y/dt^2 to get an equation similar to the spring equation solved in Section 8.1.]

(c) Show that the path is a right circular helix. What are the radius and axis of the cylinder on which it lies? (The dimensions of the helix followed by a particle in a magnetic field in a bubble chamber are used to measure the charge to mass ratio of the particle.)

14. In Exercise 13, how does the geometry of the helix change if (a) m is doubled, (b) q is doubled, (c) $\|\boldsymbol{\sigma}'(0)\|$ is doubled?

15. Show that a circle of radius r has constant curvature $1/r$.

16. Compute the curvature and principal normal vector for the helix $(r\cos wt, r\sin wt, ct)$ in terms of r, w, and c.

17. Find the curvature of the ellipse $x^2 + 2y^2 = 1$, $z = 0$. (Choose a suitable parametrization.)

18. Compute the curvature and principal normal vector of the elliptical helix $(\cos t, 2\sin t, t)$.

19. Show that if the curvature of a curve is identically zero, then the curve is a straight line.

★20. Derive formula (5) by using the methods used to derive (4).

★21. A particle is moving along a curve at constant speed. Express the magnitude of the force on the particle in terms of the mass of the particle, the speed of the particle, and the curvature of the curve.

★22. Let $\mathbf{T}$ and $\mathbf{N}$ be the unit tangent and principal normal vectors to a space curve $\mathbf{r} = \boldsymbol{\sigma}(t)$. Define a third unit vector perpendicular to them by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. This is called the *binormal* vector. Together, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed system of mutually orthogonal unit vectors which may be thought of as moving along the curve (see Fig. 14.7.4.)

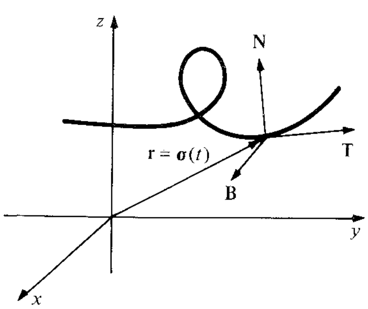

Figure 14.7.4. The vectors **T**, **N**, and **B** form a "moving basis" along the curve.

(a) Show that
$$\mathbf{B} = (\mathbf{v} \times \mathbf{a})/\|\mathbf{v} \times \mathbf{a}\| = (\mathbf{v} \times \mathbf{a})/(k\|\mathbf{v}\|^3),$$
where $\mathbf{a}$ is the acceleration vector.

(b) Show that $(d\mathbf{B}/dt) \cdot \mathbf{B} = 0$. [*Hint:* $\|\mathbf{B}\|^2 = 1$ is constant.]

(c) Show that $(d\mathbf{B}/dt) \cdot \mathbf{T} = 0$. [*Hint:* Take derivatives in $\mathbf{B} \cdot \mathbf{T} = 0$.]

(d) Show that $d\mathbf{B}/dt$ is a scalar multiple of $\mathbf{N}$.

(e) Using part (d) we can define a scalar-valued function τ called the *torsion* by $d\mathbf{B}/dt = -\tau\|\mathbf{v}\|\mathbf{N}$. Show that
$$\tau = \frac{[\boldsymbol{\sigma}'(t) \times \boldsymbol{\sigma}''(t)] \cdot \boldsymbol{\sigma}'''(t)}{\|\boldsymbol{\sigma}'(t) \times \boldsymbol{\sigma}''(t)\|^2}.$$

★23. (a) Show that if a curve lies in a plane, then the torsion τ is identically zero. [*Hint*: The vector function $\boldsymbol{\sigma}(t)$ must satisfy an equation of the form $\boldsymbol{\sigma}(t) \cdot \mathbf{n} = 0$. By taking successive derivatives show that $\boldsymbol{\sigma}'$, $\boldsymbol{\sigma}''$, and $\boldsymbol{\sigma}'''$ all lie in the same plane through the origin. What does this do to the triple product in Exercise 22(e)?]

(b) Show that $\mathbf{B}$ is then constant and is a normal vector to the plane in which the curve lies.

★24. If the torsion is not zero, it gives a measure of how fast the curve is tending to twist out of the plane. Compute the binormal vector and the torsion for the helix of Example 8.

★25. Using the results of Exercises 22 and 23, prove the following *Frenet formulas* for a curve parametrized by arc length:
$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= \phantom{-k\mathbf{T}} \quad k\mathbf{N}, \\ \frac{d\mathbf{N}}{ds} &= -k\mathbf{T} \qquad\quad + \tau\mathbf{B}, \\ \frac{d\mathbf{B}}{ds} &= \phantom{-k\mathbf{T}} \quad -\tau\mathbf{N}. \end{aligned}$$
[*Hint:* To get the second formula from the others, note that $\mathbf{N} \cdot \mathbf{N}$, $\mathbf{N} \cdot \mathbf{B}$, and $\mathbf{N} \cdot \mathbf{T}$ are constant. Take derivatives and use earlier formulas to get $(d\mathbf{N}/ds) \cdot \mathbf{B}$ and $(d\mathbf{N}/ds) \cdot \mathbf{T}$.]

★26. Kepler's first law of planetary motion states that *the orbit of each planet is an ellipse with the sun as one focus*. The origin $(0,0)$ is placed at the sun, and polar coordinates (r,θ) are introduced. The planet's motion is $r = r(t)$, $\theta = \theta(t)$, and these are related by $r(t) = l/[1 + e\cos\theta(t)]$, where $l = k^2/GM$ and $e^2 = 1 - (2k^2E/G^2M^2m)$; k is a constant, G is the universal gravitation constant, E is the energy of the system, M and m are the masses of the sun and planet, respectively

(a) Assume $e < 1$. Change to rectangular coordinates to verify that the planet's orbit is an ellipse.

(b) Let $\mu = 1/r$. Verify the *energy equation*
$$(d\mu/d\theta)^2 + \mu^2 = (2/k^2m)(GMm\mu - E).$$

Supplement to Chapter 14: Rotations and the Sunshine Formula

Rotations described in terms of the cross product are used to derive the sunshine formula.

The purpose of this section is to derive the sunshine formula, which has been stated and used in the supplements to Chapters 5 and 10. Before we begin the actual derivation, we will study some properties of rotations in preparation for the description of the earth rotating on its axis. The cross product, introduced in Section 13.5, will be used extensively here.

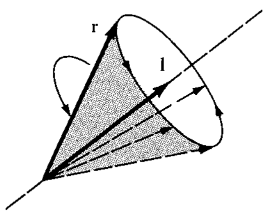

Figure 14.S.1. If $\mathbf{r}$ rotates about $\mathbf{l}$, its tip describes a circle.

Consider two unit vectors $\mathbf{l}$ and $\mathbf{r}$ in space with the same base point. If we rotate $\mathbf{r}$ about the axis through $\mathbf{l}$, then the tip of $\mathbf{r}$ describes a circle (Fig. 14.S.1). (Imagine $\mathbf{l}$ and $\mathbf{r}$ glued rigidly at their base points and then spun about the axis through $\mathbf{l}$.) Assume that the rotation is at a uniform rate counterclockwise (when viewed from the tip of $\mathbf{l}$), making a complete revolution in T units of time. The vector $\mathbf{r}$ now is a vector function of time, so we may write $\mathbf{r} = \boldsymbol{\sigma}(t)$. Our first aim is to find a convenient formula for $\boldsymbol{\sigma}(t)$ in terms of its starting position $\mathbf{r}_0 = \boldsymbol{\sigma}(0)$.

Let λ denote the angle between $\mathbf{l}$ and $\mathbf{r}_0$; we can assume that $\lambda \neq 0$ and $\lambda \neq \pi$, i.e., $\mathbf{l}$ and $\mathbf{r}_0$ are not parallel, for otherwise $\mathbf{r}$ would not rotate. Construct the unit vector $\mathbf{m}_0$ as shown in Fig. 14.S.2. From this figure we see that

$$\mathbf{r}_0 = \cos\lambda\mathbf{l} + \sin\lambda\mathbf{m}_0 . \tag{1}$$

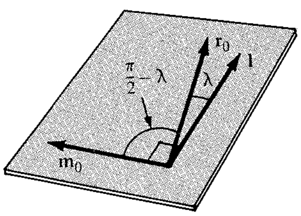

Figure 14.S.2. The vector $\mathbf{m}_0$ is in the plane of $\mathbf{r}_0$ and $\mathbf{l}$, is orthogonal to $\mathbf{l}$, and makes an angle of $(\pi/2) - \lambda$ with $\mathbf{r}_0$.

(In fact, formula (1) can be taken as the algebraic definition of $\mathbf{m}_0$ by writing $\mathbf{m}_0 = (1/\sin\lambda)\mathbf{r}_0 - (\cos\lambda/\sin\lambda)\mathbf{l}$. We assumed that $\lambda \neq 0$, and $\lambda \neq \pi$, so $\sin\lambda \neq 0$.)

Now add to this figure the unit vector $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0$. (See Fig. 14.S.3.) The triple $(\mathbf{l}, \mathbf{m}_0, \mathbf{n}_0)$ consists of three mutually orthogonal unit vectors, just like $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

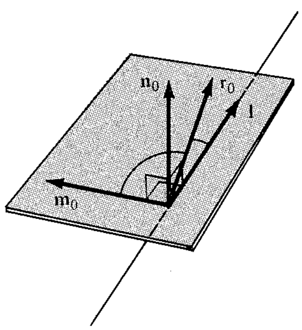

Figure 14.S.3. The triple $(\mathbf{l}, \mathbf{m}_0, \mathbf{n}_0)$ is a right-handed orthogonal set of unit vectors.

Example 1 Let $\mathbf{l} = (1/\sqrt{3})(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r}_0 = \mathbf{k}$. Find $\mathbf{m}_0$ and $\mathbf{n}_0$.

Solution The angle between $\mathbf{l}$ and $\mathbf{r}_0$ is given by $\cos\lambda = \mathbf{l} \cdot \mathbf{r}_0 = 1/\sqrt{3}$. This was determined by dotting both sides of formula (1) by $\mathbf{l}$ and using the fact that $\mathbf{l}$ is a unit vector. Thus $\sin\lambda = \sqrt{1 - \cos^2\lambda} = \sqrt{2/3}$, and so from formula (1) we get

$$\begin{aligned}
\mathbf{m}_0 &= \frac{1}{\sin\lambda}\boldsymbol{\sigma}(0) - \frac{\cos\lambda}{\sin\lambda}\mathbf{l} \\
&= \sqrt{\frac{3}{2}}\,\mathbf{k} - \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}} \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\
&= \frac{2}{\sqrt{6}}\mathbf{k} - \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j})
\end{aligned}$$

and

$$\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{6}} \end{vmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}. \blacktriangle$$

Return to Fig. 14.S.3 and rotate the whole picture about the axis $\mathbf{l}$. Now $\mathbf{m}$ and $\mathbf{n}$ will vary with time as well. Since the angle λ remains constant, formula (1) applied after time t to $\mathbf{r}$ and $\mathbf{l}$ gives

$$\mathbf{m} = \frac{1}{\sin\lambda}\mathbf{r} - \frac{\cos\lambda}{\sin\lambda}\mathbf{l} \tag{1'}$$

(See Fig. 14.S.4.)

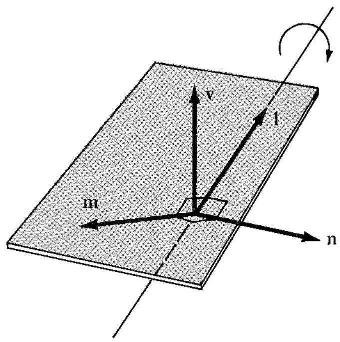

Figure 14.S.4. The three vectors $\mathbf{v}$, $\mathbf{m}$, and $\mathbf{n}$ all rotate about $\mathbf{l}$.

On the other hand, since $\mathbf{m}$ is perpendicular to $\mathbf{l}$, it rotates in a circle in the plane of $\mathbf{m}_0$ and $\mathbf{n}_0$. It goes through an angle 2π in time T, so it goes through an angle $2\pi t/T$ in t units of time, and so

$$\mathbf{m} = \cos\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\left(\frac{2\pi t}{T}\right)\mathbf{n}_0 .$$

Inserting this in formula (1′) and rearranging gives

$$\mathbf{r} = \boldsymbol{\sigma}(t) = (\cos\lambda)\mathbf{l} + \sin\lambda\cos\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\lambda\sin\left(\frac{2\pi t}{T}\right)\mathbf{n}_0 . \tag{2}$$

This formula expresses explicitly how $\mathbf{r}$ changes in time as it is rotated about $\mathbf{l}$, in terms of the basic trihedral $(\mathbf{l}, \mathbf{m}_0, \mathbf{n}_0)$.

Example 2 Express the function $\boldsymbol{\sigma}(t)$ explicitly in terms of $\mathbf{l}$, $\mathbf{r}_0$, and T.

Solution We have $\cos\lambda = \mathbf{l}\cdot\mathbf{r}_0$ and $\sin\lambda = \|\mathbf{l}\times\mathbf{r}_0\|$. Furthermore $\mathbf{n}_0$ is a unit vector perpendicular to both $\mathbf{l}$ and $\mathbf{r}_0$, so we must have

$$\mathbf{n}_0 = \frac{\mathbf{l}\times\mathbf{r}_0}{\|\mathbf{l}\times\mathbf{r}_0\|} .$$

Thus $(\sin\lambda)\mathbf{n}_0 = \mathbf{l}\times\mathbf{r}_0$. Finally, from formula (1), we obtain $(\sin\lambda)\mathbf{m}_0 = \mathbf{r}_0 - (\cos\lambda)\mathbf{l} = \mathbf{r}_0 - (\mathbf{r}_0\cdot\mathbf{l})\mathbf{l}$. Substituting all this into formula (2),

$$\mathbf{r} = (\mathbf{r}_0\cdot\mathbf{l})\mathbf{l} + \cos\left(\frac{2\pi t}{T}\right)\left[\mathbf{r}_0 - (\mathbf{r}_0\cdot\mathbf{l})\mathbf{l}\right] + \sin\left(\frac{2\pi t}{T}\right)(\mathbf{l}\times\mathbf{r}_0). \blacktriangle$$

Example 3 Show by a direct geometric argument that the speed of the tip of $\mathbf{r}$ is $(2\pi/T)\sin\lambda$. Verify that equation (2) gives the same formula.

Solution The tip of $\mathbf{r}$ sweeps out a circle of radius $\sin\lambda$, so it covers a distance $2\pi\sin\lambda$ in time T. Its speed is therefore $(2\pi\sin\lambda)/T$ (Fig. 14.S.5). From formula (2), we find the velocity vector to be

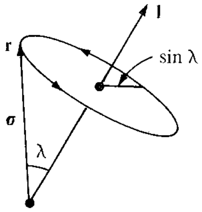

Figure 14.S.5. The tip of $\mathbf{r}$ sweeps out a circle of radius $\sin\lambda$.

$$\frac{d\mathbf{r}}{dt} = -\sin\lambda\cdot\frac{2\pi}{T}\sin\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\lambda\cdot\frac{2\pi}{T}\cos\left(\frac{2\pi t}{T}\right)\mathbf{n}_0,$$

and its length is (since $\mathbf{m}_0$ and $\mathbf{n}_0$ are unit orthogonal vectors)

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\sin^2\lambda\cdot\left(\frac{2\pi}{T}\right)^2\sin^2\left(\frac{2\pi t}{T}\right) + \sin^2\lambda\cdot\left(\frac{2\pi}{T}\right)^2\cos^2\left(\frac{2\pi t}{T}\right)}$$
$$= \sin\lambda\cdot\left(\frac{2\pi}{T}\right), \text{ as above. } \blacktriangle$$

Now we apply our study of rotations to the motion of the earth about the sun, incorporating the rotation of the earth about its own axis as well. We will use a simplified model of the earth–sun system, in which the sun is fixed at the origin of our coordinate system and the earth moves at uniform speed around a circle centered at the sun. Let $\mathbf{u}$ be a unit vector pointing *from* the sun *to* the earth; we have $\mathbf{u} = \cos(2\pi t/T_y)\mathbf{i} + \sin(2\pi t/T_y)\mathbf{j}$, where T_y is the length of a year (t and T_y measured in the same units). See Fig. 14.S.6. Notice that the unit vector pointing from the earth to the sun is $-\mathbf{u}$ and that we have oriented our axes so that $\mathbf{u} = \mathbf{i}$ when $T = 0$.

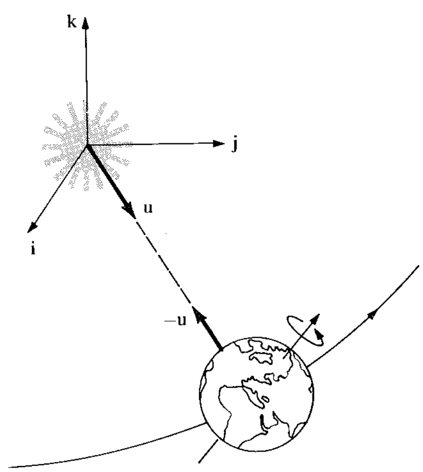

Figure 14.S.6. The unit vector $\mathbf{u}$ points from the sun to the earth at time t.

Next we wish to take into account the rotation of the earth. The earth rotates about an axis which we represent by a unit vector $\mathbf{l}$ pointing from the center of the earth to the North Pole. We will assume that $\mathbf{l}$ is fixed[4] with respect to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; astronomical measurements show that the inclination of $\mathbf{l}$ (the angle between $\mathbf{l}$ and $\mathbf{k}$) is presently about 23.5°. We will denote this angle by α. If we measure time so that the first day of summer in the northern hemisphere occurs when $t = 0$, then the axis $\mathbf{l}$ must tilt in the direction $-\mathbf{i}$, and so we must have $\mathbf{l} = \cos\alpha\mathbf{k} - \sin\alpha\mathbf{i}$. (See Fig. 14.S.7.)

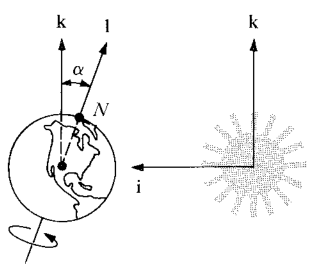

Figure 14.S.7. At $t = 0$, the earth's axis is tilted toward the sun.

Now let $\mathbf{r}$ be the unit vector at time t from the center of the earth to a fixed point P on the earth's surface. Notice that if $\mathbf{r}$ is located with its base

[4] Actually, the axis $\mathbf{l}$ is known to rotate about $\mathbf{k}$ once every 21,000 years. This phenomenon, called precession or wobble, is due to the irregular shape of the earth and may play a role in long-term climatic changes, such as ice ages. See pp. 130–134 of *The Weather Machine* by Nigel Calder, Viking (1974).

point at P, then it represents the local *vertical* direction. We will assume that P is chosen so that at $t = 0$, it is noon at the point P; then $\mathbf{r}$ lies in the plane of $\mathbf{l}$ and $\mathbf{i}$ and makes an angle of less than 90° with $-\mathbf{i}$. Referring to Fig. 14.S.8, we introduce the unit vector $\mathbf{m}_0 = -(\sin\alpha)\mathbf{k} - (\cos\alpha)\mathbf{i}$ orthogonal to $\mathbf{l}$. We then have $\mathbf{r}_0 = (\cos\lambda)\mathbf{l} + (\sin\lambda)\mathbf{m}_0$, where λ is the angle between $\mathbf{l}$ and $\mathbf{r}_0$. Since $\lambda = 90° - l$, where l is the latitude of the point P, we obtain the expression $\mathbf{r}_0 = (\sin l)\mathbf{l} + (\cos l)\mathbf{m}_0$. As in Fig. 14.S.3, let $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0$.

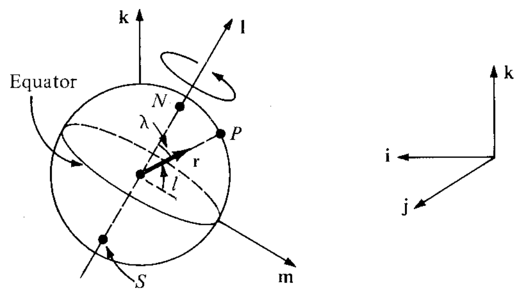

Figure 14.S.8. The vector $\mathbf{r}$ is the vector from the center of the earth to a fixed location P. The latitude of P is l and the colatitude is $\lambda = 90° - l$. The vector $\mathbf{m}_0$ is a unit vector in the plane of the equator (orthogonal to $\mathbf{l}$) and in the plane of $\mathbf{l}$ and $\mathbf{r}_0$.

Example 4 Prove that $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0 = -\mathbf{j}$.

Solution Geometrically, $\mathbf{l} \times \mathbf{m}_0$ is a unit vector orthogonal to $\mathbf{l}$ and $\mathbf{m}_0$ pointing in the sense given by the right-hand rule. But $\mathbf{l}$ and $\mathbf{m}_0$ are both in the $\mathbf{ik}$ plane, so $\mathbf{l} \times \mathbf{m}_0$ points orthogonal to it in the direction $-\mathbf{j}$ (see Fig. 14.S.8).

Algebraically, $\mathbf{l} = (\cos\alpha)\mathbf{k} - (\sin\alpha)\mathbf{i}$ and $\mathbf{m}_0 = -(\sin\alpha)\mathbf{k} - (\cos\alpha)\mathbf{i}$, so

$$\mathbf{l} \times \mathbf{m}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\alpha & 0 & \cos\alpha \\ -\cos\alpha & 0 & -\sin\alpha \end{vmatrix} = -\mathbf{j}(\sin^2\alpha + \cos^2\alpha) = -\mathbf{j}. \ \blacktriangle$$

Now we apply formula (2) to get

$$\mathbf{r} = (\cos\lambda)\mathbf{l} + \sin\lambda\cos\left(\frac{2\pi t}{T_d}\right)\mathbf{m}_0 + \sin\lambda\sin\left(\frac{2\pi t}{T_d}\right)\mathbf{n}_0,$$

where T_d is the length of time it takes for the earth to rotate once about its axis (with respect to the "fixed stars"—i.e., our $\mathbf{i}, \mathbf{j}, \mathbf{k}$ vectors).[5] Substituting the expressions derived above for λ, $\mathbf{l}$, $\mathbf{m}_0$, and $\mathbf{n}_0$, we get

$$\mathbf{r} = \sin l(\cos\alpha\mathbf{k} - \sin\alpha\mathbf{i}) + \cos l\cos\left(\frac{2\pi t}{T_d}\right)(-\sin\alpha\mathbf{k} - \cos\alpha\mathbf{i}) - \cos l\sin\left(\frac{2\pi t}{T_d}\right)\mathbf{j}.$$

Hence

$$\begin{aligned}\mathbf{r} = &-\left[\sin l\sin\alpha + \cos l\cos\alpha\cos\left(\frac{2\pi t}{T_d}\right)\right]\mathbf{i} - \cos l\sin\left(\frac{2\pi t}{T_d}\right)\mathbf{j} \\ &+\left[\sin l\cos\alpha - \cos l\sin\alpha\cos\left(\frac{2\pi t}{T_d}\right)\right]\mathbf{k}.\end{aligned} \tag{3}$$

Example 5 What is the speed (in kilometers per hour) of a point on the equator due to the rotation of the earth? A point at latitude 60°? (The radius of the earth is 6371 kilometers.)

Solution From Example 3, the speed is $s = (2\pi R/T_d)\sin\lambda = (2\pi R/T_d)\cos l$, where R is the radius of the earth and l is the latitude. (The factor R is inserted since $\mathbf{r}$ is a *unit* vector; the actual vector from the earth's center to a point P is $R\mathbf{r}$).

[5] T_d is called the length of the sidereal day. It differs from the ordinary, or solar, day by about 1 part in 365. (Can you explain why?) In fact, $T_d \approx 23.93$ hours.

Using $T_d = 23.93$ hours and $R = 6371$ kilometers, we get $s = 1673 \cos l$ kilometers per hour. At the equator $l = 0$, so the speed is 1673 kilometers per hour; at $l = 60°$, $s = 836.4$ kilometers per hour. ▲

With formula (3) at our disposal, we are now ready to derive the sunshine formula. The intensity of light on a portion of the earth's surface (or at the top of the atmosphere) is proportional to $\sin A$, where A is the angle of elevation of the sun above the horizon (see Fig. 14.S.9). (At night $\sin A$ is negative, and the intensity then is of course zero.)

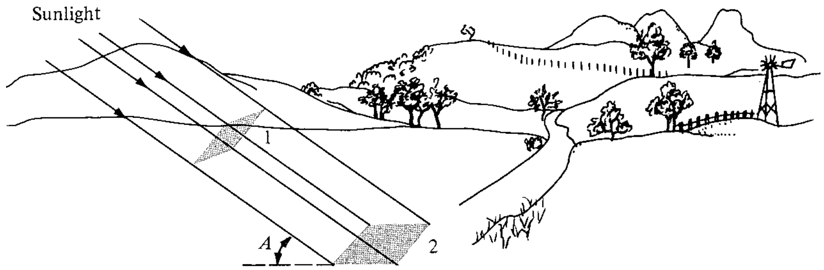

Figure 14.S.9. The intensity of sunlight is proportional to $\sin A$. The ratio of area 1 to area 2 is $\sin A$.

Thus we want to compute $\sin A$. From Fig. 14.S.10 we see that $\sin A = -\mathbf{u} \cdot \mathbf{r}$. Substituting $\mathbf{u} = \cos(2\pi t/T_y)\mathbf{i} + \sin(2\pi t/T_y)\mathbf{j}$ and formula (3) into this formula for $\sin A$ and taking the dot product gives

$$\begin{aligned}\sin A &= \cos\left(\frac{2\pi t}{T_y}\right)\left[\sin l \sin\alpha + \cos l \cos\alpha \cos\left(\frac{2\pi t}{T_d}\right)\right]\\ &\quad + \sin\left(\frac{2\pi t}{T_y}\right)\left[\cos l \sin\left(\frac{2\pi t}{T_d}\right)\right]\\ &= \cos\left(\frac{2\pi t}{T_y}\right)\sin l \sin\alpha\\ &\quad + \cos l\left[\cos\left(\frac{2\pi t}{T_y}\right)\cos\alpha\cos\left(\frac{2\pi t}{T_d}\right) + \sin\left(\frac{2\pi t}{T_y}\right)\sin\left(\frac{2\pi t}{T_d}\right)\right]. \qquad (4)\end{aligned}$$

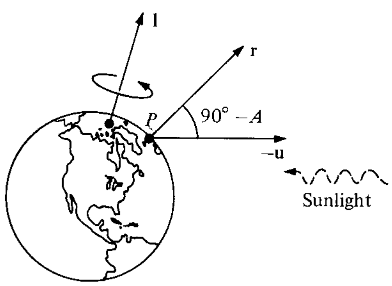

Figure 14.S.10. The geometry for the formula $\sin A = \cos(90° - A)$ $= -\mathbf{u} \cdot \mathbf{r}$.

Example 6 Set $t = 0$ in formula (4). For what l is $\sin A = 0$? Interpret your result.

Solution With $t = 0$ we get

$$\sin A = \sin l \sin\alpha + \cos l \cos\alpha = \cos(l - \alpha).$$

This is zero when $l - \alpha = \pm\pi/2$. Now $\sin A = 0$ corresponds to the sun on the horizon (sunrise or sunset), when $A = 0$ or π. Thus, at $t = 0$, this occurs when $l = \alpha \pm (\pi/2)$. The case $\alpha + (\pi/2)$ is impossible, since l lies between $-\pi/2$ and $\pi/2$. The case $l = \alpha - (\pi/2)$ corresponds to a point on the Antarctic Circle; indeed at $t = 0$ (corresponding to noon on the first day of northern summer) the sun is just on the horizon at the Antarctic Circle. ▲

Our next goal is to describe the variation of $\sin A$ with time *on a particular day*. For this purpose, the time variable t is not very convenient; it will be better to measure time from noon on the day in question.

To simplify our calculations, we will assume that the expressions $\cos(2\pi t/T_y)$ and $\sin(2\pi t/T_y)$ are constant over the course of any particular day; since T_y is 365 times as large as the change in t, this is a reasonable approximation. On the nth day (measured from June 21), we may replace $2\pi t/T_y$ by $2\pi n/365$, and formula (4) gives

$$\sin A = (\sin l)P + (\cos l)\left[Q\cos\left(\frac{2\pi t}{T_d}\right) + R\sin\left(\frac{2\pi t}{T_d}\right)\right], \tag{5}$$

where $P = \cos(2\pi n/365)\sin\alpha$, $Q = \cos(2\pi n/365)\cos\alpha$, and $R = \sin(2\pi n/365)$.

We will write the expression $Q\cos(2\pi t/T_d) + R\sin(2\pi t/T_d)$ in the form $U\cos[2\pi(t - t_n)/T_d]$, where t_n is the time of noon of the nth day. To find U, we use the addition formula to expand the cosine:

$$U\cos\left(\frac{2\pi t}{T_d} - \frac{2\pi t_n}{T_d}\right) = U\left[\cos\left(\frac{2\pi t}{T_d}\right)\cos\left(\frac{2\pi t_n}{T_d}\right) + \sin\left(\frac{2\pi t}{T_d}\right)\sin\left(\frac{2\pi t_n}{T_d}\right)\right].$$

Setting this equal to $Q\cos(2\pi t/T_d) + R\sin(2\pi t/T_d)$ and comparing coefficients of $\cos 2\pi t/T_d$ and $\sin 2\pi t/T_d$ gives

$$U\cos\frac{2\pi t_n}{T_d} = Q \quad \text{and} \quad U\sin\frac{2\pi t_n}{T_d} = R.$$

Squaring the two equations and adding gives

$$U^2 = Q^2 + R^2 \quad \text{or} \quad U = \sqrt{Q^2 + R^2}\,,$$ [6]

while dividing the second equation by the first gives $\tan(2\pi t_n/T_d) = R/Q$. We are interested mainly in the formula for U; substituting for Q and R gives

$$\begin{aligned} U &= \sqrt{\cos^2\left(\frac{2\pi n}{365}\right)\cos^2\alpha + \sin^2\left(\frac{2\pi n}{365}\right)} \\ &= \sqrt{\cos^2\left(\frac{2\pi n}{365}\right)(1 - \sin^2\alpha) + \sin^2\left(\frac{2\pi n}{365}\right)} \\ &= \sqrt{1 - \cos^2\left(\frac{2\pi n}{365}\right)\sin^2\alpha}\,. \end{aligned}$$

Letting τ be the time in hours from noon on the nth day so that $(t - t_n)/T_d = \tau/24$, we may substitute into formula (5) to obtain the final formula:

$$\sin A = \sin l\cos\left(\frac{2\pi n}{365}\right)\sin\alpha + \cos l\sqrt{1 - \cos^2\left(\frac{2\pi n}{365}\right)\sin^2\alpha}\;\cos\left(\frac{2\pi\tau}{24}\right), \tag{6}$$

which is identical (after some changes in notation) to formula (1) on page 301.

Example 7 How high is the sun in the sky in Edinburgh (latitutde 56°) at 2 P.M. on February 1?

Solution We plug into formula (6): $\alpha = 23.5°$, $l = 90 - 56 = 34°$, n = number of days after June 21 = 225, and τ = 2 hours. We get

$$\sin A = 0.5196,$$

so $A = 31.3°$. ▲

[6] We take the positive square root because $\sin A$ should have a local maximum when $t = t_n$.

Exercises for The Supplement to Chapter 14

1. Let $\mathbf{l} = (\mathbf{j} + \mathbf{k})/\sqrt{2}$ and $\mathbf{r}_0 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$. (a) Find $\mathbf{m}_0$ and $\mathbf{n}_0$. (b) Find $\mathbf{r} = \boldsymbol{\sigma}(t)$ if $T = 24$. (c) Find the equation of the line tangent to $\boldsymbol{\sigma}(t)$ at $t = 12$ and $T = 24$.
2. From formula (2), verify that $\boldsymbol{\sigma}(T/2) \cdot \mathbf{n} = 0$. Also, show this geometrically. For what values of t is $\boldsymbol{\sigma}(t) \cdot \mathbf{n} = 0$?
3. If the earth rotated in the opposite direction about the sun, would T_d be longer or shorter than 24 hours? (Assume the solar day is fixed at 24 hours.)
4. Show by a direct geometric construction that $\mathbf{r} = \boldsymbol{\sigma}(T_d/4) = -\sin l \sin \alpha \mathbf{i} - \cos l \mathbf{j} + \sin l \cos \alpha \mathbf{k}$. Does this formula agree with formula (3)?
5. Derive an "exact" formula for the time of sunset from formula (4).
6. Why does formula (6) for $\sin A$ not depend on the radius of the earth? The distance of the earth from the sun?
7. How high is the sun in the sky in Paris at 3 P.M. on January 15? (The latitude of Paris is 49°N).
8. How much solar energy (relative to a summer day at the equator) does Paris receive on January 15? (The latitude of Paris is 49° N).
9. How would your answer in Exercise 8 change if the earth were to roll to a tilt of 32° instead of 23.5°?

Review Exercises for Chapter 14

Sketch the graphs of the conics in Exercises 1–8.

1. $4x^2 + 9y^2 = 36$
2. $x = 2y^2$
3. $x^2 - 4y^2 = 16$
4. $4x^2 + 16y^2 = 81$
5. $100x^2 + 100y^2 = 1$
6. $y^2 = 16 + 4x^2$
7. $x^2 - y = 14$
8. $2x^2 + 2y^2 = 80$

Sketch the graphs of the conics in Exercises 9–12.

9. $9x^2 - 18x + y^2 - 4y + 4 = 0.$
10. $9x^2 + 18x - y^2 + 2y - 8 = 0.$
11. $x^2 + 2xy + 3y^2 = 14.$
12. $x^2 - 2xy - 3y^2 = 14.$

Sketch or describe the level curves for the functions and values in Exercises 13–16.

13. $f(x, y) = 3x - 2y;\ c = 2$
14. $f(x, y) = x^2 - y^2;\ c = -1$
15. $f(x, y) = x^2 + xy;\ c = 2$
16. $f(x, y) = x^2 + 4;\ c = 85$

Describe the level surfaces $f(x, y, z) = c$ for each of the functions in Exercises 17–20. Sketch for $c = 1$ and $c = 5$.

17. $f(x, y, z) = x - y - z$
18. $f(x, y, z) = x + y - 2z$
19. $f(x, y, z) = x^2 + y^2 + z^2 + 1$
20. $f(x, y, z) = x^2 + 2y^2 + 3z^2$

Sketch and describe the surfaces in Exercises 21–28.

21. $x^2 + 4y^2 + z^2 = 1$
22. $x^2 + 4y^2 - z^2 = 0$
23. $x^2 + 4y^2 - z^2 = 1$
24. $x^2 + 4y^2 + z^2 = 0$
25. $x^2 + 4y^2 - z = 1$
26. $x^2 + 4y^2 - z = 0$
27. $x^2 + 4y^2 + z = 1$
28. $x^2 + 4y^2 + z = 0$

29. This exercise concerns the *elliptic hyperboloid of one sheet*. An example of this type was studied in Example 6, Section 14.4. A standard form for the equation is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (a, b, \text{ and } c \text{ positive}).$$
   (a) What are the horizontal cross sections obtained by holding z constant?
   (b) What are the vertical cross sections obtained by holding either x or y constant?
   (c) Sketch the surface defined by
$$\frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{1} = 1.$$
   In a yz plane, sketch the cross-section curves obtained from this surface when x is held constantly equal to 0, 1, 2, and 3 ($x = 2$ is especially interesting).
30. (a) Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 - z^2$. In particular, discuss the surface $f(x, y, z) = c$ when c is positive, when c is negative, and when c is zero.
   (b) Several level surfaces of f are sketched in Fig. 14.R.1. Find the value of c associated with each.
   (c) Describe how the appearance of the level surfaces changes if we consider instead the function $g(x, y, z) = x^2 + 2y^2 - z^2$.
31. Let $f(x, y) = x^2 + 2y^2 + 1$
   (a) Sketch the level curves $f(x, y) = c$ for $c = -10, -1, 0, 1, 2$, and 10.
   (b) Describe the intersection of the graph of f with the vertical planes $x = 1$, $x = -1$, $x = 2$, $y = 1$, $y = 2$, $y = -1$.
   (c) Sketch the graph of f.

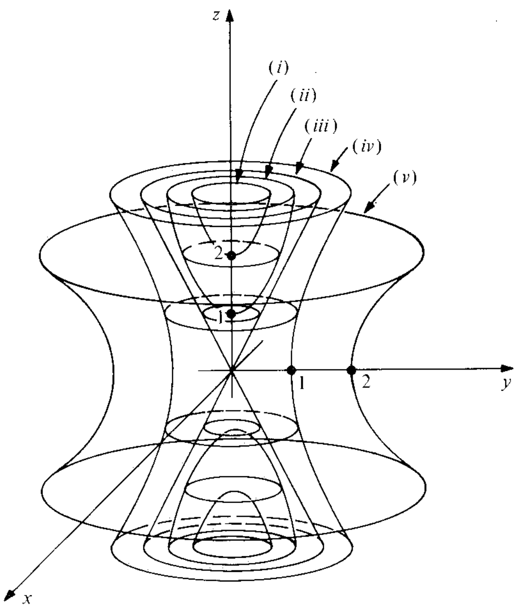

Figure 14.R.1. Level surfaces of $x^2 + y^2 - z^2$.

32. Do as in Exercise 31 for $f(x, y) = y/x$, and describe the intersection of the graph of f with the cylinder of radius R (that is, $r = R$ in cylindrical coordinates).

In Exercises 33–38, fill in the blanks and plot.

	Rectangular coordinates	*Cylindrical coordinates*	*Spherical coordinates*
33.	$(1, -1, 1)$		
34.	$(1, 0, 3)$		
35.		$(5, \pi/12, 4)$	
36.		$(8, 3\pi/2, 2)$	
37.			$(3, -\pi/6, \pi/4)$
38.			$(10, \pi/4, \pi/2)$

39. A surface is described in cylindrical coordinates by $3r^2 = z^2 + 1$. Convert to rectangular coordinates and plot.
40. Show that a surface described in spherical coordinates by $f(\rho, \phi) = 0$ is a surface of revolution.
41. Describe the geometric meaning of replacing (ρ, θ, ϕ) by $(\rho, \theta + \pi, \phi + \pi/2)$ in spherical coordinates.
42. Describe the geometric meaning of replacing (ρ, θ, ϕ) by $(4\rho, \theta, \phi)$ in spherical coordinates.
43. Describe by means of cylindrical coordinates a solenoid consisting of a copper rod of radius 5 centimeters and length 15 centimeters wound on the outside with copper wire to a thickness of 1.2 centimeters. Give separate descriptions of the rod and the winding.
44. A gasoline storage tank has two spherical cap ends of arc length 56.55 feet. The cylindrical part of the tank has length 16 feet and circumference 113.10 feet. Let $(0, 0, 0)$ be the geometric center of the tank. See Fig. 14.R.2.

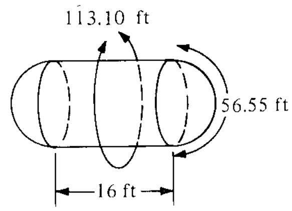

Figure 14.R.2. The gasoline storage tank for Exercise 44.

(a) Describe the cylindrical part of the tank via cylindrical coordinates.
(b) Describe the hemispherical end caps with spherical coordinates. (Set up spherical coordinates using the centers of the cap ends as the origin.)

Sketch the curves or surfaces given by the equations in Exercises 45–52.

45. $z = x + y$
46. $x^2 + 2xz + z^2 = 0$
47. $\boldsymbol{\sigma}(t) = 3\sin t\,\mathbf{i} + t\,\mathbf{j} + \cos t\,\mathbf{k}$
48. $\boldsymbol{\sigma}(t) = (\sin t, t + 1, 2t - 1)$
49. $z^2 = -(x^2 + y^2/4)$
50. $z = -(x^2 + y^2)$
51. $z^2 = -x^2 - 3y^2 + 2$
52. $z^2 = x^2 - 4y^2$

Find the equation of the line tangent to each of the curves at the indicated point in Exercises 53 and 54.

53. $(t^3 + 1, e^{-t}, \cos(\pi t/2));\ t = 1$
54. $(t^2 - 1, \cos t^2, t^4);\ t = \sqrt{\pi}$

Find the velocity and acceleration vectors for the curves in Exercises 55–58.

55. $\boldsymbol{\sigma}_1(t) = e^t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$.
56. $\boldsymbol{\sigma}_2(t) = \dfrac{t^2}{1 + t^2}\mathbf{i} + t\mathbf{j} + \mathbf{k}$.
57. $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_1(t) + \boldsymbol{\sigma}_2(t)$, where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are given in Exercises 55 and 56.
58. $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_1(t) \times \boldsymbol{\sigma}_2(t)$, where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are given in Exercises 55 and 56.
59. Write in parametric form the curve described by the equations $x - 1 = 2y + 1 = 3z + 2$.
60. Write the curve $x = y^3 = z^2 + 1$ in parametric form.
61. Find the arc length of $\boldsymbol{\sigma}(t) = t\,\mathbf{i} + \ln t\,\mathbf{j} + 2\sqrt{2t}\,\mathbf{k}$; $1 \leqslant t \leqslant 2$.
62. Express as an integral the arc length of the curve $x^2 = y^3 = z^5$ between $x = 1$ and $x = 4$. (Find a parametrization.)
63. A particle moving on the curve $\boldsymbol{\sigma}(t) = 3t^2\mathbf{i} - \sin t\mathbf{j} - e^t\mathbf{k}$ is released at time $t = \frac{1}{2}$ and flies off on a tangent. What are its coordinates at time $t = 1$?
64. A particle is constrained to move around the unit circle in the xy plane according to the formula $(x, y, z) = (\cos(t^2), \sin(t^2), 0)$, $t \geqslant 0$.

(a) What are the velocity vector and speed of the particle as functions of t?
(b) At what point on the circle should the particle be released to hit a target at $(2,0,0)$? (Be careful about which direction the particle is moving around the circle.)
(c) At what time t should the release take place? (Use the smallest $t > 0$ which will work.)
(d) What are the velocity and speed at the time of release?
(e) At what time is the target hit?

65. A particle of mass m is subject to the force law $\mathbf{F} = -k\mathbf{r}$, where k is a constant.
(a) Write down differential equations for the components of $\mathbf{r}(t)$.
(b) Solve the equations in (a) subject to the initial conditions $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{r}'(0) = 2\mathbf{j} + \mathbf{k}$.

66. Show that the quantity

$$\frac{m}{2}\left\|\frac{d\mathbf{r}}{dt}\right\|^2 + \frac{k}{2}\mathbf{r}\cdot\mathbf{r}$$

is independent of time when a particle moves under the force law in Exercise 65.

67. Find the curvature of the ellipse $4x^2 + 9y^2 = 16$.

★ 68. Let $\mathbf{r} = \boldsymbol{\sigma}(t)$ be a curve in space and $\mathbf{N}$ be its principal normal vector. Consider the "parallel curve" $\mathbf{r} = \boldsymbol{\mu}(t) = \boldsymbol{\sigma}(t) + \mathbf{N}(t)$, where $\boldsymbol{\sigma}(t)$ is the displacement vector to $P(t)$ from a fixed origin.
(a) Under what conditions does $\boldsymbol{\mu}(t)$ have zero velocity for some t_0? [Hint: Use the Frenet formulas, Exercise 25, Section 14.7]
(b) Find the parametric equation of the parallel curve to the ellipse $(\frac{1}{4}\cos t, 4\sin t, 0)$.

69. Find a formula for the curvature of the graph $y = f(x)$ in terms of f and its derivatives.

70. The contour lines on a topographical map are the level curves of the function giving height above sea level as a function of position. Figure 14.R.3 is a portion of the U.S. Geological Survey map of Yosemite Valley. There is a heavy contour line for every 200 feet of elevation and a lighter line at each 40-foot interval between these.
(a) What does it mean in terms of the terrain when these contour lines are far apart?
(b) What if they are close together?
(c) What does it mean when several contour lines seem to merge for a distance? Is eleva-

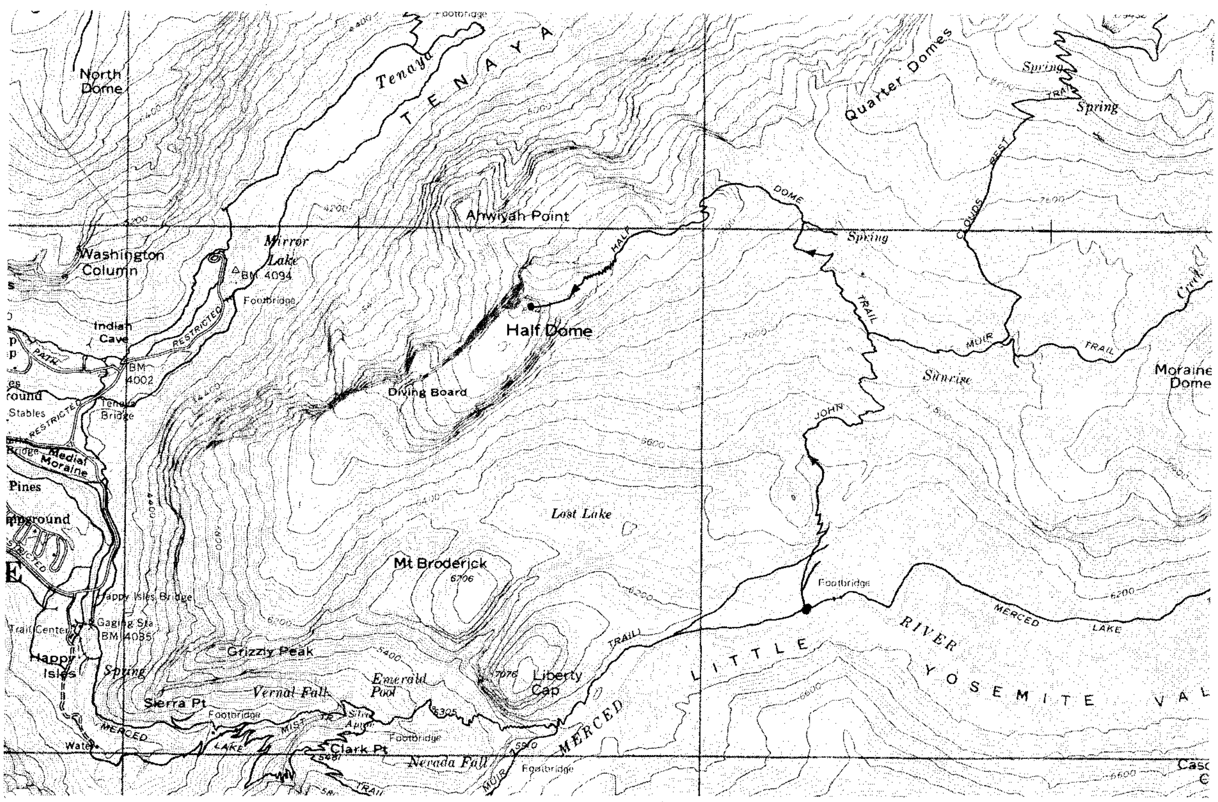

Figure 14.R.3. Yosemite Valley (portion). (U.S. Department of Interior Geological Survey.)

tion really a function of position at such points? (Look at the west face of Half Dome. Does this seem like a good direction from which to climb it?)

(d) Sketch a cross section of terrain along a north–south line through the top of Half Dome.

(e) A hiker is not likely to follow the straight north–south path in part (d) and is probably more interested in the behavior of the terrain along the trail he or she will follow. The 3-mile route from the Merced River (altitude approximately 6100 feet) to the top of Half Dome (approximately 8842 feet) along the John Muir and Half Dome Trails has been emphasized in Fig. 14.R.3. Show how a cross section of the terrain along this trail behaves by plotting altitude above sea level as a function of miles along the trail. (A piece of string or flexible wire may be of aid in measuring distances along the trail.)

★71. Find the curvature of the "helical spiral" $(t, t\cos t, t\sin t)$ for $t > 0$. Sketch.

★72. Describe the level curves $f(x, y) = c$ for each of the following functions. In particular, discuss any special values of c at which the behavior of the level curves changes suddenly. Sketch the curves for $c = -1, 0$, and 1.

(a) $f(x, y) = x + 2y$;
(b) $f(x, y) = x^2 - y^2$;
(c) $f(x, y) = y^2 - x^2$;
(d) $f(x, y) = x^2 + y^2$;
(e) $f(x, y) = xy$;
(f) $f(x, y) = y - 2x^2$.

★73. (a) Write in parametric form the curve which is the intersection of the surfaces $x^2 + y^2 + z^2 = 3$ and $y = 1$.

(b) Find the equation of the line tangent to this curve at $(1, 1, 1)$.

(c) Write an integral expression for the arc length of this curve. What is the value of this integral?

★74. Let n be a positive integer and consider the curve

$$\left.\begin{aligned} x &= \cos t\cos(4nt) \\ y &= \cos t\sin(4nt) \\ z &= \sin t \end{aligned}\right\}\ -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2},$$

(a) Show that the path traced out lies on the surface of the sphere of radius 1 centered at the origin.

(b) How many times does the curve wind around the z axis?

(c) Where does the curve cross the xy plane?

(d) Sketch the curve when $n = 1$ and when $n = 2$.

★75. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be unit vectors, and define the curve $\boldsymbol{\sigma}(t)$ by

$$\boldsymbol{\sigma}(t) = \frac{\mathbf{u}_1\cos t + \mathbf{u}_2\sin t}{\|\mathbf{u}_1\cos t + \mathbf{u}_2\sin t\|}, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}.$$

(a) Find $\boldsymbol{\sigma}(0)$ and $\boldsymbol{\sigma}(\pi/2)$.

(b) On what surfaces does $\boldsymbol{\sigma}(t)$ lie for all t?

(c) Find, by geometry, the arc length of the curve $\boldsymbol{\sigma}(t)$ for $0 \leqslant t \leqslant \pi/2$.

(d) Express the arc length of the curve $\boldsymbol{\sigma}(t)$ for $0 \leqslant t \leqslant \pi/2$ as an integral.

(e) Find a curve $\boldsymbol{\sigma}_1(t)$ which traverses the same path as $\boldsymbol{\sigma}(t)$ for $0 \leqslant t \leqslant \pi/2$ and such that the speed $\|\boldsymbol{\sigma}_1'(t)\|$ is constant.

★76 (a) Show that the hyperboloid $x^2 + y^2 - z^2 = 4$ is a ruled surface by finding two straight lines lying in the surface through each point. [*Hint:* Let (x_0, y_0, z_0) lie on the surface; write the equation of the line in the form $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + t$; write out $x^2 + y^2 - z^2 = 4$ using $x_0^2 + y_0^2 - z_0^2 = 4$ to obtain two equations for a and b representing a line and a circle in the (a, b) plane. Show that these equations have two solutions by showing that the distance from the origin to the line is less than the radius of the circle.]

(b) Is the hyperboloid $x^2 + y^2 - z^2 = -4$ a ruled surface? Explain.

(c) Generalize the results of parts (a) and (b).