11 The Integral

In this chapter we define the integral in terms of transitions; i.e., by the method of exhaustion. The reader is assumed to be familiar with the summation notation and its basic properties, as presented in most calculus texts.

Piecewise Constant Functions

In the theory of differentiation, the simplest functions were the linear functions $f(x) = ax + b$. We knew that the derivative of $ax + b$ should be a, and we defined the derivative for more general functions by comparison with the linear functions, using the notion of overtaking to make the comparisons.

For integration theory, the comparison functions are the *piecewise constant functions.* Roughly speaking, a function f on $[a, b]$ is piecewise constant if $[a, b]$ can be broken into a finite number of subintervals such that f is constant on each subinterval.

Definition A *partition* of the interval $[a, b]$ is a sequence of numbers $(t_0, t_1, \ldots, t_n)$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

We think of the numbers $t_0, t_1, \ldots, t_n$ as dividing $[a, b]$ into the subintervals $[t_0, t_1]$, $[t_1, t_2]$, ... , $[t_{n-1}, t_n]$. (See Fig. 11-1.) The number n of intervals can be as small as 1 or as large as we wish.

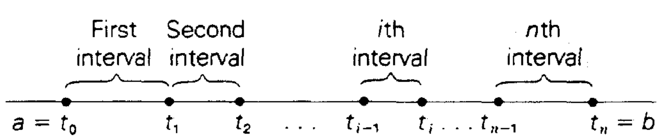

Fig. 11-1 A partition of $[a, b]$.

Definition A function f defined on an interval $[a, b]$ is called *piecewise constant* if there is a partition $(t_0, \ldots, t_n)$ of $[a, b]$ and real numbers $k_1, \ldots, k_n$ such that

$$f(t) = k_i \quad \text{for all } t \text{ in } (t_{i-1}, t_i)$$

The partition $(t_0, \ldots, t_n)$ is then said to be *adapted* to the piecewise constant function f.

Notice that we put no condition on the value of f at the points $t_0, t_1, \ldots, t_n$, but we require that f be constant on each of the *open* intervals between successive points of the partition. As we will see in Worked Example 1, more than one partition may be adapted to a given piecewise constant function. Piecewise constant functions are sometimes called *step functions* because their graphs often resemble staircases (see Fig. 11-2).

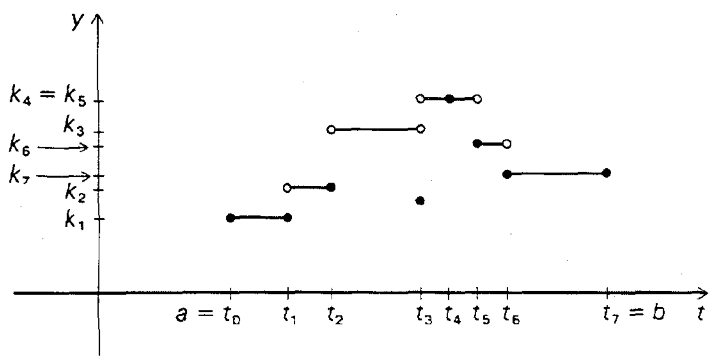

Fig. 11-2 A step function.

Worked Example 1 Draw a graph of the piecewise constant function f on $[0,1]$ defined by

$$f(t) = \begin{cases} -2 & \text{if } 0 \leqslant t < \frac{1}{3} \\ 3 & \text{if } \frac{1}{3} \leqslant t < \frac{1}{2} \\ 3 & \text{if } \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ 1 & \text{if } \frac{3}{4} < t \leqslant 1 \end{cases}$$

Give three different partitions which are adapted to f and one partition which is not adapted to f.

Solution The graph is shown in Fig. 11-3. As usual, an open circle indicates a point which is not on the graph. (The resemblance of this graph to a staircase is rather faint.)

One partition adapted to f is $(0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1)$. (That is, $t_0 = 0$, $t_1 = \frac{1}{3}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$, $t_4 = 1$. Here $k_1 = -2, k_2 = k_3 = 3, k_4 = 1$.) If we delete $\frac{1}{2}$, the

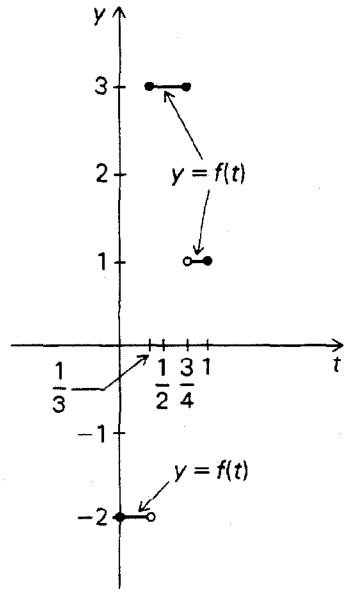

Fig. 11-3 Another step function.

partition $(0, \frac{1}{3}, \frac{3}{4}, 1)$ is still adapted to f because f is constant on $(\frac{1}{3}, \frac{3}{4})$. Finally, we can always add extra points to an adapted partition and it will still be adapted. For example, $(0, \frac{1}{8}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{8}{9}, 1)$ is an adapted partition.

A partition which is not adapted to f is $(0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1)$. Even though the intervals in this partition are very short, it is not adapted to f because f is not constant on $(0.7, 0.8)$. (Can you find another open interval in the partition on which f is not constant?)

Motivated by our physical example, we define the integral of a piecewise constant function as a sum.

Definition Let f be a piecewise constant function on $[a, b]$. Let $(t_0, \ldots, t_n)$ be a partition of $[a, b]$ which is adapted to f and let k_i be the value of f on (t_{i-1}, t_i). The sum

$$k_1(t_1 - t_0) + k_2(t_2 - t_1) + \cdots + k_n(t_n - t_{n-1})$$

is called the *integral* of f on $[a, b]$ and is denoted by $\int_a^b f(t)\, dt$; that is,

$$\int_a^b f(t)\, dt = \sum_{i=1}^{n} k_i \Delta t_i$$

where $\Delta t_i = t_i - t_{i-1}$.

We shall verify shortly that the definition is independent of the choice of adapted partition.

If all the k_i's are nonnegative, the integral $\int_a^b f(t)\,dt$ is precisely the area "under" the graph of f—that is, the area of the set of points (x, y) such that $a \leqslant x \leqslant b$ and $0 \leqslant y \leqslant f(x)$. The region under the graph in Fig. 11-2 is shaded in Fig. 11-4.

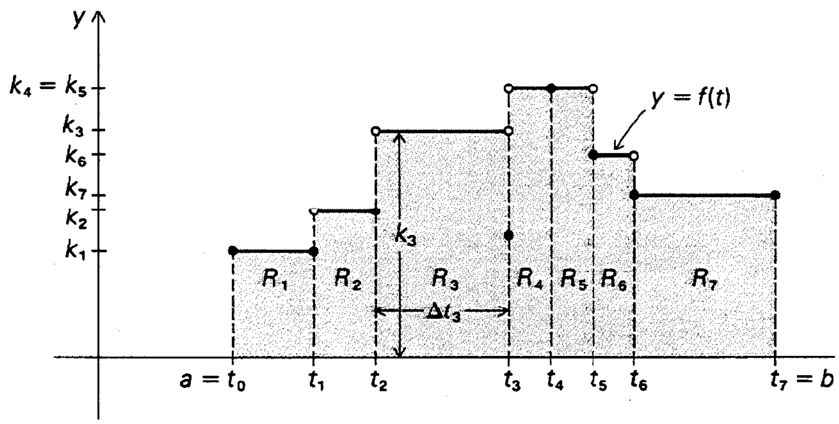

Fig. 11-4 The shaded area is the sum of the areas of the rectangles R_i.

The notation $\int_a^b f(t)\,dt$ for the integral is due to Leibniz. The symbol $\int$, called an *integral sign*, is an elongated S which replaces the Greek Σ of ordinary summation. Similarly, the dt replaces the Δt_i's in the summation formula. The function $f(t)$ which is being integrated is called the *integrand*. The endpoints a and b are also called *limits of integration.*

Worked Example 2 Compute $\int_0^1 f(t)\,dt$ for the function in Worked Example 1, first using the partition $(0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1)$ and then using the partition $(0, \frac{1}{3}, \frac{3}{4}, 1)$.

Solution With the first partition, we have

$$\begin{aligned} k_1 &= -2 & \Delta t_1 &= t_1 - t_0 = \tfrac{1}{3} - 0 = \tfrac{1}{3} \\ k_2 &= 3 & \Delta t_2 &= t_2 - t_1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6} \\ k_3 &= 3 & \Delta t_3 &= t_3 - t_2 = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4} \\ k_4 &= 1 & \Delta t_4 &= t_4 - t_3 = 1 - \tfrac{3}{4} = \tfrac{1}{4} \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^1 f(t)\,dt = \sum_{i=1}^{4} k_i \Delta t_i &= (-2)(\tfrac{1}{3}) + (3)(\tfrac{1}{6}) + (3)(\tfrac{1}{4}) + (1)(\tfrac{1}{4}) \\ &= -\tfrac{2}{3} + \tfrac{1}{2} + \tfrac{3}{4} + \tfrac{1}{4} = \tfrac{5}{6} \end{aligned}$$

Using the second partition, we have

$$\begin{aligned} k_1 &= -2 \quad \Delta t_1 = t_1 - t_0 = \tfrac{1}{3} - 0 = \tfrac{1}{3} \\ k_2 &= 3 \quad \Delta t_2 = t_2 - t_1 = \tfrac{3}{4} - \tfrac{1}{3} = \tfrac{5}{12} \\ k_3 &= 1 \quad \Delta t_3 = t_3 - t_2 = 1 - \tfrac{3}{4} = \tfrac{1}{4} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 f(t)\,dt = \sum_{i=1}^{3} k_i \Delta t_i &= (-2)(\tfrac{1}{3}) + (3)(\tfrac{5}{12}) + (1)(\tfrac{1}{4}) \\ &= -\tfrac{2}{3} + \tfrac{5}{4} + \tfrac{1}{4} = \tfrac{5}{6} \end{aligned}$$

which is the same answer we obtained from the first partition.

Theorem 1 *Let f be a piecewise constant function on $[a, b]$. Suppose that $(t_0, t_1, \ldots, t_n)$ and $(s_0, s_1, \ldots, s_m)$ are adapted partitions, with $f(t) = k_i$ on (t_{i-1}, t_i) and $f(t) = j_i$ on (s_{i-1}, s_i). Then*

$$\sum_{i=1}^{n} k_i \Delta t_i = \sum_{i=1}^{m} j_i \Delta s_i$$

Proof The idea is to reduce the problem to the simplest case, where the second partition is obtained from the first by the addition of a single point. Let us begin by proving the proposition for this case. Assume, then, that $m = n + 1$, and that $(s_0, s_1, \ldots, s_m) = (t_0, t_1, \ldots, t_l, t_*, t_{l+1}, \ldots, t_n)$; i.e., the s-partition is obtained from the t-partition by inserting an extra point t^* between t_l and t_{l+1}. The relation between s's, t's, j's, and k's is illustrated in Fig. 11-5. The sum obtained from the s-partition is

$$\begin{aligned} \sum_{i=1}^{m} j_i \Delta s_i = j_1(s_1 - s_0) + \cdots + j_{l+1}(s_{l+1} - s_l) \\ + j_{l+2}(s_{l+2} - s_{l+1}) + \cdots + j_{n+1}(s_{n+1} - s_n) \end{aligned}$$

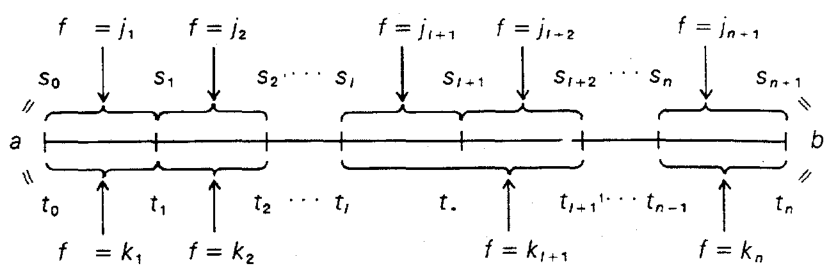

Fig. 11-5 The s-partition is finer than the t-partition.

Substituting the appropriate k's and t's for the j's and s's converts the sum to

$$k_1(t_1 - t_0) + \cdots + k_{l+1}(t_* - t_l) + k_{l+1}(t_{l+1} - t_*) + \cdots + k_n(t_n - t_{n-1})$$

We can combine the two middle terms:

$$\begin{aligned} k_{l+1}(t_* - t_l) + k_{l+1}(t_{l+1} - t_*) &= k_{l+1}(t_* - t_l + t_{l+1} - t_*) \\ &= k_{l+1}(t_{l+1} - t_l) \end{aligned}$$

The sum is now

$$k_1(t_1 - t_0) + \cdots + k_{l+1}(t_{l+1} - t_l) + \cdots + k_n(t_n - t_{n-1}) = \sum_{i=1}^{n} k_i \, \Delta t_i$$

which is the sum obtained from the t-partition. (For a numerical illustration, see Worked Example 2.) This completes the proof for the special case.

To handle the general case, we observe first that, given two partitions $(t_0, \ldots, t_n)$ and $(s_0, \ldots, s_m)$, we can find a partition $(u_0, \ldots, u_p)$ which contains both of them taking all the points $t_0, \ldots, t_n, s_0, \ldots, s_m$, eliminating duplications, and putting the points in the correct order. (See Solved Exercise 1, immediately following the end of the proof.) Adding points to an adapted partition produces another adapted partition, since if a function is constant on an interval, it is certainly constant on any subinterval. It follows that the u-partition is adapted to f if the s- and t-partitions are. Now we can get from the t-partition to the u-partition by adding points one at a time. By the special case above, we see that the sum is unchanged each time we add a point, so the sum obtained from the u-partition equals the sum obtained from the t-partition. In a similar way, we can get from the s-partition to the u-partition by adding one point at a time, so the sum from the u-partition equals the sum from the s-partition. Since the sums from the t- and s-partitions are both equal to the sum from the u-partition, they are equal to each other, which is what we wanted to prove.

Solved Exercises

1. Consider s- and t-partitions of $[1,8]$ as follows. Let the s-partition be $(1,2,3,4,7,8)$, and let the t-partition be $(1,4,5,6,8)$. Find the corresponding u-partition, and show that you can get from the s- and t-partitions to the u-partition by adding one point at a time.

2. Let $f(t)$ be defined by

$$f(t) = \begin{cases} 2 & \text{if } 0 \leqslant t < 1 \\ 0 & \text{if } 1 \leqslant t < 3 \\ -1 & \text{if } 3 \leqslant t \leqslant 4 \end{cases}$$

For any number x in $(0,4]$, $f(t)$ is piecewise constant on $[0,x]$.

(a) Find $\int_0^x f(t)\,dt$ as a function of x. (You will need to use different formulas on different intervals.)

(b) Let $F(x) = \int_0^x f(t)dt$, for $x \in (0,4]$. Draw a graph of F.

(c) At which points is F differentiable? Find a formula for $F'(x)$.

Exercises

1. Let $f(t)$ be the greatest integer function: $f(t) = n$ on $(n, n+1)$, where n is any integer. Compute $\int_0^8 f(t)dt$ using each of the partitions (1,2,3,4,5,6,7,8) and (1,2,2.5,3,3.5,4,5,6,7,8). Sketch.
2. Show that, given any two piecewise constant functions on the same interval, there is a partition which is adapted to both of them.

Upper and Lower Sums and the Definition

Having defined the integral for piecewise constant functions, we will now define the integral for more general functions. You should compare the definition with the way in which we passed from linear functions to general functions when we defined the derivative in Chapters 1 and 2. We begin with a preliminary definition.

Definition Let f be a function defined on $[a, b]$. If g is any piecewise constant function on $[a, b]$ such that $g(t) \leqslant f(t)$ for all t in the open interval (a, b), we call the number $\int_a^b g(t)dt$ a *lower sum* for f on $[a, b]$. If h is a piecewise constant function and $f(t) \leqslant h(t)$ for all t in (a, b), the number $\int_a^b h(t)\,dt$ is called an *upper sum* for f on $[a, b]$.

Worked Example 3 Let $f(t) = t^2 + 1$ for $0 \leqslant t \leqslant 2$. Let

$$g(t) = \begin{cases} 0 & 0 \leqslant t \leqslant 1 \\ 2 & 1 < t \leqslant 2 \end{cases} \quad \text{and} \quad h(t) = \begin{cases} 2 & 0 \leqslant t \leqslant \frac{2}{3} \\ 4 & \frac{2}{3} < t \leqslant \frac{4}{3} \\ 5 & \frac{4}{3} < t \leqslant 2 \end{cases}$$

Draw a graph showing $f(t)$, $g(t)$, and $h(t)$. What upper and lower sums for f can be obtained from g and h?

Solution The graph is shown in Fig. 11-6.

Since $g(t) \leqslant f(t)$ for all t in the open interval $(0, 2)$ (the graph of g lies below that of f), we have a lower sum

$$\int_0^2 g(t)\,dt = 0 \cdot 1 + 2 \cdot 1 = 2$$

Since the graph of h lies above that of f, $h(t) \geqslant f(t)$ for all t in the interval $(0, 2)$, so we have the upper sum

$$\int_0^2 h(t)\,dt = 2 \cdot \tfrac{2}{3} + 4 \cdot \tfrac{2}{3} + 5 \cdot \tfrac{2}{3} = \tfrac{22}{3} = 7\tfrac{1}{3}$$

The integral of a function should lie between the lower sums and the upper sums. For instance, the integral of the function in Worked Example 3 should lie in the interval $[2, 7\frac{1}{3}]$. If we could find upper and lower sums which are arbitrarily close together, then the integral would be pinned down to a single point. This idea leads to the formal definition of the integral.

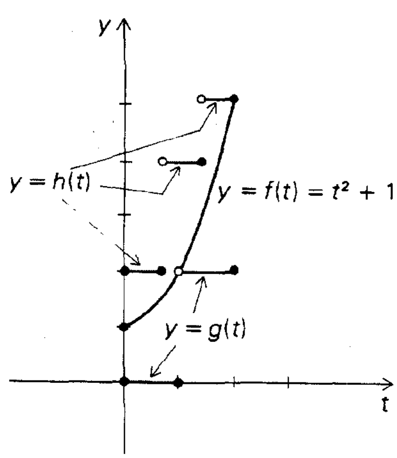

Fig. 11-6 $g(t) \leqslant f(t) \leqslant h(t)$.

Definition Let f be a function defined on $[a, b]$, and let L_f and U_f be the sets of lower and upper sums for f, as defined above. If there is a transition point from L_f to U_f, then we say that f is *integrable* on $[a, b]$. The transition point is called *the integral of f on $[a, b]$* and is denoted by

$$\int_a^b f(t)\,dt$$

Remark The letter t is called the *variable of integration*; it is a dummy variable in that we can replace it by any other letter without changing the value of the integral; a and b are called the *endpoints* for the integral.

The next theorem contains some important facts about upper and lower sums.

Theorem 2

1. *Every lower sum for f is less than or equal to every upper sum.*
2. *Every number less than a lower sum is a lower sum; every number greater than an upper sum is an upper sum.*

Proof To prove part 1 we must show that, if g and h are piecewise constant functions on $[a, b]$ such that $g(t) \leqslant f(t) \leqslant h(t)$ for all t in (a, b), then

$$\int_a^b g(t)\,dt \leqslant \int_a^b h(t)\,dt$$

The function f will play no role in this proof; all we use is the fact that $g(t) \leqslant h(t)$ for all $t \in (a, b)$.

By Theorem 1 we can use any adapted partition we want to compute the integrals of g and h. In particular, we can combine partitions which are adapted to g and h to obtain a partition which is adapted to both g and h. See Exercise 2. Let $(t_0, t_1, \ldots, t_n)$ be such a partition. Then we have constants k_i and l_i such that $g(t) = k_i$ and $h(t) = l_i$ for t in (t_{i-1}, t_i). By hypothesis, we have $k_i \leqslant l_i$ for each i. (See Fig. 11-7.) Now, each $\Delta t_i = t_i - t_{i-1}$ is positive, so $k_i \Delta t_i \leqslant l_i \Delta t_i$ for each i. Therefore,

$$\sum_{i=1}^{n} k_i \Delta t_i \leqslant \sum_{i=1}^{n} l_i \Delta t_i$$

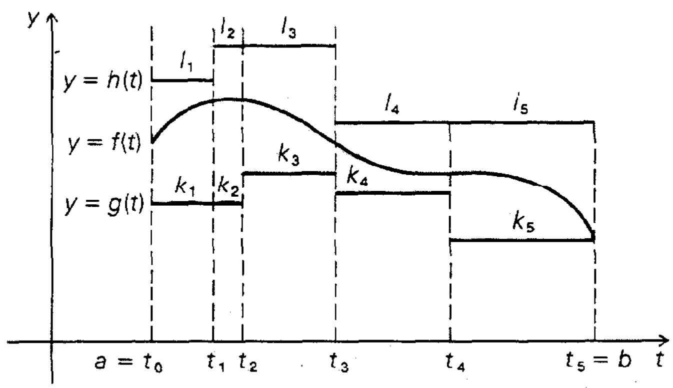

Fig. 11·7 Upper sums are larger than lower sums.

But these sums are just the integrals of g and h, so we are done with part 1.

We will now prove the first half of part 2, that every number lower than a lower sum is a lower sum. (The other half is virtually identical; we leave it to the reader as an exercise.) Let β be a lower sum and let $c < \beta$ be any number. β is the integral $\int_a^b g(t)\,dt$ of a piecewise constant function g with $g(t) \leqslant f(t)$ for all $t \in (a, b)$; we wish to show that c is such an integral as well. To do this, we choose an adapted partition $(t_0, t_1, \ldots, t_n)$ for g and create a new function e by lowering g on the interval (t_0, t_1). Specifically, we let $e(t) = g(t)$ for all t not in (t_0, t_1), and we put $e(t) = g(t) - p$ for t in (t_0, t_1), where p is a positive constant to be chosen in such a way that the integral comes out right. Specifically, if $g(t) = k_i$ for t in (t_{i-1}, t_i), then $e(t) = k_1 - p$ for t in (t_0, t_1), and $e(t) = k_i$ for t in (t_{i-1}, t_i), $i \geqslant 2$. We have

$$\begin{aligned}\int_a^b e(t)\,dt &= (k_1 - p)\,\Delta t_1 + k_2\,\Delta t_2 + \cdots + k_n\,\Delta t_n \\ &= k_1\,\Delta t_1 + \cdots + k_n\,\Delta t_n - p\,\Delta t_1 \\ &= \int_a^b g(t)\,dt - p\,\Delta t_1\end{aligned}$$

Setting this equal to c and solving for p gives

$$p = \frac{1}{\Delta t_1}\left[\int_a^b g(t)\,dt - c\right]$$

which is positive, since we assumed $c < \beta = \int_a^b g(t)\,dt$. With this value of p, we define e as we just described (it is important that p be positive, so that $e(t) \leqslant f(t)$ for all t in (a, b)), and so we have $\int_a^b e(t)\,dt = c$, as desired.

It follows from the completeness axiom in Chapter 4 that L_f and U_f are intervals which, if nonempty, are infinite in opposite directions.

In Fig. 11-8 we show the possible configurations for L_f and U_f. If there is a transition point from L_f to U_f, there is exactly one, so the integral, if it exists, is well defined.

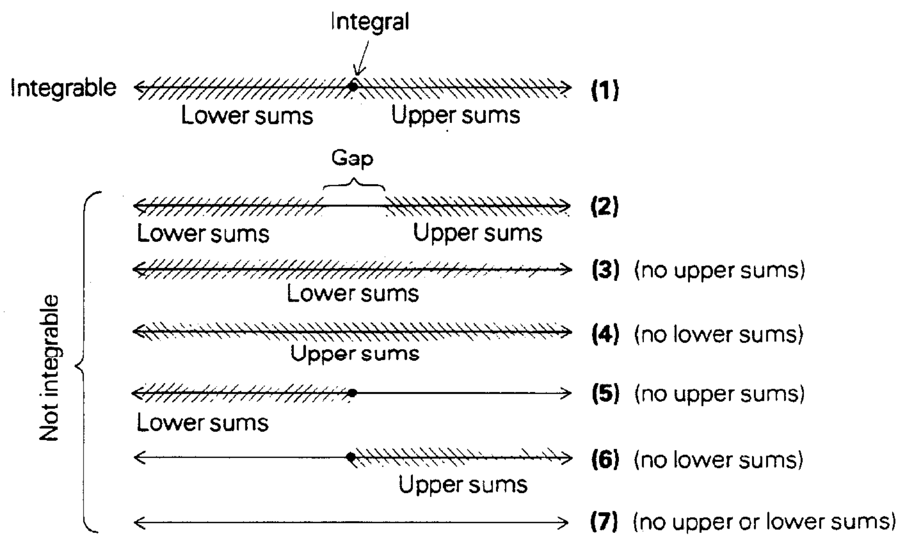

Fig. 11-8 Possible configurations of lower and upper sums.

Do not conclude from Fig. 11-8 that most functions are nonintegrable. In fact, cases (3), (4), (5), (6), and (7) can occur only when f is unbounded (see Worked Example 4). The functions for which L_f and U_f has a gap between them (case (2)) are quite "pathological" (see Solved Exercise 3). In fact, Theorem 3 in the next section tells us that integrability is even more common than differentiability.

Worked Example 4 Show that the set of upper sums for f is nonempty if and only if f is bounded above on $[a, b]$; i.e., if and only if there is a number M such that $f(t) \leqslant M$ for all t in $[a, b]$.

Solution Suppose first that $f(t) \leqslant M$ for all t in $[a, b]$. Then we can consider the piecewise constant function g defined by $g(t) = M$ for all t in $[a, b]$. We have $f(t) \leqslant g(t)$ for all t in $[a, b]$, so

$$\int_a^b h(t)\,dt = M(b-a)$$

is an upper sum for f.

The converse is a little more difficult. Suppose that there exists an upper sum for f. This upper sum is the integral of a piecewise constant function h on $[a, b]$ such that $f(t) \leqslant h(t)$ for all t in $[a, b]$. It suffices to show that the piecewise constant function h is bounded above on $[a, b]$. To do this, we choose a partition $(t_0, t_1, \ldots, t_n)$ for h. For every t in $[a, b]$, the value of $h(t)$ belongs to the finite list

$$h(t_0), k_1, h(t_1), k_2, \ldots, h(t_{n-1}), k_n, h(t_n)$$

where k_i is the value of h on (t_{i-1}, t_i). If we let M be the largest number in the list above, we may conclude that $h(t)$, and hence $f(t)$, is less than or equal to M for all $t \in [a, b]$. (See Fig. 11-9.)

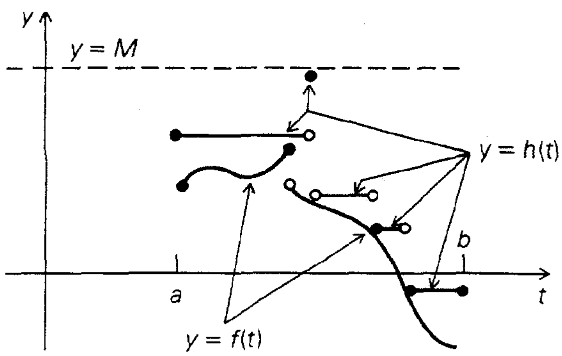

Fig. 11-9 The piecewise constant function h is bounded above by M on $[a, b]$, so f is bounded above too.

Solved Exercises

3. Let

$$f(t) = \begin{cases} 0 & \text{if } t \text{ is irrational} \\ 1 & \text{if } t \text{ is rational} \end{cases}$$

for $0 \leqslant t \leqslant 1$. Show that f is not integrable.

4. Show that every piecewise constant function is integrable and that its integral as defined on p. 155 is the same as its integral as originally defined on p. 149.

Exercises

3. Let f be defined on $[0, 1]$ by

$$f(t) = \begin{cases} 1 & \text{if } t \text{ is rational} \\ 0 & \text{if } t \text{ is irrational} \end{cases}$$

Find the sets of upper and lower sums for f and show that f is not integrable.

4. Let f be a function defined on $[a, b]$ and let S_0 be a real number. Show that S_0 is *the integral of f on* $[a, b]$ if and only if:
   1. Every number $S < S_0$ is a lower sum for f on $[a, b]$.
   2. Every number $S > S_0$ is an upper sum for f on $[a, b]$.

The Integrability of Continuous Functions

We will now prove that a continuous function on a closed interval is integrable. Since every differentiable function is continuous, we conclude that every differentiable function is integrable; however, some noncontinuous functions can be integrable. For instance, piecewise constant functions are integrable (see Solved Exercise 4), but they are not continuous. On the other hand, the wild function described in Solved Exercise 3 shows that there is really something to prove. Since the proof is similar in spirit to those in Chapter 5, it may be useful to review that section before proceeding.

Theorem 3 *If f is continuous on* $[a, b]$, *then f is integrable on* $[a, b]$.

Proof Since the continuous function f is bounded, it has both upper and lower sums. To show that there is no gap between the upper and lower sums, we will prove that, for any positive number ϵ, there are lower and upper sums within ϵ of one another. (The result of Problem 13, Chapter 4 implies that f is integrable.) To facilitate the proof, if $[a, x]$ is any subinterval of $[a, b]$, and ϵ is a positive real number, we will say that f is ϵ-*integrable* on $[a, x]$ if there are piecewise constant functions g and h on $[a, x]$ with $g(t) < f(t) < h(t)$ for all t in (a, x), such that $\int_a^x h(t)\,dt - \int_a^x g(t)\,dt < \epsilon$. What we wish to show, then, is that the continuous function f is ϵ-integrable on $[a, b]$, for any positive ϵ.

We define the set S_ϵ (for any $\epsilon > 0$) to consist of those x in $(a, b]$ for which f is ϵ-integrable on $[a,x]$. We wish to show that $b \in S_\epsilon$.

We will use the completeness axiom, so we begin by showing that S_ϵ is convex. If $x_1 < x < x_2$, and x_1 and x_2 are in S_ϵ, then x is certainly in $(a, b]$. To show that f is ϵ-integrable on $(a, x]$, we take the piecewise constant functions which do the job on $[a, x_2]$ (since $x_2 \in S_\epsilon$) and restrict them to $[a, x]$. The completeness axiom implies that S_ϵ is an interval. We analyze the interval S_ϵ in two lemmas, after which we will end the proof of Theorem 3.

Lemma 1 S_ϵ *contains all* x *in* (a, c) *for some* $c > a$.

Proof We use the continuity of f at a. Let $\delta = \epsilon/[2(b - a)]$. Since $f(a) - \delta < f(a) < f(a) + \delta$, there is an interval $[a, c)$ such that $f(a) - \delta < f(t) < f(a) + \delta$ for all t in $[a, c)$. Now for x in $[a, c]$, if we restrict f to $[a, x]$, we can contain it between the constant function $g(t) = f(a) - \delta$ and $h(t) = f(a) + \delta$. Therefore,

$$\int_a^x h(t)\,dt - \int_a^x g(t)\,dt = (f(a) + \delta)(x - a) - (f(a) - \delta)(x - a)$$

$$= (f(a) + \delta - f(a) + \delta)(x - a)$$

$$= 2\delta(x - a) = 2 \cdot \frac{\epsilon}{2(b - a)}(x - a)$$

$$= \epsilon\,\frac{x - a}{b - a} < \epsilon \quad \text{since } x < c \leqslant b,$$

and we have shown that f is ϵ-integrable on $[a, x]$. Therefore, x belongs to S_ϵ as required.

Lemma 2 *If* $x_0 < a$, *and* x_0 *belongs to* S_ϵ, *then* S_ϵ *contains all* x *in* $(x_0, c]$ *for some* $c > x_0$.

Proof This is a slightly more complicated variation of the previous proof. By the hypothesis concerning x_0, there are piecewise constant functions g_0 and h_0 on $[a, x_0]$ with $g_0(t) \leqslant f(t) \leqslant h_0(t)$ for all t in (a, x_0), such that the difference $\delta = \int_a^{x_0} h_0(t)\,dt - \int_a^{x_0} g_0(t)\,dt$ is less than ϵ. By the continuity of f at x_0 there is a number $c > x_0$ such that

$$f(x_0) - \frac{\epsilon - \delta}{2(b - a)} < f(t) < f(x_0) + \frac{\epsilon - \delta}{2(b - a)}$$

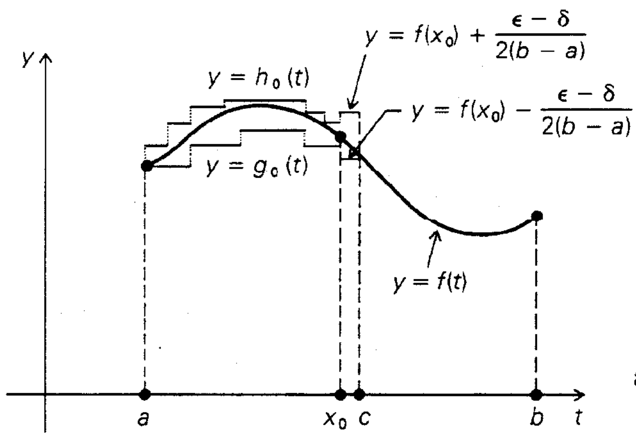

Fig. 11-10 Extending g_0 and h_0 from $[a, x_0]$ to $[a, c]$.

for all t in $[x_0, c)$.

Now we extend the functions g_0 and h_0 to the interval $[a, c]$ by defining g and h as follows (Fig. 11-10):

$$g(t) = \begin{cases} g_0(t) & a \leqslant t \leqslant x_0 \\ f(x_0) - \dfrac{\epsilon - \delta}{2(b-a)} & x_0 < t \leqslant c \end{cases}$$

$$h(t) = \begin{cases} h_0(t) & a \leqslant t \leqslant x_0 \\ f(x_0) + \dfrac{\epsilon - \delta}{2(b-a)} & x_0 < t < c \end{cases}$$

Clearly, we have $g(t) \leqslant f(t) \leqslant h(t)$ for all t in $[a, c]$, We claim now that $\int_a^c h(t)\,dt - \int_a^c g(t)\,dt < \epsilon$. To show this, we observe first that

$$\int_a^c g(t)\,dt = \int_a^{x_0} g(t)\,dt + [f(x_0) - \frac{\epsilon - \delta}{2(b-a)}](c - x_0)$$

(In fact, the sum for $\int_a^c g(t)\,dt$ is just that for $\int_a^{x_0} g(t)\,dt$ with one more term added for the interval $[x_0, c]$.) Similarly, $\int_a^c h(t)\,dt = \int_a^{x_0} h(t)\,dt + [f(x_0) + (\epsilon - \delta)/2(b-a)](c - x_0)$. Subtracting the last two integrals gives

$$\int_a^c h(t)\,dt - \int_a^c g(t)\,dt = \int_a^{x_0} h(t)\,dt - \int_a^{x_0} g(t)\,dt$$

$$+ [f(x_0) + \frac{\epsilon - \delta}{2(b-a)}](c - x_0) - [f(x_0) - \frac{\epsilon - \delta}{2(b-a)}](c - x_0)$$

$$= \int_a^{x_0} h(t)\,dt - \int_a^{x_0} g(t)\,dt + \frac{(\epsilon - \delta)(c - x_0)}{(b-a)}$$

Since $(c - x_0)/(b - a) < 1$, the last expression is less than $\int_a^{x_0} h(t)\,dt - \int_a^{x_0} g(t)\,dt + \epsilon - \delta = \delta + \epsilon - \delta = \epsilon$, and we have shown that f is ϵ-integrable on $[a, c]$, so c belongs to S_ϵ. Since S_ϵ is an interval, S_ϵ contains all of $(a, c]$.

Proof of Theorem 3 (completed) By Lemma 1, S_ϵ is nonempty for every ϵ. By Lemma 2, the right-hand endpoint of S_ϵ cannot be less than b, so $S_\epsilon = [a, b)$ or $[a, b]$ for every $\epsilon > 0$. To show that S_ϵ is actually $[a, b]$ and not $[a, b)$ we use once more an argument like that in the lemmas. By continuity of f at b, we can find a number $c < b$ such that $f(b) - \epsilon/2(b - a) < f(t) < f(b) + \epsilon/2(b - a)$ for all t in $[c, b]$. Now we use the fact that $c \in S_{\epsilon/2}$, i.e., f is $\epsilon/2$-integrable on $[a, c]$. (Remember, we have established that f is ϵ-integrable on $[a, c]$ for *all* numbers ϵ, so we can use $\epsilon/2$ for ϵ.) As we did in Lemma 2, we can put together piecewise constant functions g_0 and h_0 for f on $[a, c]$ with the constant functions $f(b) \pm \epsilon/2(b - a)$ on $(c, b]$ to establish that f is ϵ-integrable on $[a, b]$, i.e., that $S_\epsilon = [a, b]$.

We have shown that f has lower and upper sums on $[a, b]$ which are arbitrarily close together, so there is no gap between the intervals L_f and U_f and f is integrable on $[a, b]$.

Solved Exercises

5. Show that, if f is ϵ-integrable on $[a, x_0]$, and $a < x < x_0$, then f is ϵ-integrable on $[a, x]$.
6. Show that every monotonic function is integrable.
7. Find $\int_1^2 (1/t)\,dt$ to within an error of no more than $\frac{1}{10}$.
8. If you used a method analogous to that in Solved Exercise 7, how many steps would it take to calculate $\int_1^2 (1/t)\,dt$ to within $\frac{1}{100}$?

Exercises

5. Let f be the nonintegrable function of Solved Exercise 3.
   (a) For which values of ϵ is f ϵ-integrable on $[0, 1]$?
   (b) Let x belong to $[0, 1]$. For which values of ϵ is f ϵ-integrable on $[0, x]$?

6. Prove that if g is piecewise constant on $[a, c]$, and $b \in (a, c)$, then

$$\int_a^c g(t)\,dt = \int_a^b g(t)\,dt + \int_b^c g(t)\,dt$$

[*Hint:* Choose a partition which includes c as one of its points.]

7. Prove that if g_1 and g_2 are piecewise constant on $[a, b]$, and s_1 and s_2 are constants, then $s_1 g_1 + s_2 g_2$ is also piecewise constant, and

$$\int_a^b (s_1 g_1(t) + s_2 g_2(t))\,dt = s_1 \int_a^b g_1(t)\,dt + s_2 \int_a^b g_2(t)\,dt$$

[*Hint:* Choose a partition adapted to both g_1 and g_2, and write out all the sums.]

8. Let f be defined on $[0, 1]$ by

$$f(t) = \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{1}{t} & \text{if } 0 < t \leqslant 1 \end{cases}$$

Is f integrable on $[0, 1]$? On $[\frac{1}{8}, 1]$?

Properties of the Integral

We now establish some basic properties of the integral. These properties imply, for example, that the integral of a piecewise constant function must be defined as we did.

Theorem 4

1. *If $a < b < c$ and f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$ and*

$$\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$$

2. *If f and g are integrable on $[a, b]$ and if $f(t) \leqslant h(t)$ for all $t \in (a, b)$, then*

$$\int_a^b f(t)\,dt \leqslant \int_a^b h(t)\,dt$$

3. *If $f(t) = k$ for all $t \in (a, b)$, then*

$$\int_a^b f(t)\,dt = k(b-a)$$

Proof We want to show that the sum

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = I$$

is a transition point between the lower and upper sums for f on $[a, b]$. Let $m < I$; we will show that m is a lower sum for f on $[a, b]$. Since $m < I$, we can write $m = m_1 + m_2$, where $m_1 < \int_a^b f(t)\,dt$ and $m_2 < \int_b^c f(t)\,dt$. (See Solved Exercise 9.) Thus m_1 and m_2 are lower sums for f on $[a, b]$ and $[b, c]$, respectively. Thus, there is a piecewise constant function g_1 on $[a, b]$ with $g_1(t) < f(t)$ for all t in (a, b), such that $\int_a^b g_1(t)\,dt = m_1$, and there is a piecewise constant function g_2 on $[a, b]$ with $g_2(t) < f(t)$ for all t in (b, c) such that $\int_b^c g_2(t) = m_2$. Put together g_1 and g_2 to obtain a function g on $[a, c]$ by the definition

$$g(t) = \begin{cases} g_1(t) & a \leqslant t < b \\ f(b) - 1 & t = b \\ g_2(t) & b < t \leqslant c \end{cases}$$

The function g is piecewise constant on $[a, c]$, with $g(t) < f(t)$ for all t in (a, c). The sum which represents the integral for g on $[a, c]$ is the sum of the sums representing the integrals of g_1 and g_2, so we have

$$\int_a^c g(t)\,dt = \int_a^b g_1(t)\,dt + \int_b^c g_2(t)\,dt = m_1 + m_2 = m$$

and m is a lower sum. Similarly, any number M greater than I is an upper sum, so I is the integral of f on $[a, b]$.

We leave the proof of part 2 for the reader (Exercise 9). Part 3 follows from Solved Exercise 4.

Solved Exercises

9. If $m < p_1 + p_2$, prove that $m = m_1 + m_2$ for some numbers $m_1 < p_1$ and $m_2 < p_2$.
10. Let

$$f(x) = \begin{cases} 0 & 0 \leqslant x \leqslant 1 \\ x^2 & 1 < x \leqslant 2 \end{cases}$$

Prove that f is integrable on $[0,2]$.

Exercises

9. Prove that, if f and g are integrable on $[a, b]$, and if $f(t) \leqslant h(t)$ for all t in (a, b), then $\int_a^b f(t)\,dt \leqslant \int_a^b h(t)\,dt$. [*Hint:* Relate the lower and upper sums for f and h.]
10. Let m, p_1, and p_2 be positive numbers such that $m < p_1 p_2$. Prove that m can be written as a product $m_1 m_2$, where $0 < m_1 < p_1$ and $0 < m_2 < p_2$.

Calculating Integrals by Hand

We have just given an elaborate definition of the integral and proved that continuous functions are integrable, but we have not yet computed the integral of any functions except piecewise constant functions. You may recall that in our treatment of differentiation it was quite difficult to compute derivatives by using the definitions; instead, we used the algebraic rules of calculus to compute most derivatives. The situation is much the same for integration. In Chapter 12 we will develop the machinery which makes the calculation of many integrals quite simple. Before doing this, however, we will calculate one integral "by hand" in order to illustrate the definition.

After the constant functions, the simplest continuous function is $f(t) = t$. We know by Theorem 1 that the integral $\int_a^b t\,dt$ exists; to calculate this integral, we will find upper and lower sums which are closer and closer together, reducing to zero the possible error in the estimate of the integral.

Let $f(t) = t$. We divide the interval $[a, b]$ into n equal parts, using the partition

$$(t_0, \ldots, t_n) = \left(a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{(n-1)(b-a)}{n}, b\right)$$

Note that $\Delta t_i = (b-a)/n$ for each i.

Now we define the piecewise constant function g on $[a, b]$ by setting $g(t) = t_{i-1}$ for $t_{i-1} \leqslant t < t_i$. (See Fig. 11-11.)

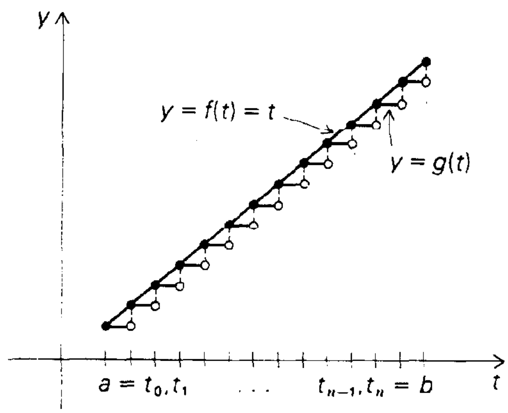

Fig. 11-11 Estimating $\int_a^b t\,dt$ from below.

Note that for $t_{i-1} \leqslant t < t_i$, we have $g(t) = t_{i-1} \leqslant t = f(t)$, so $\int_a^b g(t)\,dt$ is a lower sum for f on $[a, b]$. The definition of the integral of a piecewise constant function gives

$$\int_a^b g(t)\,dt = \sum_{i=1}^{n} k_i \Delta t_i = \sum_{i=1}^{n} t_{i-1} \Delta t_i$$

since t_{i-1} is the constant value of g on (t_{i-1}, t_i). We know that $\Delta t_i = (b-a)/n$ for each i. To find out what t_i is, we note that $t_0 = a$, $t_1 = a + (b-a)/n$, $t_2 = a + [2(b-a)/n]$, and so on so the general formula is $t_i = a + [i(b-a)/n]$. Substituting for t_{i-1} and Δt_i gives

$$\int_a^b g(t)\,dt = \sum_{i=1}^{n} \left[a + \frac{(i-1)(b-a)}{n}\right]\left(\frac{b-a}{n}\right)$$

$$= \sum_{i=1}^{n} \left[\frac{a(b-a)}{n} + \frac{i(b-a)^2}{n^2} - \frac{(b-a)^2}{n^2}\right]$$

We may rewrite this as:

$$\sum_{i=1}^{n} \frac{a(b-a)}{n} + \frac{(b-a)^2}{n^2} \sum_{i=1}^{n} i - \sum_{i=1}^{n} \frac{(b-a)^2}{n^2}$$

The outer terms do not involve i, so each may be summed by adding the summand to itself n times, i.e., by multiplying the summand by n. The middle term is summed by the formula $\sum_{i=1}^{n} i = n(n+1)/2$. The result is

$$a(b-a) + \frac{(b-a)^2}{n^2} \frac{n(n+1)}{2} - \frac{(b-a)^2}{n}$$

which simplifies to

$$\frac{b^2 - a^2}{2} - \frac{1}{2n}(b-a)^2$$

(You should carry out the simplification.) We have, therefore, shown that $[(b^2 - a^2)/2] - (1/2n)(b-a)^2$ is a lower sum for f.

We now move on to the upper sums. Using the same partition as before, but with the function $h(t)$ defined by $h(t) = t_i$ for $t_{i-1} \leqslant t < t_i$, $\int_a^b h(t)\,dt$ is an upper sum. Some algebra (see Solved Exercise 11) gives

$$\int_a^b h(t)\,dt = \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n}$$

Our calculations of upper and lower sums therefore show that

$$\frac{b^2 - a^2}{2} - \frac{(b-a)^2}{2n} \leqslant \int_a^b t\,dt \leqslant \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n}$$

These inequalities, which hold for all n, show that the integral can be neither larger nor smaller than $(b^2 - a^2)/2$, so it must be equal to $(b^2 - a^2)/2$.

Worked Example 5 Find $\int_0^3 x\,dx$.

Solution We first note that since the variable of integration is a "dummy variable,"

$$\int_0^3 x\,dx = \int_0^3 t\,dt$$

Using the formula just obtained, we can evaluate this integral as $\frac{1}{2}(3^2 - 0^2) = \frac{9}{2}$.

In the next chapter you will find, to your relief, a much easier way to compute integrals. Evaluating integrals like $\int_a^b t^2\,dt$ or $\int_a^b t^3\,dt$ by hand is possible but rather tedious. The methods of the next chapter will make these integrals simple to evaluate.

Solved Exercises

11. Draw a graph of the function $h(t)$ used above to find an upper sum for $f(t) = t$ and show that

$$\int_a^b h(t)\,dt = \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n}$$

12. Using the definition of the integral, find $\int_a^b 5t\,dt$.
13. (a) Sketch the region under the graph of $f(t) = t$ on $[a, b]$, if $0 < a < b$.
    (b) Compare the area of this region with $\int_a^b t\,dt$.

Exercises

11. Using the definition of the integral, find a formula for $\int_a^b (t+3)\,dt$.
12. Using the definition of the integral, find a formula for $\int_a^b (-t)\,dt$.
13. Explain the relation between $\int_{-2}^1 t\,dt$ and an area.

Problems for Chapter 11

1. Let f be the function defined by

$$f(t) = \begin{cases} 2 & 1 \leqslant t < 4 \\ 5 & 4 \leqslant t < 7 \\ 1 & 7 \leqslant t \leqslant 10 \end{cases}$$

   (a) Find $\int_1^{10} f(t)\,dt$.
   (b) Find $\int_2^9 f(t)\,dt$.
   (c) Suppose that g is a function on $[1, 10]$ such that $g(t) \leqslant f(t)$ for all t in $[1, 10]$. What inequality can you derive for $\int_1^{10} g(t)\,dt$?

(d) With $g(t)$ as in part (c), what inequalities can you obtain for $\int_1^{10} 2g(t)\,dt$ and $\int_1^{10} -g(t)\,dt$. [*Hint:* Find functions like f with which you can compare g.]

2. Let $f(t)$ be the "greatest integer function"; that is, $f(t)$ is the greatest integer which is less than or equal to t—for example, $f(n) = n$ for any integer, $f(5\frac{1}{2}) = 5$, $f(-5\frac{1}{2}) = -6$, and so on.

   (a) Draw a graph of $f(t)$ on the interval $[-4, 4]$.

   (b) Find $\int_0^1 f(t)\,dt$, $\int_0^6 f(t)\,dt$, $\int_{-2}^2 f(t)\,dt$, and $\int_0^{4.5} f(t)\,dt$.

   (c) Find a general formula for $\int_0^n f(t)\,dt$, where n is any positive integer.

   (d) Let $F(x) = \int_0^x f(t)\,dt$, where $x > 0$. Draw a graph of F for $x \in [0,4]$, and find a formula for $F'(x)$, where it is defined.

3. Suppose that $f(t)$ is piecewise constant on $[a, b]$. Let $g(t) = f(t) + k$, where k is a constant.

   (a) Show that $g(t)$ is piecewise constant.

   (b) Find $\int_a^b g(t)\,dt$ in terms of $\int_a^b f(t)\,dt$.

4. For $t \in [0, 1]$ let $f(t)$ be the first digit after the decimal point in the decimal expansion of t.

   (a) Draw a graph of f. (b) Find $\int_0^1 f(t)\,dt$.

5. Using the definition of the integral, find $\int_0^1 (1 - x)\,dx$.

6. Suppose that $f(t)$ is piecewise constant on $[a, b]$. Let $F(x) = \int_a^x f(t)\,dt$. Prove that if f is continuous at $x_0 \in [a, b]$, then F is differentiable at x_0 and $F'(x) = f(x)$. (See Solved Exercise 2.)

7. Show that $-3 \leqslant \int_1^2 (t^3 - 4)\,dt \leqslant 4$.

8. (a) Show that, if f is piecewise constant on $[a, b]$, then there is an adapted partition for f which is "coarsest" in the sense that it is contained in any other adapted partition.

   (b) Use the result of (a) to give a new proof of Theorem 1.

9. Let F be defined on $[0, 1]$ by

$$f(t) = \begin{cases} 1 & \text{if } t = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that f is integrable on $[0, 1]$.

   (b) What is $\int_0^t f(s)\,ds = F(t)$?

   (c) For which values of t is $F'(t) = f(t)$?

10. Show that if f_1 and f_2 are integrable on $[a, b]$, and s_1 and s_2 are any real numbers, then $s_1 f_1 + s_2 f_2$ is integrable on $[a, b]$, and

$$\int_a^b [s_1 f_1(t) + s_2 f_2(t)]\, dt = s_1 \int_a^b f_1(t)\, dt + s_2 \int_a^b f_2(t)\, dt$$

11. (a) Show that, if f is continuous on $[a, b]$, where $a < b$, and $f(t) > 0$ for all t in $[a, b]$, and then $\int_a^b f(t)\, dt > 0$.
    (b) Show that the result in (a) still holds if f is continuous, $f(t) \geqslant 0$ for all t in $[a, b]$, and $f(t) > 0$ for *some* t in $[a, b]$.
    (c) Are the results in (a) and (b) still true if the hypothesis of continuity is replaced by integrability?

12. Find functions f_1 and f_2, neither of which is integrable on $[0, 1]$ such that:
    (a) $f_1 + f_2$ is integrable on $[0, 1]$.
    (b) $f_1 - f_2$ is not integrable on $[0, 1]$.

13. Let f be defined on $[0, 1]$ by

$$f(t) = \begin{cases} 0 & t = 0 \\ \dfrac{1}{\sqrt{t}} & 0 < t \leqslant 1 \end{cases}$$

    (a) Show that there are no upper sums for f on $[0, 1]$ and hence that f is not integrable.
    (b) Show that every number less than 2 is not lower sum. [*Hint*: Use step functions which are zero on an interval $[0, \epsilon)$ and approximate f very closely on $[\epsilon, 1]$. Take ϵ small and use the integrability of f on $[\epsilon, 1]$.]
    (c) Show that no number greater than or equal to 2 is a lower sum. [*Hint*: Show $\epsilon f(\epsilon) + \int_\epsilon^1 f(t)\, dt < 2$ for all ϵ in $(0, 1)$.]
    (d) If you had to assign a value to $\int_0^1 f(t)\, dt$, what value would you assign?

14. Modeling your discussion after Problem 13, find the upper and lower sums for each of the following functions on $[0, 1]$. How would you define $\int_0^1 f(t)\, dt$ in each case?

    (a) $f(t) = \begin{cases} 0 & t = 0 \\ -\dfrac{1}{\sqrt[3]{t}} & t > 0 \end{cases}$

    (b) $f(t) = \begin{cases} 0 & t = 0 \\ \dfrac{1}{t^2} & t > 0 \end{cases}$