12 The Fundamental Theorem of Calculus

The fundamental theorem of calculus reduces the problem of integration to antidifferentiation, i.e., finding a function F such that $F' = f$. We shall concentrate here on the proof of the theorem, leaving extensive applications for your regular calculus text.

Statement of the Fundamental Theorem

Theorem 1 Fundamental Theorem of Calculus: Suppose that the function F is differentiable everywhere on $[a, b]$ and that F' is integrable on $[a, b]$. Then

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

In other words, if f is integrable on $[a, b]$ and F is an *antiderivative* for f, i.e., if $F' = f$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Before proving Theorem 1, we will show how easy it makes the calculation of some integrals.

Worked Example 1 Using the fundamental theorem of calculus, compute $\int_a^b t^2\,dt$.

Solution We begin by finding an antiderivative $F(t)$ for $f(t) = t^2$; from the power rule, we may take $F(t) = \frac{1}{3}t^3$. Now, by the fundamental theorem, we have

$$\int_a^b t^2\,dt = \int_a^b f(t)\,dt = F(b) - F(a) = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3$$

We conclude that $\int_a^b t^2\,dt = \frac{1}{3}(b^3 - a^3)$ It is possible to evaluate this integral "by hand," using partitions of $[a, b]$ and calculating upper and lower sums, but the present method is much more efficient.

According to the fundamental theorem, it does not matter which antiderivative we use. But in fact, we do not need the fundamental theorem to tell us that if F_1 and F_2 are both antiderivatives of f on $[a, b]$, then

$$F_1(b) - F_1(a) = F_2(b) - F_2(a)$$

To prove this we use the fact that any two antiderivatives of a function differ by a constant. (See Corollary 3 of the mean value theorem, Chapter 7.) We have, therefore, $F_1(t) = F_2(t) + C$, where C is a constant, and so

$$F_1(b) - F_1(a) = [F_2(b) + C] - [F_2(a) + C]$$

The C's cancel, and the expression on the right is just $F_2(b) - F_2(a)$.

Expressions of the form $F(b) - F(a)$ occur so often that it is useful to have a special notation for them: $F(t)\big|_a^b$ means $F(b) - F(a)$. One also writes $F(t) = \int f(t)\,dt$ for the antiderivative (also called an indefinite integral). In terms of this new notation, we can write the formula of the fundamental theorem of calculus in the form:

$$\int_a^b f(t)\,dt = F(t)\Big|_a^b \quad \text{or} \quad \int_a^b f(t)\,dt = \left(\int f(t)\,dt\right)\Big|_a^b$$

where F is an antiderivative of f on $[a, b]$.

Worked Example 2 Find $(t^3 + 5)\big|_2^3$.

Solution Here $F(t) = t^3 + 5$ and

$$\begin{aligned}(t^3 + 5)\big|_2^3 &= F(3) - F(2)\\ &= 3^3 + 5 - (2^3 + 5)\\ &= 32 - 13 = 19\end{aligned}$$

Worked Example 3 Find $\int_2^6 (t^2 + 1)\,dt$.

Solution By the sum and power rules for antiderivatives, an antiderivative for $t^2 + 1$ is $\frac{1}{3}t^3 + t$. By the fundamental theorem

$$\int_2^6 (t^2+1)\,dt = \left(\frac{1}{3}t^3+t\right)\Big|_2^6$$
$$= \frac{6^3}{3}+6-\left(\frac{2^3}{3}+2\right)$$
$$= 78-4\tfrac{2}{3} = 73\tfrac{1}{3}$$

Solved Exercises

1. Evaluate $\int_0^1 x^4\,dx$.
2. Find $\int_0^3 (t^2+3t)\,dt$.
3. Suppose that $v = f(t)$ is the velocity at time t of an object moving along a line. Using the fundamental theorem of calculus, interpret the integral $\int_a^b v\,dt = \int_a^b f(t)\,dt$.

Exercises

1. Find $\int_a^b s^4\,ds$.
2. Find $\int_{-1}^1 (t^4+t^{917})\,dt$.
3. Find $\displaystyle\int_0^{10} \left(\frac{t^4}{100}-t^2\right)\,dt$

Proof of the Fundamental Theorem

We will now give a complete proof of the fundamental theorem of calculus. The basic idea is as follows: Letting F be an antiderivative for f on $[a,b]$, we will show that if L_f and U_f are any lower and upper sums for f on $[a,b]$, then $L_f \leqslant F(b)-F(a) \leqslant U_f$. Since f is assumed to be integrable on $[a,b]$, the only number which can separate the lower sums from the upper sums in this way is the integral $\int_a^b f(t)\,dt$. It will follow that $F(b)-F(a)$ must equal $\int_a^b f(t)\,dt$.

To show that every lower sum is less than or equal to $F(b)-F(a)$, we must take any piecewise constant g on $[a,b]$ such that $g(t) \leqslant f(t)$ for all t in (a,b) and show that $\int_a^b g(t)\,dt \leqslant F(b)-F(a)$. Let $(t_0, t_1, \ldots, t_n)$ be a partition adapted to g and let k_i be the value of g on (t_{i-1}, t_i). Since $F' = f$, we have

$$k_i = g(t) \leqslant f(t) = F'(t)$$

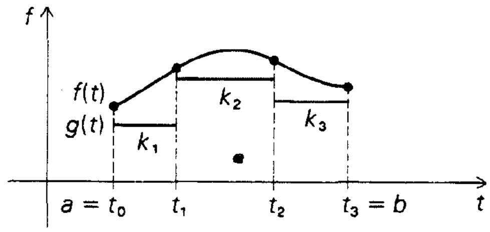

Fig. 12-1 A lower sum for the integral of f.

Hence

$$k_i \leqslant F'(t)$$

for all t in (t_{i-1}, t_i). (See Fig. 12-1). It follows from Corollary 1 of the mean value theorem (see p. 174) that

$$k_i \leqslant \frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}}$$

Hence

$$k_i \, \Delta t_i \leqslant F(t_i) - F(t_{i-1})$$

Summing from $i = 1$ to n, we get

$$\sum_{i=1}^{n} k_i \, \Delta t_i \leqslant \sum_{i=1}^{n} [F(t_i) - F(t_{i-1})]$$

The left-hand side is just $\int_a^b g(t)\,dt$, by the definition of the integral of a step function. The right-hand side is a telescoping sum equal to $F(t_n) - F(t_0)$. (See Fig. 12-2.) Thus we have

$$\int_a^b g(t)\,dt \leqslant F(b) - F(a)$$

which is what we wanted to prove.

In the same way (see Exercise 4), we can show that if $h(t)$ is a piecewise constant function such that $f(t) \leqslant h(t)$ for all t in (a, b), then

$$F(b) - F(a) \leqslant \int_a^b h(t)\,dt$$

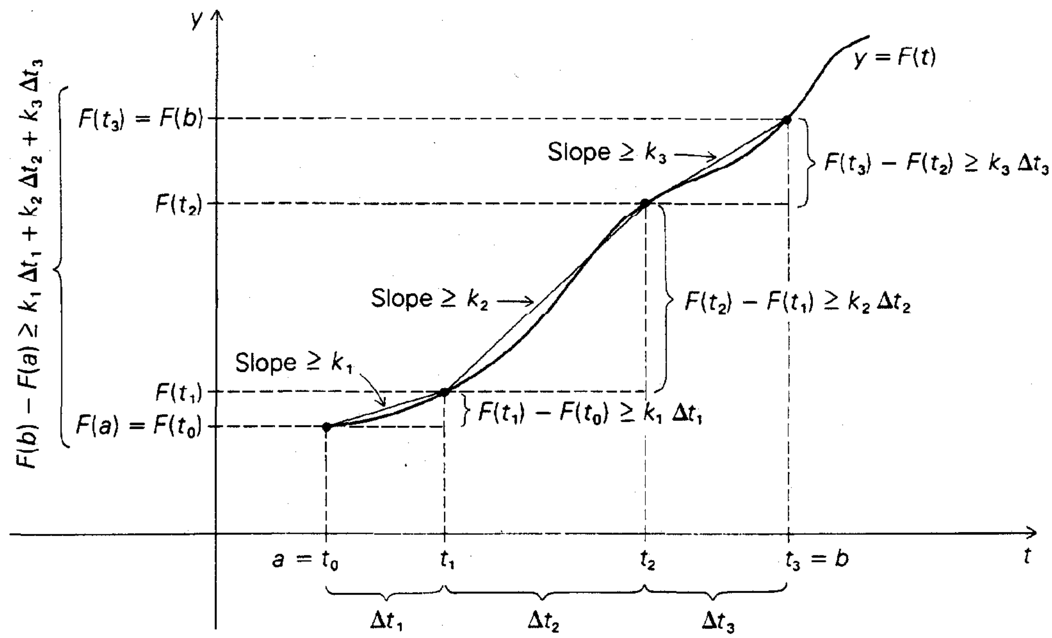

Fig. 12-2 The total change of F on $[a,b]$ is the sum of changes on the subintervals $[t_{i-1}, t_i]$. The change on the ith subinterval is at least $k_i \Delta t_i$.

as required. This completes the proof of the fundamental theorem.

Solved Exercises

4. Suppose that F is continuous on $[0,2]$, that $F'(x) < 2$ for $0 \leqslant x \leqslant \frac{1}{3}$, and that $F'(x) < 1$ for $\frac{1}{3} \leqslant x \leqslant 2$. What can you say about $F(2) - F(0)$?

Exercises

4. Prove that if $h(t)$ is a piecewise constant function on $[a,b]$ such that $f(t) \leqslant h(t)$ for all $t \in (a,b)$, then $F(b) - F(a) \leqslant \int_a^b h(t)\,dt$, where F is any antiderivative for f on $[a,b]$.

5. Let $a_0, \dots, a_n$ be any numbers and let $\delta_i = a_i - a_{i-1}$. Let $b_k = \Sigma_{i=1}^{h} \delta_i$ and let $d_i = b_i - b_{i-1}$. Express the b's in terms of the a's and the d's in terms of the δ's.

Alternative Version of the Fundamental Theorem

We have seen that the fundamental theorem of calculus enables us to compute integrals by using antiderivatives. The inverse relationship between integration and differentiation is completed by the following alternative version of the fundamental theorem, which enables us to build up an antiderivative for a function by taking definite integrals and letting the endpoint vary.

Theorem 2 Fundamental Theorem of Calculus: Alternative Version. *Let f be continuous on the interval I and let a be a number in I. Define the function F on I by*

$$F(t) = \int_a^t f(s)\,ds$$

Then $F'(t) = f(t)$; that is

$$\frac{d}{dt}\int_a^t f(s)\,ds = f(t)$$

In particular, every continuous function has an antiderivative.

Proof We use the method of rapidly vanishing functions from Chapter 3. We need to show that the function

$$r(t) = F(t) - F(t_0) - f(t_0)(t - t_0)$$

is rapidly vanishing at t_0. Substituting the definition of F and using additivity of the integral, we obtain

$$\begin{aligned} r(t) &= \int_a^t f(s)\,ds - \int_a^{t_0} f(s)\,ds - f(t_0)(t - t_0) \\ &= \int_{t_0}^t f(s)\,ds - f(t_0)(t - t_0) \end{aligned}$$

Given any number $\epsilon > 0$, there is an interval I about t_0 such that $f(t_0) - (\epsilon/2) < f(s) < f(t_0) + (\epsilon/2)$ for all s in I. (Here we use the continuity of f at t_0.) For $t > t_0$ in I, we thus have

$$\left(f(t_0) - \frac{\epsilon}{2}\right)(t - t_0) \leqslant \int_{t_0}^{t} f(s)\,ds \leqslant \left(f(t_0) + \frac{\epsilon}{2}\right)(t - t_0)$$

Subtracting $f(t_0)(t - t_0)$ everywhere gives

$$-\frac{\epsilon}{2}(t - t_0) \leqslant r(t) \leqslant \frac{\epsilon}{2}(t - t_0)$$

or

$$\left|r(t)\right| \leqslant \frac{\epsilon}{2}\left|t - t_0\right| \quad (\text{since for } t > t_0,\ |t - t_0| = t - t_0)$$

For $t < t_0$ in I, we have

$$\left(f(t_0) - \frac{\epsilon}{2}\right)(t_0 - t) \leqslant \int_{t}^{t_0} f(s)\,ds \leqslant \left(f(t_0) + \frac{\epsilon}{2}\right)(t_0 - t)$$

We may rewrite the integral as $-\int_{t_0}^{t} f(s)ds$. Subtracting $f(t_0)(t_0 - t)$ everywhere in the inequalities above gives

$$-\frac{\epsilon}{2}(t_0 - t) \leqslant -r(t) \leqslant \frac{\epsilon}{2}(t_0 - t)$$

so, once again,

$$\left|r(t)\right| \leqslant \frac{\epsilon}{2}\left|t - t_0\right| \quad (\text{since for } t \leqslant t_0,\ |t - t_0| = t_0 - t)$$

We have shown that, for $t \neq t_0$ in I, $|r(t)| \leqslant (\epsilon/2)|t - t_0|$; since $(\epsilon/2)\ |t - t_0| < \epsilon\,|t - t_0|$, Theorem 2 of Chapter 3 tells us that $r(t)$ is rapidly vanishing at t_0.

Since the proof used the continuity of f only at t_0, we have the following more general result.

Corollary *Let f be integrable on the interval $I = [a, b]$ and let t_0 be a number in (a, b). If f is continuous at the point t_0, then the function $F(t) = \int_a^t f(s)\,ds$ is differentiable at t_0, and $F'(t_0) = f(t_0)$.*

This corollary is consistent with the results of Problem 9, Chapter 11.

Solved Exercises

5. Suppose that f is continuous on the real line and that g is a differentiable function. Let $F(x) = \int_0^{g(x)} f(t)\,dt$. Calculate $F'(x)$.
6. Let $F(x) = \int_1^{x^2} (1/t)\,dt$. What is $F'(x)$?

Exercises

6. (a) Without using logarithms, show that $\int_1^{x^2} (1/t)\,dt = 2\int_1^x (1/t)\,dt$.

   (b) What is the relation between $\int_1^{x^n}(1/t)\,dt$ and $\int_1^x (1/t)\,dt$?
7. Prove a fundamental theorem for $G(t) = \int_t^b f(s)\,ds$.
8. Find $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt.$

Problems for Chapter 12

1. Evaluate the following definite integrals:

   (a) $\int_1^3 t^3\,dt$ (b) $\int_{-1}^2 (t^4 + 8t)\,dt$

   (c) $\int_0^{-4} (1 + x^2 - x^3)\,dx$ (d) $\int_1^2 4\pi r^2\,dr$

   (e) $\int_2^{-1} (1 + t^2)^2\,dt$
2. If f is integrable on $[a, b]$, show by example that $F(t) = \int_a^t f(t)\,dt$ is continuous but need not be differentiable.
3. Evaluate:

   (a) $(d/dt)\int_0^t 3/(x^4 + x^3 + 1)^6\,dx$ (b) $(d/dt)\int_t^3 x^2(1 + x)^5\,dx$

   (c) $\dfrac{d}{dt}\displaystyle\int_t^4 \frac{u^4}{(u^2 + 1)^3}\,du$
4. Let f be continuous on the interval I and let a_1 and a_2 be in I. Define the functions:

$$F_1(t) = \int_{a_1}^{t} f(s)\,ds \quad \text{and} \quad F_2(t) = \int_{a_2}^{t} f(s)\,ds$$

   (a) Show that F_1 and F_2 differ by a constant.

   (b) Express the constant $F_2 - F_1$ as an integral.

5. Suppose that

$$f(t)=\begin{cases} t^2 & 0\leqslant t\leqslant 1 \\ 1 & 1\leqslant t<5 \\ (t+6)^2 & 5\leqslant t\leqslant 6 \end{cases}$$

   (a) Draw a graph of f on the interval $[0,6]$.
   (b) Find $\int_0^6 f(t)\,dt$.
   (c) Find $\int_0^6 f(x)\,dx$.
   (d) Let $F(t) = \int_0^t f(s)\,ds$. Find a formula for $F(t)$ in $[0,6]$ and draw a graph of F.
   (e) Find $F'(t)$ for t in $(0,6)$.

6. Prove Theorem 2 without using rapidly vanishing functions, by showing directly that $f(t_0)$ is a transition point between the slopes of lines overtaking and overtaken by the graph of F at t_0.

7. (a) Find $(d/dx)\int_1^{ax}(1/t)\,dt$, where a is a positive constant.
   (b) Show that $\int_1^{ax}(1/t)\,dt - \int_1^x(1/t)\,dt$ is a constant.
   (c) What is the constant in (b)?
   (d) Show without using logarithms that if $F(x)=\int_1^x(1/t)\,dt$, then $F(xy)=F(x)+F(y)$.
   (e) Show that if $F_c(x)=\int_1^x(c/t)\,dt$, where c is a constant, then $F_c(xy)=F_c(x)+F_c(y)$, both with and without using logarithms.