2 Transitions and Derivatives

In this chapter we reformulate the definition of the derivative in terms of the concept of transition point. Other concepts, such as change of sign, can also be expressed in terms of transitions. In addition to the new language, a few new basic properties of change of sign and overtaking will be introduced.

In what follows the reader is assumed to be familiar with the interval notation and the containment symbol $\in$. Thus $x \in (a,b]$ means $a < x \leqslant b$, $x \in (-\infty,b)$ means $x < b$, etc. Intervals of the form (a,b) are called open, while those of the form $[a,b]$ are called closed.

Transition Points

Some changes are sudden, or definitive, and are marked by a transition point. The time of sunrise marks the transition from night to day, and the summer solstice marks the transition from spring to summer. Not all transitions take place in time, though. For example, let T denote the temperature of some water. For certain values of T, the water is in a liquid state; for other values of T, the water is in a solid state (ice) or a gaseous state (water vapor). Between these states are two transition temperatures, the freezing point and the boiling point.

Here is another example. A tortoise and a hare are running a race. Let T denote the period of time during which the tortoise is in the lead; let H denote the period of time during which the hare is ahead. A moment at which the hare overtakes the tortoise is a transition point from T to H. When the tortoise overtakes the hare, the transition is from H to T.

In order to do mathematics with the concept of transition, we must give a formal definition. The following definition has been chosen for its intuitive content and for technical convenience.

Definition Let A and B be two sets of real numbers. A number x_0 is called a *transition point from A to B* if there is an open interval I containing x_0 such that

1. If $x \in I$ and $x < x_0$, then x is in A but not in B.
2. If $x \in I$ and $x > x_0$, then x is in B but not in A.

There are several remarks to be made concerning this definition. The first one concerns definitions in general; the others concern the particular definition above.

Remark 1 Definitions play an important role in mathematics. They set out in undisputable terms what is meant by a certain phrase, such as "x_0 is a transition point from A to B." Definitions are usually made to reflect some intuitive idea, and our intuition is usually a reliable guide to the use of the defined expression. Still, if we wish to establish that "x_0 is a point of transition from A to B" in a given example, the definition is the final authority; we must demonstrate that the conditions set out in the definition are met. Partial or approximate compliance is not acceptable; the conditions must be met fully and exactly.

There is no disputing the correctness of a definition, but only its usefulness. Over a long period of mathematical history, the most useful definitions have survived. Thousands have been discarded as inappropriate or useless.

Remark 2 There are a couple of specific points to be noted in the definition above. First of all, we do not specify to what set the point x_0 itself belongs. It may belong to A, B, both, or neither. The second thing to notice is the role of the interval I. Its inclusion in the definition corresponds to the intuitive notion that transitional change may be temporary. For instance, in the transition from ice to water, the interval I must be chosen so that its right-hand endpoint is less than or equal to the boiling point. Reread the definition now to be sure that you understand this remark.

Remark 3 In the example of the tortoise and the hare, suppose that the tortoise is behind for $t < t_1$, that they run neck and neck from t_1 to t_2, and that the tortoise leads for $t > t_2$. There is no transition *point* in this case, but rather a transition *period*. In our definition, we are only concerned with transition points. In nonmathematical situations, it is not always clear when a transition is abrupt and when it occurs over a period. Consider, for example, the transition of power from one government to another, or the transition of an embryo from pre-life to life.

Worked Example 1 Let A be the set of real numbers r for which the point $(1,2)$ lies *outside* the circle $x^2 + y^2 = r^2$. Let B be the set of r for which $(1,2)$ lies inside $x^2 + y^2 = r^2$. Find the transition point from A to B. Does it belong to A, B, both, or neither?

Solution The point $(1,2)$ lies at a distance $\sqrt{1^2 + 2^2} = \sqrt{5}$ from the origin, so $A = (0,\sqrt{5})$ and $B = (\sqrt{5},\infty)$. The transition point is $\sqrt{5}$, which belongs to neither A nor B. (The point $(1,2)$ lies *on* the circle of radius $\sqrt{5}$, not inside or outside it.)

If A and B are intervals, then a transition point from A to B must be a common endpoint. (See Fig. 2-1.) Notice that, when A and B are intervals, there is at most one transition point between them, which may or may not belong to A or B.

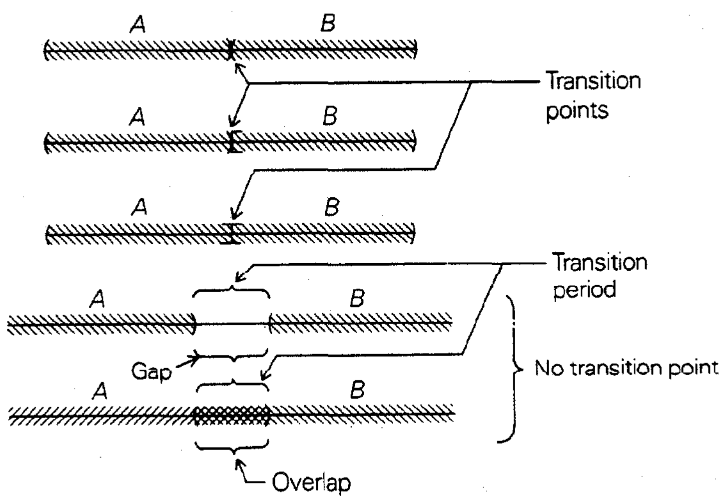

Fig. 2-1 If A and B are intervals, transition points are common endpoints.

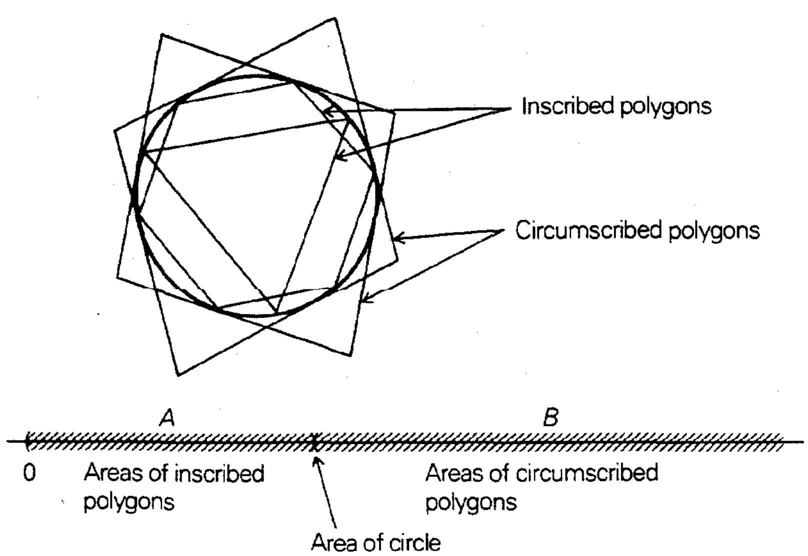

Fig. 2-2 The area of a circle can be described as a transition point.

Some transitions occur along curves or surfaces. For example, the coastline is the transition curve between land and sea, while your skin is the transition surface between your body and the atmosphere. We give a mathematical example of a transition curve in Chapter 6, but we will usually be dealing with transition points.

The notion of transition occurs in the answer given by the ancient Greeks to the question, "What is the area enclosed by a circle of radius r?" We may consider the set A consisting of the areas of all possible polygons inscribed in the circle and the set B consisting of the areas of all possible circumscribed polygons. The transition point from A to B is the area of the circle. (See Fig. 2-2.) By using inscribed and circumscribed regular polygons with sufficiently many sides, Archimedes was able to locate the transition point quite accurately.

Solved Exercises*

1. Let B be the set consisting of those x for which $x^2 - 1 > 0$, and let $A = (-1, 1]$. Find the transition points from B to A and from A to B.
2. Let $A = (-\infty, -1/1000)$, $B = (1/1000, \infty)$, $C = [-1/1000, 1/1000]$. Find the transition points from each of these sets to each of the others.
3. Let x be the distance from San Francisco on a road crossing the United States. If A consists of those x for which the road is in California at mile x, and B consists of those x for which the road is in Nevada at mile x, what is the transition point called?

Exercises

1. Describe the following as transition points:

   (a) Vernal equinox.
   (b) Entering a house.
   (c) A mountain top.
   (d) Zero.
   (e) Outbreak of war.
   (f) Critical mass.
   (g) A window shattering.
   (h) Revolution.

2. What transition points can you identify in the following phenomena? Describe them.

   (a) The movement of a pendulum.
   (b) Diving into water.
   (c) A traffic accident.
   (d) Closing a door.
   (e) Drinking a glass of water.
   (f) Riding a bicycle.

*Solutions appear in the Appendix.

3. For each of the following pairs of functions, $f(x)$ and $g(x)$, let A be the set of x where $f(x) > g(x)$, and let B be the set of x where $f(x) < g(x)$. Find the transition points, if there are any, from A to B and from B to A.

   (a) $f(x) = 2x - 1$; $g(x) = -x + 2$ (b) $f(x) = x^2 + 2$; $g(x) = 3x + 6$

   (c) $f(x) = x^3 - x$; $g(x) = x$ (d) $f(x) = x^2 - 1$; $g(x) = -x^2 + 1$

4. Find the transition points from A to B and from B to A in each of the following cases:

   (a) $A = [1,3)$
   $B =$ the set of x for which $x^2 - 4x + 3 > 0$

   (b) $A =$ the set of x for which $-3 < x \leqslant 1$ or $10 < x \leqslant 15$
   $B =$ the set of x for which $x < -3$, $0 \leqslant x \leqslant 10$, or $16 < x$

Change of Sign and Overtaking

The concept of change of sign was defined in Chapter 1. Now we express it in terms of transitions. Let f be any function, N the set of x for which x is in the domain of f and $f(x) < 0$, and P the set of x for which x is in the domain of f and $f(x) > 0$.

Theorem 1 *(a) x_0 is a point of transition from N to P if and only if f changes sign from negative to positive at x_0. (b) x_0 is a point of transition from P to N if and only if f changes sign from positive to negative at x_0.*

Proof (a) Suppose that x_0 is a transition point from N to P. By definition, there is an open interval I containing x_0 such that (i) if $x \in I$ and $x < x_0$, then x is in N but not in P, and (ii) if $x \in I$ and $x > x_0$, then x is in P but not in N. Letting $I = (a, b)$ and noting that $x \in (a, b)$ and $x < x_0$ is the same as saying $a < x < x_0$, we see that (i) reads: if $a < x < x_0$ then x is in the domain of f and $f(x) < 0$; and (ii) reads: if $x_0 < x < b$ then x is in the domain of f and $f(x) > 0$. Thus, by the definition of change of sign given in Chapter 1, f changes sign from negative to positive. Conversely, we can reverse this argument to show that if f changes sign from negative to positive, then x_0 is a transition point from P to N.

The proof of (b) is similar.

Let us return to the race between the tortoise and the hare. Denote by $f(t)$ the tortoise's position at time t and by $g(t)$ the hare's position at time t. The transition "*the tortoise overtakes the hare at time t_0*" means that there is an open interval I about t_0 such that:

1. If $t \in I$ and $t < t_0$, then $f(t) < g(t)$.
2. If $t \in I$ and $t > t_0$, then $f(t) > g(t)$.

If we graph f and g, this means that the graph of f lies below that of g for t just to the left of t_0 and above that of g for t just to the right of t_0. (See Fig. 2-3.) Notice the role of the interval I. (We could have taken a slightly larger one.) It appears in the definition because the hare may overtake the tortoise at a later time t_1.

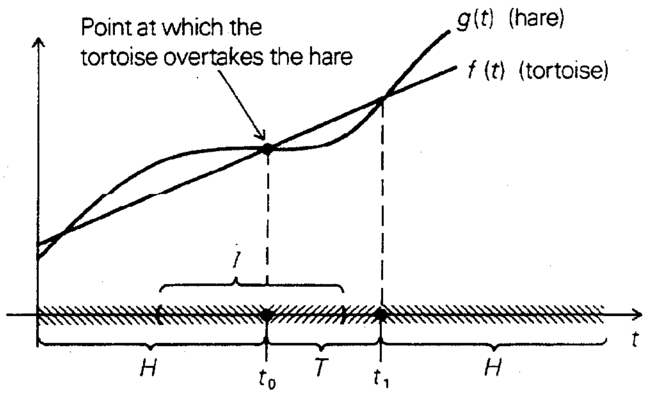

Fig. 2-3 The tortoise overtakes the hare at t_0 and the hare overtakes the tortoise at t_1.

We now state the definition of overtaking for general functions.

Definition Let f and g be two functions, A the set of x (in the domain of f and of g) such that $f(x) < g(x)$, and B the set of x (in the domain of f and g) such that $f(x) > g(x)$. If x_0 is a transition point from A to B, we say that f *overtakes* g at x_0.

In other words, f overtakes g at x_0 if there is an interval I containing x_0 such that (f and g are defined on I, except possibly at x_0) and

1. For x in I and $x < x_0$, $f(x) < g(x)$.

2. For x in I and $x > x_0$, $f(x) > g(x)$.

We call an open interval I about x_0 for which conditions 1 and 2 are true *an interval which works* for f and g at x_0; i.e., I is small enough so that in I to the left of x_0, f is below g, while in I to the right of x_0, f is above g. (See Fig. 2-4.) Clearly, if a certain interval I works for f and g at x_0, so does any open interval J contained in I, as long as it still contains x_0.

If the tortoise and hare both fall asleep and start running the wrong way when they wake up, we still say that "the tortoise overtakes the hare" if the hare passes the tortoise when going in the wrong direction. (See Fig. 2-5.) In the general situation, when f overtakes g at x_0, the graph of f may actually be going downward. It is only the change in f as compared with the change in g which is important.

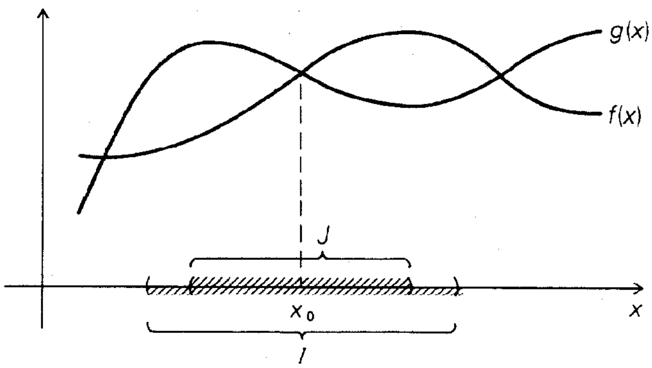

Fig. 2-4 If I is an interval that works (for f overtaking g at x_0), so is any smaller interval J.

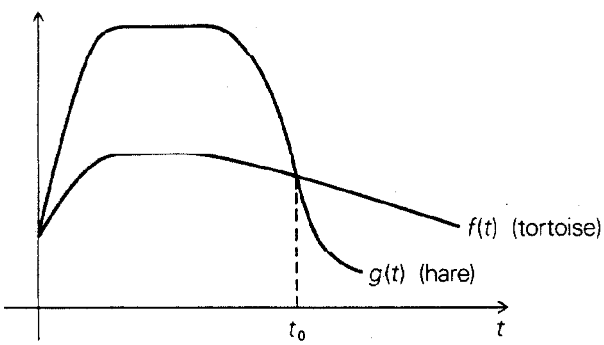

Fig. 2-5 The tortoise overtakes the hare (f overtakes g) at t_0.

Worked Example 2 For the functions f and g in Fig. 2-6, tell whether f overtakes g, g overtakes f; or neither, at each of the points x_1, x_2, x_3, x_4, and x_5. When overtaking takes place, indicate an interval which works.

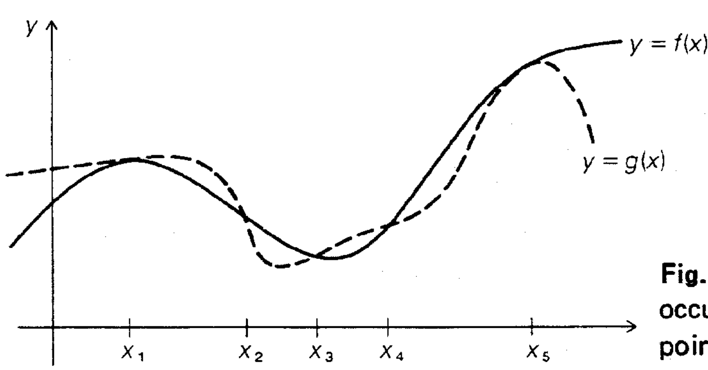

Fig. 2-6 What overtakings occur at the indicated points?

Solution Neither function overtakes the other at x_1. In fact, in the interval $(0,x_2)$ about x_1, $g(x) > f(x)$ for x both to the right and to the left of x_1. At x_2, f overtakes g; an interval which works is $I = (x_1, x_3)$. (Note that the number x_2 is what we called x_0 in the definition.) At x_3, g overtakes f; an interval which works is (x_2, x_4). At x_4, f overtakes g again; an interval which works is (x_3, x_5). Neither function overtakes the other at x_5.

The relations between the concepts of overtaking and change of sign are explored in Problems 8 and 9 at the end of the chapter.

If, while the tortoise is overtaking the hare at t_0, a snail overtakes the tortoise at t_0, then we may conclude that the snail overtakes the hare at t_0. Let us state this as a theorem about functions.

Theorem 2 Suppose f, g, and h are functions such that f overtakes g at x_0 and g overtakes h at x_0. Then f overtakes h at x_0.

Proof Let I_1 be an interval which works for f and g at x_0, and let I_2 be an interval which works for g and h at x_0. That these intervals exist follows from the assumptions of the theorem. Choose I to be any open interval about x_0 which is contained in both I_1 and I_2. For instance, you could choose I to consist of all points which belong to both I_1 and I_2. (Study Fig. 2-7, where a somewhat smaller interval is chosen. Although the three graphs intersect in a complicated way, notice that the picture looks quite simple in the shaded region above the interval I. You should return to the figure frequently as you read the rest of this proof.)

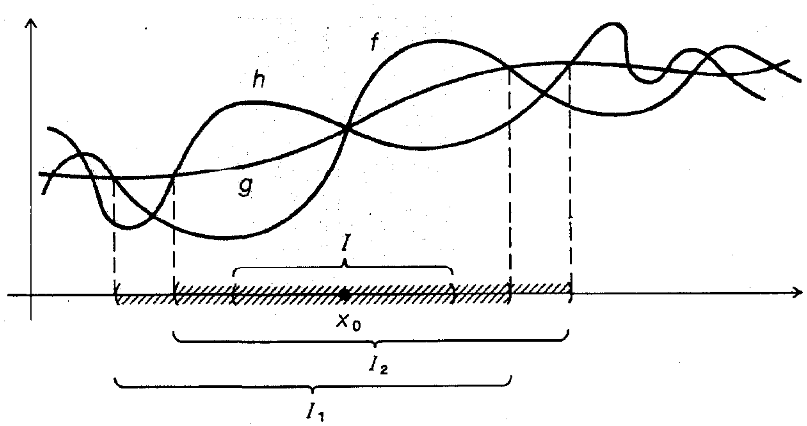

Fig. 2-7 f overtakes g and g overtakes h at x_0.

We will show that the interval I works for f and h. We begin by assuming that

$$x \in I \text{ and } x < x_0 \tag{A}$$

Since I is contained in both I_1 and I_2, (A) implies:

$$x \in I_1 \quad \text{and} \quad x < x_0 \tag{A$_1$}$$
$$x \in I_2 \quad \text{and} \quad x < x_0 \tag{A$_2$}$$

Since I_1 works for f and g at x_0 and I_2 works for g and h at x_0, $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ imply

$$f(x) < g(x) \text{ (and } x \text{ is in the domain of } f \text{ and } g) \qquad (\mathbf{B}_1)$$
$$g(x) < h(x) \text{ (and } x \text{ is in the domain of } g \text{ and } h) \qquad (\mathbf{B}_2)$$

$(\mathbf{B}_1)$ and $(\mathbf{B}_2)$ together imply that

$$f(x) < h(x) \text{ (and } x \text{ is in the domain of } f \text{ and } h) \qquad (\mathbf{B})$$

This chain of reasoning began with **(A)** and concluded with **(B)**, so we have proven that if $x \in I$ and $x < x_0$, then $f(x) < h(x)$. Similarly, one proves that if $x \in I$ and $x > x_0$ then $f(x) > h(x)$, so I works.

The next result provides a link between the concepts of linear change and transition. We know that a faster object overtakes a slower one when they meet. Here is the formal version of that fact for uniform motion. Its proof is a good exercise in the algebra of inequalities.

Theorem 3 *Let $f_1(x) = m_1x + b_1$ and $f_2(x) = m_2x + b_2$ be linear functions whose graphs both pass through the point (x_0, y_0). If $m_2 > m_1$, then f_2 overtakes f_1 at x_0.*

Proof Since $y_0 = f_1(x_0) = m_1x_0 + b_1$, we can solve for b_1 to get $b_1 = y_0 - m_1x_0$. Substituting this into the formula for $f_1(x)$, we have $f_1(x) = m_1x + y_0 - m_1x_0$, which we can rewrite as $f_1(x) = m_1(x - x_0) + y_0$. Similarly, we have $f_2(x) = m_2(x - x_0) + y_0$. From Fig. 2-8, we guess that it is possible to take $I = (-\infty, \infty)$. To finish the proof, we must show that $x < x_0$ implies $f_2(x) < f_1(x)$ and that $x > x_0$ implies $f_2(x) > f_1(x)$.

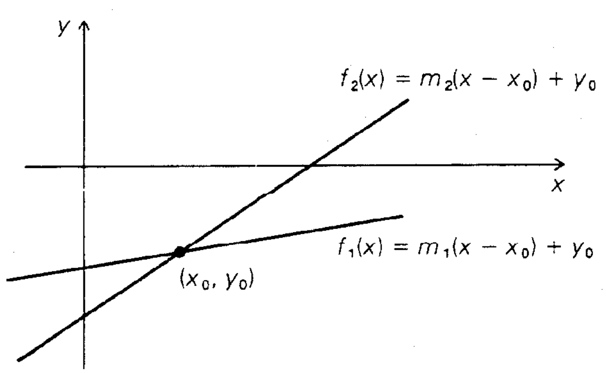

Fig. 2-8 f_1 and f_2 both pass through (x_0, y_0) and f_2 has larger slope.

Assume that $x < x_0$. Then $x - x_0 < 0$. Since $m_2 - m_1 > 0, x - x_0$ and $m_2 - m_1$ have opposite signs so that

$$(m_2 - m_1)(x - x_0) < 0$$

so

$$m_2(x - x_0) - m_1(x - x_0) < 0$$

so

$$m_2(x - x_0) < m_1(x - x_0)$$

and

$$m_2(x - x_0) + y_0 < m_1(x - x_0) + y_0$$

which says exactly that $f_2(x) < f_1(x)$.

If $x > x_0$, we have $x - x_0 > 0$, so $(m_2 - m_1)(x - x_0) > 0$. A chain of manipulations like the one above (which you should write out yourself) leads to the conclusion $f_2(x) > f_1(x)$. That finishes the proof.

Terminology It is useful to be able to speak of graphs overtaking one another. If G_1 and G_2 are curves in the plane which are the graphs of functions $f_1(x)$ and $f_2(x)$, we will sometimes say that G_2 overtakes G_1 at a point when what is really meant is that f_2 overtakes f_1. Thus, Theorem 3 may be rephrased as follows (see Fig. 2-9):

> If the line l_1 with slope m_1 meets the line l_2 with slope m_2 at (x_0, y_0), and if $m_2 > m_1$, then l_2 overtakes l_1 at x_0.

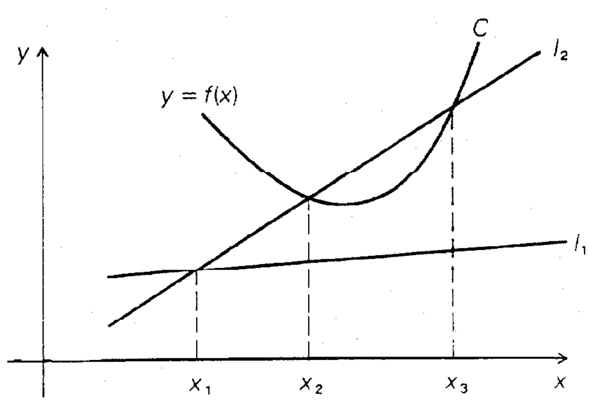

Fig. 2-9 l_2 overtakes l_1 at x_1; l_2 overtakes C at x_2; C overtakes l_2 at x_3.

(This statement only applies to those lines which are graphs of functions—we may not speak of a vertical line overtaking or being overtaken by anything.)

Solved Exercises

4. Where does $1/x$ change sign?
5. Let $f(x) = 3x + 2$, $g(x) = x + 2$. Show in two ways that f overtakes g at 0: (a) by the definition of overtaking; (b) by Theorem 3.
6. Let $f(x) = -3$ and $g(x) = -x^2$. At what point or points does f overtake g? Construct an interval which works. Sketch.
7. Let $f(x) = 2x^2$ and $g(x) = 5x - 3$. Show that g overtakes f at $x = 1$. Is $(-\infty, \frac{5}{4})$ an interval which works? Is it the largest one?

Exercises

5. Describe the change of sign at $x = 0$ of the function $f(x) = mx$ for various values of m. Can you find a transition point on the "m axis" where a certain change takes place?
6. Let $f(x) = x^2 - 2x - 3$ and $g(x) = 2x - 2$. Where does g overtake f? Find an interval which works. Sketch.
7. Let $f(x) = x^3 - x$ and $g(x) = 2x$. At what point or points does f overtake g? Does $(-3, 0)$ work for any of these points? If not, why not? Sketch.
8. Let $f(x) = -x^2 + 4$ and $g(x) = 3x - 2$. At what point or points does f overtake g? Find the largest interval which works. Sketch.
9. Let $f(x) = 1/(1 - x)$ and $g(x) = -x + 1$.
   (a) Show that f overtakes g at $x = 0$. Find an interval which works. Sketch.
   (b) Show that f overtakes g at $x = 2$. Is $(1, 3)$ an interval which works? What is the largest interval which works?

The Derivative

The derivative was defined in Chapter 1. To rephrase that definition using the language of transitions, we shall use the following terminology. Let f be a function whose domain contains an open interval about x_0. Let A be the set of numbers m such that the linear function $f(x_0) + m(x - x_0)$ (whose graph is the line through $(x_0, f(x_0))$ with slope m) is overtaken by f at x_0. Let B be the set of numbers m such that the linear function $f(x_0) + m(x - x_0)$ overtakes f at x_0.

Theorem 4 *A number m_0 is a transition point from A to B if and only if m_0 is the derivative of f at x_0.*

Proof First, suppose that m_0 is a transition point from A to B. Thus there is an open interval I about m_0 such that (i) if $m_1 \in I$ and $m_1 < m_0$, then m_1 is in A but not in B, and (ii) if $m_1 \in I$ and $m_1 > m_0$, then m_1 is in B but not in A. Let $m < m_0$. Choose $m_1 \in I$ such that $m \leqslant m_1 < m_0$. (Why can we do this?) By (i), the function $f(x_0) + m_1(x - x_0)$ is overtaken by f at x_0. By Theorems 2 and 3, $f(x_0) + m(x - x_0)$ is also overtaken by f at x_0; i.e. $f(x) - [f(x_0) + m(x - x_0)]$ changes sign from negative to positive at x_0. This gives condition 1 of the definition of the derivative, and condition 2 is proved in the same way.

Conversely, if m_0 is the derivative of f at x_0, then the definition of the derivative shows that $A = (-\infty, m_0)$ and $B = (m_0, \infty)$, and so m_0 is the transition point from A to B.

In Chapter 1 we proved that the derivative is unique if it exists. In the present terminology, this means that there is at most one transition point from A to B. (In Problems 18-20, the reader is invited to prove this directly from the definition of A and B).

Solved Exercise

8. Let $f(x) = x^2 + \frac{3}{2}x + 2$ and $x_0 = 2$. Construct the sets A and B and use them to calculate $f'(0)$.

Exercises

10. Calculate the derivative of $f(x) = x^2 - x$ at $x = 1$ using Theorem 4.

11. For each of the following functions, find the sets A and B involved in Theorem 4 and show that the derivative does not exist at the specific point:

(a) $f(x) = \left\{ \begin{array}{ll} x & \text{if } x \leqslant 1 \\ 2x - 1 & \text{if } x > 1 \end{array} \right\}, \qquad x_0 = 1$

(b) $f(x) = \begin{cases} 2x+4 & \text{if } x \leqslant -1 \\ 2x+3 & \text{if } x > -1 \end{cases}, \qquad x_0 = -1$

(c) $f(x) = \begin{cases} x^2 & \text{if } x \leqslant 0 \\ x & \text{if } x > 0 \end{cases}, \qquad x_0 = 0$

12. Is the following an acceptable definition of the tangent line? "The tangent line through a point on a graph is the one line which neither overtakes nor is overtaken by the graph." If so, discuss. If not, give an example.

Problems for Chapter 2

1. Describe the following as transition points.
   (a) Turning on a light.
   (b) 100° Centigrade.
   (c) An aircraft landing.
   (d) Signing a contract.

2. What transition points can you identify in the following phenomena?
   (a) Breathing.
   (b) A heart beating.
   (c) Blinking.
   (d) Walking.
   (e) A formal debate.
   (f) Firefighting.
   (g) Marriage.
   (h) Receiving exam results.
   (i) Solving homework problems.

3. Let A be the set of areas of *triangles* inscribed in a circle of radius 1, B the set of areas of circumscribed triangles. Is there a transition point from A to B? (You may assume that the largest inscribed and smallest circumscribed triangles are equilateral.) What happens if you use quadrilaterals instead of triangles? Octagons?

4. For each of the following pairs of sets, find the transition points from A to B and from B to A.
   (a) $A = (0,1)$; $B = (-\infty, \frac{1}{2})$ and $(1,3)$.
   (b) $A =$ those x for which $x^2 < 3$; $B =$ those x for which $x^2 > 3$.
   (c) $A =$ those x for which $x^3 \leqslant 4$; $B =$ those x for which $x^3 \geqslant 4$.
   (d) $A =$ those a for which the equation $x^2 + a = 0$ has two real roots;
   $B =$ those a for which the equation $x^2 + a = 0$ has less than two real roots.
   (e) $A =$ those x for which $\sqrt{x^2 - 4}$ exists (as a real number):
   $B =$ those x for which $\sqrt{x^2 - 4}$ does not exist (as a real number).

5. For which values of n (positive and negative) does x^n change sign at 0?

6. For each of the following pairs of functions, find:
   1. The set A where $f(x) < g(x)$.
   2. The set B where $f(x) > g(x)$.
   3. The transition points from A to B and from B to A.
   4. Intervals which work for f overtaking g and g overtaking f.

   Make a sketch in each case.

   (a) $f(x) = 7x + 2$; $g(x) = 2x - 4$
   (b) $f(x) = x^2 + 2x + 2$; $g(x) = -x + 3$
   (c) $f(x) = -2x^2 + 4x - 3$; $g(x) = 2x - 5$
   (d) $f(x) = -x^2 + 2x + 1$; $g(x) = x^2 - 1$
   (e) $f(x) = -x^3 + 4x$; $g(x) = x^2 - 2x$
   (f) $f(x) = 1/x^2$ $(x \neq 0)$; $g(x) = -x^2 + 1$
   (g) $f(x) = -1/x^2$ $(x \neq 0)$; $g(x) = [1/(x+2)] - 2$ $(x \neq -2)$

7. For which values of m does $f(x) = m(x - 1) + 1$ overtake $g(x) = x^2$ at 1?

8. Show that f changes sign at x_0 if and only if f overtakes or is overtaken by the zero function ($g(x) = 0$ for all x) at x_0. (This problem and the next one show that the concepts of overtaking and sign change can be defined in terms of one another.)

9. Let f and g be functions, and define h by $h(x) = f(x) - g(x)$, for all x such that $f(x)$ and $g(x)$ are both defined. Prove that f overtakes g at x_0 if and only if h changes sign from negative to positive at x_0.

In Problems 10 to 17, let

$f(x) = 2x^2 - 5x + 2$, $\quad g(x) = \frac{3}{4}x^2 + 2x$,
$h(x) = -3x^2 + x + 3$, $\quad k(x) = 3x - 4$, and
$l(x) = -2x + 3$

10. Find the derivative of $f(x) + g(x)$ at $x = 1$.

11. Find the derivative of $3f(x) - 2h(x)$ at $x = 0$.

12. Find the equation of the tangent line to
    (a) $f(x)$ at $x = 1$ (b) $g(x)$ at $x = -2$
    (c) $h(x)$ at $x = 100$ (d) $k(x)$ at $x = -10^8$ (sketch)

13. Where does $l(x)$ overtake $k(x)$?

14. Where does $l(x)$ overtake the tangent line to $h(x)$ at $x = -1$?

15. For what real number c is the line $y = ck(x)$ parallel to the tangent line to $f(x)$ at $x = 2$?

16. Where does $g(x) + l(x)$ overtake $f(x) + k(x)$? What are the derivatives of $g + l$ and $f + k$ at these points?

17. Let $f(t) = 2t^2 - 5t + 2$ be the position of object A, and let $h(t) = -3t^2 + t + 3$ be the position of object B.
    (a) When is A moving faster than B?
    (b) How fast is B going when A stops?
    (c) When does B turn around?
18. A set of numbers S is called *convex* if, whenever x_1 and x_2 lie in S and $x_1 < y < x_2$, then y lies in S too. Prove that the sets A and B defined in the preamble to Theorem 4 are convex.
19. Let A and B be sets of real numbers such that A is convex (see Problem 18). Prove that there is *at most* one point of transition from A to B.
20. Use Problems 18 and 19 to prove the uniqueness of derivatives.
21. (a) Find sets A, B, and C such that there are transition points from A to B, from B to C, and from C to A.
    (b) Prove that this cannot happen if A, B, and C are convex (see Problem 18).
22. Let $f(x)$ and $g(x)$ be functions which are differentiable at x_0.
    (a) Prove that, if $f'(x_0) > g'(x_0)$, then $f(x)$ overtakes $g(x)$ at x_0. [*Hint*: Use a line with slope in the interval $(g'(x_0), f'(x_0))$].
    (b) Prove that, if $f'(x_0) < g'(x_0)$, then $f(x)$ is overtaken by $g(x)$ at x_0.
    (c) Give examples to show that, if $f'(x_0) = g'(x_0)$, then it is possible that $f(x)$ overtakes $g(x)$ at x_0, or $f(x)$ is overtaken by $g(x)$ at x_0, or neither.
    (d) Solve Problem 6 by using (a) and (b).
23. Explain how Fig. 2-10 illustrates the definition of the derivative of $y = x^2$.

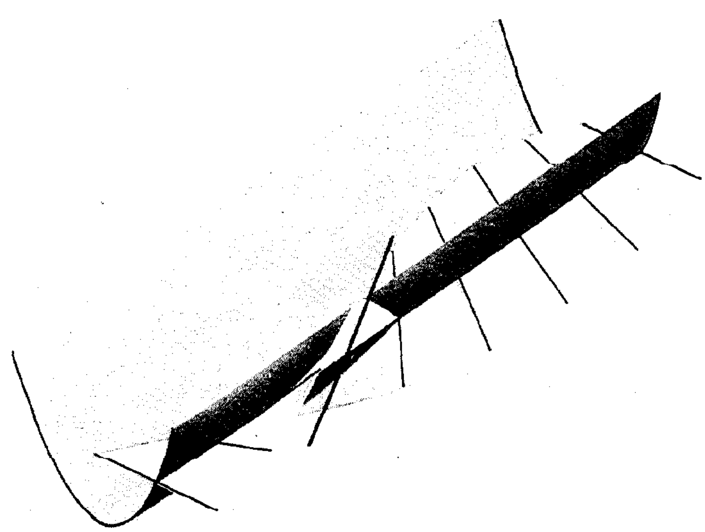

Fig. 2-10 This illustrates the derivative?