## SECTION 8

## BIFURCATION THEOREMS FOR PARTIAL DIFFERENTIAL EQUATIONS

As we have seen in earlier sections, there are two methods generally available for proving bifurcation theorems. The first is the original method of Hopf, and the second is using invariant manifold techniques to reduce one to the finite (often two) dimensional case.

For partial differential equations, such as the Navier-Stokes equations (see Section 1) the theorems as formulated by Hopf (see Section 5) or by Ruelle-Takens (see Sections 3,4) do not apply as stated. The difficulty is precisely that the vector fields generating the flows are usually not smooth functions on any reasonable Banach space.

For partial differential equations, Hopf's method can be pushed through, provided the equations are of a certain "parabolic" type. This was done by Judovich [11], Iooss [3], Joseph and Sattinger [1] and others. In particular, the methods do apply to the Navier-Stokes equations. The result

is that if the spectral conditions of Hopf's theorem are fulfilled, then indeed a periodic solution will develop, and moreover, the stability analysis given earlier, applies. The crucial hypothesis needed in this method is analyticity of the solution in $t$.

Here we wish to outline a different method for obtaining results of this type. In fact, the earlier sections were written in such a way as to make this method fairly clear: instead of utilizing smoothness of the generating vector field, or $t$-analyticity of the solution, we make use of smoothness of the flow $F_t^\mu$. This seems to have technical advantages when one considers the next bifurcation to invariant tori; analyticity in $t$ is not enough to deal with the Poincaré map of a periodic solution (see Section 2B).

It is useful to note that there are general results applicable to concrete evolutionary partial differential equations which enable the determination of the smoothness of their flows on convenient Banach spaces. These results are found in Dorroh-Marsden [1]. We have reproduced some of the relevant parts of this work along with useful background material in Section 8A for the reader's convenience.

We shall begin by formulating the results in a general manner and then in Section 9 we will describe how this procedure can be effected for the Navier-Stokes equations. In the course of doing this we shall establish basic existence, uniqueness and smoothness results for the Navier-Stokes equations by using the method of Kato-Fujita [1] and results of Dorroh-Marsden [1] (§8A).

It should be noted that bifurcation problems for partial

differential equations other than the Navier-Stokes equations are fairly common. For instance in chemical reactions (see Kopell-Howard [1,2,5]) and in population dynamics (see Section 10). Problems in other subjects are probably of a similar type, such as in electric circuit theory and elastodynamics (see Stern [1], Ziegler [1] and Knops and Wilkes [1]). It seems likely that the real power of bifurcation theorems and periodic solutions is only beginning to be realized in applications.

The General Set-Up and Assumptions.

We shall be considering a system of evolution equations of the general form

$$dx/dt = X_\mu(x), \quad x(0) \text{ given},$$

where $X_\mu$ is a densely defined nonlinear operator on a suitable function space $E$, a Banach space, and depends on a parameter $\mu$. For example, $X_\mu$ may be the Navier-Stokes operator and $\mu$ the Reynolds number (see Section 1). This system is assumed to define unique local solutions $x(t)$ and thereby a semiflow $F_t$ which maps $x(0)$ to $x(t)$, for $\mu$ fixed, $t \geq 0$.

The key thing we need to know about the flow $F_t$ of our system is that, for each fixed $t,\mu$, $F_t$ is a $C^\infty$ mapping on the Banach space $E$ ($F_t$ is only locally defined in general). We note (see Section 8A at the end of this section) the properties that one usually has for $F_t$ and which we shall assume are valid:

(a) $F_t$ is defined on an open subset of

$$R^+ \times E, \quad R^+ = \{t \in R | t \geq 0\};$$

(b) $F_{t+s} = F_t \circ F_s$ (where defined );

(c) $F_t(x)$ is separately (hence jointly [§ 8A]) continuous in $t, x \in R^+ \times E$.

We shall make two standing assumptions on the flow. The first of these is

(8.1) <u>Smoothness Assumption</u>. <u>Assume that for each fixed</u> $t$, $F_t$ <u>is a</u> $C^\infty$ <u>map of</u> (<u>an open set in</u>) $E$ <u>to</u> $E$.

This is what we mean by a smooth semigroup. Of course we cannot have smoothness in $t$ since, in general, the generator $X_\mu$ of $F_t$ will only be densely defined and is not a smooth map of $E$ to $E$. However, as explained in Section 8A it is not unreasonable to expect smoothness in $\mu$, $t$ if $t > 0$. (This is the nonlinear analogue of "analytic semigroups" and holds for "parabolic type" equations). We shall need this below.

In Section 9 we shall outline how one can check this assumption for the Navier-Stokes equations by using general criteria applicable to a wide variety of systems. (For systems such as nonlinear wave equations, this is well known through the work of Segal [1] and others.)

The second condition is

(8.2) <u>Continuation Assumption</u>. <u>Let</u> $F_t(x)$, <u>for fixed</u> $x$ <u>lie in a bounded set in</u> $E$ <u>for all</u> $t$ <u>for which</u> $F_t(x)$

*is defined. Then* $F_t(x)$ *is defined for all* $t \geq 0$.

This merely states that our existence theorem for $F_t$ is strong enough to guarantee that the only way an orbit can fail to be defined is if it tends to infinity in a finite time. This assumption is valid for most situations and in particular for the Navier-Stokes equations.

Suppose we have a fixed point of $F_t$, which we may assume to be $0 \in E$; i.e., $F_t(0) = 0$ for all $t \geq 0$. Letting $DF_t$ denote the Fréchet derivative of $F_t$ for fixed $t$, $G_t = DF_t(0)$ is clearly a linear semigroup on $E$. Its generator, which is formally $DX(0)$, is therefore a densely defined closed linear operator which represents the linearized equations.* Our hypotheses below will be concerned with the spectrum of the linear semigroup $G_t$, which, under suitable conditions (Hille-Phillips [1]) is the exponential of the spectrum of $DX(0)$. (Compare Section 2A).

The third assumption is:

(8.3) *Hypotheses on the Spectrum. Assume we have a family* $F_t^\mu$ *of smooth nonlinear semigroups defined for* $\mu$ *in an interval about* $0 \in R$. *Suppose* $F_t^\mu(x)$ *is jointly smooth in* $t,x,\mu$, *for* $t > 0$. *Assume:*

(a) $0$ is a fixed point for $F_t^\mu$;

(b) for $\mu < 0$, the spectrum of $G_t^\mu$ is contained in

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*Even if a semigroup is not smooth, it may make sense to linearize the equations and the flow. For example the flow of the Euler equations is $C^1$ from $H^s$ to $H^{s-1}$, but the derivative extends to a bounded operator on $H^{s-1}$; cf. Dorroh-Marsden [1].

$D = \{z \in \mathbb{C}: |z| < 1\}$, where $G_t^\mu = D_x F_t^\mu(x)|_{x=0}$;

(c) for $\mu = 0$ (resp. $\mu > 0$) the spectrum of $G_1^\mu$ at the origin has two isolated simple eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ with $|\lambda(\mu)| = 1$ (resp. $|\lambda(\mu)| > 1$) and the rest of the spectrum is in $D$ and remains bounded away from the unit circle.

(d) $(d/dt)|\lambda(\mu)|\big|_{\mu=0} > 0$ (the eigenvalues move steadily across the unit circle).

Under these conditions, bifurcation to periodic orbits takes place. For their stability we make:

(8.4) Stability Assumption. The condition $V'''(0) < 0$ holds, where $V'''(0)$ is calculated according to the procedures of Section 4 (see Section 4A).

This calculation may be done directly on the vector field $X$, since the computations are finite dimensional; unboundedness of the generator $X$ causes no problems.

Bifurcation to Periodic Orbits.

Let us recap the result:

(8.5) Theorem. Under the above hypotheses, there is a fixed neighborhood $U$ of $0$ in $E$ and an $\varepsilon > 0$ such that $F_t^\mu(x)$ is defined for all $t \geq 0$ for $\mu \in [-\varepsilon, \varepsilon]$ and $x \in V$. There is a one-parameter family of closed orbits for $F_t^\mu$ for $\mu > 0$, one for each $\mu > 0$ varying continuously with $\mu$. They are locally attracting and hence stable. Solutions near them are defined for all $t \geq 0$. There is a neighborhood $U$ of the origin such that any closed orbit in $U$ is

one of the above orbits.

Note especially that near the periodic orbit, solutions are defined for all $t \geq 0$. This is an important criterion for global existence of solutions (see also Sattinger [1,2]).

Of course one can consider generalizations: for instance, when the system depends on many parameters with multiple eigenvalues crossing or to a system with symmetry as was previously described. Also, the bifurcation of periodic orbits to invariant tori can be proved in the same way.

Proof of Theorem (Outline). From our work in Section 2 we know that the center manifold theorem applies to flows. Thus, for the smooth flow $F_t(x,\mu) = (F_t^\mu(x),\mu)$ we can deduce the existence of a locally invariant center manifold $C$; a three-manifold tangent to the $\mu$ axis and the two eigendirections of $G_t^0(0)$. (The invariant manifold is attracting and contains all the local recurrence, but $F_t$ still is only a local flow on this center).

Now there is a remarkable property of smooth semiflows (going back to Bochner-Montgomery [1]; cf. Chernoff-Marsden [2]) which is proved in Section 8A: this is that the semiflow $F_t$ is generated by a $C^\infty$ vector field on the finite dimensional manifold $C$; i.e., the original $X$ restricts to a $C^\infty$ vector field (defined at all points) on $C$.

This trick then immediately reduces us to the Hopf theorem in two dimensions and the proof can then be referred back to Section 3. $\square$

Bifurcation to Invariant Tori.

This can be carried out exactly as in Section 6. However, as explained in Section 2B, we need to know that $F_t^\mu(x)$ is smooth in $t,\mu,x$ for $t > 0$. Then the Poincaré map for the closed orbit will be well defined and smooth and after we reduce to finite dimensions via the center manifold theorem as in Section 6, it will be a diffeomorphism by the corollary on p. 265. Therefore we can indeed use exactly the same bifurcation theorems as in Section 6 for bifurcation to tori. To check the hypothesis of smoothness, one uses results of Section 8A and Section 9.

## SECTION 8A

## NOTES ON NONLINEAR SEMIGROUPS

In this section we shall assemble some tools which are useful in the proofs of bifurcation theorems. We begin with some general properties of flows and semiflows ($\equiv$ nonlinear groups and nonlinear semigroups) following Chernoff-Marsden [1,2]. These include various important continuity and smoothness properties. Next, we give a basic criterion for when a semiflow consists of smooth mappings following Dorroh-Marsden [1].

### Flows and Semiflows.

(8A.1) Definitions. Let $D$ be a set. A flow on $D$ is a collection of maps $F_t : D \to D$ defined for all $t \in \mathbb{R}$ such that:

1) $F_0 = \text{Identity}$

and

2) $F_{t+s} = F_t \circ F_s$ for all $t,s \in \mathbb{R}$.

Note that for fixed $t$, $F_t$ is one to one and onto, since $F_{-t} \circ F_t = \mathrm{Id}$, and $F_t \circ F_{-t} = \mathrm{Id}$; i.e., $F_t^{-1} = F_{-t}$.

A <u>semiflow on D</u> is a collection of maps $F_t: D \to D$ defined for $t \geq 0$, also satisfying 1) and 2) for $t,s \geq 0$.

<u>Warning</u>: a semiflow need not consist of bijections.

(8A.2) <u>Definition</u>. Let $N$ be a topological space and $D \subset N$. A <u>local flow on D</u> is a map $F: \mathscr{D} \subset \mathbb{R} \times D \to D$, where $\mathscr{D}$ is open in the $N$ topology induced on $\mathbb{R} \times D$, such that for all $x \in D$, $(0,x) \in \mathscr{D}$ and if $\mathscr{D}_t = \{x \in D \mid (t,x) \in \mathscr{D}\}$ so we can define $F_t: \mathscr{D}_t \to D$, then $F_t$ satisfies (1) and (2) where defined. The flow is <u>maximal</u> if $(t,x) \in \mathscr{D}$, $(s,F_t(x)) \in \mathscr{D} \Rightarrow (s+t,x) \in \mathscr{D}$. Similarly one defines a <u>local semiflow</u> and a <u>maximal semiflow</u>.

Now let $N$ be a Banach manifold. <u>A vector field with domain D</u> is a map $X: D \to T(N)$ such that $X(x) \in T_x(N)$ for all $x \in D$. ($T_xN$ is the tangent space to $N$ at $x \in D \subset N$.) An <u>integral curve for X</u> is a curve $c$: $(a,b) \subset \mathbb{R} \to D$ such that $c$ is differentiable as a map from $(a,b)$ to $N$ and $c'(t) = X(c(t))$. A <u>flow</u> (resp. semiflow, local flow) <u>for X</u> is a flow (resp. semiflow, local flow) on $D$ such that for all $x \in D$, the map $t \mapsto F_t(x)$ is an integral curve of $X$.

If $F_t$ is a flow on $N$ such that $F: \mathbb{R} \times N \to N$ via $(t,x) \mapsto F_t(x)$ is a $C^0$ map, we say $F$ is a <u>$C^0$ flow on N</u>.

If $F_t$ is a flow for $X$ and $F_t$ extends to a continuous map $F_t: N \to N$ and the extension is a $C^0$ flow on $M$, then $F_t$ is a <u>$C^0$ flow for X</u>.

If $F_t$ is a $C^0$ flow on $N$ and for all $t$ fixed

$F_t: N \to N$ is of class $C^k$ (resp. $T^k$), then $F_t$ is a <u>flow of class $C^k$</u>. (resp. $T^k$) Here $F_t: N \to N$ is $T^k$ means that the $j^{th}$ tangent map $T^jF_t: T^j(N) \to T^j(N)$ exists and is continuous for $j \leq k$; $F_t: N \to N$ is $C^k$ means that in each chart, the map $x \to d_x^jF$ $j \leq k$ is continuous in the norm topology. ($d_x^jF$ is the $j^{th}$ total derivative of $F$ at $x$. It is a $j$-multilinear map on the model space for $N$.) The $T^k$ case differs from the $C^k$ case in that norm continuity is replaced by strong continuity.

<u>Warning</u>. A $C^k$ flow is not assumed to be $C^k$ in the $t$-variable, only in the $x$-variable. A flow will be $C^k$ in the $t$-variable only if it is generated by a smooth everywhere defined vector field; see however the Bochner-Montgomery theorem stated below.

<u>Separate and Joint Continuity</u>.

(8A.3) <u>Theorem</u>. (<u>Chernoff-Marsden</u> [2]). <u>Let</u> $N$ <u>be a Banach manifold</u>. <u>Let</u> $F_t$ <u>be a flow (or local flow) on</u> $N$, <u>and let</u> $F$ <u>be separately continuous in</u> $x$ <u>and</u> $t$ (<u>i.e.</u>, $t \mapsto F_t(x)$ <u>is continuous for fixed</u> $x$ <u>and</u> $x \mapsto F_t(x)$ <u>is continuous for fixed</u> $t$), <u>then</u> $F_t$ <u>is a</u> $C^0$ <u>flow</u>; <u>i.e.</u> $F_t$ <u>is jointly continuous</u>.

For the proof, we shall use the following.

(8A.4) <u>Lemma</u>. (Bourbaki [1] Chapter 9, page 18; Choquet, [1] Vol. 1, page 127). <u>Let</u> $E$ <u>be a Baire space</u>. <u>Let</u> $F$, $G$ <u>be metric spaces</u>. <u>Let</u> $\phi: E \times F \to G$ <u>be separately continuous</u>, <u>then for all</u> $f \in F$, <u>there is a dense set</u>

$S_f \subseteq E$ <u>whose complement is first category such that if</u> $e \in S_f$, <u>then</u> $\phi$ <u>is continuous at</u> $(e,f)$.

<u>Proof of the Theorem (8A.3)</u>. Since this is a local theorem, we may work in a chart. Therefore, we may assume that $N$ is a Banach space and $F_t$ is a local flow on $N$. We let $E = \mathbb{R}$, $F = G = N$. Let $x \in U \subset N$, $t \in (-\varepsilon,\varepsilon)$. There is a dense set of $t_x \in (-\varepsilon,\varepsilon)$ such that $F$ is continuous at $(t_x,x)$. Since the domain of definition of $F$ is assumed open in $\mathbb{R} \times N$ we can choose $t_x$ close to $t$ so that the various compositions are defined. Let $t_n \to t$ and $x_n \to x$, and write $F_{t_n}(x_n) = F_{t-t_x} \circ F_{t_x+t_n-t}(x_n)$. Since $t_x + t_n - t \to t_x$ and $F$ is continuous at $(t_x,x)$, $F_{t_x+t_n-t}(x_n) = y_n \to F_{t_x}(x)$. Since for fixed $t$, $x \mapsto F_t(x)$ is continuous, we have that $F_{t_n}(x_n) = F_{t-t_x}(y_n) \to F_{t-t_x}(F_{t_x}(x)) = F_t(x)$. $\square$

(8A.5) <u>Remarks</u>.

1) Let $G$ be a topological group which is also a Baire space. Let $\Phi: G \times N \to N$ be a separately continuous group action of $G$ on a metric space $N$, then the above argument also shows that $\Phi$ is jointly continuous.

2) Suppose that $D \subseteq N$ is dense and that $F_t$ is a flow on $D$ which extends by continuity to a flow on $M$ such that $t \mapsto F_t(x)$ is continuous for each $x \in D$. Then the same is true for each $x \in N$ and the extended flow is $C^0$. Indeed, let $x_n \to x$, where $x_n \in D$ and $x \in N$. Then for fixed $t$, $F_t(x_n) \to F_t(x)$, so that $t \mapsto F_t(x)$ is the pointwise limit of continuous functions. Therefore, for each $x \in N$, there is a second category set $S_x \subseteq \mathbb{R}$ such that if

$t \in S_x$, then $t \mapsto F_t(x)$ is continuous. The argument used in the proof of Theorem 8A.3 shows that $S_x = \mathbb{R}$ for all $x \in N$.

3) Many of these results can be generalized to the case in which $N$ is not locally metrizable; e.g. a manifold modelled on a topological vector space (e.g.: a manifold modelled on a Banach space with the weak topology; - a "weak manifold"). c.f. Ball [1].

4) The same argument also works for semiflows, at least for $t > 0$. If $N$ is locally compact, joint continuity is also true at $t = 0$ (Dorroh [1], but one can give a more direct argument). In general, however, joint continuity may fail at $t = 0$ so it has to be postulated.

Using these methods we can obtain an interesting result on the $t$-continuity of the derivatives of a differentiable flow.

(8A.6) <u>Theorem.</u> <u>Let</u> $N$ <u>be a Banach manifold</u>. <u>Let</u> $F_t$ <u>be a</u> $C^0$ <u>flow</u> (<u>or local flow</u>, <u>or semiflow</u>) <u>on</u> $N$. <u>Let</u> $F_t$ <u>be of class</u> $T^k$ <u>for</u> $k \geq 1$. <u>Then for each</u> $j \leq k$, $T^jF_t : T^j(N) \to T^j(N)$ <u>is jointly continuous in</u> $t \in \mathbb{R}$ <u>and</u> $x \in T^j(N)$. (<u>Only</u> $t > 0$ <u>for semiflows</u>.)

<u>Proof</u>. By induction and Theorem 8A.3 we are reduced immediately to the case $k = 1$. We may also assume that we are working in a chart. Therefore, $TF_t(x,v) = (F_t(x), D_xF_t(x)\cdot v)$. By assumption, this is continuous in the space variable $x$, so we need to show it is continuous in $t$. But clearly $D_xF_t(x)\cdot v = \lim_{n\to\infty} n(F_t(x + \frac{v}{n}) - F_t(x))$. Thus $t \mapsto D_xF_t(x)\cdot v$ is the pointwise limit of continuous functions so has a dense set of points of $t$-continuity. The rest of

the proof is as in Remark 2 of (8A.5). □

The Generalized Bochner-Montgomery Theorem.

For simplicity we will give the next result for the case of flat manifolds. But it holds for general manifolds M, as one sees by working in local charts.

(8A.7) Theorem. (Chernoff-Marsden). *Let $F_t$ be a jointly continuous flow on a Banach space $\mathbb{E}$. Suppose that, for each $t$, $F_t$ is a $C^k$ mapping, $k \geq 1$. Assume also that, for each $x \in \mathbb{E}$, $||DF_t(x)-I|| \to 0$ as $t \to 0$, where $||\cdot||$ is the operator norm. Then $F_t(x)$ is jointly of class $C^k$ in $t$ and $x$. Moreover the generator $X$ of the flow is an everywhere-defined vector field of class $C^{k-1}$ on $\mathbb{E}$.*

Proof. Under the stated hypotheses, we can show that $DF_t(x)$ is jointly continuous as a mapping from $R \times \mathbb{E}$ into $\mathscr{L}(\mathbb{E}, \mathbb{E})$, the latter being all bounded linear maps of $\mathbb{E}$ to $\mathbb{E}$ equipped with the norm topology. In fact, if we write $\phi(t,x)$ for $DF_t(x)$. The chain rule implies the relation

$$\phi(s+t,x) = \phi(s,F_t(x))\cdot\phi(t,x). \tag{8A.1}$$

We have separate continuity of $\phi$ by assumption, and then we can apply Baire's argument as in Theorem (8A.3), together with the identity (8A.1) to deduce joint continuity.

Now let $\phi(t)$ be a $C^\infty$ function on $\mathbb{R}$ with compact support. Define $J_\phi: \mathbb{E} \to \mathbb{E}$ by

$$J_\phi(x) = \int_{-\infty}^{\infty} \phi(t)F_t(x)dt. \tag{8A.2}$$

By joint continuity, we can differentiate under the integral

sign in (8A.2), thus obtaining

$$DJ_\phi(x) = \int_{-\infty}^{\infty} \phi(t) DF_t(x)dt. \tag{8A.3}$$

Now if $\phi$ approximates the $\delta$-function then $||DJ_\phi(x)-I||$ is small; in particular $DJ_\phi(x)$ is invertible. By the inverse function theorem it follows that $J_\phi$ is a local $C^k$ diffeomorphism.

Moreover,

$$\begin{aligned} J_\phi(F_t(x)) &= \int_{-\infty}^{\infty} \phi(s) F_{s+t}(x)ds \\ &= \int_{-\infty}^{\infty} \phi(s-t) F_s(x)ds. \end{aligned}$$

The latter is differentiable in $t$ and $x$. Since $J_\phi$ is a local $C^k$ diffeomorphism, $F_t x$ is jointly $C^k$ for $t$ near 0. But then the flow identity shows that the same is true for all $t$. $\square$

(8A.8) Remarks.

1) The above result is a non-linear generalization of the fact well known in linear theory that a norm-continuous linear semigroup has a bounded generator (and hence is defined for all $t \in \mathbb{R}$, not merely $t \geq 0$).

Furthermore, the same argument as above applies to semiflows and to local flows. This has the amusing consequence that a semiflow which is $C^k$ and the derivative is norm continuous in $t$ at $t = 0$ has integral curves which are locally uniformly extendable backwards in time (since the generator is $C^{k-1}$). This is most significant when combined with the next remark.

2) If $\mathbb{E}$ is finite dimensional, the norm convergence of $DF_t(x)$ to $I$ follows automatically from the smoothness hypothesis. Indeed, Theorem (8A.6) implies that $DF_t(x) \to I$ in the strong operator topology, i.e., $DF_t(x)v \to v$ for each $v$; but for a finite-dimensional space this is the same as norm convergence.

Accordingly if $M$ is a finite-dimensional manifold, a flow on $M$ which is jointly continuous and $C^k$ in the space variable is jointly $C^k$. The latter is a classical result of Montgomery. There is a generalization, due to Bochner and Montgomery [1] for actions of finite dimensional Lie groups. This generalization can also be obtained by the methods used to prove Theorem (8A.7) (cf. Chernoff-Marsden [2]).

Let us summarize a consequence of remarks (8A.8) that is useful.

(8A.9) <u>Corollary</u>. <u>Let</u> $F_t$ <u>be a local</u> $C^k$ <u>semiflow on a Banach manifold</u> $N$. <u>Suppose that</u> $F_t$ <u>leaves invariant a finite dimensional submanifold</u> $M \subset N$. <u>Then on</u> $M$, $F_t$ <u>is locally reversible, is jointly</u> $C^k$ <u>in</u> $t$ <u>and</u> $x$ <u>and is generated by a</u> $C^{k-1}$ <u>vector field on</u> $M$.

Another fact worth pointing out is a result of Dorroh [1]. Namely, under the conditions of Theorem (8A.7), $F_t$ is actually locally conjugate to a flow with a $C^k$ generator (rather than $C^{k-1}$).

Lipschitz Flows.

(8A.10) Definitions. Let $F_t$ be a flow (or a semi-flow) on a metric space $M$, e.g. a Banach manifold. We say that $F_t$ is Lipschitz provided that for each $t$ there is a constant $M_t$ such that

$$d(F_t x, F_t y) \leq M_t d(x,y), \quad \forall\, x,y \in M.$$

The least such constant is called the Lipschitz norm, $||F_t||_{Lip}$.

We say that $F_t$ is locally Lipschitz provided that, for every $x_0 \in M$ and $t_0 \in \mathbb{R}$, there is a neighborhood $\mathscr{U}$ of $x_0$ and a number $\varepsilon > 0$, such that

$$d(F_t x, F_t y) \leq M(t_0, x_0) d(x,y)$$

for all $x,y \in \mathscr{U}$ and $t \in [t_0-\varepsilon, t_0+\varepsilon]$. If $\mathscr{U}$ can be taken to be any bounded set, we say that the flow $F_t$ is semi-Lipschitz. (This term was introduced by Segal.) Note that $C^1$ flows are locally Lipschitz.

Let $F_t$ be a continuous Lipschitz flow, and let $M_t = ||F_t||_{Lip}$. Then (just as in the linear case) we have an estimate of the form

$$M_t \leq Me^{\beta|t|} \tag{8A.4}$$

where $M$, $\beta$ are constants. Indeed, note that $M_t$ is submultiplicative: $M_{s+t} \leq M_s \cdot M_t$; this is an immediate consequence of the flow identity. Moreover, we know

$$M_t = \sup_{x \neq y} d(F_t x, F_t y)/d(x,y).$$

Thus $M_t$ is lower semicontinuous, being the supremum of a family of continuous functions. In particular, $M_t$ is meas-

urable. But then an argument of Hille-Phillips [1, Thm. 7.6.5] shows that (8A.4) holds for some constants $M$, $\beta$.

Uniqueness of Integral Curves.

It is a familiar fact that integral curves of Lipschitz vector fields are uniquely determined by their initial values, but that there are continuous vector fields for which this is not the case.* On the other hand, it is known that integral curves for generators of linear semigroups are unique. The following result shows that such uniqueness is a consequence of the local Lipschitz nature of the flow (cf. van Kampen's Theorem; Hartman [1, p. 35]).

(8A.11) Theorem. Let $X$ be a vector field on the Banach manifold $M$, with domain $D$. Assume that $X$ has a locally Lipschitz flow $F_t$. More precisely, assume that:

(a) $F_t$ is a group of bijections on $D$, and, for each $x \in D$, $t \mapsto F_t x$ is differentiable in $M$, with

$$\frac{d}{dt} F_t(x) = X(F_t(x))$$

(b) For each $x_0 \in M$ and $t_0 \in \mathbb{R}$ there is a neighborhood $\mathscr{U}$ of $x_0$ in $M$ and an $\varepsilon > 0$, such that in local charts,

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*The famous example is $X(x) = x^{2/3}$ on the line. In a Fréchet space $E$, a continuous linear vector field $S: X \to X$ may have infinitely many integral curves with given initial data; viz. $S(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$ on $E$ the space of real sequences under pointwise convergence, or may have no integral curves with given initial data; viz. $S(f) = df/dx$ on $E = C^\infty$ functions on $[0,1]$ which vanish to all orders at $0$ and $1$. The result (8A.11) is generalized significantly in Dorroh-Marsden [1].

$$d(F_t x, F_t y) \leq Cd(x,y)$$

*for* $x,y \in \mathscr{U}$, *and* $t \in [t_0-\varepsilon, t_0+\varepsilon]$. *Here the constant* $C$ *is supposed to be independent of* $x$, $y$ *and* $t$. (*In other words, the local Lipschitz constant is supposed to be locally bounded in* $t$. *This is the case for a globally Lipschitz flow, for example.*)

*Conclusion: if* $c(t)$ *is a curve in* $D$ *such that* $c'(t) = X(c(t))$, *then* $c(t) \equiv F_t(c(0))$.

*Proof*. We can work in a local chart (see (8A.13)), so we assume $M = \mathbb{E}$, a Banach space. Given $t_0$, let $x_0 = c(t_0)$. Then choose $\varepsilon > 0$ and a neighborhood $\mathscr{U}$ of $x_0$ as in hypothesis (b); in addition, $\varepsilon$ should be small enough so that $c(t) \in \mathscr{U}$ if $|t-t_0| \leq \varepsilon$.

Define $h(t) = F_{t_0-t}c(t)$. Then, for $t$ near $t_0$, and $\tau$ small,

$$\begin{aligned} ||h(t+\tau) - h(t)|| &= ||F_{t_0-t-\tau}c(t+\tau) - F_{t_0-t}c(t)|| \\ &= ||F_{t_0-t-\tau}c(t+\tau) - F_{t_0-t-\tau}F_\tau c(t)|| \\ &\leq C||c(t+\tau) - F_\tau c(t)||. \end{aligned}$$

Moreover, $\frac{1}{\tau}[c(t+\tau) - F_\tau c(t)] = \frac{1}{\tau}[c(t+\tau) - c(t)] + \frac{1}{\tau}[c(t) - F_\tau c(t)] \to X(c(t)) - X(c(t)) = 0$ as $\tau \to 0$. Thus $h$ is differentiable, and $h'(t) \equiv 0$. It follows that $h(t)$ is constant, whence $c(t) = F_{t-t_0}c(t_0)$ for $t$ near $t_0$. From this the relation $c(t) = F_t c(0)$ follows easily. $\square$

(8A.12) *Corollary*. *The conclusion of Theorem (8A.11) applies to* $C^1$ *flows* $F_t$.

Proof. We shall verify condition (b) of the hypothesis. In a local chart, our results (see (8A.6)) on joint continuity show that $DF_t(x)\cdot y$ is continuous jointly in $t$, $x$, and $y$. Hence, by the Banach-Steinhaus Theorem, for a given $x_0$ and $t_0$ there is a convex neighborhood $\mathscr{U}$ of $x_0$ and an $\varepsilon > 0$ so that $||DF_t(x)|| \leq C$ if $x \in \mathscr{U}$ and $|t-t_0| \leq \varepsilon$. The mean value theorem then shows that $||F_t(x) - F_t(y)|| \leq C||x-y||$ if $x,y \in \mathscr{U}$ and $|t-t_0| \leq \varepsilon$. $\square$

The above results generalize classical theorems of Kneser and Van Kampen. They easily generalize to semi-flows.

Note. An explicit example of a continuous vector field with a jointly continuous flow $F_t$ for which the conclusion of Theorem (8A.11) fails is the following well known example. On $\mathbb{R}$ let $X$ be defined by

$$X(x) = \frac{3}{2}\,|x|^{1/3}.$$

Define $\phi(y) = |y|^{3/2}\,\mathrm{sgn}\,y$. Then $\phi$ is differentiable, with $\phi'(y) = \frac{3}{2}\,|y|^{1/2}$. It is easy to check that $F_t(x) = \phi(t+\phi^{-1}(x))$ is a flow for $X$. In particular $F_t(0) = |t|^{3/2}\,\mathrm{sgn}\,t$. But $c(t) \equiv 0$ is another integral curve with $c(0) = 0$. See Hartman [1] for more examples.

(8A.13) Remarks.

1) In case one wishes to work globally on $M$ and not in charts, one should use the proper sort of metric as follows:

Definition. Let $N$ be a Banach manifold modelled on a Banach space $\mathbb{E}$. Let $d$ be a metric on $N$. We say $d$ is compatible with the structure of $N$, if $d$ gives the

topology of $N$ and if given any $x_0 \in N$, there is, about $x_0$, a chart $(\mathscr{U},\phi)$ and constants $\alpha(x_0)$, $\beta(x_0)$ such that for all $x,y \in \mathscr{U}$, $d(x,y) \leq \alpha||\phi(x) - \phi(y)|| \leq \beta d(x,y)$.

2) This method is not the one usually employed to prove uniqueness of integral curves, for example, in $\mathbb{R}^n$. One usually assumes that the vector field is locally Lipschitz and then uses integration to prove this result. Let us recall how this goes. Let $X$ be a locally Lipschitz vector field on $\mathbb{R}^n$ (or any Banach space). Let $d(t)$ and $c(t)$ be two integral curves of $X$ such that $d(0) = c(0)$. Then $|d(t)-c(t)| = |\int_0^t X(d(s)) - X(c(s))ds| \leq K \int_0^t |d(s)-c(s)|ds$.

One then uses the fact (called <u>Gronwall's inequality</u>) that if $\alpha$ is such that $\alpha(t) \leq \int_0^t K\alpha(s)ds$, then $\alpha(t) \leq \alpha(0)e^{Kt}$. Therefore, we have $d(t) = c(t)$. For purposes of partial differential equations, however, it is important to have the result as stated in (8A.11) because one is often able to find Lipschitz bounds for the constructed flows, but rarely on the given generator.

3) Another method sometimes used to prove uniqueness of integral curves for a vector field $X$ with domain $D \subseteq N$ is called the <u>energy method</u>. Suppose there is a smooth function $H: D \times D \to \mathbb{R}^+$ such that $H(x,y) = 0$ if and only if $x = y$ and such that for any two integral curves $c$ and $d$ for $X$, $\frac{dH(c(t),d(t))}{dt} \leq K(t,d(0),c(0))H(c(t),d(t))$ where $K$ is locally bounded in $t$. Then as in Remark 2 we can conclude that $X$ has unique integral curves. This method is directly applicable to classical solutions of the Euler and Navier-Stokes equation, for example.

Measurable flows.

Under rather general conditions, continuity of a flow in the time variable can be deduced from measurability. For example, we have the following result. See also Ball [2].

(8A.14) Theorem. Let $M$ be a separable metric space. Let $F_t$ be a flow (or local semiflow) of continuous maps on $M$. Assume that, for each $x \in M$, the map $t \mapsto F_t(x)$ is Borel measurable; that is, the inverse image of any open set is a Borel subset of $\mathbb{R}$. Then $F_t$ is jointly continuous (respectively, jointly continuous for $t > 0$).

Proof. Because $M$ is separable, the Borel function $t \mapsto F_t(x)$ is continuous when restricted to the complement of some first-category set $C \subset \mathbb{R}$ (cf. Bourbaki [1].) Given $t_0$ and a sequence $t_n \to t_0$, note that $\bigcup_{n=1}^{\infty} [C-(t_0-t_n)] = D$ is of the first category, hence there exists an $s \in \mathbb{R}$ with $s \notin D$; that is, $t_n - t_0 + s \notin C$ for all $n$. Accordingly, $F_{t_n-t_0+s}(x) \to F_s(x)$ when $n \to \infty$. Now apply the continuous map $F_{s-t_0}$ to deduce that $F_{t_n}(x) \to F_{t_0}(x)$.

Hence $F_t(x)$ is separately continuous, and the conclusion follows from Theorem 1. $\square$

Theorems of this sort are well known for linear semigroups (see Yosida [1] for instance).

Some Results on Time Dependent Linear Evolution Equations.

In order to study smoothness criteria we shall need to make use of some results about linear evolution equations. These results are taken from Kato [1,4,5]. We begin by de-

fining an evolution system. (This exposition is adapted from Dorroh-Marsden [1].)

(8A.15) Definition. Let $X$ be a Banach space and $T > 0$. A subset $\{U(t,s)\ 0 \leq s \leq t < T\}$ of $B(X) = B(X,X)$ (bounded operators on $X$) is called an evolution system in $X$ if

i) $U(t,t) = I$ for $0 \leq t < T$, and

ii) $U(t,s)U(s,r) = U(t,r)$ for $0 \leq r \leq s \leq t < T$.

An evolution system $\{U(t,x) \mid 0 \leq s \leq t < T\}$ in $X$ is said to be strongly continuous if for each $f \in X$, the function $U(\cdot,\cdot)f$ maps $[0,T) \times [0,T)$ continuously into $X$. The $X$-infinitesimal generator of $\{U(t,s)\}$ is the collection $\{A(s) \mid 0 \leq s < T\}$ of operators in $X$ defined by

$$A(s)f = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}[U(s+\varepsilon,s)f-f]$$

with $D(A(s))$ consisting of all $f$ for which this limit exists, where the limit is taken in $X$.

(8A.16) Remarks. a) If $\{U(t,s)\}$ is a strongly continuous evolution system in $X$, then it follows from the uniform boundedness principle that $||U(t,s)||_{X,X}$ is bounded for $s$ and $t$ in closed and bounded intervals.

b) Let $\{U(t,s) \mid 0 \leq s \leq t < T\}$ be a strongly continuous evolution system in $X$ with $X$-infinitesimal generator $\{A(s) \mid 0 \leq s < T\}$, and let $0 < a < T$. Then $\{A(s+a) \mid 0 \leq s < T - a\}$ is the $X$-infinitesimal generator of the strongly continuous evolution system $\{U(t+a, s+a) \mid 0 \leq s \leq t < T - a\}$.

(8A.17) Proposition. Let $\{U(t,s) \mid 0 \leq s \leq t < T\}$ be a strongly continuous evolution system in $X$ with $X$-infinite-

*simal generator* $\{A(s) \mid 0 \leq s < T\}$. *If* $f \in D(A(s))$ *for all* $0 \leq s < T$, *and* $A(\cdot)f$ *maps* $[0,T)$ *continuously into* $X$, *then*

$$(\partial/\partial s)[U(t,s)f] = -U(t,s)A(s)f \tag{8A.5}$$

*for* $0 \leq s \leq t < T$, $t > 0$.

*Proof*. If $0 \leq s < t < T$, then

$$U(t,s+\varepsilon)f - U(t,s)f = U(t,s+\varepsilon)[f-U(s+\varepsilon,s)f],$$

and therefore,

$$(\partial^+/\partial s)U(t,s)f = -U(t,s)A(s)f.$$

Thus for each $t \in (0,T)$, the function $U(t,\cdot)f$ has a continuous right derivative on $[0,t)$. Thus the function $U(t,\cdot)f$ is continuously differentiable on $[0,t)$ (see Yosida [1], p. 239), and (8A.5) holds for $0 \leq s < t < T$. Since the derivative of $U(t,\cdot)f$ has a limit from the left at $t$, it follows that

$$(\partial^-/\partial s)[U(t,s)f] = -U(t,s)A(s)f$$

for $0 < s = t < T$. But, because of the domain of $U(t,\cdot)f$, this is what (8A.5) means when $s = t$. □

(8A.18) *Corollary*. *Let* $\{U(t,s) \mid 0 \leq s \leq t < T\}$ *be a strongly continuous evolution system in* $X$ *with* $X$-*infinitesimal generator* $\{A(s) \mid 0 \leq s < T\}$. *Let* $f(s) \in D(A(s))$ *for* $0 \leq s < T$, *suppose* $f$ *is continuously differentiable from* $[0,T)$ *into* $X$, *and that* $A(\cdot)f(\cdot)$ *maps* $[0,T)$ *continuously into* $X$. *Then*

$$(\partial/\partial s)[U(t,s)f(s)] = U(t,s)f'(s) - U(t,s)A(s)f(s)$$

<u>for</u> $0 \leq s \leq t < T$, $t > 0$.

<u>Proof</u>. This follows from the Proposition, the strong continuity of $\{U(t,s)\}$, and the local boundedness of $||U(t,s)||_{X,X}$. $\square$

We call (8A.5) the <u>backward differential equation</u>. In order that the <u>forward differential equation</u>

$$(\partial/\partial t)[U(t,s)f] = A(t)U(t,s)f \tag{8A.6}$$

hold, it is necessary that $U(t,s)f \in D(A(t))$, and this is a more restrictive condition which may not be satisfied when the hypothesis of the above Proposition is satisfied.

Now suppose $Y$ is another Banach space with $Y$ densely and continuously embedded in $X$.

(8A.19) <u>Definition</u>. An evolution system $\{U(t,s)\}$ in $X$ is said to be $Y$-<u>regular</u> if each transformation $U(t,s)$ maps $Y$ continuously into $Y$, and $\{U(t,s)\}$ is strongly continuous in $Y$; i.e., if $\{U(t,s)\}$ is a strongly continuous evolution system in $Y$ as well as in $X$.

Kato in [4] and [5] gives a variety of conditions on families $\{A(s)\}$ of operators in $X$ which are sufficient for these families to be the $X$-infinitesimal generators of strongly continuous evolution systems in $X$. Some of these conditions are also sufficient for the evolution system to be $Y$-regular and for the forward differential equation to hold. He also gives several convergence theorems for evolu-

tion systems and upper bounds for the operator norms in terms of certain parameters of the infinitesimal generator.

Since these results bear directly on our results later, we quickly summarize the fundamental points here for reference. Kato's papers should be consulted for details and related remarks.

(8A.20) Definitions. Let $A \in G(X)$, the set of semigroup generators in $X$. $Y$ is said to be admissible with respect to $A$, or simply $A$-admissible, if $\{e^{tA}\}$ leaves $Y$ invariant and forms a semigroup of class $C_0$ in $Y$.

A subset $G(X)$ is said to be stable if there are constants $M$ and $\beta$ (called constants of stability) such that

$$\left\| \prod_{j=1}^{k} (\lambda I - A_j)^{-1} \right\| \leq M(\lambda-\beta)^{-k}$$

for $\lambda > \beta$ and $A_1,\ldots,A_k$ elements of the subset.

(8A.21) Theorem. (Existence Theorem). Let $T > 0$, let $A(t) \in G(X)$ for $0 \leq t < T$, and assume that

i) $\{A(t) \mid 0 \leq t < T\}$ is stable, say with constant $M$, $\beta$;

ii) $Y$ is $A(t)$-admissible for each $t$, and if $A^*(t) \in G(Y)$ is the part of $A(t)$ within $Y$, then $\{A^*(t)\}$ is stable, say with constants $M^*, \beta^*$; and

iii) $Y \subset D(A(t))$ for each $t$, and $A^-(\cdot)$ is continuous from $[0,T)$ into $B(Y,X)$, where $A^-(t)$ is the restriction of $A(t)$ to $Y$, (called the part of $A(t)$ within $Y$ to $X$).

Then there is a unique strongly continuous evolution

<u>system</u> $\{U(t,s)\}$ <u>in</u> $X$ <u>with</u> $X$-<u>infinitesimal generator extending</u> $\{A^-(T)\}$; <u>i.e.</u>, <u>with infinitesimal generator</u> $\{B(t)\}$ <u>such that</u> $B(t) \supset A^-(t)$ <u>for each</u> $t$. <u>Furthermore</u>,

$$||U(t,s)||_{X,X} \leq Me^{\beta(t-s)} \quad \text{for} \quad 0 \leq s \leq t < T.$$

(8A.22) <u>Remarks</u>. a) If $A(t)$ are independent of $t$, the stability condition for $A$ is the condition for the Hille-Yosida theorem (Yosida [1]).

b) Actually in Theorem (8A.21) Kato shows that the $X$-infinitesimal generator is precisely $\{A(t)\}$.

We can add on any bounded operator to a family $\{A(t)\}$ of generators and still get generators:

(8A.23) <u>Remark</u>. Let $\{A(t) | 0 \leq t < T\}$ satisfy the hypothesis of Theorem (8A.21), let $B(t) \in B(X,X)$ for $0 \leq t < T$, and let $B(\cdot)f$ map $[0,T)$ continuously into $X$ for each $f \in X$. Then there is a unique strongly continuous evolution system in $X$ with $X$-infinitesimal generator extending $\{A(t) + B(t)\}$.

In examples, it may be difficult to verify the stability condition (i). To this end we have a useful criterion given in the following proposition. First some notation: Let $G(X,M,\beta)$ denote the generators $A$ on $X$ with constants $M,\beta$: $||(\lambda-A)^{-k}|| \leq M/(\lambda-\beta)^k$, $\lambda > \beta$ (corresponding to the semigroup condition $||F_t|| \leq Me^{\beta t}$). In particular, if $M = 1$ we have the generator of a <u>quasi-contractive semigroup</u>, the condition being $||(\lambda-A)^{-1}|| \leq 1/(\lambda-\beta)$, $\lambda > \beta$; or on the flow $||F_t|| \leq e^{\beta t}$. Examples of this type of semigroup

are common.

(8A.24) Remark. (Trotter, Feller) For a given semigroup $F_t$ with generator $A \in G(X,M,\beta)$ the space $X$ can be renormed so that $||F_t|| \leq e^{\beta t}$. Indeed the new norm is $|||x||| = \sup_{t \geq 0} ||e^{-\beta t} F_t(x)||$.

One should note however that it is not always possible to renorm $X$ so that two semigroups simultaneously become quasi-contractive.

(8A.25) Theorem. For each $t$, let $|| \; ||_t$ be a new norm on $X$ equivalent to the original one and vary smoothly in $t$; i.e.,: satisfying

$$\frac{||x||_t}{||x||_s} \leq e^{c|t-s|}, \quad x \in X, \quad 0 \leq s, \quad t \leq T.$$

For each $t$, let $A(t)$ be the generator of a quasi-contractive, semigroup with constant $\beta$ in the norm $||\cdot||_t$. Then $\{A(t)\}$ is stable on $X$ with $M = e^{2cT}$, $0 \leq t \leq T$, with respect to any of the norms $|| \; ||_t$.

The proof is actually a simple verification; see Kato [3, Prop. 3.4].

There is another useful criterion for the hypotheses of (8A.21) to hold as follows:

(8A.26) Theorem. Let i) and iii) of (8A.21) hold and replace ii) by

ii") There is a family $\{S(t)\}$ of isomorphisms of $Y$ onto $X$ such that

$$S(t)A(t)S(t)^{-1} = A(t) + B(t),$$

$B(t) \in B(X)$ *where* $B: [0,T) \to B(X)$ *is strongly continuous. Assume* $S: [0,T) \to B(Y,X)$ *is strongly* $C^1$.

*Then the conclusions of (8A.21) hold* ((ii") $\Longrightarrow$ (ii)) *and moreover, the forward differential equation holds, and the evolution system is* Y-*regular*.

Two important approximation theorems follow (see Kato [5]).

(8A.27) *Theorem. Let* $\{A_n(t)\}$ *satisfy the hypotheses of (8A.21),* $n = 0,1,2,\ldots$ *where there are uniform stability constants in* i) *and* ii). *Assume*

$$||A_0^-(t) - A_n^-(t)||_{Y,X} \to 0 \quad \text{as} \quad n \to \infty$$

*uniformly in* $t$. *Then* $U_n(t,s) \to U_0(t,s)$ *strongly in* $B(X)$, *uniformly in* $t,s \in [0,T)$, *and* $||U_n(t,s) - U_0(t,s)||_{Y,X} \to 0$ *as* $n \to \infty$.

(8A.28) *Theorem. Let* $\{A_n(t)\}$ *satisfy the hypotheses of* (8A.26), $n = 0,1,2,\ldots$ *where the primitive constants* $M,\beta,||S||_{\infty Y,X},||S^{-1}||_{\infty,Y,X},||B||_{\infty,X,X}, ||\dot{S}||_{\infty,Y,X}$ *can be chosen independent of* $n$. *Assume* $||A_0(t) - A_n(t)||_{Y,X} \to 0$ *as* $n \to \infty$ *uniformly in* $t$, *as in (8A.26), and in addition that* $B_n(t) \to B_0(t)$ *in* $B(X)$, $S_n(t) \to S_0(t)$ *in* $B(Y,X)$, $S_n(t) \to S_0(t)$ *in* $B(Y,X)$ *uniformly in* $t$. *Then,*

$$U_n(t,s) \to U_0(t,s)$$

*strongly in* $B(Y)$ *uniformly in* $t,s \in [0,T)$.

A Criterion for Smoothness

We now give a result which tells us when a semiflow consists of smooth mappings. The result is powerful when used in conjunction with the above linear results. The present theorem is due to Dorroh-Marsden [1], to which we refer for additional results. Before proceeding, the reader should attempt Exercise 2.9 to get a feel for the situation.

We use the following notation. $X$ and $Y$ are Banach spaces with $Y$ densely and continuously embedded in $X$. $D \subset Y$ is open and $F_t$ is a continuous local semiflow on $D$. We let $G: D \to X$ be such that $F_t$ is a semiflow for $G$. For $p,q \in D$, and the line segment $\{p+r(q-p) \mid 0 \leq r \leq 1\} \subset D$ set

$$Z(q,p) = \int_0^1 DG(p+r(q-p))dr$$

the averaged derivative of $G$ along the segment.

Assumptions. a) $G: D \to X$ is $C^1$.

b) for fixed $f \in D$ and $g \in D$ sufficiently close to $f$, there is a strongly continuous evolution system $\{U^g(t,s) \mid 0 \leq s \leq t < T_g\}$ in $X$ whose $X$-infinitesimal generator is an extension of $\{Z(F_s g, F_s f): 0 \leq s < T_g\}$ (here $T_g$ denotes a time of existence for $F_s g$ and $F_s f$).

c) $||U^g(t,s) - U^f(t,s)||_{Y,X} \to 0$ as $||g-f||_Y \to 0$ (see (8A.27)).

(8A.29) Theorem. Under these assumptions a), b), c), $F_t: Y \to X$ is Frechet differentiable at $f$ with $DF_t(f) = U^f(t,0)$.

(8A.30) Remarks. The proof also shows that if

$||U^g(t,0)||_{X,X}$ is uniformly bounded, as $g$ varies $F_t$ will be $X \to X$ Lipschitz for $g$ near enough to $f$ in $Y$.

2. A translation argument shows $DF_{t-s}(F_s(f)) = U^f(t,s)$.

Further Assumptions.

d) $U^g(t,s)$ is $Y$-regular

and replace c) by

c)' $U^g(t,s)$ converges strongly in $Y$ to $U^f(t,s)$ as $g$ converges to $f$ along straight line intervals (see Theorem (8A.28)).

(8A.31) Theorem. Under a), b), c)', d), $F_t: Y \to Y$ is Gateaux differentiable at $f$ and

$$DF_t(f) = U^f(t,0).$$

(8A.32) Remarks. 1. The proof also shows that if $||U^g(t,0)||_{Y,Y}$ is locally bounded, then $F_t: Y \to Y$ is locally Lipschitz.

2. If (8A.26) is used, we see that, in fact, $DF_t(f)$ is locally bounded in $B(Y,Y)$ for $f \in Y$. We can iterate the use of (8A.31) to get that $F_t$ is twice Gateaux differentiable, etc. This will imply that $F_t$ is in fact $C^\infty$. (Gateaux differentiability and norm continuity of the derivative implies $C^1$ from $f(x) - f(y) = \int_0^1 Df(x+t(x-y))(x-y)dt$; also Gateau differentiability with locally bounded derivatives implies Lipschitz continuity).

3. The derivative $DF_t(f)$ in (8A.29) extends to a bounded operator $X \to X$.

Proof of (8A.29). Let $0 < T' < T_f$. For $||g-f||_Y$

sufficiently small, we define $w$ on $[0,T']$ by $w(s) = F_s g - F_s f$. Differentiating, and using $Z(q,p)(q-p) = G(q) - G(p)$, we have

$$w'(s) = G(F_s g) - G(F_s f) = Z(F_s g, F_s f)w(s)$$

for $0 \leq s \leq T'$. If $0 \leq s \leq t \leq T'$, then by the Corollary on p. 273.

$$(\partial/\partial s)U^g(t,s)w(s) = 0,$$

so that

$$F_t g - F_t f = U^g(t,0)(g-f)$$

for $0 \leq t \leq T'$. Thus we have the estimates

$$||F_t g - F_t f - U^f(t,0)(g-f)||_X ||g-f||_Y^{-1} \leq ||U^g(t,0) - U^f(t,0)||_{Y,X},$$

and

$$||F_t g - F_t f||_X \leq ||U^g(t,0)||_{X,X} ||g-f||_X. \qquad \square$$

<u>Proof of (8A.31)</u>. As in the proof of (8A.29), we have

$$F_t g - F_t f = U^g(t,0)(g-f),$$

so that

$$||F_t g - F_t f||_X \leq ||U^g(t,0)||_{X,X} ||g-f||_X,$$

and

$$||F_t g - F_t f||_Y \leq ||U^g(t,0)||_{Y,Y} ||g-f||_Y.$$

This establishes the claims about Lipschitz continuity. If we let $g = f + \lambda h$, then we get

$$\lambda^{-1}[F_t(f+\lambda h) - F_t f] = U^{f+\lambda h}(t,0)h,$$

from which the differentiability claim follows directly. $\square$

Using these arguments, one can also establish $Y$ differentiability of $F_t$ in $t$ and differentiability in an external parameter. We consider two such results from Dorroh-Marsden [1].

(8A.33) Corollary. Under the hypothesis of (8A.29) or (8A.31) suppose that the evolution system $\{U^f(t,s)\}$ with X-infinitesimal generator $\{DG(F_s)\}$ satisfies $U^f(t,s)X \subset Y$ for $0 \leq s < t < T_f$. Then for $f \in D$, $G(F_t f) \in Y$ for $0 < t < T_f$. If $U^f(\cdot,0)g$ is Y-continuous on $(0,T_f)$ for each $g \in X$, then $F_{(\cdot)}f$ is continuously Y-differentiable on $(0,T_f)$.

Proof. Under these hypotheses one can establish a chain rule, so that by differentiating $F_{t+s}f = F_t(F_s(f))$ in $s$ at $s = 0$, we get

$$\begin{aligned} G(F_t f) &= DF_t(f)\cdot G(f) \\ &= U^f(t,0)\cdot G(f) \in Y \end{aligned}$$

which proves the first part. The second part follows because

$$\begin{aligned} F_t f - F_s f &= \int_s^t G(F_\tau f)d\tau \\ &= \int_s^t U^f(\tau,0)\cdot G(f)d\tau. \quad \square \end{aligned}$$

The condition that $F_t f$ be Y-differentiable for $t > 0$ is a nonlinear analogue of what one has for linear analytic semigroups (see Yosida [1]).

For the dependence on a parameter, we assume $G(f,z)$, $F_t^z$ depend on a parameter $z \in V \subset Z$ where $V$ is open in a

Banach space. Assume at the outset that $F_t^z(f)$ is continuous in all variables, and for each $z$, $F_t^z$ is as above.

To determine differentiability of $F_t^z(f)$ in $(z,f)$ we can use a simple suspension trick. Namely, consider the semiflow $H_t$ on $D \times V$ defined by

$$H_t(f,z) = (F_t^z(f),z).$$

The generator is $K: D \times V \to X \times Z$,

$$K(f,z) = (G(f,z),0).$$

If (8A.29) or (8A.30) applies to $H_t$ then we can conclude differentiability* of $F_t^z(f)$ in $(f,z)$.

One of the key ingredients in (8A.29) is hypotheses concerning the linearized equations. Here

$$DK(f,z)(g,w) = (D_1G(f,z)\cdot g + D_2G(f,z)\cdot w,0)$$

so we would be required to solve, according to (8A.29) or (8A.31), the system

$$\frac{dw}{dt} = 0 \quad \text{i.e.} \quad w = \text{constant}$$

$$\frac{dg(t)}{dt} = D_1G(F_t^z(f),z)\cdot g(t) + D_2G(F_t^z(f),z)\cdot w$$

(similarly for systems involving the averaged generators $Z$). This is a linear system in $g$ with an inhomogeneous term $D_2G(F_t^z(f),z)\cdot w$. The solution can be written down in terms of the evolution system for $D_1G(F_t^z(f),z)$ via Duhamel's formula in the usual way. For systems of this type there are theorems

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*A more direct analysis, obtaining refined results, is given by Dorroh-Marsden [1].

available to guarantee that we have an evolution system and to study its properties. Note, for example, Theorem 7.2 of Kato [4]. In this way, one can check the hypotheses of (8A.29) or (8A.31) for $H_t$. We would conclude, respectively, differentiability of $F_t^{(\cdot)}(\cdot)$ from $D \times V$ to $X \times V$ and (under the stronger conditions of (8A.31)), from $D \times V$ to $Y \times V$.

(For holomorphic semigroups, smooth dependence on a parameter can be analyzed directly, as in Kato [3], p. 487).

(8A.34) <u>Remark and Application</u>. In Aronson-Thames [1] the following system is studied:

$$u_{xx} - q^2 u = u_t, \quad v_{xx} - q^2 v = v_t \quad \text{in} \quad (0,1) \times R^+$$

$$\left.\begin{aligned} u_x(1,t) &= v_x(0,t) = 0 \\ u_x(0,t) &= -pq(f \circ v)(0,t), \\ v_x(1,t) &= pq\{1-(f\circ u)(1,t)\} \end{aligned}\right\} \qquad t \geq 0$$

Here $p$ and $q$ are positive parameters and $f(u) = u^2/(1+u^2)$. This system is related to enzyme diffusion in biological systems. They show that the eigenvalue conditions of the Hopf theorem are met. In Dorroh-Marsden [1] it is shown, using methods described above, that the semiflow of this system is smooth. It follows at once that the Hopf theorem is valid and hence proves the existence of stable periodic solutions for this system for supercritical values of the parameters.

These equations are usually called the Glass-Kauffman equations; cf. Glass-Kauffman [1]. Recent work of the discrete analogue has also been done by Hsü.