SECTION 1
INTRODUCTION TO STABILITY AND BIFURCATION IN DYNAMICAL SYSTEMS AND FLUID MECHANICS

Suppose we are studying a physical system whose state x is governed by an evolution equation $\frac{dx}{dt} = X(x)$ which has unique integral curves. Let x_0 be a fixed point of the flow of X; i.e., $X(x_0) = 0$. Imagine that we perform an experiment upon the system at time $t = 0$ and conclude that it is then in state x_0. Are we justified in predicting that the system will remain at x_0 for all future time? The mathematical answer to this question is obviously yes, but unfortunately it is probably not the question we really wished to ask. Experiments in real life seldom yield exact answers to our idealized models, so in most cases we will have to ask whether the system will remain near x_0 if it started near x_0. The answer to the revised question is not always yes, but even so, by examining the evolution equation at hand more minutely, one can sometimes make predictions about the future behavior of a system starting near x_0. A trivial example will illustrate some of the problems involved. Consider the following two differential equations on the real line:

$$x'(t) = -x(t) \tag{1.1}$$

and

$$x'(t) = x(t). \tag{1.2}$$

The solutions are respectively:

$$x(x_0,t) = x_0 e^{-t} \tag{1.1'}$$

and

$$x(x_0,t) = x_0 e^{+t}. \tag{1.2'}$$

Note that 0 is a fixed point of both flows. In the first case, for all $x_0 \in \mathbb{R}$, $\lim_{t\to\infty} x(x_0,t) = 0$. The whole real line moves toward the origin, and the prediction that if x_0 is near 0, then $x(x_0,t)$ is near 0 is obviously justified. On the other hand, suppose we are observing a system whose state x is governed by (1.2). An experiment telling us that at time $t = 0$, $x(0)$ is approximately zero will certainly not permit us to conclude that $x(t)$ stays near the origin for all time, since all points except 0 move rapidly away from 0. Furthermore, our experiment is unlikely to allow us to make an accurate prediction about $x(t)$ because if $x(0) < 0$, $x(t)$ moves rapidly away from the origin toward $-\infty$ but if $x(0) > 0$, $x(t)$ moves toward $+\infty$. Thus, an observer watching such a system would expect sometimes to observe $x(t) \underset{t\to\infty}{\longrightarrow} -\infty$ and sometimes $x(t) \underset{t\to\infty}{\longrightarrow} +\infty$. The solution $x(t) = 0$ for all t would probably never be observed to occur because a slight perturbation of the system would destroy this solution. This sort of behavior is frequently observed in nature. It is not due to any nonuniqueness in the solution to the differential equation involved, but to the <u>instability</u> of that solution under small perturbations in initial data.

Indeed, it is only <u>stable</u> mathematical models, or features of models that can be relevant in "describing" nature.[+]

Consider the following example.[*] A rigid hoop hangs from the ceiling and a small ball rests in the bottom of the hoop. The hoop rotates with frequency ω about a vertical axis through its center (Figure 1.1a).

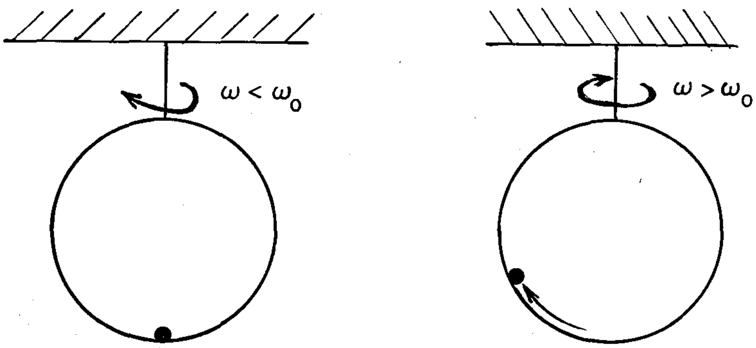

Figure 1.1a Figure 1.1b

For small values of ω, the ball stays at the bottom of the hoop and that position is stable. However, when ω reaches some critical value ω_0, the ball rolls up the side of the hoop to a new position $x(\omega)$, which is stable. The ball may roll to the left or to the right, depending to which side of the vertical axis it was initially leaning (Figure 1.1b). The position at the bottom of the hoop is still a fixed point, but it has become unstable, and, in practice, is never observed to occur. The solutions to the differential equations governing the ball's motion are unique for all values of ω,

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[+]For further discussion, see the conclusion of Abraham-Marsden [1].

[*]This example was first pointed out to us by E. Calabi.

but for $\omega > \omega_0$, this uniqueness is irrelevant to us, for we cannot predict which way the ball will roll. Mathematically, we say that the original stable fixed point has become unstable and has split into two stable fixed points. See Figure 1.2 and Exercise 1.16 below.

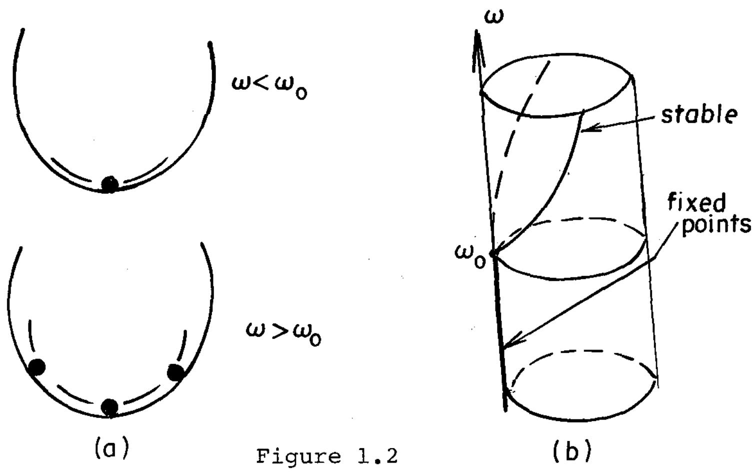


Figure 1.2

Since questions of stability are of overwhelming practical importance, we will want to define the concept of stability precisely and develop criteria for determining it.

(1.1) Definition. Let F_t be a C^0 flow (or semiflow)* on a topological space M and let A be an invariant set; i.e., $F_t(A) \subset A$ for all t. We say A is stable (resp. asymptotically stable or an attractor) if for any neighborhood U of A there is a neighborhood V of A such

*i.e., $F_t: M \to M$, F_0 = identity, and $F_{t+s} = F_s \circ F_t$ for all $t, s \in \mathbb{R}$. C^0 means $F_t(x)$ is continuous in (t,x). A semiflow is one defined only for $t \geq 0$. Consult, e.g., Lang [1], Hartman [1], or Abraham-Marsden [1] for a discussion of flows of vector fields. See Section 8A, or Chernoff-Marsden [1] for the infinite dimensional case.

that the flow lines (integral curves) $x(x_0,t) \equiv F_t(x_0)$ belong to U if $x_0 \in V$ (resp. $\bigcap_{t\geq 0} F_t(V) = A$).

Thus A is stable (resp. attracting) when an initial condition slightly perturbed from A remains near A (resp. tends towards A). (See Figure 1.3).

If A is not stable it is called unstable.

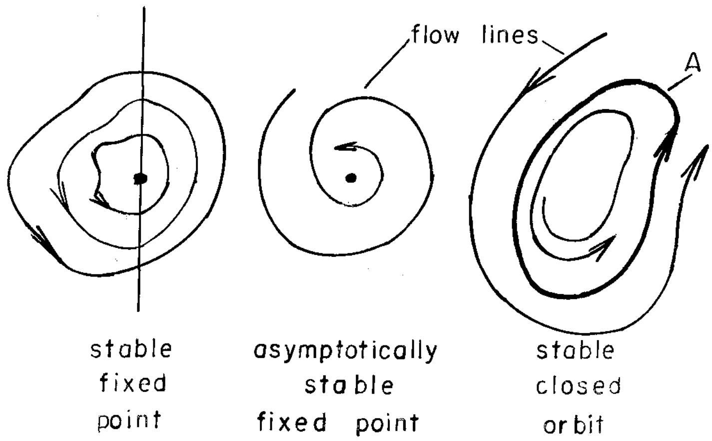


Figure 1.3

(1.2) Exercise. Show that in the ball in the hoop example, the bottom of the hoop is an attracting fixed point for $\omega < \omega_0 = \sqrt{g/R}$ and that for $\omega > \omega_0$ there are attracting fixed points determined by $\cos\theta = g/\omega^2 R$, where θ is the angle with the negative vertical axis, R is the radius of the hoop and g is the acceleration due to gravity.

The simplest case for which we can determine the stability of a fixed point x_0 is the finite dimensional, linear case. Let $X\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The flow of

X is $x(x_0,t) = e^{tX}(x_0)$. Clearly, the origin is a fixed point. Let $\{\lambda_j\}$ be the eigenvalues of X. Then $\left\{e^{\lambda_j t}\right\}$ are the eigenvalues of e^{tX}. Suppose $\mathrm{Re}\,\lambda_j < 0$ for all j. Then $\left|e^{\lambda_j t}\right| = e^{\mathrm{Re}\,\lambda_j t} \to 0$ as $t \to \infty$. One can check, using the Jordan canonical form, that in this case 0 is asymptotically stable and that if there is a λ_j with positive real part, 0 is unstable. More generally, we have:

(1.3) Theorem. Let $X\colon E \to E$ be a continuous, linear map on a Banach space E. The origin is a stable attracting fixed point of the flow of X if the spectrum $\sigma(X)$ of X is in the open left-half plane. The origin is unstable if there exists $z \in \sigma(X)$ such that $\mathrm{Re}(z) > 0$.

This will be proved in Section 2A, along with a review of some relevant spectral theory.

Consider now the nonlinear case. Let P be a Banach manifold* and let X be a C^1 vector field on P. Let $X(p_0) = 0$. Then $dX(p_0)\colon T_{p_0}(P) \to T_{p_0}(P)$ is a continuous linear map on a Banach space. Also in Section 2A we shall demonstrate the following basic theorem of Liapunov [1].

(1.4) Theorem. Let X be a C^1 vector field on a Banach manifold P and let p_0 be a fixed point of X, i.e., $X(p_0) = 0$. Let F_t be the flow of X i.e., $\frac{\partial}{\partial t} F_t(x) = X(F_t(x))$, $F_0(x) = x$. (Note that $F_t(p_0) = p_0$ for all t.) If the spectrum of $dX(p_0)$ lies in the left-half plane; i.e., $\sigma(dX(p_0)) \subset \{z \in \mathbb{C} \mid \mathrm{Re}\, z < 0\}$, then p_0 is asymptotically

*We shall use only the most elementary facts about manifold theory, mostly because of the convenient geometrical language. See Lang [1] or Marsden [4] for the basic ideas.

stable.

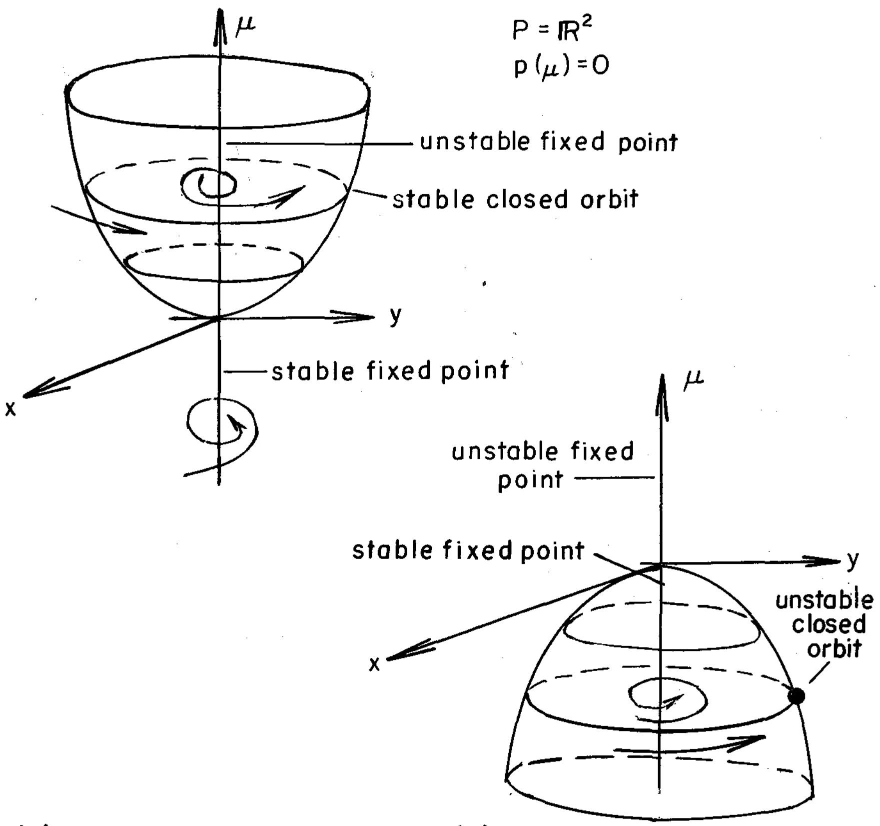
If there exists an isolated $z \in \sigma(dX(p_0))$ such that $\text{Re } z > 0$, p_0 is unstable. If $\sigma(dX(p_0)) \subset \{z | \text{Re } z \leq 0\}$ and there is a $z \in \sigma(dX(p_0))$ such that $\text{Re } z = 0$, then stability cannot be determined from the linearized equation.

(1.5) Exercise. Consider the following vector field on $\mathbb{R}^2$: $X(x,y) = (y, \mu(1-x^2)y-x)$. Decide whether the origin is unstable, stable, or attracting for $\mu < 0$, $\mu = 0$, and $\mu > 0$.

Many interesting physical problems are governed by differential equations depending on a parameter such as the angular velocity ω in the ball in the hoop example. Let $X_\mu: P \to TP$ be a (smooth) vector field on a Banach manifold P. Assume that there is a continuous curve $p(\mu)$ in P such that $X_\mu(p(\mu)) = 0$, i.e., $p(\mu)$ is a fixed point of the flow of X_μ. Suppose that $p(\mu)$ is attracting for $\mu < \mu_0$ and unstable for $\mu > \mu_0$. The point $(p(\mu_0), \mu_0)$ is then called a bifurcation point of the flow of X_μ. For $\mu < \mu_0$ the flow of X_μ can be described (at least in a neighborhood of $p(\mu)$) by saying that points tend toward $p(\mu)$. However, this is not true for $\mu > \mu_0$, and so the character of the flow may change abruptly at μ_0. Since the fixed point is unstable for $\mu > \mu_0$, we will be interested in finding stable behavior for $\mu > \mu_0$. That is, we are interested in finding bifurcation above criticality to stable behavior.

For example, several curves of fixed points may come together at a bifurcation point. (A curve of fixed points is a curve $\alpha: I \to P$ such that $X_\mu(\alpha(\mu)) = 0$ for all μ. One such curve is obviously $\mu \mapsto p(\mu)$.) There may be curves of stable fixed points for $\mu > \mu_0$. In the case of the ball in the hoop, there are two curves of stable fixed points for $\omega \geq \omega_0$, one moving up the left side of the hoop and one moving up the right side (Figure 1.2).

Another type of behavior that may occur is bifurcation to periodic orbits. This means that there are curves of the form $\alpha: I \to P$ such that $\alpha(\mu_0) = p(\mu_0)$ and $\alpha(\mu)$ is on a closed orbit γ_μ of the flow of X_μ. (See Figure 1.4). The Hopf bifurcation is of this type. Physical examples in fluid mechanics will be given shortly.



(a) Supercritical Bifurcation (Stable Closed Orbits)

(b) Subcritical Bifurcation (Unstable Closed Orbits)

Figure 1.4

The General Nature of the Hopf Bifurcation

The appearance of the stable closed orbits (= periodic solutions) is interpreted as a "shift of stability" from the original stationary solution to the periodic one, i.e., a point near the original fixed point now is attracted to and becomes indistinguishable from the closed orbit. (See Figures 1.4 and 1.5).

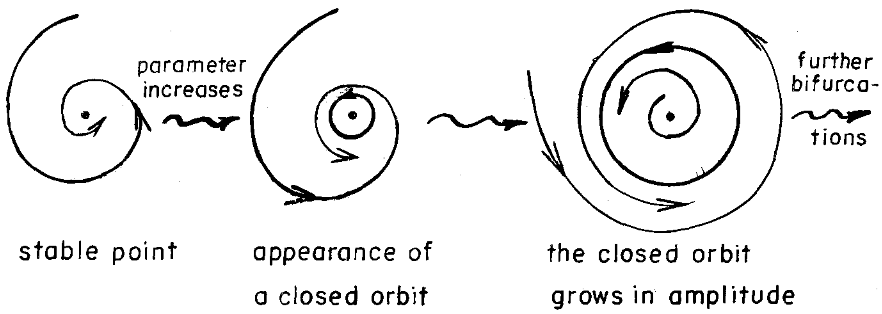


Figure 1.5
The Hopf Bifurcation

Other kinds of bifurcation can occur; for example, as we shall see later, the stable closed orbit in Figure 1.4 may bifurcate to a stable 2-torus. In the presence of symmetries, the situation is also more complicated. This will be treated in some detail in Section 7, but for now we illustrate what can happen via an example.

(1.6) <u>Example</u>: <u>The Ball in the Sphere</u>. A rigid, hollow sphere with a small ball inside it hangs from the ceiling and rotates with frequency ω about a vertical axis through its center (Figure 1.6).

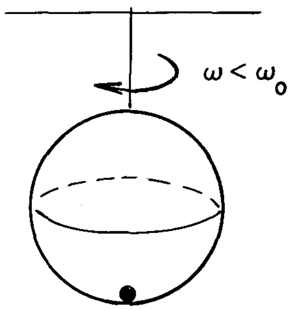


Figure 1.6

For small ω, the bottom of the sphere is a stable point, but for $\omega > \omega_0$ the ball moves up the side of the sphere to a new fixed point. For each $\omega > \omega_0$, there is a stable, invariant circle of fixed points (Figure 1.7). We get a circle of fixed points rather than isolated ones because of the symmetries present in the problem.

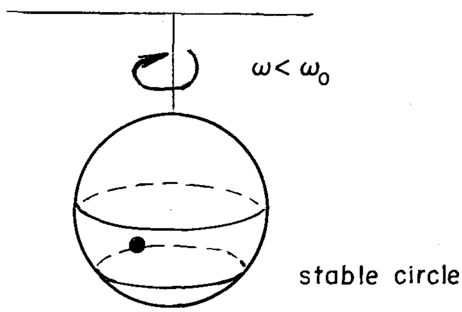


Figure 1.7

Before we discuss methods of determining what kind of bifurcation will take place and associated stability questions, we shall briefly describe the general basin bifurcation picture of R. Abraham [1,2].

In this picture one imagines a rolling landscape on which water is flowing. We picture an attractor as a basin into which water flows. Precisely, if F_t is a flow on M and A is an attractor, the basin of A is the set of all $x \in M$ which tend to A as $t \to +\infty$. (The less picturesque phrase "stable manifold" is more commonly used.)

As parameters are tuned, the landscape undulates and the flow changes. Basins may merge, new ones may form, old ones may disappear, complicated attractors may develop, etc.

The Hopf bifurcation may be pictured as follows. We begin with a simple basin of parabolic shape; i.e., a point attractor. As our parameter is tuned, a small hillock forms and grows at the center of the basin. The new attractor is, therefore, circular (viz the periodic orbit in the Hopf theorem) and its basin is the original one minus the top point of the hillock.

Notice that complicated attractors can spontaneously appear or dissappear as mesas are lowered to basins or basins are raised into mesas.

Many examples of bifurcations occur in nature, as a glance at the rest of the text and the bibliography shows. The Hopf bifurcation is behind oscillations in chemical and biological systems (see e.g. Kopell-Howard [1-6], Abraham [1,2] and Sections 10, 11), including such things as "heart flutter".*
One of the most studied examples comes from fluid mechanics, so we now pause briefly to consider the basic ideas of

*That "heart flutter" is a Hopf bifurcation is a conjecture told to us by A. Fischer; cf. Zeeman [2].

the subject.

The Navier-Stokes Equations

Let $D \subseteq R^3$ be an open, bounded set with smooth boundary. We will consider D to be filled with an incompressible homogeneous (constant density) fluid. Let u and p be the velocity and pressure of the fluid, respectively. If the fluid is viscous and if changes in temperature can be neglected, the equations governing its motion are:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u = -\text{grad } p \text{ (+ external forces)} \tag{1.3}$$

$$\text{div } u = 0 \tag{1.4}$$

The boundary condition is $u|_{\partial D} = 0$ (or $u|_{\partial D}$ prescribed, if the boundary of D is moving) and the initial condition is that $u(x,0)$ is some given $u_0(x)$. The problem is to find $u(x,t)$ and $p(x,t)$ for $t > 0$. The first equation (1.3) is analogous to Newton's Second Law $F = ma$; the second (1.4) is equivalent to the incompressibility of the fluid.*

Think of the evolution equation (1.3) as a vector field and so defines a flow, on the space $\mathfrak{X}$ of all divergence free vector fields on D. (There are major technical difficulties here, but we ignore them for now - see Section 8.)

The Reynolds number of the flow is defined by $R = \frac{UL}{\nu}$, where U and L are a typical speed and a length associated with the flow, and ν is the fluid's viscosity. For example, if we are considering the flow near a sphere toward which fluid is projected with constant velocity $U_\infty \vec{i}$

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*See any fluid mechanics text for a discussion of these points. For example, see Serrin [1], Shinbrot [1] or Hughes-Marsden [3].

(Figure 1.8), then L may be taken to be the radius of the sphere and $U = U_\infty$.

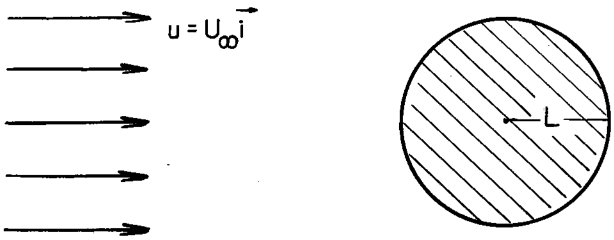

Figure 1.8

If the fluid is not viscous ($\nu = 0$), then $R = \infty$, and the fluid satisfies Euler's equations:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\text{grad } p \tag{1.5}$$

$$\text{div } u = 0. \tag{1.6}$$

The boundary condition becomes: $u|_{\partial D}$ is parallel to ∂D, or $u||\partial D$ for short. This sudden change of boundary condition from $u = 0$ on ∂D to $u||\partial D$ is of fundamental significance and is responsible for many of the difficulties in fluid mechanics for R very large (see footnote below).

The Reynolds number of the flow has the property that, if we rescale as follows:

$$u^* = \frac{U^*}{U} u$$

$$x^* = \frac{L^*}{L} x$$

$$t^* = \frac{T^*}{T} t$$

$$p^* = \left(\frac{U^*}{U}\right)^2 p$$

then if $T = L/U$, $T^* = L^*/U^*$ and provided $R^* = U^*L^*/\nu^* = R = UL/\nu$, u^* satisfies the same equations with respect to x^* and t^* that u satisfies with respect to x and t; i.e.,

$$\frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla^*)u^* - \nu^* \nabla u^* = -\text{grad } p^* \tag{1.7}$$

$$\text{div } u^* = 0 \tag{1.8}$$

with the same boundary condition $u^*\big|_{\partial D} = 0$ as before. (This is easy to check and is called Reynolds' law of similarity.) Thus, the nature of these two solutions of the Navier-Stokes equations is the same. The fact that this rescaling can be done is essential in practical problems. For example, it allows engineers to test a scale model of an airplane at low speeds to determine whether the real airplane will be able to fly at high speeds.

(1.7) <u>Example</u>. Consider the flow in Figure 1.8. If the fluid is not viscous, the boundary condition is that the velocity at the surface of the sphere is parallel to the sphere, and the fluid slips smoothly past the sphere (Figure 1.9).

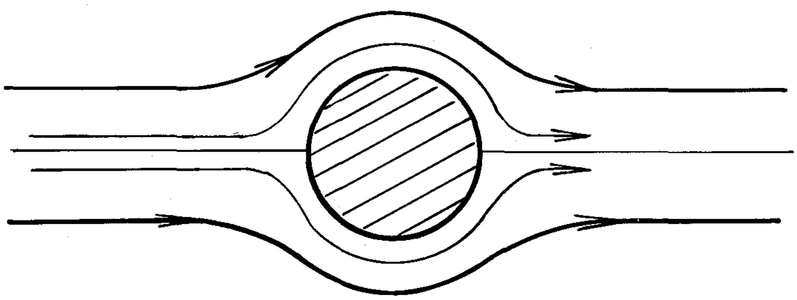

Figure 1.9

Now consider the same situation, but in the viscous case. Assume that R starts off small and is gradually increased. (In the laboratory this is usually accomplished by increasing the velocity $U_\infty \vec{i}$, but we may wish to think of it as $\nu \to 0$, i.e., molasses changing to water.) Because of the no-slip condition at the surface of the sphere, as U_∞ gets larger, the velocity gradient increases there. This causes the flow to become more and more complicated (Figure 1.10).*

For small values of the Reynolds number, the velocity field behind the sphere is observed to be stationary, or approximately so, but when a critical value of the Reynolds number is reached, it becomes periodic. For even higher values of the Reynolds number, the periodic solution loses stability and further bifurcations take place. The further bifurcation illustrated in Figure 1.10 is believed to represent a bifurcation from an attracting periodic orbit to a periodic orbit on an attracting 2-torus in $\mathfrak{X}$. These further bifurcations may eventually lead to turbulence. See Remark 1.15 and Section 9 below.

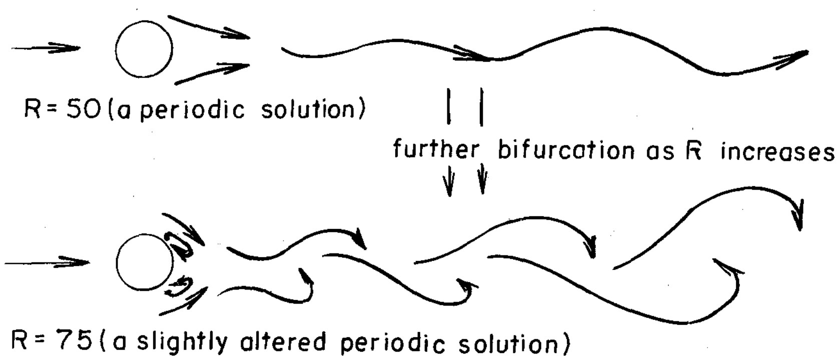

Figure 1.10

*These large velocity gradients mean that in numerical studies, finite difference techniques become useless for interesting flows. Recently A. Chorin [1] has introduced a brilliant technique for overcoming these difficulties and is able to simulate numerically for the first time, the "Karmen vortex sheet", illustrated in Figure 1.10. See also Marsden [5] and Marsden-McCracken [2].

(1.8) Example. Couette Flow. A viscous,* incompressible, homogeneous fluid fills the space between two long, coaxial cylinders which are rotating. For example, they may rotate in opposite directions with frequency ω (Figure 1.11). For small values of ω, the flow is horizontal, laminar and stationary.

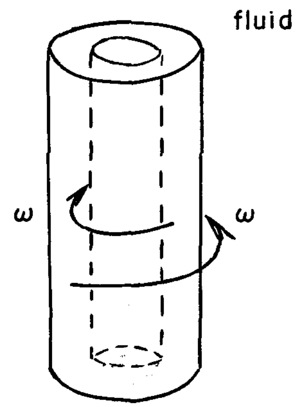

Figure 1.11

If the frequency is increased beyond some value ω_0, the fluid breaks up into what are called Taylor cells (Figure 1.12).

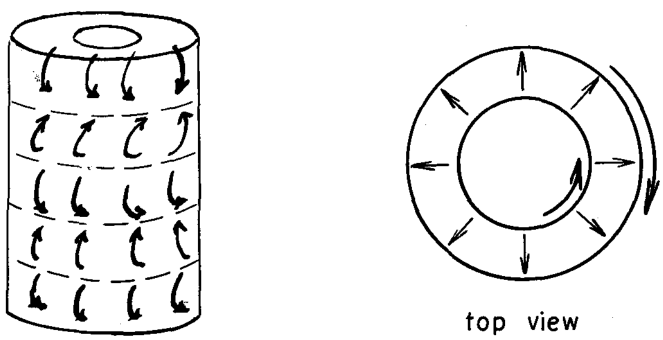

Figure 1.12

*Couette flow is studied extensively in the literature (see Serrin [1], Coles [1]) and is a stationary flow of the Euler equations as well as of the Navier-Stokes equations (see the following exercise).

Taylor cells are also a stationary solution of the Navier-Stokes equations. For larger values of ω, bifurcations to periodic, doubly periodic and more complicated solutions may take place (Figure 1.13).

Figure 1.13

For still larger values of ω, the structure of the Taylor cells becomes more complex and eventually breaks down completely and the flow becomes turbulent. For more information, see Coles [1] and Section 7.

(1.9) Exercise. Find a stationary solution $\vec{u}$ to the Navier-Stokes equations in cylindrical coordinates such that $\vec{u}$ depends only on r, $u_r = u_z = 0$, the external force $f = 0$ and the angular velocity ω satisfies $\omega|_{r=A_1} = -\rho_1$, and $\omega|_{r=A_2} = +\rho_2$ (i.e., find Couette flow). Show that $\vec{u}$ is also a solution to Euler equations.

(Answer: $\omega = \dfrac{\alpha}{r^2} + \beta$ where $\alpha = \dfrac{-(\rho_1+\rho_2)A_1^2A_2^2}{A_2^2-A_1^2}$ and $\beta = \dfrac{\rho_1^2A_1^2+\rho_2^2A_2^2}{A_2^2-A_1^2}$).

Another important place in fluid mechanics where an instability of this sort occurs is in flow in a pipe. The flow is steady and laminar (Poiseuille flow) up to Reynolds numbers around 4,000, at which point it becomes unstable and transition to chaotic or turbulent flow occurs. Actually if the experiment is done carefully, turbulence can be delayed until rather large R. It is analogous to balancing a ball on the tip of a rod whose diameter is shrinking.

Statement of the Principal Bifurcation Theorems

Let $X_\mu : P \to T(P)$ be a C^k vector field on a manifold P depending smoothly on a real parameter μ. Let F_t^μ be the flow of X_μ. Let p_0 be a fixed point for all μ, an attracting fixed point for $\mu < \mu_0$, and an unstable fixed point for $\mu > \mu_0$. Recall (Theorem 1.4) that the condition for stability of p_0 is that $\sigma(dX_\mu(p_0)) \subset \{z \mid \mathrm{Re}\, z < 0\}$. At $\mu = \mu_0$, some part of the spectrum of $dX_\mu(p_0)$ crosses the imaginary axis. The nature of the bifurcation that takes place at the point (p_0, μ_0) depends on how that crossing occurs (it depends, for example, on the dimension of the generalized eigenspace* of $dX_{\mu_0}(p_0)$ belonging to the part of the spectrum that crosses the axis). If P is a finite dimensional space, there are bifurcation theorems giving necessary conditions for certain kinds of bifurcation to occur. If P is not finite dimensional, we may be able, nevertheless, to reduce the problem to a finite dimensional one via the center manifold theorem by means of the following simple but crucial suspension trick. Let ψ be the time 1 map of the flow $F_t = (F_t^\mu, \mu)$ on $P \times \mathbb{R}$. As we shall show in Section 2A, $\sigma(d\psi(p_0,\mu_0)) = e^{\sigma(dX(p_0,\mu_0))}$. That is, $\sigma(d\psi(p_0,\mu_0)) = e^{\sigma(dX_{\mu_0}(p_0))} \cup \{1\}$.

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*The definition and basic properties are reviewed in Section 2A.

The following theorem is now applicable to ψ (see Sections 2-4 for details).

(1.10) Center Manifold Theorem (Kelley [1], Hirsch-Pugh-Shub [1], Hartman [1], Takens [2], etc.). *Let ψ be a mapping from a neighborhood of α_0 in a Banach manifold P to P. We assume that ψ has k continuous derivatives and that $\psi(\alpha_0) = \alpha_0$. We further assume that $d\psi(\alpha_0)$ has spectral radius 1 and that the spectrum of $d\psi(\alpha_0)$ splits into a part on the unit circle and the remainder, which is at a non-zero distance from the unit circle. Let Y denote the generalized eigenspace of $d\psi(\alpha_0)$ belonging to the part of the spectrum on the unit circle; assume that Y has dimension $d < \infty$. Then there exists a neighborhood V of α_0 in P and a C^{k-1} submanifold M, called a center manifold for ψ, of V of dimension d, passing through α_0 and tangent to Y at α_0, such that:*

(a) (Local Invariance): *If $x \in M$ and $\psi(x) \in V$, then $\psi(x) \in M$.*

(b) (Local Attractivity): *If $\psi^n(x) \in V$ for all $n = 0,1,2,\ldots$, then as $n \to \infty$, $\psi^n(x) \to M$.*

(1.11) Remark. It will be a corollary to the proof of the Center Manifold Theorem that if ψ is the time 1 map of F_t defined above then the center manifold M can be chosen so that properties (a) and (b) apply to F_t for all $t > 0$.

(1.12) Remark. The Center Manifold Theorem is not always true for C^∞ ψ in the following sense: since $\psi \in C^k$ for all k, we get a sequence of center manifolds M^k, but their intersection may be empty. See Remarks 2.6

regarding the differentiability of M.

We will be particularly interested in the case in which bifurcation to stable closed orbits occurs. With X_μ as before, assume that for $\mu = \mu_0$ (resp. $\mu > \mu_0$), $\sigma(dX_\mu(p_0))$ has two isolated nonzero, simple complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ such that $\mathrm{Re}\,\lambda(\mu) = 0$ (resp. > 0) and such that $\left.\frac{d(\mathrm{Re}\,\lambda(\mu))}{d\mu}\right|_{\mu=\mu_0} > 0$. Assume further that the rest of $\sigma(dX_\mu(p_0))$ remains in the left-half plane at a nonzero distance from the imaginary axis. Using the Center Manifold Theorem, we obtain a 3-manifold $M \subset P$, tangent to the eigenspace of $\lambda(\mu_0), \overline{\lambda(\mu_0)}$ and to the μ-axis at $\mu = \mu_0$, locally invariant under the flow of X, and containing all the local recurrence. The problem is now reduced to one of a vector field in two dimensions $\hat{X}_\mu: \mathbb{R}^2 \to \mathbb{R}^2$. The Hopf Bifurcation Theorem in two dimensions then applies (see Section 3 for details and Figures 1.4, 1.5):

(1.13) Hopf Bifurcation Theorem for Vector Fields (Poincaré [1], Andronov and Witt [1], Hopf [1], Ruelle-Takens [1], Chafee [1], etc.). Let X_μ be a C^k ($k \geq 4$) vector field on $\mathbb{R}^2$ such that $X_\mu(0) = 0$ for all μ and $X = (X_\mu, 0)$ is also C^k. Let $dX_\mu(0,0)$ have two distinct, simple* complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ such that for $\mu < 0$, $\mathrm{Re}\,\lambda(\mu) < 0$, for $\mu = 0$, $\mathrm{Re}\,\lambda(\mu) = 0$, and for $\mu > 0$, $\mathrm{Re}\,\lambda(\mu) > 0$. Also assume $\left.\frac{d\,\mathrm{Re}\,\lambda(\mu)}{d\mu}\right|_{\mu=0} > 0$. Then there is a C^{k-2} function $\mu: (-\varepsilon, \varepsilon) \to R$ such that

*Simple means that the generalized eigenspace (see Section 2A) of the eigenvalue is one dimensional.

$(x_1, 0, \mu(x_1))$ _is on a closed orbit of period_ $\approx \frac{2\pi}{|\lambda(0)|}$ _and radius growing like_ $\sqrt{\mu}$, _of the flow of_ X _for_ $x_1 \neq 0$ _and such that_ $\mu(0) = 0$. _There is a neighborhood_ U _of_ $(0,0,0)$ _in_ $\mathbb{R}^3$ _such that any closed orbit in_ U _is one of the above._ _Furthermore,_ (c) _if_ 0 _is a "vague attractor"*_ _for_ X_0, _then_ $\mu(x_1) > 0$ _for all_ $x_1 \neq 0$ _and the orbits are attracting_ (see Figures 1.4, 1.5).

If, instead of a pair of conjugate eigenvalues crossing the imaginary axis, a real eigenvalue crosses the imaginary axis, two stable fixed points will branch off instead of a closed orbit, as in the ball in the hoop example. See Exercise 1.16.

After a bifurcation to stable closed orbits has occurred, one might ask what the next bifurcation will look like. One can visualize an invariant 2-torus blossoming out of the closed orbit (Figure 1.14). In fact, this phenomenon can occur. In order to see how, we assume we have a stable closed orbit for F_t^μ. Associated with this orbit is a Poincaré map. To define the Poincaré map, let x_0 be a point on the orbit, let N be a codimension one manifold through x_0 transverse to the orbit. The Poincaré map P_μ takes each point $x \in U$, a small neighborhood of x_0 in N, to the next point at which $F_t^\mu(x)$ intersects N (Figure 1.15). The Poincaré map is a diffeomorphism from U to $V = P_\mu(U) \subseteq N$, with $P_\mu(x_0) = x_0$

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*This condition is spelled out below, and is reduced to a specific hypothesis on X in Section 4A. See also Section 4C. The case in which $d \operatorname{Re} \lambda(\mu)/d\mu = 0$ is discussed in Section 3A. In Section 3B it is shown that "vague attractor" can be replaced by "asymptotically stable". For a discussion of what to expect generically, see Ruelle-Takens [1], Sotomayer [1], Newhouse and Palis [1] and Section 7.

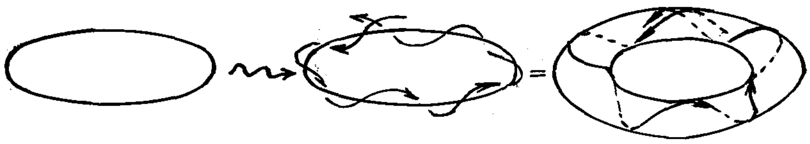

Figure 1.14

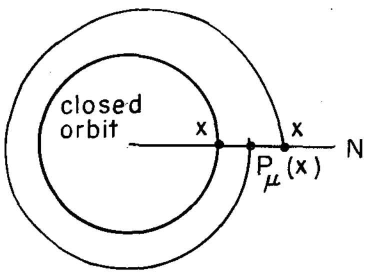

Figure 1.15

(see Section 2B for a summary of properties of the Poincaré map). The orbit is attracting if $\sigma(dP_\mu(x_0)) \subset \{z \mid |z| < 1\}$ and is not attracting if there is some $z \in \sigma(dP_\mu(x_0))$ such that $|z| > 1$.

We assume, as above, that $X_\mu: P \to TP$ is a C^k vector field on a Banach manifold P with $X_\mu(p_0) = 0$ for all μ. We assume that p_0 is stable for $\mu < \mu_0$, and that p_0 becomes unstable at μ_0, at which point bifurcation to a stable, closed orbit $\gamma(\mu)$ takes place. Let P_μ be the Poincaré map associated with $\gamma(\mu)$ and let $x_0(\mu) \in \gamma(\mu)$. We further assume that at $\mu = \mu_1$, two isolated, simple, complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ of $dP_\mu(x_0(\mu))$ cross the unit circle such that $\left.\frac{d|\lambda(\mu)|}{d\mu}\right|_{\mu=\mu_1} > 0$ and such that the rest of $\sigma(dP_\mu(x_0(\mu)))$ remains inside the unit circle, at a nonzero distance from it. We then apply the Center Manifold Theorem to the map $P = (P_\mu, \mu)$ to obtain, as before, a locally invariant 3-manifold for P. The Hopf Bifurcation

Theorem for diffeomorphisms (in (1.14) below) then applies to yield a one parameter family of invariant, stable circles for P_μ for $\mu > \mu_1$. Under the flow of X_μ, these circles become stable invariant 2-tori for F_t^μ (Figure 1.16).

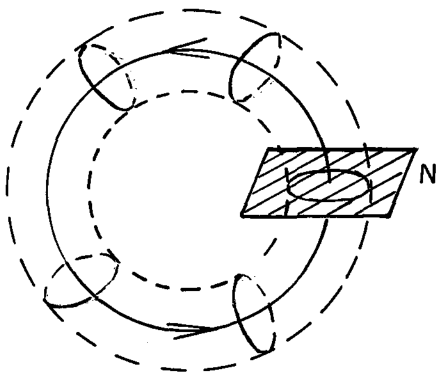

Figure 1.16

(1.14) Hopf Bifurcation Theorem for Diffeomorphisms (Sacker [1], Naimark [2], Ruelle-Takens [1]). *Let* $P_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ *be a one-parameter family of* C^k $(k \geq 5)$ *diffeomorphisms satisfying:*

(a) $P_\mu(0) = 0$ *for all* μ

(b) *For* $\mu < 0$, $\sigma(dP_\mu(0)) \subset \{z \mid |z| < 1\}$

(c) *For* $\mu = 0$ $(\mu > 0)$, $\sigma(dP_\mu(0))$ *has two isolated, simple, complex conjugate eigenvalues* $\lambda(\mu)$ *and* $\overline{\lambda(\mu)}$ *such that* $|\lambda(\mu)| = 1$ $(|\lambda(\mu)| > 1)$ *and the remaining part of* $\sigma(dP_\mu(0))$ *is inside the unit circle, at a nonzero distance from it.*

(d) $\left.\dfrac{d|\lambda(\mu)|}{d\mu}\right|_{\mu=0} > 0.$

Then (under two more "vague attractor" hypotheses which will be explained during the proof of the theorem), there is a *continuous one parameter family of invariant attracting circles of* P_μ, *one for each* $\mu \in (0,\varepsilon)$ *for small* $\varepsilon > 0$.

(1.15) *Remark.* In Sections 8 and 9 we will discuss how these bifurcation theorems yielding closed orbits and invariant tori can actually be applied to the Navier-Stokes equations. One of the principal difficulties is the smoothness of the flow, which we overcome by using general smoothness results (Section 8A). Judovich [1-11], Iooss [1-6], and Joseph and Sattinger have used Hopf's original method for these results. Ruelle and Takens [1] have speculated that further bifurcations produce higher dimensional stable, invariant tori, and that the flow becomes turbulent when, as an integral curve in the space of all vector fields, it becomes trapped by a "strange attractor" (strange attractors are shown to be abundant on k-tori for $k \geq 4$); see Section 9. They can also arise spontaneously (see 4B.8 and Section 12). The question of how one can explicitly follow a fixed point through to a strange attractor is complicated and requires more research. Important papers in this direction are Takens [1,2], Newhouse [1] and Newhouse and Peixoto [1].

(1.16) *Exercise.* (a) Prove the following:

Theorem. Let $\mathbb{H}$ be a Hilbert space (or manifold) and $\Phi_\mu: \mathbb{H} \to \mathbb{H}$ a map defined for each $\mu \in \mathbb{R}$ such that the map $(\mu,x) \mapsto \Phi_\mu(x)$ is a C^k map, $k \geq 1$, from $\mathbb{R} \times \mathbb{H}$ to $\mathbb{H}$, and for all $\mu \in \mathbb{R}$, $\Phi_\mu(0) = 0$. Define $L_\mu = D\Phi_\mu(0)$ and suppose the spectrum of L_μ lies inside the unit circle for $\mu < 0$. Assume further there is a real, simple, isolated eigenvalue $\lambda(\mu)$ of L_μ such that $\lambda(0) = 1$, $(d\lambda/d\mu)(0) > 0$, and L_0^*

has the eigenvalue 1 (Figure 1.17); then there is a C^{k-1} curve ℓ of fixed points of $\Phi: (x,\mu) \mapsto (\Phi_\mu(x),\mu)$ near $(0,0) \in \mathbb{H} \times \mathbb{R}$. The curve is tangent to $\mathbb{H}$ at $(0,0)$ in $\mathbb{H} \times \mathbb{R}$ (Figure 1.18). These points and the points of $(0,\mu)$ are the only fixed points of Φ in a neighborhood of $(0,0)$.

(b) Show that the hypotheses apply to the ball in the hoop example (see Exercise 1.2).

<u>Hint</u>: Pick an eigenvector $(z,0)$ for $(L_0,0)$ in $\mathbb{H} \times \mathbb{R}$ with eigenvalue 1. Use the center manifold theorem

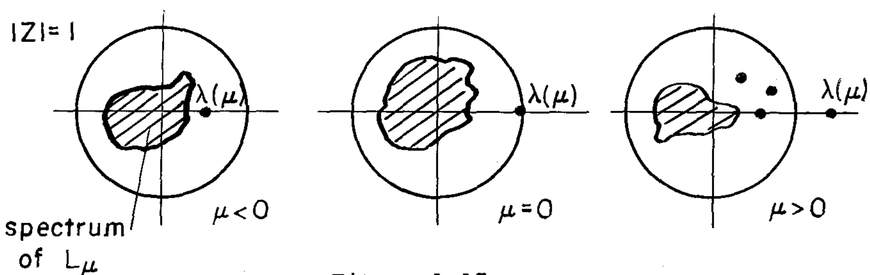

Figure 1.17

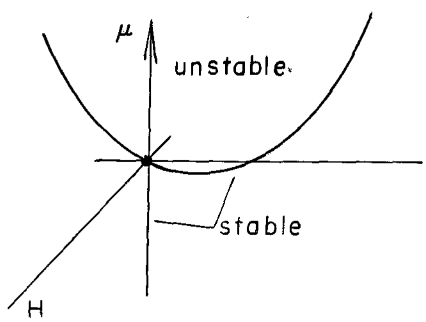

Figure 1.18

to obtain an invariant 2-manifold C tangent to $(z,0)$ and the μ axis for $\Phi(x,\mu) = (\Phi_\mu(x),\mu)$. Choose coordinates (α,μ) on C where α is the projection to the normalized eigenvector $z(\mu)$ for L_μ. Set $\Phi(x,\mu) = (f(\alpha,\mu),\mu)$ in these coordinates. Let $g(\alpha,\mu) = \frac{f(\alpha,\mu)}{\alpha} - 1$ and we use the implicit function theorem to get a curve of zeros of g in C. (See Ruelle-Takens [1, p. 190]).

(1.17) Remark. The closed orbits which appear in the Hopf theorem need not be globally attracting, nor need they persist for large values of the parameter μ. See remarks (3A.3).

(1.18) Remark. The reduction to finite dimensions using the center manifold theorem is analogous to the reduction to finite dimensions for stationary bifurcation theory of elliptic type equations which goes under the name "Lyapunov-Schmidt" theory. See Nirenberg [1] and Vainberg-Trenogin [1,2].

(1.19) Remark. Bifurcation to closed orbits can occur by other mechanisms than the Hopf bifurcation. In Figure 1.19 is shown an example of S. Wan.

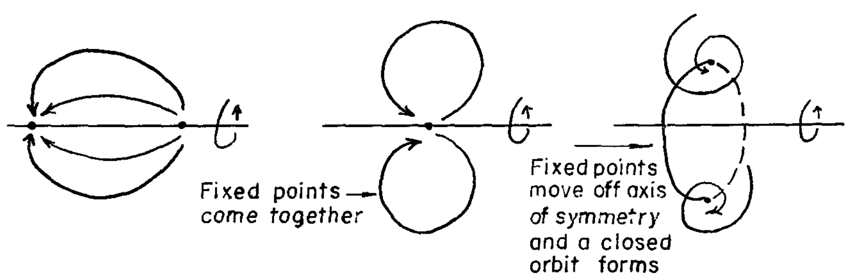

Figure 1.19