10 The Exponential and Logarithm Functions

Some texts define e^x to be the inverse of the function $\ln x = \int_1^x 1/t\,dt$. This approach enables one to give a quick definition of e^x and to overcome a number of technical difficulties, but it is an unnatural way to define exponentiation. Here we give a complete account of how to define $\exp_b(x) = b^x$ as a continuation of rational exponentiation. We prove that $\exp_b$ is differentiable and show how to introduce the number e.

Powers of a Number

If n is a positive integer and b is a real number, the power b^n is defined as the product of b with itself n times:

$$b^n = b \cdot b \cdot \ldots \cdot b \ (n \text{ times})$$

If b is unequal to 0, so is b^n, and we define

$$b^{-n} = \frac{1}{b^n} = \frac{1}{b} \cdots \frac{1}{b} \ (n \text{ times}).$$

We also set

$$b^0 = 1$$

If b is positive, we define $b^{1/2} = \sqrt{b}$, $b^{1/3} = \sqrt[3]{b}$, etc., since we know how to take roots of numbers. Recall that $\sqrt[n]{b}$ is the unique positive number such that $(\sqrt[n]{b})^n = b$; i.e., $\sqrt[n]{y}$ is the inverse function of x^n. Formally, for n a positive integer, we define

$$b^{1/n} = \sqrt[n]{b} \quad \text{(the positive } n\text{th root of } b)$$

and we define

$$b^{-1/n} = \frac{1}{b^{1/n}}$$

Worked Example 1 Express $9^{-1/2}$ and $625^{-1/4}$ as fractions.

Solution $9^{-1/2} = 1/9^{1/2} = 1/\sqrt{9} = \frac{1}{3}$ and $625^{-1/4} = 1/\sqrt[4]{625} = \frac{1}{5}$.

Worked Example 2 Show that, if we assume the rule $b^{x+y} = b^x b^y$, we are *forced* to define $b^0 = 1$ and $b^{-x} = 1/b^x$.

Solution If we set $x = 1$ and $y = 0$, we get $b^{1+0} = b^1 \cdot b^0$, i.e., $b = b \cdot b^0$ so $b^0 = 1$. Next, if we set $y = -x$, we get $b^{x-x} = b^x b^{-x}$, i.e., $1 = b^0 = b^x b^{-x}$, so $b^{-x} = 1/b^x$. (Notice that this is an argument for *defining* b^0, $b^{-1/n}$, and b^{-n} the way we did. It does not prove it. Once powers are defined, and only then, can we claim that rules like $b^{x+y} = b^x b^y$ are true.)

Finally, if r is a rational number, we define b^r by expressing r as a quotient m/n of positive integers and defining

$$b^r = (b^m)^{1/n}$$

We leave it to the reader (Exercise 8) to verify that the result is independent of the way in which r is expressed as a quotient of integers. Note that $b^{m/n}$ is always positive, even if m or n is negative.

Thus the laws of exponents,

$$b^n b^m = b^{n+m} \quad \text{and} \quad b^n/b^m = b^{n-m} \tag{i}$$

$$(b^n)^m = b^{nm} \tag{ii}$$

$$(bc)^n = b^n c^n \tag{iii}$$

which are easily seen for integer powers from the definition of power, may now be extended to rational powers.

Worked Example 3 Prove (i) for rational exponents, namely,

$$b^{m_1/n_1}\, b^{m_2/n_2} = b^{(m_1/n_1)+(m_2/n_2)} \tag{i$'$}$$

Solution From (iii) we get

$$(b^{m_1/n_1}\, b^{m_2/n_2})^{n_1 n_2} = (b^{m_1/n_1})^{n_1 n_2}(b^{m_2/n_2})^{n_1 n_2}$$

By (ii) this equals

$$((b^{m_1/n_1})^{n_1})^{n_2}((b^{m_2/n_2})^{n_2})^{n_1}$$

By definition of $b^{m/n}$, we have $(b^{m/n})^n = b^m$, so the preceding expression is

$$(b^{m_1})^{n_2}(b^{m_2})^{n_1} = b^{m_1 n_2}\, b^{m_2 n_1}$$

again by (ii), which equals

$$b^{m_1 n_2 + m_2 n_1}$$

by (i).

Hence

$$(b^{m_1/n_1}\, b^{m_2/n_2})^{n_1 n_2} = b^{m_1 n_2 + m_2 n_1}$$

so

$$b^{m_1/n_1}\, b^{m_2/n_2} = (b^{(m_1 n_2 + m_2 n_1)})^{1/n_1 n_2}$$

By the definition $b^{m/n} = (b^m)^{1/n}$, this equals

$$b^{(m_1 n_2 + m_2 n_1)/n_1 n_2} = b^{(m_1/n_1) + (m_2/n_2)}$$

as required.

Similarly, we can prove (ii) and (iii) for rational exponents:

$$(b^{m_1/n_1})^{m_2/n_2} = b^{m_1 m_2/n_1 n_2} \quad \textbf{(ii')}$$
$$(bc)^{m/n} = b^{m/n} c^{m/n} \quad \textbf{(iii')}$$

Worked Example 4 Simplify $(x^{2/3}(x^{-3/2}))^{8/3}$.

Solution $(x^{2/3}x^{-3/2})^{8/3} = (x^{(2/3)-(3/2)})^{8/3} = (x^{-5/6})^{8/3} = x^{-20/9} = 1/\sqrt[9]{x^{20}}$.

Worked Example 5 If $b > 1$ and p and q are rational numbers with $p < q$, prove that $b^p < b^q$.

Solution By the laws of exponents, $b^q/b^p = b^{q-p}$. Let $z = q - p > 0$. We shall show that $b^z > 1$, so $b^q/b^p > 1$ and thus $b^q > b^p$.

Suppose that $z = m/n$. Then $b^z = (b^m)^{1/n}$. However, $b^m = b \cdot b \cdot \ldots \cdot b$ (m times) > 1 since $b > 1$, and $(b^m)^{1/n} > 1$ since $b^m > 1$. (The nth root $c^{1/n}$ of a number $c > 1$ is also greater than 1, since, if $c^{1/n} \leqslant 1$, then $(c^{1/n})^n = c \leqslant 1$ also.) Thus $b^z > 1$ if $z > 0$, and the solution is complete.

As a consequence, we can say that if $b \geqslant 1$ and $p \leqslant q$, then $b^p \leqslant b^q$.

Solved Exercises*

1. Find $8^{-3/2}$ and $8^{1/2}$.

*Solutions appear in the Appendix.

2. Find $9^{3/2}$.
3. Simplify $(x^{2/3})^{5/2}/x^{1/4}$.
4. Verify **(ii)** if either p or q is zero.

Exercises

1. Simplify by writing with rational exponents:

   (a) $\left[\dfrac{\sqrt[4]{ab^3}}{\sqrt{b}}\right]^6$ (b) $\sqrt[3]{\dfrac{\sqrt{a^3b^9}}{\sqrt[4]{a^6b^6}}}$

2. Factor (i.e., write in the form $(x^a \pm y^b)(x^c \pm y^d)$, a, b, c, d rational numbers):

   (a) $x - \sqrt{xy} - 2y$ (b) $x - y$
   (c) $\sqrt[3]{xy^2} + \sqrt[3]{yx^2} + x + y$ (d) $x - 2\sqrt{x} - 8$
   (e) $x + 2\sqrt{3x} + 3$

3. Solve for x:

   (a) $10^x = 0.001$ (b) $5^x = 1$
   (c) $2^x = 0$ (d) $x - 2\sqrt{x} - 3 = 0$ (factor)

4. Do the following define the same function on (a) $(-1, 1)$, (b) $(0, 3)$?

   $f_1(x) = x^{1/2}$
   $f_2(x) = \sqrt[4]{x^2}$
   $f_3(x) = (\sqrt[4]{x})^2$ (which, if any, are the same?)

5. Based on the laws of exponents which we want to hold true, what would be your choice for the value of 0^0? Discuss.

6. Using rational exponents and the laws of exponents, verify the following root formulas.

   (a) $\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}$ (b) $\sqrt[ac]{x^{ab}} = \sqrt[c]{x^b}$

7. Find all real numbers x which satisfy the following inequalities.

   (a) $x^{1/3} > x^{1/2}$ (b) $x^{1/2} > x^{1/3}$
   (c) $x^{1/p} > x^{1/q}$, p, q positive odd integers, $p > q$
   (d) $x^{1/q} > x^{1/p}$, p, q positive odd integers, $p > q$

8. Suppose that $b > 0$ and that $p = m/n = m'/n'$. Show, using the definition of rational powers, that $b^{m/n} = b^{m'/n'}$; i.e., b^p is unambiguously defined.

The Function $f(x) = b^x$

Having defined $f(x) = b^x$ if x is rational, we wish to extend the definition to allow x to range through all real numbers. If we take, for example, $b = 2$ and compute some values, we get:

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2^x	0.25	0.354...	0.5	0.707...	1	1.414...	2	2.828...	4

These values may be plotted to get an impression of the graph (Fig. 10-1). It seems natural to conjecture that the graph can be filled in with a smooth curve, i.e., that b^x makes sense for all x.

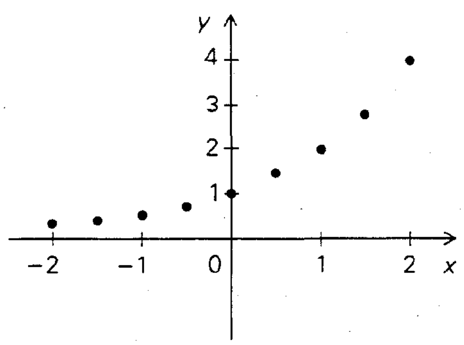

Fig. 10-1 The plot of some points $(x, 2^x)$ for rational x.

To calculate a number like $2^{\sqrt{3}}$, we should be able to take a decimal approximation to $\sqrt{3} \approx 1.732050808$ ..., say, 1.7320, calculate the rational power $2^{1.7320} = 2^{17320/10000}$, and hope to get an approximation to $2^{\sqrt{3}}$. Experimentally, this leads to reasonable data. On a calculator, one finds the following:

x	2^x
1	2
1.7	3.24900958
1.73	3.31727818
1.732	3.32188010
1.73205	3.32199523
1.7320508	3.32199707
1.732050808	3.32199708

The values of 2^x as x gets closer to $\sqrt{3}$ seem to be converging to some definite number. By doing more and more calculations, we could approximate this number to as high a degree of accuracy as we wished. We thus have a method for generating the decimal expansion of a number which could be called $2^{\sqrt{3}}$. To define $2^{\sqrt{3}}$ and other irrational powers, we shall use the transition idea.

Let b be positive and let x be irrational. Let A be the set of all real numbers α which are less than or equal to b^p, where p is some rational number and $p < x$. Similarly, let B be the set of numbers $\beta \geqslant b^q$ where q is some rational number and $q > x$ (Fig. 10-2).

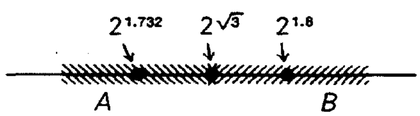

Fig. 10-2 Powers of 2 with rational exponents less than $\sqrt{3}$ go into set A (along with all numbers less than such powers) and all powers of 2 with rational exponents larger than $\sqrt{3}$ go into set B (along with all numbers larger than such powers).

Theorem 1 There is exactly one transition point from A to B if $b > 1$ (and from B to A if $0 < b < 1$). This transition point is called b^x (if $b = 1$, we define $1^x = 1$ for all x).

The function b^x so obtained is a continuous function of x.

The proof is given in the next two sections. (We shall assume it for now.) There we shall also show that the laws of exponents for rational numbers remain true for arbitrary real exponents. A specific case follows.

Worked Example 6 Simplify $(\sqrt{(3^\pi)})(3^{-\pi/4})$.

Solution $\sqrt{3^\pi}\, 3^{-\pi/4} = (3^\pi)^{1/2} 3^{-\pi/4} = 3^{(\pi/2)-(\pi/4)} = 3^{\pi/4}$.

Sometimes the notation $\exp_b x$ is used for b^x, exp standing for "exponential." One reason for this is typographical: an expression like

$$\exp_b \left(\frac{x^2}{2} + 3x\right)$$

is easier on the eyes and on the printer than $b^{(x^2/2)+3x}$. Another reason is mathematical: when we write $\exp_b x$, we indicate that we are thinking of b^x *as a function of* x.

We saw in Worked Example 5 that, for $b > 1$ and p and q rational with $p < q$ we had $b^p < b^q$. We can prove the same thing for real exponents: if $x < y$, we can choose rational numbers, p and q, such that $x < p < q < y$. By the definition of b^x and b^y as transition points, we must have $b^x \leqslant b^p$ and $b^q \leqslant b^y$, so $b^x \leqslant b^p < b^q \leqslant b^y$, and thus $b^x < b^y$.

In functional notation, if $b > 1$, we have $\exp_b x < \exp_b y$ whenever $x < y$; in the language of Chapter 5, $\exp_b$ is an increasing function. Similarly, if $0 < b < 1$, $\exp_b$ is a decreasing function.

It follows from Theorem 1 of Chapter 8 that for $b > 1$, b^x has a unique inverse function with domain $(0, \infty)$ and range $(-\infty, \infty)$. This function is denoted $\log_b$. Thus $x = \log_b y$ is the number such that $b^x = y$.

Worked Example 7 Find $\log_3 9$, $\log_{10}(10^a)$, and $\log_9 3$.

Solution Let $x = \log_3 9$. Then $3^x = 9$. Since $3^2 = 9$, x must be 2. Similarly, $\log_{10} 10^a$ is a and $\log_9 3 = \frac{1}{2}$ since $9^{1/2} = 3$.

The graph of $\log_b x$ for $b > 1$ is sketched in Fig. 10-3 and is obtained by flipping over the graph of $\exp_b x$ along the diagonal $y = x$. As usual with inverse functions, the label y in $\log_b y$ is only temporary to stress the fact that $\log_b y$ is the inverse of $y = \exp_b x$. From now on we shall usually use the variable name x and write $\log_b x$.

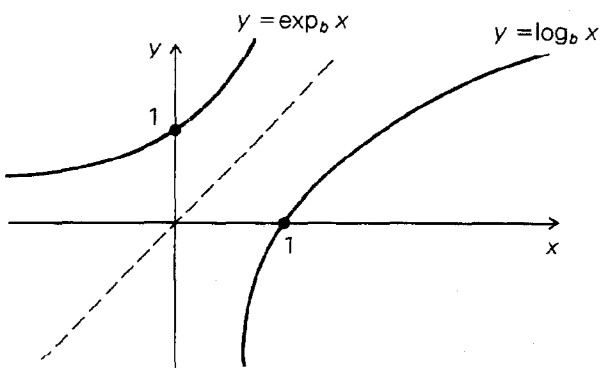

Fig. 10-3 The graphs of $y = \exp_b x$ and $y = \log_b x$ compared.

Notice that for $b > 1$, $\log_b x$ is increasing. If $b < 1$, $\exp_b x$ is decreasing and so is $\log_b x$. However, while $\exp_b x$ is always positive, $\log_b x$ can be either positive or negative.

From the laws of exponents we can read off corresponding laws for $\log_b x$:

$$\log_b (xy) = \log_b x + \log_b y \text{ and } \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y \tag{i}$$

$$\log_b (x^y) = y \log_b x \tag{ii}$$

$$\log_b x = \log_b (c) \log_c (x) \tag{iii}$$

For instance, to prove (i), we remember that $\log_b x$ is the number such that $\exp_b (\log_b x) = x$. So we must check that

$$\exp_b (\log_b x + \log_b y) = \exp_b (\log_b xy).$$

But the left-hand side is $\exp_b (\log_b x) \exp_b (\log_b y) = xy$ as is the right-hand side. The other laws are proved in the same way.

Solved Exercises

5. How is the graph of $\exp_{1/b} x$ related to that of $\exp_b x$?
6. Simplify: $(2^{\sqrt{3}} + 2^{-\sqrt{3}})(2^{\sqrt{3}} - 2^{-\sqrt{3}})$.
7. Match the graphs and functions in Fig. 10-4.

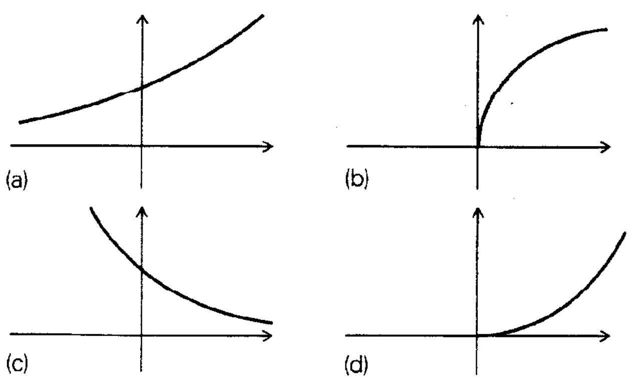

Fig. 10-4 Match the graphs and functions:

(a) $y = x^{\sqrt{3}}$ (b) $y = x^{1/\sqrt{3}}$
(c) $y = (\sqrt{3})^x$ (d) $y = (1/\sqrt{3})^x$

8. Find $\log_2 4$, $\log_3 81$ and $\log_{10} 0.01$.
9. (a) Simplify $\log_b (b^{2x}/2b)$
   (b) Solve for x: $\log_2 x = \log_2 5 + 3 \log_2 3$.

Exercises

9. Simplify: $\dfrac{(\sqrt{3})^{\pi} - (\sqrt{2})^{\sqrt{5}}}{\sqrt[4]{3^{\pi}} + 2^{\sqrt{5}/4}}$

10. Give the domains and ranges of the following functions and graph them:

    (a) $y = 2^{(x^2)}$ (b) $y = 2^{\sqrt{x}}$ (c) $y = 2^{1/x}$

11. Graph $y = 3^{x+2}$ by "shifting" the graph of $y = 3^x$ two units to the left. Graph $y = 9(3^x)$ by "stretching" the graph of $y = 3^x$ by a factor of 9 in the y-axis direction. Compare the two results. In general, how does shifting the graph $y = 3^x$ by k units to the left compare with stretching the graph by a factor of 3^k in the y-axis direction?

12. Consider $f(x) = (-3)^x$. For which fractions x is $f(x)$ defined? Not defined? How might this affect your ability to define $f(\pi)$?

13. Graph the following functions on one set of axes.

    (a) $f(x) = 2^x$ (b) $g(x) = x^2 + 1$ (c) $h(x) = x + 1$

    Can you make an estimate of $f'(1)$?

14. Solve for x:

    (a) $\log_x 5 = 0$ (b) $\log_2(x^2) = 4$ (c) $2 \log_3 x + \log_3 4 = 2$

15. Use the definition of $\log_b x$ to prove:

    (a) $\log_b (x^y) = y \log_b x$ (b) $\log_b x = \log_b (c) \log_c (x)$

Convex Functions*

We shall use the following notion of convexity to prove Theorem 1.

> ***Definition*** Let $f(x)$ be a function defined for every rational [real] x. We call f *convex* provided that for every pair of rational [real] numbers x_1 and x_2 with $x_1 \leqslant x_2$, and rational [real] λ with $0 \leqslant \lambda \leqslant 1$ we have

*See "To e Via Convexity" by H. Samelson, *Am. Math. Monthly*, November 1974, p. 1012. Some valuable remarks were also given us by Peter Renz.

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda)f(x_2)$$

If $\leqslant$ can be replaced by $<$ throughout, we say that f is *strictly convex.*

Notice that $\lambda x_1 + (1-\lambda)x_2$ lies between x_1 and x_2; for example, if $\lambda = \frac{1}{2}$, $\lambda x_1 + (1-\lambda)x_2 = \frac{1}{2}(x_1 + x_2)$ is the midpoint. Thus convexity says that at any point z between x_1 and x_2, $(z, f(z))$ lies beneath the chord joining $(x_1, f(x_1))$ to $(x_2, f(x_2))$. (See Fig. 10-5.) To see this, notice that the equation of the chord is

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

Setting $x = z = \lambda x_1 + (1-\lambda)x_2$, we get

$$\begin{aligned} y &= f(x_1) + \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(\lambda x_1 + (1-\lambda)x_2 - x_1) \\ &= f(x_1) + (f(x_2) - f(x_1))(1-\lambda) \\ &= \lambda f(x_1) + (1-\lambda)f(x_2) \end{aligned}$$

So the condition in the definition says exactly that $f(z) \leqslant \lambda f(x_1) + (1-\lambda)f(x_2)$, the y value of the point on the chord above z.

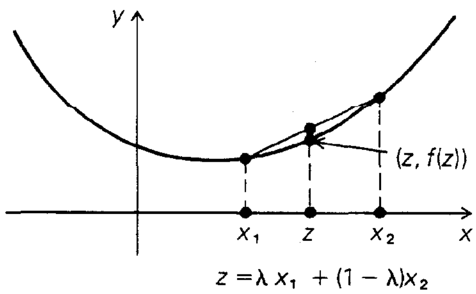

Fig. 10-5 For a convex function, the chord lies above the graph.

Theorem 2 *If $b > 1$, $f(x) = b^x$ defined for x rational, is (strictly) convex.*

Proof First of all, we prove that for $x_1 < x_2$,

$$f\left(\frac{x_1 + x_2}{2}\right) < \tfrac{1}{2}(f(x_1) + f(x_2))$$

i.e.,

$$b^{(x_1+x_2)/2} < \tfrac{1}{2}(b^{x_1} + b^{x_2})$$

Indeed, this is the same as

$$b^{(x_1+x_2)/2} - b^{x_1} < b^{x_2} - b^{(x_1+x_2)/2}$$

i.e.,

$$b^{x_1}(b^{(x_2-x_1)/2} - 1) < (b^{(x_2-x_1)/2} - 1)b^{(x_1+x_2)/2}$$

Since $x_1 < (x_1 + x_2)/2$, this is indeed true, as $b^{x_1} < b^{(x_1+x_2)/2}$ (see Worked Example 5).

Having taken care of $\lambda = \frac{1}{2}$, we next assume $0 < \lambda < \frac{1}{2}$. Proceeding as above,

$$b^{\lambda x_1+(1-\lambda)x_2} < \lambda b^{x_1} + (1-\lambda)b^{x_2}$$

is the same as

$$\lambda(b^{\lambda x_1+(1-\lambda)x_2} - b^{x_1}) < (1-\lambda)(b^{x_2} - b^{\lambda x_1+(1-\lambda)x_2})$$

i.e.,

$$\lambda b^{x_1}(b^{(1-\lambda)(x_2-x_1)} - 1) < (1-\lambda)(b^{(1-\lambda)(x_2-x_1)} - 1)b^{\lambda x_2+(1-\lambda)x_1}$$

But if $0 < \lambda < \frac{1}{2}$, then $\lambda < (1-\lambda)$, and since $x_1 < \lambda x_2 + (1-\lambda)x_1$, $b^{x_1} < b^{\lambda x_2+(1-\lambda)x}$. Hence the inequality is true. If we replace λ by $(1-\lambda)$ everywhere in this argument we get the desired inequality for $\frac{1}{2} < \lambda < 1$.

One can prove that b^x is convex for $b < 1$ in exactly the same manner.

Note The inequality obtained in Solved Exercise 11 is important and will be used in what follows.

Solved Exercises

10. (a) Prove that $y = x^2$ is strictly convex.

    (b) Find a convex function that is not strictly convex.

11. Suppose f is convex and $x_1 < x_2 < x_3$. Show that

$$f(x_1) \geqslant f(x_2) + \{[f(x_3) - f(x_2)]/(x_3 - x_2)\}(x_1 - x_2)$$

Sketch. What if f is strictly convex? Make up a similar inequality of the form $f(x_3) \geqslant$ something.

Exercises

16. If $f(x)$ is twice differentiable on $(-\infty, \infty)$ with $f''(x)$ continuous and $f''(x) > 0$, prove that f is convex. [*Hint*: Consider $g(x) = \lambda f(x) + (1 - \lambda)f(x_1) - f(\lambda x + (1 - \lambda)x_1)$ and show that g is increasing, and $g(x_1) = 0$.]
17. Show that $f(x) = |x|$ is convex.

Proof of Theorem 1

Let us suppose that $b > 1$ and that x is a given irrational number (the case $b < 1$ is dealt with similarly). Let A be the set of α such that $\alpha \leqslant b^p$, where p is rational and $p < x$, and let B be the set of $\beta \geqslant b^q$, where q is rational and $q > x$.

Lemma 1 *A and B are convex and hence intervals; $A = (-\infty, \alpha_0)$ or $(-\infty, \alpha_0]$ and $B = (\beta_0, \infty)$ or $[\beta_0, \infty)$ for some $\alpha_0 \leqslant \beta_0$.*

Proof Suppose y_1 and y_2 belong to A, and $y_1 < y < y_2$. Thus $y_2 \leqslant b^p$ for some rational $p < x$. Hence $y \leqslant b^p$ for the same p, so y belongs to A. This shows A is convex and, if y_2 belongs to A and $y < y_2$, then y belongs to A. Hence $A = (-\infty, \alpha_0)$ or $(-\infty, \alpha_0]$ for some number α_0.

Similarly, $B = (\beta_0, \infty)$ or $[\beta_0, \infty)$ for some number β_0.

Either $\alpha_0 \leqslant \beta_0$ or $\alpha_0 > \beta_0$. If $\alpha_0 > \beta_0$, then β_0 belongs to A so $\beta_0 \leqslant b^p$ for some $p < x$. This implies that b^p belongs to B, so $b^p \geqslant b^q$ for some $q > x$. But if $p < x < q$, $b^p < b^q$. Thus $\alpha_0 > \beta_0$ is impossible, so it must be that $\alpha_0 \leqslant \beta_0$, as required.

The next step uses convexity.

Lemma 2 *The numbers α_0 and β_0 given in Lemma 1 are equal.*

Proof Suppose $\alpha_0 < \beta_0$, the only possibility other than $\alpha_0 = \beta_0$ (see Lemma 1). Pick p and q rational with $p < x$ and $q > x$. Then b^p belongs to A, so $b^p \leqslant \alpha_0$ and similarly $b^q \geqslant \beta_0$. Picking a smaller p and larger q will insure that $b^p < \alpha_0$ and $b^q > \beta_0$.

If we choose λ such that λ is rational and $0 < (b^q - \beta_0)/(b^q - b^p) < \lambda < (b^q - \alpha_0)/(b^q - b^p) < 1$, then we will have

$$\alpha_0 < \lambda b^p + (1 - \lambda)b^q < \beta_0$$

(Why is it possible to choose such a λ?)

Suppose that $\lambda p + (1 - \lambda)q > x$. By Solved Exercise 11 with $x_1 = p$, $x_2 = \lambda p + (1 - \lambda)q$, and $x_3 = q$, we get

$$b^p \geqslant b^{\lambda p+(1-\lambda)q} + \frac{(b^q - b^{\lambda p+(1-\lambda)q}}{\lambda(q - p)}(1 - \lambda)(p - q)$$

i.e.,

$$\lambda b^p + (1 - \lambda)b^q \geqslant b^{\lambda p+(1-\lambda)q} \geqslant \beta_0$$

which is impossible, since $\lambda b^p + (1 - \lambda)b^q < \beta_0$. Similarly, if $\lambda p + (1 - \lambda)q < x$, the inequality (also from Solved Exercise 11),

$$b^q \geqslant b^{\lambda p+(1-\lambda)q} + \frac{b^p - b^{\lambda p+(1-\lambda)q}}{\lambda(p - q)}(1 - \lambda)(q - p)$$

leads to a contradiction. Since λ, p, q are rational and x is irrational, we cannot have $\lambda p + (1 - \lambda)q = x$. Hence $\alpha_0 < \beta_0$ is impossible.

Lemmas 1 and 2 can be summarized as follows: The sets A and B are as shown in Fig. 10-6 and the endpoints may or may not belong to A or B. This means that $\alpha_0 = \beta_0$ is the transition point from A to B. Thus, b^x is defined. From the construction, note that if $p < x < q$ and p, q are rational, then $b^p < b^x < b^q$.

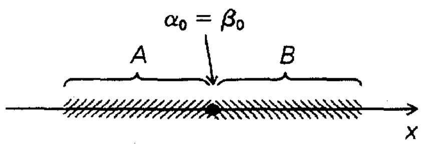

Fig. 10-6 The configuration of A, B, α_0, and β_0.

Lemma 3 *The function b^x is increasing* $(b > 1)$.

Proof If $x_1 < x_2$ and x_1, x_2 are irrational, then pick a rational r with $x_1 < r < x_2$. Then $b^{x_1} < b^r < b^{x_2}$ (see the comment just before the statement of the lemma). If x_1, x_2 are rational, see Worked Example 5.

Lemma 4 *The laws of exponents hold for b^x.*

Proof We prove $b^x b^y = b^{x+y}$. The rest are similar.

Assume $b^x b^y > b^{x+y}$, and let $\epsilon = b^x b^y - b^{x+y}$. Pick a rational number $r > x + y$ such that $b^r - b^{x+y} < \epsilon$ (why is this possible?). Write $r = p_1 + p_2$ where $p_1 > x$ and $p_2 > y$. Since the laws of exponents are true for rationals, we get

$$b^r = b^{p_1} b^{p_2} > b^x b^y$$

Hence

$$b^x b^y < b^r < b^{x+y} + \epsilon = b^{x+y} + (b^x b^y - b^{x+y}) = b^x b^y,$$

which is a contradiction. Similarly, $b^x b^y < b^{x+y}$ is impossible, so we must have equality.

Lemma 5 *b^x is a (strictly) convex function (defined for every real x).*

Proof Since we know that b^x is increasing and that the laws of exponents hold, our proof given in Theorem 2 is valid for arbitrary x_1, x_2, and λ, rational or not.

It only remains to prove that b^x is continuous. The following might surprise you.

Theorem 3 *Any convex function $f(x)$ (defined for all real x) is continuous.*

Proof Fix a number x_0 and let $c > f(x_0)$. Refer to Fig. 10-7 and the definitions on pp. 54 and 31. Pick $x_1 < x_0 < x_2$.

By convexity,

$$f(\lambda x_2 + (1 - \lambda)x_0) \leqslant \lambda f(x_2) + (1 - \lambda)f(x_0)$$

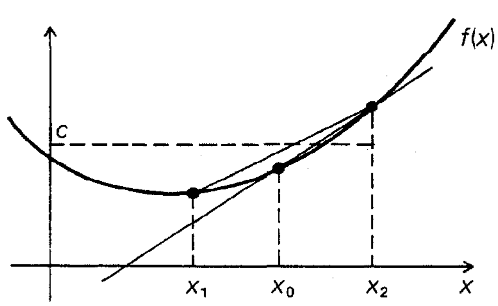

Fig. 10-7 The geometry needed for Theorem 3.

Choose ϵ_2 such that $0 < \epsilon_2 < 1$ and such that

$$\epsilon_2 f(x_2) + (1 - \epsilon_2) f(x_0) < c$$

i.e.,

$$\epsilon_2 (f(x_2) - f(x_0)) < c - f(x_0)$$

(if $f(x_2) - f(x_0) < 0$, any ϵ_2 will do; if $f(x_2) - f(x_0) > 0$, we need $\epsilon_2 < (c - f(x_0))/[f(x_2) - f(x_0)]$). Then if $0 \leqslant \lambda \leqslant \epsilon_2, f(\lambda x_2 + (1-\lambda)x_0) \leqslant \lambda f(x_2) + (1-\lambda) f(x_0) < c$. If $x_0 \leqslant x \leqslant x_0 + \epsilon_2(x_2 - x_0)$, we can write $x = \lambda x_2 + (1-\lambda)x_0$, where $\lambda = (x - x_0)/(x_2 - x_0) \leqslant \epsilon_2$. Thus

$$f(x) = f(\lambda x_2 + (1-\lambda)x_0) \leqslant \lambda f(x_2) + (1-\lambda) f(x_0) < c$$

whenever $x_0 \leqslant x \leqslant x_0 + \epsilon_2(x_2 - x_0)$. Similarly, by considering the line through $(x_1, f(x_1))$ and $(x_0, f(x_0))$ we can find ϵ_1 such that if $x_0 - \epsilon_1(x_0 - x_1) < x < x_0$, then $f(x) < c$. If $I = (x_0 - \epsilon_1(x_0 - x_1), x_0 + \epsilon_2(x_2 - x_0))$, then for any x in $I, f(x) < c$.

If $d < f(x_0)$, we can show that if $x_1 < x < x_0$ but x is sufficiently close to x, then $f(x) > d$ by using the inequality

$$f(x) \geqslant f(x_0) + \frac{f(x_2) - f(x_0)}{x_2 - x_0}(x - x_0)$$

and an argument like the one just given. The case $x > x_0$ is similar. Thus there is an open interval J about x_0 such that $f(x) > d$ if x is in J.

Thus, by the definition of continuous function, f is continuous at x_0.

Solved Exercises

12. Suppose that $f(x)$ is convex, $a < b$, and $f(a) < f(x)$ for every x in $(a, b]$. Prove that f is increasing on $[a, b]$.

13. A certain function $f(x)$ defined on $(-\infty, \infty)$ satisfies $f(xy) = (f(y))^x$ for all real numbers x and y. Show that $f(x) = b^x$ for some $b > 0$.

Exercises

18. Give the details of the part of the proof of Theorem 3 dealing with the case $d < f(x_0)$.

19. Prove that if f is strictly convex and $f(0) < 0$, then the equation $f(x) = 0$ has at least one real root.

20. Suppose that f satisfies $f(x + y) = f(x)f(y)$ for all real numbers x and y. Suppose that $f(x) \neq 0$ for some x. Prove the following.

    (a) $f(0) = 1$

    (b) $f(x) \neq 0$ for all x

    (c) $f(x) > 0$ for all x

    (d) $f(-x) = \dfrac{1}{f(x)}$

Differentiation of the Exponential Function

Now we turn our attention to the differentiability of b^x. Again, convexity will be an important tool.

Theorem 4 *If $b > 0$, then $f(x) = b^x$ is differentiable and*

$$f'(x) = f'(0)f(x), \quad \text{i.e.,} \quad \frac{d}{dx} b^x = f'(0) \cdot b^x$$

Proof From the equation $f(x) = f(x - x_0)f(x_0)$ and the chain rule, all we need to do is show that $f(x)$ is differentiable at $x = 0$.

Refer to Fig. 10-8. Let $x_2 > 0$ and consider the line through $(0, 1)$ and $(x_2, f(x_2))$, i.e.,

$$y = 1 + \frac{f(x_2) - 1}{x_2} \cdot x = l_2(x)$$

This line overtakes $f(x)$ at $x = 0$. Indeed, if $0 < x < x_2$, then $f(x) < l_2(x)$ since f is (strictly) convex. If $x < 0 < x_2$, then $f(x) > l_2(x)$ by Solved Exercise 11, with $x_1, 0, x_2$ replacing x_1, x_2, and x_3.

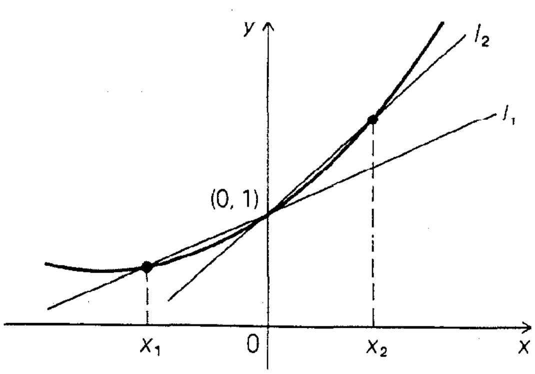

Fig. 10-8 The geometry needed for Theorem 4.

In exactly the same way, we see that for $x_1 < 0$, the line l_1 passing through $(x_1, f(x_1))$ and $(0, 1)$ is overtaken by f at $x = 0$.

Going back to the definition of derivative in terms of transitions (Theorem 4, p. 27), we let

$A =$ the set of slopes of lines l_1 which are overtaken by f at $x = 0$

and

$B =$ the set of slopes of lines l_2 which overtake f at $x = 0$

Let $A = (-\infty, \alpha)$ or $(-\infty, \alpha]$ and $B = (\beta, \infty)$ or $[\beta, \infty)$. We know that $\alpha \leqslant \beta$. We want to prove that $\alpha = \beta$.

Our remarks above on convexity imply that the slope of the line $y = 1 + \{[f(x_2) - 1]/x_2\}x$ belongs to B, i.e.,

$$\beta \leqslant \frac{f(x_2) - 1}{x_2}$$

Similarly,

$$\frac{f(x_1) - 1}{x_1} \leqslant \alpha$$

In particular, set $x_1 = -x_2 = -t$, where $t > 0$. Then

$$\begin{aligned}\beta - \alpha \leqslant \frac{f(x_2) - 1}{x_2} - \frac{f(x_1) - 1}{x_1} &= \frac{b^t - 1}{t} - \frac{b^{-t} - 1}{-t} \\ &= \frac{b^t + b^{-t} - 2}{t} = \frac{b^{-t}}{t}(b^{2t} - 2b^t + 1) \\ &= \frac{b^{-t}}{t}(b^t - 1)^2\end{aligned}$$

Now we may use the convexity inequality which tells us that $f(t) < l_2(t)$ if $0 < t < x_2$, i.e.,

$$b^t < 1 + \frac{b^{x_2} - 1}{x_2}t, \quad \text{i.e.,} \quad \frac{b^t - 1}{t} < \frac{b^{x_2} - 1}{x_2}$$

This gives

$$\beta - \alpha < b^{-t}(b^t - 1) \cdot \left(\frac{b^{x_2} - 1}{x_2}\right)$$

Suppose that $\beta - \alpha$ is positive. Then, letting $c = (\beta - \alpha)x_2/(b^{x_2} - 1)$, we have

$$b^{-t}(b^t - 1) > c$$

But $g(t) = b^{-t}(b^t - 1)$ is continuous, and $g(0) = 0$. Thus if t is near enough to zero we would have $b^{-t}(b^t - 1) < c$, a contradiction. Thus $\beta = \alpha$ and so $f'(0)$ exists.

We still need to find $f'(0)$. It would be nice to be able to adjust b so that $f'(0) = 1$, for then we would have simply $f'(x) = f(x)$. To be able to keep track of b, we revert to the $\exp_b(x)$ notation, so Theorem 4 reads as follows: $\exp_b'(x) = \exp_b'(0)\exp_b(x)$.

Let us start with the base 10 of common logarithms and try to find another base b for which $\exp_b'(0) = 1$. By definition of the logarithm,

$$b = 10^{\log_{10} b} \quad \text{(see p. 129)}$$

Therefore

$$b^x = (10^{\log_{10} b})^x = 10^{x \log_{10} b}$$

Hence

$$\exp_b(x) = \exp_{10}(x \log_{10} b)$$

Differentiate by using the chain rule:

$$\exp_b'(x) = \exp_{10}'(x \log_{10} b) \cdot \log_{10} b$$

Set $x = 0$:

$$\exp_b'(0) = \exp_{10}'(0) \cdot \log_{10} b$$

If we pick b so that

$$\exp_{10}'(0) \cdot \log_{10} b = 1 \tag{1}$$

then we will have $\exp_b'(x) = \exp_b(x)$, as desired. Solving (1) for b, we have

$$\log_{10} b = \frac{1}{\exp_{10}'(0)}$$

That is,

$$b = \exp_{10}\left[\frac{1}{\exp_{10}'(0)}\right]$$

We denote the number $\exp_{10}[1/\exp_{10}'(0)]$ by the letter e. Its numerical value is approximately 2.7182818285, and we have

$$\exp_e'(x) = \exp_e(x)$$

Although we started with the arbitrary choice of 10 as a base, it is easy to show (see Solved Exercise 15) that any initial choice of base leads to the same value for e. Since the base e is so special, we write $\exp(x)$ for $\exp_e(x) = e^x$.

Logarithms to the base e are called *natural logarithms*. We denote $\log_e x$ by $\ln x$. (The notation $\log x$ is generally used in calculus books for the common logarithm $\log_{10} x$.) Since $e^1 = e$, we have $\ln e = 1$.

Worked Example 8 Simplify $\ln(e^5) + \ln(e^{-3})$.

Solution By the laws of logarithms, $\ln(e^5) + \ln(e^{-3}) = \ln(e^5 \cdot e^{-3}) = \ln(e^2) = 2$.

We can now complete our differentiation formula for the general exponential function $\exp_b x$. Since $b = e^{\ln b}$, we have $b^x = e^{x \ln b}$. Using the chain rule, we find

$$\begin{aligned} \frac{d}{dx} b^x &= \frac{d}{dx} e^{x \ln b} \\ &= e^{x \ln b} \frac{d}{dx}(x \ln b) \\ &= e^{x \ln b} \ln b \\ &= b^x \ln b \end{aligned}$$

Thus the mysterious factor $\exp_b'(0)$ turns out to be just the natural logarithm of b.

Worked Example 9 Differentiate: (a) $f(x) = e^{3x}$; (b) $g(x) = 3^x$.

Solution

(a) Let $u = 3x$ so $e^{3x} = e^u$ and use the chain rule:

$$\begin{aligned} \frac{d}{dx} e^u &= \left(\frac{d}{du} e^u\right) \frac{du}{dx} \\ &= e^u \cdot 3 = 3e^{3x} \end{aligned}$$

(b) $\dfrac{d}{dx} 3^x = 3^x \ln 3$.

This expression cannot be simplified further; one can find the value $\ln 3 \approx 1.0986$ in a table or with a calculator.

Solved Exercises

14. Differentiate the following functions.

(a) e^{2x} (b) 2^x

(c) xe^{3x} (d) $\exp(x^2 + 2x)$

(e) x^2

15. Show that, for any base b, $\exp_b(1/\exp_b'(0)) = e$.

16. Differentiate:

(a) $e^{\sqrt{x}}$ (b) $e^{\sin x}$

(c) $2^{\sin x}$ (d) $(\sin x)^2$

17. Prove that for $t > 0$ and $b > 1$, we have $b^t - 1 < (b^t \log_e b)t$.

Exercises

21. Differentiate the following functions.

(a) e^{x^2+1} (b) $\sin(e^x)$ (c) $3^x - 2^{x-1}$

(d) $e^{\cos x}$ (e) $\tan(3^{2x})$ (f) $e^{1-x^2} + x^3$

22. Differentiate (assume f and g differentiable where necessary):

(a) $(x^3 + 2x - 1)e^{x^2 + \sin x}$ (b) $e^{2x} - \cos(x + e^{2x})$

(c) $(e^{3x^3+x})(1 - e^x)$ (d) $(e^{x+1} + 1)(e^{x-1} - 1)$

(e) $f(x) \cdot e^x + g(x)$ (f) $e^{f(x)+x^2}$

(g) $f(x) \cdot e^{g(x)}$ (h) $f(e^x + g(x))$

23. Show that $f(x) = e^x$ is an increasing function.

24. Find the critical points of $f(x) = x^2 e^{-x}$.

25. Find the critical points of $f(x) = \sin x e^x$, $-4\pi \leqslant x \leqslant 4\pi$.

26. Simplify the following expressions:

(a) $\ln(e^{x+1}) + \ln(e^2)$ (b) $\ln(e^{\sin x}) - \ln(e^{\cos x})$

The Derivative of the Logarithm

We can differentiate the logarithm function by using the inverse function rule of Chapter 8. If $y = \ln x$, then $x = e^y$ and

$$\frac{dy}{dx}=\frac{1}{dx/dy}=\frac{1}{e^y}=\frac{1}{x}$$

Hence

$$\frac{d}{dx}\ln x=\frac{1}{x}$$

For other bases, we use the same process:

$$\frac{d}{dx}\log_b x=\frac{1}{\dfrac{d}{dy}b^y}=\frac{1}{\ln b\cdot b^y}=\frac{1}{\ln b\cdot x}$$

That is,

$$\frac{d}{dx}\log_b x=\frac{1}{(\ln b)x}$$

The last formula may also be proved by using law 3 of logarithms:

$$\ln x=\log_e x=\log_b x\cdot\ln b$$

so

$$\frac{d}{dx}\log_b x=\frac{d}{dx}\left(\frac{1}{\ln b}\ln x\right)=\frac{1}{\ln b}\,\frac{d}{dx}\ln x=\frac{1}{(\ln b)\cdot x}$$

Worked Example 10 Differentiate: (a) $\ln(3x)$; (b) $xe^x\ln x$; (c) $8\log_3 8x$.

Solution

(a) By the chain rule, setting $u=3x$,

$$\frac{d}{dx}\ln 3x=\frac{d}{du}(\ln u)\cdot\frac{du}{dx}=\frac{1}{3x}\cdot 3=\frac{1}{x}$$

Alternatively, $\ln 3x=\ln 3+\ln x$, so the derivative with respect to x is $1/x$.

(b) By the product rule:

$$\frac{d}{dx}(xe^x\ln x)=x\frac{d}{dx}(e^x\ln x)+e^x\ln x=xe^x\ln x+e^x+e^x\ln x$$

(c) From the formula $(d/dx)\log_b x=1/(\ln b)x$ with $b=3$,

$$\frac{d}{dx}8\log_3 8x\ =\ 8\frac{d}{dx}\log_3 8x$$

$$= 8\left(\frac{d}{du}\log_3 u\right)\frac{du}{dx}\ (u = 8x)$$

$$= 8 \cdot \frac{1}{\ln 3 \cdot u} \cdot 8$$

$$= \frac{64}{(\ln 3)8x} = \frac{8}{(\ln 3)x}$$

In order to differentiate certain expressions it is sometimes convenient to begin by taking logarithms.

Worked Example 11 Differentiate the function $y = x^x$.

Solution We take natural logarithms,

$$\ln y = \ln(x^x) = x \ln x$$

Next we differentiate, remembering that y is a function of x:

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

Hence

$$\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x)$$

In general, $(d/dx)\ln f(x) = f'(x)/f(x)$ is called the *logarithmic derivative* of f. Other applications are given in the exercises which follow.

Solved Exercises

18. Differentiate:
    (a) $\ln 10x$ (b) $\ln u(x)$ (c) $\ln(\sin x)$
    (d) $(\sin x)\ln x$ (e) $(\ln x)/x$ (f) $\log_5 x$

19. (a) If n is any *real number*, prove that

    $$\frac{d}{dx}x^n = nx^{n-1} \quad \text{for } x > 0$$

    (b) Find $(d/dx)(x^\pi)$.

20. Use logarithmic differentiation to calculate dy/dx, if y is given by $y = (2x + 3)^{3/2}/\sqrt{x^2 + 1}$.

21. Differentiate $y = x^{(x^x)}$.

Exercises

27. Differentiate:

(a) $\ln(2x+1)$ (b) $\ln(x^2-3x)$ (c) $\ln(\tan x)$
(d) $(\ln x)^3$ (e) $(x^2-2x)\ln(2x+1)$ (f) $e^{x+\ln x}$
(g) $[\ln(\tan 3x)]/(1+\ln x^2)$

28. Use logarithmic differentiation to differentiate:

(a) $y = x^{3x}$ (b) $y = x^{\sin x}$
(c) $y = (\sin x)^{\cos x}$ (d) $y = (x^3+1)^{x^2-2}$
(e) $y = (x-2)^{2/3}(4x+3)^{8/7}$

Problems for Chapter 10

1. Simplify:
(a) $e^{4x}[\ln(e^{3x-1}) - \ln(e^{1-x})]$ (b) $e^{(x\ln 3 + \ln 2^x)}$

2. Differentiate:
(a) $e^{x\sin x}$ (b) x^e
(c) $14^{x^2-8\sin x}$ (d) x^{x^2}
(e) $\ln(x^{-5}+x)$ (f) $(\ln x)^{\exp x}$
(g) $\sin(x^4+1)\cdot\log_8(14x-\sin x)$

3. Sketch the graph of $y = xe^{-x}$; indicate on your graph the regions where y is increasing, decreasing, concave upward or downward.

4. Find the minimum of $y = x^x$ for x in $(0,\infty)$.

5. Suppose that f is continuous and that $f(x+y) = f(x)f(y)$ for all x and y. Show that $f(x) = b^x$ for some b. [*Hint*: Try showing that f is actually differentiable at 0.]

6. Let f be a twice differentiable convex function. Prove that $f''(x) \geqslant 0$.

7. Let f be an increasing continuous convex function. Let f^{-1} be the inverse function. Show that $-f^{-1}$ is convex. Apply this result to $\log_b x$.

8. Suppose that $f(x)$ is a function defined for all real x. If $x_1 < x_2 < x_3$ and $f(x_1) = f(x_2) = f(x_3)$, prove that f is not strictly convex. Give an example to show that f may be convex.

9. Suppose that $f(x)$ is a strictly convex differentiable function defined on $(-\infty,\infty)$. Show that the tangent line to the graph of $y = f(x)$ at (x_0,y_0) does not intersect the graph at any other point. Here, x_0 is any real number and $y_0 = f(x_0)$. What can we say if we only assume f to be convex?

10. Suppose that $f(x)$ is defined for all real x and that f is strictly convex on

$(-\infty, 0)$ and strictly convex on $(0, \infty)$. Prove that if f is convex on $(-\infty, \infty)$, then f is strictly convex on $(-\infty, \infty)$.

11. Prove that $e^x > 1 + x^2$, for $x \geqslant 1$. [*Hint*: Note that $e > 2$ and show that the difference between these two functions is increasing.]

12. We have seen that the exponential function $\exp(x)$ satisfies the following relations: $\exp(x) > 0$, $\exp(0) = 1$, and $\exp'(x) = \exp(x)$. Let $f(x)$ be a function such that

    $0 \leqslant f'(x) \leqslant f(x)$ for all $x \geqslant 0$

    Prove that $0 \leqslant f(x) \leqslant f(0)\exp(x)$ for all $x \geqslant 0$. (*Hint*: Consider $g(x) = f(x)/\exp(x)$.)

13. Let $f(x)$ be an increasing continuous function. Given x_0, let

    A = the set of $f(x)$ where $x < x_0$

    B = the set of $f(x)$ where $x > x_0$

    Show that $f(x_0)$ is the transition point from A to B.

14. Show that the sum $f(x) + g(x)$ of two convex functions $f(x)$ and $g(x)$ is convex. Show that if f is strictly convex, then so is $f + g$. Use this to show that $f(x) = ax^2 + bx + c$ is strictly convex if $a > 0$, where a, b, and c are constants.

15. Suppose $f(x)$ is defined and differentiable for all real x.

    (a) Does $f''(x) \geqslant 0$ for all x imply that $f(x)$ is convex?

    (b) Does $f''(x) > 0$ for all x imply that $f(x)$ is convex? Strictly convex?