SECTION 2
THE CENTER MANIFOLD THEOREM

In this section we will start to carry out the program outlined in Section 1 by proving the center manifold theorem. The general invariant manifold theorem is given in Hirsch-Pugh-Shub [1]. Most of the essential ideas are also in Kelley [1] and a treatment with additional references is contained in Hartman [1]. However, we shall follow a proof given by Lanford [1] which is adapted to the case at hand, and is direct and complete. We thank Professor Lanford for allowing us to reproduce his proof.

The key job of the center manifold theorem is to enable one to reduce to a finite dimensional problem. In the case of the Hopf theorem, it enables a reduction to two dimensions without losing any information concerning stability. The outline of how this is done was presented in Section 1 and the details are given in Sections 3 and 4.

In order to begin, the reader should recall some results about basic spectral theory of bounded linear operators by consulting Section 2A. The proofs of Theorems 1.3

and 1.4 are also found there.

Statement and Proof of the Center Manifold Theorem

We are now ready for a proof of the center manifold theorem. It will be given in terms of an invariant manifold for a map Ψ, not necessarily a local diffeomorphism. Later we shall use it to get an invariant manifold theorem for flows. Remarks on generalizations are given at the end of the proof.

(2.1) Theorem. Center Manifold Theorem. *Let Ψ be a mapping of a neighborhood of zero in a Banach space Z into Z. We assume that Ψ is C^{k+1}, $k \geq 1$ and that $\Psi(0) = 0$. We further assume that $D\Psi(0)$ has spectral radius 1 and that the spectrum of $D\Psi(0)$ splits into a part on the unit circle and the remainder which is at a non-zero distance from the unit circle.** *Let Y denote the generalized eigenspace of $D\Psi(0)$ belonging to the part of the spectrum on the unit circle; assume that Y has dimension $d < \infty$.*

Then there exists a neighborhood V of 0 in Z and a C^k submanifold M of V of dimension d, passing through 0 and tangent to Y at 0, such that

a) *(Local Invariance): If $x \in M$ and $\Psi(x) \in V$, then $\Psi(x) \in M$*

b) *(Local Attractivity): If $\Psi^n(x) \in V$ for all $n = 0, 1, 2, \ldots$, then, as $n \to \infty$, the distance from $\Psi^n(x)$ to M goes to zero.*

We begin by reformulating (in a slightly more general way) the theorem we want to prove. We have a mapping Ψ of a neighborhood of zero in a Banach space Z into Z, with

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*This holds automatically if Z is finite dimensional or, more generally, if $D\Psi(0)$ is compact.

$\Psi(0) = 0$. We assume that the spectrum of $D\Psi(0)$ splits into a part on the unit circle and the remainder, which is contained in a circle of radius strictly less than one, about the origin. The basic spectral theory discussed in Section 2A guarantees the existence of a spectral projection P of Z belonging to the part of the spectrum on the unit circle with the following properties:

i) P commutes with $D\Psi(0)$, so the subspaces PZ and $(I-P)Z$ are mapped into themselves by $D\Psi(0)$.

ii) The spectrum of the restriction of $D\Psi(0)$ to PZ lies on the unit circle, and

iii) The spectral radius of the restriction of $D\Psi(0)$ to $(I-P)Z$ is strictly less than one.

We let X denote $(I-P)Z$, Y denote PZ, A denote the restriction of $D\Psi(0)$ to X and B denote the restriction of $D\ (0)$ to Y. Then $Z = X \oplus Y$ and

$$\Psi(x,y) = (Ax+X^*(x,y),\ \ By + Y^*(x,y)),$$

where

A is bounded linear operator on X with spectral radius strictly less than one.

B is a bounded operator on Y with spectrum on the unit circle. (All we actually need is that the spectral radius of B^{-1} is no larger than one.)

X^* is a C^{k+1} mapping of a neighborhood of the origin in $X \oplus Y$ into X with a second-order zero at the origin, i.e. $X(0,0) = 0$ and $DX(0,0) = 0$, and

Y is a C^{k+1} mapping of a neighborhood of the origin in $X \oplus Y$ into Y with a second-order zero at the origin.

We want to find an invariant manifold for Ψ which is tangent to Y at the origin. Such a manifold will be the graph of a mapping u which maps a neighborhood of the origin in Y into X, with $u(0) = 0$ and $Du(0) = 0$.

In the version of the theorem we stated in 2.1, we assumed that Y was finite-dimensional. We can weaken this assumption, but not eliminate it entirely.

(2.2) Assumption. There exists a C^{k+1} real-valued function ϕ on Y which is 1 on a neighborhood of the origin and zero for $||y|| > 1$. Perhaps surprisingly, this assumption is actually rather restrictive. It holds trivially if Y is finite-dimensional or if Y is a Hilbert space; for a more detailed discussion of when it holds, see Bonic and Frampton [1].

We can now state the precise theorem we are going to prove.

(2.3) Theorem. *Let the notation and assumptions be as above. Then there exist* $\varepsilon > 0$ *and a* C^k*-mapping* u^* *from* $\{y \in Y: ||y|| < \varepsilon\}$ *into* X, *with a second-order zero at zero, such that*

a) *The manifold* $\Gamma_{u^*} = \{(x,y) | \ x = u^*(y)$ and $||y|| < \varepsilon\} \subset X \oplus Y$, i.e. *the graph of* u^* *is invariant for* Ψ *in the sense that, if* $||y|| < \varepsilon$ *and if* $\Psi(u^*(y),y) = (x_1,y_1)$ *with* $||y_1|| < \varepsilon$ *then* $x_1 = u^*(y_1)$.

b) *The manifold* Γ_{u^*} *is locally attracting for* Ψ *in the sense that, if* $||x|| < \varepsilon$, $||y|| < \varepsilon$, *and if* $(x_n,y_n) = \Psi^n(x,y)$ *are such that* $||x_n|| < \varepsilon$, $||y_n|| < \varepsilon$ *for all* $n > 0$, *then*

$$\lim_{n\to\infty} ||x_n - u^*(y_n)|| = 0.$$

Proceeding with the proof, it will be convenient to assume that $||A|| < 1$ and that $||B^{-1}||$ is not much greater than 1. This is not necessarily true but we can always make it true by replacing the norms on X, Y by equivalent norms. (See Lemma 2A.4). We shall assume that we have made this change of norm. It is unfortunately a little awkward to explicitly set down exactly how close to one $||B^{-1}||$ should be taken. We therefore carry out the proof as if $||B^{-1}||$ were an adjustable parameter; in the course of the argument, we shall find a finite number of conditions on $||B^{-1}||$. In principle, one should collect all these conditions and impose them at the outset.

The theorem guarantees the existence of a function u^* defined on what is perhaps a very small neighborhood of zero. Rather than work with very small values of x,y, we shall scale the system by introducing new variables x/ε, y/ε (and calling the new variables again x and y). This scaling does not change A,B, but, by taking ε very small, we can make X^*, Y^*, together with their derivatives of order $\leq k+1$, as small as we like on the unit ball. Then by multiplying $X^*(x,y)$, $Y^*(x,y)$ by the function $\phi(y)$ whose existence is asserted in the assumption preceding the statement of the theorem, we can also assume that $X^*(x,y)$, $Y^*(x,y)$ are zero when $||y|| > 1$. Thus, if we introduce

$$\lambda = \sup_{\substack{||x||\leq 1 \\ y \text{ unrestricted}}} \sup_{\substack{j_1,j_2 \\ j_1+j_2\leq k+1}} \{||D_x^{j_1} D_y^{j_2} X^*(x,y)|| + ||D_x^{j_1} D_y^{j_2} Y^*(x,y)||\},$$

we can make λ as small as we like by choosing ε very small. The only use we make of our technical assumption on Y is to arrange things so that the supremum in the definition of λ may be taken over all y and not just over a bounded set.

Once we have done the scaling and cutting off by ϕ, we can prove a global center manifold theorem. That is, we shall prove the following.

(2.4) Lemma. *Keep the notation and assumptions of the center manifold theorem. If λ is sufficiently small (and if $||B^{-1}||$ is close enough to one), there exists a function u^*, defined and k times continuously differentiable on all of Y, with a second-order zero at the origin, such that*

a) *The manifold $\Gamma_{u^*} = \{(x,y)|x = u^*(y), y \in Y\}$ is invariant for Ψ in the strict sense.*

b) *If $||x|| < 1$, and y is arbitrary then $\lim_{n\to\infty} ||x_n - u^*(y_n)|| = 0$ (where $(x_n, y_n) = \Psi^n(x,y)$).*

As with $||B^{-1}||$, we shall treat λ as an adjustable parameter and impose the necessary restrictions on its size as they appear. It may be worth noting that λ depends on the choice of norm; hence, one must first choose the norm to make $||B^{-1}||$ close to one, then do the scaling and cutting off to make λ small. To simplify the task of the reader who wants to check that all the required conditions on $||B^{-1}||$ can be satisfied simultaneously, we shall note these conditions with a * as with (2.3)* on p. 34.

The strategy of proof is very simple. We start with a manifold M of the form $\{x = u(y)\}$ (this stands for the graph of u); we let M denote the image of M under Ψ. With some mild restrictions on u, we first show that the manifold

ΨM again has the form

$$\{x = \hat{u}(y)\}$$

for a new function $\hat{u}$. If we write $\mathscr{F}u$ for $\hat{u}$ we get a (non-linear) mapping

$$u \mapsto \mathscr{F}u$$

from functions to functions. The manifold M is invariant if and only if $u = \mathscr{F}u$, so we must find a fixed point of $\mathscr{F}$. We do this by proving that $\mathscr{F}$ is a contraction on a suitable function space (assuming that λ is small enough).

More explicitly, the proof will be divided into the following steps:

I) Derive heuristically a "formula" for $\mathscr{F}$.

II) Show that the formula obtained in I) yields a well-defined mapping of an appropriate function space U into itself.

III)† Prove that $\mathscr{F}$ is a contraction on U and hence has a unique fixed point u^*.

IV) Prove that b) of Lemma (2.4) holds for u^*.

We begin by considering Step I).

I) To construct $u(y)$, we should proceed as follows

i) Solve the equation

$$y = B\tilde{y} + Y^*(u(\tilde{y}),\tilde{y}) \tag{2.1}$$

for $\tilde{y}$. This means that y is the Y-component of $\Psi(u(\tilde{y}),\tilde{y})$.

ii) Let $u(y)$ be the X-component of $\Psi(u(\tilde{y}),\tilde{y})$, i.e.,

$$u(y) = Au(\tilde{y}) + X^*(u(\tilde{y}),\tilde{y}). \tag{2.2}$$

II) We shall somewhat arbitrarily choose the space of functions u we want to consider to be

†One could use the implicit function theorem at this step. For this approach, see Irwin [1].

$U = \{u: Y \to X \mid D^{k+1}u$ continuous; $||D^j u(y)|| < 1$ for $j = 0,1,\ldots,k+1$, all y; $u(0) = Du(0) = 0\}$.

We must carry out two steps:

i) Prove that, for any given $u \in U$, equation (1) has a unique solution $\tilde{y}$ for each $y \in Y$. And

ii) Prove that $\mathscr{F}u$, defined by (2.2) is in U.

To accomplish (i), we rewrite (2.1) as a fixed-point problem:

$$\tilde{y} = B^{-1}y - B^{-1}Y^*(u(\tilde{y}),\tilde{y}).$$

It suffices, therefore, to prove that the mapping

$$\tilde{y} \mapsto B^{-1}y - B^{-1}Y^*(u(\tilde{y}),\tilde{y})$$

is a contraction on Y. We do this by estimating its derivative:

$$||D_{\tilde{y}}[B^{-1}y-B^{-1}Y^*(u(\tilde{y}),\tilde{y})]|| \le ||B^{-1}|| \quad ||D_1Y^*(u(\tilde{y}),\tilde{y})Du(\tilde{y}) + D_2Y^*(u(\tilde{y}),\tilde{y})|| \le 2\lambda||B^{-1}||$$

by the definitions of λ and U. If we require

$$2\lambda||B^{-1}|| < 1, \tag{2.3}^*$$

equation (2.1) has a unique solution $\tilde{y}$ for each y. Note that $\tilde{y}$ is a function of y, depending also on the function u. By the inverse function theorem, $\tilde{y}$ is a C^{k+1} function of y.

Next we establish (ii). By what we have just proved, $\mathscr{F}u \in C^{k+1}$. Thus to show $\mathscr{F}u \in U$, what we must check is that

$$||D^j\mathscr{F}u(y)|| \le 1 \quad \text{for all } y, \quad j = 0,1,2,\ldots,k+1 \tag{2.4}$$

and $\mathscr{F}u(0) = 0$, $D\mathscr{F}u(0) = 0$. (2.5)

First take $j = 0$:

$$||\mathscr{F}u(y)|| \le ||A||\cdot||u(\tilde{y})|| + ||X^*(u(\tilde{y}),\tilde{y})|| \le ||A|| + \lambda,$$

so if we require

$$||A|| + \lambda \le 1, \tag{2.6}^*$$

then $||\mathscr{F}u(y)|| \le 1$ for all y.

To estimate $D\mathscr{F}u$ we must first estimate $D\tilde{y}(y)$. By differentiating (2.1), we get

$$I = [B+DY^u(\tilde{y})]D\tilde{y}, \tag{2.7}$$

where $Y^u\colon Y \to Y$ is defined by

$$Y^u(y) = Y^*(u(y),y).$$

By a computation we have already done,

$$||DY^u(\tilde{y})|| \leq 2\lambda \quad \text{for all} \quad \tilde{y}.$$

Now $B + DY^u = B[I+B^{-1}DY^u]$ and since $2\lambda||B^{-1}|| < 1$ (by (2.3)*), $B + DY^u$ is invertible and

$$||(B+DY^u)^{-1}|| \leq ||B^{-1}||(1-2\lambda||B^{-1}||)^{-1}.$$

The quantity on the right-hand side of this inequality will play an important role in our estimates, so we give it a name:

$$\gamma \equiv ||B^{-1}||(1-\lambda||B^{-1}||)^{-1}. \tag{2.8}$$

Note that, by first making $||B^{-1}||$ very close to one and then by making λ small, we can make γ as close to one as we like.

We have just shown that

$$||Dy(\tilde{y})|| \leq \gamma \quad \text{for all} \quad y. \tag{2.9}$$

Differentiating the expression (2.2) for $\mathscr{F}u(y)$ yields

$$\left.\begin{array}{l} D\mathscr{F}u(y) = [A\,Du(\tilde{y}) + DX^u(\tilde{y})]\,D\tilde{y}(y); \\ \text{and} \\ (X^u(\tilde{y}) = X^*(u(\tilde{y}),\tilde{y})). \end{array}\right\} \tag{2.10}$$

Thus

$$||D\mathscr{F}u(y)|| \leq (||A|| + 2\lambda)\cdot\gamma, \tag{2.11}$$

so if we require

$$(||A|| + 2\lambda)\gamma \leq 1, \qquad (2.12)^*$$

we get

$$||D\mathscr{F}u(y)|| \leq 1 \quad \text{for all} \quad y.$$

We shall carry the estimates just one step further. Differentiating (2.7) yields

$$0 = (B+DY^u(\tilde{y}))D^2\tilde{y} + D^2Y^u(D\tilde{y})^2.$$

By a straightforward computation,

$$||D^2Y^u(\tilde{y})|| \leq 5\lambda \quad \text{for all} \quad \tilde{y},$$

so

$$||D^2y(\tilde{y})|| = ||(B + DY^u(\tilde{y}))^{-1}D^2Y^u(D\tilde{y})^2||$$

$$\leq \gamma \cdot 5\lambda \cdot \gamma^2 = 5\lambda\gamma^3.$$

Now, by differentiating the formula (2.10) for $D\mathscr{F}u$, we get

$$D^2\mathscr{F}u(y) = [A\ D^2u(\tilde{y}) + D^2X^u(\tilde{y})](D\tilde{y})^2 + [A\ Du(\tilde{y}) + DX^u(\tilde{y})]D^2\tilde{y},$$

so

$$||D^2\mathscr{F}u(y)|| \leq (||A|| + 5\lambda)\gamma^2 + (||A|| + 2\lambda)\cdot 5\lambda\gamma^3.$$

If we require

$$(||A|| + 5\lambda)\gamma^2 + (||A|| + 2\lambda)\cdot 5\lambda\gamma^3 \leq 1, \qquad (2.13)^*$$

we have

$$||D^2\mathscr{F}u(y)|| \leq 1 \quad \text{for all} \quad y.$$

At this point it should be plausible by imposing a sequence of stronger and stronger conditions on γ,λ, that we can arrange

$$||D^j\mathscr{F}(y)|| \leq 1 \quad \text{for all} \quad y, \quad j = 3,4,\ldots,k+1.$$

The verification that this is in fact possible is left to the reader.

To check (2.5), i.e. $\mathscr{F}u = 0$, $D\mathscr{F}u = 0$ (assuming $u = 0$, $Du = 0$) we note that

$$\begin{aligned}
\tilde{y}(0) &= 0 \quad \text{since } 0 \text{ is a solution of } 0 = B\tilde{y} + Y(u(\tilde{y}),\tilde{y}) \\
\mathscr{F}u(0) &= Au(0) + X(u(0),0) = 0 \quad \text{and} \\
D\mathscr{F}u(0) &= [A\, Du(0) + D_1X(0,0)Du(0) + D_2X(0,0)]\cdot D\tilde{y}(0) \\
&= [A\cdot 0 + 0 + 0]\cdot D\tilde{y}(0) = 0.
\end{aligned}$$

This completes step II). Now we turn to III)

III) We show that $\mathscr{F}$ is a contraction and apply the contraction mapping principle. What we actually do is slightly more complicated.

i) We show that $\mathscr{F}$ is a contraction in the supremum norm. Since U is not complete in the supremum norm, the contraction mapping principle does not imply that $\mathscr{F}$ has a fixed point in U, but it does imply that $\mathscr{F}$ has a fixed point in the completion of U with respect to the supremum norm.

ii) We show that the completion of U with respect to the supremum norm is contained in the set of functions u from Y to X with Lipschitz-continuous k^{th} derivatives and with a second-order zero at the origin. Thus, the fixed point u^* of $\mathscr{F}$ has the differentiability asserted in the theorem.

We proceed by proving i).

i) Consider $u_1, u_2 \in U$, and let $||u_1-u_2||_0 = \sup_y ||u_1(y)-u_2(y)||$. Let $\tilde{y}_1(y)$, $\tilde{y}_2(y)$ denote the solution of

$$y = B\tilde{y}_i + Y(u_i(\tilde{y}_i),\tilde{y}_i) \quad i = 1,2.$$

We shall estimate successively $||\tilde{y}_1-\tilde{y}_2||_0$, and $||\mathscr{F}u_1-\mathscr{F}u_2||_0$. Subtracting the defining equations for $\tilde{y}_1,\tilde{y}_2$, we get

$$B(\tilde{y}_1-\tilde{y}_2) = Y(u_2(\tilde{y}_2),\tilde{y}_2) - Y(u_1(\tilde{y}_1),\tilde{y}_1),$$

so that

$$||\tilde{y}_1-\tilde{y}_2|| \leq ||B^{-1}||\cdot\lambda\cdot[||u_2(\tilde{y}_2)-u_1(\tilde{y}_1)|| + ||\tilde{y}_2-\tilde{y}_1||]. \qquad (2.14)$$

Since $||Du_1||_0 \leq 1$, we can write

$$\left.\begin{aligned} ||u_2(\tilde{y}_2) - u_1(\tilde{y}_1)|| &\leq ||u_2(\tilde{y}_2) - u_1(\tilde{y}_2)|| + \\ ||u_1(\tilde{y}_2) - u_1(\tilde{y}_1)|| &\leq ||u_2-u_1||_0 + ||\tilde{y}_2-\tilde{y}_1||. \end{aligned}\right. \qquad (2.15)$$

Inserting (2.15) in (2.14) and rearranging, yields

$$\left.\begin{aligned} (1-2\lambda\cdot||B^{-1}||)||\tilde{y}_1-\tilde{y}_2|| &\leq \lambda\cdot||B^{-1}||\cdot||u_2-u_1||_0, \\ \text{i.e.} \qquad ||\tilde{y}_1-\tilde{y}_2||_0 &\leq \lambda\cdot\gamma\cdot||u_2-u_1||. \end{aligned}\right\} \qquad (2.16)$$

Now insert estimates (2.15) and (2.16) in

$$\begin{aligned} \mathscr{F}u_1(y) - \mathscr{F}u_2(y) = {} & A[u_1(\tilde{y}_1) - u_2(\tilde{y}_2)] \\ & + [X(u_1(\tilde{y}_1),\tilde{y}_1) - X(u_2(\tilde{y}_2),\tilde{y}_2)] \end{aligned}$$

to get

$$\begin{aligned} ||\mathscr{F}u_1 - \mathscr{F}u_2||_0 &\leq ||A||[||u_2-u_1||_0 + ||\tilde{y}_2-\tilde{y}_1||_0] \\ &\quad + \lambda[||u_2-u_1||_0 + 2\cdot||\tilde{y}_2-\tilde{y}_1||_0] \\ &\leq ||u_2-u_1||_0\{||A||(1+\gamma\lambda) + \lambda(1+2\gamma\lambda)\}. \end{aligned}$$

If we now require

$$\alpha = ||A||(1+\gamma\lambda) + \lambda(1+2\gamma\lambda) < 1, \qquad (2.17)^*$$

$\mathscr{F}$ will be a contraction in the supremum norm.

ii) The assertions we want all follow directly from the following general result.

(2.5) Lemma. *Let (u_n) be a sequence of functions on a Banach space* Y *with values on a Banach space* X. *Assume that, for all* n *and* $y \in Y$,

$$||D^j u_n(y)|| \leq 1 \qquad j = 0,1,2,\ldots,k,$$

and that each $D^k u_n$ *is Lipschitz continuous with Lipschitz constant one*. *Assume also that for each* y, *the sequence* $(u_n(y))$ *converges weakly (i.e., in the weak topology on* X) *to a unit vector* $u(y)$. *Then*

a) u *has a Lipschitz continuous* k^{th} *derivative with Lipschitz constant one*.

b) $D^j u_n(y)$ *converges weakly to* $D^j u(y)$* *for all* y *and* $j = 1,2,\ldots,k$.

If X, Y are finite dimensional, all the Banach space technicalities in the statement of the proposition disappear, and the proposition becomes a straightforward consequence of the Arzela-Ascoli Theorem. We postpone the proof for a moment, and instead turn to step IV).

IV) We shall prove the following: Let $x \in X$ with $||x|| \leq 1$ and let $y \in Y$ be arbitrary. Let $(x_1,y_1) = \Psi(x,y)$.

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*This statement may require some interpretation. For each $n,y,D^j u_n(y)$ is a bounded symmetric j-linear map from Y^j to X. What we are asserting is that, for each $y,y_1,\ldots,y_j$, the sequence $(D^j u_n(y)(y_1,\ldots,y_j))$ of elements of X converges in the weak topology on X to $D^j u(y)(y_1,\ldots,y_j)$.

Then

$$||x_1|| \le 1$$

and

$$||x_1-u^*(y_1)|| \le \alpha\cdot||x-u^*(y)||, \tag{2.18}$$

where α is as defined in (2.17). By induction,

$$||x_n-u^*(y_n)|| \le \alpha^n||x-u^*(y)|| \to 0 \quad \text{as} \quad n \to \infty,$$

as asserted.

To prove $||x_1|| \le 1$, we first write

$$x_1 = Ax + X(x,y), \quad \text{so that}$$

$$||x_1|| \le ||A||\cdot||X|| + \lambda \le ||A|| + \lambda \le 1 \quad \text{by} \quad (2.6)$$

To prove (2.18), we essentially have to repeat the estimates made in proving that $\mathscr{F}$ is a contraction. Let $\tilde{y}_1$ be the solution of

$$y_1 = B\tilde{y}_1 + Y(u^*(\tilde{y}_1),\tilde{y}_1).$$

On the other hand, by the definition of y_1 we have

$$y_1 = By + Y(x,y).$$

Subtracting these equations and proceeding exactly as in the derivation of (2.16), we get

$$||\tilde{y}_1-y|| \le \lambda\cdot\gamma\cdot||u^*(y)-x||.$$

Next, we write

$$u^*(y_1) = \mathscr{F}u^*(y_1) = Au^*(y_1) + X(u^*(\tilde{y}_1),\tilde{y}_1)$$

$$x_1 = Ax + X(x,y).$$

Subtracting and making the same estimates as before, we get

$$||x_1-u^*(y_1)|| \le \alpha\cdot||x-u^*(y)||$$

as desired. This completes step IV).

Let us finish the argument by supplying the details for Lemma (2.5).

Proof of Lemma (2.5)

We shall give the argument only for $k = 1$; the generalization to arbitrary k is a straightforward induction argument.

We start by choosing $y_1, y_2 \in Y$ and $\phi \in X^*$ and consider the sequence of real-valued functions of a real variable

$$t \to \phi(u_n(y_1+ty_2)) \equiv \psi_n(t).$$

From the assumptions we have made about the sequence (u_n), it follows that

$$\lim_{n\to\infty} \psi_n(t) = \phi(u(y_1+ty_2)) \equiv \psi(t)$$

for all t, that $\psi_n(t)$ is differentiable, that

$$|\psi_n'(t)| \le ||\phi|| \cdot||y_1|| \quad \text{for all } n, t$$

and that

$$|\psi_n'(t_1) - \psi_n'(t_2)| \le ||\phi||\cdot||y_2||^2|t_1-t_2|$$

for all n, t_1, t_2. By this last inequality and the Arzela-Ascoli Theorem, there exists a subsequence $\psi_{n_j}'(t)$ which converges uniformly on every bounded interval. We shall temporarily denote the limit of this subsequence by $X(t)$. We have

$$\psi_{n_j}(t) = \psi_{n_j}(0) + \int_0^t \psi_{n_j}'(\tau)\, d\tau;$$

hence, passing to the limit $j \to \infty$, we get

$$\psi(t) = \psi(0) + \int_0^t X(\tau)\, d\tau ,$$

which implies that $\psi(t)$ is continuously differentiable and that

$$\psi'(t) = X(t).$$

To see that

$$\lim_{n\to\infty} \psi_n'(t) = \psi'(t)$$

(i.e., that it is not necessary to pass to a subsequence), we note that the argument we have just given shows that <u>any</u> subsequence of $(\psi_n'(t))$ has a subsequence converging to $\psi'(t)$; this implies that the original sequence must converge to this limit.

Since

$$\psi_n'(0) = \phi(Du_n(y_1)(y_2)),$$

we conclude that the sequence

$$Du_n(y_1)(y_2)$$

converges in the weak topology on X^{**} to a limit, which we shall denote by $Du(y_1)(y_2)$; this notation is at this point only suggestive. By passage to a limit from the corresponding property of $Du_n(y_1)(y_2)$, we see that

$$y_2 \mapsto Du_n(y_1)(y_2)$$

is a bounded linear mapping of norm ≤ 1 from Y to X^{**} for each y_1. We denote this linear operator by $Du(y_1)$. Since

$$||Du_n(y_1)(y_2) - Du_n(y_1')(y_2)|| \leq ||y_1 - y_1'||\cdot||y_2||,$$

we have

$$||Du(y_1) - Du(y_1')|| \leq ||y_1 - y_1'||,$$

i.e., the mapping $y \mapsto Du(y)$ is Lipschitz continuous from Y to $L(Y, X^{**})$.

The next step is to prove that

$$u(y_1+y_2) - u(y_1) = \int_0^1 Du(y_1+\tau y_2)(y_2)\, d\tau; \qquad (2.19)$$

this equation together with the norm-continuity of $y \mapsto Du(y)$ will imply that u is (Fréchet)-differentiable. The integral in (2.19) may be understood as a vector-valued Riemann integral. By the first part of our argument,

$$\phi(u(y_1+y_2)) - \phi(u(y_1)) = \int_0^1 \phi(Du(y_1+\tau y_2)(y_2))\, d\tau,$$

for all $\phi \in X^{**}$, and taking Riemann integrals commutes with continuous linear mappings, so that

$$\phi([u(y_1+y_2) - u(y_1) - \int_0^1 Du(y_1+\tau y_2)(y_2)])\, d\tau = 0$$

for all $\phi \in X^*$. Therefore (2.19) is proved.

The situation is now as follows: We have shown that, if we regard u as a mapping into X^{**}, which contains X, then it is Fréchet differentiable with derivative Du. On the other hand, we know that u actually takes values in X and want to conclude that it is differentiable as a mapping into X. This is equivalent to proving that $Du(y_1)(y_2)$ belongs to X for all y_1, y_2. But

$$Du(y_1)(y_2) = \text{norm} \lim_{t \to 0} \frac{u(y_1+ty_2)-u(y_1)}{t};$$

the difference quotients on the right all belong to X, and X is norm closed in X^{**}. Thus, $Du(y_1)(y_2)$ is in X and the proof is complete. $\square$

(2.6) Remarks on the Center Manifold Theorem

1. It may be noted that we seem to have lost some differentiability in passing from Ψ to u^*, since we assumed that Ψ is C^{k+1} and only concluded that u^* is C^k. In fact, however, the u^* we obtain has a Lipschitz continuous k^{th} derivative, and our argument works just as well if we only assume that Ψ has a Lipschitz continuous k^{th} derivative, so in this class of maps, no loss of differentiability occurs. Moreover, if we make the weaker assumption that the k^{th} derivative of Ψ is uniformly continuous on some neighborhood of zero, we can show that the same is true of u^*. (Of course, if X and Y are finite dimensional, continuity on a neighborhood of zero implies uniform continuity on a neighborhood of zero, but this is no longer true if X or Y is infinite dimensional).

2. As C. Pugh has pointed out, if Ψ is infinitely differentiable, the center manifold cannot, in general, be taken to be infinitely differentiable. It is also not true that, if Ψ is analytic there is an analytic center manifold. We shall give a counterexample in the context of equilibrium points of differential equations rather than fixed points of maps; cf. Theorem 2.7 below. This example, due to Lanford, also shows that the center manifold is not unique; cf. Exercise 2.8.

Consider the system of equations:

$$\frac{dy_1}{dt} = - y_2, \quad \frac{dy_2}{dt} = 0, \quad \frac{dx}{dt} = - x + h(y_1), \tag{2.20}$$

where h is analytic near zero and has a second-order zero at zero. We claim that, if h is not analytic in the whole complex plane, there is no function $u(y_1,y_2)$, analytic in a neighborhood of $(0,0)$ and vanishing to second order at $(0,0)$, such that the manifold

$$\{x = u(y_1,y_2)\}$$

is locally invariant under the flow induced by the differential equation near $(0,0)$. To see this, we assume that we have an invariant manifold with

$$u(y_1,y_2) = \sum_{\substack{j_1,j_2 \\ j_1+j_2\geq 1}} c_{j_1 j_2} y_1^{j_1} y_2^{j_2} .$$

Straightforward computation shows that the expansion coefficients c_{j_1,j_2} are uniquely determined by the requirement of invariance and that

$$c_{j_1,j_2} = \frac{(j_1+j_2)!}{(j_1)!} h_{j_1+j_2},$$

where

$$h(y_1) = \sum_{j\geq 2} h_j y_1^j .$$

If the series for h has a finite radius of convergence, the series for $u(o,y_2)$ diverges for all non-zero y_2.

The system of differential equations has nevertheless many infinitely differentiable center manifolds. To construct one, let $\tilde{h}(y_1)$ be a bounded infinitely differentiable function agreeing with h on a neighborhood of zero. Then the manifold defined by

$$u(y_1,y_2) = \int_{-\infty}^{0} e^{\sigma}\tilde{h}(y_1-\sigma y_2)\, d\sigma \qquad (2.21)$$

is easily verified to be globally invariant for the system

$$\frac{dy_1}{dt} = -y_2, \quad \frac{dy_2}{dt} = 0, \quad \frac{dx}{dt} = -x + \tilde{h}(y_1) \tag{2.22}$$

and hence locally invariant at zero for the original system.

(To make the expression for u less mysterious, we sketch its derivation. The equations for y_1, y_2 do not involve x and are trivial to solve explicitly. A function u defining an invariant manifold for the modified system (2.22) must satisfy

$$\frac{d}{dt} u(y_1 - ty_2, y_2) = -u(y_1 - ty_2) + \tilde{h}(y_1 - ty_2)$$

for all t, y_1, y_2. The formula (2.21) for u is obtained by solving this ordinary differential equation with a suitable boundary condition at $t = -\infty$.)

3. Often one wishes to replace the fixed point 0 of Ψ by an invariant manifold V and make spectral hypotheses on a normal bundle of V. We shall need to do this in Section 9. This general case follows the same procedure; details are found in Hirsch-Pugh-Shub [1].

The Center Manifold Theorem for Flows

The center manifold theorem for maps can be used to prove a center manifold theorem for flows. We work with the time t maps of the flow rather than with the vector fields themselves because, in preparation for the Navier Stokes equations, we want to allow the vector field generating the flow to be only densely defined, but since we can often prove that the time t-maps are C^∞ this is a reasonable hypothesis for many partial differential equations (see Section 8A for details).

(2.7) Theorem. Center Manifold Theorem for Flows. Let Z be a Banach space admitting a C^∞ norm* away from 0 and let F_t be a C^0 semiflow defined in a neighborhood of zero for $0 \le t \le \tau$. Assume $F_t(0) = 0$, and that for $t > 0$, $F_t(x)$ is C^{k+1} jointly in t and x. Assume that the spectrum of the linear semigroup $DF_t(0): Z \to Z$ is of the form $e^{t(\sigma_1 \cup \sigma_2)}$ where $e^{t\sigma_1}$ lies on the unit circle (i.e. σ_1 lies on the imaginary axis) and $e^{t\sigma_2}$ lies inside the unit circle a nonzero distance from it, for $t > 0$; i.e. σ_2 is in the left half plane. Let Y be the generalized eigenspace corresponding to the part of the spectrum on the unit circle. Assume $\dim Y = d < \infty$.

Then there exists a neighborhood V of 0 in Z and a C^k submanifold $M \subset V$ of dimension d passing through 0 and tangent to Y at 0 such that

(a) If $x \in M$, $t > 0$ and $F_t(x) \in V$, then $F_t(x) \in M$.

(b) If $t > 0$ and $F_t^n(x)$ remains defined and in V for all $n = 0,1,2,\ldots$, then $F_t^n(x) \to M$ as $n \to \infty$.

This way of formulating the result is the most convenient for it applies to ordinary as well as to partial differential equations, the reason is that we do not need to worry about "unboundedness" of the generator of the flow. Instead we have used a smoothness assumption on the flow.

The center manifold theorem for maps, Theorem 2.1, applies to each F_t, $t > 0$. However, we are claiming that V and M can be chosen independent of t. The basic reason

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*Notice that this assumption on Z was not required above, but it is needed here. The reason will be evident below. Such a Banach space is often called "smooth".

for this is that the maps $\{F_t\}$ commute: $F_s \circ F_t = F_{t+s} = F_t \circ F_s$, where defined. However, this is somewhat oversimplified. In the proof of the center manifold theorem we would require the F_t to remain globally commuting after they have been cut off by the function ϕ. That is, we need to ensure that in the course of proving lemma 2.4, λ can be chosen small (independent of t) and the F_t's are globally defined and commute.

The way to ensure this is to first cut off the F_t in Z outside a ball B in such a way that the F_t are not disturbed in a small ball about 0, $0 \leq t \leq \tau$, and are the identity outside of B. This may be achieved by joint continuity of F_t and use of a C^∞ function f which is one on a neighborhood of 0 and is 0 outside B. Then defining

$$G_t(x) = F_\tau(x) \qquad \text{where} \quad \tau = \int_0^t f(F_s(x))ds, \tag{2.23}$$

it is easy to see that G_t extends to a global semiflow[†] on Z which coincides with F_t, $0 \leq t \leq \tau$ on a neighborhood of zero, and which is the identity outside B. Moreover, G_t remains a C^{k+1} semiflow. (For this to be true we required the smoothness of the norm on Z and that for $t > 0$ F_t has smoothness in t and x jointly[*]).

Now we can rescale and chop off <u>simultaneously</u> the G_t outside B as in the above proof. Since this does not affect F_t on a small neighborhood of zero, we get our result.

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[*]In linear semigroup theory this corresponds to analyticity of the semigroup; it holds for the heat equation for instance. For the Navier Stokes eqations, see Sections 8,9.

[†]See Renz [1] for further details.

(2.8) Exercise. (nonuniqueness of the center manifold for flows). Let $X(x,y) = (-x,y^2)$. Solve the equation $\frac{d(x,y)}{dt} = X(x,y)$ and draw the integral curves.

Show that the flow of X satisfies the conditions of Theorem 2.7 with Y the y-axis. Show that the y-axis is a center manifold for the flow. Show that each integral curve in the lower half plane goes toward the origin as $t \to \infty$ and that the curve becomes parallel to the y-axis as $t \to -\infty$. Show that the curve which is the union of the positive y-axis with any integral curve in the lower half plane is a center manifold for the flow of X. (see Kelley [1], p. 149).

(2.9) Exercise. (Assumes a knowledge of linear semi-group theory).

Consider a Hilbert space H (or a "smooth" Banach space) and let A be the generator of an analytic semigroup. Let $K : H \to H$ be a C^{k+1} mapping and consider the evolution equations

$$\frac{dx}{dt} = Ax + K(x), \quad x(0) = x_0 \tag{2.24}$$

(a) Show that these define a local semiflow $F_t(x)$ on H which is C^{k+1} in (t,x) for $t > 0$. (Hint: Solve the Duhamel integral equation $x(t) = e^{At}x_0 + \int_0^t e^{(t-s)A}K(x(s))ds$ by iteration or the implicit function theorem on a suitable space of maps (references: Segal [1], Marsden [1], Robbin [1]).

(b) Assume $K(0) = 0$, $DK(0) = 0$. Show the existence of invariant manifolds for (2.24) under suitable spectral hypotheses on A by using Theorem 2.7.

SECTION 2A
SOME SPECTRAL THEORY

In this section we recall quickly some relevant results in spectral theory. For details, see Rudin [1] or Dunford-Schwartz [1,2]. Then we go on and use these to prove Theorems 1.3 and 1.4.

Let $T: E \to E$ be a bounded linear operator on a Banach space E and let $\sigma(T)$ denote its spectrum, $\sigma(T) = \{\lambda \in \mathbb{C} | T - \lambda I$ is not invertible on the complexification of $E\}$. Then $\sigma(T)$ is non-empty, is compact, and for $\lambda \in \sigma(T), |\lambda| \leq ||T||$. Let $r(T)$ denote its spectral radius defined by $r(T) = \sup\{|\lambda| \; |\lambda \in \sigma(T)\}$. $r(t)$ is also given by the spectral radius formula:

$$r(T) = \operatorname*{limit}_{n\to\infty} ||T^n||^{1/n} .$$

(The proof is analogous to the formula for the radius of convergence of a power series and can be supplied without difficulty; cf. Rudin [1, p.355].)

The following two lemmas are also not difficult and are proven in the references given:

(2A.1) Lemma. *Let* $f(z) = \sum_0^\infty a_n z^n$ *be an entire function and define* $\tilde{f}(T) = \sum_0^\infty a_n T^n$, *then* $\sigma(\tilde{f}(T)) = f(\sigma(T))$.

(2A.2) Lemma. *Suppose* $\sigma(T) = \sigma_1 \cup \sigma_2$ *where* $d(\sigma_1,\sigma_2) > 0$, *then there are unique T-invariant subspaces* E_1 *and* E_2 *such that* $E = E_1 \oplus E_2$ *and if* $T_i = T|_{E_i}$, *then* $\sigma(T_i) = \sigma_i$. E_i *is called the generalized eigenspace of* σ_i.

Lemma 2A.2 is done as follows: Let γ_1 be a closed curve with σ_1 in its interior and σ_2 in its exterior, then $T_1 = \frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{zI-T}$.

Note that the eigenspace of an eigenvalue λ is not always the same as the generalized eigenspace of λ. In the finite dimensional case, the generalized eigenspace of λ is the subspace corresponding to all the Jordan blocks containing λ in the Jordan canonical form.

(2A.3) Lemma. *Let* T, σ_1, *and* σ_2 *be as in Lemma 2A.2 and assume that* $d(e^{\sigma_1},e^{\sigma_2}) > 0$. *Then for the operator* e^{tT}, *the generalized eigenspace of* e^{tT_i} *is* E_i.

Proof. Write, according to Lemma 2A.2, $E = E_1 \oplus E_2$. Thus,

$$e^{tT}(e_1,e_2) = \sum_0^\infty \frac{t^nT^n}{n!}(e_1,e_2) = \left(\sum_0^\infty \frac{t^nT^n}{n!}e_1, \frac{t^nT^n}{n!}e_2\right)$$

$$= \left(\sum_0^\infty \frac{t^nT_1^n}{n!}e_1, \frac{t^nT_2^n}{n!}e_2\right) = \left(e^{tT_1}e_1, e^{tT_2}e_2\right).$$

From this the result follows easily. □

(2A.4) Lemma. *Let* $T: E \to E$ *be a bounded, linear* operator on a Banach space E. Let r be any number greater than $r(T)$, the spectral radius of T. Then there is a norm $| \; |$ on E equivalent to the original norm such that $|T| \leq r$.

Proof. We know that $r(t)$ is given by $r(T) = \lim ||T^n||^{1/n}$. Therefore, $\sup_{n\geq 0} \frac{||T^n||}{r^n} < \infty$. If we define $|x| = \sup_{n\geq 0} \frac{||T^n(x)||}{r^n}$, we see that $| \; |$ is a norm and that $||x|| \leq |x| \leq \left(\sup_{n\geq 0} \frac{||T^n||}{r^n}\right) ||x||$. Hence $|T(x)| = \sup_{n\geq 0} \frac{||T^{n+1}(x)||}{r^n} = r \sup_{n\geq 0} \frac{||T^{n+1}(x)||}{r^{n+1}} \leq r|x|$. □

(2A.5) Lemma. Let $A: E \to E$ be a bounded operator on E and let $r > \sigma(A)$ (i.e. if $\lambda \in \sigma(A)$, Re $\lambda < r$). Then there is an equivalent norm $| \; |$ on E such that for $t \geq 0$,

$$|e^{tA}| \leq e^{rt}.$$

Proof. Note that (see Lemma 2A.1) e^{rt} is $\geq$ spectral radius of e^{tA}; i.e. $e^{rt} \geq \lim_{n\to\infty} ||e^{ntA}||^{1/n}$. Set

$$|x| = \sup_{\substack{n\geq 0\\ t\geq 0}} \frac{||e^{ntA}(x)||}{e^{rnt}}$$

and proceed as in Lemma 2A.4. □

There is an analogous lemma if $r < \sigma(A)$, giving $|e^{tA}| \geq e^{rt}$. With this machinery we now turn to Theorems 1.3 and 1.4 of Section 1:

(1.3) Theorem. Let $T: E \to E$ be a bounded linear operator. Let $\sigma(T) \subset \{z | \text{Re } z < 0\}$ (resp. $\sigma(T) \supset \{z | \text{Re } z > 0\}$), then the origin is an attracting (resp. repelling) fixed point for the flow $\phi_t = e^{tT}$ of T.

Proof. This is immediate from Lemma 2A.5 for if $\sigma(T) \supset \{z \mid \operatorname{Re} z < 0\}$, there is an $r < 0$ with $\sigma(T) < r$, as $\sigma(T)$ is compact. Thus $|e^{tA}| \leq e^{rt} \to 0$ as $t \to +\infty$. □

Next we prove the first part of Theorem 1.4 from Section 1.

(1.4) Theorem. *Let X be a C^1 vector field on a Banach manifold P and be such that $X(p_0) = 0$. If $\sigma(dX(p_0)) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$, then p_0 is asymptotically stable.*

Proof. We can assume that P is a Banach space E and that $p_0 = 0$. As above, renorm E and find $\varepsilon > 0$ such that $\|e^{tA}\| \leq e^{-\varepsilon t}$, where $A = dX(0)$.

From the local existence theory of ordinary differential equations*, there is a r-ball about 0 for which the time of existence is uniform if the initial condition x_0 lies in this ball. Let

$$R(x) = X(x) - DX(0)\cdot x.$$

Find $r_2 \leq r_1$ such that $\|x\| \leq r_2$ implies $\|R(x)\| \leq \alpha\|x\|$ where $\alpha = \varepsilon/2$.

Let B be the open $r_2/2$ ball about 0. We shall show that if $x_0 \in B$, then the integral curve starting at x_0 remains in B and $\to 0$ exponentially as $t \to +\infty$. This will prove the result.

Let $x(t)$ be an integral curve of X starting at x_0. Suppose $x(t)$ remains in B for $0 \leq t < T$. Then noting that

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*See for instance, Hale [1], Hartman [1], Lang [1], etc.

$$\begin{aligned} x(t) &= x_0 + \int_0^t X(x(s))\, ds \\ &= x_0 + \int_0^t A(x(s)) + R(x(s))\, ds \end{aligned}$$

gives (the Duhamel formula; Exercise 2.9),

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}R(x(s))\, ds$$

and so

$$||x(t)|| \leq e^{-t\varepsilon}x_0 + \alpha \int_0^t e^{-(t-s)\varepsilon}||x(s)||ds.$$

Thus, letting $f(t) = e^{t\varepsilon}||x(t)||$,

$$f(t) \leq x_0 + \alpha \int_0^t f(s)\, ds$$

and so

$$f(t) \leq ||x_0||e^{\alpha t}.$$

(This elementary inequality is usually called Gronwall's inequality cf. Hartman [1].)

Thus

$$||x(t)|| \leq ||x_0||e^{(\alpha-\varepsilon)t} = ||x_0||e^{-\varepsilon t/2}.$$

Hence $x(t) \in B$, $0 \leq t < T$ so $x(t)$ may be indefinitely extended in t and the above estimate holds. It shows $x(t) \to 0$ as $t \to +\infty$. □

The instability part of Theorem 1.4 requires use of the invariant manifold theorems, splitting the spectrum into two parts in the left and right half planes. The above analysis shows that on the space corresponding to the spectrum in the right half plane, p_0 is repelling, so is unstable.

(2A.6) Remarks. Theorem 1.3 is also true by the same proof if T is an unbounded operator, provided we know $\sigma(e^{tT}) = e^{t\sigma(T)}$ which usually holds for decent T, (e.g.: if the spectrum is discrete) but need not always be true (See Hille-Phillips [1]). This remark is important for applications to partial differential equations. Theorem 1.4's proof would require $R(x) = o(||x||)$ which is unrealistic for partial differential equations. However, the following holds:

<u>Assume</u> $\sigma(DF_t(0)) = e^{t\sigma(DX(0))}$ <u>and we have a</u> C^0 <u>flow</u> F_t <u>and each</u> F_t <u>is</u> C^1 <u>with derivative strongly continuous in</u> t <u>(See Section 8A)</u>, <u>then the conclusion of 1.4 is true</u>.

This can be proved as follows: in the notation above, we have, by Taylor's theorem:

$$F_t(x) - 0 = F_t(x) - F_t(0) = DF_t(0)\cdot x + R(t,x)$$

where $R(t,x)$ is the remainder. Now we will have

$$||R(t,x)|| = 0(||x||)e^{-\varepsilon t},$$

as long as x remains in a small neighborhood of 0; this is because $||DF_t(0)|| \leq e^{-2\varepsilon t}$ and hence $||DF_t(x)|| \leq e^{-\varepsilon t}$ for some $\varepsilon > 0$ and x in some neighborhood of 0 $\cdots$ remember $R(t,x) = \int_0^1 \{DF_t(sx)\cdot x - DF_t(0)\cdot x\}\, ds$. Therefore, arguing as in Theorem 1.4, we can conclude that x remains close to 0 and $F_t(x) \to 0$ exponentially as $t \to \infty$. $\square$

(2A.7) <u>Exercise</u>. Let E be a Banach space and $F: E \to E$ a C^1 map with $F(0) = 0$ and the spectrum of $DF(0)$ inside the unit circle. Show that there is a neighborhood U of 0 such that if $x \in U$, $F^{(n)}(x) \to 0$ as $n \to \infty$; i.e. 0 is a stable point of F.

SECTION 2B

THE POINCARÉ MAP

We begin by recalling the definition of the Poincaré map. In doing this, one has to prove that the mapping exists and is differentiable. In fact, one can do this in the context of C^0 flows $F_t(x)$ such that for each t, F_t is C^k, as was the case for the center manifold theorem for flows, but here with the additional assumption that $F_t(x)$ is smooth in t as well for $t > 0$. Again, this is the appropriate hypothesis needed so that the results will be applicable to partial differential equations. However, let us stick with the ordinary differential equation case where F_t is the flow of a C^k vector field X at first.

First of all we recall that a closed orbit γ of a flow F_t on a manifold M is a non-constant integral curve $\gamma(t)$ of X such that $\gamma(t+\tau) = \gamma(t)$ for all $t \in \mathbb{R}$ and some $\tau > 0$. The least such τ is the period of γ. The image of γ is clearly diffeomorphic to a circle.

(2B.1) Definition. Let γ be a closed orbit, let

$m \in \gamma$, say $m = \gamma(0)$ and let S be a local transversal section; i.e. a submanifold of codimension one transverse to γ(i.e. $\gamma'(0)$ is not tangent to S). Let $\mathcal{D} \subset M \times \mathbb{R}$ be the domain (assumed open) on which the flow is defined.

A Poincaré map of γ is a mapping $P: W_0 \to W_1$ where:

(PM 1) $W_0, W_1 \subset S$ are open neighborhoods of $m \in S$, and P is a C^k diffeomorphism;

(PM 2) there is a function $\delta: W_0 \to \mathbb{R}$ such that for all $x \in W_0$, $(x, \tau-\delta(x)) \in \mathcal{D}$ and $P(x) = F(x, \tau-\delta(x))$; and finally,

(PM 3) if $t \in (0, \tau-\delta(x))$, then $F(x,t) \notin W_0$ (see Figure 2B.1).

(2B.2) *Theorem* (*Existence and uniqueness of Poincaré maps*).

(i) *If* X *is a* C^k *vector field on* M, *and* γ *is a closed orbit of* X, *then there exists a Poincaré map of* γ.

(ii) *If* $P: W_0 \to W_1$ *is a Poincaré map of* γ (*in a local transversal section* S *at* $m \in \gamma$) *and* P' *also* (*in* S' *at* $m' \in \gamma$), *then* P *and* P' *are locally conjugate*. *That is, there are open neighborhoods* W_2 *of* $m \in S$, W_2' *of* $m' \in S'$, *and a* C^k *diffeomorphism* $H: W_2 \to W_2'$, *such that* $W_2 \subset W_0 \cap W_1$, $W_2' \subset W_0'$ *and the diagram*

$$\begin{tikzcd} \Theta^{-1}(W_2) \cap W_2 \arrow[r, "\Theta"] \arrow[d, "H"'] & W_2 \cap \Theta W_2 \arrow[d, "H"] \\ W_2' \arrow[r, "\Theta'"] & S' \end{tikzcd}$$

commutes.

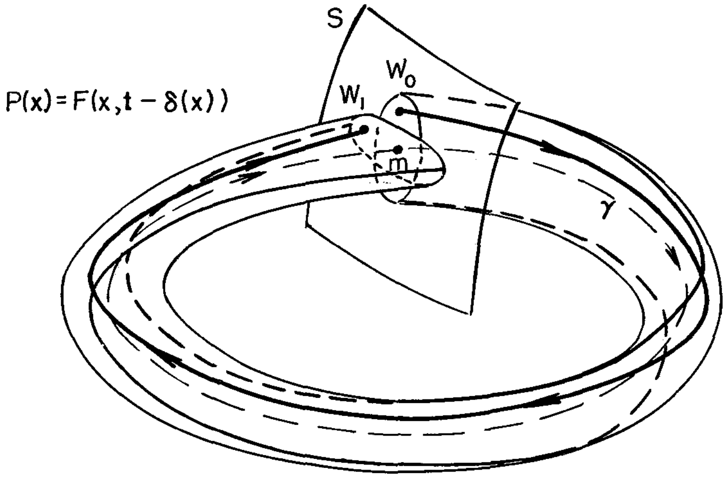

Figure 2B.1

Here is the idea of the proof of existence of P (for further details see Abraham-Marsden [1]). Choose S arbitrarily. By continuity, F_τ is a homeomorphism of a neighborhood U_0 of m to another neighborhood U_2 of m. By assumption, $F_{\tau+t}(x)$ is t-differentiable at $t = 0$ and is transverse to S at $x = m$ and hence also in a neighborhood of m. Therefore, there is a unique number $\delta(x)$ near zero such that $F_{\tau-\delta(x)}(x) \in S$. This is $P(x)$, and by construction P will be as differentiable as F is. The derivative of P at m is seen to be just the projection of the derivative of F_τ on T_mS (this is done below). Hence if F_τ is a diffeomorphism, P will be as well. $\square$

For partial differential equations, F_t is often just a semi-flow i.e. defined for $t \geq 0$ (See Section 8A). In particular, this means F_t is not generally a diffeomorphism.

(For instance if F_t is the flow of the heat equation on L_2, F_t is not surjective). For the Poincaré mapping this means that we will have a P (if $F_t(x)$ is differentiable in t, x for $t > 0$), but P is just a map, not necessarily a diffeomorphism. This is one of the technical reasons why it is important to have the center manifold theorem for maps and not just diffeomorphisms.

As we outlined in Section 1, there is a Hopf theorem for diffeomorphisms and this is to be eventually applied to the Poincaré map P to deduce the existence of an invariant circle for P and hence an invariant torus for F_t.

For partial differential equations, this seems like a dilemma since P need not be a diffeomorphism. However, this can be overcome by a trick: first apply the center manifold theorem to reduce everything to finite dimensions--this does not require diffeomorphisms; then, as proved in Section 8A, (see 8A.9) in finite dimensions F_t and hence P will automatically become (local) diffeomorphisms. Thus the dilemma is only apparent.

Let us now prove the fundamental results concerning the derivative of P so we can relate the spectrum of P to that of F_t. It suffices to do the computation in a Banach space E, and we can let $m = 0$, the origin of E.

We begin by calculating $dP(0)$ in terms of F_t.

(2B.3) Lemma. *Let $F_t: E \to E$ be a C^1 flow on a Banach space. Let 0 be on a closed orbit γ with period $\tau \neq 0$. Let $\frac{\partial F_t}{\partial t}(0)\big|_{t=0} = V$. Let V be the subspace generated by V and let F be a complementary subspace, so that $E = F \oplus V$ and $F_t(x,y) = (F_t^1(x,y), F_t^2(x,y))$. Let $P: F \to F$*

<u>be the Poincaré map associated with</u> γ <u>at the point</u> 0. <u>Then</u> $dP(0) = d_1F_\tau^2(0)$.

<u>Proof</u>. Let $\tau(x)$ be the time at $P(x)$ ($\tau-\delta(x)$ in the above notation). Then by definition of P,

$$P(x) = F^1_{\tau(x)}(x,0)$$

so

$$dP(x) = \frac{\partial F^1_{\tau(x)}}{\partial t}(x,0)d\tau(x) + d_1F^1_{\tau(x)}(x,0).$$

Letting $(x,0) = 0$, we get

$$dP(0) = \frac{\partial F^1}{\partial t}(0)d\tau(0) + d_1F^1_\tau(0).$$

However $\frac{\partial F_\tau}{\partial t}(0) = \left(\frac{\partial F^1_\tau}{\partial t}(0), \frac{\partial F^2_\tau}{\partial t}(0)\right) = (0,V)$ by construction, so $\frac{\partial F^1_\tau}{\partial t}(0) = 0$. Thus, $dP(0) = d_1F^1_\tau(0)$. $\square$

(2B.4) <u>Lemma</u>. $d_2F_\tau(0,0)V = V$.

<u>Proof</u>. $\left.\frac{dF_{\tau+s}}{ds}(0,0)\right|_{s=0} = \left.\frac{dF_t}{dt}(0,0)\right|_{t=\tau} = V$

$$\left.\frac{dF_{\tau+s}}{ds}(0,0)\right|_{s=0} = \left.\frac{dF_\tau \circ F_s}{ds}(0,0)\right|_{s=0} = dF_\tau(0,0)\left.\frac{dF_s}{ds}(0,0)\right|_{s=0}$$

$= dF_\tau(0,0)(0,V) = (d_1F_\tau(0,0)\cdot 0 + d_2F_\tau(0,0)V) = d_2F_\tau(0,0)V$.
So, $d_2F_\tau(0,0)V = V$. $\square$

(2B.5) <u>Lemma</u>. $\sigma(dF_\tau(0,0)) = \sigma(P(0)) \cup \{1\}$. (<u>This is true of the point spectrum, too</u>).

<u>Proof</u>. The matrix of $dF_\tau(0,0)$ is

$$\left(\begin{array}{c|c} dP(0) & 0 \\ \hline * & 1 \end{array}\right)$$

where * indicates some unspecified matrix entry.
Let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma(dF_\tau(0,0))$ iff $dF_\tau(0,0) - \lambda I$ is not

1-1 or is not onto. But

$$(dF_\tau(0,0) - \lambda I)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (dP(0)\alpha,\ *(\alpha) + \beta) + (-\lambda\alpha, -\lambda\beta)$$
$$= (dP(0)\alpha - \lambda\alpha,\ *(\alpha) + \beta(1 - \lambda)). \qquad (2B.1)$$

Clearly, $1 \in \sigma(dF_\tau(0,0))$. Assume $\lambda \neq 1$ and $\lambda \in \sigma(dF_\tau(0,0))$. Then either, there exists (α,β) such that the expression (2B.1) is zero or the map is not onto. The former implies $\lambda \in \sigma(dP(0))$. Assume the latter. Since $\lambda \neq 1$, for any α, we can choose β so that the second component is onto. Therefore, $\lambda \in \sigma(dP(0))$. On the other hand, let $1 \neq \lambda \in \sigma(dP(0))$. If, $dP(0) - \lambda I$ is not onto, then clearly neither is $dF_\tau(0,0) - \lambda I$. Suppose there exists α such that $dP(0)\alpha - \lambda\alpha = 0$. Then choosing $\beta = \frac{*(\alpha)}{\lambda-1}$, we see that $\lambda \in \alpha(dF_\tau(0,0))$. □

Consult Abraham-Marsden [1] Abraham-Robbin[1] or Hartman [1, Section IV. 6, IX.10] for additional details on this and and the associated Floquet theory.

One of the most basic uses of the Poincaré map is in the proof of the following.

(2B.6) Theorem. *Let γ be a closed orbit for F_t with period τ. Let $m \in \gamma$ and suppose $dF_\tau(m)$ has spectrum inside the unit circle except for one point 1 on the unit circle. Then γ is an attracting (stable) closed orbit.*

Proof. By Lemma 2B.5, the condition on the spectrum means that the spectrum of the derivative of the Poincaré map P at m is inside the unit circle. Hence from 2A.7 m is an attracting fixed pojnt for P. It follows from the construction of P that γ is attracting for F_t. □

(2B.7) Exercise.

(a) Give the details in the last step of the above proof.

(b) If, in 2B.6, P has an attracting invariant circle, give the details of the proof that this yields an attracting invariant 2 torus for the flow.