8 Inverse Functions and the Chain Rule

Formulas for the derivatives of inverse and composite functions are two of the most useful tools of differential calculus. As usual, standard calculus texts should be consulted for additional applications.

Inverse Functions

Definition Let the function f be defined on a set A. Let B be the range of values for f on A; that is, $y \in B$ means that $y = f(x)$ for some x in A. We say that f is *invertible on A* if, for every y in B, there is a *unique* x in A such that $y = f(x)$.

If f is invertible on A, then there is a function g, whose domain is B, given by this rule: $g(y)$ is that unique x in A for which $f(x) = y$. We call g the *inverse function* of f on A. The inverse function to f is denoted by f^{-1}.

Worked Example 1 Let $f(x) = x^2$. Find a suitable A such that f is invertible on A. Find the inverse function and sketch its graph.

Solution The graph of f on $(-\infty, \infty)$ is shown in Fig. 8-1. For y in the range of f—that is, $y \geqslant 0$—there are two values of x such that $f(x) = y$: namely, $-\sqrt{y}$ and $+\sqrt{y}$. To assure only one x we may restrict f to $A = [0, \infty)$. Then $B = [0, \infty)$ and for any y in B there is exactly one x in A such that $f(x) = y$: namely, $x = \sqrt{y} = g(y)$. (Choosing $A = (-\infty, 0]$ and obtaining $x = -\sqrt{y}$ is also acceptable.) We obtain the graph of the inverse by looking at the graph of f from the back of the page.

If we are given a formula for $y = f(x)$ in terms of x, we may be able to find a formula for its inverse $x = g(y)$ by solving the equation $y = f(x)$ for x in terms of y. Sometimes, for complicated functions f, one cannot solve $y = f(x)$ to get an explicit formula for x in terms of y. In that case, one must resort to theoretical results which guarantee the existence of an inverse function. These will be discussed shortly.

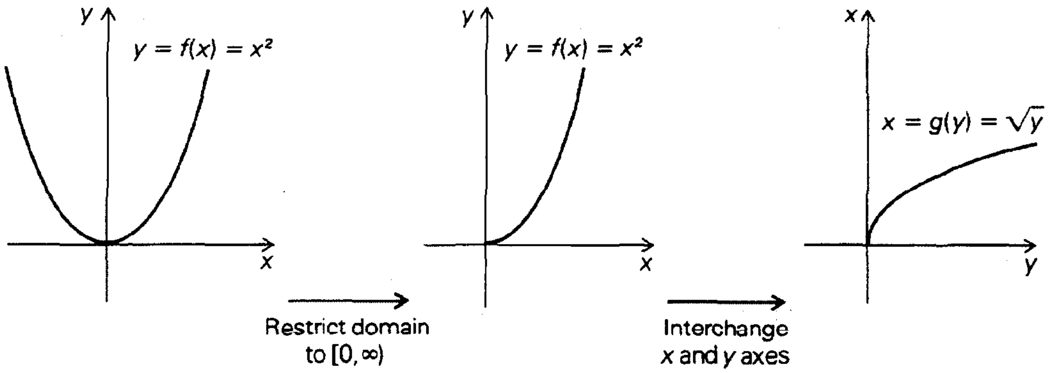

Fig. 8-1 A restriction of $f(x) = x^2$ and its inverse.

Worked Example 2 Does $f(x) = x^3$ have an inverse on $(-\infty, \infty)$? Sketch.

Solution From Fig. 8-2 we see that the range of f is $(-\infty, \infty)$ and for each $y \in (-\infty, \infty)$ there is exactly one number x such that $f(x) = y$—namely, $x = \sqrt[3]{y} = f^{-1}(y)$ (negative if $y < 0$, positive or zero if $y \geqslant 0$)—so the answer is yes. We can regain x as our independent variable by observing that if $f^{-1}(y) = \sqrt[3]{y}$, then $f^{-1}(x) = \sqrt[3]{x}$. Thus we can replace y by x so that the independent variable has a more familiar name. As shown in Fig. 8-2(*b*) and (*c*), this renaming does not affect the graph of f^{-1}.

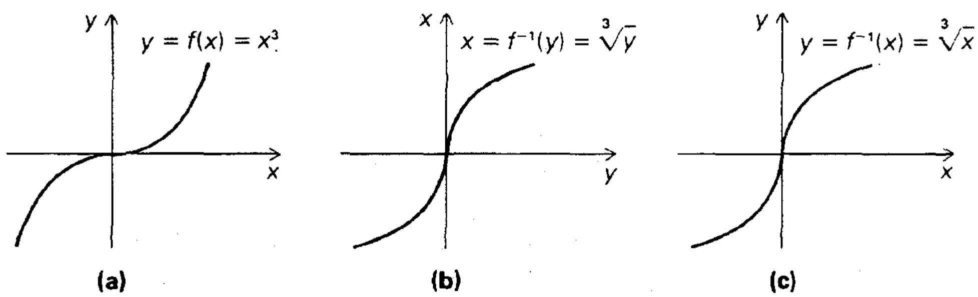

Fig. 8-2 The inverse of the cubing function is the cube root, whatever you call the variables.

Solved Exercises*

1. If $m \neq 0$, find the inverse function for $f(x) = mx + b$ on $(-\infty, \infty)$.
2. Find the inverse function for $f(x) = (ax + b)/(cx + d)$ on its domain $(c \neq 0)$.

*Solutions appear in the Appendix.

3. Find an inverse function g for $f(x) = x^2 + 2x + 1$ on some interval containing zero. What is $g(9)$? What is $g(x)$?

4. Sketch the graph of the inverse function for each function in Fig. 8-3.

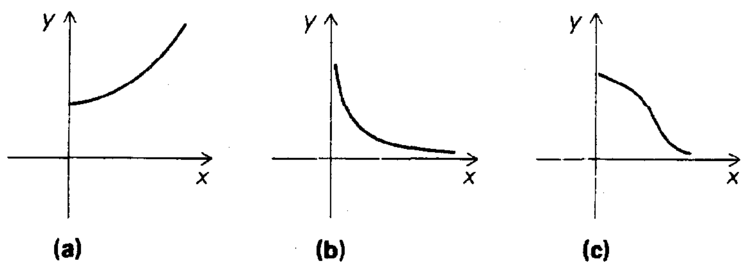

Fig. 8-3 Sketch the inverse functions.

5. Determine whether or not each function in Fig. 8-4 is invertible on its domain.

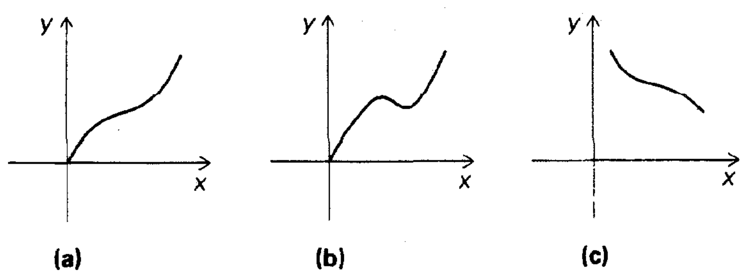

Fig. 8-4 Which functions are invertible?

Exercises

1. Find the inverse for each of the following functions on the given interval:
   (a) $f(x) = 2x + 5$ on $[-4,4]$ (b) $f(x) = -\frac{1}{3}x + 2$ on $(-\infty,\infty)$
   (c) $h(t) = t - 10$ on $[0,\pi)$
   (d) $a(s) = (2s + 5)/(-s + 1)$ on $[-\frac{1}{2},\frac{1}{2}]$
   (e) $f(x) = x^5$ on $(-\infty,\infty)$ (f) $f(x) = x^8$ on $(0,1]$

2. Determine whether each function in Fig. 8-5 has an inverse. Sketch the inverse if there is one.

3. Sketch a graph of $f(x) = x/(1 + x^2)$ and find an interval on which f is invertible.

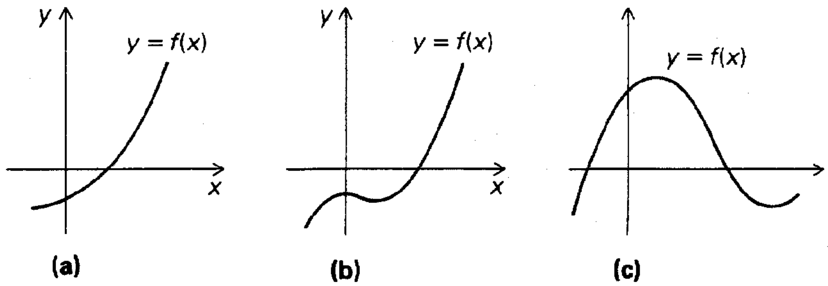

Fig. 8-5 Which functions are invertible?

4. Enter the number 2.6 on a calculator, then push the x^2 key followed by the $\sqrt{x}$ key. Is there any round-off error? Try the $\sqrt{x}$ key, then the x^2 key. Also try a sequence such as $x^2, \sqrt{x}, x^2, \sqrt{x}, \ldots$. Do the errors build up? Try pushing the x^2 key five times, then the $\sqrt{x}$ key five times. Do you get back the original number? Try these experiments with different starting numbers.

5. If we think of a French-English dictionary as defining a function from the set of French words to the set of English words (does it really?), how is the inverse function defined? Discuss.

A Test for Invertibility

A function may be invertible even though we cannot find an explicit formula for the inverse function. This fact gives us a way of obtaining "new functions." There is a useful calculus test for finding intervals on which a function is invertible.

Theorem 1 *Suppose that f is continuous on $[a,b]$ and that f is increasing at each point of (a,b). (For instance, this holds if $f'(x) > 0$ for each x in (a,b).) Then f is invertible on $[a,b]$, and the inverse f^{-1} is defined on $[f(a), f(b)]$.*

If f is decreasing rather than increasing at each point of (a,b), then f is still invertible; in this case, the domain of f^{-1} is $[f(b), f(a)]$.

Proof If f is continuous on $[a,b]$ and increasing at each point of (a,b), we know by the results of Chapter 5 on increasing functions that f is increasing on $[a,b]$; that is, if $a \leqslant x_1 < x_2 \leqslant b$, then $f(x_1) < f(x_2)$. In particular, $f(a) < f(b)$. If y is any number in $(f(a), f(b))$, then by the intermediate value theorem there is an x in (a, b) such that $f(x) = y$. If $y = f(a)$ or $f(b)$, we can choose $x = a$ or $x = b$. Since f is increasing on $[a, b]$ for any y in $[f(a), f(b)]$, there can only be one x such that $y = f(x)$. In fact, if we have $f(x_1) = f(x_2) = y$, with $x_1 < x_2$, we would have $y < y$, which is impossible. Thus, by definition, f is invertible on $[a, b]$ and the domain of f^{-1} is the range $[f(a), f(b)]$ of values of f on $[a, b]$. The proof of the second assertion is similar.

Worked Example 3 Verify, using Theorem 1, that $f(x) = x^2$ has an inverse if f is defined on $[0, b]$ for a given $b > 0$.

Solution Since f is differentiable on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$ and hence on $[0, b]$. But $f'(x) = 2x > 0$ for $0 < x < b$. Thus f is increasing. Hence Theorem 1 guarantees that f has an inverse defined on $[0, b^2]$.

In general, a function is not monotonic throughout the interval on which it is defined. Theorem 1 shows that the turning points of f divide the domain of f into subintervals on each of which f is monotonic and invertible.

Solved Exercises

6. Let $f(x) = x^5 + x$.
   (a) Show that f has an inverse on $[-2, 2]$. What is the domain of this inverse?
   (b) Show that f has an inverse on $(-\infty, \infty)$.
   (c) What is $f^{-1}(2)$?
   (d) Numerically calculate $f^{-1}(3)$ to two decimal places of accuracy.
7. Find intervals on which $f(x) = x^5 - x$ is invertible.
8. Show that, if n is odd, $f(x) = x^n$ is invertible on $(-\infty, \infty)$. What is the domain of the inverse function?
9. Discuss the invertibility of $f(x) = x^n$ for n even.

Exercises

6. Show that $f(x) = -x^3 - 2x + 1$ is invertible on $[-1, 2]$. What is the domain of the inverse?
7. (a) Show that $f(x) = x^3 - 2x + 1$ is invertible on $[2, 4]$. What is the domain of the inverse?

(b) Find the largest possible intervals on which f is invertible.

8. Find the largest possible intervals on which $f(x) = 1/(x^2 - 1)$ is invertible. Sketch the graphs of the inverse functions.

9. Show that $f(x) = \frac{1}{3}x^3 - x$ is not invertible on any open interval containing 1.

10. Let $f(x) = x^5 + x$.

(a) Find $f^{-1}(246)$.

(b) Find $f^{-1}(4)$, correct to at least two decimal places using a calculator.

Differentiating Inverse Functions

Even though the inverse function $f^{-1}(y)$ is defined somewhat abstractly, there is a simple formula for its derivative. To motivate the formula we proceed as follows. First of all, note that if l is the linear function $l(x) = mx + b$, then $l^{-1}(y) = (1/m)y - (b/m)$ (see Solved Exercise 1), so $l'(x) = m$, and $(l^{-1})'(y) = 1/m$, the reciprocal. We can express this by $dx/dy = 1/(dy/dx)$.*

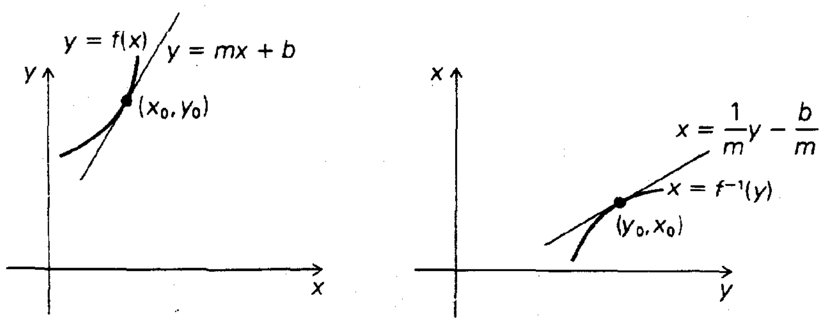

Fig. 8-6 Flipping the graphs preserves tangency.

Next, examine Fig. 8-6, where we have flipped the graph of f and its tangent line $y = mx + b$ in the drawing on the left to obtain the graph of f^{-1} together with the line $x = l^{-1}(y) = (y/m) - (b/m)$ in the drawing on the right. Since the line is tangent to the curve on the left, and flipping the drawing should preserve this tangency, the line $x = (1/m)y - (b/m)$ ought to be the tangent line to $x = f^{-1}(y)$ at (y_0, x_0), and its slope $1/m$ should be the derivative $(f^{-1})'(y_0)$. Since $m = f'(x_0)$, we conclude that

*As usual in calculus, we use the Leibniz notation dy/dx for $f'(x)$, when $y = f(x)$.

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Since the expression $(f^{-1})'$ is awkward, we sometimes revert to the notation $g(y)$ for the inverse function and write

$$g'(y_0) = \frac{1}{f'(x_0)}$$

Notice that although dy/dx is not an ordinary fraction, the rule

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

is valid.

Theorem 2 Suppose that $f'(x) > 0$ or $f'(x) < 0$ for all x in an open interval I containing x_0, so that by Theorem 1 there is an inverse function g to f, defined on an open interval containing $f(x_0) = y_0$, with $g(y_0) = x_0$. Then g is differentiable at y_0 and

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}$$

Proof We will consider the case $f'(x) > 0$. The case $f'(x) < 0$ is entirely similar (in fact, one may deduce the case $f' < 0$ from the case $f' > 0$ by considering $-f$).

From Theorem 1, f is increasing on I and the domain of g is an open interval J. The graphs of typical f and g are drawn in Fig. 8-7, which you should consult while following the proof.

Let $x = l(y)$ be a line through $(y_0, g(y_0)) = (y_0, x_0)$ (Fig. 8-7) with slope $m > 1/f'(x_0) > 0$. We will prove that l overtakes g at y_0. We may write $l(y) = my + b$, so $l(y_0) = x_0$. Define $\tilde{l}(x) = (x/m) - (b/m)$ which is the inverse function of $l(y)$. The line $y = \tilde{l}(x)$ passes through $(x_0, f(x_0))$ and has slope $1/m < f'(x_0)$.

By definition of the derivative, $\tilde{l}$ is overtaken by f at x_0. This means that there is an interval I containing x_0 such that

$$f(x) > \tilde{l}(x) \quad \text{if } x \text{ is in } I_1, x > x_0$$

and

$$f(x) < \tilde{l}(x) \quad \text{if } x \text{ is in } I_1, x < x_0$$

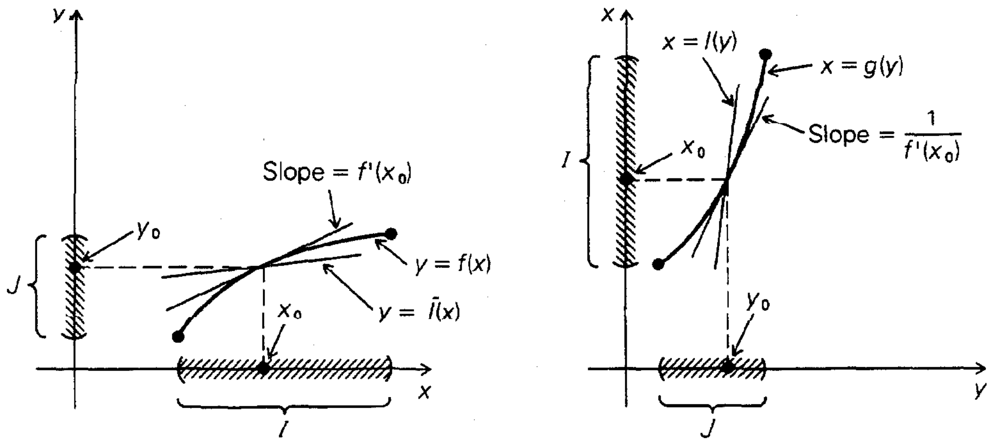

Fig. 8-7 Illustrating the proof of the inverse function theorem.

Let J_1 be the corresponding y interval, an open interval containing y_0 (see Theorem 1). Since f is increasing, $x > x_0$ corresponds to $y > y_0$ if $y = f(x)$, and likewise $x < x_0$ corresponds to $y < y_0$.

Now let y be in J_1, $y > y_0$, and let $y = f(x)$ so $x > x_0$ and x is in I_1. Hence

$$f(x) > \tilde{l}(x) = \frac{x}{m} - \frac{b}{m}$$

Since $y = f(x)$ and $x = g(y)$, this becomes

$$y > \frac{g(y)}{m} - \frac{b}{m}$$

i.e.,

$$l(y) = my + b > g(y)$$

since $m > 0$. Similarly, one shows that if $y < y_0$, and y is in I,

$$l(y) < g(y)$$

Hence l overtakes g at y_0, which proves what we promised.

A similar argument shows that a line with slope $m < 1/f'(x_0)$ is overtaken by g at y_0. Thus, by definition of the derivative, g is differentiable at y_0, and its derivative there is $1/f'(x_0)$ as required.

Worked Example 4 Use the inverse function rule to compute the derivative of $\sqrt{x}$. Evaluate the derivative at $x = 2$.

Solution Let us write $g(y) = \sqrt{y}$. This is the inverse function to $f(x) = x^2$. Since $f'(x) = 2x$,

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}}$$

so $(d/dy)\sqrt{y} = 1/(2\sqrt{y})$. We may substitute any letter for y in this result, including x, so we get the formula

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

When $x = 2$, the derivative is $1/(2\sqrt{2})$.

Generalizing this example, we can show that if $f(x) = x^r$, r a rational number, then

$$f'(x) = rx^{r-1}$$

(First consider the case where $1/r$ is an integer.)

Solved Exercises

10. Verify the inverse function rule for $y = (ax + b)/(cx + d)$ by finding dy/dx and dx/dy directly. (See Solved Exercise 2.)
11. If $f(x) = x^3 + 2x + 1$, show that f has an inverse on $[0, 2]$. Find the derivative of the inverse function at $y = 4$.
12. Give an example of a differentiable increasing function $f(x)$ on $(-\infty, \infty)$ which has an inverse $g(y)$, but such that g is not differentiable at $y = 0$.
13. Check the self-consistency of the inverse function theorem in this sense: if it is applied to the inverse of the inverse function, we recover something we know to be true.

Exercises

11. Let $y = x^3 + 2$. Find dx/dy when $y = 3$.
12. If $f(x) = x^5 + x$, find the derivative of the inverse function when $y = 34$.
13. Find the derivative of $\sqrt[3]{x}$ on $I = (0, \infty)$.
14. For each function f below, find the derivative of the inverse function g at the points indicated:

(a) $f(x) = 3x + 5$; find $g'(2), g'(\frac{3}{4})$.
(b) $f(x) = x^5 + x^3 + 2x$; find $g'(0), g'(4)$.
(c) $f(x) = \frac{1}{12}x^3 - x$ on $[-1, 1]$; find $g'(0), g'(\frac{11}{12})$.

15. Carry out the details of the proof of Theorem 2 in the case $m < 1/f'(x_0)$. [*Hint*: First consider the case $m > 0$.]

Composition of Functions

The idea behind the definition of composition of functions is that one variable depends on another through an intermediate one.

Definition Let f and g be functions with domains D_f and D_g. Let D be the set consisting of those x in D_g for which $g(x)$ belongs to D_f. For x in D, we can evaluate $f(g(x))$, and we call the result $h(x)$. The resulting function $h(x) = f(g(x))$, with domain D, is called the *composition* of f and g. It is often denoted by $f \circ g$.

Worked Example 5 If $f(u) = u^3 + 2$ and $g(x) = \sqrt{x^2 + 1}$, what is $h = f \circ g$?

Solution We calculate $h(x) = f(g(x))$ by writing $u = g(x)$ and substituting in $f(u)$. We get $u = \sqrt{x^2 + 1}$ and

$$h(x) = f(u) = u^3 + 2 = (\sqrt{x^2 + 1})^3 + 2 = (x^2 + 1)^{3/2} + 2$$

Since each of f and g is defined for all numbers, the domain D of h is $(-\infty, \infty)$.

Solved Exercises

14. Let $g(x) = x + 1$ and $f(u) = u^2$. Find $f \circ g$ and $g \circ f$.
15. Let $h(x) = x^{24} + 3x^{12} + 1$. Write $h(x)$ as a composite function $f(g(x))$.
16. Let $f(x) = x - 1$ and $g(x) = \sqrt{x}$.
    (a) What are the domains D_f and D_g?
    (b) Find $f \circ g$ and $g \circ f$. What are their domains?
    (c) Find $(f \circ g)(2)$ and $(g \circ f)(2)$.
    (d) Sketch graphs of $f, g, f \circ g$, and $g \circ f$.
17. Let i be the identity function $i(x) = x$. Show that $i \circ f = f$ and $f \circ i = f$ for any function f.

Exercises

16. Find $f \circ g$ and $g \circ f$ in each of the following cases.
    (a) $g(x) = x^3$; $f(x) = \sqrt{x-2}$
    (b) $g(x) = x^r$; $f(x) = x^s$ (r, s rational)
    (c) $g(x) = 1/(1-x)$; $f(x) = \frac{1}{2} - \sqrt{3x}$
    (d) $g(x) = (3x-2)/(4x+1)$; $f(x) = (2x-7)/(9x+3)$
17. Write the following as compositions of simpler functions:
    (a) $h(x) = 4x^2/(x^2-1)$
    (b) $h(r) = (r^2+6r+9)^{3/2} + (1/\sqrt{r^2+6r+9})$
    (c) $h(u) = \sqrt{(1-u)/(1+u)}$
18. Show that the inverse f^{-1} of any function f satisfies $f \circ f^{-1} = i$ and $f^{-1} \circ f = i$, where i is the identity function of Solved Exercise 17.

The Chain Rule

***Theorem 3** Chain Rule. Suppose that g is differentiable at x_0 and that f is differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at x_0 and its derivative there is $f'(g(x_0)) \cdot g'(x_0)$.*

As with the algebraic rules of differentiation, to prove the chain rule, we reduce the problem to one about rapidly vanishing functions (See Chapter 3). Write $y_0 = g(x_0)$ and

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + r(y)$$

where $r(y)$ is rapidly vanishing at y_0. Set $y = g(x)$ in this formula to get

$$f(g(x)) = f(g(x_0)) + f'(g(x_0))(g(x) - g(x_0)) + r(g(x))$$

Suppose we knew that $r(g(x))$ vanished rapidly at x_0. Then we could differentiate the right-hand side with respect to x at $x = x_0$ to obtain what we want (using the sum rule, the constant multiple rule, and the fact that the derivative of a constant is zero). In other words, the proof of Theorem 3 is reduced to the following result.

Lemma *Let $r(y)$ be rapidly vanishing at y_0 and suppose that $g(x)$ is differentiable at x_0, where $y_0 = g(x_0)$. Then $r(g(x))$ is rapidly vanishing at x_0.*

Proof We will use the characterization of rapidly vanishing functions given by Theorem 1 of Chapter 3. First of all, we have $r(g(x_0)) = r(y_0) = 0$, since r vanishes at y_0. Now, given any number $\epsilon > 0$, we must find an open interval I about x_0 such that, for all x in I with $x \neq x_0$, we have $|r(g(x))| < \epsilon|x - x_0|$.

Pick a number M such that $M > |g'(x_0)|$. Since r vanishes rapidly at y_0 and ϵ/M is a positive number, there is an open interval J about y_0 such that, for all y in J with $y \neq y_0$, we have $|r(y)| < (\epsilon/M)\ |y - y_0|$.

Next, let I be an open interval about x_0 such that, for $x \neq x_0$ in I, we have $|g(x) - g(x_0)| < M|x - x_0|$. Such an interval exists because $-M < g'(x_0) < M$. (See Fig. 8-8.) By making I sufficiently small (see Solved Exercise 19), we can be sure that $g(x)$ is in J for all x in I.

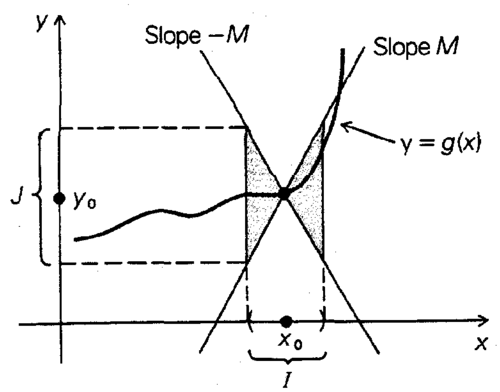

Fig. 8-8 The graph $y = g(x)$ lies in the "bow-tie" region for x in I.

Finally, let $x \in I, x \neq x_0$. To show that $|r(g(x))| < \epsilon|x - x_0|$, we consider separately the cases $g(x) \neq y_0$ and $g(x) = y_0$. In the first case, we have

$$|r(g(x))| < \frac{\epsilon}{M}|g(x) - y_0| = \frac{\epsilon}{M}|g(x) - g(x_0)| < \frac{\epsilon}{M}\cdot M|x - x_0| = \epsilon|x - x_0|$$

In the second case, $r(g(x)) = 0$ and $\epsilon|x - x_0| > 0$, so we are done.

Worked Example 6 Verify the chain rule for $f(u) = u^2$ and $g(x) = x^3 + 1$.

Solution Let $h(x) = f(g(x)) = [g(x)]^2 = (x^3 + 1)^2 = x^6 + 2x^3 + 1$. Thus $h'(x) = 6x^5 + 6x^2$. On the other hand, since $f'(u) = 2u$ and $g'(x) = 3x^2$,

$$f'(g(x)) \cdot g'(x) = (2 \cdot (x^3 + 1))\, 3x^2 = 6x^5 + 6x^2$$

Hence the chain rule is verified in this case.

Solved Exercises

18. Show that the rule for differentiating $1/g(x)$ follows from that for $1/x$ and the chain rule.

19. How should we choose I in the proof of the lemma so that $g(x)$ is in J for all x in I?

Exercises

19. Suppose f is differentiable on (a, b) with $f'(x) > 0$ for all x, and g is differentiable on (c, d) with $g'(x) < 0$ for x in (c, d). Let (c, d) be the image set of f, so $g \circ f$ is defined. Show that $g \circ f$ has an inverse and calculate the derivative of the inverse.

20. Let f, g, h all be differentiable. State a theorem concerning the differentiability of $k(x) = h(g(f(x)))$. Think of a *specific* example where you would apply this result.

21. Prove that $\frac{d}{dx}[f(x)]^r = r[f(x)]^{r-1} f'(x)$ (rational power of a function rule).

Problems for Chapter 8

1. Prove that, if f is continuous and increasing (or decreasing) on an open interval I, then the inverse function f^{-1} is continuous as well.

2. Let f be differentiable on an open interval I. Assume f' is continuous and $f'(x_0) \neq 0$. Prove that, on an interval about x_0, f has a differentiable inverse g.

3. Suppose that f is differentiable on an open interval I and that f' is continuous. Assume $f'(x) \neq 0$ for all x in I. Prove f has an inverse which is differentiable.

4. Let $f^{(n)}(x) = f \circ \cdots \circ f$ (n times). Express $f^{(n)\prime}(0)$ in terms of $f'(0)$ if $f(0) = 0$.

5. Show that the inverse function rule can be derived from the chain rule if you assume that the inverse function is differentiable (use the relation $f^{-1}(f(x)) = x$).

6. Let $f(x) = x^3 - 3x + 7$.
   (a) Find an interval containing zero on which f is invertible. Denote the inverse by g.
   (b) What is the domain of g?
   (c) Calculate $g'(7)$.
7. Let $f(x) = (ax + b)/(cx + d)$, and let $g(x) = (rx + s)/(tx + u)$.
   (a) Show that $f \circ g$ and $g \circ f$ are both of the form $(kx + l)/(mx + n)$ for some k, l, m, and n.
   (b) Under what conditions on a, b, c, d, r, s, t, u does $f \circ g = g \circ f$?
8. If f is a given differentiable function and $g(x) = f(\sqrt{x})$, what is $g'(x)$?
9. If f is differentiable and has an inverse, and $g(x) = f^{-1}(\sqrt{x})$, what is $g'(x)$?
10. Find a formula for the second derivative of $f \circ g$ in terms of the first and second derivatives of f and g.
11. Find a formula for the second derivative of $g(y)$ if $g(y)$ is the inverse function of $f(x)$.
12. Derive a formula for differentiating the composition $f_1 \circ f_2 \circ \dots \circ f_n$ of n functions.