SECTION 4
COMPUTATION OF THE STABILITY CONDITION

Seeing if the Hopf theorem applies in any given situation is a matter of analysis of the spectrum of the linearized equations; i.e. an eigenvalue problem. This procedure is normally straightforward. (For partial differential equations, consult Section 8.)

It is less obvious how to determine the stability of the resulting periodic orbits. We would now like to develop a method which is applicable to concrete examples. In fact we give a specific computational algorithm which is summarized in Section 4A below. (Compare with similar formulas based on Hopf's method discussed in Section 5A.) The results here are derived from McCracken [1].

Reduction to Two Dimensions

We begin by examining the reduction to two dimensions in detail.

Suppose $X : N \to T(N)$ is a C^k vector field, depending smoothly on μ, on a Banach manifold N such that $X_\mu(a(\mu)) = 0$

for all μ, where $a(\mu)$ is a smooth one-parameter family of zeros of X_μ. Suppose that for $\mu < \mu_0$, the spectrum $\sigma(dX_\mu(a(\mu))) \subset \{z \mid \operatorname{Re} z < 0\}$, so that $a(\mu)$ is an attracting fixed point of the flow of X_μ. To decide whether the Hopf Bifurcation Theorem applies, we compute $dX_\mu(a(\mu))$. If two simple, complex conjugate nonzero eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ cross the imaginary axis with nonzero speed at $\mu = \mu_0$ and if the rest of $\sigma(dX_\mu(a(\mu)))$ remains in the left-half plane bounded away from the imaginary axis, then bifurcation to periodic orbits occurs.

However, since unstable periodic orbits are observed in nature only under special conditions (see Section 7), we will be interested in knowing how to decide whether or not the resultant periodic orbits are stable. In order to apply Theorem 3.1, we must reduce the problem to a two dimensional one. We assume that we are working in a chart, i.e., that $N = E$, a Banach space. For notational convenience we also assume that $\mu_0 = 0$ and $a(\mu) = 0$ for all μ. Let $X_\mu = (X_\mu^1, X_\mu^2, X_\mu^3)$ where X_μ^1 and X_μ^2 are coordinates in the eigenspace of $dX_0(0)$ corresponding to the eigenvalues $\lambda(0)$ and $\overline{\lambda(0)}$, and X_μ^3 is a coordinate in a subspace F complementary to this eigenspace. We assume that coordinates in the eigenspace have been chosen so that

$$dX_0(0,0,0) = \begin{pmatrix} 0 & |\lambda(0)| & 0 \\ -|\lambda(0)| & 0 & 0 \\ 0 & 0 & d_3X_3(0,0,0) \end{pmatrix} . \qquad (4.1)$$

This can always be arranged by splitting E into the subspaces corresponding to the splitting of the spectrum of

$dX_0(0)$ into $\{z|\ \mathrm{Re}\ z < 0\} \subset \{\lambda(0), \lambda(0)\}$, as in 2A.2. By the center manifold theorem there is a center manifold for the flow of $X = (X_\mu, 0)$ tangent to the eigenspace of $\lambda(0)$ and $\overline{\lambda(0)}$ and to the μ-axis at the point $(0,0,0,0)$. The center manifold may be represented locally as the graph of a function, that is, as $\{(x_1,x_2,f(x_1,x_2,\mu),\mu)$ for (x_1,x_2,μ) in some neighborhood of $(0,0,0)\}$. Also, $f(0,0,0) = df(0,0,0) = 0$ and the projection map $P(x_1,x_2,f(x_1,x_2,\mu),\mu) = (x_1,x_2,\mu)$ is a local chart for the center manifold. In a neighborhood of the origin X is tangent to the center manifold because the center manifold is locally invariant under the flow of X. We consider the push forward of X: $\hat{X}(x_1,x_2,\mu) =$ $TP\circ X(x_1,x_2,f(x_1,x_2,\mu),\mu) = (X^1_\mu(x_1,x_2,f(x_1,x_2,\mu)),$ $X^2_\mu(x_1,x_2,f(x_1,x_2,\mu)),0)$ by linearity of P. If we let $\hat{X}_\mu(x_1,x_2) = (X^1(x_1,x_2,f(x_1,x_2,\mu)),\ X^2_\mu(x_1,x_2,f(x_1,x_2,\mu)))$, then $\hat{X}_\mu$ is a smooth one-parameter family of vector fields on R^2 such that $\hat{X}_\mu(0,0) = 0$ for all μ. We will show that $\hat{X}_\mu$ satisfies the conditions (except, of course, the stability condition) of the Hopf Bifurcation Theorem. If ϕ_t and $\hat{\phi}_t$ are the flows of X and $\hat{X}$ respectively, then $P\circ\phi_t = \hat{\phi}_t\circ P$ for points on the center manifold. Therefore, if the resultant closed orbits of $\hat{\phi}_t$ are not attracting, those of ϕ_t will not be either. We will also show that if the origin is a vague attractor for $\hat{\phi}_t$ at $\mu = 0$, then the closed orbits of ϕ_t are attracting.

Since the center manifold has the property that it contains all the local recurrence of ϕ_t, the points $(0,0,0,\mu)$ are on it for small μ and so $f(0,0,\mu) = 0$ for small μ. Thus, $\hat{X}_\mu(0,0) = (X^1(0,0,f(0,0,\mu)),\ X^2_\mu(0,0,f(0,0,\mu))) =$ $(X^1(0,0,0),\ X^2(0,0,0)) = 0$. $P\circ X = \hat{X}\circ P$ on the center manifold, so $P \circ dX = d\hat{X} \circ P$ for vectors tangent to the center manifold. A typical tangent vector to the center manifold has the form $(u,v,d_1 f(x_1,x_2,\mu)u + d_2 f(x_1,x_2,\mu)v + d_3 f(x_1,x_2,\mu)w,w)$, where $(x_1,x_2,f(x_1,x_2,\mu),\mu)$ is the base point of the vector. Because we wish to calculate $\sigma(d\hat{X}_\mu(0,0))$, we will be interested in the case $w = 0$. Now $P \circ dX(0,0,0,\mu)(u,v,d_1 f(0,0,\mu)u + d_2 f(0,0,\mu)v,0) = d\hat{X}(0,0,\mu)(u,v,0)$. That is, $dX^i_\mu(0,0,0)(u,v,d_1 f(0,0,\mu)u + d_2 f(0,0,\mu)v) = dX^i_\mu(0,0)(u,v)$ for $i = 1,2$. Let $\lambda \in \sigma(d\hat{X}_\mu(0,0))$. Since $d\hat{X}_\mu(0,0)$ is a two-by-two matrix, λ is an eigenvalue and there is a complex vector (u,v) such that $d\hat{X}_\mu(0,0)(u,v) = (\lambda u,\lambda v)$. We will show that λ is a eigenvalue of $d\hat{X}_\mu(0,0,0)$ and $(u,v,d_1 f(0,0,\mu)u + d_2 f(0,0,\mu)v)$ is an eigenvector. Because X is tangent to the center manifold,

$$X^3(x_1,x_2,f(x_1,x_2,\mu),\mu) = d_1 f(x_1,x_2,\mu)X^1(x_1,x_2,f(x_1,x_2,\mu),\mu) + d_2 f(x_1,x_2,\mu)X^2(x_1,x_2,f(x_1,x_2,\mu),\mu).$$

Therefore,

$$\begin{aligned} d_1 X^3(x_1,x_2,f(x_1,x_2,\mu),\mu)u &+ d_3 X^3(x_1,x_2,\mu),\mu) \circ d_1 f(x_1,x_2,\mu)u \\ &= d_1 d_1 f(x_1,x_2,\mu) \circ X^1(x_1,x_2,f(x_1,x_2,\mu),\mu)u \\ &+ d_1 f(x_1,x_2,\mu) \circ d_1 X^1(x_1,x_2,f(x_1,x_2,\mu),\mu)u \\ &+ d_1 f(x_1,x_2,\mu) \circ d_3 X^1(x_1,x_2,f(x_1,x_2,\mu),\mu) \circ d_1 f(x_1,x_2,\mu)u \\ &+ d_1 d_2 f(x_1,x_2,\mu)X^2(x_1,x_2,f(x_1,x_2,\mu),\mu)u \\ &+ d_2 f(x_1,x_2,\mu) \circ d_1 X^2(x_1,x_2,f(x_1,x_2,\mu),\mu)u \\ &+ d_2 f(x_1,x_2,\mu) \circ d_3 X^2(x_1,x_2,f(x_1,x_2,\mu),\mu) \circ d_1 f(x_1,x_2,\mu)u \end{aligned}$$

and

$$d_2X^3(x_1,x_2,f(x_1,x_2,f(x_1,x_2,\mu),\mu)v$$
$$+ d_3X^3(x_1,x_2,f(x_1,x_2,\mu),\mu)\circ d_2f(x_1,x_2,\mu)v$$
$$= d_2d_1f(x_1,x_2,\mu)\circ X^1(x_1,x_2,f(x_1,x_2,\mu),\mu)v$$
$$+ d_1f(x_1,x_2,\mu)\circ d_2X^1(x_1,x_2,f(x_1,x_2,\mu),\mu)v +$$
$$+ d_1f(x_1,x_2,\mu)\circ d_3X^1(x_1,x_2,f(x_1,x_2,\mu),\mu)\circ d_2f(x_1,x_2,\mu)v$$
$$+ d_2d_2f(x_1,x_2,\mu)X^2(x_1,x_2,f(x_1,x_2,\mu),\mu)v$$
$$+ d_2f(x_1,x_2,\mu)\circ d_2X^2(x_1,x_2,f(x_1,x_2,\mu),\mu)v$$
$$+ d_2f(x_1,x_2,\mu)\circ d_3X^2(x_1,x_2,f(x_1,x_2,\mu),\mu)\circ d_2f(x_1,x_2,\mu)v.$$

At the point $(0,0,0,\mu)$, $X^1 = X^2 = 0$ and so we get

$$dX^3(0,0,0,\mu)(u,v,d_1f(0,0,\mu)u + d_2f(0,0,\mu)v)$$
$$= d_1X^3(0,0,0,\mu)u + d_2X^3(0,0,0,\mu)v + d_3X^3(0,0,0,\mu)\circ d_1f(0,0,\mu)u$$
$$+ d_3X^3(0,0,0,\mu)\circ d_2f(0,0,\mu)v$$
$$= d_1f(0,0,\mu)\circ(d_1X^1(0,0,0,\mu)u + d_2X^1(0,0,0,\mu)v$$
$$+ d_3X^1(0,0,0,\mu)\circ d_1f(0,0,\mu)u + d_3X^1(0,0,0,\mu)\circ d_2f(0,0,\mu)v)$$
$$+ d_2f(0,0,\mu)\circ(d_1X^2(0,0,0,\mu)u + d_2X^2(0,0,0,\mu)v$$
$$+ d_3X^2(0,0,0,\mu)\circ d_1f(0,0,\mu)u + d_3X^2(0,0,0,\mu)\circ d_2f(0,0,\mu)v)$$
$$= d_1f(0,0,\mu)\lambda u + d_2f(0,0,\mu)\lambda v$$
$$= \lambda(d_1f(0,0,\mu)u + d_2f(0,0,\mu)v)$$

by the assumption that (u,v) is an eigenvector of $d\hat{X}_\mu(0,0)$ with eigenvalue λ.

When $\mu = 0$, $df = 0$ and we have that

$$d\hat{X}_0(0,0) = \begin{pmatrix} 0 & |\lambda(0)| \\ -|\lambda(0)| & 0 \end{pmatrix} .$$

The eigenvalues of $d\hat{X}_\mu(0,0)$ are continuous in μ because they are roots of a quadratic polynomial. Let these roots be $\alpha_1(\mu)$ and $\alpha_2(\mu)$ so that $\alpha_1(0) = \lambda(0)$ and $\alpha_2(0) = \overline{\lambda(0)}$. Because $\alpha_1(\mu)$ and $\alpha_2(\mu) \in \sigma(dX_\mu(0,0,0))$, if $\alpha_1(\mu) \neq \lambda(\mu)$ and $\alpha_2(\mu) \neq \overline{\lambda(\mu)}$, then $\operatorname{Re} \alpha_i(\mu)$ would be bounded away from zero for small μ. Since this is not true, $\alpha_1(\mu) = \lambda(\mu)$ and $\alpha_2(\mu) = \overline{\lambda(\mu)}$. Furthermore, since $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ are simple eigenvalues of $\sigma(dX_\mu(0,0,0))$, we must have that the center manifold is tangent to the eigenspace of $\{\lambda(\mu),\overline{\lambda(\mu)}\}$ at the point $(0,0,0,\mu)$.

We show now that if $V'''(0) < 0$ for $\hat{X}$, then the closed orbits of ϕ_t are attracting. The map $Q(x_1,x_2,x_3,\mu) = (x_1,x_2,x_3-f(x_1,x_2,\mu),\mu)$ is a diffeomorphism from a neighborhood $\mathscr{U}$ of $(0,0,0,0)$ onto a neighborhood V of $(0,0,0,0)$ where we have chosen $\mathscr{U}$ small enough so that X is tangent to the center manifold M for $(x_1,x_2,f(x_1,x_2,\mu),\mu) \in \mathscr{U}$. Clearly $Q|_M = P$, $\tilde{X}|_{\{x_3=0\}} = \hat{X}$, and $\tilde{\phi}_t\Big|_{\{x_3=0\}} = \hat{\phi}_t$. Therefore, we are immediately reduced to the case of Y_μ a vector field on $R^2 \oplus F$ where R^2 is invariant under Y_μ and Y_μ satisfies the conditions for Hopf Bifurcation with R^2 being the eigenspace of $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ at $(0,0,0,\mu)$. The center manifold for y is $\{(x_1,x_2,0,\mu)\}$. Assume that $V'''(0) < 0$ for $Y = Y_{\{x_3=0\}}$ and let the point $(x_1,0,0,\mu(x_1))$ be on a closed orbit of the flow ϕ_t of Y. Because R^2 is invariant,

$$d\phi_{T(x_1)}(x_1,0,\mu(x_1)) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & d_3\phi^3_{T(x_1)}(x_1,0,0,\mu(x_1)) \end{pmatrix}.$$

By assumption, $(d_3\phi^3_{T(0)}(0,0,0,0)) = \left(e^{T(0)d_3X^3(0,0,0)}\right) = e^{\sigma(T(0)d_3X^3(0,0,0))}$ is inside the unit circle. By continuity, so is $\sigma(d_3\phi^3_{T(x_1)}(x_1,0,0,\mu(x_1)))$. Since $V'''(0) < 0$, the eigenvalues of the Poincaré map in R^2 have absolute value less than 1, so all the eigenvalues of the Poincaré map are inside the unit circle and so the orbit is attracting (see Section 2B).

Summarizing: We have shown that the stability problem for the closed orbits of the flow of X_μ is the same as that for the closed orbits of the flow of $\hat{X}_\mu$, where $\hat{X}_\mu(x_1,x_2) = (X^1_\mu(x_1,x_2,f(x_1,x_2,\mu)),\ X^2_\mu(x_1,x_2,f(x_1,x_2,\mu)))$. Coordinates are chosen so that x_1,x_2 are coordinates in the eigenspace of $dX_0(0,0,0)$ and the third component is in a complementary subspace F. The set $\{(x_1,x_2,f(x_1,x_2,\mu),\mu)$ for (x_1,x_2,μ) in a neighborhood of $(0,0,0)\}$ is the center manifold.

Outline of the Stability Calculation

From the proof of Theorem 4.5, we know that the closed orbits of $\hat{X}_\mu$ will be attracting if $V'''(0) < 0$ (or more generally, see Section 3B, if the first nonzero derivative of V at the origin is negative). The derivatives of V at $(0,0)$ can be computed from those of X_0 at $(0,0,0)$. We do this in two steps. First we compute $V'''(0)$ from the derivatives of $\hat{X}_0$, the vector field pushed to the center manifold, at $(0,0)$ using the equation:

$$V(x_1) = \int_0^{T(x_1)} \hat{X}^1(a_t(x_1,0),b_t(x_1,0))dt \qquad (4.2)$$

where (a_t,b_t) is the flow of $\hat{X}$. (Note that in the two-dimensional case, $X = \hat{X}$.) Then we compute the derivatives of $\hat{X}_0$ at $(0,0)$ from those of X_0 at $(0,0,0)$. Since $\hat{X}_\mu(x_1,x_2) = (X_\mu^1(x_1,x_2,f(x_1,x_2,\mu)),\ X_\mu^2(x_1,x_2,f(x_1,x_2,\mu)))$, what we need to know is the derivatives of f at the point $(0,0,0)$. We can find these using the local invariance of the center manifold under the flow of X. We use the equation (see page 107):

$$\begin{aligned} X^3(x_1,x_2,f(x_1,x_2,\mu)) &= d_1 f(x_1,x_2,\mu)\circ X^1(x_1,x_2,f(x_1,x_2,\mu)) \\ &\quad + d_2 f(x_1,x_2,\mu)\circ X^2(x_1,x_2,f(x_1,x_2,\mu)). \end{aligned} \tag{4.3}$$

Calculation of $V'''(0)$ in Terms of $\hat{X}$

We now calculate $V'''(0)$ from the derivatives of $\hat{X}_0$ at $(0,0)$ using (4.2). We assume that coordinates have been chosen so that

$$d\hat{X}_0(0,0) = \begin{pmatrix} \dfrac{\partial \hat{X}_0^1}{\partial x_1}(0,0,0) & \dfrac{\partial \hat{X}_0^1}{\partial x_2}(0,0,0) \\ \dfrac{\partial \hat{X}_0^2}{\partial x_1}(0,0,0) & \dfrac{\partial \hat{X}_0^2}{\partial x_2}(0,0,0) \end{pmatrix} = \begin{pmatrix} 0 & |\lambda(0)| \\ -|\lambda(0)| & 0 \end{pmatrix}. \tag{4.4}$$

This change of variables is not necessary, but it simplifies the computations considerably and, although our method for finding $V'''(0)$ will work if the change of variable has not been made, our formula will not be correct in that case.

From (4.2) we see that

$$\begin{aligned} V'(x_1) &= \int_0^{T(x_1)} \frac{d}{dx_1}\,[\hat{X}^1(a_t(x_1,0),b_t(x_1,0))]\,dt \\ &\quad + T'(x_1)\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0)) \end{aligned}$$

$$V''(x_1) = \int_0^{T(x_1)} \frac{d^2}{dx_1^2} [\hat{X}^1(a_t(x_1,0),b_t(x_1,0))]dt$$

$$+ T'(x_1) \frac{d}{dx_1} [\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0))]$$

$$+ T''(x_1)\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0))$$

$$+ T'(x_1) \frac{d}{dx_1} [\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0))].$$

Using the chain rule, we get:

$$V''(x_1) = \int_0^{T(x_1)} \frac{d^2}{dx_1^2} [\hat{X}^1(a_t(x_1,0),b_t(x_1,0))]dt$$

$$+ T'(x_1)\left(\frac{\partial\hat{X}^1}{\partial a}\frac{\partial a_t}{\partial x_1}\bigg|_{(T(x_1),x_1,0)} + \frac{\partial\hat{X}^1}{\partial b}\frac{\partial b_t}{\partial x_1}\bigg|_{(T(x_1),x_1,0)}\right)$$

$$+ T''(x_1)\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0))$$

$$+ T'(x_1)\left(T'(x_1)\frac{\partial\hat{X}_1}{\partial a}\frac{\partial a_t}{\partial t}\bigg|_{(T(x_1),x_1,0)}\right.$$

$$+ \frac{\partial\hat{X}^1}{\partial a}\frac{\partial a_t}{\partial x_1}\bigg|_{(T(x_1),x_1,0)} + T'(x_1)\frac{\partial\hat{X}^1}{\partial b}\frac{\partial b_t}{\partial t}\bigg|_{(T(x_1),x_1,0)}$$

$$\left.+ \frac{\partial\hat{X}^1}{\partial b}\frac{\partial b_t}{\partial x_1}\bigg|_{(T(x_1),x_1,0)}\right) + T'(x_1)^2\left(\frac{\partial\hat{X}^1}{\partial a}\frac{\partial a_t}{\partial t}\;(T(x_1),x_1,0)\right.$$

$$\left.+ \frac{\partial\hat{X}^1}{\partial b}\frac{\partial b_t}{\partial t}\bigg|_{(T(x_1),x_1,0)}\right) + T''(x_1)\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0)).$$

Differentiating once more,

$$V'''(x_1) = \int_0^{T(x_1)} \frac{d^3}{dx_1^3} [\hat{X}^1(a_t(x_1,0),b_t(x_1,0))]dt$$

$$+ T'(x_1) \frac{d^2}{dx_1^2} [\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0)]$$

$$+ 2T''(x_1)\left\{\frac{\partial \hat{X}^1}{\partial a}\frac{\partial a_t}{\partial x_1}\Bigg|_{(T(x_1),x_1,0)} + \frac{\partial \hat{X}}{\partial b}\frac{\partial b_t}{\partial x_1}\Bigg|_{(T(x_1),x_1,0)}\right\}$$

$$+ 2T'(x_1)\Bigg[\frac{\partial^2 \hat{X}^1}{\partial a^2}\frac{\partial a_t}{\partial x_1}\left\{\frac{\partial a_t}{\partial t}T'(x_1) + \frac{\partial a_t}{\partial x_1}\right\}$$

$$+ \frac{\partial^2 \hat{X}^1}{\partial a \partial b}\frac{\partial a_t}{\partial x_1}\left\{\frac{\partial b_t}{\partial t}T'(x_1) + \frac{\partial b_t}{\partial x_1}\right\} + \frac{\partial \hat{X}^1}{\partial a}\left\{\frac{\partial^2 a_t}{\partial t \partial x_1}T'(x_1) + \frac{\partial^2 a_t}{\partial x_1^2}\right\}$$

$$+ \frac{\partial^2 \hat{X}^1}{\partial b^2}\frac{\partial b_t}{\partial x_1}\left\{\frac{\partial b_t}{\partial t}T'(x_1) + \frac{\partial b_t}{\partial x_1}\right\}$$

$$+ \frac{\partial \hat{X}^1}{\partial b}\left\{\frac{\partial^2 b_t}{\partial t \partial x_1}T'(x_1) + \frac{\partial^2 b_t}{\partial x_1^2}\right\}\Bigg]\Bigg|_{(T(x_1),x_1,0)}$$

$$+ 2T'(x_1)T''(x_1)\left\{\frac{\partial \hat{X}^1}{\partial a}\frac{\partial a_t}{\partial t} + \frac{\partial \hat{X}^1}{\partial b}\frac{\partial b_t}{\partial t}\right\}\Bigg|_{(T(x_1),x_1,0)}$$

$$+ T'(x_1)^2\Bigg[\frac{\partial^2 \hat{X}^1}{\partial a^2}\frac{\partial a_t}{\partial t}\left\{\frac{\partial a_t}{\partial t}T'(x_1) + \frac{\partial a_t}{\partial x_1}\right\}$$

$$+ \frac{\partial^2 \hat{X}^1}{\partial a \partial b}\frac{\partial a_t}{\partial t}\left\{\frac{\partial b_t}{\partial t}T'(x_1) + \frac{\partial b_t}{\partial x_1}\right\} + \frac{\partial \hat{X}^1}{\partial a}\left\{\frac{\partial^2 a_t}{\partial t^2}T'(x_1) + \frac{\partial^2 a_t}{\partial x_1 \partial t}\right\}$$

$$+ \frac{\partial^2 \hat{X}^1}{\partial b^2}\frac{\partial b_t}{\partial t}\left\{\frac{\partial b_t}{\partial t}T'(x_1) + \frac{\partial b_t}{\partial x_1}\right\}$$

$$+ \frac{\partial \hat{X}^1}{\partial b}\left\{\frac{\partial^2 b_t}{\partial t^2}T'(x_1) + \frac{\partial^2 b_t}{\partial x_1 \partial t}\right\}\Bigg]\Bigg|_{(T(x_1),x_1,0)}$$

$$+ T'''(x_1)\hat{X}^1(a_{T(x_1)}(x_1,0),b_{T(x_1)}(x_1,0))$$

$$+ T''(x_1)\left[\frac{\partial\hat{X}^1}{\partial a}\frac{\partial a_t}{\partial t}T'(x_1) + \frac{\partial\hat{X}^1}{\partial a}\frac{\partial a_t}{\partial x_1} + \frac{\partial\hat{X}^1}{\partial b}\frac{\partial b_t}{\partial t}T'(x_1) + \frac{\partial\hat{X}^1}{\partial b}\frac{\partial b_t}{\partial x_1}\right]\Bigg|_{(T(x_1),x_1,0)} .$$

In the case $x_1 = 0$, we can considerably simplify this equation. We know the following about the point $(0,0)$:

$$a_t(0,0) = b_t(0,0) = 0 \quad \text{for all} \quad t \tag{4.5}$$

$$\frac{\partial a_t}{\partial t}(0,0) = \frac{\partial b_t}{\partial t}(0,0) = 0 \tag{4.6}$$

$$d\hat{X}(a_t(0,0),\ b_t(0,0)) = d\hat{X}(0,0) = \begin{pmatrix} 0 & |\lambda(0)| \\ -|\lambda(0)| & 0 \end{pmatrix} \tag{4.7}$$

$$d\hat{\phi}_t(0,0) = e^{t d\hat{X}(0,0)} = \begin{pmatrix} \cos|\lambda(0)|t & \sin|\lambda(0)|t \\ -\sin|\lambda(0)|t & \cos|\lambda(0)|t \end{pmatrix} \tag{4.8}$$

$$T(0) = 2\pi/|\lambda(0)| \tag{4.9}$$

and

$$T'(0) = 0. \tag{4.10}$$

<u>Proof of (4.10)</u>. Let $S(x_1) = T(x_1,\mu(x_1))$. Then $S'(0) = 0$ because given small $x > 0$, there is a small $y < 0$ such that $S(x) = S(y)$. Thus, $\frac{S(x) - S(0)}{x}$ and $\frac{S(y) - S(0)}{y}$ have opposite signs. Choosing $x_n \downarrow 0$, we get the result. $S'(0) = \frac{\partial T}{\partial x_1}(0,0) + \mu'(0)\frac{\partial T}{\partial \mu}(0,0)$. But $\mu'(0) = 0$, as was shown in the Proof of Theorem 3.1 (see p. 65).

Therefore, $V'''(0) = \int_0^{2\pi/|\lambda(0)|} \frac{\partial^3\hat{X}^1}{\partial x_1^3}(a_t(x_1,0),b_t(x_1,0))dt.$

We now evaluate $\left.\frac{d^3X_1}{dx_1^3}\right|_{a_t(0,0),b_t(0,0)}$ and get:

$$V'''(0) = \int_0^{2\pi/|\lambda(0)|} \Bigg[\frac{\partial^3 \hat{X}}{\partial a^3}(0,0)\cos^3|\lambda(0)|t - \frac{\partial^3 \hat{X}}{\partial b^3}(0,0)\sin^3|\lambda(0)|t$$

$$- 3\frac{\partial^3 \hat{X}^1}{\partial a^2 \partial b}(0,0)\cos^2|\lambda(0)|t\ \sin|\lambda(0)|t$$

$$+ 3\frac{\partial^3 \hat{X}^1}{\partial a \partial b^2}(0,0)\cos|\lambda(0)|t\ \sin^2|\lambda(0)|t$$

$$+ 3\frac{\partial^2 \hat{X}^1}{\partial a^2}(0,0)\ \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t$$

$$- 3\frac{\partial^2 \hat{X}^1}{\partial b^2}(0,0)\ \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t$$

$$+ 3\frac{\partial^2 \hat{X}^1}{\partial a \partial b}(0,0)\Bigg\{\frac{\partial^2 b_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t$$

$$- \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t\Bigg\} + |\lambda(0)|\ \frac{\partial^3 b_t}{\partial x_1^3}(0,0)\Bigg]dt$$

$$= \int_0^{2\pi/|\lambda(0)|} \Bigg[3\frac{\partial^2 \hat{X}^1}{\partial a^2}(0,0)\ \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t$$

$$- 3\frac{\partial^2 \hat{X}^1}{\partial b^2}(0,0)\ \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t$$

$$+ 3\frac{\partial^2 \hat{X}^1}{\partial a \partial b}(0,0)\Bigg\{\frac{\partial^2 b_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t - \frac{\partial^2 a_t}{\partial x_1^2}\sin|\lambda(0)|t\Bigg\}$$

$$+ |\lambda(0)|\ \frac{\partial^3 b_t}{\partial x_1^3}(0,0)\Bigg]dt.$$

In order to get a formula for $V'''(0)$ depending only on the derivatives of $\hat{X}$ at the origin, we must evaluate the derivatives of the flow (e.g., $\frac{\partial^3 b_t}{\partial x_1^3}(0,0)$) from those of $\hat{X}$

at $(0,0)$. This can be done because the origin is a fixed point of the flow of $\hat{X}$. Because this idea is important, we state it in a more general case.

(4.1) Theorem. Let X be a C^k vector field on R^n such that $X(0) = 0$ (or $X(p) = 0$). Let Φ_t be the time t map of the flow of X. The first three (or, the first j) derivatives of Φ_t at 0 can be calculated from the first three (or, the first j) derivatives of X at 0.

Proof. Consider $\dfrac{\partial \Phi_t^i}{\partial x_j}(0)$:

$$\frac{\partial}{\partial t}\frac{\partial \Phi_t^i}{\partial x_j}(0) = \frac{\partial}{\partial x_j}\frac{\partial \Phi_t^i}{\partial t}(0) = \frac{\partial}{\partial x_j} X^i \circ \Phi_t(0) = \frac{\partial X^i}{\partial x_k} \circ \Phi_t(0)\, \frac{\partial \Phi_t^k}{\partial x_j}(0)$$

$$= \frac{\partial X^i}{\partial x_k}(0)\, \frac{\partial \Phi_t^k}{\partial x_j}(0)$$

because $\Phi_t(0) = 0$. Furthermore, $\dfrac{\partial \Phi_0^i}{\partial x_j}(0) = \delta_{ij}$ because $\Phi_0(x) = x$ for all x. So $d\Phi_t(0)$ satisfies the differential equation $\dfrac{\partial}{\partial t}(d\Phi_t(0)) = dX(0)$. $d\Phi_t(0)$ and $d\Phi_0(0) = I$. Thus, $d\Phi_t(0) = e^{tdX(0)}$.

Consider $\dfrac{\partial^2 \Phi_t^i}{\partial x_j \partial x_k}(0)$:

$$\frac{\partial}{\partial t}\frac{\partial^2 \Phi_t^i}{\partial x_j \partial x_k}(0) = \frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial \Phi_t^i}{\partial t}(0) = \frac{\partial^2}{\partial x_j \partial x_k} X^i \circ \Phi_t(0)$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial X^i}{\partial x_\ell} \circ \Phi_t\, \frac{\partial \Phi_t^\ell}{\partial x_k}\right)(0) = \frac{\partial^2 X^i}{\partial x_p \partial x_\ell} \circ \Phi_t\, \frac{\partial \Phi_t^p}{\partial x_j}\, \frac{\partial \Phi_t^\ell}{\partial x_k}$$

$$+ \frac{\partial X^i}{\partial x_\ell} \circ \Phi_t\, \frac{\partial^2 \Phi_t^\ell}{\partial x_j \partial x_k}(0)$$

$$= \frac{\partial^2 X^i}{\partial x_p \partial x_\ell}(0)\, \frac{\partial \Phi_t^p}{\partial x_j}(0)\, \frac{\partial \Phi_t^\ell}{\partial x_k}(0) + \frac{\partial X^i}{\partial x_\ell}(0)\, \frac{\partial^2 \Phi_t^\ell}{\partial x_j \partial x_k}(0).$$

Furthermore, $\frac{\partial^2 \Phi_0^i}{\partial x_j \partial x_k}(0) = 0$. We get the differential equation:

$$\frac{\partial^2 \Phi_t^i}{\partial x_j \partial x_k}(0) = d^2X(0)\left(\frac{\partial \Phi_t}{\partial x_j}(0), \frac{\partial \Phi_t}{\partial x_k}(0)\right) + dX(0) \cdot \frac{\partial^2 \Phi_t}{\partial x_j \partial x_k} \cdot$$

The solution is:

$$\frac{\partial^2 \Phi_t}{\partial x_j \partial x_k}(0) = e^{tdX(0)} \int_0^t e^{-sdX(0)} d^2X(0)\left(\frac{\partial \Phi_s}{\partial x_j}(0), \frac{\partial \Phi_s}{\partial x_k}(0)\right) ds$$
$$+ e^{tdX(0)} \frac{\partial^2 \Phi_0}{\partial x_j \partial x_k}(0).$$

$$\frac{\partial^2 \Phi_t}{\partial x_j \partial x_k}(0) = e^{tdX(0)} \int_0^t e^{-sdX(0)} d^2X(0)\left(\frac{\partial \Phi_s}{\partial x_j}(0), \frac{\partial \Phi_s}{\partial x_k}(0)\right) ds.$$

Finally consider $\frac{\partial^3 \Phi_t^i}{\partial x_j \partial x_k \partial x_h}(0)$:

$$\frac{\partial}{\partial t}\left(\frac{\partial^3 \Phi_t^i}{\partial x_j \partial x_k \partial x_h}\right)(0) = \frac{\partial^3}{\partial x_j \partial x_k \partial x_h}\left(\frac{\partial \Phi_t^i}{\partial t}\right)(0) = \frac{\partial^3}{\partial x_j \partial x_k \partial x_h} X^i \circ \Phi_t(0)$$

$$= \frac{\partial}{\partial x_h}\left(\frac{\partial^2 X^i}{\partial x_p \partial x_\ell} \circ \Phi_t \frac{\partial \Phi_t^p}{\partial x_j} \frac{\partial \Phi_t^\ell}{\partial x_k} + \frac{\partial X^i}{\partial x_\ell} \circ \Phi_t \frac{\partial^2 \Phi_t^\ell}{\partial x_j \partial x_k}\right)(0)$$

$$= \left[\frac{\partial^3 X^i}{\partial x_p \partial x_\ell \partial x_q} \circ \Phi_t \frac{\partial \Phi_t^q}{\partial x_h} \frac{\partial \Phi_t^p}{\partial x_j} \frac{\partial \Phi_t^\ell}{\partial x_k}\right.$$

$$+ \frac{\partial^2 X^i}{\partial x_p \partial x_\ell} \circ \Phi_t \left(\frac{\partial^2 \Phi_t^p}{\partial x_j \partial x_h} \frac{\partial \Phi_t^\ell}{\partial x_k} + \frac{\partial \Phi_t^p}{\partial x_j} \frac{\partial^2 \Phi_t^\ell}{\partial x_k \partial x_h}\right)$$

$$\left. + \frac{\partial^2 X^i}{\partial x_p \partial x_\ell} \circ \Phi_t \frac{\partial \Phi_t^p}{\partial x_h} \frac{\partial^2 \Phi_t^\ell}{\partial x_j \partial x_k} + \frac{\partial X^i}{\partial x_\ell} \circ \Phi_t \frac{\partial^3 \Phi_t^\ell}{\partial x_j \partial x_k \partial x_h}\right](0)$$

$$= \frac{\partial^3 X^i}{\partial x_p \partial x_\ell \partial x_q}(0) \frac{\partial \Phi_t^q}{\partial x_h}(0) \frac{\partial \Phi_t^p}{\partial x_j}(0) \frac{\partial \Phi_t^\ell}{\partial x_k}(0)$$

$$+ \frac{\partial^2 X^i}{\partial x_p \partial x_\ell}(0) \left[\frac{\partial^2 \Phi_t^p}{\partial x_j \partial x_h}(0) \frac{\partial \Phi_t^\ell}{\partial x_k}(0) + \frac{\partial \Phi_t^p}{\partial x_j}(0) \frac{\partial^2 \Phi_t^\ell}{\partial x_k \partial x_h}(0) \right.$$

$$\left. + \frac{\partial^2 \Phi_t^\ell}{\partial x_j \partial x_k}(0) \frac{\partial \Phi_t^p}{\partial x_h}(0) \right] + \frac{\partial X^i}{\partial x_\ell}(0) \frac{\partial^3 \Phi_t^\ell}{\partial x_j \partial x_k \partial x_h}(0).$$

Furthermore, $\frac{\partial^3 \Phi_0^i}{\partial x_j \partial x_k \partial x_h}(0) = 0$. We get the differential equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial^3 \Phi_t^i}{\partial x_j \partial x_k \partial x_h}(0) \right) = d^3X(0)\left(\frac{\partial \Phi_t}{\partial x_j}(0), \frac{\partial \Phi_t}{\partial x_k}(0), \frac{\partial \Phi_t}{\partial x_h}(0) \right)$$

$$+ d^2X(0)\left(\frac{\partial \Phi_t}{\partial x_k}(0), \frac{\partial^2 \Phi_t}{\partial x_j \partial x_h}(0) \right) + d^2X(0)\left(\frac{\partial \Phi_t}{\partial x_j}(0), \frac{\partial^2 \Phi_t}{\partial x_k \partial x_h}(0) \right)$$

$$+ d^2X(0)\left(\frac{\partial \Phi_t}{\partial x_h}(0), \frac{\partial^2 \Phi_t}{\partial x_j \partial x_k}(0) \right) + dX(0)\left(\frac{\partial^3 \Phi_t}{\partial x_j \partial x_k \partial x_h}(0) \right)$$

and

$$\frac{\partial^3 \Phi_0}{\partial x_j \partial x_k \partial x_h}(0) = 0.$$

The solution is:

$$\frac{\partial^3 \Phi_t}{\partial x_j \partial x_k \partial x_h}(0)$$

$$= e^{tdX(0)} \int_0^t e^{-sdX(0)} \left\{ d^3X(0)\left(\frac{\partial \Phi_t}{\partial x_j}(0), \frac{\partial \Phi_t}{\partial x_k}(0), \frac{\partial \Phi_t}{\partial x_h}(0) \right) \right.$$

$$+ d^2X(0)\left(\frac{\partial \Phi_t}{\partial x_k}(0), \frac{\partial^2 \Phi_t}{\partial x_j \partial x_h}(0) \right) + d^2X(0)\left(\frac{\partial \Phi_t}{\partial x_j}(0), \frac{\partial^2 \Phi_t}{\partial x_k \partial x_h}(0) \right)$$

$$+ \, d^2X(0)\left(\frac{\partial\Phi_t}{\partial x_h}(0), \frac{\partial^2\Phi_t}{\partial x_j\, x_k}(0)\right)\Bigg\}ds.$$

In the case we are considering, $d\hat{X}(0,0) = \begin{pmatrix} 0 & |\lambda(0)| \\ -|\lambda(0)| & 0 \end{pmatrix}$

and so $d\hat{\phi}_t(0,0) = \begin{pmatrix} \cos|\lambda(0)|t & \sin|\lambda(0)|t \\ -\sin|\lambda(0)|t & \cos|\lambda(0)|t \end{pmatrix}$. We wish to

calculate $\dfrac{\partial^2\hat{\phi}_t}{\partial x_1^2}(0,0) = \left(\dfrac{\partial^2 a_t}{\partial x_1^2}(0,0), \dfrac{\partial^2 b_t}{\partial x_1^2}(0,0)\right)$ and

$\dfrac{\partial^3 b_t}{\partial x_1^3}(0,0)$. First note that $e^{-sdX(0)} = e^{-s\begin{pmatrix} 0 & |\lambda(0)| \\ -|\lambda(0)| & 0 \end{pmatrix}}$

$= \begin{pmatrix} \cos|\lambda(0)|s & -\sin|\lambda(0)|s \\ \sin|\lambda(0)|s & \cos|\lambda(0)|s \end{pmatrix}$. Also,

$$d^2\hat{X}(0)\left(\frac{\partial\hat{\phi}_s}{\partial x_1}(0), \frac{\partial\hat{\phi}_s}{\partial x_1}(0)\right) = \left(\frac{\partial^2\hat{X}^1}{\partial x_1^2}(0,0)\cos^2|\lambda(0)|s\right.$$

$$- 2\,\frac{\partial^2\hat{X}^1}{\partial x_1\partial x_2}(0,0)\cos|\lambda(0)|s\,\sin|\lambda(0)|s$$

$$+ \frac{\partial^2\hat{X}^1}{\partial x_2^2}(0,0)\sin^2|\lambda(0)|s, \; \frac{\partial^2\hat{X}^2}{\partial x_1^2}(0,0)\cos^2|\lambda(0)|s$$

$$- 2\,\frac{\partial^2\hat{X}^2}{\partial x_1\partial x_2}(0,0)\cos|\lambda(0)|s\,\sin|\lambda(0)|s$$

$$\left. + \; \frac{\partial^2\hat{X}^2}{\partial x_2^2}(0,0)\sin^2|\lambda(0)|s\right).$$

Consequently,

$$e^{-sd\hat{X}(0)}\, d^2\hat{X}(0)\left(\frac{\partial\hat{\phi}_s}{\partial x_1}(0), \frac{\partial\hat{\phi}_s}{\partial x_1}(0)\right) =$$

$$= \left[\frac{\partial^2 \hat{X}^1}{\partial x_1^2}(0)\cos^3|\lambda(0)|s - \frac{\partial^2 \hat{X}^2}{\partial x_2^2}(0)\sin^3|\lambda(0)|s \right.$$

$$+ \left(-2\frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2}(0) - \frac{\partial^2 \hat{X}^2}{\partial x_1^2}(0) \right) \sin|\lambda(0)|s \cos^2|\lambda(0)|s$$

$$+ \left(2\frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2}(0) + \frac{\partial^2 \hat{X}^1}{\partial x_2^2}(0) \right) \cos|\lambda(0)|s \sin^2|\lambda(0)|s,\ \frac{\partial^2 \hat{X}^2}{\partial x_1^2}(0)\cos^3|\lambda(0)|s$$

$$+ \frac{\partial^2 \hat{X}^1}{\partial x_2^2}(0)\ \sin^3|\lambda(0)|s + \left(-2\frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2}(0) + \frac{\partial^2 \hat{X}^1}{\partial x_1^2}(0) \right) \sin|\lambda(0)|s \cos^2|\lambda(0)|s$$

$$\left. + \left(-2\frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2}(0) + \frac{\partial^2 \hat{X}^2}{\partial x_2^2}(0) \right) \cos|\lambda(0)|s \sin^2|\lambda(0)|s \right],$$

and thus

$$\int_0^t e^{-sd\hat{X}(0)}\ d^2\hat{X}(0)\left(\frac{\partial \hat{\phi}_s}{\partial x_1}(0),\ \frac{\partial \hat{\phi}_s}{\partial x_1}(0) \right) ds$$

$$= \frac{1}{3|\lambda(0)|} \left[\left(-2\frac{\partial^2 \hat{X}^2}{\partial x_2^2} - 2\frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^2}{\partial x_1^2} \right) + 3\frac{\partial^2 \hat{X}^1}{\partial x_1^2}\sin|\lambda(0)|t \right.$$

$$+ 3\frac{\partial^2 \hat{X}^2}{\partial x_2^2}\cos|\lambda(0)|t + \left(-\frac{\partial^2 \hat{X}^1}{\partial x_1^2} + 2\frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^1}{\partial x_2^2} \right) \sin^3|\lambda(0)|t$$

$$+ \left(-\frac{\partial^2 \hat{X}^2}{\partial x_2^2} + 2\frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^2}{\partial x_1^2} \right) \cos^3|\lambda(0)|t,$$

$$\left(2\frac{\partial^2 \hat{X}^1}{\partial x_2^2} - 2\frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^1}{\partial x_1^2} \right)$$

$$+ 3 \frac{\partial^2 \hat{X}^1}{\partial x_1^2} \sin|\lambda(0)|t - 3 \frac{\partial^2 \hat{X}^1}{\partial x_2^2} \cos|\lambda(0)|t$$

$$+ \left(- \frac{\partial^2 \hat{X}^2}{\partial x_1^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^2}{\partial x_2^2} \right) \sin^3|\lambda(0)|t$$

$$\left. + \left(\frac{\partial^2 \hat{X}^1}{\partial x_2^2} + 2 \frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^1}{\partial x_1^2} \right) \cos^3|\lambda(0)|t \right]$$

(where all derivatives are evaluated at the origin).

Putting this all together,

$$e^{td\hat{X}(0)} \int_0^t e^{-sd\hat{X}(0)} d^2\hat{X}(0) \left(\frac{\partial \hat{\phi}_s}{\partial x_1}(0), \frac{\partial \hat{\phi}_s}{\partial x_1}(0) \right) ds = \left(\frac{\partial^2 a_t}{\partial x_1^2}(0), \frac{\partial^2 b_t}{\partial x_1^2}(0) \right)$$

$$= \frac{1}{3|\lambda(0)|} \left[\left(- 2 \frac{\partial^2 \hat{X}^2}{\partial x_2^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^2}{\partial x_1^2} \right) \cos|\lambda(0)|t \right.$$

$$+ \left(2 \frac{\partial^2 \hat{X}^1}{x_2^2} - 2 \frac{\partial^2 \hat{X}^2}{x_1 \; x_2} + \frac{\partial^2 \hat{X}^1}{x_1^2} \right) \sin|\lambda(0)|t$$

$$+ \left(3 \frac{\partial^2 \hat{X}^1}{\partial x_1^2} - 3 \frac{\partial^2 \hat{X}^1}{\partial x_2^2} \right) \sin|\lambda(0)|t \cos|\lambda(0)|t$$

$$+ 3 \frac{\partial^2 \hat{X}^2}{\partial x_2^2} \cos^2|\lambda(0)|t + 3 \frac{\partial^2 \hat{X}^2}{\partial x_1^2} \sin^2|\lambda(0)|t$$

$$+ \left(- \frac{\partial^2 \hat{X}^1}{\partial x_1^2} + 2 \frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^1}{\partial x_2^2} \right) \cos|\lambda(0)|t \sin^3|\lambda(0)|t$$

$$+ \left(\frac{\partial^2 \hat{X}^1}{\partial x_2^2} + 2 \frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^1}{\partial x_1^2} \right) \sin|\lambda(0)|t \cos^3|\lambda(0)|t$$

$$+ \left(- \frac{\partial^2 \hat{X}^2}{\partial x_2^2} + 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^2}{\partial x_1^2}\right) \cos^4 |\lambda(0)| t$$

$$+ \left(- \frac{\partial^2 \hat{X}^2}{\partial x_1^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^2}{\partial x_2^2}\right) \sin^4 |\lambda(0)| t,$$

$$\left(2 \frac{\partial^2 \hat{X}^1}{\partial x_2^2} - 2 \frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^1}{\partial x_1^2}\right) \cos |\lambda(0)| t$$

$$+ \left(2 \frac{\partial^2 \hat{X}^2}{\partial x_2^2} + 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^2}{\partial x_1^2}\right) \sin |\lambda(0)| t$$

$$+ \left(3 \frac{\partial^2 \hat{X}^2}{\partial x_1^2} - 3 \frac{\partial^2 \hat{X}^2}{\partial x_2^2}\right) \sin |\lambda(0)| t \cos |\lambda(0)| t$$

$$- 3 \frac{\partial^2 \hat{X}^1}{\partial x_1^2} \sin^2 |\lambda(0)| t - 3 \frac{\partial^2 \hat{X}^1}{\partial x_2^2} \cos^2 |\lambda(0)| t$$

$$+ \left(\frac{\partial^2 \hat{X}^2}{\partial x_2^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^2}{\partial x_1^2}\right) \sin |\lambda(0)| t \cos^3 |\lambda(0)| t$$

$$+ \left(- \frac{\partial^2 \hat{X}^2}{\partial x_1^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial x_1 \partial x_2} + \frac{\partial^2 \hat{X}^2}{\partial x_2^2}\right) \cos |\lambda(0)| t \sin^3 |\lambda(0)| t$$

$$+ \left(\frac{\partial^2 \hat{X}^1}{\partial x_1^2} - 2 \frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^1}{\partial x_2^2}\right) \sin^4 |\lambda(0)| t$$

$$+ \left(\frac{\partial^2 \hat{X}^1}{\partial x_2^2} + 2 \frac{\partial^2 \hat{X}^2}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{X}^1}{\partial x_1^2}\right) \cos^4 |\lambda(0)| t \Bigg] .$$

Before computing $\dfrac{\partial^3 b_t}{\partial x_1^3}(0)$, which is a lengthy calculation, we will use the information above to simplify our expression for $V'''(0)$. To do this, we must compute

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t\,dt,$$

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t\,dt,$$

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t\,dt,$$

and

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t\,dt.$$

The results are:

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 a_t}{\partial x_1^2}(0,0)dt = \frac{\pi}{|\lambda(0)|^2}\left[\frac{\partial^2\hat{X}^2}{\partial a^2} + \frac{\partial^2\hat{X}^2}{\partial b^2}\right]$$

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t\,dt = \frac{\pi}{3|\lambda(0)|^2}\left(-2\frac{\partial^2\hat{X}^2}{\partial b^2} - 2\frac{\partial^2\hat{X}^1}{\partial a\partial b} - \frac{\partial^2\hat{X}^2}{\partial a^2}\right)$$

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t\,dt = \frac{\pi}{3|\lambda(0)|^2}\left(2\frac{\partial^2\hat{X}^1}{\partial b^2} - 2\frac{\partial^2\hat{X}^2}{\partial a\partial b} + \frac{\partial^2\hat{X}^1}{\partial a^2}\right)$$

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t\,dt = \frac{\pi}{3|\lambda(0)|^2}\left(2\frac{\partial^2\hat{X}^1}{\partial b^2} - 2\frac{\partial^2\hat{X}^2}{\partial a\partial b} + \frac{\partial^2\hat{X}^1}{\partial a^2}\right)$$

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t\,dt = \frac{\pi}{3|\lambda(0)|^2}\left(2\frac{\partial^2\hat{X}^2}{\partial b^2} + 2\frac{\partial^2\hat{X}^1}{\partial a\partial b} + \frac{\partial^2\hat{X}^2}{\partial a^2}\right)$$

Therefore,

$$V'''(0) = |\lambda(0)|\int_0^{2\pi/|\lambda(0)|} \frac{\partial^2 b_t}{\partial x_1^3}(0,0)dt$$

$$+ 3\int_0^{2\pi/|\lambda(0)|}\left[\frac{\partial^2\hat{X}^1}{\partial a^2}(0,0)\ \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t\right.$$

$$- \frac{\partial^2 \hat{X}^1}{\partial b^2}(0,0) \; \frac{\partial^2 b_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t$$

$$+ \frac{\partial^2 \hat{X}^1}{\partial a \partial b}(0,0) \left[\frac{\partial^2 b_t}{\partial x_1^2}(0,0)\cos|\lambda(0)|t \right.$$

$$\left. \left. - \frac{\partial^2 a_t}{\partial x_1^2}(0,0)\sin|\lambda(0)|t \right) \right] dt$$

i.e.

$$V'''(0) = |\lambda(0)| \int_0^{2\pi/|\lambda(0)|} \frac{\partial^3 b_t}{\partial x_1^3}(0,0)dt$$

$$+ \frac{\pi}{|\lambda(0)|^2} \frac{\partial^2 \hat{X}^1}{\partial a^2} \left(-2 \frac{\partial^2 \hat{X}^2}{\partial b^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial a \partial b} - \frac{\partial^2 \hat{X}^2}{\partial a^2} \right)$$

$$+ \frac{\pi}{|\lambda(0)|^2} \frac{\partial^2 \hat{X}^1}{\partial b^2} \left(-2 \frac{\partial^2 \hat{X}^2}{\partial b^2} - 2 \frac{\partial^2 \hat{X}^1}{\partial a \partial b} - \frac{\partial^2 \hat{X}^2}{\partial a^2} \right)$$

$$= |\lambda(0)| \int_0^{2\pi/|\lambda(0)|} \frac{\partial^3 b_t}{\partial x_1^3}(0,0)dt$$

$$- \frac{\pi}{|\lambda(0)|^2} \left(2 \frac{\partial^2 \hat{X}^1}{\partial a^2} \frac{\partial^2 \hat{X}^2}{\partial b^2} + \frac{\partial^2 \hat{X}^1}{\partial b^2} \frac{\partial^2 \hat{X}^2}{\partial a^2} + 2 \frac{\partial^2 \hat{X}^1}{\partial a^2} \frac{\partial^2 \hat{X}^1}{\partial a \partial b} + 2 \frac{\partial^2 \hat{X}^1}{\partial b^2} \frac{\partial^2 \hat{X}^1}{\partial a \partial b} \right.$$

$$\left. + \frac{\partial^2 \hat{X}^1}{\partial a^2} \frac{\partial^2 \hat{X}^2}{\partial a^2} + 2 \frac{\partial^2 \hat{X}^1}{\partial b^2} \frac{\partial^2 \hat{X}^2}{\partial b^2} \right)$$

(where all derivatives are taken at the origin).

To compute $\frac{\partial^3 b_t}{\partial x_1^3}(0,0)$, we use the equation:

$$\left(\frac{\partial^3 a_t}{\partial x_1^3}(0,0), \frac{\partial^3 b_t}{\partial x_1^3}(0,0) \right)$$

$$= e^{td\hat{X}(0)} \int_0^t e^{-sd\hat{X}(0)} \left\{ d^3\hat{X}(0) \left(\frac{\partial \hat{\phi}_s}{\partial x_1}(0,0), \frac{\partial \hat{\phi}_s}{\partial x_1}(0,0), \frac{\partial \hat{\phi}_s}{\partial x_1}(0,0) \right) \right.$$

$$\left. + 3d^2\hat{X}(0) \left(\frac{\partial \hat{\phi}_s}{\partial x_1}(0,0), \frac{\partial^2 \hat{\phi}_s}{\partial x_1^2}(0,0) \right) \right\} ds.$$

The calculation involved is quite long,* but is straightforward, so we will merely indicate how it is done and then state the results. The lengthy computation alluded to is:

$$\int_0^{2\pi/|\lambda(0)|} e^{td\hat{X}(0)} \int_0^t e^{-sdX(0)} d^2\hat{X}(0)\left(\frac{\partial\Phi_s}{\partial x_1}(0,0), \frac{\partial^2\Phi_s}{\partial x_1^2}(0,0)\right) ds\, dt$$

$$= \left\{\text{something}, \frac{2\pi}{|\lambda(0)|^3}\frac{\partial^2\hat{X}_1}{\partial x_1^2}\frac{\partial^2\hat{X}_2}{\partial x_2^2} + \frac{5\pi}{4|\lambda(0)|^3}\frac{\partial^2\hat{X}_1}{\partial x_1^2}\frac{\partial^2\hat{X}_1}{\partial x_1\partial x_2}\right.$$

$$+ \frac{7\pi}{4|\lambda(0)|^3}\frac{\partial^2\hat{X}_1}{\partial x_1^2}\frac{\partial^2\hat{X}_2}{\partial x_1^2} + \frac{5\pi}{4|\lambda(0)|}\frac{\partial^2\hat{X}_1}{\partial x_1\partial x_2}\frac{\partial^2\hat{X}_1}{\partial x_2^2}$$

$$+ \frac{3\pi}{4|\lambda(0)|^3}\frac{\partial^2\hat{X}_2}{\partial x_1\partial x_2}\frac{\partial^2\hat{X}_2}{\partial x_1^2} + \frac{5\pi}{4|\lambda(0)|^3}\frac{\partial^2\hat{X}_1}{\partial x_2^2}\frac{\partial^2\hat{X}_2}{\partial x_2^2}$$

$$\left. + \frac{\pi}{|\lambda(0)|^3}\frac{\partial^2\hat{X}_1}{\partial x_2^2}\frac{\partial^2\hat{X}_2}{\partial x_1^2} + \frac{3\pi}{4|\lambda(0)|^3}\frac{\partial^2\hat{X}_2}{\partial x_1\partial x_2}\frac{\partial^2\hat{X}_2}{\partial x_2^2}\right\}; \text{ (at } (0,0))$$

and one easily sees that

$$\int_0^{2\pi/|\lambda(0)|} e^{td\hat{X}(0)} \int_0^t e^{-sd\hat{X}(0)} d^3\hat{X}(0)\left(\frac{\partial\Phi_s}{\partial x_1}(0,0), \frac{\partial\Phi_s}{\partial x_1}(0,0), \frac{\partial\Phi_s}{\partial x_1}(0,0)\right) ds\, dt$$

$$= \frac{3\pi}{4|\lambda(0)|^2}\left(\frac{\partial^3\hat{X}_1}{\partial x_1^3} + \frac{\partial^3\hat{X}_1}{\partial x_1\partial x_2^2} + \frac{\partial^3\hat{X}_1}{\partial x_1^2\partial x_2} + \frac{\partial^3\hat{X}_2}{\partial x_2^2}\right).$$

The final result of the computations is, therefore,

---

*We are not joking! One has to be prepared to shack up with the previous calculations for several days. Details will be sent only on serious request.

$$\int_0^{2\pi/|\lambda(0)|} \frac{\partial^3 b_t}{\partial x_1^3}(0,0)dt$$

$$= \frac{3\pi}{4|\lambda(0)|^2}\left[\frac{\partial^3\hat{X}^1}{\partial a^3}(0,0) + \frac{\partial^3\hat{X}^1}{\partial a\partial b^2}(0,0) + \frac{\partial^3\hat{X}^2}{\partial a^2\partial b}(0,0) + \frac{\partial^3\hat{X}^2}{\partial b^3}(0,0)\right]$$

$$+ \frac{\pi}{|\lambda(0)|^3}\left\{2\,\frac{\partial^2\hat{X}^1}{\partial a^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial b^2}(0,0) + \frac{\partial^2\hat{X}^1}{\partial b^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a^2}(0,0)\right.$$

$$+ \frac{5}{4}\frac{\partial^2\hat{X}^1}{\partial a^2}(0,0)\,\frac{\partial^2\hat{X}^1}{\partial a\partial b}(0,0) + \frac{3}{4}\frac{\partial^2\hat{X}^2}{\partial b^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a\partial b}(0,0)$$

$$+ \frac{7}{4}\frac{\partial^2\hat{X}^1}{\partial a^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a^2}(0,0) + \frac{5}{4}\frac{\partial^2\hat{X}^2}{\partial b^2}(0,0)\,\frac{\partial^2\hat{X}^1}{\partial b^2}(0,0)$$

$$\left.+ \frac{5}{4}\frac{\partial^2\hat{X}^1}{\partial a\partial b}(0,0)\,\frac{\partial^2\hat{X}^1}{\partial b^2}(0,0) + \frac{3}{4}\frac{\partial^2\hat{X}^2}{\partial a\partial b}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a^2}(0,0)\right].$$

Thus we get our formula:

(4.2) <u>Formula</u>

$$V'''(0) = \frac{3\pi}{4|\lambda(0)|}\left\{\frac{\partial^3\hat{X}^1}{\partial a^3}(0,0) + \frac{\partial^3\hat{X}^1}{\partial a\partial b^2}(0,0)\right.$$

$$\left.+ \frac{\partial^3\hat{X}^2}{\partial a^2\partial b}(0,0) + \frac{\partial^3\hat{X}^2}{\partial b^3}(0,0)\right\}$$

$$+ \frac{3\pi}{4|\lambda(0)|^2}\left\{-\frac{\partial^2\hat{X}^1}{\partial a^2}(0,0)\,\frac{\partial^2\hat{X}^1}{\partial a\partial b}(0,0)\right.$$

$$+ \frac{\partial^2\hat{X}^2}{\partial b^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a\partial b}(0,0)$$

$$+ \frac{\partial^2\hat{X}^2}{\partial a^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a\partial b}(0,0) - \frac{\partial^2\hat{X}^1}{\partial b^2}(0,0)\,\frac{\partial^2\hat{X}^1}{\partial a\partial b}(0,0)$$

$$\left.+ \frac{\partial^2\hat{X}^1}{\partial a^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial a^2}(0,0) - \frac{\partial^2\hat{X}^1}{\partial b^2}(0,0)\,\frac{\partial^2\hat{X}^2}{\partial b^2}(0,0)\right\}.$$

In two dimensions, where $X = \hat{X}$, this can be used directly to test stability if $d\hat{X}_0(0,0)$ is in the form on p. 108.

<u>Expressing the Stability Condition in Terms of</u> X.

We now use the equation

$$X_\mu^3(x_1,x_2,f(x_1,x_2,\mu)) = d_1f(x_1,x_2,\mu) \circ X^1(x_1,x_2,f(x_1,x_2,\mu)) + d_2f(x_1,x_2,\mu) \circ X^2(x_1,x_2,f(x_1,x_2,\mu))$$

near $(0,0,0,0)$

to compute the derivatives of $\hat{X}_0$ at $(0,0)$ in terms of those of X_0 at $(0,0,0)$. Since no differentiation with respect to μ occurs, we drop all reference to μ. Upon differentiating this with respect to x_1 and to x_2, we get:

$$\begin{aligned}
d_1X^3(x_1,x_2,f(x_1,x_2)) &+ d_3X^3(x_1,x_2,f(x_1,x_2)) \circ d_1f(x_1,x_2) \\
&= d_1d_1f(x_1,x_2) \circ X^1(x_1,x_2,f(x_1,x_2)) \\
&+ d_1f(x_1,x_2) \circ d_1X^1(x_1,x_2,f(x_1,x_2)) \\
&+ d_1f(x_1,x_2) \circ d_3X^1(x_1,x_2,f(x_1,x_2)) \circ d_1f(x_1,x_2) \\
&+ d_1d_2f(x_1,x_2) \circ X^2(x_1,x_2,f(x_1,x_2)) \\
&+ d_2f(x_1,x_2) \circ d_1X^2(x_1,x_2,f(x_1,x_2)) \\
&+ d_2f(x_1,x_2) \circ d_3X^2(x_1,x_2,f(x_1,x_2)) \circ d_1f(x_1,x_2)
\end{aligned}$$

and

$$\begin{aligned}
d_2X^3(x_1,x_2,f(x_1,x_2)) &+ d_3X^3(x_1,x_2,f(x_1,x_2)) \circ d_2f(x_1,x_2) \\
&= d_2d_2f(x_1,x_2) \circ X^2(x_1,x_2,f(x_1,x_2))
\end{aligned}$$

$$+ d_2f(x_1,x_2) \circ d_2X^2(x_1,x_2,f(x_1,x_2))$$

$$+ d_2f(x_1,x_2) \circ d_3X^2(x_1,x_2,f(x_1,x_2)) \circ d_2f(x_1,x_2)$$

$$+ d_1d_2f(x_1,x_2) \circ X^1(x_1,x_2,f(x_1,x_2))$$

$$+ d_1f(x_1,x_2) \circ d_2X^1(x_1,x_2,f(x_1,x_2))$$

$$+ d_1f(x_1,x_2) \circ d_3X^1(x_1,x_2,f(x_1,x_2)) \circ d_2f(x_1,x_2).$$

We now differentiate the first equation with respect to x_1 and x_2 and the second with respect to x_2 only. We get three expressions. These expressions are easy to write down and to evaluate at the point $x_1 = x_2 = 0$, where

$$f = df = 0 \quad \text{and} \quad dX = \begin{pmatrix} 0 & |\lambda(0)| & 0 \\ -|\lambda(0)| & 0 & 0 \\ 0 & 0 & d_3X^3 \end{pmatrix}.$$

This procedure yields

$$d_1d_1X^3(0,0,0) + d_3X^3(0,0,0) \circ d_1d_1f(0,0) = -2|\lambda(0)|d_1d_2f(0,0)$$

$$d_1d_2X^3(0,0,0) + d_3X^3(0,0,0) \circ d_1d_2f(0,0)$$

$$= |\lambda(0)|d_1d_1f(0,0) - |\lambda(0)|d_2d_2f(0,0)$$

$$d_2d_2X^3(0,0,0) + d_3X^3(0,0,0) \circ d_2d_2f(0,0) = 2|\lambda(0)|d_1d_2f(0,0)$$

i.e.

$$\begin{bmatrix} d_3X^3(0,0,0) & 2|\lambda(0)| & 0 \\ -|\lambda(0)| & d_3X^3(0,0,0) & |\lambda(0)| \\ 0 & -2|\lambda(0)| & d_3X^3(0,0,0) \end{bmatrix} \begin{pmatrix} d_1d_1f(0,0) \\ d_1d_2f(0,0) \\ d_2d_2f(0,0) \end{pmatrix}$$

$$= \begin{pmatrix} -d_1d_1X^3(0,0,0) \\ -d_1d_2X^3(0,0,0) \\ -d_2d_2X^3(0,0,0) \end{pmatrix}$$

See formula (4A.6) on p. 134 for the expression for d_id_jf obtained by inverting the 3×3 matrix on the left. Note that since the determinant is $d_3X^3(0,0,0)\ (d_3X^3(0,0,0)^2 + 4|\lambda(0)|^2)$, and since $\sigma(d_3X^3(0,0,0)) \subset \sigma(dX(0,0,0)) \subset \{z|\text{Re } z < 0\} \cup \{\lambda(0),\overline{\lambda(0)}\}$ implies both $d_3X^3(0,0,0)$ and $d_3X^3(0,0,0)^2 + 4|\lambda(0)|^2 = (d_3X^3(0,0,0) + 2|\lambda(0)|i)(d_3X^3(0,0,0) -2|\lambda(0)|i)$ are invertible, the matrix is invertible.

Finally we must compute the first three derivatives of $\hat{X}_0$ at $(0,0)$ in terms of those of X_0 at $(0,0,0)$. Remember that

$$\hat{X}^i(x_1,x_2) = X^i(x_1,x_2,f(x_1,x_2)) \quad \text{for} \quad i = 1,2.$$

Therefore,

$$\begin{aligned} d_j\hat{X}^i(x_1,x_2) &= d_jX^i(x_1,x_2,f(x_1,x_2) \\ &\quad + d_3X^i(x_1,x_2,f(x_1,x_2)) \circ d_jf(x_1,x_2) \\ &\quad \text{for} \quad i,j = 1,2. \end{aligned}$$

So

$$d\hat{X}(0,0) = \begin{pmatrix} 0 & |\lambda(0)| \\ -|\lambda(0)| & 0 \end{pmatrix}.$$

Differentiating again, we get:

$$\begin{aligned}
d_k d_j \hat{X}^i(x_1,x_2) &= d_k d_j X^i(x_1,x_2,f(x_1,x_2)) \\
&+ d_3 d_j X^i(x_1,x_2,f(x_1,x_2)) \circ d_k f(x_1,x_2) \\
&+ d_k d_3 X^i(x_1,x_2,f(x_1,x_2)) \circ d_j f(x_1,x_2)) \\
&+ d_3 d_3 X^i(x_1,x_2,f(x_1,x_2)) \circ d_k f(x_1,x_2) \circ d_j f(x_1,x_2) \\
&+ d_3 X^i(x_1,x_2,f(x_1,x_2)) \circ d_k d_j f(x_1,x_2), \quad i,j,k = 1,2.
\end{aligned}$$

Evaluating at $t = 0$, we get:

$$d_k d_j \hat{X}^i(0,0) = d_k d_j X^i(0,0,0), \qquad i,j,k = 1,2.$$

We differentiate once more at evaluate at 0:

$$\begin{aligned}
d_\ell d_k d_j \hat{X}^i(0,0) &= d_\ell d_k d_j X^i(0,0,0) + d_3 d_j X^i(0,0,0) \circ d_\ell d_k f(0,0) \\
&+ d_k d_3 X^i(0,0,0) \circ d_\ell d_j f(0,0) \\
&+ d_\ell d_3 X^i(0,0,0) \circ d_k d_j f(0,0), \quad i,j,k, \ = 1,2.
\end{aligned}$$

This can be inserted into the previous results to give an explicit expression for $V'''(0)$ on p.

Below in Section 4A, we shall summarize the results algorithmically so that this proof need not be traced through, and in Section 4B examples and exercises illustrating the method are given.

(4.3) Exercise. In Exercise 1.16 make a stability analysis for the pair of bifurcated fixed points. In that proof, write $f(\alpha,0) = \alpha + A\alpha^3 + \cdots$ and show that we have stability if $A < 0$ and supercritical bifurcation and instability with subcritical bifurcation if $A > 0$. Develop an explicit formula for A and apply it to the ball in the hoop example. (Reference: Ruelle-Takens [1], p. 189-191.)

SECTION 4A

HOW TO USE THE STABILITY FORMULA; AN ALGORITHM

The above calculations are admittedly a little long, but they are not difficult. Here we shall summarize the results of the calculation in the form of a specific algorithm that can be followed for any given vector field. In the two dimensional case the algorithm ends rather quickly. In general, it is much longer. Examples will be given in Section 4B following.

Stability is determined by the sign of $V'''(0)$, so our object is to calculate this number. We assume there is no difficulty in calculating the spectrum of the linearized problem.

Before stating the procedure for calculation of $V'''(0)$, let us recall the set up and the overall operation.

Let $X_\mu: E \to E$ be a C^k $(k \geq 5)$ vector field on a Banach space E (if X_μ is a vector field on a manifold, one must work in a chart to compute the stability condition). Assume $X_\mu(a(\mu)) = 0$ for all μ and let the spectrum of

$dX_\mu(a(\mu))$ satisfy:

For $\mu < \mu_0$, $\sigma(dX_\mu(a(\mu)) \subset \{z \mid \mathrm{Re}\ z < 0\}$. $dX_\mu(a(\mu))$ has two complex conjugate, simple eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$. At $\mu = \mu_0$, $\lambda(\mu)$ and $\lambda(\mu)$ cross the imaginary axis with non-zero speed and $\lambda(\mu_0) \neq 0$. The rest of $\sigma(dX_\mu(a(\mu))$ remains in the left half plane bounded away from the imaginary axis.

I. Under these circumstances

(A) Bifurcation to periodic orbits takes place, as described in Theorem 3.1.

Choose coordinates so that $X_{\mu_0} = (X^1_{\mu_0}, X^2_{\mu_0}, X^3_{\mu_0})$ where $X^1_{\mu_0}$ and $X^2_{\mu_0}$ are coordinates in the eigenspace to $\lambda(\mu_0)$ and $\overline{\lambda(\mu_0)}$ and $X^3_{\mu_0}$ is the coordinate in some complementary subspace. Choose the coordinates so that*

$$dX_{\mu_0}(a(\mu_0)) = \begin{pmatrix} 0 & |\lambda(\mu_0)| & 0 \\ -|\lambda(\mu_0)| & 0 & 0 \\ 0 & 0 & d_3X^3_{\mu_0}(a(\mu_0)) \end{pmatrix}. \quad (4.1)$$

(B) If the coefficient $V'''(0)$ computed in (II) below is negative, the periodic orbits occur for $\mu > \mu_0$ and are attracting. If $V'''(0) > 0$, the orbits occur for $\mu < \mu_0$, are repelling on the center manifold, and so are unstable in general.

(C) If $V'''(0) = 0$, the test yields no information

---

*See Examples 4B.2 and 4B.8. Computer programs are available for this step. See for instance, "A Program to Compute the Real Schur Form of a Real Square Matrix" by B.N. Parlett and R. Feldman; ERL Memorandum M 526 (1975), Univ. of Calif., Berkeley.

and the procedures outlined in Section 4 must be used to compute $V^{(5)}(0)$. Good luck.

II. Write out the expression

$$
\begin{aligned}
V'''(0) = {} & \frac{3\pi}{4|\lambda(\mu_0)|}\left(\frac{\partial^3 \hat{X}^1_{\mu_0}}{\partial x_1^3}(a_1(\mu_0),a_2(\mu_0))\right. \\
& + \frac{\partial^3 \hat{X}^1_{\mu_0}}{\partial x_1 \partial x_2^2}(a_1(\mu_0),a_2(\mu_0)) + \frac{\partial^3 \hat{X}^2_{\mu_0}}{\partial x_1^2\, \partial x_2}(a_1(\mu_0),a_2(\mu_0)) \\
& \left. + \frac{\partial^3 \hat{X}^2_{\mu_0}}{\partial x_2^3}(a_1(\mu_0),a_2(\mu_0))\right) \\
& + \frac{3\pi}{4|\lambda(\mu_0)|^2}\left(-\frac{\partial^2 X^1_{\mu_0}}{\partial x_1^2}(a_1(\mu_0),a_2(\mu_0))\,\frac{\partial^2 X^1_{\mu_0}}{\partial x_1 \partial x_2}(a_1(\mu_0),a_2(\mu_0))\right. \\
& + \frac{\partial^2 X^2_{\mu_0}}{\partial x_2^2}(a_1(\mu_0),a_2(\mu_0))\,\frac{\partial^2 X^2_{\mu_0}}{\partial x_1 \partial x_2}(a_1(\mu_0),a_2(\mu_0)) \\
& + \frac{\partial^2 X^2_{\mu_0}}{\partial x_1^2}(a_1(\mu_0),a_2(\mu_0))\,\frac{\partial^2 X^2_{\mu_0}}{\partial x_1 \partial x_2}(a_1,(\mu_0),a_2(\mu_0)) \\
& - \frac{\partial^2 X^1_{\mu_0}}{\partial x_2^2}(a_1(\mu_0),a_2(\mu_0))\,\frac{\partial^2 X^1_{\mu_0}}{\partial x_1 \partial x_2}(a_1(\mu_0),a_2(\mu_0)) \\
& + \frac{\partial^2 X^1_{\mu_0}}{\partial x_1^2}(a_1(\mu_0),a_2(\mu_0))\,\frac{\partial^2 X^2_{\mu_0}}{\partial x_1^2}(a_1(\mu_0),a_2(\mu_0)) \\
& \left. - \frac{\partial^2 X^1_{\mu_0}}{\partial x_2^2}(a_1(\mu_0),a_2(\mu_0))\,\frac{\partial^2 X^2_{\mu_0}}{\partial x_2^2}(a_1(\mu_0),a_2(\mu_0))\right). \qquad (4A.2)
\end{aligned}
$$

(A) If your space is two dimensional, let $\hat{X}^1 = X^1$, $\hat{X}^2 = X^2$. Take off the hats; you are done with the computation of $V'''(0)$ and the results may be read off from I.

Otherwise, go to Step B.

(B) In expression (4A.2) fill in

$$d_j\hat{X}^i_{\mu_0}(a_1(\mu_0),a_2(\mu_0)) = d_jX^i_{\mu_0}(a(\mu_0)) \quad \text{for } i,j = 1,2. \tag{4A.3}$$

$$d_kd_j\hat{X}^i_{\mu_0}(a_1(\mu_0),a_2(\mu_0)) = d_kd_jX^i_{\mu_0}(a(\mu_0)), \quad \text{for } i,j,k = 1,2. \tag{4A.4}$$

and

$$\begin{aligned} d_\ell d_k d_j\hat{X}^i_{\mu_0}(a_1(\mu_0),a_2(\mu_0)) &= d_\ell d_k d_j X^i_{\mu_0}(a(\mu_0)) \\ &+ d_3d_jX^i_{\mu_0}(a(\mu_0)) \circ d_\ell d_k f(a_1(\mu_0),a_2(\mu_0)) \\ &+ d_3d_kX^i_{\mu_0}(a(\mu_0)) \circ d_\ell d_j f(a_1(\mu_0),a_2(\mu_0)) \\ &+ d_3d_\ell X^i_{\mu_0}(a(\mu_0)) \circ d_k d_j f(a_1(\mu_0),a_2(\mu_0)) \quad \text{for } i,j,k,\ell = 1,2. \end{aligned} \tag{4A.5}$$

(C) In this expression you now have, fill in d_id_jf given by:

$$\begin{pmatrix} d_1d_1f(a_1(\mu_0),a_2(\mu_0)) \\ d_1d_2f(a_1(\mu_0),a_2(\mu_0)) \\ d_2d_2f(a_1(\mu_0),a_2(\mu_0)) \end{pmatrix} =$$

$$\Delta^{-1}\begin{bmatrix} 2|\lambda(\mu_0)|^2+(d_3X^3_{\mu_0}(a(\mu_0)))^2 & -2|\lambda(\mu_0)|d_3X^3_{\mu_0}(a(\mu_0)) & 2|\lambda(\mu_0)|^2 \\ |\lambda(\mu_0)|d_3X^3_{\mu_0}(a(\mu_0)) & (d_3X^3_{\mu_0}(a(\mu_0)))^2 & -|\lambda(\mu_0)|d_3X^3_{\mu_0}(a(\mu_0)) \\ 2|\lambda(\mu_0)|^2 & 2|\lambda(\mu_0)|d_3X^3_{\mu_0}(a(\mu_0)) & 2|\lambda(\mu_0)|^2+(d_3X^3_{\mu_0}(a(\mu_0)))^2 \end{bmatrix} \begin{pmatrix} -d_1d_1X^3_{\mu_0}(a(\mu_0)) \\ -d_1d_2X^3_{\mu_0}(a(\mu_0)) \\ -d_2d_2X^3_{\mu_0}(a(\mu_0)) \end{pmatrix} \tag{4A.6}$$

where $\Delta = (d_3 X^3_{\mu_0}(a(\mu_0)))((d_3 X^3_{\mu_0}(a(\mu_0)))^2 + 4|\lambda(0)|^2)$.

(Note: if X^3 is linear, all derivatives of f are zero.)

(D) If you have done it correctly your expression for $V'''(0)$ is now entirely in terms of known quantities and in an explicit example, is a known real number and you may go to Step I to read off the results.

Remarks. S. Wan has recently obtained a proof of the stability formula using complex notation, which is somewhat simpler. It also yields information on the period (it is closely related to the expression $\beta_0 + ia_0 b_0$ from Section 3C; stability being the real part, the period being the imaginary part). The formulas have been programmed and interesting numerical work is being done by B. Hassard (SUNY at Buffalo).

SECTION 4B
EXAMPLES

We now consider a few examples to illustrate how the above procedure works. The first few examples are all simple, designed to illustrate basic points. We finish in Example 4B.8 with a fairly intricate example from fluid mechanics (the Lorenz equations).

(4B.1) Example (see Hirsch-Smale [1], Chapter 10 and Zeeman [2] for motivation). Consider the differential equation $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 - a\frac{dx}{dt} + x = 0$, a special case of Lienard's equation. Before applying the Hopf Bifurcation theorem we make this into a first order differential equation on R^2. Let $y = \frac{dx}{dt}$. Then we get the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -y^3 + ay - x.$$

Let $X_a(x,y) = (y,-y^3+ay-x)$. Now $X_a(0,0) = 0$ for all a and

$$dX_a(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & +a \end{pmatrix}.$$

The eigenvalues are $\frac{a \pm \sqrt{a^2-4}}{2}$. Consider a such that $|a| < 2$. In this case $\operatorname{Im} \lambda(a) \neq 0$, where $\lambda(a) = \frac{a + \sqrt{a^2-4}}{2} = \frac{a}{2} + i\frac{\sqrt{4-a^2}}{2}$. Furthermore, for $-2 < a < 0$, $\operatorname{Re} \lambda(a) < 0$ and for $a = 0$, $\operatorname{Re} \lambda(a) = 0$ and for $2 > a > 0$, $\operatorname{Re} \lambda(a) > 0$ and $\left.\frac{d(\operatorname{Re} \lambda(a))}{da}\right|_{a=0} = \frac{1}{2}$. Therefore, the Hopf Bifurcation theorem applies and we conclude that there is a one parameter family of closed orbits of $X = (X_a, 0)$ in a neighborhood of $(0,0,0)$. To find out if these orbits are stable and if they occur for $a > 0$, we look at $X_0(x,y) = (y, -y^3-x)$. $dX_0(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\lambda(0) = i$. Recall that to use the stability formula developed in the above section we must choose coordinates so that

$$dX_0(0,0) = \begin{pmatrix} 0 & \operatorname{Im} \lambda(0) \\ -\operatorname{Im} \lambda(0) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, the original coordinates are appropriate to the calculation; an example where this is not true will be given below. We calculate the partials of X_0 at $(0,0)$ up to order three:

$$\frac{\partial^n X_1}{\partial x^j \partial y^{n-j}}(0,0) = 0 \quad \text{for all} \quad n > 1$$

since $X_1(x,y) = y$.

$$\frac{\partial^2 X_2}{\partial x_1^2}(0,0) = 0, \quad \frac{\partial^2 X_2}{\partial x_1 \partial x_2}(0,0) = 0, \quad \frac{\partial^2 X_2}{\partial x_2^2}(0,0) = 0,$$

$$\frac{\partial^3 X_2}{\partial x_1^3}(0,0) = 0, \quad \frac{\partial^3 X_2}{\partial x_1^2 \partial x_2}(0,0) = 0, \quad \frac{\partial^3 X_2}{\partial x_1 \partial x_2^2}(0,0) = 0, \quad \frac{\partial^3 X_2}{\partial x_2^3}(0,0) = -6.$$

Thus, $V'''(0) = \frac{3\pi}{4}(-6) < 0$, so the periodic orbits are attracting and bifurcation takes place above criticality.

(4B.2) Example. On R^2 consider the vector field

$$X_\mu(x,y) = (x+y, -x^3-x^2y+(\mu-2)x+(\mu-1)y).$$

$$X_\mu(0,0) = 0$$

and

$$dX_\mu(0,0) = \begin{pmatrix} 1 & 1 \\ \mu-2 & \mu-1 \end{pmatrix}.$$

The eigenvalues are $\frac{\mu \pm \sqrt{\mu^2-4}}{2} = \frac{\mu \pm i\sqrt{4-\mu^2}}{2}$. Let $-1 < \mu < 1$, then the conditions for Hopf Bifurcation to occur at $\mu = 0$ are fulfilled with $\lambda(0) = 1$. Consider $X_0(x,y) = (x+y, -x^3-x^2y-2x-y)$. $dX_0(0,0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, which is not in the required form. We must make a change of coordinates so that $dX_0(0,0)$ becomes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. That is, we must find vectors $\hat{e}_1$ and $\hat{e}_2$ so that $dX_0(0,0)\hat{e}_1 = -\hat{e}_2$ and $dX_0(0,0)\hat{e}_2 = \hat{e}_1$. The vectors $\hat{e}_1 = (1,-1)$ and $\hat{e}_2 = (0,1)$ will do. (A procedure for finding $\hat{e}_1$ and $\hat{e}_2$ is to find α and $\overline{\alpha}$ the complex eigenvectors; we may then take $\hat{e}_1 = \alpha + \overline{\alpha}$ and $\overline{e}_2 = i(\alpha-\overline{\alpha})$. See Section 4, Step 1 for details.) $X_0(x\hat{e}_1+y\hat{e}_2) = X_0(x,y-x) = (y,-x^3-x^2(y-x)-2x-(y-x)) = (y,-x^2y-x-y) = y\hat{e}_1 + (-x^2y-x)\hat{e}_2$. Therefore in the new coordinate system, $X_0(x,y) = (y,-x^2y-x)$.

$$\frac{\partial^n X_1}{\partial x^j \partial y^{n-j}}(0,0) = 0 \quad \text{for all } n > 1.$$

$$\frac{\partial^2 X_2}{\partial y^2}(0,0) = 0, \quad \frac{\partial^2 X_2}{\partial x \partial y}(0,0) = 0, \quad \frac{\partial^2 X_2}{\partial y^2}(0,0) = 0,$$

$$\frac{\partial^3 X_2}{\partial x^3}(0,0) = 0, \quad \frac{\partial^3 X_2}{\partial x^2 \partial y}(0,0) = -2, \quad \frac{\partial^3 X_2}{\partial x \partial y^2}(0,0) = 0, \quad \frac{\partial^3 X_2}{\partial y^3}(0,0) = 0.$$

Therefore, $V'''(0) = \frac{3\pi}{4|\lambda(0)|}(-2) < 0$. The orbits are stable and bifurcation takes place above criticality.

(4B.3) Example. The van der Pol Equation

The van der Pol equation $\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$ is important in the theory of the vacuum tube. (See Minorsky [1], LaSalle-Lefschetz [1] for details.) As is well known, for all $\mu > 0$, there is a stable oscillation for the solution of this equation. It is easy to check that the eigenvalue conditions for the Hopf Bifurcation theorem are met so that bifurcation occurs at the right for $\mu = 0$. However, if $\mu = 0$ the equation is a linear rotation, so $V^{(n)}(0) = 0$ for all n. For $\mu = 0$, all circles centered at the origin are closed orbits of the flow. By uniqueness, these are the closed orbits given by the Hopf Theorem. Thus, we cannot use the Hopf Theorem on the problem as stated here to get the existence of stable oscillations for $\mu > 0$. In fact, the closed orbits bifurcate off the circle of radius two (see LaSalle-Lefschetz [1], p. 190 for a picture). In order to obtain them from the Hopf Theorem one needs to make a change of coordinates bringing the circle of radius 2 into the origin. In fact the general van der Pol equation $u'' + f(u)u' + g(u) = 0$ can be transformed into the general Lienard equation $x' = y - F(x)$, $y' = -g(x)$ by means of $x = u$, $y = u' + F(u)$. This change of coordinates reduces the present example to 4B.1. See Brauer-Nohel [1, p. 219 ff.] for general information on these matters.)

(4B.4) Example. On R^3, let $X_\mu(x,y,z) = (\mu x+y+6x^2, -x+\mu y+yz, (\mu^2-1)y-x-z+x^2)$. Then $X_\mu(0,0,0) = 0$ and

$$dX_\mu(0,0,0) = \begin{pmatrix} \mu & 1 & 0 \\ -1 & \mu & 0 \\ -1 & \mu^2-1 & -1 \end{pmatrix}$$ which has eigenvalues -1 and $\mu \pm i$. For $\mu = 0$, the eigenspace of $\pm i$ for $dX_0(0,0,0)$ is spanned by $\{(1,0,-1),(0,1,0)\}$. The complementary subspace is spanned by $(0,0,1)$. With respect to this basis $X_\mu(x,y,z) = (\mu x+y+6x^2,-x+\mu y+yz,\mu x+\mu^2 y-z+x^2)$. We now compute the stability condition. $|\lambda(0)| = 1$ and $d_3X_0^3(0,0,0) = -1$.

$$\begin{pmatrix} d_1d_1f(0,0) \\ d_1d_2f(0,0) \\ d_2d_2f(0,0) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ -1 & -1 & 1 \\ 2 & -2 & 3 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6/5 \\ -2/5 \\ 4/5 \end{pmatrix}$$

$$d_\ell d_j d_k \hat{X}_0^1(0,0) = 0$$

$$d_1d_1d_1\hat{X}_0^2(0,0) = 0$$

$$d_1d_1d_2\hat{X}_0^2(0,0) = 1\cdot(6/5) = 6/5$$

$$d_1d_2d_2\hat{X}_0^2(0,0) = 3\cdot 1\cdot 4/5 = 12/5.$$

Therefore, $V'''(0) = \frac{3\pi}{4}(6/5 + 12/5) > 0$, so the orbits are unstable.

The next two exercises discuss some easy two dimensional examples.

(4B.5) <u>Exercise</u>. Let $X(x,y) = A_\mu\binom{x}{y} + B(x,y)$ where $B(x,y) = (ax^2+cy^2,dx^2+fy^2)$ and $A = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$. Show that $\mu_0 = 0$ is a bifurcation point and that a stable periodic orbit develops for $\mu > 0$ provided that $cf > ad$. (This example is a two dimensional prototype of the Navier-Stokes equations; note that X is linear plus quadratic.)

(4B.6) Exercise (see Arnold [2]). Let $\dot{z} = z(i\omega+\mu+cz\bar{z})$ in R^2 using complex notation. Show that bifurcation to periodic orbits takes place at $z = \mu = 0$. Show that these orbits are stable if $c < 0$.

For some additional easy two dimensional examples, see Minorsky [1], p. 173-177. These include an oscillator instability of an amplifier in electric circuit theory and oscillations of ships. The reader can also study the example $\ddot{x} + \sin x + \varepsilon\dot{x} = M$ for a pendulum with small friction and being acted on by a torque M. See Arnold [1], p. 94 and Andronov-Chailkin [1].

The following is a fairly simple three-dimensional exercise to warm the reader up for the following example.

(4B.7) Exercise. Let $X_\mu(x,y,z,w) = (\mu x+y+z-w,-x+\mu y,-z,-w+y^3)$. Show that bifurcation to attracting closed orbits takes place at $(x,y,z,w) = (0,0,0,0)$ and $\mu = 0$. (Answer: $V'''(0) = -9\pi/4$).

The following example, the most intricate one we shall discuss, has many interesting features. For example, change in the physical parameters can alter the bifurcation from sub to supercritical; in the first case, complicated "Lorenz attractors" (see Section 12) appear in plcace of closed orbits.

(4B.8) Example (suggested by J.A. Yorke and D. Ruelle). The Lorenz Equations (see Lorenz [1]).

The Lorenz equations are an idealization of the equations of motion of the fluid in a layer of uniform depth when the temperature difference between the top and the bottom is maintained at a constant value. The equations are

$$\frac{dx}{dt} = -\sigma x + \sigma y$$

$$\frac{dy}{dt} = -xz + rx - y$$

$$\frac{dz}{dt} = xy - bz.$$

Lorenz [1] says that "... x is proportional to the intensity of the convective motion, while y is proportional to the temperature difference between the ascending and descending currents, similar signs of x and y denoting that warm fluid is rising and cold fluid is descending. The variable z is proportional to the distortion of the vertical temperature profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries". $\sigma = K^{-1}\nu$ is the Prandtl number, where K is the coefficient of thermal expansion and ν is the viscosity; r, the Rayleigh number, is the bifurcation parameter.

For $r > 1$, the system has a pair of fixed points at $x = y = \pm\sqrt{b(r-1)}$, $z = r - 1$. The linearization of the vector field at the fixed point $x = y = +\sqrt{b(r-1)}$, $z = r - 1$ is

$$M \equiv \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix} = dX_r(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1).$$

The characteristic polynomial of this matrix is

$$x^3 + (\sigma+b+1)x^2 + (r+\sigma)bx + 2\sigma b(r-1) = 0,$$

which has one negative and two complex conjugate roots. For $\sigma > b + 1$, a Hopf bifurcation occurs at $r = \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$. We shall now prove this and determine the stability.

Let the characteristic polynomial be written

$$(x-\lambda)(x-\bar{\lambda})(x-\alpha) = 0, \text{ where } \lambda = \lambda_1 + i\lambda_2,$$

i.e. $x^3 - (2\lambda_1+\alpha)x^2 + (|\lambda|^2+2\lambda_1\alpha)x - |\lambda|^2\alpha = 0.$

Clearly, this has two pure imaginary roots iff the product of the coefficients of x^2 and x equals the constant term. That is, iff $(\sigma+b+1)(r_0+\sigma) = 2\sigma b(r_0-1)$ or $r_0 = \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$. Thus, we have the bifurcation value. We now wish to calculate $\lambda_1'(r_0)$. Equating coefficients of like powers of x, we get

$$(\sigma+b+1) = 2\lambda_1 + \alpha$$

$$(r+\sigma)b = |\lambda|^2 + 2\lambda_1\alpha$$

and

$$-2\sigma b(r-1) = |\lambda|^2\alpha.$$

Thus, $\alpha = -(\sigma+b+1+2\lambda_1)$ and $(r+\sigma)b\alpha = 2\lambda_1\alpha^2 - 2\lambda b(r-1)$, so that $-(\sigma+b+1+2\lambda_1)(r+\sigma)b = -2\sigma b(r-1) + 2\lambda_1(\sigma+b+1+2\lambda_1)^2$. Differentiating with respect to r, setting $r = r_0$, and recalling that $\lambda_1(r_0) = 0$, we obtain

$$\lambda_1'(r_0) = \frac{b(\sigma-b-1)}{2[b(r_0+\sigma) + (\sigma+b+1)^2]} > 0 \quad \text{for} \quad \sigma > b + 1.$$

Thus, the eigenvalues cross the imaginary axis with non-zero speed, so a Hopf bifurcation occurs at $r_0 = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$. We will compute $V'''(0)$ for arbitrary σ, b and will evaluate it at the physically significant values $\sigma = 10$, $b = 8/3$. At r_0, α is minus the coefficient of x^2, so $\alpha = -(\sigma+b+1)$; and at r_0, $|\lambda|^2$ is the coefficient of x, so $|\lambda|^2 = \frac{2\sigma b(\sigma+1)}{\sigma-b-1}$.

Following I(A) of Section 4A, must compute a basis for R^3 in which

$$dX_{r_0}(\sqrt{b(r_0-1)},\sqrt{b(r_0-1)},r_0-1) =$$

$$M = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r_0-1)} \\ \sqrt{b(r_0-1)} & \sqrt{b(r_0-1)} & -b \end{bmatrix}$$

becomes

$$\begin{bmatrix} 0 & \sqrt{\dfrac{2\sigma b(\sigma+1)}{\sigma-b-1}} & 0 \\ -\sqrt{\dfrac{2\sigma b(\sigma+1)}{\sigma-b-1}} & & 0 \\ 0 & 0 & -(\sigma+b+1) \end{bmatrix}.$$

The basis vectors will be u,v,w where $Mu = -|\lambda|v$, $Mv = -|\lambda|u$, $Mw = \alpha w$. An eigenvector of M with eigenvalue α is $\left(-\sigma, b+1, \sqrt{\dfrac{(\sigma+b+1)(\sigma-b-1)b}{\sigma+1}}\right)$. The eigenspace of M corresponding to the eigenvalues $\lambda,\bar{\lambda}$ is the orthogonal complement of the eigenvector of M^t corresponding to the eigenvalue α. This eigenvector is $\left(\sigma+b-1, -(\sigma-b-1), -\sqrt{\dfrac{b(\sigma+b+1)(\sigma-b-1)}{\sigma+1}}\right)$. We will choose $u = (-(\sigma-b-1), -(\sigma+b-1), 0)$. Because $M^2 = \begin{pmatrix} -|\lambda|^2 & 0 \\ 0 & -|\lambda|^2 \end{pmatrix}$ on the eigenspace of $\lambda,\bar{\lambda}$, we may choose $v = -\dfrac{1}{|\lambda|}Mu =$ $\left(\sqrt{\dfrac{2\sigma b(\sigma-b-1)}{\sigma+1}}, -\sqrt{\dfrac{2b(\sigma-b-1)}{\sigma(\sigma+1)}}, (\sigma-1)\sqrt{\dfrac{2(\sigma+b+1)}{\sigma}}\right)$. We now have our new basis and, as in Example 4B.4, after writing the differential equations in this basis to get $\hat{X}^1,\hat{X}^2,\hat{X}^3$, we are ready to compute $V'''(0)$. This is a very lengthy computation so we give only the results.* Following (II) of Section 4A,

*Details will be sent on request.

the second derivative terms of $V'''(0)$ are

$$\frac{3\pi}{4|\lambda|^2}\left(-\frac{\partial^2 X^1}{\partial x_1^2}\frac{\partial^2 X^1}{\partial x_1 \partial x_2} + \frac{\partial^2 X^2}{\partial x_2^2}\frac{\partial^2 X^2}{\partial x_1 \partial x_2} + \frac{\partial^2 X^2}{\partial x_1^2}\frac{\partial^2 X^2}{\partial x_1 \partial x_2}\right.$$

$$\left. - \frac{\partial^2 X^1}{\partial x_1^2}\frac{\partial^2 X^1}{\partial x_1 \partial x_2} + \frac{\partial^2 X^1}{\partial x_1^2}\frac{\partial^2 X^1}{\partial x_2^2}\frac{\partial^2 X^2}{\partial x_2^2}\right) =$$

$$\frac{3\pi(\sigma-b-1)^2}{4\sigma b(\sigma+1)^3\omega^2}\sqrt{\frac{2b(\sigma-b-1)}{\sigma(\sigma+1)}}\Big\{[2\sigma^2 b^2(\sigma+b-1) - 2\sigma b^2(\sigma-b-1)$$

$$+ 2\sigma b(\sigma-1)(\sigma-b-1)(\sigma+b+1) + 2\sigma(\sigma-1)^2(\sigma+1)(\sigma+b+1)][b(\sigma+1)(\sigma+b-1)(\sigma-b-1)$$

$$- 2b^2(\sigma-b-1) + 2b(\sigma-1)(\sigma-b-1)(\sigma+b+1) + 2(\sigma-1)^2(\sigma+1)(\sigma+b+1)]$$

$$+ [b(\sigma+1)(\sigma-b-1)^2 + (\sigma^2-1)(\sigma-b-1)(\sigma+b+1) - \sigma(\sigma^2-1)(\sigma+b-1)(\sigma+b+1)$$

$$- \sigma b(\sigma+1)(\sigma+b-1)(\sigma-b-1) - (\sigma^2-1)(\sigma+b+1)(\sigma-b-1)^2][(\sigma^2-1)(\sigma+b-1)(\sigma+b+1)$$

$$+ b(\sigma+1)(\sigma+b-1)(\sigma-b-1) - 2b(\sigma-1)(\sigma+b+1) - 2B^2(\sigma-b-1)$$

$$+ 2B(\sigma-1)(\sigma-b-1)(\sigma+b+1)] + 2b\sigma(\sigma-1)(\sigma+1)^2(\sigma-b-1)(\sigma+b+1)(\sigma+b-1)^2$$

$$+ 2b^2\sigma(\sigma+1)^2(\sigma-b-1)^2(\sigma+b-1)^2 - [8b^2\sigma(\sigma-1)(\sigma-b-1)(\sigma+b+1)$$

$$+ 8b\sigma(\sigma-1)^2(\sigma+1)(\sigma+b+1) - 8b^3\sigma(\sigma-b-1)][(\sigma-1)(\sigma-b-1)(\sigma+b+1)$$

$$- b(\sigma-b-1) - (\sigma-1)(\sigma+b+1)]\Big\} \equiv A_1\xi.$$

The third derivative terms are

$$\frac{3\pi}{4|\lambda|}\left(\frac{\partial^3 \hat{X}^1}{\partial x_1^3} + \frac{\partial^3 \hat{X}^1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 \hat{X}^2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 \hat{X}^2}{\partial x_2^3}\right)$$

$$= \frac{3\pi}{4|\lambda|}\left\{\frac{\partial^2 f}{\partial x_1^2}\left(3\frac{\partial^2 X^1}{\partial x_1 \partial x_2} + \frac{\partial^2 X^2}{\partial x_2 \partial x_3}\right)\right.$$

$$\left. + \frac{\partial^2 f}{\partial x_2^2}\left(3\frac{\partial^2 X^2}{\partial x_2 \partial x_3} + \frac{\partial^2 X^1}{\partial x_1 \partial x_3}\right) + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\left(\frac{\partial^2 X^1}{\partial x_1 \partial x_2} + \frac{\partial^2 X^2}{\partial x_1 \partial x_3}\right)\right\}$$

$$\frac{3\pi(\sigma-1)(\sigma-b-1)^2}{2\sigma b(\sigma+1)^3\omega^2}\sqrt{\frac{2b(\sigma-b-1)}{\sigma(\sigma+1)}}\frac{1}{[(\sigma+b+1)^2(\sigma-b-1) + 8\sigma b(\sigma+1)]}$$

$$\Big\{-4\sigma^2 b^2(\sigma^2-1)(\sigma-b+1) + \sigma b(\sigma+b-1)(\sigma-b-1)(\sigma+b+1)^2$$

$$+ 2\sigma b(\sigma+1)(\sigma-b-1)(\sigma+b+1)(\sigma^2+b-1)$$

$$- 2\sigma^2 b^2(\sigma+1)(\sigma+b-1)(\sigma+b+1)][3\sigma b(\sigma+1)(\sigma+b-1) - b(\sigma+1)(2b+1)(\sigma-b-1)$$

$$- 2b(\sigma-b-1)^2(\sigma+b+1) - 4(\sigma^2-1)(\sigma-b-1)(\sigma+b+1) + (\sigma^2-1)(b+2)(\sigma+b+1)]$$

$$+ [4\sigma^2 b^2(\sigma+1)^2(\sigma+b-1) - 2\sigma b(\sigma+1)(\sigma+b+1)(\sigma-b-1)(\sigma^2+b-1)$$

$$+ 2\sigma^2 b^2(\sigma+1)(\sigma+b+1)(\sigma+b-1) - 8\sigma^2 b^2(\sigma+1)(\sigma^2+b-1)$$

$$- 2\sigma b^2(\sigma+b+1)^2(\sigma-b-1)(\sigma^2+b-1)][2b(\sigma+b+1)(\sigma-b-1)^2$$

$$- 4(\sigma^2-1)(\sigma-b-1)\sigma+b+1) + \sigma b(\sigma+1)(\sigma+b-1)$$

$$+ b(\sigma+1)(2b+5)(\sigma-b-1) + 3(b+2)(\sigma^2-1)(\sigma+b+1)]$$

$$- [2\sigma b^2(\sigma+1)(\sigma+b-1)(\sigma+b+1) + 4\sigma b(\sigma+b+1)(\sigma^2+b-1)$$

$$- (\sigma+b+1)^2(\sigma-b-1)(\sigma^2+b-1) + \sigma b(\sigma+b+1)^2(\sigma+b-1)]$$

$$[2\sigma b^2(\sigma+1)(b+2)(\sigma-b-1) + 2\sigma b^2(\sigma-b-1)^2(\sigma+b+1)$$

$$- \sigma(\sigma-1)(\sigma+1)^2(\sigma+b+1)(\sigma-b-1) + \sigma b(\sigma+1)^2(\sigma-b-1)(\sigma+b-1)$$

$$- (\sigma-1)(\sigma+1)^2(b+1)(\sigma-b-1)(\sigma+b+1) - b(b+1)(\sigma+1)^2(\sigma-b-1)^2$$

$$- b(\sigma+1)(\sigma+b+1)(\sigma-b-1)^3]\Big\} \equiv A_2\xi$$

where

$$\omega = \frac{(\sigma-1)}{(\sigma+1)}\sqrt{\frac{2(\sigma+b+1)}{\sigma}}\,[(b+1)(\sigma+1)(\sigma-b-1) + \sigma(\sigma+1)(\sigma+b-1) + b(\sigma+b+1)(\sigma-b-1)]$$

and

$$\xi = \frac{3\pi(\sigma-b-1)^2}{2\sigma b(\sigma+1)^3\omega^2}\sqrt{\frac{2b(\sigma-b-1)}{\sigma(\sigma+1)}}.$$

Since $V'''(0) = (A_1+A_2)\xi$ and $\xi > 0$, the periodic orbits resulting from the Hopf bifurcation are stable if $A_1 + A_2 < 0$, and unstable if $A_1 + A_2 > 0$. For $\sigma = 10$, $b = 8/3$, $A_1 \simeq 1.63 \times 10^9$, $A_2 \simeq 0.361 \times 10^9$, $A_1 + A_2 \simeq 1.99 \times 10^9$;

therefore, the orbits are unstable, i.e. the bifurcation is subcritical.

Our calculations thus prove the conjecture of Lorenz [1], who believed the orbits to be unstable because of numerical work he had done. For different σ or b however, the sign may change, so one cannot conclude that the closed orbits are always unstable. A simple computer program determines the regions of stability and instability in the b-σ plane. See Figure 4B.1

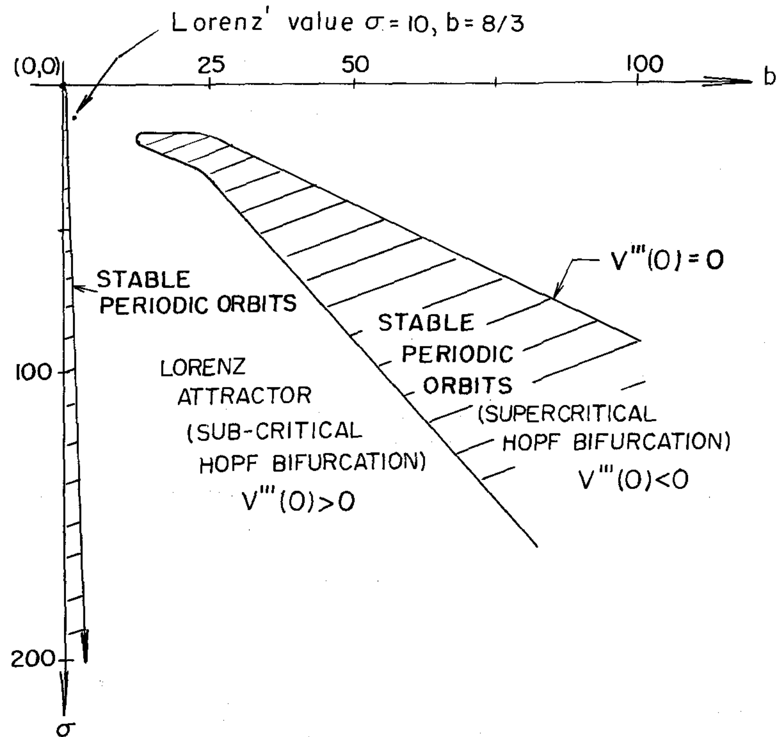

Figure 4B.1

We also investigate the behavior of the system for fixed $b > 0$ as $\sigma \to \infty$. This has also been done by Martin and McLaughlin [1]. Our result agrees with theirs. We proceed as follows:

$$\{(\sigma+b+1)^2(\sigma-b-1) + 8\sigma b(\sigma+1)\}A_1 + A_2 = p(b,\sigma)$$

is a polynomial of degree 11 in σ. For b fixed, the highest order term is $(8b^2+12b)\sigma^{11}$. If $b > 0$ this coefficient is positive, so for large positive σ (with b fixed), $V'''(0) > 0$ and the bifurcation is subcritical.

This example may be important for understanding eventual theorems of turbulence (see discussion in Section 9). The idea is shown in Figure 4B.2. For further information on the behavior of solutions above criticality, see Lorenz [1] and Section 12. (L. Howard has built a device to simulate the dynamics of these equations.)

(4B.9) <u>Exercise</u>. Analyze the behavior of $V'''(0)$ as $b \to \infty$ for fixed σ and as $b \to \infty$ for $\sigma = \beta b$, for various $\beta > 0$.

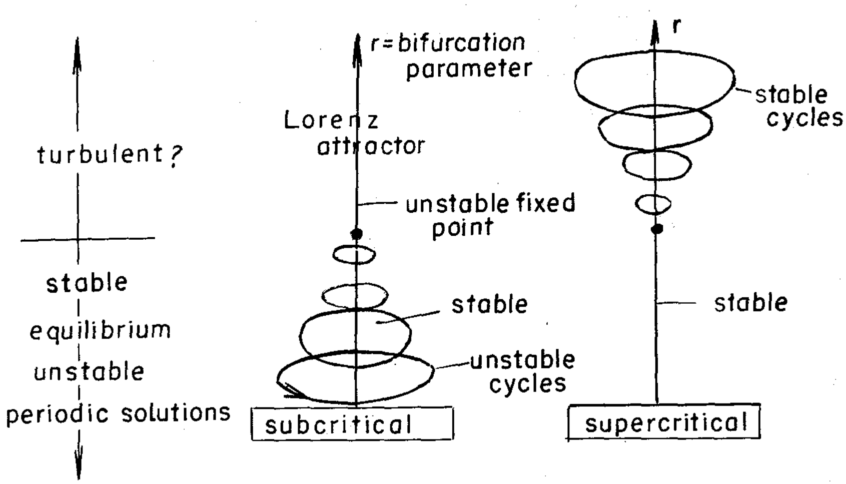

Figure 4B.2

(4B.10) Exercise. By a change of variable, analyze the stability of the fixed point $x = y = -b(r-1)$, $z = r - 1$. Show that Hopf bifurcation occurs at $r_0 = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ and that the orbits obtained are attracting iff those obtained from the analysis above are.

(4B.11) Exercise. Prove that for $r > 1$, the matrix

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -1 \end{bmatrix}$$

has one negative and two complex conjugate roots.

(4B.12) Exercise. Let F denote the vector field on R^3 defined by the right hand side of the Lorentz equations.

(a) Note that $\operatorname{div} F = -\sigma - 1 - b$, a constant. Use this to estimate the order of magnitude of the contractions in the principal directions at the fixed points.

(b) If $V = (x,y,z)$, show that the inner product $\langle F,V\rangle$ is a quadratic function of x,y,z. By considering $\frac{d}{dt}\langle V,V\rangle$, show that this implies that solutions of the Lorenz equations are globally defined in t. [Note that most quadratic equations, eq: $\dot{x} = x^2$ do not have global t solutions.]

(4B.13) Exercise. The following equations arise in the oscillatory Zhabotinskii reaction (cf. Hastings-Murray [1]):

$$\dot{x} = s(y-xy+x-qx^2)$$

$$\dot{y} = \frac{1}{s}(fz-y-xy)$$

$$\dot{z} = w(x-y)$$

(compare the Lorenz equations!). Let f be the bifurcation parameter and let, eg: $s = 7.7 \times 10$, $q = 8.4 \times 10^{-6}$, $w = 1.61 \times 10^{-1}$. Show that a Hopf bifurcation occurs at $f = f_c$ where

$$2q(2+3f_c) = (2f_c+q-1)[(1-f_c-q) + \{(1-f_c-q)^2 + 4q(1+f_c)\}^{1/2}].$$

Show that for the above values, the bifurcation is subcritical. S. Hastings informs us that the bifurcation picture looks like that in Figure 4B.3 (the existence of stable closed orbits for supercritical values is proven in Hastings-Murray [1]).

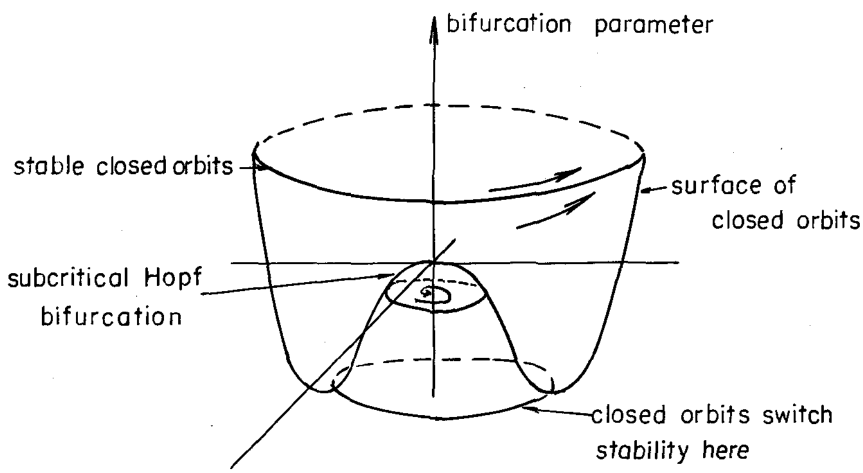

Figure 4B.3

SECTION 4C

HOPF BIFURCATION AND THE METHOD OF AVERAGING

by

S. Chow and J. Mallet-Paret

The method of averaging* provides an algorithm for preparing a bifurcation problem, that is, putting it into a normal form. Once this is done, one may more readily determine certain qualitative features of the bifurcation, by means of the implicit function theorem (or contraction mapping principle) and the center manifold theorem.

Consider first the Hopf bifurcation problem

$$\dot{z} = f(z,\alpha) \qquad (4C.1)$$

about the equilibrium $z = 0$. Thus assume $z \in R^n$, $\alpha \in (-\alpha_0,\alpha_0)$, f takes values in R^n and $f(0,\alpha) = 0$. For simplicity, an ordinary differential equation is considered although we could just as well consider a partial differential equation. In

*The method has been used by a number of authors, such as Halanay, Hale, Meyer, Diliberto, etc. cf. Kurzweil [1] and articles in Lefschetz [1].

this case z and $\dot{z}$ belong to (generally different) Banach spaces X_1 and X_2 and f is smooth from $X_1 \times (-\alpha_0,\alpha_0)$ to X_2. We may write (1) in the form

$$\begin{aligned} \dot{z} &= A(\alpha)z + g(z,\alpha) \\ |g(z,\alpha)| &= O(|z|^2). \end{aligned} \tag{4C.2}$$

The standard hypotheses (see Sections 1,3) on the spectrum of $A(\alpha)$ hold, namely that it posses a pair $\lambda(\alpha),\overline{\lambda(\alpha)}$ of complex conjugate eigenvalues of the form

$$\lambda(\alpha) = \gamma(\alpha) + i\omega(\alpha)$$

where

$$\gamma(0) = 0, \ \nu \overset{\text{def}}{=} \gamma'(0) \neq 0$$

and

$$\omega \overset{\text{def}}{=} \omega(0) \neq 0$$

and that the remainder of the spectrum of $A(\alpha)$ stays uniform positive distance away from the imaginary axis. Decompose $z \in R^n$ as

$$z = (x,y) \qquad P \oplus Q = R^2 \oplus R^{n-2}$$

according to the spectrum of $A(0)$, so that P is the eigenspace corresponding to $\lambda(0),\overline{\lambda(0)}$ and Q is its complement. With this decomposition we may assume

$$A(\alpha) = \begin{pmatrix} A_P(\alpha) & O(\alpha) \\ O(\alpha) & A_Q(\alpha) \end{pmatrix}$$

where

$$A_P(\alpha) = \begin{pmatrix} \gamma(\alpha) & -\omega(\alpha) \\ \omega(\alpha) & \gamma(\alpha) \end{pmatrix}$$

and the spectrum of $A_Q(\alpha)$ stays uniformly away from the imaginary axis. We may also represent x in polar coordinates

$$x = (r \cos \theta,\ r \sin \theta).$$

Consider a periodic solution bifurcating from $(x,y,\alpha) = (0,0,0)$. The differential equation for r is

$$\dot{r} = \gamma(\alpha)r + O(r^2) = \alpha\nu r + O(\alpha^2 r) + O(r^2),$$

and it follows that when r attains its maximum on the solution, $\dot{r} = 0$, and hence

$$\alpha = O(r). \tag{4C.3}$$

Now the periodic solution also lies on the center manifold Σ, described by

$$\Sigma\colon\ y = \phi(r,\theta,\alpha). \tag{4C.4}$$

The fixed point $(r,y) = (0,0)$ lies on Σ for all α; moreover, Σ is tangent to $Px(-\alpha_0,\alpha_0)$ at $(r,\alpha) = (0,0)$ which implies (4C.4) has the form

$$\Sigma\colon\ y = r\psi(r,\theta,\alpha)$$
$$\psi(0,\theta,0) = 0.$$

Thus, we have on the periodic solution

$$y = O(r^2) + O(r\alpha) = O(r^2). \tag{4C.5}$$

Choosing $\varepsilon > 0$ of the same order as the amplitude of the solution, we may scale the equation by making the replacements

$$r \to \varepsilon r, \quad \alpha \to \varepsilon\alpha, \quad y \to \varepsilon y.$$

The estimates (4C.3), (4C.5) imply then

$$\left.\begin{aligned} r &= O(1) \\ \dot{x} &= O(1) \\ y &= O(\varepsilon) \end{aligned}\right\} \text{ in scaled coordinates.} \tag{4C.6}$$

The precise relation between ε and α will be determined later when α will be chosen as a particular function of ε. We will in fact show then that

$$\alpha = 0 \quad \text{in scaled coordinates.}$$

Expand the differential equation (4C.2) in a Taylor series, in scaled coordinates. It is not difficult to show the estimates (4C.6) imply the equation takes the form

$$\begin{aligned} \dot{x} &= A_P(\varepsilon\alpha) + B_2x^2 + \varepsilon^3B_3x^3 + Gxy + O(\varepsilon^2\alpha) + O(\varepsilon^3) \\ \dot{y} &= A_Qy + \varepsilon Jx^2 + O(\varepsilon\alpha) + O(\varepsilon^2) \end{aligned} \tag{4C.7}$$

where

$B_j = (B_j^1, B_j^2)$ = homogeneous polynomial of degree j in $x \in R^2$, taking values in R^2

$G: R^2 \times R^{n-2} \to R^2$ bilinear

$A_Q = A_Q(0)$

$J = R^2 \times R^2 \to R^2$ symmetric, bilinear.

In polar coordinates (4C.7) becomes

$$\left.\begin{aligned} \dot{r} &= \varepsilon\alpha\nu r + \varepsilon r^2C_3(\theta) + \varepsilon^2r^3C_4(\theta) + \varepsilon rG_2(\theta)y \\ &\qquad + O(\varepsilon^2\alpha) + O(\varepsilon^3) \\ \dot{\theta} &= \omega + \varepsilon rD_3(\theta) + O(\varepsilon\alpha) + O(\varepsilon^2) \\ \dot{y} &= A_Qy + \varepsilon r^2J(\cos\theta, \sin\theta)^2 + O(\varepsilon\alpha) + O(\varepsilon^2) \end{aligned}\right\} \tag{4C.8}$$

where

$$C_j(\theta) = (\cos\theta)B^1_{j-1}(\cos\theta, \sin\theta) + (\sin\theta)B^2_{j-1}(\cos\theta, \sin\theta)$$

$$D_j(\theta) = (\cos\theta)B^2_{j-1}(\cos\theta, \sin\theta) - (\sin\theta)B^1_{j-1}(\cos\theta, \sin\theta)$$

$= $ homogeneous trigonometric polynomials of degree j

$G_2(\theta) = $ homogeneous trigonometric polynomial of degree 2 taking values in Q^* the dual of Q.

The goal of the method of averaging is to "average out" the dependence of $\dot{r}$ on θ and y, that is, to find a new radial coordinate $\bar{r}$ in which the equation for $\dot{\bar{r}}$ is

$$\dot{\bar{r}} = F(\bar{r}, \varepsilon).$$

If this were done, then all periodic solutions would simply be circles $\bar{r} = \bar{r}(\varepsilon)$ satisfying $F(\bar{r}(\varepsilon), \varepsilon) = 0$. Actually it is not necessary to entirely eliminate dependence on ε and y; generally all that is required is the absence of ε and y from a finite number of terms in the Taylor series expression in ε and y. For example, in (4C.8), generically it is enough to average the $\varepsilon, \varepsilon y$ and ε^2 terms.

More precisely, consider any differential equation

$$\dot{r} = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \varepsilon^j R_{jk}(r, \theta, \alpha) y^k$$

$$\dot{\theta} = \omega + O(\varepsilon)$$

$$\dot{y} = A_Q y + O(\varepsilon).$$

The series for $\dot{r}$ may be a finite Taylor series with remainder. In order to average a given term, say R_{pq}, define a new coordinate $\bar{r}$ by

$$\bar{r} = r + \varepsilon^p u(r, \theta, \alpha) y^q.$$

In the new coordinates, the coefficient of $\varepsilon^p y^q$ becomes $\bar{R}_{pq}$

where

$$\overline{R}_{pq}(r,\theta,\alpha) = \frac{\partial u}{\partial \theta}\omega + quy^q A_Q + R_{pq}(r,\theta,\alpha).$$

Two cases are considered.

Case I, $q = 0$. In this case we choose u to be

$$u(r,\theta,\alpha) = -\frac{1}{\omega}\int_0^\theta R_{p0}(r,\xi,\alpha)d\xi + \frac{\theta}{2\pi\omega}\int_0^{2\pi} R_{p0}(r,\xi,\alpha)d\xi.$$

Observe that u is 2π-periodic in θ and $\overline{R_{p0}}$ is independent of θ and, in fact, is the mean value

$$\overline{R}_{p0}(r,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} R_{p0}(r,\xi,\alpha)d\xi.$$

Therefore, we have averaged the coefficient of $\varepsilon^p y^0$.

Case II, $q > 0$. Here we wish to choose u so that $\overline{R_{pq}}$ is identically zero thus eliminating the $\varepsilon^p y^q$ term. Therefore, we seek a 2π-periodic function $u(r,\theta,\alpha)$ satisfying

$$\frac{\partial u}{\partial \theta}\omega + quA_Q + R_{pq}(r,\theta,\alpha) = 0. \qquad (4C.9)$$

By considering R_{pq} as a forcing term in (4C.9), it follows that such a unique u always exists if and only if the homogeneous equation

$$\frac{\partial u}{\partial \theta}\omega + quA_Q = 0$$

has no nontrivial solution 2π-periodic solution. It can be shown that this is the case provided that

$$\frac{1}{\omega_i}\sum_{j=1}^{n-2} n_j\lambda_j \neq \text{integer}$$

$$\text{for all integers} \quad n_j \geq 0, \quad \sum_{j=1}^{n-2} n_j = q$$

$$\lambda_1, \ldots, \lambda_{n-2} = \text{eigenvalues of } A_Q.$$

In particular this can always be done if either $q = 1$, or A_Q is stable.

We return to the bifurcation problem (4C.8) and now average the $\varepsilon, \varepsilon y$ and ε^2 in the above fashion by means of the transformation

$$\bar{r} = r + \varepsilon u(r,\theta,\alpha) + \varepsilon w(r,\theta,\alpha)y + \varepsilon^2 v(r,\theta,\alpha).$$

In fact, the transformation has the form

$$\bar{r} = r + \varepsilon r^2 u(\theta) + \varepsilon r w(\theta) y + \varepsilon^2 r^3 v(\theta),$$

and this yields the equation for $\bar{r}$

$$\left.\begin{aligned} \dot{\bar{r}} &= \varepsilon[\alpha\nu\bar{r} + \bar{r}^2 C_3(\theta) + \bar{r}^2 u'(\theta)\omega] \\ &+ \varepsilon\bar{r}[G_2(\theta) + w(\theta)A_Q + w'(\theta)\omega]y \\ &+ \varepsilon^2\bar{r}^3[C_4(\theta) + u'(\theta)D_3(\theta) + w(\theta)J(\cos\theta, \sin\theta)^2 \\ &- 2u(\theta)u'(\theta)\omega + v'(\theta)\omega] \\ &+ O(\varepsilon^2\alpha) + O(\varepsilon^3). \end{aligned}\right\} \quad (4C.10)$$

Since C_3 has mean value zero, we choose

$$u(\theta) = -\frac{1}{\omega}\int_0^\theta C_3(\xi)d\xi$$

so that the coefficient of ε in (10) is $\alpha\nu\bar{r}$. Set $w(\theta)$ equal to the unique solution of

$$G_2(\theta) + w(\theta)A_Q + w'(\theta)\omega = 0$$
$$w(\theta) \quad \text{of period} \quad 2\pi.$$

so the εy term vanishes. Finally, we may choose $v(\theta)$ to make the coefficient of $\varepsilon^2\bar{r}^3$ the constant

$$\begin{aligned} K &= \mathrm{mean}[C_4(\theta) + u'(\theta)D_3(\theta) + w(\theta)J(\cos\theta, \sin\theta) \\ &\quad -2u(\theta)u'(\theta)\omega] \\ &= \frac{1}{2\pi}\int_0^{2\pi} C_4(\theta) - \frac{1}{\omega} C_3(\theta)D_3(\theta) + w(\theta)J(\cos\theta, \sin\theta)^2 . \end{aligned}$$

Thus in the new coordinates (r,θ,y), (4C.8) becomes

$$\left.\begin{aligned} \dot{\bar{r}} &= \varepsilon\alpha\nu\bar{r} + \varepsilon^2\bar{r}^3K + O(\varepsilon^2\alpha) + O(\varepsilon^3) \\ \dot{\theta} &= \omega + O(\varepsilon) \\ \dot{y} &= A_Q y + O(\varepsilon). \end{aligned}\right\} \qquad (4C.11)$$

The equation for $\dot{y}$ may be neglected now as we restrict to the center manifold $y = r\psi(\varepsilon r,\theta,\varepsilon\alpha)$. Moreover, it is not difficult to show that the unique branch of periodic solutions bifurcating from the origin has the form

$$\left.\begin{aligned} \bar{r} &= \left|\frac{\nu}{K}\right|^{1/2} + O(\varepsilon) \\ \alpha &= -\varepsilon\,\mathrm{sgn}(\nu K) \end{aligned}\right\} \text{ in scaled, averaged coordinates}$$

that is,

$$\left.\begin{aligned} r &= \left|\frac{\nu}{K}\right|^{1/2}\varepsilon + O(\varepsilon^2) \\ y &= O(\varepsilon^2) \\ \alpha &= -\varepsilon^2\mathrm{sgn}(\nu K) \end{aligned}\right\} \text{ in original coordinates.} \qquad (4C.12)$$

The amplitude of the bifurcating solution is therefore approximately $\left(-\frac{\alpha\nu}{K}\right)^{1/2}$ and one sees the period is near $\frac{2\pi}{\omega}$.

In case $K = 0$, one simply averages higher order terms in ε and y in the same manner. The possible normal forms one arrives at in this case are

$$\dot{\bar{r}} = \varepsilon\alpha\nu\bar{r} + \varepsilon^{2p}\bar{r}^{2p+1}K' + O(\varepsilon^2\alpha) + O(\varepsilon^{2p+1})$$

$$\dot{\theta} = \omega + O(\varepsilon)$$

for integers $p \geq 2$ and $K' \neq 0$. The bifurcating solution in this case has the form

$$\left.\begin{aligned} r &= \left|\frac{\nu}{K'}\right|^{1/2p}\varepsilon + O(\varepsilon^2) \\ y &= 0(\varepsilon^2) \\ \alpha &= -\varepsilon^{2p}\operatorname{sgn}(\nu K') \end{aligned}\right\} \text{ in original coordinates}$$

so has amplituded near $\left(-\frac{\alpha\nu}{K'}\right)^{1/2p}$ and period near $\frac{2\pi}{\omega}$. Observe that in all these cases bifurcation takes place only on one side of $\alpha = 0$. For cases in which all bifurcating solutions occur at $\alpha = 0$ (for example in the proof of the Lyapunov center theorem - see Section 3C), the method of averaging gives no information.

For more details of the above method, as well as several applications, see Chow and Mallet-Paret [1]. We mention here two examples treated in this paper using averaging.

(1) Delay Differential Equations (Wright's Equation). The equation

$$\dot{z}(t) = -az(t-1)[1+z(t)]$$

arises in such diverse areas as population models and number theory, and is one of the most deeply studied delay equations.

For $a > \frac{\pi}{2}$, topological fixed point techniques prove the existence of a periodic solution. Using averaging techniques, one can analyze the behavior of this solution near $a = \frac{\pi}{2}$. In particular, for $\frac{\pi}{2} < a < \frac{\pi}{2} + \varepsilon$ the solution bifurcates from $z = 0$, is stable, and has the asymptotic form

$$z(t) = K(a - \tfrac{\pi}{2})^{1/2}\cos(\tfrac{\pi}{2}\, t) + O(a - \tfrac{\pi}{2})$$

$$K = \left(\frac{40}{3\pi-2}\right)^{1/2} \approx 2.3210701.$$

(2) Diffusion Equations. Linear equations with non-linear boundary conditions, such as

$$u_t = u_{xx} \qquad t \geq 0 \qquad 0 < x < 1$$

$$u_x(0,t) = 0 \qquad u_x(1,t) = ag(u(0,t),\, u(1,t))$$

occur in various problems in biology and chemical reactions. (See, for example, Aronson [2].) Here we take

$$g(u,v) = \alpha u + \beta v + O(u^2+v^2),$$

so the linearized equation around $u = 0$ has the boundary conditions

$$u_x(0,t) = 0 \qquad u_x(1,t) = a[\alpha u(0,t) + \beta u(1,t)].$$

For appropriate parameter values (α,β) a pair of eigenvalues of this problem crosses the imaginary axis with non-zero speed as a passes the critical value a_0. The stability of the resulting Hopf bifurcation can be determined by averaging.

The power of the averaging method is that it can handle a rather wide variety of bifurcation problems. We mention two here.

(3) Almost Periodic Equations. Consider

$$\dot{z}(t) = A(\alpha)z + g(z,t,\alpha) \tag{4C.13}$$

where $A(\alpha)$ is as before, g is almost periodic in t uniformly for (z,α) in compact sets and $g = O(|z|^2)$. Suppose in addition that the periods $\frac{2\pi}{N\omega}$ for $N = 1,2,3,4$ are bounded away from the fundamental periods for g. Then an averaging procedure similar to that described above yields a normal form in scaled coordinates given by (4C.11). Here however the higher order terms (but not the constant K) are almost periodic in t. Thus this manifold can be thought of as a cylinder in the $(x,t) \in R^n \times R^1$ space, where each section $t = \text{const.}$ is a circle near $x = 0$. The cylinder is almost periodic in t with the same fundamental periods as those in g.

(4) A special case of (3) occurs in studying the bifurcation of an invariant torus from a periodic orbit of an autonomous equation. In an appropriate local coordinate system around the orbit, the autonomous equation takes the form (4C.13) where t represents the (periodic) coordinate around the orbit and z the normal to the orbit. The condition on the fundamental periods of g reduces to the

standard condition that the periodic orbit have no characteristic multipliers which are N^{th} roots of unity, for $N = 1,2,3,4$. The invariant cylinder that is obtained if $K \neq 0$ is periodic in t and thus is actually a two dimensional torus around the periodic orbit.