3 Algebraic Rules of Differentiation

We shall now prove the sum, constant multiple, product, and quotient rules of differential calculus. (Practice with these rules must be obtained from a standard calculus text.) Our proofs use the concept of "rapidly vanishing functions" which we will develop first. (The reader will be assumed to be familiar with absolute values.)

Rapidly Vanishing Functions

We say that a function f *vanishes at* x_0, or that x_0 is a *root of* f, if $f(x_0) = 0$. If we compare various functions which vanish near a point, we find that some vanish "more rapidly" than others. For example, let $f(x) = x - 2$, and let $g(x) = 20(x - 2)^2$. They both vanish at 2, but we notice that the derivative of g vanishes at 2 as well, while that of f does not. Computing numerically, we find

$f(3) = 1$	$g(3) = 20$
$f(1) = -1$	$g(1) = 20$
$f(2.1) = 0.1$	$g(2.1) = 0.2$
$f(1.9) = -0.1$	$g(1.9) = 0.2$
$f(2.01) = 0.01$	$g(2.01) = 0.002$
$f(1.99) = -0.01$	$g(1.99) = 0.002$
$f(2.001) = 0.001$	$g(2.001) = 0.00002$
$f(1.999) = -0.001$	$g(1.999) = 0.00002$
$f(2.0001) = 0.0001$	$g(2.0001) = 0.0000002$
$f(1.9999) = -0.0001$	$g(1.9999) = 0.0000002$

As x approaches 2, $g(x)$ appears to be dwindling away more rapidly than $f(x)$. Guided by this example, we make the following definition.

Definition We say that a function $r(x)$ *vanishes rapidly* at x_0 if $r(x_0) = 0$ and $r'(x_0) = 0$.

The following theorem shows that rapidly vanishing functions, as we have defined them, really do vanish quickly. This theorem will be useful in proving

properties of rapidly vanishing functions, as well as in establishing the connection between transitions and limits (see Chapter 13).

Theorem 1 *Let r be a function such that $r(x_0) = 0$. Then $r(x)$ vanishes rapidly at $x = x_0$ if and only if, for every positive number ϵ, there is an open interval I about x_0 such that, for all $x \neq x_0$ in I, $|r(x)| < \epsilon|x - x_0|$.*

Proof The theorem has two parts. First of all, suppose that $r(x)$ vanishes rapidly at x_0, and that ϵ is a positive number. We will find an interval I with the required properties. Since $-\epsilon < r'(x_0) = 0 < \epsilon$, the line through $(x_0, r(x_0))$ with slope $-\epsilon$ [slope ϵ] is overtaken by [overtakes] the graph of f at x_0. Since $r(x_0) = 0$, the equations of these two lines are $y = -\epsilon(x - x_0)$ and $y = \epsilon(x - x_0)$. (Refer to Fig. 3-1.)

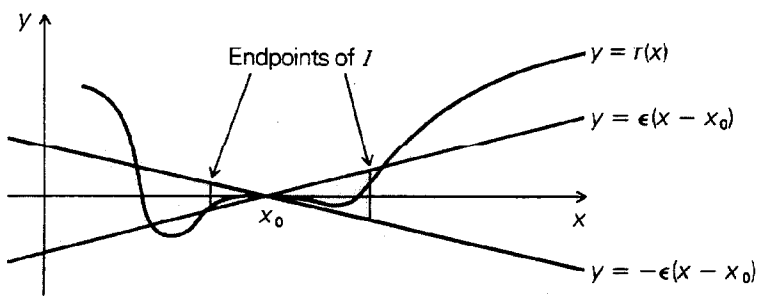

Fig. 3-1 The graph $y = r(x)$ is squeezed between the lines $y = -\epsilon(x - x_0)$ and $y = \epsilon(x - x_0)$.

Let I_1 be an interval which works for $r(x)$ overtaking $-\epsilon(x - x_0)$ at x_0, and I_2 an interval which works for $\epsilon(x - x_0)$ overtaking $r(x)$ at x_0. Choose an open interval I containing x_0 and contained in both I_1 and I_2.

For $x < x_0$ and x in I, we have $r(x) < -\epsilon(x - x_0)$, since $r(x)$ overtakes $-\epsilon(x - x_0)$, and $r(x) > \epsilon(x - x_0)$, since $\epsilon(x - x_0)$ overtakes $r(x)$. We may rewrite these two inequalities as $-\epsilon(x_0 - x) < r(x) < \epsilon(x_0 - x)$, i.e., $-\epsilon|x - x_0| < r(x) < \epsilon|x - x_0|$, since $|x - x_0| = x_0 - x$ when $x < x_0$.

Now we assume $x > x_0$ in I. Our overtakings imply that $r(x) > -\epsilon(x - x_0)$ and $r(x) < \epsilon(x - x_0)$, so $-\epsilon(x - x_0) < r(x) < \epsilon(x - x_0)$, and, once again, $-\epsilon|x - x_0| < r(x) < \epsilon|x - x_0|$.

We have shown that, for $x \neq x_0$ and x in I, $-\epsilon|x - x_0| < r(x) < \epsilon|x - x_0|$. But this is the same as $|r(x)| < \epsilon|x - x_0|$. We have finished half of our "if and only if" proof. Geometrically speaking, we have shown that the graph of $r(x)$ for $x \in I$ lies inside the shaded "bow-tie" region in Fig. 3-1.

For the second half of the proof, we assume that, for any $\epsilon > 0$, there exists an interval I such that, for all x in I, $x \neq x_0$, $|r(x)| < \epsilon|x - x_0|$.

Reversing the steps in the preceding argument shows that, for all $\epsilon > 0$, the line $y = -\epsilon(x - x_0)$ is overtaken by the graph of $r(x)$ at x_0, while the line $y = \epsilon(x - x_0)$ overtakes the graph. Thus the set A in the definition of the derivative contains all negative numbers, and B contains all positive numbers; therefore, 0 is a point of transition from A to B. Thus $r'(x_0) = 0$, so r vanishes rapidly at x_0.

The next theorem shows the importance of rapidly vanishing functions in the study of differentiation.

Theorem 2 *A function f is differentiable at x_0 and $m_0 = f'(x_0)$ if and only if the function $r(x)$, defined by*

$$r(x) = f(x) - [f(x_0) + m_0(x - x_0)]$$

vanishes rapidly at x_0.

The function $r(x)$ represents the *error,* or the *remainder* involved in approximating f by its tangent line at x_0. Another way of stating Theorem 2 is that f is differentiable at x_0, with derivative $f'(x_0) = m_0$, if and only if f can be written as a sum $f(x) = f(x_0) + m_0(x - x_0) + r(x)$, where $r(x)$ vanishes rapidly at x_0. Once we know how to recognize rapidly vanishing functions, Theorem 2 will provide a useful test for differentiability and a tool for computing derivatives.

Notice that Theorem 2, like Theorem 1, is of the "if and only if" type. Thus, it has two independent parts. We must prove that if $f(x)$ is differentiable at x_0 and $m_0 = f'(x_0)$, then $r(x)$ vanishes rapidly at x_0, and we must prove as well that if $r(x)$ vanishes rapidly at x_0, then $f(x)$ is differentiable at x_0 and $m_0 = f'(x_0)$. (Each of the two parts of the theorem is called the *converse* of the other.) In the proof which follows, we prove the second part first, since it is convenient to do so.

Proof of Theorem 2 Suppose that $r(x) = f(x) - f(x_0) - m_0(x - x_0)$ vanishes rapidly at x_0. We will show that $f'(x_0) = m_0$. We first show that, for $m > m_0$, the line through $(x_0, f(x_0))$ overtakes the graph of f at x_0. (Refer to Fig. 3-2 as you read the following argument.) To begin, note that if $m > m_0$, then $m - m_0 > 0$. Since $r'(x_0) = 0$, the line through $(x_0, r(x_0))$ with slope $m - m_0$ overtakes the graph of r at x_0. Since $r(x_0) = 0$, the equation of this line is $y = (m - m_0)(x - x_0)$. By the definition of overtaking, there is an open interval I around x_0 such that (i) $x < x_0$ in I implies $(m - m_0)(x - x_0) < r(x)$, i.e.,

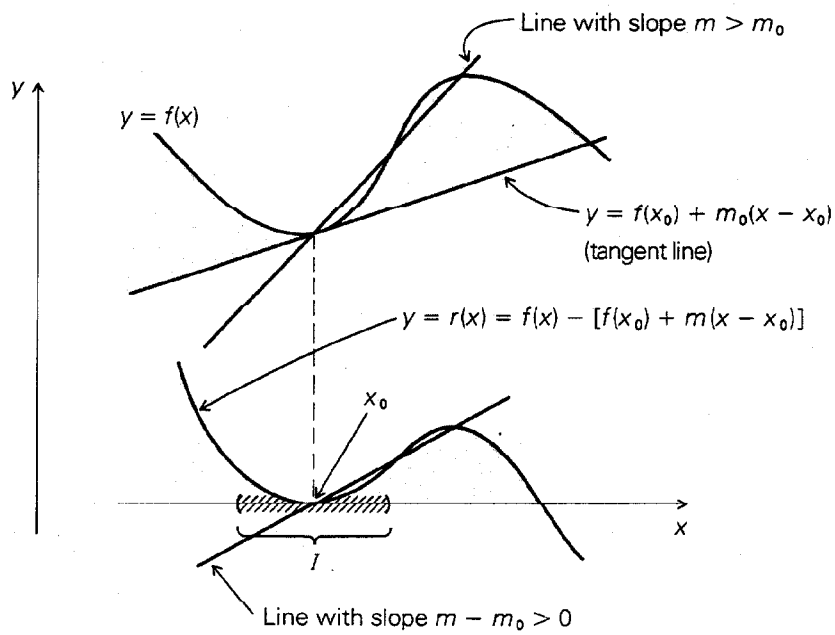

Fig. 3-2 Illustrating the proof of Theorem 2.

$$(m - m_0)(x - x_0) < f(x) - f(x_0) - m_0(x - x_0) \qquad (\mathrm{A}_1)$$

and (ii) $x > x_0$ in I implies

$$(m - m_0)(x - x_0) > f(x) - f(x_0) - m_0(x - x_0) \qquad (\mathrm{A}_2)$$

Adding $f(x_0) + m_0(x - x_0)$ to both sides of (A_1) and (A_2) gives

$$f(x_0) + m(x - x_0) < f(x), \quad \text{for } x < x_0 \text{ in } I$$

and

$$f(x_0) + m(x - x_0) > f(x), \quad \text{for } x > x_0 \text{ in } I$$

In other words, the line $y = f(x_0) + m(x - x_0)$ overtakes the graph of f at x_0. Recall that m was any number greater than m_0.

Similarly, one proves that, for $m < m_0$, the line $y = f(x_0) + m(x - x_0)$ is overtaken by the graph of f at x_0, so f is differentiable at x_0, and m_0 must be the derivative of f at x_0 by the definition on p. 6 and Theorem 4, p. 27. This completes the proof of half of Theorem 2.

Next, we assume that f is differentiable at x_0 and that $m_0 = f'(x_0)$. Reversing the steps above shows that the line $y = c(x - x_0)$ overtakes the graph of r for $c > 0$ and is overtaken by it for $c < 0$. Clearly, $r(x_0) = 0$, so r vanishes rapidly at x_0.

Solved Exercises*

1. Let $g_1(x) = x$, $g_2(x) = x^2$, $g_3(x) = x^3$. Compute their values at $x = 0.1$, 0.01, 0.002, and 0.0004. Discuss.
2. Prove:
   (a) If f and g vanish at x_0, so does $f + g$.
   (b) If f vanishes at x_0, and g is any function which is defined at x_0, then fg vanishes at x_0.
3. Prove that $(x - x_0)^2$ vanishes rapidly at x_0.

Exercises

1. Fill in the details of the last two paragraphs of the proof of Theorem 2.
2. Do Solved Exercise 3 by using Theorem 1.
3. Prove that $f(x) = x^3$ is rapidly vanishing at $x = 0$.
4. Let $g(x)$ be a quadratic polynomial such that $g(0) = 5$.
   (a) Can $g(x)$ vanish rapidly at some integer?
   (b) Can $g(x)$ vanish rapidly at more than one point?
5. The polynomials $x^3 - 5x^2 + 8x - 4$, $x^3 - 4x^2 + 5x - 2$, and $x^3 - 3x^2 + 3x - 1$ all vanish at $x_0 = 1$. By evaluating these polynomials for values of x very near 1, on a calculator, try to guess which of the polynomials vanish rapidly at 1. (Factoring the polynomials may help you to understand what is happening.)

The Sum and Constant Multiple Rules

The sum rule states that $(f + g)'(x) = f'(x) + g'(x)$. To prove this, we must show that the remainder for $f + g$, namely

$$\{f(x) + g(x)\} - [\{f(x_0) + g(x_0)\} - \{f'(x_0) + g'(x_0)\}(x - x_0)]$$

vanishes rapidly at x_0, for then Theorem 2 would imply that $f(x) + g(x)$ is dif-

*Solutions appear in the Appendix.

ferentiable at x_0 with derivative $m_0 = f'(x_0) + g'(x_0)$. We may rewrite the remainder as

$$[f(x) - f(x_0) - f'(x_0)(x - x_0)] + [g(x) - g(x_0) - g'(x_0)(x - x_0)].$$

By Theorem 2, each of the expressions in square brackets represents a function which vanishes rapidly at x_0, so we need to show that the sum of two rapidly vanishing functions is rapidly vanishing.

The constant multiple rule states that $(af)'(x) = af'(x)$. The proof of this rests upon the fact that a constant multiple of a rapidly vanishing function is again rapidly vanishing. Theorem 3 proves these two basic properties of rapidly vanishing functions.

Theorem 3

1. *If $r_1(x)$ and $r_2(x)$ vanish rapidly at x_0, then so does $r_1(x) + r_2(x)$.*
2. *If $r_1(x)$ vanishes rapidly at x_0, and a is any real number, then $ar_1(x)$ vanishes rapidly at x_0.*

Proof 1. Let $r(x) = r_1(x) + r_2(x)$, where $r_1(x)$ and $r_2(x)$ vanish rapidly at x_0. By Solved Exercise 2, $r(x_0) = 0$; we will use Theorem 1 to show that $r(x)$ vanishes rapidly at x_0. Given $\epsilon > 0$, we must find an interval I such that $|r(x)| < \epsilon|x - x_0|$ for all $x \neq x_0$ in I. Since $r_1(x)$ and $r_2(x)$ vanish rapidly at x_0, there are intervals I_1 and I_2 about x_0 such that $x \neq x_0$ in I_1 implies that $|r_1(x)| < (\epsilon/2)\ |x - x_0|$, while $x \neq x_0$ in I_2 implies $|r_2(x)| < (\epsilon/2)\ |x - x_0|$. (We can apply Theorem 1 with any positive number, including $\epsilon/2$, in place of ϵ.) Let I be an interval containing x_0 and contained in both I_1 and I_2. For $x \neq x_0$ in I, we have both inequalities: $|r_1(x)| < (\epsilon/2)\ |x - x_0|$ and $|r_2(x)| < (\epsilon/2)\ |x - x_0|$. Adding these inequalities gives

$$|r_1(x)| + |r_2(x)| < \frac{\epsilon}{2}|x - x_0| + \frac{\epsilon}{2}|x - x_0| = \epsilon|x - x_0|$$

(You should now be able to see why we used $\epsilon/2$.) The triangle inequality for absolute values states that $|r_1(x) + r_2(x)| \leqslant |r_1(x)| + |r_2(x)|$, so we have $|r(x)| < \epsilon|x - x_0|$ for $x \neq x_0$ in I.

2. Let $r(x) = ar_1(x)$. By Solved Exercise 2, $r(x_0) = 0$. Given $\epsilon > 0$, we apply Theorem 1 to r_1 vanishing rapidly at x_0 to obtain an interval I about x_0 such that $|r_1(x)| < (\epsilon/|a|)\ |x - x_0|$ for $x \neq x_0$ in I. (If $a = 0$, $ar(x) = 0$ is obviously rapidly vanishing, so we need only deal with the case $a \neq 0$.) Now we have, for $x \neq x_0$ in I,

$$|r(x)| = |ar_1(x)| = |a|\,|r_1(x)| < |a|\,\frac{\epsilon}{|a|}\,|x - x_0| = \epsilon|x - x_0|$$

or $|r(x)| < \epsilon|x - x_0|$, and we are done.

Theorem 4

1. *(**Sum Rule**). If the functions $f(x)$ and $g(x)$ are differentiable at x_0, then so is the function $f(x) + g(x)$, and its derivative at x_0 is $f'(x_0) + g'(x_0)$.*
2. *(**Constant Multiple Rule**). If $f(x)$ is differentiable at x_0, and a is any real number, then the function $af(x)$ is differentiable at x_0, and its derivative there is $af'(x_0)$.*

Proof 1. Since $f(x)$ and $g(x)$ are differentiable at x_0, Theorem 2 tells us that

$$r_1(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

and

$$r_2(x) = g(x) - g(x_0) - g'(x_0)(x - x_0)$$

vanish rapidly at x_0. Adding these two equations, we conclude by Theorem 3 that

$$\begin{aligned} r_1(x) + r_2(x) = {} & [f(x) + g(x)] - [f(x_0) + g(x_0)] \\ & - [f'(x_0) + g'(x_0)](x - x_0) \end{aligned}$$

is also rapidly vanishing at x_0. Hence, by Theorem 2, $f(x) + g(x)$ is differentiable at x_0 with derivative equal to $f'(x_0) + g'(x_0)$.

2. See Solved Exercise 5.

Solved Exercises

4. Prove that, if $r_1(x)$ and $r_2(x)$ vanish rapidly at x_0, then so does $r_1(x) - r_2(x)$.
5. Prove part (2) of Theorem 4 (the constant multiple rule).

Exercises

6. Let a be a nonzero constant, and assume that af overtakes g at x_0. Prove that f overtakes $(1/a)\,g$ at x_0 if $a > 0$, while f is overtaken by $(1/a)\,g$ at x_0 if $a < 0$.

7. Show that, if you *assume* the sum and constant multiple rules, then Theorem 3 is an easy consequence. (This means that Theorem 3 is a *special case* of the sum and constant multiple rules. Our proof of Theorem 4 proceeded from this special case to the general case.)

8. Prove that, if $f_1(x)$ overtakes $g_1(x)$ at x_0, and $f_2(x)$ overtakes $g_2(x)$ at x_0, then $f_1(x) + f_2(x)$ overtakes $g_1(x) + g_2(x)$ at x_0.

9. If $f_1(x) + f_2(x)$ is differentiable at x_0, are $f_1(x)$ and $f_2(x)$ necessarily differentiable there? Can just one of them be nondifferentiable at x_0?

10. Show, by calculating the derivative, that the quadratic polynomial $ax^2 + bx + c$ vanishes rapidly at x_0 if and only if x_0 is a *double root* of the equation $ax^2 + bx + c = 0$. What does the graph look like in this case?

The Product Rule

The sum rule depended on the fact that the sum of two rapidly vanishing functions is again rapidly vanishing. For the product rule, we need a similar result for products, where only one factor is known to be rapidly vanishing.

Theorem 5 *If $r(x)$ vanishes rapidly at x_0 and $f(x)$ is differentiable at x_0, then $f(x)r(x)$ vanishes rapidly at x_0.*

Proof Note that part 2 of Theorem 3 is a special case of this theorem, where $f(x)$ is constant. We prove Theorem 5 in two steps, the first of which shows that $f(x)$ can be "sandwiched" between two constant values.

Step 1 We will prove that there is a constant $B > 0$ and an interval I_1 about x_0 such that, for x in I_1, we have

$$-B < f(x) < B, \quad \text{i.e., } |f(x)| < B.$$

(Refer to Fig. 3-3.) The number B is called a *bound* for $|f(x)|$ near x_0. Through the point $(x_0, f(x_0))$, we draw the two lines with slope $f'(x_0) + 1$ and $f'(x_0) - 1$. The first of them overtakes the graph of f at x_0; the second

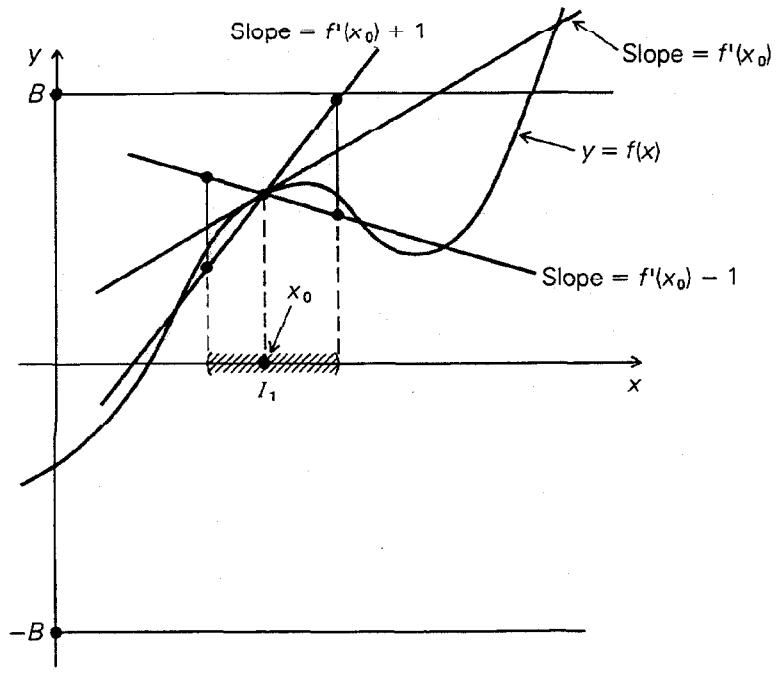

Fig. 3-3 The graph of f on the interval I_1 lies between the lines $y = -B$ and $y = B$.

is overtaken. (We could just as well have used slopes $f'(x_0) + \frac{1}{2}$ and $f'(x_0) - \frac{1}{2}$, or any other numbers bracketing $f'(x_0)$.) On a sufficiently small interval I_1, the graph of f lies between these two lines. (I_1 is any interval contained in intervals which work for both overtakings.) Now choose B large enough so that the bow-tie region between the lines and above I_1 lies between the lines $y = -B$ and $y = B$. The reader may fill in the algebra required to determine a possible choice for B. (See Solved Exercise 6.)

Step 2. Clearly $f(x)r(x)$ vanishes at x_0. We now apply Theorem 1 just as we did in proving part (ii) of Theorem 3.

Given $\epsilon > 0$, since $r(x)$ vanishes rapidly at x_0, we can find an interval I about x_0 such that, for $x \neq x_0$ in I, $|r(x)| < (\epsilon/B)\ |x - x_0|$, where B is the bound from step 1. Now we have, for $x \neq x_0$ in I, $|f(x)r(x)| = |f(x)||r(x)| < B\ (\epsilon/B)\ |x - x_0| = \epsilon|x - x_0|$. Thus, by Theorem 1, $f(x)r(x)$ vanishes rapidly at x_0.

We can now deduce the product rule from Theorem 5 by a computation.

Theorem 6 Product Rule. *If the functions $f(x)$ and $g(x)$ are differentiable at x_0, then so is the function $f(x)g(x)$, and its derivative at x_0 is*

$$f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Proof By Theorem 2, $f(x) = f(x_0) + f'(x_0)(x - x_0) + r_1(x)$ and $g(x) = g(x_0) + g'(x_0)(x - x_0) + r_2(x)$ where $r_1(x)$ and $r_2(x)$ vanish rapidly at x_0. Multiplying the two expressions gives

$$\begin{aligned} f(x)g(x) &= f(x_0)[g(x_0) + g'(x_0)(x - x_0) + r_2(x)] \\ &\quad + f'(x_0)(x - x_0)[g(x_0) + g'(x_0)(x - x_0) + r_2(x)] \\ &\quad + r_1(x)[g(x_0) + g'(x_0)(x - x_0) + r_2(x)] \\ &= f(x_0)g(x_0) + [f'(x_0)g(x_0) + f(x_0)g'(x_0)](x - x_0) \\ &\quad + f(x_0)r_2(x) + r_1(x)g(x_0) + f'(x_0)g'(x_0)(x - x_0)^2 \\ &\quad + f'(x_0)(x - x_0)r_2(x) + r_1(x)g'(x_0)(x - x_0) + r_1(x)r_2(x) \end{aligned}$$

Now each of the last two lines in the preceding sum vanishes rapidly at x_0: $f(x_0)r_2(x) + r_1(x)g(x_0)$ is a sum of constant multiples of rapidly vanishing functions (apply Theorem 3); $f'(x_0)g'(x_0)(x - x_0)^2$ is a constant multiple of $(x - x_0)^2$, which is rapidly vanishing by Solved Exercise 3; each of the next two terms is the product of a linear function and a rapidly vanishing function (apply Theorem 4), and the last term is the product of two rapidly vanishing functions (apply Theorem 4 again). Applying Theorem 3 to the sum of the last two lines, we conclude that

$$f(x)g(x) = f(x_0)g(x_0) + [f'(x_0)g(x_0) + f(x_0)g'(x_0)](x - x_0) + r(x),$$

where $r(x)$ vanishes rapidly at x_0. Theorem 2 therefore shows that $f(x)g(x)$ is differentiable at x_0 with derivative $f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Note that the correct formula for the derivative of a product appeared as the coefficient of $(x - x_0)$ in our computation; there was no need to know it in advance.

Now that we have proven the product rule, we may use one of its important consequences: the derivative of x^n is nx^{n-1} (see Solved Exercise 7).

Solved Exercises

6. In step 1 of the proof above, estimate how large B must be so that $-B < f(x) < B$ for all x in I.
7. Prove that $f(x) = x^4$ vanishes rapidly at 0.
8. Find a function which vanishes rapidly at both 3 and 7.

Exercises

11. Find a function which vanishes rapidly at 1, 2, and 3. Sketch a graph of this function.
12. Show that $(x - a)^n$ vanishes rapidly at a if n is any positive integer greater than or equal to 2.
13. Recall that, if $g(x)$ is a polynomial and $g(a) = 0$, then $g(x) = (x - a)h(x)$, where $h(x)$ is another polynomial. We call a *a multiple root* of $g(x)$ if $h(a) = 0$. Prove that a is a multiple root of $g(x)$ if and only if $g(x)$ vanishes rapidly at a.
14. If $f(x)$ and $g(x)$ are defined in an interval about x_0, and $f'(x_0)$ and $(fg)'(x_0)$ both exist, does $g'(x_0)$ necessarily exist? (Compare Exercise 9.)

The Quotient Rule

Theorem 7 Suppose that g is differentiable at x_0 and that $g(x_0) \neq 0$. Then there is an interval I about x_0 on which $g(x)$ is never zero, and $1/g$ is differentiable at x_0 with derivative $-g'(x_0)/g(x_0)^2$.

Proof Suppose $g(x_0) > 0$. (The case $g(x_0) < 0$ is discussed at the end of this proof.) Then a bow-tie argument like that on p. 39 shows that $g(x) > \frac{1}{2}g(x_0)$ for x in some interval I about x_0. (See Fig. 3-4 and Solved Exercise 9.) Since $0 < \frac{1}{2}g(x_0) < g(x)$ for x in I, we have $0 < 1/g(x) < 2/g(x_0)$ for x in I. In Theorem 5, we actually proved that the product of a rapidly vanishing function and a function bounded between two values is again rapidly vanishing. By Theorem 2, we must show that

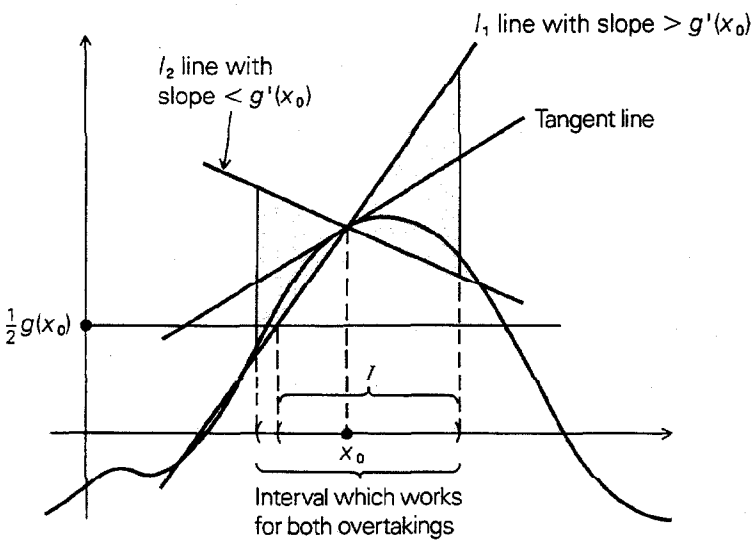

Fig. 3-4 $g(x)$ is bounded below near x_0 by $\frac{1}{2}g(x_0)$.

$$r(x) = \frac{1}{g(x)} - \frac{1}{g(x_0)} + \frac{g'(x_0)}{g(x_0)^2}(x - x_0)$$

vanishes rapidly at x_0. Collecting terms over a common denominator, we get

$$r(x) = \frac{1}{g(x)g(x_0)^2}\{g(x_0)^2 - g(x)g(x_0) + g'(x_0)g(x)(x - x_0)\}$$

Since $1/[g(x)g(x_0)^2]$ lies between the values 0 and $[2/g(x_0)]\,[1/g(x_0)^2]$, it suffices to prove that the expression in braces,

$$r_1(x) = g(x_0)^2 - g(x)g(x_0) + g'(x_0)g(x)(x - x_0)$$

vanishes rapidly at zero. Setting $x = x_0$ in $r_1(x)$, we get $r_1(x_0) = 0$. We must show that $r_1'(x_0) = 0$. However, now that we have proven the sum, constant multiple, and product rules, we may use them:

$$r_1'(x) = -g'(x)g(x_0) + g'(x_0)g'(x)(x - x_0) + g'(x_0)g(x)$$

wherever g is differentiable. Setting $x = x_0$, we get $r_1'(x_0) = 0$, so $r_1(x)$ vanishes rapidly at x_0, and so does $r(x)$. That finishes the proof for the case $g(x_0) > 0$.

If $g(x_0) < 0$, we can apply the previous argument to $-g$. Namely, write

$$\frac{1}{g(x)} = -\frac{1}{-g(x)}$$

Since $g(x)$ is differentiable at x_0, so is $-g(x)$. But $-g(x_0)$ is positive, so the argument above implies that $1/[-g(x)]$ is differentiable at x_0. Now the constant multiple rule gives the differentiability of $-1/[-g(x)]$.

Corollary ***Quotient Rule.*** *Suppose that f and g are differentiable at x_0 and that $g(x_0) \neq 0$. Then f/g is differentiable at x_0 with derivative*

$$\frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$$

Proof Write $f/g = f \cdot (1/g)$ and apply the product rule and Theorem 7.

With these basic results in hand we can now readily differentiate any rational function. (See your regular calculus text.)

Solved Exercises

9. Using Fig. 3-4, locate the left-hand endpoint of the interval I.
10. Find a function g such that $g(0) \neq 0$, but $g(x) = 0$ for some point x in every interval about 0. Could such a g be differentiable at 0?

Exercises

15. Let a and b be real numbers. Suppose that $f(x)$ vanishes rapidly at a and that $g(x)$ is differentiable at a. Find a necessary and sufficient condition for $f(x)/(g(x) - b)$ to vanish rapidly at a.
16. Could f/g be differentiable at x_0 without f and g themselves being differentiable there?

Problems for Chapter 3

1. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geqslant 0 \\ -x^2 & \text{if } x \leqslant 0 \end{cases}$$

Does $f(x)$ vanish rapidly at $x = 0$?

2. Find a function $g(x)$ defined for all x such that $x^2g(x)$ does not vanish rapidly at $x = 0$. Sketch the graphs of $g(x)$ and $x^2g(x)$.

3. A function f is called *locally bounded* at x_0 if there is an open interval I about x_0 and a constant B such that $|f(x)| \leqslant B$ for x in I. Prove that if r is rapidly vanishing at x_0 and f is locally bounded at x_0, then fr vanishes rapidly at x_0.

4. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ -x^2 & \text{if } x \text{ is rational} \end{cases}$$

Show that $f(x)$ is not differentiable at any nonzero value of x_0. Is $f(x)$ differentiable at 0? (Prove your answer.)

5. Suppose that $f(x)$ and $g(x)$ are differentiable at x_0 and that they both vanish there. Prove that their product vanishes rapidly at x_0. [*Hint*: Use the product rule.]

6. Find functions $f(x)$ and $g(x)$, defined for all x, such that $f(0) = g(0) = 0$, but $f(x)g(x)$ does not vanish rapidly at x_0.

7. Suppose that $f(x)$ and $g(x)$ are differentiable at x_0 and that $f(x_0) < g(x_0)$. Prove that there is an interval I about x_0 such that $f(x) < g(x)$ for all x in I. [*Hint*: Consider $g(x) - f(x)$ and look at the proof of Theorem 7.]

8. Prove that, if $r_1(x)$ and $r_1(x) + r_2(x)$ vanish rapidly at x_0, then so does $r_2(x)$.

9. Prove that, if $f(x)g(x)$ vanishes rapidly at x_0, and $g(x)$ is differentiable at x_0 with $g(x_0) \neq 0$, then $f(x)$ vanishes rapidly at x_0. What happens if $g(x_0) = 0$?

10. Find a function which vanishes rapidly at every integer.