Chapter 16

Gradients, Maxima, and Minima

The gradient of a function of several variables vanishes at a maximum or a minimum.

The gradient of a function f is a vector whose components are the partial derivatives of f. Derivatives in any direction can be found in terms of the gradient, using the chain rule. The gradient will be used to find the equations for tangent planes to level surfaces. The last two sections of the chapter extend our earlier studies of maxima and minima (Chapter 3) to functions of several variables.

16.1 Gradients and Directional Derivatives

The directional derivative is the dot product of the gradient and the direction vector.

The right-hand side of the chain rule

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

has the appearance of a dot product—in fact it is the dot product of the vectors

$$\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k} \quad \text{and} \quad \frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}.$$

We recognize the first vector as the *velocity vector* of a parametric curve; if $\boldsymbol{\sigma}(t)$ is the vector representation of the curve, it is just $\boldsymbol{\sigma}'(t)$. The second vector is something new: it depends upon the function $u=f(x,y,z)$ and contains in vector form all three partial derivatives of f. This is called the *gradient* of f and is denoted ∇f. Thus

$$\nabla f(x,y,z)=f_x(x,y,z)\mathbf{i}+f_y(x,y,z)\mathbf{j}+f_z(x,y,z)\mathbf{k}.$$

Example 1 (a) Find ∇f if $u=f(x,y,z)=xy-z^2$.
(b) Find ∇f for the function $f(x,y,z)=e^{xy}-x\cos(yz^2)$.

Solution (a) Substituting the partial derivatives of f into the formula for the gradient of f, we find $\nabla f(x,y,z)=y\mathbf{i}+x\mathbf{j}-2z\mathbf{k}$.

(b) Here $f_x(x, y, z) = ye^{xy} - \cos(yz^2)$, $f_y(x, y, z) = xe^{xy} + xz^2\sin(yz^2)$, and $f_z(x, y, z) = 2xyz\sin(yz^2)$, so

$$\nabla f(x, y, z) = \left[ye^{xy} - \cos(yz^2)\right]\mathbf{i} + \left[xe^{xy} + xz^2\sin(yz^2)\right]\mathbf{j} + \left[2xyz\sin(yz^2)\right]\mathbf{k}. \blacktriangle$$

Notice that the vector $\nabla f(x, y, z)$ is a function of the point (x, y, z) in space; in other words, ∇f is a function of the point in space where the partial derivatives are evaluated. A rule $\mathbf{\Phi}$ which assigns a vector $\mathbf{\Phi}(x, y, z)$ in space to each point (x, y, z) of some domain in space is called a *vector field*. Thus, for a given function f, ∇f is a vector field. Similarly, a vector field in the xy plane is a rule $\mathbf{\Phi}$ which assigns to each point (x, y) a vector $\mathbf{\Phi}(x, y)$ in the plane.

The Gradient

If $z = f(x, y)$ is a function of two variables, its gradient vector field ∇f is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j}.$$

If $u = f(x, y, z)$ is a function of three variables, its gradient vector field ∇f is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}.$$

We may sketch a vector field $\mathbf{\Phi}(x, y)$ in the plane by choosing several values for (x, y), evaluating $\mathbf{\Phi}(x, y)$ at each point, and drawing the vector $\mathbf{\Phi}(x, y)$ with its tail at the point (x, y). The same thing may be done for vector fields in space, although they are more difficult to visualize.

Example 2 Sketch the gradient vector field of the function $f(x, y) = x^2/10 + y^2/6$.

Solution The partial derivatives are $f_x(x, y) = x/5$ and $f_y(x, y) = y/3$. Evaluating these for various values of x and y and plotting, we obtain the sketch in Fig. 16.1.1. For instance, $f_x(2, 2) = \frac{2}{5}$ and $f_y(2, 2) = \frac{2}{3}$; thus the vector $\frac{2}{5}\mathbf{i} + \frac{2}{3}\mathbf{j}$ is plotted at the point $(2, 2)$, as indicated in the figure. $\blacktriangle$

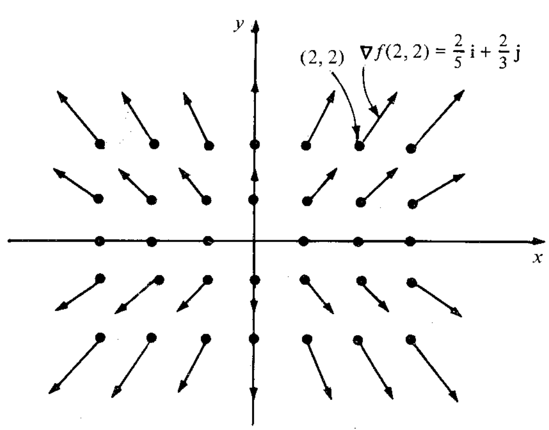


Figure 16.1.1. The gradient vector field ∇f, where $f(x, y) = (x^2/10) + (y^2/6)$.

In sketching a vector field $\mathbf{\Phi}(x, y)$, we sometimes find that the vectors are so long that they overlap one another, making the drawing confusing. In this case, it is better to sketch $\varepsilon\mathbf{\Phi}(x, y)$, where ε is a small positive number. This is illustrated in the next example.

Example 3 (a) Illustrate the vector field $\mathbf{\Phi}(x, y) = 3y\mathbf{i} - 3x\mathbf{j}$ by sketching $\frac{1}{6}\mathbf{\Phi}(x, y)$.
(b) Using the law of equality of mixed partial derivatives, show that the vector field in (a) is *not* the gradient vector field of any function.

Solution (a) If we sketched $\mathbf{\Phi}(x, y) = 3y\mathbf{i} - 3x\mathbf{j}$ itself, the vectors at different base points would overlap. Instead we sketch $\frac{1}{6}\mathbf{\Phi}(x, y) = \frac{1}{2}y\mathbf{i} - \frac{1}{2}x\mathbf{j}$ in Fig. 16.1.2.

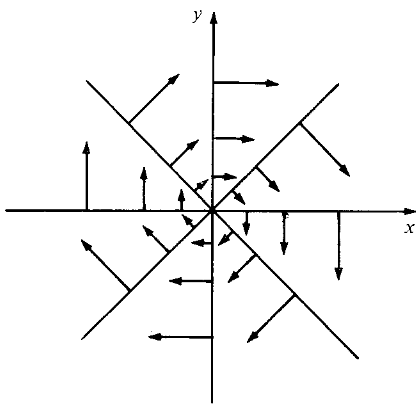


Figure 16.1.2. The vector field $3y\mathbf{i} - 3x\mathbf{j}$ is not the gradient of a function.

(b) If $\mathbf{\Phi}(x, y) = 3y\mathbf{i} - 3x\mathbf{j}$ were the gradient of a function $z = f(x, y)$, we would have $\partial z/\partial x = 3y$ and $\partial z/\partial y = -3x$. By the equality of mixed partial derivatives, $\partial^2 z/\partial x\,\partial y = -3$ and $\partial^2 z/\partial y\,\partial x = 3$ would have to be equal; but $3 \neq -3$, so our vector field cannot be a gradient. ▲

In a number of situations later in the book, the vector $\mathbf{r}$ from the origin to a point (x, y, z) plays a basic role. The next example illustrates its use.

Example 4 Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}, \qquad r \neq 0.$$

What is $\|\nabla(1/r)\|$?

Solution By definition of the gradient,

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\mathbf{k}.$$

Now

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{x}{r^3},$$

and, similarly,

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3}, \qquad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}.$$

Thus

$$\nabla\left(\frac{1}{r}\right) = -\frac{x}{r^3}\mathbf{i} - \frac{y}{r^3}\mathbf{j} - \frac{z}{r^3}\mathbf{k} = -\frac{1}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -\frac{\mathbf{r}}{r^3},$$

as required. Finally,

$$\left\|\nabla\left(\frac{1}{r}\right)\right\| = \left|-\frac{1}{r^3}\right|\|\mathbf{r}\| = \frac{r}{r^3} = \frac{1}{r^2} = \frac{1}{x^2+y^2+z^2}. \quad \blacktriangle$$

In the next box we restate the chain rule from Section 15.3 in terms of gradients.

> **The Chain Rule for Functions and Curves**
>
> Let f be a function of two (three) variables, $\boldsymbol{\sigma}(t)$ a parametric curve in the plane (in space), and $h(t) = f(\boldsymbol{\sigma}(t))$ the composite function. Then
>
> $$h'(t) = \nabla f(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t); \qquad \text{that is,} \quad \frac{d}{dt} f(\boldsymbol{\sigma}(t)) = \nabla f(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t).$$

In this form, the chain rule looks more like it did for functions of one variable:

$$\frac{d}{dt}(f(g(t))) = f'(g(t))g'(t).$$

Example 5 Verify the chain rule for $u = f(x, y, z) = xy - z^2$ and $\boldsymbol{\sigma}(t) = (\sin t, \cos t, e^t)$.

Solution The gradient vector field of f is $y\mathbf{i} + x\mathbf{j} - 2z\mathbf{k}$; the velocity vector is given by $\boldsymbol{\sigma}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + e^t\mathbf{k}$. By the chain rule,

$$\frac{du}{dt} = \nabla f \cdot \boldsymbol{\sigma}' = (y\mathbf{i} + x\mathbf{j} - 2z\mathbf{k}) \cdot (\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + e^t\mathbf{k})$$
$$= y\cos t - x\sin t - 2ze^t = \cos^2 t - \sin^2 t - 2e^{2t}.$$

To verify this directly, we first compute the composition as $f(\boldsymbol{\sigma}(t)) = \sin t \cos t - e^{2t}$. Then by one-variable calculus, we find

$$\frac{d}{dt} f(\boldsymbol{\sigma}(t)) = -\sin^2 t + \cos^2 t - 2e^{2t}.$$

Thus the chain rule is verified in this case. ▲

Example 6 Suppose that f takes the value 2 at all points on a curve $\boldsymbol{\sigma}(t)$. What can you say about $\nabla f(\boldsymbol{\sigma}(t))$ and $\boldsymbol{\sigma}'(t)$?

Solution If $f(\boldsymbol{\sigma}(t))$ is always equal to 2, the derivative $(d/dt) f(\boldsymbol{\sigma}(t))$ is zero. By the chain rule, $0 = \nabla f(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t)$, so the gradient vector $\nabla f(\boldsymbol{\sigma}(t))$ and the velocity vector $\boldsymbol{\sigma}'(t)$ are perpendicular at all points on the curve. ▲

Let $u = f(x, y, z)$ be a function (with continuous partial derivatives) and $\boldsymbol{\sigma}(t)$ a parametrized curve in space. The derivative with respect to t of the composite function $f(\boldsymbol{\sigma}(t))$ may be thought of as "the derivative of f *along the curve* $\boldsymbol{\sigma}(t)$." According to the chain rule, the value of this derivative at $t = t_0$ is $\nabla f(\boldsymbol{\sigma}(t_0)) \cdot \boldsymbol{\sigma}'(t_0)$. We may write this dot product as

$$\|\nabla f(\boldsymbol{\sigma}(t_0))\|\,\|\boldsymbol{\sigma}'(t_0)\|\cos\theta,$$

where θ is the angle between the gradient vector $\nabla f(\boldsymbol{\sigma}(t_0))$ and the velocity vector $\boldsymbol{\sigma}'(t_0)$ (Fig. 16.1.3). If we fix the function f and differentiate it along various curves through a given point $\mathbf{r}$ (here, as usual, we identify a point with the vector from the origin to the point), the derivative will be proportional to the *speed* $\|\boldsymbol{\sigma}'(t_0)\|$ and to the cosine of the angle between the gradient and

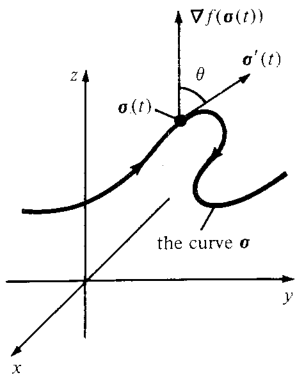


Figure 16.1.3. The derivative of f along the curve $\sigma(t)$ is

$$\frac{d}{dt} f(\sigma(t)) = \nabla f(\sigma(t)) \cdot \sigma'(t) = \|\nabla f(\sigma(t))\| \, \|\sigma'(t)\| \cos\theta.$$

velocity vectors. To describe how the derivative of f varies as we change the direction of the curve along which it is differentiated, we fix $\mathbf{r}$ and choose $\sigma(t) = \mathbf{r} + t\mathbf{d}$ for $\mathbf{d}$ a unit vector. (Note that since $\mathbf{d}$ is a unit vector, the speed of the curve $\sigma(t)$ is 1, so 1 unit of time corresponds to 1 unit of distance along the curve.)

We make the following definition: Let $f(x, y, z)$ be a function of three variables, $\mathbf{r}$ a point in its domain, and $\mathbf{d}$ a unit vector. Define the parametric curve $\sigma(t)$ by $\sigma(t) = \mathbf{r} + t\mathbf{d}$. The derivative $(d/dt) f(\sigma(t))|_{t=0}$ is called the *directional derivative* of f at $\mathbf{r}$ in the direction of $\mathbf{d}$.

Since $\sigma'(t) = \mathbf{d}$ and $\|\mathbf{d}\| = 1$, we see that if f has continuous partial derivatives, the directional derivative at $\mathbf{r}$ in the direction of $\mathbf{d}$ is

$$\nabla f(\mathbf{r}) \cdot \mathbf{d} = \|\nabla f(\mathbf{r})\| \cos\theta.$$

Notice that the directional derivatives in the directions of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are just the partial derivatives. For instance, choosing $\mathbf{d} = \mathbf{i}$, $\nabla f \cdot \mathbf{i} = (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) \cdot \mathbf{i} = f_x$. Similarly, $\nabla f \cdot \mathbf{j} = f_y$ and $\nabla f \cdot \mathbf{k} = f_z$.

As we let $\mathbf{d}$ vary, the directional derivative takes its maximum value when $\cos\theta = 1$, that is, when $\mathbf{d}$ points in the direction of $\nabla f(\mathbf{r})$. The maximum value of the directional derivative is just the length $\|\nabla f(\mathbf{r})\|$.

The following box summarizes our findings.

Gradients and Directional Derivatives

The *directional derivative* at $\mathbf{r}$ in the direction of a unit vector $\mathbf{d}$ is the rate of change of f along the straight line through $\mathbf{r}$ in direction $\mathbf{d}$; i.e., along $\sigma(t) = \mathbf{r} + t\mathbf{d}$.

The directional derivative at $\mathbf{r}$ in the direction $\mathbf{d}$ equals $\nabla f(\mathbf{r}) \cdot \mathbf{d}$. It is greatest (for fixed $\mathbf{r}$) when $\mathbf{d}$ points in the direction of the gradient $\nabla f(\mathbf{r})$ and least when $\mathbf{d}$ points in the same direction as $-\nabla f(\mathbf{r})$.

Example 7 Compute the directional derivatives of the following functions at the indicated points in the given directions.

(a) $f(x, y) = x + 2x^2 - 3xy$; $(x_0, y_0) = (1, 1)$; $\mathbf{d} = (\frac{3}{5}, \frac{4}{5})$.
(b) $f(x, y) = \ln(\sqrt{x^2 + y^2})$; $(x_0, y_0) = (1, 0)$; $\mathbf{d} = (2\sqrt{5}/5, \sqrt{5}/5)$.
(c) $f(x, y, z) = xyz$; $(x_0, y_0, z_0) = (1, 1, 1)$; $\mathbf{d} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{k}$.
(d) $f(x, y, z) = e^x + yz$; $(x_0, y_0, z_0) = (1, 1, 1)$; $\mathbf{d} = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} + \mathbf{k})$.

Solution (a) $\nabla f(x, y) = (1 + 4x - 3y, -3x)$. At $(1, 1)$ this is equal to $(2, -3)$. The directional derivative is $\nabla f(x_0, y_0) \cdot \mathbf{d} = (2, -3) \cdot (\frac{3}{5}, \frac{4}{5}) = -\frac{6}{5}$.

(b) $\nabla f(x, y) = (x/(x^2 + y^2), y/(x^2 + y^2))$, so $\nabla f(1, 0) = (1, 0)$. Thus, the directional derivative in direction $(2\sqrt{5}/5, \sqrt{5}/5)$ is $2\sqrt{5}/5$.

(c) $\nabla f(x, y, z) = (yz, xz, xy)$, which equals $(1, 1, 1)$ at $(1, 1, 1)$. For $\mathbf{d}$ equal to $(1/\sqrt{2}, 0, 1/\sqrt{2})$, the directional derivative is $1/\sqrt{2} + 0 + 1/\sqrt{2} = \sqrt{2}$.

(d) $\nabla f(x, y, z) = (e^x, z, y)$, which equals $(e, 1, 1)$ at $(1, 1, 1)$. For $\mathbf{d}$ equal to $(1/\sqrt{3}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k})$, the directional derivative is $e(1/\sqrt{3}) + 1(-1\sqrt{3}) + 1(1/\sqrt{3}) = e/\sqrt{3}$. ▲

If one wishes to move from $\mathbf{r} = (x, y, z)$ in a direction in which f is *increasing* most quickly, one should move in the direction $\nabla f(\mathbf{r})$. This is because $\nabla f(\mathbf{r}) \cdot \mathbf{d} = \|\nabla f(\mathbf{r})\| \cos\theta$ is maximum when $\theta = 0$, i.e., $\cos\theta = 1$, so $\mathbf{d}$ is in the direction of $\nabla f(\mathbf{r})$. Likewise, $-\nabla f(\mathbf{r})$ is the direction in which f is *decreasing* at the fastest rate.

Example 8 Let $u = f(x, y, z) = (\sin xy)e^{-z^2}$. In what direction from $(1, \pi, 0)$ should one proceed to increase f most rapidly?

Solution We compute the gradient:

$$\begin{aligned}\nabla f &= \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\\ &= y\cos(xy)e^{-z^2}\mathbf{i} + x\cos(xy)e^{-z^2}\mathbf{j} + (-2z\sin xy)e^{-z^2}\mathbf{k}.\end{aligned}$$

At $(1, \pi, 0)$ this becomes

$$\pi\cos(\pi)\mathbf{i} + \cos(\pi)\mathbf{j} = -\pi\mathbf{i} - \mathbf{j}.$$

Thus one should proceed in the direction of the vector $-\pi\mathbf{i} - \mathbf{j}$. ▲

Example 9 Captain Astro is drifting in space near the sunny side of Mercury and notices that the hull of her ship is beginning to melt. The temperature in her vicinity is given by $T = e^{-x} + e^{-2y} + e^{3z}$. If she is at $(1, 1, 1)$, in what direction should she proceed in order to *cool* fastest?

Solution In order to cool the fastest, the captain should proceed in the direction in which T is decreasing the fastest; that is, in the direction $-\nabla T(1, 1, 1)$. However,

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} = -e^{-x}\mathbf{i} - 2e^{-2y}\mathbf{j} + 3e^{3z}\mathbf{k}.$$

Thus,

$$-\nabla T(1, 1, 1) = e^{-1}\mathbf{i} + 2e^{-2}\mathbf{j} - 3e^{3}\mathbf{k}$$

is the direction required. ▲

Directional derivatives are also defined for functions of two variables. In this case, we have a geometric interpretation of the directional derivatives of $f(x, y)$ in terms of the graph $z = f(x, y)$. Given a point (x_0, y_0) in the plane and a unit vector $\mathbf{d} = a\mathbf{i} + b\mathbf{j}$, we can intersect the graph with the plane $\mathscr{P}$ in space which lies above the line through (x_0, y_0) with direction $\mathbf{d}$. (See Fig. 16.1.4.)

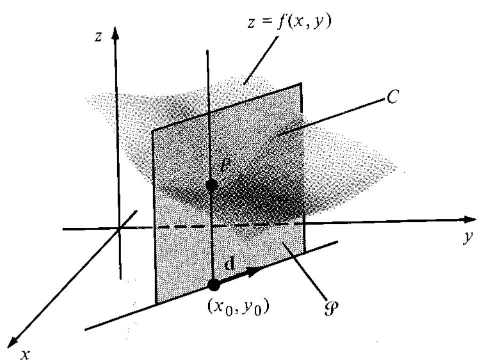


Figure 16.1.4. The slope at the point P of the curve C in the plane $\mathscr{P}$ is the directional derivative at (x_0, y_0) of f in the direction $\mathbf{d}$.

The result is a curve C which may be parametrized by the formula $(x, y, z) = (x_0 + at, y_0 + bt, f(x_0 + at, y_0 + bt))$. The tangent vector to this curve at $P = (x_0, y_0, f(x_0, y_0))$ is

$$\begin{aligned}\mathbf{v} &= a\mathbf{i} + b\mathbf{j} + \frac{d}{dt}f(x_0 + at, y_0 + bt)\Big|_{t=0}\mathbf{k}\\ &= a\mathbf{i} + b\mathbf{j} + \left[af_x(x_0, y_0) + bf_y(x_0, y_0)\right]\mathbf{k}.\end{aligned}$$

The slope of C in the plane $\mathscr{P}$ at P is the ratio of the vertical component

$af_x(x_0, y_0) + bf_y(x_0, y_0)$ of $\mathbf{v}$ to the length $\sqrt{a^2 + b^2}$ of the horizontal component; but $\sqrt{a^2 + b^2} = 1$, since $\mathbf{d} = a\mathbf{i} + b\mathbf{j}$ is a unit vector. Hence the slope of C in the plane $\mathscr{P}$ is just $af_x(x_0, y_0) + bf_y(x_0, y_0) = \mathbf{d} \cdot \nabla f(x_0, y_0)$, which is precisely the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{d}$.

If we let the vector $\mathbf{d}$ rotate in the xy plane, then the plane $\mathscr{P}$ will rotate about the vertical line through (x_0, y_0) and the curve C will change. The slopes at P of all these curves are determined by the two numbers $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, and the tangent lines to all these curves lie in the tangent plane to $z = f(x, y)$ at P.

Example 10 Let $f(x, y) = x^2 - y^2$. In what direction from $(0, 1)$ should one proceed in order to increase f the fastest? Illustrate your answer with a sketch.

Solution The required direction is

$$\begin{aligned}\nabla f(0,1) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \qquad \text{at } (0,1)\\ &= 2x\mathbf{i} - 2y\mathbf{j} \qquad \text{at } (0,1)\\ &= -2\mathbf{j}.\end{aligned}$$

Thus one should head toward the origin along the y axis. The graph of f, sketched in Fig. 16.1.5, illustrates this. ▲

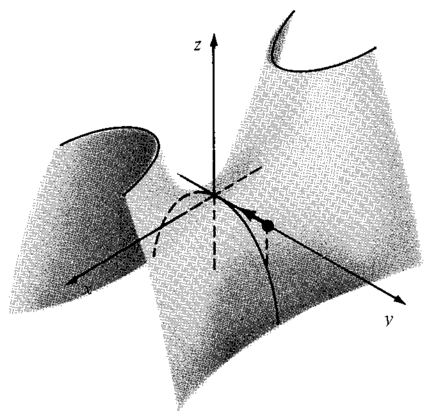


Figure 16.1.5. Starting from $(0, 1)$, moving in along the y axis makes the graph rise the steepest.

Our final example concerns the "position vector" $\mathbf{r}$; see Example 4.

Example 11 Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\|$. Compute ∇r. In what direction is r increasing the fastest? Interpret your answer geometrically.

Solution We know that $r = \sqrt{x^2 + y^2 + z^2}$, so

$$\nabla r = \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k},$$

since $\partial r/\partial x = \frac{1}{2} \cdot 2x/\sqrt{x^2 + y^2 + z^2} = x/r$ and so forth. Thus

$$\nabla r = \frac{1}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\mathbf{r}}{r}.$$

Thus r is increasing fastest in the direction of $\mathbf{r}/r$, which is a unit vector pointing outward from the origin. This makes sense since r is the distance from the origin. ▲

Exercises for Section 16.1

Compute the gradients of the functions in Exercises 1–8.

1. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
2. $f(x, y, z) = xy + yz + xz$.
3. $f(x, y, z) = x + y^2 + z^3$.
4. $f(x, y, z) = xy^2 + yz^2 + zx^2$.
5. $f(x, y) = \ln(\sqrt{x^2 + y^2})$.
6. $f(x, y) = (x^2 + y^2)\ln\sqrt{x^2 + y^2}$.
7. $f(x, y) = xe^{x^2+y^2}$.
8. $f(x, y) = x\exp(xy^3 + 3)$.

9. Sketch the gradient vector field of $f(x, y) = x^2/8 + y^2/12 + 6$.
10. Sketch the gradient vector field of $f(x, y) = x^2/8 - y^2/12$.
11. (a) Illustrate the vector field $\mathbf{\Phi}(x, y) = x\mathbf{j} - y\mathbf{i}$ by sketching $\frac{1}{3}\mathbf{\Phi}(x, y)$ instead. (b) Show that $\mathbf{\Phi}$ is not a gradient vector field.
12. (a) Sketch the vector field $\mathbf{\Phi}(x, y) = \frac{1}{4}\mathbf{i} + [1/(9 + x^2 + y^2)]\mathbf{j}$. (b) Explain why $\mathbf{\Phi}$ is or is not a gradient vector field.
13. Show that $\nabla(1/r^2) = -2\mathbf{r}/r^4$ $(r \neq 0)$.
14. Find $\nabla(1/r^3)$ $(r \neq 0)$.

Verify the chain rule for the functions and curves in Exercises 15–18.

15. $f(x, y, z) = xz + yz + xy$; $\boldsymbol{\sigma}(t) = (e^t, \cos t, \sin t)$.
16. $f(x, y, z) = e^{xyz}$; $\boldsymbol{\sigma}(t) = (6t, 3t^2, t^3)$.
17. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $\boldsymbol{\sigma}(t) = (\sin t, \cos t, t)$.
18. $f(x, y, z) = xy + yz + xz$; $\boldsymbol{\sigma}(t) = (t, t, t)$.

19. Suppose that $f(\boldsymbol{\sigma}(t))$ is an increasing function of t. What can you say about the angle between the gradient ∇f and the velocity vector $\boldsymbol{\sigma}'$?
20. Suppose that $f(\boldsymbol{\sigma}(t))$ attains a minimum at the time t_0. What can you say about the angle between $\nabla f(\boldsymbol{\sigma}(t_0))$ and $\boldsymbol{\sigma}'(t_0)$?

In Exercises 21–28, compute the directional derivative of each function at the given point in the given direction.

21. $f(x, y) = x^2 + y^2 - 3xy^3$; $(x_0, y_0) = (1, 2)$; $\mathbf{d} = (1/2, \sqrt{3}/2)$.
22. $f(x, y) = e^x\cos y$; $(x_0, y_0) = (0, \pi/4)$; $\mathbf{d} = (\mathbf{i} + 3\mathbf{j})/\sqrt{10}$.
23. $f(x, y) = 17x^y$; $(x_0, y_0) = (1, 1)$; $\mathbf{d} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.
24. $f(x, y) = e^{x^2\cos y}$; $(x_0, y_0) = (1, \pi/2)$; $\mathbf{d} = (3\mathbf{i} + 4\mathbf{j})/5$.
25. $f(x, y, z) = x^2 - 2xy + 3z^2$; $(x_0, y_0, z_0) = (1, 1, 2)$; $\mathbf{d} = (\mathbf{i} + \mathbf{j} - \mathbf{k})/\sqrt{3}$.
26. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$; $(x_0, y_0, z_0) = (1, 10, 100)$; $\mathbf{d} = (1, -1, -1)/\sqrt{3}$.
27. $f(x, y, z) = \sin(xyz)$; $(x_0, y_0, z_0) = (1, 1, \pi/4)$; $\mathbf{d} = (1/\sqrt{2}, 0, -1/\sqrt{2})$.
28. $f(x, y, z) = 1/(x^2 + y^2 + z^2)$; $(x_0, y_0, z_0) = (2, 3, 1)$; $\mathbf{d} = (\mathbf{j} - 2\mathbf{k} + \mathbf{i})/\sqrt{6}$.

In Exercises 29–32 determine the direction in which each of the functions is increasing fastest at (1, 1).

29. $f(x, y) = x^2 + 2y^2$
30. $g(x, y) = x^2 - 2y^2$
31. $h(x, y) = e^x\sin y$
32. $l(x, y) = e^x\sin y - e^{-x}\cos y$

33. Captain Astro is once again in trouble near the sunny side of Mercury. She is at location (1, 1, 1), and the temperature of the ship's hull when she is at location (x, y, z) will be given by $T(x, y, z) = e^{-x^2-2y^2-3z^2}$, where x, y and z are measured in meters.
    (a) In what direction should she proceed in order to decrease the temperature most rapidly?
    (b) If the ship travels at e^8 meters per second, how fast will be the temperature decrease if she proceeds in that direction?
    (c) Unfortunately, the metal of the hull will crack if cooled at a rate greater than $\sqrt{14}\,e^2$ degrees per second. Describe the set of possible directions in which she may proceed to bring the temperature down at no more than that rate.
34. Suppose that a mountain has the shape of an elliptic paraboloid $z = c - ax^2 - by^2$, where a, b, and c are positive constants, x and y are the east-west and north-south map coordinates, and z is the altitude above sea level (x, y, and z are all measured in meters). At the point (1, 1), in what direction is the altitude increasing most rapidly? If a marble were released at (1, 1), in what direction would it begin to roll?
35. An engineer wishes to build a railroad up the mountain of Exercise 34. Straight up the mountain is much too steep for the power of the engines. At the point (1, 1), in what directions may the track be laid so that it will be climbing with a 3% grade—that is, an angle whose tangent is 0.03. (There are two possibilities.) Make a sketch of the situation indicating the two possible directions for a 3% grade at (1, 1).
36. The height h of the Hawaiian volcano Mauna Loa is (roughly) described by the function $h(x, y) = 2.59 - 0.00024y^2 - 0.00065x^2$, where h is the height above sea level in miles and x and y measure east-west and north-south distances in miles from the top of the mountain.
    At $(x, y) = (-2, -4)$:
    (a) How fast is the height increasing in the direction (1, 1) (that is, northeastward)? Express your answer in miles of height per mile of horizontal distance travelled.
    (b) In what direction is the steepest upward path?

(c) In what direction is the steepest downward path?
(d) In what direction(s) is the path level?
(e) If you proceed south, are you ascending or descending? At what rate?
(f) If you move northwest, are you ascending or descending? At what rate?
(g) In what direction(s) may you proceed in order to be climbing with a grade of 3%?

37. In what direction from $(1,0)$ does the function $f(x, y) = x^2 - y^2$ increase the fastest? Illustrate with a sketch.
38. In what direction from $(-1,0)$ does the function $f(x, y) = x^2 - y^2$ increase fastest? Sketch.
39. In what direction is the length of $\mathbf{r} + \mathbf{j}$ increasing fastest at the point $(1,0,1)$? ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$).
40. In what direction should you travel from the point $(2,4,3)$ to make the length of $\mathbf{r} + \mathbf{k}$ decrease as fast as possible?
41. Suppose that f and g are real-valued functions (with continuous partial derivatives). Show that:
    (a) $\nabla f = \mathbf{0}$ if f is constant;
    (b) $\nabla(f + g) = \nabla f + \nabla g$;
    (c) $\nabla(cf) = c\nabla f$ if c is a constant;
    (d) $\nabla(fg) = f\nabla g + g\nabla f$;
    (e) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ at points where $g \neq 0$.
42. What rate of change does $\nabla f(x, y, z) \cdot (-\mathbf{j})$ represent?
43. (a) In what direction is the directional derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $(1,1)$ equal to zero?
    (b) How about at an arbitrary point (x_0, y_0) in the first quadrant?
    (c) Describe the level curves of f. In particular, discuss them in terms of the result of (b).
44. Suppose that $f(x, y)$ is given (and has continuous partial derivatives). At $(1,1)$ the directional derivative in the direction toward $(2,4)$ is 2 and in the direction toward $(2,2)$ it is 3. Find the gradient of f at $(1,1)$ and the directional derivative there in the direction toward $(2,3)$.
45. A function $f(x, y)$ has, at the point $(1,3)$, directional derivatives of $+2$ in the direction toward $(2,3)$ and -2 in the direction toward $(1,4)$. Determine the gradient vector at $(1,3)$ and compute the directional derivative in the direction toward $(3,6)$.
46. In electrostatistics, the force $\mathbf{P}$ of attraction between two particles of opposite charge is given by $\mathbf{P} = k(\mathbf{r}/\|\mathbf{r}\|^3)$ (*Coulomb's law*), where k is a constant and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\mathbf{P}$ is the gradient of $f = -k/\|\mathbf{r}\|$.
47. The potential V due to two infinite parallel filaments of charge of linear densities λ and $-\lambda$ is $V = (\lambda/2\pi\varepsilon_0)\ln(r_2/r_1)$, where $r_1^2 = (x - x_0)^2 + y^2$ and $r_2^2 = (x + x_0)^2 + y^2$. We think of the filaments as being in the z direction, passing through the xy plane at $(-x_0, 0)$ and $(x_0, 0)$.
    (a) Find $\nabla V(x, y)$, using the chain rule.
    (b) Verify the *flux law* $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$.
48. For each of the following find the maximum and minimum values attained by the function f along the curve $\boldsymbol{\sigma}(t)$:
    (a) $f(x, y) = xy$; $\boldsymbol{\sigma}(t) = (\cos t, \sin t)$; $0 \leqslant t \leqslant 2\pi$.
    (b) $f(x, y) = x^2 + y^2$; $\boldsymbol{\sigma}(t) = (\cos t, 2\sin t)$; $0 \leqslant t \leqslant 2\pi$.

★49. What conditions on the function $f(x, y)$ hold if the vector field $\mathbf{k} \times \nabla f$ is a gradient vector field?

★50. (a) Let F be a function of one variable and f a function of two variables. Show that the gradient vector of $g(x, y) = F(f(x, y))$ is parallel to the gradient vector of $f(x, y)$.
    (b) Let $f(x, y)$ and $g(x, y)$ be functions such that $\nabla f = \lambda \nabla g$ for some function $\lambda(x, y)$. What is the relation between the level curves of f and g? Explain why there might be a function F such that $g(x, y) = F(f(x, y))$.

16.2 Gradients, Level Surfaces, and Implicit Differentiation

The gradient of a function of three variables is perpendicular to the surfaces on which the function is constant.

Recall that the tangent plane to a graph $z = f(x, y)$ was defined as the graph of the linear approximation to f. We found (Section 15.3) that the tangent plane at a point could also be characterized as the plane containing the tangent *lines* to all curves on the surface through the given point. For a general surface, we take this as a definition.

Definition: Tangent Plane to a Surface

Let S be a surface in space, $\mathbf{r}_0$ a point of S. If there is a plane which contains the tangent lines at $\mathbf{r}_0$ to all curves through $\mathbf{r}_0$ in S, then this plane is called *the tangent plane to S at $\mathbf{r}_0$*. A normal to the tangent plane is sometimes said to be *perpendicular to S*.

The next box tells how to find the tangent plane to a level surface.

Gradients and Tangent Planes

Let $\mathbf{r}_0$ lie on the level surface S defined by $f(x, y, z) = c$, and suppose that $\nabla f(\mathbf{r}_0) \neq \mathbf{0}$. Then $\nabla f(\mathbf{r}_0)$ is normal to the tangent plane to S at $\mathbf{r}_0$. (See Fig. 16.2.1.)

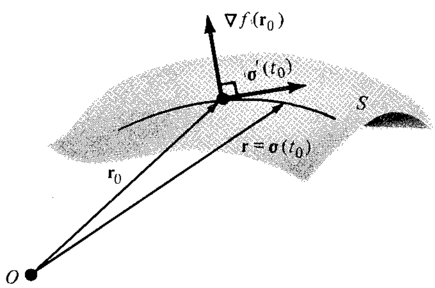


Figure 16.2.1. The gradient of f at $\mathbf{r}_0$ is perpendicular to the tangent vector of any curve in the level surface.

To prove this assertion, first observe that $f(\boldsymbol{\sigma}(t)) = c$ if the curve $\boldsymbol{\sigma}(t)$ lies in S. Hence

$$\frac{d}{dt} f(\boldsymbol{\sigma}(t)) = 0.$$

By the chain rule in terms of gradients, this gives

$$\nabla f(\boldsymbol{\sigma}(t)) \cdot \boldsymbol{\sigma}'(t) = 0.$$

Setting $t = t_0$, we have $\nabla f(\mathbf{r}_0) \cdot \boldsymbol{\sigma}'(t_0) = 0$ for every curve $\boldsymbol{\sigma}$ in S, so $\nabla f(\mathbf{r}_0)$ is normal to the tangent plane. (We required $\nabla f(\mathbf{r}_0) \neq \mathbf{0}$ so there would be a well-defined plane orthogonal to $\nabla f(\mathbf{r}_0)$.)

Example 1 Let $u = f(x, y, z) = x^2 + y^2 - z^2$. Find $\nabla f(0, 0, 1)$. Plot this on the level surface $f(x, y, z) = -1$.

Solution We have

$$\nabla f = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}.$$

At $(0, 0, 1)$, $\nabla f(0, 0, 1) = -2\mathbf{k}$.

The level surface $x^2 + y^2 - z^2 = -1$ is a hyperboloid of two sheets (Section 14.4). If we plot $\nabla f(0, 0, 1)$ on it (Fig. 16.2.2), we see that it is indeed perpendicular to the surface. ▲

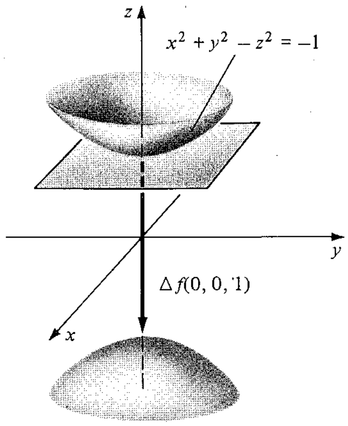


Figure 16.2.2. $\nabla f(0,0,1)$ is perpendicular to the surface.

Example 2 Find a unit normal to the surface $\sin(xy) = e^z$ at $(1, \pi/2, 0)$.

Solution Let $f(x, y, z) = \sin(xy) - e^z$, so the surface is $f(x, y, z) = 0$. A normal is $\nabla f = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j} - e^z\mathbf{k}$. At $(1, \pi/2, 0)$, we get $-\mathbf{k}$. Thus $-\mathbf{k}$ (or $\mathbf{k}$) is the required unit normal. (It already has length 1, so there is no need to normalize.) ▲

Example 3 The gravitational force exerted on a mass m at (x, y, z) by a mass M at the origin is, by Newton's law of gravitation,

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}, \qquad \text{where} \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \text{and} \quad r = \|\mathbf{r}\|.$$

Write $\mathbf{F}$ as the negative gradient of a function V (called the gravitational potential) and verify that F is orthogonal to the level surfaces of V.

Solution By Example 4, Section 16.1, $\nabla(1/r) = -(\mathbf{r}/r^3)$. Therefore we can choose $V = -GMm/r$ to give $\mathbf{F} = -\nabla V$. The vector $\mathbf{F}$ points toward the origin. The level surfaces of V are $1/r = c$—that is, $r = 1/c$, a sphere. Therefore, $\mathbf{F}$ is orthogonal to these surfaces. ▲

The gradient enables us to compute the equation of the tangent plane to the level surface S at $\mathbf{r}_0$. Indeed, $\nabla f(\mathbf{r}_0)$ will be a normal to this plane, which passes through $\mathbf{r}_0$. Therefore its equation can be read off immediately. (See Section 13.4.)

Example 4 Compute the equation of the plane tangent to the surface $3xy + z^2 = 4$ at $(1, 1, 1)$.

Solution Here $f(x, y, z) = 3xy + z^2$ and $\nabla f = (3y, 3x, 2z)$, which at $(1, 1, 1)$ is the vector $3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Thus the tangent plane is

$$3(x-1) + 3(y-1) + 2(z-1) = 0 \quad \text{or} \quad 3x + 3y + 2z = 8. ▲$$

Example 5 (a) Find a unit normal to the ellipsoid $x^2 + 2y^2 + 3z^2 = 10$ at each of the points $(\sqrt{10}, 0, 0)$, $(-\sqrt{10}, 0, 0)$, $(1, 0, \sqrt{3})$, and $(-1, 0, -\sqrt{3})$.
(b) Do the vectors you have found point to the inside or outside of the ellipsoid?
(c) Give equations for the tangent planes to the surface at the two points of the surface with $x_0 = y_0 = 1$.

Solution (a) Letting $f(x, y, z) = x^2 + 2y^2 + 3z^2 = 10$, we find $\nabla f(x, y, z) = (2x, 4y, 6z)$. At $(\sqrt{10}, 0, 0)$, a unit normal to the ellipsoid is

$$\frac{\nabla f(\sqrt{10}, 0, 0)}{\|\nabla f(\sqrt{10}, 0, 0)\|} = \frac{(2\sqrt{10}, 0, 0)}{\left((2\sqrt{10})^2 + 0^2 + 0^2\right)^{1/2}} = (1, 0, 0).$$

At $(-\sqrt{10}, 0, 0)$, it is $(-1, 0, 0)$. At $(1, 0, \sqrt{3})$, it is

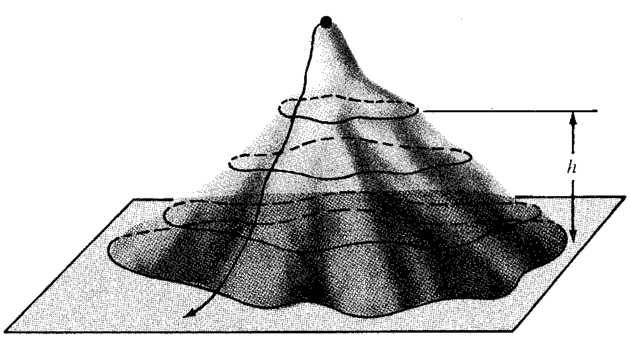
$$\frac{\nabla f(1, 0, \sqrt{3})}{\|\nabla f(1, 0, \sqrt{3})\|} = \left(\frac{1}{\sqrt{28}}, 0, \frac{3\sqrt{3}}{\sqrt{28}}\right),$$

and at $(-1, 0, -\sqrt{3})$ it is $(-1/\sqrt{28}, 0, -3\sqrt{3}/\sqrt{28})$.

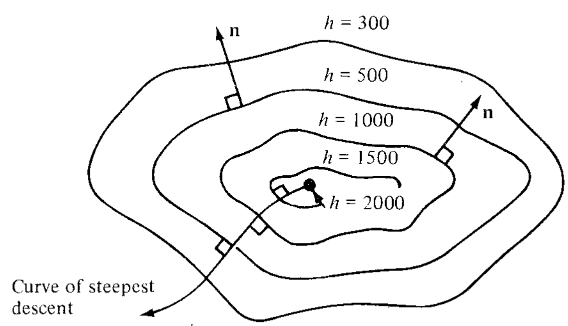
(b) The vectors are pointing to the outside of the ellipsoid.

(c) The two points are $(1, 1, \sqrt{7/3})$, and $(1, 1, -\sqrt{7/3})$. Evaluating the gradient, $\nabla f(1, 1, \sqrt{7/3}) = (2, 4, 2\sqrt{21})$ and $\nabla f(1, 1, -\sqrt{7/3}) = (2, 4, -2\sqrt{21})$, so the tangent planes to the surface at the points $(1, 1, \sqrt{7/3})$ and $(1, 1, -\sqrt{7/3})$ are given by $2(x-1) + 4(y-1) + 2\sqrt{21}(z - \sqrt{7/3}) = 0$ and $2(x-1) + 4(y-1) - 2\sqrt{21}(z + \sqrt{7/3}) = 0$, respectively. ▲

There is also a connection between gradients and tangents for functions of two variables: the tangent line to a level *curve* of a function $f(x, y)$ is perpendicular to the gradient of f at each point. Combining this fact with the box on p. 801, we see that the direction in which the function f is increasing or decreasing most rapidly is perpendicular to the level curves of f. For example, to get down most directly from the top of a hill, one should proceed in a direction perpendicular to the level contours. (See Fig. 16.2.3.)



(a) Steepest descent of a hill



(b) Contour map of hill 2000 feet high

Figure 16.2.3. The curve of steepest descent is perpendicular to the level curves. (a) Steepest descent of a hill. (b) Contour map of hill 2000 feet high.

Gradients, Level Surfaces, and Level Curves

The normal to the tangent plane at $\mathbf{r}_0 = (x_0, y_0, z_0)$ of the level surface $f(x, y, z) = c$ is $\nabla f(\mathbf{r}_0)$. The equation of the plane is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The equation of the tangent line at (x_0, y_0) to the curve $f(x, y) = c$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

Example 6 Find the equation of the tangent line to $xy = 6$ at $x = 1, y = 6$.

Solution With $f(x, y) = xy$, we have $f_x(x, y) = y$ and $f_y(x, y) = x$. Then $f_x(1,6) = 6$ and $f_y(1,6) = 1$, so from the preceding box, the equation of the tangent line through $(1,6)$ is

$$6(x-1) + 1(y-6) = 0 \quad \text{or} \quad y = -6x + 12. \ \blacktriangle$$

In the next example we check that the equation given in Section 15.2 for the tangent plane to a graph is consistent with that given here.

Example 7 Let $z = g(x, y)$. The graph of g may be defined as the level surface $f(x, y, z) = 0$, where $f(x, y, z) = z - g(x, y)$. Compute the gradient of f and verify that it is perpendicular to the tangent plane of the graph $z = g(x, y)$ as defined in Section 15.2.

Solution With $f(x, y, z) = z - g(x, y)$,

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}\\ &= -g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}.\end{aligned}$$

This is exactly the normal to the tangent plane at (x, y) to the graph of g. ▲

Many functions of several variables are built by combining functions of one variable. We actually found partial derivatives of such functions in our earlier work on implicit differentiation and related rates. For instance, suppose that $y = f(x)$ and that x and y satisfy the relation

$$x^3 + 8x \sin y = 0.$$

Then differentiating with respect to x, using the chain rule for functions of *one* variable, gives

$$3x^2 + 8 \sin y + 8x \cos y \frac{dy}{dx} = 0.$$

which we can solve for dy/dx to obtain

$$\frac{dy}{dx} = -\frac{3x^2 + 8 \sin y}{8x \cos y}.$$

From the point of view of multivariable calculus, we may say that the graph $y = f(x)$ lies on the level curve $F(x, y) = 0$, where

$$F(x, y) = x^3 + 8x \sin y.$$

A normal vector to this curve at (x, y) is

$$\nabla F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} = (3x^2 + 8 \sin y)\mathbf{i} + (8x \cos y)\mathbf{j},$$

so a tangent vector is given by any vector perpendicular to ∇F, such as

$$(-8x \cos y)\mathbf{i} + (3x^2 + 8 \sin y)\mathbf{j}.$$

Thus dy/dx, the slope of the tangent line, is

$$\frac{3x^2 + 8 \sin y}{-8x \cos y}.$$

The general procedure is indicated in the following box.

Implicit Differentiation and Partial Derivatives

If $y = f(x)$ is a function satisfying the relation $z = F(x, y) = 0$, then

$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}, \tag{1}$$

i.e.

$$f'(x) = -\frac{F_x(x, f(x))}{F_y(x, f(x))}. \tag{1'}$$

Indeed, differentiating $F(x, y) = 0$ with respect to x using the chain rule gives

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

i.e.,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Solving for dy/dx gives the result in the box. Notice that in (1) it is *incorrect* to "cancel the ∂z's," because the minus sign would be left.

Example 8 Suppose that y is defined implicitly in terms of x by $e^{x-y} + x^2 - y = 1$. Find dy/dx at $x = 0, y = 0$ using formula (1).

Solution Here $z = F(x, y) = e^{x-y} + x^2 - y - 1$, so

$$\frac{\partial z}{\partial x} = e^{x-y} + 2x \quad \text{and} \quad \left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}} = 1.$$

Likewise

$$\frac{\partial z}{\partial y} = -e^{x-y} - 1 \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{\substack{x=0\\y=0}} = -2.$$

Therefore

$$-\frac{\partial z/\partial x}{\partial z/\partial y} = 1/2.$$

and so, by (1), $dy/dx = 1/2$. ▲

Formula (1) makes sense as long as $\partial z/\partial y \neq 0$. In fact there is a result called the *implicit function theorem*[1] which guarantees that $F(x, y) = 0$ does indeed define y as a function of x, provided that $\partial z/\partial y \neq 0$. The values of x and y may have to be restricted, as we found when studying implicit differentiation in Section 2.3 (see Figure 2.3.1).

Example 9 Discuss what happens to y as a function of x if $\partial z/\partial y = 0$ in (1) for the example $x - y^3 = 0$.

Solution The equation $z = F(x, y) = x - y^3 = 0$ implicitly defines the function $y = f(x) = \sqrt[3]{x}$. We have $\partial z/\partial x = 1$ and $\partial z/\partial y = -3y^2$; so $\partial z/\partial y$ vanishes

[1] For a proof based on the mean value and intermediate value theorems, see J. Marsden and A. Tromba, *Vector Calculus*, Second Edition, Freeman (1981), Section 4.4.

when $y = 0$; this is just the point on the graph $y = \sqrt[3]{x}$ where the cube-root function is not differentiable and the tangent line becomes vertical. ▲

In related rate problems, we have a parametric curve $(x, y) = (g(t), h(t))$ which lies on a level curve $F(x, y) = 0$. Differentiating with respect to t by the chain rule, we get

$$0 = F_x(x, y)\frac{dx}{dt} + F_y(x, y)\frac{dy}{dt} \tag{2}$$

which is a relation between the rates dx/dt and dy/dt. Such relations were obtained in Section 2.5 using one variable calculus.

Example 10 Suppose that $x = g(t)$ and $y = h(t)$ satisfy the relation $x^2 - y^2 = xy$. Find a relation between dx/dt and dy/dt:
(a) by one-variable calculus;
(b) by formula (2).

Solution (a) Differentiating the relation $x^2 - y^2 = xy$ with respect to t by one-variable calculus, we obtain $2x(dx/dt) - 2y(dy/dt) = y(dx/dt) + x(dy/dt)$ or, equivalently, $(2x - y)(dx/dt) - (2y + x)(dy/dt) = 0$.
(b) To apply formula (2), we set $F(x, y) = x^2 - y^2 - xy$. Then $F_x(x, y) = 2x - y$ and $F_y(x, y) = -(2y + x)$, so (2) gives the same relation between dx/dt and dy/dt: $(2x - y)(dx/dt) - (2y + x)(dy/dt) = 0$. ▲

Example 11 Suppose that $x = g(t)$ and $y = h(t)$ satisfy the relation $x^y = 2$. Find a relation between dx/dt and dy/dt.

Solution Let $z = F(x, y) = x^y - 2$. Then $\partial z/\partial x = yx^{y-1}$ and $\partial z/\partial y = x^y \ln x$, so the relation is

$$yx^{y-1}\frac{dx}{dt} + x^y \ln x \frac{dy}{dt} = 0.$$

Using the fact that $x^y = 2$, we can simplify this to

$$\frac{y}{x}\frac{dx}{dt} + \ln x \frac{dy}{dt} = 0. \ ▲$$

Exercises for Section 16.2

In Exercises 1–4, find $\nabla f(0, 0, 1)$ and plot it on the level surface $f(x, y, z) = c$ passing through $(0, 0, 1)$.

1. $f(x, y, z) = x^2 + y^2 + z^2$
2. $f(x, y, z) = z - x^2 - y^2$
3. $f(x, y, z) = z - x + y$
4. $f(x, y, z) = z^2 - x - y$

In Exercises 5–8, find a unit normal to the given surface at the given point.

5. $xyz = 8$; $(1, 1, 8)$.
6. $x^2y^2 + y - z + 1 = 0$ at $(0, 0, 1)$.
7. $\cos(xy) = e^z - 2$ at $(1, \pi, 0)$.
8. $e^{xyz} = e$ at $(1, 1, 1)$.

9. Coulomb's law states that the electric force on a charge q at (x, y, z) produced by a charge Q at the origin is $\mathbf{F} = Qq\mathbf{r}/r^3$. Find V so that $\mathbf{F} = -\nabla V$ and verify that $\mathbf{F}$ is orthogonal to the level surfaces of V.
10. Joe Perverse has invented a new law of gravitation. In this theory, the force exerted on a mass m at (x, y, z) by a mass M at the origin is $\mathbf{F} = -JMm\mathbf{r}/r^5$, where J is Joe's constant. Find V such that $\mathbf{F} = -\nabla V$ and verify that $\mathbf{F}$ is orthogonal to the level surfaces of V.

In Exercises 11–16, find the equation for the tangent plane to each surface at the indicated point.

11. $x^2 + 2y^2 + 3z^2 = 10$; $(1, \sqrt{3}, 1)$.
12. $xyz^2 = 1$; $(1, 1, 1)$.
13. $x^2 + 2y^2 + 3xz = 10$; $(1, 2, \frac{1}{3})$.
14. $y^2 - x^2 = 3$; $(1, 2, 8)$.
15. $xyz = 1$; $(1, 1, 1)$.
16. $xy/z = 1$; $(1, 1, 1)$.

Find the equation for the tangent line to each curve at the indicated point in Exercises 17–20.

17. $x^2 + 2y^2 = 3$; $(1, 1)$.

18. $xy = 17$; $(x_0, 17/x_0)$.
19. $\cos(x + y) = 1/2$; $x = \pi/2, y = 0$.
20. $e^{xy} = 2$; $(1, \ln 2)$.

Find the equation of the line normal to the given surface at the given point in Exercises 21–24.

21. $e^{-(x^2+y^2+z^2)} = e^{-3}$; $(1, 1, 1)$
22. $2x^2 + 3y^2 + z^2 = 9$; $(1, 1, 2)$
23. $x/yz = 1$; $(1, 1, 1)$
24. $xyz^2 = 4$; $(1, 1, 2)$

In Exercises 25–30, suppose that y is defined implicitly in terms of x by the given equation. Find dy/dx using formula (1).

25. $x^2 + 2y^2 = 3$
26. $x^2 - y^2 = 7$
27. $x/y = 10$
28. $y - \sin x^3 + x^2 - y^2 = 1$
29. $x^3 - \sin y + y^4 = 4$
30. $e^{x+y^2} + y^3 = 0$

In Exercises 31–34, find dy/dx at the indicated point using formula (1).

31. $3x^2 + y^2 - e^x = 0$; $x = 0, y = 1$.
32. $x^2 + y^4 = 1$; $x = 1, y = 1$.
33. $\cos(x + y) = x + 1/2$; $x = 0, y = \pi/3$.
34. $\cos(xy) = 1/2$; $x = 1, y = \pi/3$.

In Exercises 35–38, discuss what happens to y as a function of x if $\partial z/\partial y = 0$ in (1) for the given equation.

35. $x - y^2 = 0$
36. $x - \cos y = 0$
37. $x - y^5 = 0$
38. $x - \sin y = 0$

In Exercises 39–42, suppose that x and y are functions of t satisfying the given relation. Find a relation between dx/dt and dy/dt using formula (2).

39. $x \ln y = 1$
40. $\sin(xy) + \cos(xy) = 1$
41. $x^4 + y^4 = 1$
42. $x^2 + 3y^2 = 2$
43. (a) Derive a formula like (1) for dx/dy when x and y are related by $F(x, y) = 0$. (b) Use your result in (a) to find dx/dy for the functions in Exercises 29 and 30.
44. Let y be a function of x satisfying $F(x, y, x + y) = 0$, where $F(x, y, z)$ is a given function. Find a formula for dy/dx.

Suppose that $x = g(t)$ and $y = h(t)$ satisfy the equations in Exercises 45 and 46. Relate dx/dt and dy/dt.

45. $\ln(x \cos y) = x$
46. $\cos(x - 2y^2 + y^3) = y$
47. (a) Find the plane which is tangent to the surface $z = x^2 + y^2$ at the point $(1, -2, 5)$.
    ★(b) Letting $f(x, y) = x^2 + y^2$, define the "slope" of the tangent plane relative to the xy plane and show that it equals $\|\nabla f(1, -2)\|$.
48. (a) Show that the curve $x^2 - y^2 = c$, for *any* value of c, satisfies the differential equation $dy/dx = x/y$.
    (b) Draw in a few of the curves $x^2 - y^2 = c$, say for $c = \pm 1$. At several points (x, y) along each of these curves, draw a short segment of slope x/y; check that these segments appear to be tangent to the curve. What happens when $y = 0$? What happens when $c = 0$?
49. Suppose that a particle is ejected from the surface $x^2 + y^2 - z^2 = -1$ at the point $(1, 1, \sqrt{3})$ in a direction normal to the surface at time $t = 0$ with a speed of 10 units per second. When and where does it cross the xy plane?

★50. Let V be a function defined on a domain in space. The force field associated with V is $\mathbf{F} = \mathbf{\Phi}(x, y, z) = -\nabla V(x, y, z)$; we call V the *potential* of $\mathbf{\Phi}$. Let a point with mass m move on a parametric curve $\boldsymbol{\sigma}(t)$ and satisfy Newton's second law $m\mathbf{a} = \mathbf{F}$, where $\mathbf{a}$ is the acceleration of the curve. Use the chain rule to prove the law of conservation of energy: $E = \frac{1}{2} m\|\boldsymbol{\sigma}'(t)\|^2 + V[\boldsymbol{\sigma}(t)]$ is constant, where $\boldsymbol{\sigma}(t)$ is the position vector of the curve.

★51. The level surfaces of a potential function V are called *equipotential surfaces*.
    (a) What is the relation between the force vector and the equipotential surfaces?
    (b) Explain why "sea level" is approximately an equipotential surface for the earth's gravitational field. What spoils the approximation?

16.3 Maxima and Minima

First and second derivative tests are developed for locating maximum and minimum points for functions of two variables.

In studying maxima and minima for functions of one variable, we found that the basic tests involved the vanishing of the first derivative and the sign of the second derivative. In this section we develop tests involving first and second partial derivatives for locating maxima and minima of functions of two variables.

The definitions of maxima and minima for functions of two variables are similar to those in the one-variable case, except that we use disks instead of intervals. Recall that the disk of radius r about (x_0, y_0) consists of all (x, y) such that the distance $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ is less than r. (See Fig. 15.1.2.)

Definition of Maxima and Minima

Let $f(x, y)$ be a function of two variables. We say that (x_0, y_0) is a *local minimum point* for f if there is a disk (of positive radius) about (x_0, y_0) such that $f(x, y) \geqslant f(x_0, y_0)$ for all (x, y) in the disk.

Similarly, if $f(x, y) \leqslant f(x_0, y_0)$ for all (x, y) in some disk (of positive radius) about (x_0, y_0), we call (x_0, y_0) a *local maximum point* for f.

A point which is either a local maximum or minimum point is called a *local extremum*.

We may also define *global* maximum and minimum points to be those at which a function attains the greatest and least values for all points in its domain.

Example 1 Refer to Fig. 16.3.1, a computer-drawn graph of $z = 2(x^2 + y^2)e^{-x^2-y^2}$. Where are the maximum and minimum points?

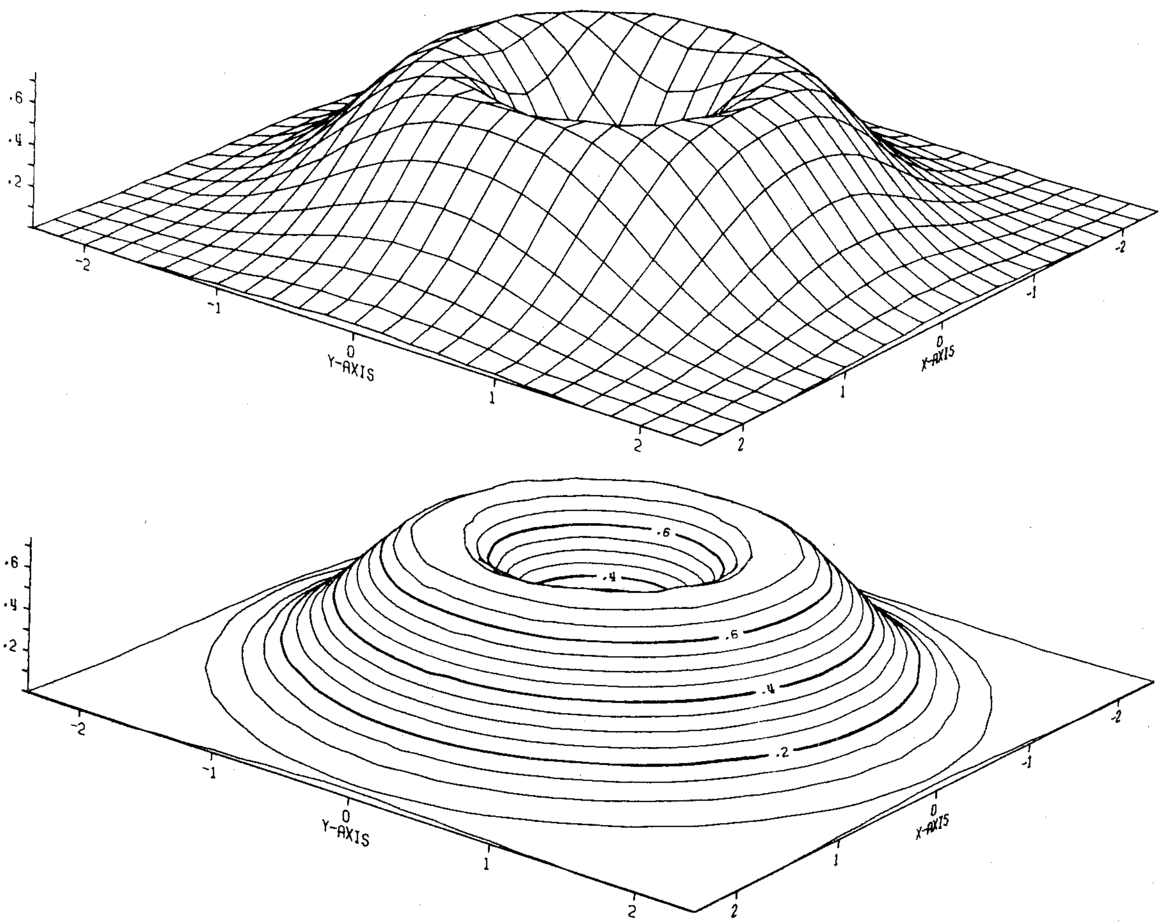


Figure 16.3.1. The volcano: $z = 2(x^2 + y^2)\exp(-x^2 - y^2)$. (a) Coordinate grid lifted to the surface. (b) Level curves lifted to the surface.

Solution There is a local (in fact, global) minimum at the volcano's center $(0,0)$, where $z=0$. There are maximum points all around the crater's rim (the circle $x^2+y^2=1$). ▲

The following is the analog in two variables of the first derivative test for one variable (see Section 3.2).

First Derivative Test

Suppose that (x_0, y_0) is a local extremum of f and that the partial derivatives of f exist at (x_0, y_0). Then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

We consider the case of a local minimum; the proof for a local maximum is essentially the same.

By assumption, there is a disk of radius r about (x_0, y_0) on which $f(x, y) \geqslant f(x_0, y_0)$. In particular, if $|x - x_0| < r$, then $f(x, y_0) \geqslant f(x_0, y_0)$, so the function $g(x) = f(x, y_0)$ has a local minimum at x_0. By the first derivative test of one-variable calculus, $g'(x_0) = 0$; but $g'(x_0)$ is just $f_x(x_0, y_0)$. Similarly, the function $h(y) = f(x_0, y)$ has a local minimum at y_0, so $f_y(x_0, y_0) = 0$.

The first derivative test has a simple geometric interpretation: at a local extremum of f, the tangent plane to the graph $z = f(x, y)$ is horizontal (that is, parallel to the xy plane.)

Points at which f_x and f_y both vanish are called *critical points* of f. As in one-variable calculus, finding critical points is only the first step in finding local extrema. A critical point could be a local maximum, local minimum, or neither. After looking at some examples, we will present the second derivative test for functions of two variables.

Example 2 Verify that the critical points of the function in Example 1 occur at $(0,0)$ and on the circle $x^2 + y^2 = 1$.

Solution Since $z = 2(x^2 + y^2)e^{-x^2-y^2}$, we have

$$\begin{aligned}\frac{\partial z}{\partial x} &= 4x(e^{-x^2-y^2}) + 2(x^2+y^2)e^{-x^2-y^2}(-2x)\\ &= 4x(e^{-x^2-y^2})(1 - x^2 - y^2)\end{aligned}$$

and

$$\frac{\partial z}{\partial y} = 4y(e^{-x^2-y^2})(1 - x^2 - y^2).$$

These vanish when $x = y = 0$ or when $x^2 + y^2 = 1$. ▲

Example 3 Let $z = x^2 - y^2$. Show that $(0,0)$ is a critical point. Is it a local extremum?

Solution The partial derivatives $\partial z/\partial x = 2x$ and $\partial z/\partial y = -2y$ vanish at $(0,0)$, so the origin is a critical point. It is neither a local maximum nor minimum since $f(x, y) = x^2 - y^2$ is zero at $(0,0)$ and can be either positive (on the x axis) or negative (on the y axis) arbitrarily near the origin. This is also clear from the graph (see Fig. 16.1.5), which shows a saddle point at $(0,0)$. ▲

If we know in advance that a function has a minimum point, and that the partial derivatives exist there, then we can use the first derivative test to locate the point.

Example 4 (a) Find the minimum distance from the origin to a point on the plane $x + 3y - z = 6$.
(b) Find the minimum distance from $(1,2,0)$ to the cone $z^2 = x^2 + y^2$.

Solution (a) Geometric intuition tells us that any plane contains a point which is closest to the origin. To find that point, we must minimize the distance $d = \sqrt{x^2 + y^2 + z^2}$, where $z = x + 3y - 6$. It is equivalent but simpler to minimize $d^2 = x^2 + y^2 + (x + 3y - 6)^2$. By the first derivative test, we must have

$$\frac{\partial(d^2)}{\partial x} = 0 \qquad \text{that is,} \quad 2x + 2(x + 3y - 6) = 0$$

and

$$\frac{\partial(d^2)}{\partial y} = 0 \qquad \text{that is,} \quad 2y + 6(x + 3y - 6) = 0.$$

Solving these equations gives $y = \frac{18}{11}$, $x = \frac{6}{11}$. Thus $z = x + 3y - 6 = -\frac{6}{11}$, and so the minimum distance is $d = \sqrt{x^2 + y^2 + z^2} = 6\sqrt{11}/11$. (See Fig. 16.3.2.)

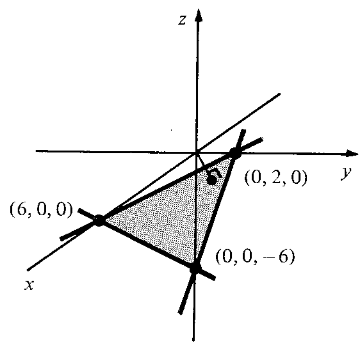


Figure 16.3.2. The point nearest to the origin on the plane $z = x + 3y - 6$ is $(6/11, 18/11, -6/11)$.

(b) We minimize the square of the distance: $d^2 = (x-1)^2 + (y-2)^2 + z^2$. Substituting $z^2 = x^2 + y^2$, we have the problem of minimizing

$$\begin{aligned} f(x,y) &= (x-1)^2 + (y-2)^2 + x^2 + y^2 \\ &= 2x^2 + 2y^2 - 2x - 4y + 5. \end{aligned}$$

Now

$$f_x(x,y) = 4x - 2 \quad \text{and} \quad f_y(x,y) = 4y - 4.$$

Thus the critical point, obtained by setting these equal to zero, is $x = \frac{1}{2}$, $y = 1$. This is the minimum point. The minimum distance is

$$\begin{aligned} d &= \sqrt{(1/2 - 1)^2 + (1-2)^2 + (1/2)^2 + 1} \\ &= \sqrt{1/4 + 1 + 1/4 + 1} = \sqrt{5/2} \approx 1.581. \ \blacktriangle \end{aligned}$$

Example 5 A rectangular box, open at the top, is to hold 256 cubic centimeters of cat food. Find the dimensions for which the surface area (bottom and four sides) is minimized.

Solution Let x and y be the lengths of the sides of the base. Since the volume of the box is to be 256, the height must be $256/xy$. Two of the sides have area $x(256/xy)$, two sides have area $y(256/xy)$, and the base has area xy, so the total surface area is $A = 2x(256/xy) + 2y(256/xy) + xy = 512/y + 512/x + xy$. To minimize A, we must have

$$0 = \frac{\partial A}{\partial x} = -\frac{512}{x^2} + y, \qquad 0 = \frac{\partial A}{\partial y} = -\frac{512}{y^2} + x.$$

The first equation gives $y = 512/x^2$; substituting this into the second equation gives $0 = -512(x^2/512)^2 + x = -x^4/512 + x$. Discarding the extraneous root $x = 0$, we have $x^3/512 = 1$, or $x = \sqrt[3]{512} = 8$. Thus $y = 512/x^2 = 8$, and the height is $256/xy = 4$, so the optimal box has a square base and is half as high as it is wide. (We have really shown only that the point $(8,8)$ is a critical point for f, but if there is any minimum point this must be it.) ▲

We now turn to the second derivative test for functions of two variables. Let us begin with an example.

Example 6 Captain Astro is being held captive by Jovians who are studying human intelligence. She is in a room where a loudspeaker emits a piercing noise. There are two knobs on the wall, whose positions, x and y, seem to affect the loudness of the noise. The knobs are initially at $x = 0$ and $y = 0$ and, when the first knob is turned, the noise gets even louder for $x < 0$ and for $x > 0$. So the captain leaves $x = 0$ and turns the second knob both ways, but, alas, the noise gets louder. Finally, she sees the formula $f(x, y) = x^2 + 3xy + y^2 + 16$ printed on the wall. What to do?

Solution First of all, she notices that $f(x, 0) = x^2 + 16$ and $f(0, y) = y^2 + 16$, so the function f, like the loudness of the noise, increases if either x or y is moved away from zero. But look! If we set $y = -x$, then the "$3xy$" term becomes negative. In fact, $f(x, -x) = x^2 - 3x^2 + x^2 + 16 = -x^2 + 16$. Captain Astro rushes to the dials and turns them both at once, in opposite directions. (Why?) The noise subsides (and the Jovians cheer). ▲

The function $f(x, y) = x^2 + 3xy + y^2 + 16$ has a critical point at $(0,0)$, and the functions $g(x) = f(x, 0)$ and $h(y) = f(0, y)$ both have zero as a local minimum point; but $(0,0)$ is not a local minimum point for f, because $f(x, -x) = -x^2 + 16$ is less than $f(0,0) = 16$ for arbitrarily small x. This example shows us that to tell whether a critical point (x_0, y_0) of a function $f(x, y)$ is a local extremum, we must look at the behavior of f along lines passing through (x_0, y_0) in all directions, not just those parallel to the axes.

The following test enables us to determine the nature of the critical point $(0,0)$ for any function of the form $Ax^2 + 2Bxy + Cy^2$.

Maximum–Minimum Test for Quadratic Functions

Let $g(x, y) = Ax^2 + 2Bxy + Cy^2$, where A, B, and C are constants.

1. If $AC - B^2 > 0$, and $A > 0$, [respectively $A < 0$], then $g(x, y)$ has a minimum [respectively maximum] at $(0,0)$.
2. If $AC - B^2 < 0$, then $g(x, y)$ takes both positive and negative values for (x, y) near $(0,0)$, so $(0,0)$ is not a local extremum for g.

To prove these assertions, we consider the two cases separately.

1. If $AC - B^2 > 0$, then A cannot be zero (why?), so we may write

$$g(x, y) = A\left(x^2 + \frac{2B}{A}xy + \frac{C}{A}y^2\right) = A\left(x^2 + \frac{2B}{A}xy + \frac{B^2y^2}{A^2} + \frac{C}{A}y^2 - \frac{B^2y^2}{A^2}\right)$$

$$= A\left(x + \frac{B}{A}y\right)^2 + \frac{1}{A}(AC - B^2)y^2. \qquad (1)$$

Both terms on the right-hand side of (1) have the same sign as A, and they are both zero only when $x + (B/A)y = 0$ and $y = 0$—that is, when $(x, y) = (0,0)$. Thus $(0,0)$ is a minimum point for g if $A > 0$ (since $g(x, y) > 0$ if $(x, y) \neq (0,0)$) and a maximum point if $A < 0$ (since $g(x, y) < 0$ if $(x, y) \neq (0,0)$).

2. If $AC - B^2 < 0$ and $A \neq 0$, then formula (1) still applies, but now the terms on the right-hand side have opposite signs. By suitable choices of x and y (see Exercise 49), we can make either term zero and the other nonzero. If $A = 0$, then $g(x, y) = y(2Bx + Cy)$, so we can again achieve both signs. ■

In case 2 of the preceding box, $(0,0)$ is called a *saddle point* for $g(x, y)$. (See Exercises 49 and 50 for a further discussion of this case and the case $AC - B^2 = 0$.)

Example 7 (a) Apply the maximum–minimum test to $f(x, y) = x^2 + 3xy + y^2 + 16$.
(b) Determine whether $(0,0)$ is a maximum point, a minimum point, or neither, of $g(x, y) = 3x^2 - 5xy + 3y^2$.

Solution (a) We may write this as $g(x, y) + 16$, where $g(x, y) = x^2 + 3xy + y^2$ has the form used in the test, with $A = 1$, $B = \frac{3}{2}$, and $C = 1$. Since $AC - B^2 = 1 - \frac{9}{4}$ is negative, there exist choices of x and y making $g(x, y)$ both positive and negative, so f has a saddle point at $(0,0)$. (Equation (1) gives $g(x, y) = (x + \frac{3}{2}y)^2 - \frac{9}{4}y^2$, so moving along the line $x = -\frac{3}{2}y$ makes g negative, while moving along $y = 0$ makes g positive.)
(b) $A = 3$, $B = -\frac{5}{2}$, and $C = 3$, so $A = 3 > 0$ and $AC - B^2 = 9 - \frac{25}{4} > 0$. Thus $(0,0)$ is a minimum point by part 1 of the maximum–minimum test. ▲

Note that for the quadratic function $g(x, y)$ in the preceding box, the constants A, B, and C can be recovered from g by the formulas

$$A = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}, \qquad B = \frac{1}{2}\frac{\partial^2 g}{\partial x\,\partial y}, \qquad C = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}$$

so that the signs of A and $AC - B^2$ are the same as those of $\partial^2 g/\partial x^2$ and $(\partial^2 g/\partial x^2)(\partial^2 g/\partial y^2) - (\partial^2 g/\partial x\,\partial y)^2$. The second derivative test for general functions involves just these combinations of partial derivatives.

Second Derivative Test

Let $f(x, y)$ have continuous second partial derivatives, and suppose that (x_0, y_0) is a critical point for f:

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

Let $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$, and $C = f_{yy}(x_0, y_0)$.

If:	then:
$A > 0$, $AC - B^2 > 0$	(x_0, y_0) is a local minimum;
$A < 0$, $AC - B^2 > 0$	(x_0, y_0) is a local maximum;
$AC - B^2 < 0$	(x_0, y_0) is a saddle point;
$AC - B^2 = 0$	the test is inconclusive.

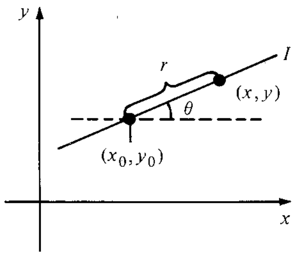


Figure 16.3.3. The point (x, y) is $(x_0 + r\cos\theta, y_0 + r\sin\theta)$.

To prove these assertions, we look at f along straight lines through (x_0, y_0). Specifically, for each fixed θ in $[0, 2\pi]$, we will consider the function $h(r) = f(x_0 + r\cos\theta, y_0 + r\sin\theta)$, which describes the behavior of f along the line through (x_0, y_0) in the direction of $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. (See Fig. 16.3.3.)

For each θ, $h(r)$ is a function of one variable with a critical point at $r = 0$. To analyze the behavior of $h(r)$ near $r = 0$ by using the second derivative test for functions of one variable, we differentiate $h(r)$ using the chain rule of Section 15.3. Let $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$; then

$$h'(r) = f_x(x, y)\frac{dx}{dr} + f_y(x, y)\frac{dy}{dr} = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

We differentiate again, applying the chain rule to f_x and f_y:

$$\begin{aligned} h''(r) &= f_{xx}(x, y)\cos\theta\frac{dx}{dr} + f_{xy}(x, y)\cos\theta\frac{dy}{dr} \\ &\quad + f_{yx}(x, y)\sin\theta\frac{dx}{dr} + f_{yy}(x, y)\sin\theta\frac{dy}{dr}. \end{aligned}$$

Since $f_{xy} = f_{yx}$ by equality of mixed partials, this becomes

$$h''(r) = f_{xx}(x, y)\cos^2\theta + 2f_{xy}(x, y)\cos\theta\sin\theta + f_{yy}(x, y)\sin^2\theta$$

or

$$\begin{aligned} h''(r) &= f_{xx}(x_0 + r\cos\theta, y_0 + r\sin\theta)\cos^2\theta \\ &\quad + 2f_{xy}(x_0 + r\cos\theta, y_0 + r\sin\theta)\cos\theta\sin\theta \\ &\quad + f_{yy}(x_0 + r\cos\theta, y_0 + r\sin\theta)\sin^2\theta. \end{aligned} \tag{2}$$

Setting $r = 0$, we get

$$h''(0) = f_{xx}(x_0, y_0)\cos^2\theta + 2f_{xy}(x_0, y_0)\cos\theta\sin\theta + f_{yy}(x_0, y_0)\sin^2\theta,$$

which has the form $Ax^2 + 2Bxy + Cy^2$, with $x = \cos\theta$, $y = \sin\theta$, and with $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$, $C = f_{yy}(x_0, y_0)$. Let $AC - B^2 = D$.

Now suppose that $D > 0$ and $f_{xx}(x_0, y_0) > 0$. By the maximum–minimum test for quadratic functions, $h''(0) > 0$, so h has a local minimum at $r = 0$. Since this is true for all values of θ, f has a local minimum along each line through (x_0, y_0). It is thus plausible that f has a local minimum at (x_0, y_0), so we will end the proof at this point. (Actually, more work is needed; for further details, see Exercises 51 and 52.)

If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum along every line through (x_0, y_0); if $D < 0$, then f has a local minimum along some lines through (x_0, y_0) and a local maximum along others. ■

Example 8 Find the maxima, minima, and saddle points of $z = (x^2 - y^2)e^{(-x^2-y^2)/2}$.

Solution First we locate the critical points by setting $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$. Here

$$\frac{\partial z}{\partial x} = \left[2x - x(x^2 - y^2)\right]e^{(-x^2-y^2)/2}$$

and

$$\frac{\partial z}{\partial y} = \left[-2y - y(x^2 - y^2)\right]e^{(-x^2-y^2)/2},$$

so the critical points are the solutions of

$$x\left[2-(x^2-y^2)\right]=0, \qquad y\left[-2-(x^2-y^2)\right]=0.$$

This has solutions $(0,0)$, $(\pm\sqrt{2},0)$, and $(0,\pm\sqrt{2})$.

The second derivatives are

$$\frac{\partial^2 z}{\partial x^2}=\left[2-5x^2+x^2(x^2-y^2)+y^2\right]e^{(-x^2-y^2)/2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=xy(x^2-y^2)e^{(-x^2-y^2)/2},$$

$$\frac{\partial^2 z}{\partial y^2}=\left[5y^2-2+y^2(x^2-y^2)-x^2\right]e^{(-x^2-y^2)/2}.$$

Using the second derivative test results in the following data:

Point	A	B	C	$AC-B^2$	Type
$(0,0)$	2	0	-2	-4	saddle
$(\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$	maximum
$(-\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$	maximum
$(0,\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$	minimum
$(0,-\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$	minimum

The results of this example are confirmed by the computer-generated graph in Fig. 16.3.4. ▲

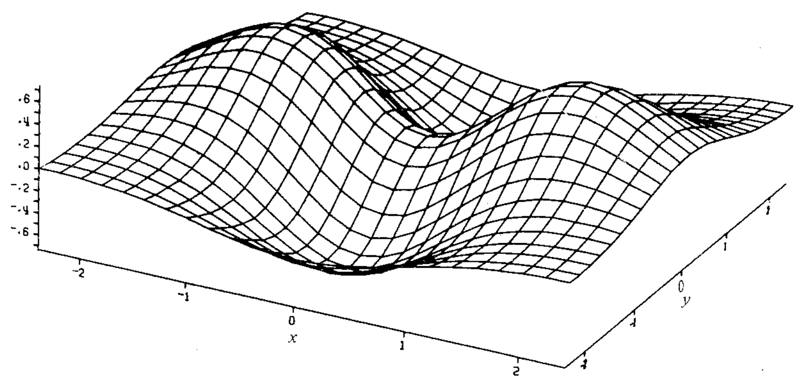


Figure 16.3.4. Computer-generated graph of $z=(x^2-y^2)e^{(-x^2-y^2)/2}$.

Example 9 Let $z=(x^2+y^2)\cos(x+2y)$. Show that $(0,0)$ is a critical point. Is it an extremum?

Solution We compute:

$$\frac{\partial z}{\partial x}=2x\cos(x+2y)-(x^2+y^2)\sin(x+2y),$$

$$\frac{\partial z}{\partial y}=2y\cos(x+2y)-2(x^2+y^2)\sin(x+2y).$$

These vanish at $(0,0)$, so $(0,0)$ is a critical point.

The second derivatives are:

$$\frac{\partial^2 z}{\partial x^2}=2\cos(x+2y)-4x\sin(x+2y)-(x^2+y^2)\cos(x+2y),$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = -4x\sin(x+2y) - 2y\sin(x+2y) - 2(x^2+y^2)\cos(x+2y),$$

$$\frac{\partial^2 z}{\partial y^2} = 2\cos(x+2y) - 8y\sin(x+2y) - 4(x^2+y^2)\cos(x+2y).$$

Evaluating at $x=0$, $y=0$, we get $A=2$, $B=0$, and $C=2$, so $A>0$, $AC-B^2>0$, and thus $(0,0)$ is a local minimum. ▲

Example 10 Find the point or points on the elliptic paraboloid $z=4x^2+y^2$ closest to $(0,0,8)$.

Solution The typical point on the paraboloid is $(x, y, 4x^2+y^2)$; its distance from $(0,0,8)$ is $\sqrt{x^2+y^2+(4x^2+y^2-8)^2}$. It is convenient to minimize the square of the distance:

$$f(x,y) = x^2+y^2+(4x^2+y^2-8)^2.$$

We begin by locating the critical points of f. The partial derivatives of f are

$$f_x(x,y) = 2x + 2(4x^2+y^2-8)\cdot 8x = 2x(32x^2+8y^2-63),$$

$$f_y(x,y) = 2y + 2(4x^2+y^2-8)\cdot 2y = 2y(8x^2+2y^2-15).$$

For $f_x(x,y)$ to be zero we must have $x=0$ or $32x^2+8y^2-63=0$. For $f_y(x,y)$ to be zero we must have $y=0$ or $8x^2+2y^2-15=0$. Thus there are four possibilities:

Case I. $x=0$ and $y=0$.

Case II. $x=0$ and $8x^2+2y^2-15=0$. Then $2y^2-15=0$ or $y=\pm\sqrt{15/2}$.

Case III. $32x^2+8y^2-63=0$ and $y=0$. Then $32x^2-63=0$ and so $x=\pm\sqrt{63/32}$.

Case IV. $32x^2+8y^2-63=0$ and $8x^2+2y^2-15=0$. Subtracting four times the second equation from the first gives $-3=0$, which is impossible, so case IV does not occur.

A simple way to see which of the points in cases I, II, and III minimizes the distance is to compute $f(x,y)$ in each case and choose the smallest value. We leave this method to the reader and, instead, use the second derivative test. The second derivatives are

$$f_{xx} = 2(32x^2+8y^2-63) + 2x\cdot 64x = 192x^2+16y^2-126,$$

$$f_{yy} = 2(8x^2+2y^2-15) + 2y\cdot 4y = 16x^2+12y^2-30,$$

$$f_{xy} = f_{yx} = 32xy.$$

Case I. $f_{xx}=-126$, $f_{yy}=-30$, $f_{xy}=0$. Thus $f_{xx}f_{yy}-f_{xy}^2 = 126\cdot 30>0$, so this point is *local maximum* for f.

Case II. $f_{xx}=16\cdot\frac{15}{2}-126<0$, $f_{yy}=12\cdot\frac{15}{2}-30>0$, $f_{xy}=0$. Therefore $f_{xx}f_{yy}-f_{xy}^2<0$, so these two points are *saddles* for f.

Case III. $f_{xx}=192\cdot\frac{63}{32}-126>0$, $f_{yy}=16\cdot\frac{63}{32}-30>0$, $f_{xy}=0$. Therefore $f_{xx}f_{yy}-f_{xy}^2>0$, so these two points are *local minima* for f. Thus the closest points to $(0,0,8)$ on the paraboloid are

$$\left(\tfrac{3}{4}\sqrt{\tfrac{7}{2}}, 0, \tfrac{63}{8}\right) \text{ and } \left(-\tfrac{3}{4}\sqrt{\tfrac{7}{2}}, 0, \tfrac{63}{8}\right). ▲$$

Supplement to Section 16.3: Astigmatism

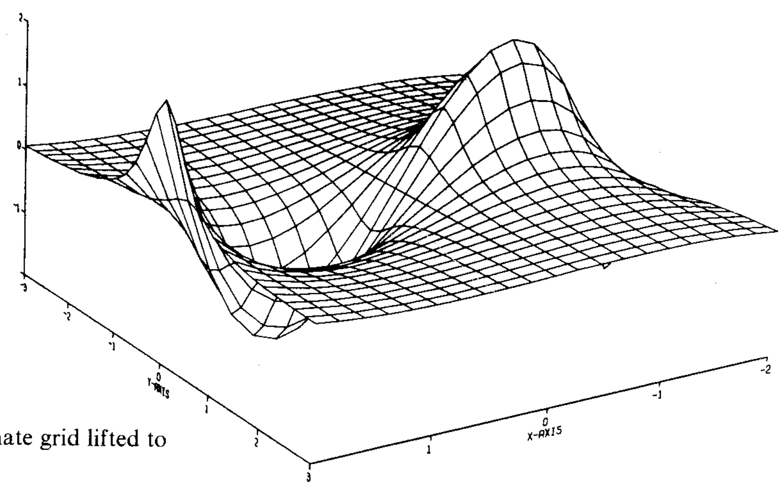
The visual problem called *astigmatism* results from a deviation from circular symmetry in the shape of the lens in your eye. Correcting astigmatism requires a compensating eyeglass or contact lens with the "opposite" deviation.

A piece of the lens surface may be described by a function $z = f(x, y) = Ax^2 + 2Bxy + Cy^2$, for x and y small. The lens is symmetric about the z axis when $B = 0$ and $A = C$. In general, if we slice the lens by a plane of the form $-x\sin\theta + y\cos\theta = 0$, which contains the z axis and the vector $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, the slice is bounded by a curve through the origin whose curvature there is $2(A\cos^2\theta + 2B\sin\theta\cos\theta + C\sin^2\theta)$ (see Section 14.7 and Exercise 56). The maxima and minima of curvature occur when $\tan 2\theta = 2B/(A - C)$. Notice that the direction of maximum and minima curvature differ by 90°; this means that an optometrist must know only one of these directions in order to orient corrective lenses properly.

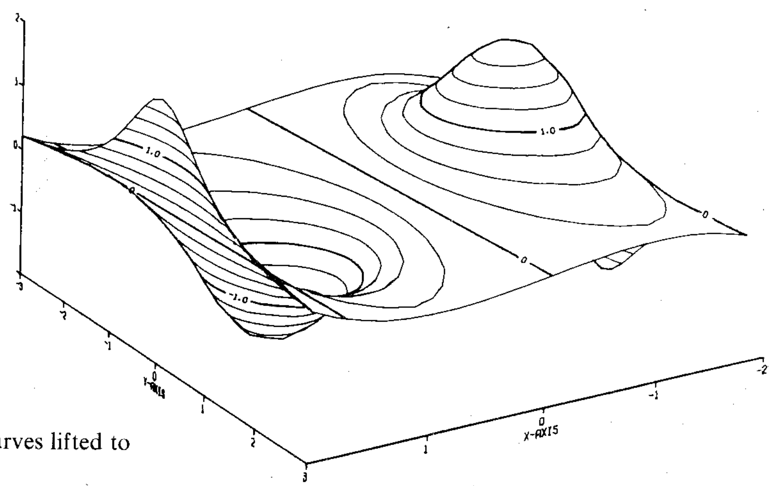
Exercises for Section 16.3

1. Refer to Fig. 16.3.5, a computer-generated graph of $z = (x^3 - 3x)/(1 + y^2)$. Where are the maximum and minimum points?

Figure 16.3.5. Computer-generated graph of $z = (x^3 - 3x)/(1 + y^2)$.



(a) Coordinate grid lifted to the graph.



(b) Level curves lifted to the graph.

2. Refer to Fig. 16.3.6, a computer-generated graph of $z = \sin(\pi x)/(1 + y^2)$. Where are the maximum and minimum points?

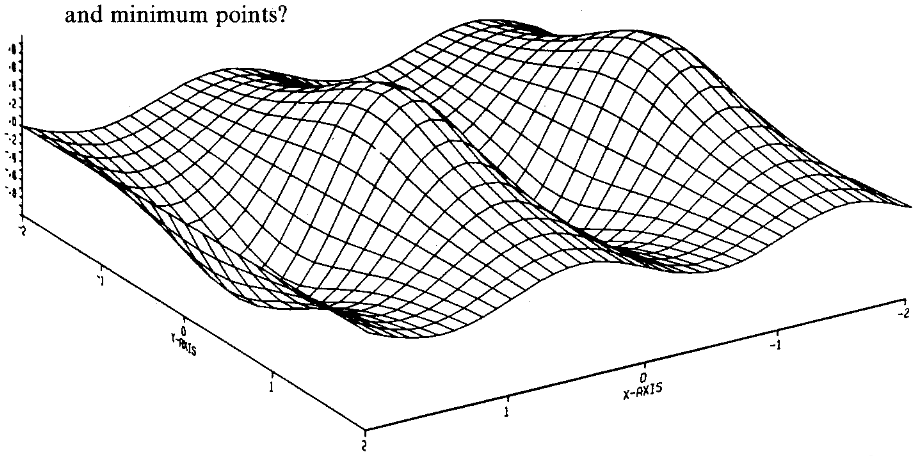


Figure 16.3.6. Computer-generated graph of $\sin(\pi x)/(1 + y^2)$.

Find the critical points of each of the functions in Exercises 3–6. Decide by inspection whether each of the critical points is a local maximum, minimum, or neither.

3. $f(x, y) = x^2 + 2y^2$
4. $f(x, y) = x^2 - 2y^2$
5. $f(x, y) = \exp(-x^2 - 7y^2 + 3)$
6. $f(x, y) = \exp(x^2 + 2y^2)$

7. Minimize the distance to the origin from the plane $x - y + 2z = 3$.
8. Find the distance from the plane given by $x + 2y + 3z - 10 = 0$: (a) To the origin. (b) To the point $(1, 1, 1)$.
9. Suppose that the material for the bottom of the box in Example 5 costs b cents per square centimeter, while that for the sides costs s cents per square centimeter. Find the dimensions which minimize the cost of the material.
10. Drug reactions can be measured by functions of the form $R(u, t) = u^2(c - u)t^2e^{-t}$, $0 \leqslant u \leqslant c$, $t \geqslant 0$. The symbols u and t are drug units and time in hours, respectively. Find the dosage u and time t at which R is a maximum.

In Exercises 11–16 use the maximum–minimum test for quadratic functions to decide whether $(0, 0)$ is a maximum, minimum, or saddle point.

11. $f(x, y) = x^2 + xy + y^2$.
12. $f(x, y) = x^2 - xy + y^2 + 1$.
13. $f(x, y) = y^2 - x^2 + 3xy$.
14. $f(x, y) = x^2 + y^2 - xy$.
15. $f(x, y) = y^2$.
16. $f(x, y) = 3 + 2x^2 - xy + y^2$.

Find the critical points of each of the functions in Exercises 17–30 and classify them as local maxima, minima, or neither.

17. $f(x, y) = x^2 + y^2 + 6x - 4y + 13$.
18. $f(x, y) = x^2 + y^2 + 3x - 2y + 1$.
19. $f(x, y) = x^2 - y^2 + xy - 7$.
20. $f(x, y) = x^2 + y^2 + 3xy + 10$.
21. $f(x, y) = x^2 + y^2 - 6x - 14y + 100$.
22. $f(x, y) = y^2 - x^2$.
23. $f(x, y) = 2x^2 - 2xy + y^2 - 2x + 1$.
24. $f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$.
25. $f(x, y) = 3x^2 + 2xy + 2y^2 - 3x + 2y + 10$.
26. $f(x, y) = x^2 + xy^2 + y^4$.
27. $f(x, y) = e^{1 + x^2 - y^2}$.
28. $f(x, y) = (x^2 + y^2)e^{x^2 - y^2}$.
29. $f(x, y) = \ln(2 + \sin xy)$. [Consider only the critical point $(0, 0)$.]
30. $f(x, y) = \sin(x^2 + y^2)$. [Consider only the critical point $(0, 0)$.]

31. Analyze the behavior of $z = x^5y + xy^5 + xy$ at its critical points.
32. Test for extrema: $z = \ln(x^2 + y^2 + 1)$.
33. Analyze the critical point at $(0, 0)$ for the function $f(x, y) = x^2 + y^3$. Make a sketch.
34. Locate any maxima, minima, or saddle points of $f(x, y) = \ln(ax^2 + by^2 + 1)$, $a, b > 0$.
35. A computer-generated graph of
$$z = (\sin \pi r)/\pi r, \qquad r = \sqrt{x^2 + y^2},$$
is shown in Fig. 16.3.7. (a) Show, by calculation, that all critical points of the function lie on circles whose radius satisfies the equation $\pi r = \tan(\pi r)$. (b) Which points are maxima? Minima? (c) What symmetries does the graph have?
36. Show that $z = (x^3 - 3x)/(1 + y^2)$ has exactly one local maximum and one local minimum. What symmetries does the graph have? (It is computer drawn in Fig. 16.3.5.)
37. The work w done in a compressor with $k + 1$ compression cylinders is given by
$$w = c_1 y + c_2, \qquad c_1 \neq 0,$$
where
$$y = \sum_{i=0}^{k} T_i(p_i/p_{i+1})^{(n-1)/n}.$$

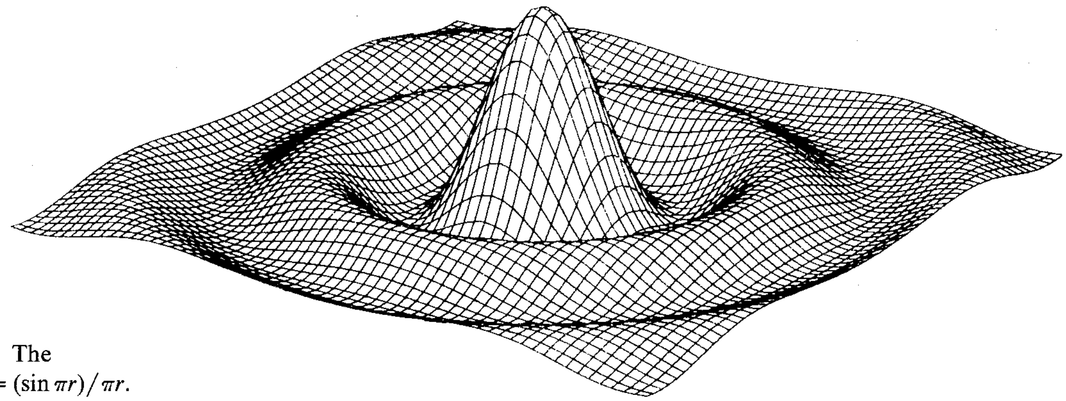


Figure 16.3.7. The sombrero: $z = (\sin \pi r)/\pi r$.

The symbols T_i and p_i stand for temperature and pressure in cylinder i, $1 \leqslant i \leqslant k+1$; the pressures $p_1, \ldots, p_k$ are the independent variables, while $p_0, p_{k+1}, T_0, \ldots, T_k$ and $n > 1$ are given.

(a) Find relations between the variables if w is a minimum.

★(b) Find p_1, p_2, p_3 explicitly for the case $k = 3$.

38. *Planck's law* gives the relationship of the energy E emitted by a blackbody to the wavelength λ and temperature T:

$$E = \frac{2\pi k^5 T^5}{h^4 c^3} \frac{x^5}{e^x - 1}, \qquad \text{where} \quad x = \frac{hc}{\lambda k T}.$$

The constants are $h = 6.6256 \times 10^{-34}$ joule seconds (Planck's constant), $k = 1.3805 \times 10^{-23}$ joule kilograms^{-1} (Boltzmann's constant), $c = 2.9979 \times 10^8$ meter second^{-1} (velocity of light). The plot of E versus λ for fixed T is called a *Planck curve*.

(a) The maximum along each Planck curve is obtained by setting $\partial E/\partial\lambda = 0$ and solving for $\lambda_{\max}$. The relationship so derived is called *Wien's displacement law*. Show that this law is just $\lambda_{\max} = hc/kTx_0$, where $5 - x_0 - 5e^{-x_0} = 0$.

(b) Clearly x_0 is close to 5. By examining the sign of $f(x) = 5 - x - 5e^{-x}$, use your calculator to complete the expansion $x_0 = 4.965\ldots$ to a full six digits.

(c) Improve upon the displacement law $\lambda_{\max} = 0.00289/T$ by giving a slightly better constant. The peak for the earth (288°K) is about 10 micrometers, the peak for the sun (6000°K) about 0.48 micrometers, so the maximum occurs in the infared and visible range, respectively.

39. Apply the second derivative test to the critical point in Example 5.

40. (a) Show that if (x_0, y_0, z_0) is a local minimum or maximum point of $w = f(x, y, z)$, then $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$ are all zero at (x_0, y_0, z_0).

(b) Find the critical points of the function $\sin(x^2 + y^2 + z^2)$.

(c) Find the point in space which minimizes the sum of the squares of the distances from $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

41. Analyze the behavior of the following functions at the indicated points:

(a) $f(x, y) = x^2 - y^2 + 3xy$; $(0,0)$.

(b) $f(x, y) = x^2 + y^2 + Cxy$; $(0,0)$. Determine what happens for various values of C. At what values of C does the behavior change qualitatively?

42. Find the local maxima and minima for $z = (x^2 + 3y^2)e^{1 - x^2 - y^2}$. (See Fig. 14.3.15.)

Exercises 43–48 deal with the *method of least squares*. It often happens that the theory behind an experiment indicates that the data should lie along a straight line of the form $y = mx + b$. The actual results, of course, will never exactly match with theory, so we are faced with the problem of finding the straight line which best fits some experimental data $(x_1, y_1), \ldots, (x_n, y_n)$ as in Fig. 16.3.8. For the straight line $y = mx + b$, each point will deviate vertically from the line by an amount $d_i = y_i - (mx_i + b)$. We would like to choose m and b in such a way as to make the total effect of these deviations as small as possible. Since some deviations are negative and some are positive, however, a better measure of the total error is the sum of the squares of these deviations; so we are led to the problem of finding m and b to minimize the function

$$\begin{aligned} s = f(m, b) &= d_1^2 + d_2^2 + \cdots + d_n^2 \\ &= \sum_{i=1}^{n} (y_i - mx_i - b)^2, \end{aligned}$$

where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are given data.

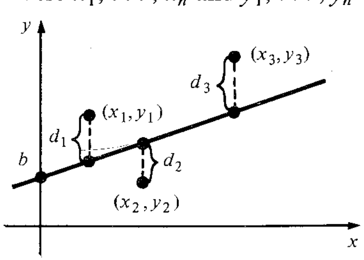


Figure 16.3.8. The method of least squares finds a straight line which "best" approximates a set of data.

43. For each set of three data points, plot the points, write down the function $f(m,b)$, find m and b to give the best straight-line fit according to the method of least squares, and plot the straight line.
    (a) $(x_1, y_1) = (1,1)$; $(x_2, y_2) = (2,3)$; $(x_3, y_3) = (4,3)$.
    (b) $(x_1, y_1) = (0,0)$; $(x_2, y_2) = (1,2)$; $(x_3, y_3) = (2,3)$.
44. Show that if only two data points (x_1, y_1) and (x_2, y_2) are given, then this method produces the line through (x_1, y_1) and (x_2, y_2).
45. Show that the equations for a critical point, $\partial s/\partial b = 0$ and $\partial s/\partial m = 0$, are equivalent to $m(\Sigma x_i) + nb = \Sigma y_i$ and $m(\Sigma x_i^2) + b(\Sigma x_i) = \Sigma x_i y_i$, where all the sums run from $i = 1$ to $i = n$.
46. If $y = mx + b$ is the best-fitting straight line to the data points $(x_1, y_1), \dots, (x_n, y_n)$ according to the least squares method, show that

$$\sum_{i=1}^{n} (y_i - mx_i - b) = 0.$$

That is, show that the positive and negative deviations cancel (see Exercise 45).
47. Use the second derivative test to show that the critical point of f is actually a minimum.
48. Use the method of least squares to find the straight line that best fits the points (0, 1), (1, 3), (2, 2), (3, 4), and (4, 5). Plot your points and line.

★49. Complete the proof of the maximum–minimum test for quadratic functions by following these steps:
    (a) If $A \neq 0$ and $AC - B^2 < 0$, show that

$$g(x,y) = A\left[\left(x + \frac{B}{A}y\right) - ey\right]\left[\left(x + \frac{B}{A}y\right) + ey\right]$$

    for some number e. What is e?
    (b) Show that the set where $g(x, y) = 0$ consists of two intersecting lines. What are their equations?
    (c) Show that $g(x, y)$ is positive on two of the regions cut out by the lines in part (b) and negative on the other two.
    (d) If $A = 0$, $g(x, y) = 2Bxy + Cy^2$. B must be nonzero. (Why?) Write $g(x, y)$ as a product of linear functions and repeat parts (b) and (c).

★50. Discuss the function $Ax^2 + 2Bxy + Cy^2$ in the case where $AC - B^2 = 0$.
    (a) If $A \neq 0$, use formula (1) in the proof of the maximum–minimum test for quadratic functions.
    (b) Sketch a graph of the function $f(x, y) = x^2 + 2xy + y^2$.
    (c) What happens if $A = 0$?

★51. Let $f(x, y) = 3x^4 - 4x^2y + y^2$.
    (a) Show that $f(x, y)$ has a critical point at the origin.
    (b) Show that for all values of θ, the function $h(r) = f(r\cos\theta, r\sin\theta)$ has a local minimum at $r = 0$.
    (c) Show that, nevertheless, the origin is not a local minimum point for f.
    (d) Find the set of (x, y) for which $f(x, y) = 0$.
    (e) Sketch the regions in the plane where $f(x, y)$ is positive and negative.
    (f) Discuss why parts (b) and (c) do not contradict one another.

★52. Complete the proof of the second derivative test by following this outline:
    (a) We begin with the case in which $D > 0$ and $f_{xx}(x_0, y_0) > 0$. Using Exercise 78, Section 15.1, show that there is a number $\varepsilon > 0$ such that whenever (x, y) lies in the disk of radius ε about (x_0, y_0),

$$f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2$$

    and $f_{xx}(x, y)$ are both positive.
    (b) Show that the function $h(r)$ is concave upward on the interval $(-\varepsilon, \varepsilon)$ for any choice of θ.
    (c) Conclude that $f(x, y) \geqslant f(x_0, y_0)$ for all (x, y) in the disk of radius ε about (x_0, y_0).
    (d) Complete the case in which $D > 0$ and $f_{xx}(x_0, y_0) < 0$.
    (e) Complete the case $D < 0$ by showing that f takes values near (x_0, y_0) which are greater and less than $f(_0, y_0)$.

★53. Find the point or points on the elliptic paraboloid $z = 4x^2 + y^2$ closest to $(0, 0, a)$ for each a. (See Example 10.) How does the answer depend upon a?

★54. Let $f(x, y) > 0$ for all x and y. Show that $f(x, y)$ and $g(x, y) = [f(x, y)]^2$ have the same critical points, with the same "type" (maximum, minimum, or saddle).

★55. Consider the general problem of finding the points on a graph $z = k(x, y)$ closest to a point (a, b, c). Show that (x_0, y_0) is a critical point for the distance from $(x, y, k(x, y))$ to (a, b, c) if and only if the line from (a, b, c) to $(x_0, y_0, k(x_0, y_0))$ is orthogonal to the graph at $(x_0, y_0, k(x_0, y_0))$.

★56. (See the supplement to this section.)
    (a) Show that if the surface $z = Ax^2 + 2Bxy +$

Cy^2 is sliced by the plane $-x\sin\theta + y\cos\theta = 0$, then the curvature of the slice at the origin is twice the absolute value of $A\cos^2\theta + 2B\sin\theta\cos\theta + C\sin^2\theta$.

(b) Show that if $B^2 - AC < 0$ and $A > 0$, then the maximum and minima of curvature occur when $\tan 2\theta = 2B/(A - C)$.

16.4 Constrained Extrema and Lagrange Multipliers

The level surfaces of two functions must cross, except where the gradients of the functions are parallel.

In studying maximum–minimum problems for a function $f(x)$ defined on an interval $[a, b]$, we found in Section 3.5 that the maximum and minimum points could occur either at critical points (where $f'(x) = 0$) or at the endpoints a and b. For a function $f(x, y)$ in the plane, it is common to replace the interval $[a, b]$ by some region D; the role of the endpoints is now played by the boundary of D, which is a curve in the plane (possibly with corners).

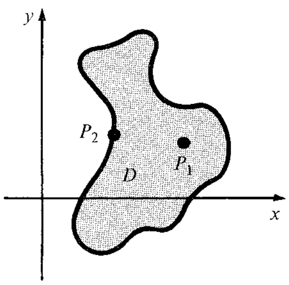


Figure 16.4.1. If the interior point P_1 is an extreme point of f on D, then the partial derivatives of f at P_1, if they exist, must be zero. If the boundary point P_2 is an extreme point, the partial derivatives there might not be zero.

The problem of finding extrema in several variables can be attacked in steps:

Step 1. Suppose that (x_0, y_0) is an extremum lying inside D, like the point P_1 in Fig. 16.4.1, and that the partial derivatives of f exist[2] at (x_0, y_0). Then our earlier analysis applies and (x_0, y_0) must be a critical point.

Step 2. The extreme point (x_0, y_0) may lie on the boundary of D, like P_2 in Fig. 16.4.1. At such a point, the partial derivatives of f might not be zero. Thus we must develop new techniques for finding candidates for the extreme points of f on the boundary.

Step 3. The function f should be evaluated at the points found in Steps 1 and 2, and the largest and smallest values should be identified.

If we can parametrize the boundary curve, say by $\boldsymbol{\sigma}(t)$ for t in $[a, b]$, then the restriction of f to the boundary[3] becomes a function of one variable, $h(t) = f(\boldsymbol{\sigma}(t))$, to which the methods of one-variable calculus apply, as in the following example.

Example 1 Find the extreme values of $z = f(x, y) = x^2 + 2y^2$ on the disk D consisting of points (x, y) satisfying $x^2 + y^2 \leqslant 1$.

Solution *Step 1.* At a critical point, $\partial z/\partial x = 2x = 0$ and $\partial z/\partial y = 4y = 0$. Thus, the only critical point is $(0, 0)$. It is clearly a minimum point for f; we may also verify this by the second derivative test:

$$\frac{\partial^2 z}{\partial x^2} = 2 > 0$$

and

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = 2 \cdot 4 - 0 > 0.$$

[2] As with functions of one variable, there may be points where the derivatives of f do not exist. If there are such points, they must be examined directly to see if they are maxima or minima.

[3] That is, the function which has the same values as f but whose domain consists only of the boundary points.

This result confirms that $(0,0)$ is a local minimum.
Step 2. The boundary of D is the unit circle, which we may parametrize by $(\cos t, \sin t)$. Along the boundary,

$$h(t) = f(\cos t, \sin t) = \cos^2 t + 2\sin^2 t = 1 + \sin^2 t.$$

Since $h'(t) = 2\sin t \cos t$, $h'(t) = 0$ at $t = 0$, $\pi/2$, π, and $3\pi/2$ (2π gives the same point as zero). Thus, the only boundary points which could possibly be local maxima and minima are $(\cos t, \sin t)$ for these values of t, i.e., $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$.
Step 3. Evaluating f at the points found in Steps 1 and 2, we obtain:

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(0,1) = 2, \quad f(-1,0) = 1, \quad f(0,-1) = 2.$$

Thus, f has a minimum point at $(0,0)$ with value 0 and maximum points at $(0,1)$ and $(0,-1)$ with value 2. (See Fig. 16.4.2.) The points $(1,0)$ and $(-1,0)$ are neither maxima nor minima for f on D even though they are minima for f on the boundary. ▲

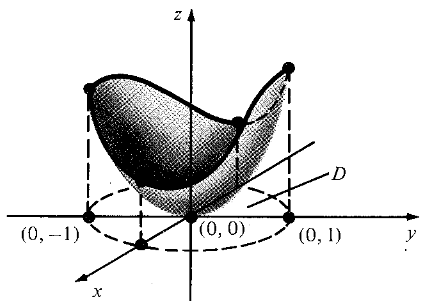


Figure 16.4.2. The function $f(x, y) = x^2 + 2y^2$ on the disk D has a minimum point at $(0,0)$ and maximum points at $(0,1)$ and $(0,-1)$.

Often it is inconvenient to find a parametrization for the curve C on which we are searching for extrema. Instead, the curve C may be given as a level curve of a function $g(x, y)$. In this case, we can still derive a first derivative test for local maxima and minima. The following result leads to the *method of Lagrange multipliers*.

First Derivative Test for Constrained Extrema

Let f and g be functions of two variables with continuous partial derivatives. Suppose that the function f, when restricted to the level curve C defined by $g(x, y) = c$, has a local extremum at (x_0, y_0) and that $\nabla g(x_0, y_0) \neq \mathbf{0}$. Then there is a number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

If $\lambda \neq 0$, this formula says that the level curves of f and g through (x_0, y_0) have the same tangent line at (x_0, y_0).

To demonstrate the result in this box, choose a parametrization $(x, y) = \boldsymbol{\sigma}(t)$ for C near (x_0, y_0), with $\boldsymbol{\sigma}(0) = (x_0, y_0)$ and $\boldsymbol{\sigma}'(0) \neq 0$.[4] Since f has a local extremum at (x_0, y_0), the function $h(t) = f(\boldsymbol{\sigma}(t))$ has a local extremum at $t = 0$, so $h'(0) = 0$. According to the chain rule (Section 16.1), we get $h'(0) = \nabla f(x_0, y_0) \cdot \boldsymbol{\sigma}'(0)$, so $\nabla f(x_0, y_0)$ is perpendicular to $\boldsymbol{\sigma}'(0)$; but we already

[4] The implicit function theorem guarantees that such a parametrization exists; see J. Marsden and A. Tromba, *Vector Calculus*, Freeman (1981), p. 237. We will not need to know the explicit parametrization for the method to be effective.

know that the gradient $\nabla g(x_0, y_0)$ is perpendicular to the tangent vector $\boldsymbol{\sigma}'(0)$ to the level curve C (Section 16.2). In the plane, any two vectors perpendicular to a given nonzero vector must be parallel, so $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some number λ. If $\lambda \neq 0$, the tangent line to the level curve of f through (x_0, y_0), which is perpendicular to $\nabla f(x_0, y_0)$, is also perpendicular to the vector $\nabla g(x_0, y_0)$; the tangent line to C is also perpendicular to $\nabla g(x_0, y_0)$, so the level curves of f and g through (x_0, y_0) must have the same tangent line. This completes the demonstration. ■

There is a nice geometric way of seeing the result above. If the level curves of f and g had different tangent lines at (x_0, y_0), then the level curves would cross one another. It would follow that the level curve C of g would intersect level curves of f for both higher and lower values of f, so the point (x_0, y_0) would not be an extremum (see Fig. 16.4.3).

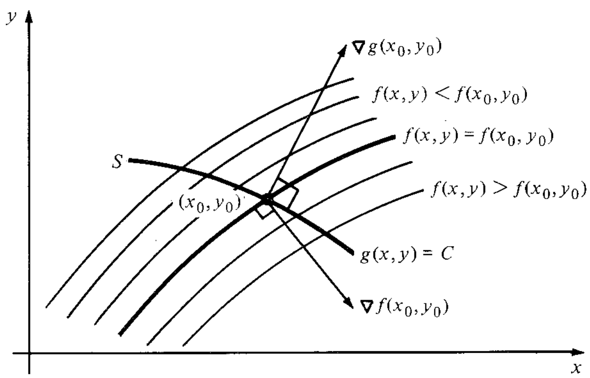


Figure 16.4.3. If $\nabla g(x_0, y_0)$ and $\nabla f(x_0, y_0)$ are not parallel, the level curve $g(x, y) = c$ cuts all nearby level curves of f.

In some problems, it is easiest to use the geometric condition of tangency directly. More often, however, we look for a point (x_0, y_0) on C and a constant λ, called a *Lagrange multiplier*, such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$. This means we wish to solve the three simultaneous equations

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y), \\ f_y(x, y) &= \lambda g_y(x, y), \\ g(x, y) &= c \end{aligned} \tag{1}$$

for the three unknown quantities x, y, and λ. Another way of looking at equations (1) is that we seek the critical points of the auxiliary function $k(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - c]$. (By a critical point of a function of three variables, we mean a point where all three of its partial derivatives vanish.) Here

$$\begin{aligned} k_x &= f_x - \lambda g_x, \\ k_y &= f_y - \lambda g_y, \\ k_\lambda &= c - g, \end{aligned}$$

and setting these equal to zero produces equations (1). We call this attack on the problem the *method of Lagrange multipliers*.

Method of Lagrange Multipliers

To find the extreme points of $f(x, y)$ subject to the constraint $g(x, y) = c$, seek points (x, y) and numbers λ such that (1) holds.

Example 2 Find the extreme values of $f(x, y) = x^2 - y^2$ along the circle S of radius 1 centered at the origin.

Solution The circle S is the level curve $g(x, y) = x^2 + y^2 = 1$, so we want x, y, and λ such that

$$f_x(x, y) = \lambda g_x(x, y),$$
$$f_y(x, y) = \lambda g_y(x, y),$$
$$g(x, y) = 1.$$

That is,

$$2x = \lambda 2x,$$
$$2y = -\lambda 2y,$$
$$x^2 + y^2 = 1.$$

From the first equation, either $x = 0$ or $\lambda = 1$. If $x = 0$, then from the third equation, $y = \pm 1$, and then from the second, $\lambda = -1$. If $\lambda = 1$, then $y = 0$ and $x = \pm 1$; so the eligible points are $(x, y) = (0, \pm 1)$ with $\lambda = -1$ and $(x, y) = (\pm 1, 0)$ with $\lambda = 1$. We must now check them to see if they really are extrema and, if so, what kind. To do this, we evaluate f:

$$f(0,1) = f(0, -1) = -1,$$
$$f(1,0) = f(-1,0) = 1,$$

so the maximum and minimum values are 1 and -1. ▲

Example 3 Find the point(s) furthest from and closest to the origin on the curve $x^6 + y^6 = 1$.

Solution We extremize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x^6 + y^6 = 1$. The Lagrange multiplier equations (1) are

$$2x = 6\lambda x^5,$$
$$2y = 6\lambda y^5,$$
$$x^6 + y^6 = 1.$$

If we rewrite the first two of these equations as

$$x(6\lambda x^4 - 2) = 0,$$
$$y(6\lambda y^4 - 2) = 0,$$

we find the solutions $(0, \pm 1)$ and $(\pm 1, 0)$ with $\lambda = \frac{1}{3}$. If x and y are both nonzero, we have $x^4 = 1/3\lambda = y^4$, so $x = \pm y$, and we get the further solutions $(\pm \sqrt[6]{1/2}, \pm \sqrt[6]{1/2})$, with $\lambda = 2^{2/3}/3$.

To tell which points are maxima and minima, we compute; $f(0, \pm 1) = f(\pm 1, 0) = 1$, while $f(\pm \sqrt[6]{1/2}, \pm \sqrt[6]{1/2}) = 2\sqrt[3]{1/2} = 2^{2/3} > 1$, so the points $(0, \pm 1)$ and $(\pm 1, 0)$ are closest to the origin, while $(\pm \sqrt[6]{1/2}, \pm \sqrt[6]{1/2})$ are farthest (see Fig. 16.4.4). ▲

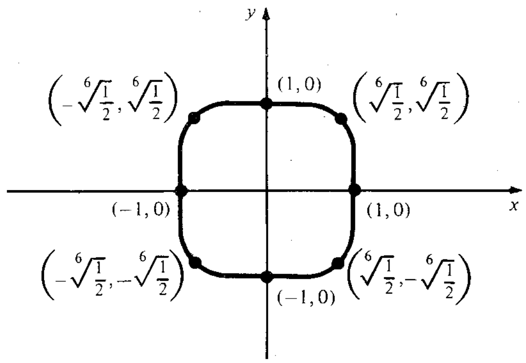


Figure 16.4.4. Extreme points of $x^2 + y^2$ on the curve $x^6 + y^6 = 1$.

For functions of three variables subject to a constraint, there is a similar method. (See Review Exercise 44 if there are two constraints.) Thus, if we are extremizing $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$, we can proceed as follows (see Exercise 23).

Method 1. Find points (x_0, y_0, z_0) and a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0),$$

and

$$g(x_0, y_0, z_0) = c,$$

or

Method 2. Find critical points of the auxiliary function of four variables given by

$$k(x, y, z, \lambda) = f(x, y, z) - \lambda[\, g(x, y, z) - c\,].$$

Example 4 The density of a metallic spherical surface $x^2 + y^2 + z^2 = 4$ is given by $\rho(x, y, z) = 2 + xz + y^2$. Find the places where the density is highest and lowest.

Solution We want to extremize $\rho(x, y, z)$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 4$. Using either method 1 or 2 above gives the equations

$$\begin{aligned} \rho_x &= \lambda g_x \\ \rho_y &= \lambda g_y \\ \rho_z &= \lambda g_z \\ g &= 4 \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} z &= 2\lambda x \\ 2y &= 2\lambda y \\ x &= 2\lambda z \\ x^2 + y^2 + z^2 &= 4 \end{aligned}$$

If $y \neq 0$ then $\lambda = 1$ from the second equation, and so $z = 2x$ and $x = 2z$ which implies $x = z = 0$. From the last equation, $y = \pm 2$. If $y = 0$ then we have

$$z = 2\lambda x, \quad x = 2\lambda z \quad \text{and} \quad x^2 + z^2 = 4.$$

Thus $z = 4\lambda^2 z$, so if $z \neq 0$ then $\lambda = \pm 1/2$, so $x = \pm z$. If $x = z$, then from the last equation $x = z = \pm\sqrt{2}$. If $x = -z$, then $x = \pm\sqrt{2}$ and $z = \mp\sqrt{2}$. The case $y = 0$ and $z = 0$ cannot occur (why?).

Thus we have six possible extrema:

$$(0, \pm 2, 0), \qquad (\pm\sqrt{2}, 0, \pm\sqrt{2}), \qquad (\pm\sqrt{2}, 0, \mp\sqrt{2}).$$

Evaluating ρ at these six points, we find that ρ is a maximum at the two points $(0, \pm 2, 0)$ (where ρ is 6) and a minimum at the two points $(\pm\sqrt{2}, 0, \mp\sqrt{2})$ (where ρ is 0). ▲

The multiplier λ was introduced as an "artificial" device enabling us to find maxima and minima, but sometimes it represents something meaningful.

Example 5 Suppose that the output of a manufacturing firm is a quantity Q of product which is a function $f(K,L)$ of the amount K of capital equipment or investment and the amount L of labor used. If the price of labor is p, the price of capital is q, and the firm can spend no more than B dollars, how do you find the amount of capital and labor to maximize the output Q?

Solution It is useful to think about the problem before applying our machinery. We would expect that if the amount of capital or labor is increased, then the output Q should also increase; that is,

$$\frac{\partial Q}{\partial K} \geqslant 0, \quad \text{and} \quad \frac{\partial Q}{\partial L} \geqslant 0.$$

We also expect that as more and more labor is added to a given amount of capital equipment, we get less and less additional output for our effort; that is,

$$\frac{\partial^2 Q}{\partial L^2} < 0.$$

Similarly,

$$\frac{\partial^2 Q}{\partial K^2} < 0.$$

It is thus reasonable to expect the level curves of output (called *isoquants*) $Q = f(K,L) = c$ to look something like the curves sketched in Fig. 16.4.5, with $c_1 < c_2 < c_3$.

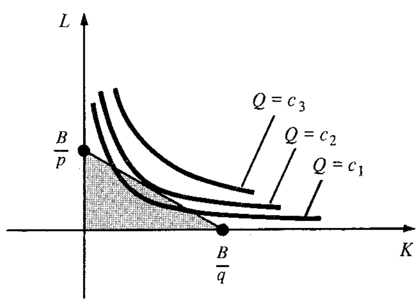


Figure 16.4.5. What is the largest value of Q in the shaded triangle?

We can interpret the convexity of the isoquants as follows. As you move to the right along a given isoquant, it takes more and more capital to replace a unit of labor and still produce the same output. The budget constraint means that we must stay inside the triangle bounded by the axes and the line $pL + qK = B$. Geometrically, it is clear that we produce the most by spending all our money in such a way as to pick the isoquant which just touches, but does not cross, the budget line.

Since the maximum point lies on the boundary of our domain, to find it we apply the method of Lagrange multipliers. To maximize $Q = f(K,L)$ subject to the constraint $pL + qK = B$, we look for critical points of the auxiliary function,

$$h(K,L,\lambda) = f(K,L) - \lambda(pL + qK - B);$$

so we want

$$\frac{\partial Q}{\partial K} = \lambda q, \qquad \frac{\partial Q}{\partial L} = \lambda p, \qquad pL + qK = B.$$

These are the conditions we must meet in order to maximize output. (We will work out a specific case in Example 6.)

In this example, λ does represent something interesting and useful. Let $k = qK$ and $l = pL$, so that k is the dollar value of the capital used and l is the dollar value of the labor used. Then the first two equations become

$$\frac{\partial Q}{\partial k} = \frac{1}{q}\frac{\partial Q}{\partial K} = \lambda = \frac{1}{p}\frac{\partial Q}{\partial L} = \frac{\partial Q}{\partial l}.$$

Thus, at the optimum production point, the marginal change in output per dollar's worth of additional capital investment is equal to the marginal change of output per dollar's worth of additional labor, and λ is this common value. At the optimum point, the exchange of a dollar's worth of capital for a dollar's worth of labor does not change the output. Away from the optimum point the marginal outputs are different, and one exchange or the other will increase the output.[5] ▲

Example 6 Carry out the analysis of Example 5 for the production function $Q(K,L) = AK^{\alpha}L^{1-\alpha}$, where A and α are positive constants and $\alpha < 1$. This *Cobb–Douglas production function* is sometimes used as a simple model for the national economy. Then Q is the output of the entire economy for a given input of capital and labor.

Solution The level curves of output are of the form $AK^{\alpha}L^{1-\alpha} = c$, or, solving for L,

$$L = \left(\frac{c}{A}\right)^{1/(1-\alpha)}K^{\alpha/(\alpha-1)}.$$

Since $\alpha/(\alpha - 1) < 0$, these curves do look like those in Fig. 16.4.5. The partial derivatives of Q are

$$\frac{\partial Q}{\partial K} = \alpha AK^{\alpha-1}L^{1-\alpha} \quad \text{and} \quad \frac{\partial Q}{\partial L} = (1-\alpha)AK^{\alpha}L^{-\alpha},$$

so there are no critical points except on the axes, where $Q = 0$. Thus the maximum must lie on the budget line $pL + qK = B$. The method of Lagrange multipliers gives the equations

$$\begin{aligned} \alpha AK^{\alpha-1}L^{1-\alpha} &= \lambda q, \\ (1-\alpha)AK^{\alpha}L^{-\alpha} &= \lambda p, \\ pL + qK &= B. \end{aligned}$$

Eliminating λ from the first two equations gives

$$\alpha pL = (1-\alpha)qK,$$

and from the third equation we obtain

$$K = \frac{\alpha B}{q} \quad \text{and} \quad L = \frac{(1-\alpha)B}{p}. \blacktriangle$$

[5] More of this type of mathematical analysis in economics can be found in *Microeconomic Theory*, by James Henderson and Richard Quandt, McGraw-Hill (1958). This reference discusses a second derivative test for the Lagrange multiplier method. [See also J. Marsden and A. Tromba, *Vector Calculus*, Second Edition, Freeman (1981).]

Exercises for Section 16.4

Use the method of Example 1 to find the extreme values of the functions in Exercises 1–4 on the disk $x^2 + y^2 \leqslant 1$.

1. $f(x, y) = 2x^2 + 3y^2$
2. $f(x, y) = xy + 5y$
3. $f(x, y) = 5x^2 - 2y^2 + 10$
4. $f(x, y) = 3xy - y + 5$

Find the extrema of f subject to the stated constraints in Exercises 5–12.

5. $f(x, y) = 3x + 2y$; $2x^2 + 3y^2 \leqslant 3$.
6. $f(x, y) = xy$; $2x + 3y \leqslant 10$, $0 \leqslant x$, $0 \leqslant y$.
7. $f(x, y) = x + y$; $x^2 + y^2 = 1$.
8. $f(x, y) = x - y$; $x^2 - y^2 = 2$.
9. $f(x, y) = xy$; $x + y = 1$.
10. $f(x, y) = \cos^2 x + \cos^2 y$; $x + y = \pi/4$.
11. $f(x, y) = x - 3y$; $x^2 + y^2 = 1$.
12. $f(x, y) = x^2 + y^2$; $x^4 + y^4 = 2$.
13. Cascade Container Company produces a cardboard shipping crate at three different plants in amounts x, y, z, respectively, producing an annual revenue of $R(x, y, z) = 8xyz^2 - 200{,}000(x + y + z)$. The company is to produce 100,000 units annually. How should production be handled to maximize the revenue?
14. The temperature T on the spherical surface $x^2 + y^2 + z^2 = 1$ satisfies the equation $T(x, y, z) = xz + yz$. Find all the hot spots.
15. A rectangular mirror with area A square feet is to have trim along the edges. If the trim along the horizontal edges costs p cents per foot and that for the vertical edges costs q cents per foot, find the dimensions which will minimize the total cost.
16. The Baraboo, Wisconsin, plant of International Widget Co. uses aluminum, iron, and magnesium to produce high-quality widgets. The quantity of widgets which may be produced using x tons of aluminum, y tons of iron, and z tons of magnesium is $Q(x, y, z) = xyz$. The cost of raw materials is aluminum, \$6 per ton; iron, \$4 per ton; and magnesium, \$8 per ton. How many tons each of aluminum, iron, and magnesium should be used to manufacture 1000 widgets at the lowest possible cost? [*Hint:* You want an extreme value for what function? Subject to what constraint?]
17. A water main consists of two sections of pipe of fixed lengths, l_1, l_2 carrying fixed amounts Q_1 and Q_2 liters per second. For a given total loss of head h, the (variable) diameters D_1, D_2 of the pipe will result in a minimum cost if
$$C = l_1(a + bD_1) + l_2(a + bD_2) = \text{minimum}$$
subject to the condition
$$h = \frac{cl_1 Q_1^2}{D_1^5} + \frac{cl_2 Q_2^2}{D_2^5}.$$
Find the ratio D_2/D_1.
18. A firm uses wool and cotton fiber to produce cloth. The amount of cloth produced is given by $Q(x, y) = xy - x - y + 1$, where x is the number of pounds of wool, y is the number of pounds of cotton, and $x > 1$ and $y > 1$. If wool costs p dollars per pound, cotton costs q dollars per pound, and the firm can spend B dollars on material, what should the mix of cotton and wool be to produce the most cloth?
19. Let $f(x, y) = x^2 + xy + y^2$.
    (a) Find the maximum and minimum points and values of f along the circle $x^2 + y^2 = 1$.
    (b) Moving counterclockwise along the circle $x^2 + y^2 = 1$, is the function increasing or decreasing at the points $(\pm 1, 0)$ and $(0, \pm 1)$?
    (c) Find extreme points and values for f in the disk D consisting of all (x, y) such that $x^2 + y^2 \leqslant 1$.
20. Locate extreme points and values for the function $f(x, y) = x^2 + y^2 - x - y + 1$ in the disk $x^2 + y^2 \leqslant 1$.
21. A transformer is built from wire of cross sections q_1 and q_2 wound with n_1 and n_2 turns onto the primary and secondary coils, respectively. The corresponding currents are i_1 and i_2. The thickness x of the primary winding and the thickness y of the secondary winding will result in minimum copper loss if
$$C = \frac{\rho n_1 \pi (D_1 + x) i_1^2}{q_1} + \frac{\rho n_2 \pi (D_2 - y) i_2^2}{q_2}$$
is a minimum. The resistivity ρ and iron core diameters D_1, D_2 are constants.
    (a) From transformer theory, $n_1 i_1 = n_2 i_2 =$ constant. By an argument involving insulation thickness, one can show that $q_1 = \alpha x h / n_1$ and $q_2 = \alpha y h / n_2$, where α and h are constants. Use these relations to simplify the expression for C.
    (b) Physical constraints give $x + y = \frac{1}{2}(D_2 - D_1)$. Apply the method of Lagrange multipliers to find x and y which minimize C subject to this condition.
22. The state of Megalomania occupies the region $x^4 + 2y^4 \leqslant 30{,}000$. The altitude at point (x, y) is $\frac{1}{8}xy + 200x$ meters above sea level. Where are the highest and lowest points in the state?

★23. Suppose that (x_0, y_0, z_0) is a critical point for the restriction of the function $f(x, y, z)$ to the surface $g(x, y, z) = c$. The method of Lagrange multipliers tells us that in this case the partial derivatives with respect to x, y, z, and λ of the function of four variables
$$k(x, y, z, \lambda) = f(x, y, z) - \lambda[g(x, y, z) - c]$$
are equal to zero.

(a) Interpret this fact as the statement about the gradient vectors of f and g at (x_0, y_0, z_0).
(b) Find the maxima and minima of xyz on the sphere $x^2 + y^2 + z^2 = 1$.
(c) Rework Example 5, Section 16.3 by minimizing a function of x, y, and h subject to the constraint $xyh = 256$.

Review Exercises for Chapter 16

Calculate the gradients of the functions in Exercises 1–4.

1. $f(x, y) = e^{xy} + \cos(xy)$
2. $f(x, y)\dfrac{x^2 - y^2}{x^2 + y^2}$
3. $f(x, y) = e^{x^2} - \cos(xy^2)$
4. $f(x, y) = \tan^{-1}(x^2 + y^2)$

In Exercises 5–8, calculate (a) the directional derivative of the function in the direction $\mathbf{d} = \mathbf{i}/\sqrt{2} - \mathbf{j}/\sqrt{2}$ and (b) the direction in which the function is increasing most rapidly at the given point.

5. $f(x, y) = \sin(x^3 - 2y^3)$; $(1, -1)$
6. $f(x, y) = \dfrac{x - y}{x + y}$; $(0, 1)$
7. $f(x, y) = \exp(x^2 - y^2 + 2)$; $(-1, 2)$
8. $f(x, y) = \sin^{-1}(x - 2y^2)$; $(0, 0)$

In Exercises 9–12, find the equation of the tangent plane to the surface at the indicated point.

9. $z = x^3 + 2y^2$; $(1, 1, 3)$
10. $z = \cos(x^2 + y^2)$; $(0, 0, 1)$
11. $x^2 + y^2 + z^2 = 1$; $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$
12. $x^3 + y^3 + z^3 = 3$; $(1, 1, 1)$

Suppose that $x = f(t)$ and $y = g(t)$ satisfy the relations in Exercises 13–16. Relate dx/dt and dy/dt.

13. $x^2 + xy + y^2 = 1$
14. $\cos(x - y) = \frac{1}{2}$
15. $(x + y)^3 + (x - y)^3 = 42e^{-\pi}$
16. $\tan^{-1}(x - y) = \pi/4$

Suppose that x and y are related by the equations given in Exercises 17–20. Find dy/dx at the indicated points.

17. $x + \cos y = 1$, $x = 1$, $y = \pi/2$.
18. $x^4 + y^4 = 17$, $x = -1$, $y = 2$.
19. $\int_x^y u^2\,du = 5\frac{1}{3}$, $x = -2$, $y = 2$.
20. $\int_x^y f(t)\,dt = 7$, $x = 2$, $y = 4$; if $\int_2^4 f(t)\,dt = 7$, $f(2) = 3$, $f'(2) = 5$, $f(4) = 11$, $f'(4) = 13$.

Find and classify (as maxima, minima or saddles) the critical points of the functions in Exercises 21–24.

21. $f(x, y) = x^2 - 6xy - y^2$
22. $f(x, y) = 2x^2 - y^2 + 5xy$
23. $f(x, y) = \exp(x^2 - y^2)$
24. $f(x, y) = \sin(x^2 + y^2)$ (consider only $(0, 0)$).
25. Prove that
$$z = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1 + 4y^2)}$$
has one local maximum, one local minimum, and one saddle point. (The computer-generated graph is shown in Fig. 16.R.1.)

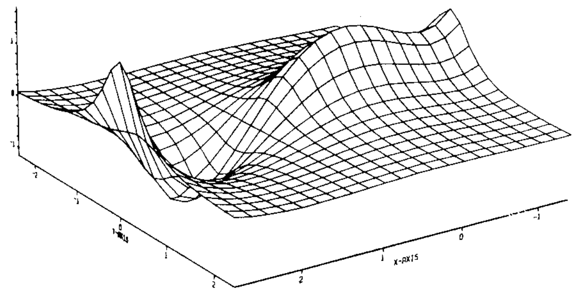


Figure 16.R.1. Computer-generated graph of $z = (3x^4 - 4x^3 - 12x^2 + 18)/12(1 + 4y^2)$.

26. Find the maxima, minima, and saddles of the function $z = (2 + \cos \pi x)(\sin \pi y)$, which is graphed in Fig. 16.R.2.

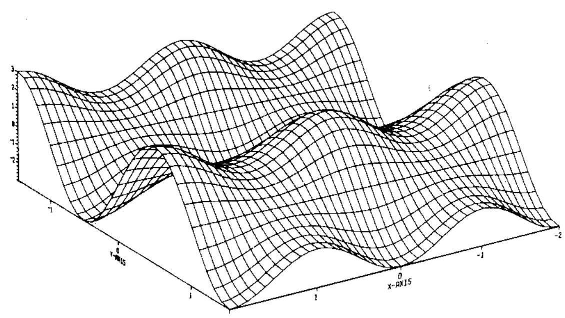


Figure 16.R.2. Computer-generated graph of $z = (2 + \cos \pi x)(\sin \pi y)$.

27. Find and describe the critical points of $f(x, y) = y \sin(\pi x)$ (See Fig. 16.R.3).

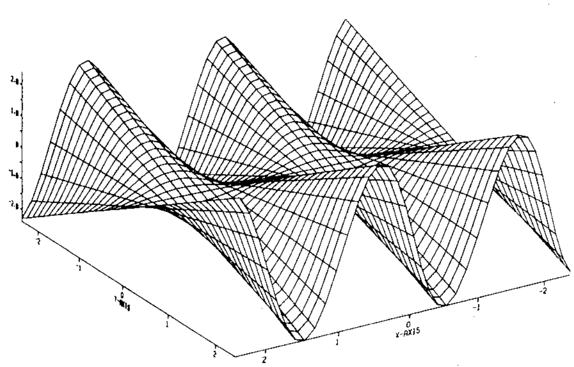


Figure 16.R.3. Computer-generated graph of $z = y \sin(\pi x)$.

28. A computer-generated graph of the function $z = \sin(\pi x)/(1 + y^2)$ is shown in Fig. 16.R.4. Verify that this function has alternating maxima and minima on the x axis with no other critical points.

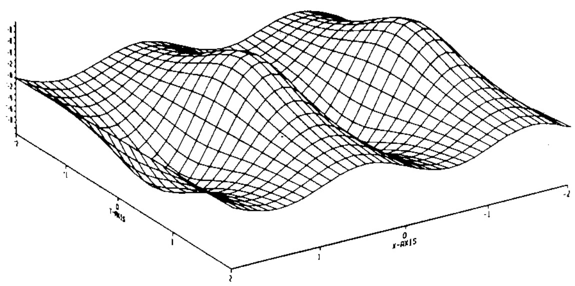


Figure 16.R.4. Computer-generated graph of $\sin(\pi x)/(1 + y^2)$.

29. In meteorology, the *pressure gradient* **G** is a vector quantity that points from regions of high pressure to regions of low pressure, normal to the lines of constant pressure (*isobars*).
    (a) In an xy coordinate system,
    $$\mathbf{G} = -\frac{\partial P}{\partial x}\mathbf{i} - \frac{\partial P}{\partial y}\mathbf{j}.$$
    Write a formula for the magnitude of the pressure gradient.
    (b) If the horizontal pressure gradient provided the only horizontal force acting on the air, the wind would blow directly across the isobars in the direction of **G**, and for a given air mass, with acceleration proportional to the magnitude of **G**. Explain, using Newton's second law.
    (c) *Buys–Ballot's law states*: "If in the Northern Hemisphere, you stand with your back to the wind, the high pressure is on your right and the low pressure on your left." Draw a figure and introduce xy coordinates so that **G** points in the proper direction.
    (d) State and graphically illustrate Buys–Ballot's law for the Southern Hemisphere, in which the orientation of high and low pressure is reversed.
30. A sphere of mass m, radius a, and uniform density has *potential* u and *gravitational force* **F**, at a distance r from the center $(0, 0, 0)$, given by
$$u = \frac{3m}{2a} - \frac{mr^2}{2a^3}, \qquad \mathbf{F} = -\frac{m}{a^3}\mathbf{r} \quad (r \leqslant a);$$
$$u = \frac{m}{r}, \qquad \mathbf{F} = -\frac{m}{r^3}\mathbf{r} \quad (r > a).$$
    Here, $r = \|\mathbf{r}\|$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
    (a) Verify that $\mathbf{F} = \nabla u$ on the inside and outside of the sphere.
    (b) Check that u satisfies Poisson's equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 =$ constant inside the sphere.
    (c) Show u satisfies Laplace's equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$ outside the sphere.
31. Minimize the distance from $(0, 0, 0)$ to each of the following surfaces. [*Hint*: Write the square of the distance as a function of x and y.]
    (a) $z = \sqrt{x^2 - 1}$;
    (b) $z = 6xy + 7$;
    (c) $z = 1/xy$.
32. Suppose that $f(x, y) = x^2 + y$. Find the maximum and minimum values of f for (x, y) on a circle of radius 1 centered at the origin in two ways:
    (a) By parametrizing the circle.
    (b) By Lagrange multipliers.

In Exercises 33–36, find the extrema of the given functions subject to the given constraints.

33. $f(x, y) = x^2 - 2xy + 2y^2$; $x^2 + y^2 = 1$.
34. $f(x, y) = xy - y^2$; $x^2 + y^2 = 1$.
35. $f(x, y) = \cos(x^2 - y^2)$; $x^2 + y^2 = 1$.
36. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$; $x + y = 1$.
37. An irrigation canal in Arizona has concrete sides and bottom with trapezoidal cross section of area $A = y(x + y\tan\theta)$ and wetted perimeter $P = x + 2y/\cos\theta$, where $x =$ bottom width, $y =$ water depth, $\theta =$ side inclination, measured from vertical. The best design for fixed inclination θ is found by solving $P =$ minimum subject to the condition $A =$ constant. Show that $y^2 = A\cos\theta/(2 - \sin\theta)$.
38. The friction in an open-air aqueduct is proportional to the wetted perimeter of the cross section. Show that the best form of a rectangular cross section is one with the width x equal to twice the depth y, by solving the problem perimeter $= 2y + x =$ minimum, area $= xy =$ constant.
39. (a) Suppose that $z = f(x, y)$ is defined, has continuous second partial derivatives, and is harmonic:
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$
    Assume that $(\partial^2 z/\partial x^2)(x_0, y_0) \neq 0$. Prove that f cannot have a local maximum or minimum at (x_0, y_0).
    (b) Conclude from (a) that if $f(x, y)$ is harmonic on the region $x^2 + y^2 < 1$ and is zero on $x^2 + y^2 = 1$, then f is zero everywhere on the unit disk. [*Hint:* Where are the maximum and minimum values of f?]

40. (a) Suppose that $u = f(x, y)$ and $v = g(x, y)$ have continuous partial derivatives which satisfy the *Cauchy–Riemann equations*:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
Show that the level curves of u are perpendicular to the level curves of v.
(b) Confirm this result for the functions $u = x^2 - y^2$ and $v = 2xy$. Sketch some of the level curves of these functions (all on the same set of axes).

41. Consider the two surfaces
$$S_1 : x^2 + y^2 + z^2 = f(x, y, z) = 6;$$
$$S_2 : 2x^2 + 3y^2 + z^2 = g(x, y, z) = 9.$$
(a) Find the normal vectors and tangent planes to S_1 and S_2 at $(1, 1, 2)$.
(b) Find the angle between the tangent planes.
(c) Find an expression for the line tangent at $(1, 1, 2)$ to the curve of intersection of S_1 and S_2. [*Hint:* It lies in both tangent planes.]

42. Repeat Exercise 41 for the surfaces $x^2 - y^2 + z^2 = 1$ and $2x^2 - y^2 + 5z^2 = 6$ at $(1, 1, -1)$.

43. (a) Consider the graph of a function $f(x, y)$ (Figure 16.R.5). Let (x_0, y_0) lie on a level curve C, so $\nabla f(x_0, y_0)$ is perpendicular to this curve. Show that the tangent plane to the graph is the plane that (i) contains the line perpendicular to $\nabla f(x_0, y_0)$ and lying in the horizontal plane $z = f(x_0, y_0)$, and (ii) has slope $\|\nabla f(x_0, y_0)\|$ relative to the xy plane. (By the slope of a plane P relative to the xy plane, we mean the tangent of the angle θ, $0 \leqslant \theta \leqslant \pi$, between the upward pointing normal $\mathbf{p}$ to P and the unit vector $\mathbf{k}$.)

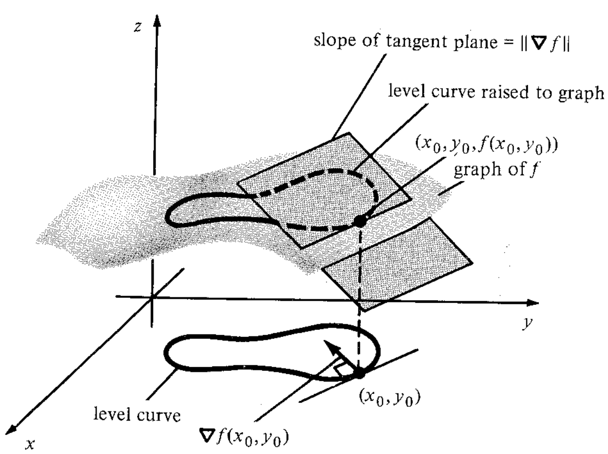


Figure 16.R.5. Relationship between the gradient of a function and the plane tangent to the function's graph (Exercise 43(a)).

(b) Use this method to show that the tangent plane of the graph of
$$f(x, y) = (x + \cos y)x^2$$
at $(1, 0, 2)$ is as sketched in Figure 16.R.6.

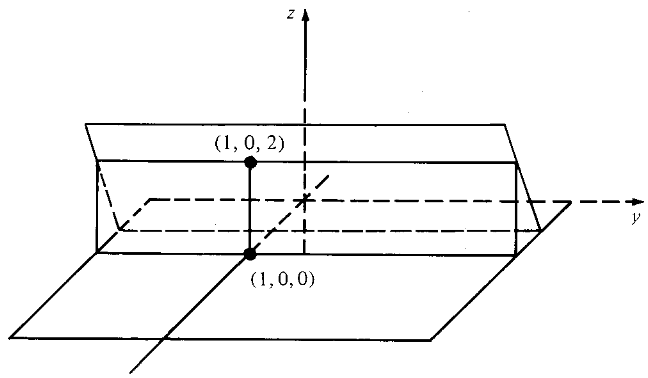


Figure 16.R.6. The plane referred to in Exercise 43(b).

44. (a) Use a geometric argument to demonstrate that if $f(x, y, z)$ is extremized at (x_0, y_0, z_0) subject to two constraints $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_2$, then there should exist λ_1 and λ_2 such that
$$\nabla f(x_0, y_0, z_0) = \lambda_1 \nabla g_1(x_0, y_0, z_0) + \lambda_2 \nabla g_2(x_0, y_0, z_0).$$
(b) Extremize $f(x, y, z) = x - y + z$ subject to the constraints $x^2 + y^2 + z^2 = 1$ and $x + y + 2z = 1$.

45. A pipeline of length l is to be constructed from one pipe of length l_1 and diameter D_1 connected to another pipe of length l_2 and diameter D_2. The finished pipe must deliver Q liters per second at pressure loss h. The expense is reduced to a minimum by minimization of the cost $C = l_1(a + bD_1) + l_2(a + bD_2)$ $(a, b = \text{constants})$ subject to the conditions
$$l_1 + l_2 = l, \quad \text{and}$$
$$h = kQ^m \left\{ \frac{l_1}{D_1^{m_1}} + \frac{l_2}{D_2^{m_2}} \right\},$$
where k, m, m_1, m_2 are constants. Show that $D_1 = D_2$ in order to achieve the minimum cost. [*Hint*: As in Exercise 44, the partials of $C - \lambda_1 l - \lambda_2 h$ with respect to l_1, l_2, D_1 and D_2 must all be zero for suitable constants λ_1 and λ_2.]

46. Ammonia, NH_3, is to be produced at fixed temperature T and pressure p. The pressures of N_2, H_2, NH_3 are labeled as u, v, w, and are known to satisfy $u + v + w = p$, $w^2 = cuv^3$ (c = positive constant). Due to the nature of the reaction $N_2 + 3H_2 \rightleftharpoons 2NH_3$, the maximium ammonia production occurs for w = maximum. Find the maximum pressure w_{max}.

For Exercises 47–50, consider the level curves for the function $f(x, y)$ shown in Figure 16.R.7. Find or estimate the maximum value of $f(x, y)$ for each of the given constraint conditions.

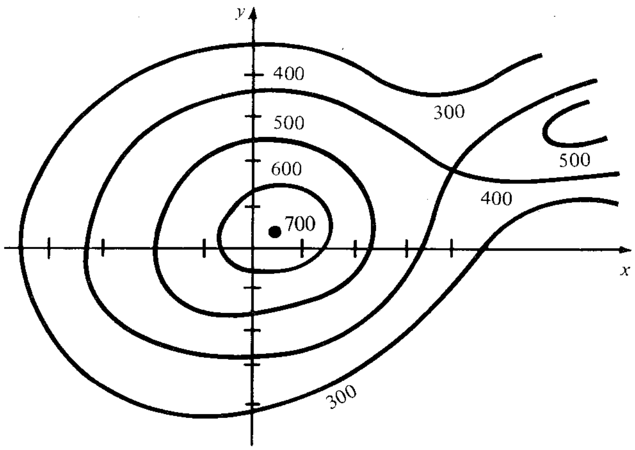


Figure 16.R.7. Level curves of a function f.

47. $x \geqslant 0, y \geqslant 0, y = -\frac{2}{3}x + 2$
48. $x^2 + y^2 \leqslant 4$
49. $x^2 + y^2 = 4$
50. $x = 4, 0 \leqslant y \leqslant 4$

51. Refer to Figure 16.R.7. The function f has exactly one saddle point. Find it.
52. Refer to Figure 16.R.7. There are two points on the graph $z = f(x, y)$ at which the tangent plane is horizontal. Give the equation of the tangent plane at each such point.
53. (a) Let y be defined implicitly by
$$x^2 + y^3 + e^y = 0.$$
Compute dy/dx in terms of x and y.
(b) Recall from p. 810 that
$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} \quad \text{if} \quad \frac{\partial F}{\partial y} \neq 0.$$
Obtain a formula analogous to this if y_1, y_2 are defined implicitly by
$$F_1(x, y_1(x), y_2(x)) = 0,$$
$$F_2(x, y_1(x), y_2(x)) = 0.$$
(c) Let y_1 and y_2 be defined by
$$x^2 + y_1^2 = \cos x,$$
$$x^2 - y_2^2 = \sin x.$$
Compute dy_1/dx and dy_2/dx using (b).
54. Thermodynamics texts[6] use the relationship
$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -1.$$
Explain the meaning of this equation and prove that it is true. [*Hint:* Start with a relationship $F(x, y, z) = 0$ that implicitly defines $x = f(y, z)$, $y = g(x, z)$, and $z = h(x, y)$ and differentiate.]
55. (a) Suppose that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. Show that if there is a function $f(x, y)$ with continuous second partial derivatives such that $\mathbf{F} = \nabla f$, then $P_y = Q_x$.
(b) Suppose that
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$
Show that if there is a function $f(x, y, z)$ with continuous second partial derivatives such that $\mathbf{F} = \nabla f$, then
$$P_y = Q_x, \qquad P_z = R_x, \qquad Q_z = R_y.$$
(c) Let $\mathbf{F} = 3xy\mathbf{i} - ye^x\mathbf{j}$. Is there an f such that $\mathbf{F} = \nabla f$?

★56. (Continuation of Exercise 55.) Suppose that P and Q have continuous partial derivatives everywhere in the xy plane and that $P_y = Q_x$. Follow the steps below to prove that there is a function f such that $\nabla f = P\mathbf{i} + Q\mathbf{j}$; that is, $f_x = P$ and $f_y = Q$.
(a) Let $g(x, y)$ be an antiderivative of Q with respect to y; that is, $g_y = Q$. Establish that $P - g_x$ is a function of x alone by showing that $(P - g_x)_y = 0$.
(b) If $P - g_x = 0$, then we may simply take $g = f$. Otherwise let $h(x)$ be an antiderivative of $P - g_x$; that is, $h'(x) = P - g_x$. Show that $f(x, y) = g(x, y) + y(x)$ satisfies $\nabla f = P\mathbf{i} + Q\mathbf{j}$.

★57. For each of the following vector fields $P\mathbf{i} + Q\mathbf{j}$, find a function f such that $f_x = P$ and $f_y = Q$ or show that no such function exists. (See Exercises 55 and 56.)
(a) $(x^2y^2 + 2x)e^{xy^2}\mathbf{i} + 2x^3ye^{xy^2}\mathbf{j}$;
(b) $(x^2y^2 + 2x)e^{xy^3}\mathbf{i} + 2x^3ye^{xy^3}\mathbf{j}$;
(c) $\dfrac{2x}{1 + x^2 + y^2}\mathbf{i} + \dfrac{2y}{1 + x^2 + y^2}\mathbf{j}$;
(d) $\dfrac{2y}{1 + x^2 + y^2}\mathbf{i} + \dfrac{2x}{1 + x^2 + y^2}\mathbf{j}$.

★58. (*The gradient and Laplacian in polar coordinates.*) Let r and θ be polar coordinates in the plane and let f be a given function of (x, y). Write
$$u = f(x, y) = f(r\cos\theta, r\sin\theta).$$
Let $\mathbf{i}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{i}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.
(a) Show that when based at $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ the vectors $\mathbf{i}_r$ and $\mathbf{i}_\theta$ are orthogonal unit vectors in the directions of increasing r and θ, respectively.
(b) Show that
$$\nabla f = \left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)\mathbf{i} + \left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)\mathbf{j}$$

[6] See S. M. Binder, "Mathematical methods in elementary thermodynamics," *J. Chem. Educ.* **43** (1966): 85–92. A proper understanding of partial differentiation can be of significant use in applications; for example, see M. Feinberg, "Constitutive equation for ideal gas mixtures and ideal solutions as consequences of simple postulates", *Chem. Eng. Sci.* **32** (1977): 75–78.

$$= \frac{\partial u}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{i}_\theta .$$

(c) Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} .$$

★59. Find a family of curves orthogonal to the level curves of $f(x, y) = x^2 - y^2$ as follows:

(a) Find an expression for a vector normal to the level curve of f through (x_0, y_0) at the point (x_0, y_0).

(b) Use this expression to find a vector tangent to the level curve of f through (x_0, y_0) at (x_0, y_0).

(c) Find a function g which has these vectors as its gradient.

(d) Explain why the level curves of g should intersect those of f orthogonally.

(e) Draw a few of the level curves of f and g to illustrate this result.

★60. (a) Figure 16.R.8 shows the graph of the function $z = (x^2 - y^2)/(x^2 + y^2)$. Show that z has different limits if we come in along the x or y axis.

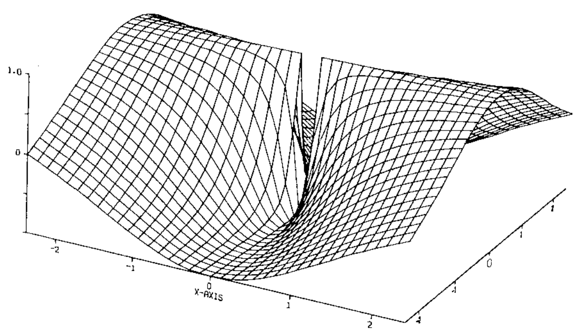


Figure 16.R.8. Computer-generated graph of $z = (x^2 - y^2)/(x^2 + y^2)$.

(b) Figure 16.R.9 shows the graph of the function $z = 2xy^2/(x^2 + y^4)$. Show that if we approach the origin on any straight line, z approaches zero, but z has different limits when $(0, 0)$ is approached along the two parabolas $x = \pm y^2$.

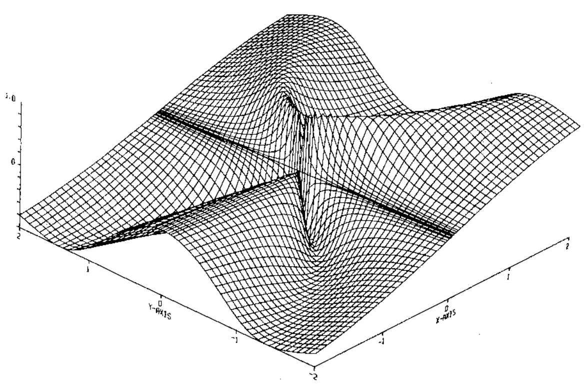


Figure 16.R.9. Computer-generated graph of $z = 2xy^2/(x^2 + y^4)$.

★61. Let $f(x, y) = y^3/(x^2 + y^2)$; $f(0, 0) = 0$.

(a) Compute f_x, f_y, $f_x(0, 0)$, and $f_y(0, 0)$.

(b) Show that, for any θ, the directional derivative $(d/dr)\, f(r\cos\theta, r\sin\theta)|_{r=0}$ exists.

(c) Show that the directional derivatives are not all given by dotting the direction vector with the gradient vector (see Fig. 16.R.10). Why does this not contradict the chain rule?

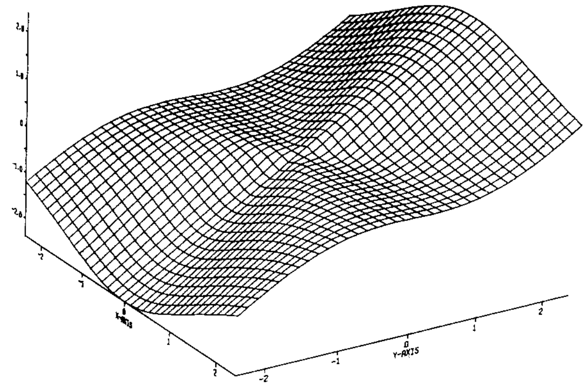


Figure 16.R.10. Computer-generated graph of $z = y^3/(x^2 + y^2)$.

★62. Do the same things as in Exercise 61 for $z = (x^3 - 3xy^2)/(x^2 + y^2)$, which is graphed in Fig. 16.R.11.

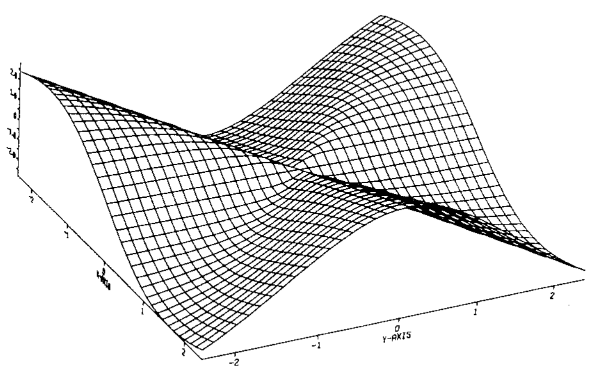


Figure 16.R.11. Computer-generated graph of $z = (x^3 - 3xy^2)/(x^2 + y^2)$.