Chapter Five

Linear Systems

Few physical elements display truly linear characteristics. For example the relation between force on a spring and displacement of the spring is always nonlinear to some degree. The relation between current through a resistor and voltage drop across it also deviates from a straight-line relation. However, if in each case the relation is reasonably *linear, then it will be found that the system behavior will be very close to that obtained by assuming an ideal, linear physical element, and the analytical simplification is so enormous that we make linear assumptions wherever we can possibly do so in good conscience.*

Robert H. Cannon, *Dynamics of Physical Systems*, 1967 [49].

In Chapters 2–4 we considered the construction and analysis of differential equation models for dynamical systems. In this chapter we specialize our results to the case of linear, time-invariant input/output systems. Two central concepts are the matrix exponential and the convolution equation, through which we can completely characterize the behavior of a linear system. We also describe some properties of the input/output response and show how to approximate a nonlinear system by a linear one.

5.1 Basic Definitions

We have seen several instances of linear differential equations in the examples in the previous chapters, including the spring–mass system (damped oscillator) and the operational amplifier in the presence of small (nonsaturating) input signals. More generally, many dynamical systems!linear can be modeled accurately by linear differential equations. Electrical circuits are one example of a broad class of systems for which linear models can be used effectively. Linear models are also broadly applicable in mechanical engineering, for example, as models of small deviations from equilibria in solid and fluid mechanics. Signal-processing systems, including digital filters of the sort used in CD and MP3 players, are another source of good examples, although these are often best modeled in discrete time (as described in more detail in the exercises).

In many cases, we *create* systems with a linear input/output response through the use of feedback. Indeed, it was the desire for linear behavior that led Harold S. Black to the invention of the negative feedback amplifier. Almost all modern signal processing systems, whether analog or digital, use feedback to produce linear or near-linear input/output characteristics. For these systems, it is often useful to represent the input/output characteristics as linear, ignoring the internal details required to get that linear response.

For other systems nonlinearities cannot be ignored, especially if one cares about the global behavior of the system. The predator–prey problem is one example of this: to capture the oscillatory behavior of the interdependent populations we must include the nonlinear coupling terms. Other examples include switching behavior and generating periodic motion for locomotion. However, if we care about what happens near an equilibrium point, it often suffices to approximate the nonlinear dynamics by their local linearization, as we already explored briefly in Section 4.3. The linearization is essentially an approximation of the nonlinear dynamics around the desired operating point.

Linearity

We now proceed to define linearity of input/output systems more formally. Consider a state space system of the form

$$\frac{dx}{dt} = f(x, u), \qquad y = h(x, u), \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. As in the previous chapters, we will usually restrict ourselves to the single-input, single-output case by taking $p = q = 1$. We also assume that all functions are smooth and that for a reasonable class of inputs (e.g., piecewise continuous functions of time) the solutions of equation (5.1) exist for all time.

It will be convenient to assume that the origin $x = 0$, $u = 0$ is an equilibrium point for this system ($\dot{x} = 0$) and that $h(0, 0) = 0$. Indeed, we can do so without loss of generality. To see this, suppose that $(x_e, u_e) \neq (0, 0)$ is an equilibrium point of the system with output $y_e = h(x_e, u_e)$. Then we can define a new set of states, inputs and outputs,

$$\tilde{x} = x - x_e, \qquad \tilde{u} = u - u_e, \qquad \tilde{y} = y - y_e,$$

and rewrite the equations of motion in terms of these variables:

$$\begin{aligned} \frac{d}{dt}\tilde{x} &= f(\tilde{x} + x_e, \tilde{u} + u_e) =: \tilde{f}(\tilde{x}, \tilde{u}), \\ \tilde{y} &= h(\tilde{x} + x_e, \tilde{u} + u_e) - y_e =: \tilde{h}(\tilde{x}, \tilde{u}). \end{aligned}$$

In the new set of variables, the origin is an equilibrium point with output 0, and hence we can carry out our analysis in this set of variables. Once we have obtained our answers in this new set of variables, we simply "translate" them back to the original coordinates using $x = \tilde{x} + x_e$, $u = \tilde{u} + u_e$ and $y = \tilde{y} + y_e$.

Returning to the original equations (5.1), now assuming without loss of generality that the origin is the equilibrium point of interest, we write the output $y(t)$ corresponding to the initial condition $x(0) = x_0$ and input $u(t)$ as $y(t; x_0, u)$. Using this notation, a system is said to be a *linear input/output system* if the following

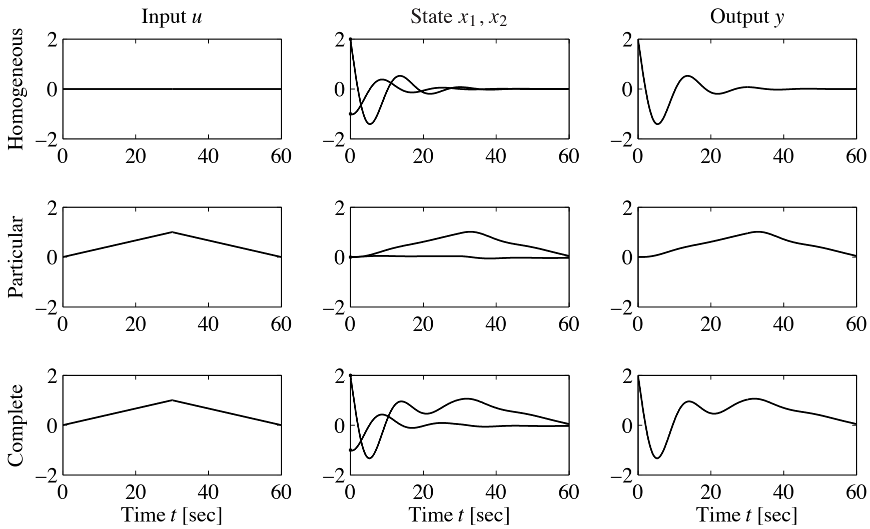


Figure 5.1: Superposition of homogeneous and particular solutions. The first row shows the input, state and output corresponding to the initial condition response. The second row shows the same variables corresponding to zero initial condition but nonzero input. The third row is the complete solution, which is the sum of the two individual solutions.

conditions are satisfied:

$$
\begin{aligned}
&\text{(i)} \quad y(t; \alpha x_1 + \beta x_2, 0) = \alpha y(t; x_1, 0) + \beta y(t; x_2, 0),\\
&\text{(ii)} \quad y(t; \alpha x_0, \delta u) = \alpha y(t; x_0, 0) + \delta y(t; 0, u),\\
&\text{(iii)} \quad y(t; 0, \delta u_1 + \gamma u_2) = \delta y(t; 0, u_1) + \gamma y(t; 0, u_2).
\end{aligned}
\tag{5.2}
$$

Thus, we define a system to be linear if the outputs are jointly linear in the initial condition response ($u = 0$) and the forced response ($x(0) = 0$). Property (iii) is a statement of the *principle of superposition*: the response of a linear system to the sum of two inputs u_1 and u_2 is the sum of the outputs y_1 and y_2 corresponding to the individual inputs.

The general form of a linear state space system is

$$
\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{5.3}
$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times p}$. In the special case of a single-input, single-output system, B is a column vector, C is a row vector and D is scalar. Equation (5.3) is a system of linear first-order differential equations with input u, state x and output y. It is easy to show that given solutions $x_1(t)$ and $x_2(t)$ for this set of equations, they satisfy the linearity conditions.

We define $x_h(t)$ to be the solution with zero input (the *homogeneous solution*) and the solution $x_p(t)$ to be the solution with zero initial condition (a *particular solution*). Figure 5.1 illustrates how these two individual solutions can be superimposed to form the complete solution.

It is also possible to show that if a finite-dimensional dynamical system is input/output linear in the sense we have described, it can always be represented by a state space equation of the form (5.3) through an appropriate choice of state variables. In Section 5.2 we will give an explicit solution of equation (5.3), but we illustrate the basic form through a simple example.

Example 5.1 Scalar system
Consider the first-order differential equation

$$\frac{dx}{dt} = ax + u, \qquad y = x,$$

with $x(0) = x_0$. Let $u_1 = A \sin \omega_1 t$ and $u_2 = B \cos \omega_2 t$. The homogeneous solution is $x_h(t) = e^{at} x_0$, and two particular solutions with $x(0) = 0$ are

$$\begin{aligned} x_{p1}(t) &= -A \frac{-\omega_1 e^{at} + \omega_1 \cos \omega_1 t + a \sin \omega_1 t}{a^2 + \omega_1^2}, \\ x_{p2}(t) &= B \frac{a e^{at} - a \cos \omega_2 t + \omega_2 \sin \omega_2 t}{a^2 + \omega_2^2}. \end{aligned}$$

Suppose that we now choose $x(0) = \alpha x_0$ and $u = u_1 + u_2$. Then the resulting solution is the weighted sum of the individual solutions:

$$\begin{aligned} x(t) = e^{at} &\left(\alpha x_0 + \frac{A\omega_1}{a^2 + \omega_1^2} + \frac{Ba}{a^2 + \omega_2^2} \right) \\ &- A \frac{\omega_1 \cos \omega_1 t + a \sin \omega_1 t}{a^2 + \omega_1^2} + B \frac{-a \cos \omega_2 t + \omega_2 \sin \omega_2 t}{a^2 + \omega_2^2}. \end{aligned} \tag{5.4}$$

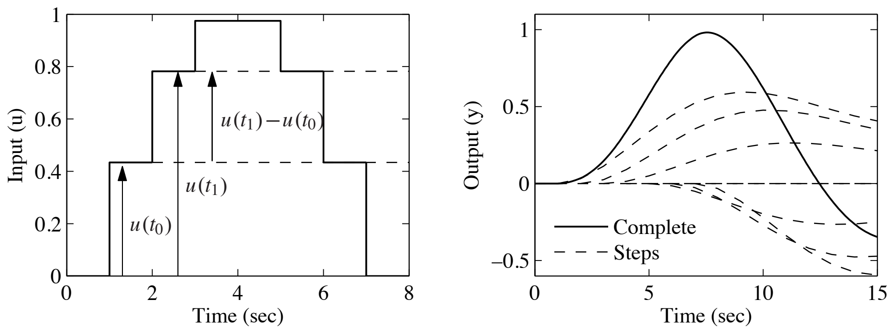
To see this, substitute equation (5.4) into the differential equation. Thus, the properties of a linear system are satisfied. ∇

Time Invariance

Time invariance is an important concept that is used to describe a system whose properties do not change with time. More precisely, for a time-invariant system if the input $u(t)$ gives output $y(t)$, then if we shift the time at which the input is applied by a constant amount a, $u(t + a)$ gives the output $y(t + a)$. Systems that are linear and time-invariant, often called *LTI systems*, have the interesting property that their response to an arbitrary input is completely characterized by their response to step inputs or their response to short "impulses."

To explore the consequences of time invariance, we first compute the response to a piecewise constant input. Assume that the system is initially at rest and consider the piecewise constant input shown in Figure 5.2a. The input has jumps at times t_k, and its values after the jumps are $u(t_k)$. The input can be viewed as a combination of steps: the first step at time t_0 has amplitude $u(t_0)$, the second step at time t_1 has amplitude $u(t_1) - u(t_0)$, etc.

Assuming that the system is initially at an equilibrium point (so that the initial condition response is zero), the response to the input can be obtained by superim-



(a) Piecewise constant input (b) Output response

Figure 5.2: Response to piecewise constant inputs. A piecewise constant signal can be represented as a sum of step signals (a), and the resulting output is the sum of the individual outputs (b).

posing the responses to a combination of step inputs. Let $H(t)$ be the response to a unit step applied at time 0. The response to the first step is then $H(t-t_0)u(t_0)$, the response to the second step is $H(t-t_1)\big(u(t_1)-u(t_0)\big)$, and we find that the complete response is given by

$$
\begin{aligned}
y(t) &= H(t-t_0)u(t_0) + H(t-t_1)\big(u(t_1)-u(t_0)\big) + \cdots \\
&= \big(H(t-t_0) - H(t-t_1)\big)u(t_0) + \big(H(t-t_1) - H(t-t_2)\big)u(t_1) + \cdots \\
&= \sum_{n=0}^{\infty} \big(H(t-t_n) - H(t-t_{n+1})\big)u(t_n) \\
&= \sum_{n=0}^{\infty} \frac{H(t-t_n) - H(t-t_{n+1})}{t_{n+1}-t_n} u(t_n)\big(t_{n+1}-t_n\big).
\end{aligned}
$$

An example of this computation is shown in Figure 5.2b.

The response to a continuous input signal is obtained by taking the limit as $t_{n+1} - t_n \to 0$, which gives

$$
y(t) = \int_0^{\infty} H'(t-\tau)u(\tau)d\tau, \tag{5.5}
$$

where H' is the derivative of the step response, also called the *impulse response*. The response of a linear time-invariant system to any input can thus be computed from the step response. Notice that the output depends only on the input since we assumed the system was initially at rest, $x(0) = 0$. We will derive equation (5.5) in a slightly different way in the Section 5.3.

5.2 The Matrix Exponential

Equation (5.5) shows that the output of a linear system can be written as an integral over the inputs $u(t)$. In this section and the next we derive a more general version of this formula, which includes nonzero initial conditions. We begin by exploring the initial condition response using the matrix exponential.

Initial Condition Response

Although we have shown that the solution of a linear set of differential equations defines a linear input/output system, we have not fully computed the solution of the system. We begin by considering the homogeneous response corresponding to the system

$$\frac{dx}{dt} = Ax. \tag{5.6}$$

For the *scalar* differential equation

$$\frac{dx}{dt} = ax, \qquad x \in \mathbb{R},\ a \in \mathbb{R},$$

the solution is given by the exponential

$$x(t) = e^{at}x(0).$$

We wish to generalize this to the vector case, where A becomes a matrix. We define the *matrix exponential* as the infinite series

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k, \tag{5.7}$$

where $X \in \mathbb{R}^{n\times n}$ is a square matrix and I is the $n \times n$ identity matrix. We make use of the notation

$$X^0 = I, \qquad X^2 = XX, \qquad X^n = X^{n-1}X,$$

which defines what we mean by the "power" of a matrix. Equation (5.7) is easy to remember since it is just the Taylor series for the scalar exponential, applied to the matrix X. It can be shown that the series in equation (5.7) converges for any matrix $X \in \mathbb{R}^{n\times n}$ in the same way that the normal exponential is defined for any scalar $a \in \mathbb{R}$.

Replacing X in equation (5.7) by At, where $t \in \mathbb{R}$, we find that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^kt^k,$$

and differentiating this expression with respect to t gives

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{1}{2}A^3t^2 + \cdots = A\sum_{k=0}^{\infty} \frac{1}{k!}A^kt^k = Ae^{At}. \tag{5.8}$$

Multiplying by $x(0)$ from the right, we find that $x(t) = e^{At}x(0)$ is the solution to the differential equation (5.6) with initial condition $x(0)$. We summarize this important result as a proposition.

Proposition 5.1. *The solution to the homogeneous system of differential equations* (5.6) *is given by*

$$x(t) = e^{At}x(0).$$

Notice that the form of the solution is exactly the same as for scalar equations, but we must put the vector $x(0)$ on the right of the matrix e^{At}.

The form of the solution immediately allows us to see that the solution is linear in the initial condition. In particular, if $x_{h1}(t)$ is the solution to equation (5.6) with initial condition $x(0) = x_{01}$ and $x_{h2}(t)$ with initial condition $x(0) = x_{02}$, then the solution with initial condition $x(0) = \alpha x_{01} + \beta x_{02}$ is given by

$$x(t) = e^{At}\big(\alpha x_{01} + \beta x_{02}\big) = \big(\alpha e^{At}x_{01} + \beta e^{At}x_{02}\big) = \alpha x_{h1}(t) + \beta x_{h2}(t).$$

Similarly, we see that the corresponding output is given by

$$y(t) = Cx(t) = \alpha y_{h1}(t) + \beta y_{h2}(t),$$

where $y_{h1}(t)$ and $y_{h2}(t)$ are the outputs corresponding to $x_{h1}(t)$ and $x_{h2}(t)$.

We illustrate computation of the matrix exponential by two examples.

Example 5.2 Double integrator

A very simple linear system that is useful in understanding basic concepts is the second-order system given by

$$\ddot{q} = u, \qquad y = q.$$

This system is called a *double integrator* because the input u is integrated twice to determine the output y.

In state space form, we write $x = (q, \dot{q})$ and

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The dynamics matrix of a double integrator is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and we find by direct calculation that $A^2 = 0$ and hence

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Thus the homogeneous solution ($u = 0$) for the double integrator is given by

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) + tx_2(0) \\ x_2(0) \end{bmatrix},$$

$$y(t) = x_1(0) + tx_2(0).$$

∇

Example 5.3 Undamped oscillator
A simple model for an oscillator, such as the spring–mass system with zero damping, is

$$\ddot{q} + \omega_0^2 q = u.$$

Putting the system into state space form, the dynamics matrix for this system can be written as

$$A = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \qquad \text{and} \qquad e^{At} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix}.$$

This expression for e^{At} can be verified by differentiation:

$$\begin{aligned} \frac{d}{dt}e^{At} &= \begin{bmatrix} -\omega_0 \sin\omega_0 t & \omega_0\cos\omega_0 t \\ -\omega_0\cos\omega_0 t & -\omega_0\sin\omega_0 t \end{bmatrix} \\ &= \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} = Ae^{At}. \end{aligned}$$

The solution is then given by

$$x(t) = e^{At}x(0) = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

If the system has damping,

$$\ddot{q} + 2\zeta\omega_0 q + \omega_0^2 q = u,$$

the solution is more complicated, but the matrix exponential can be shown to be

$$e^{-\omega_0\zeta t}\begin{bmatrix} \dfrac{\zeta e^{i\omega_d t} - \zeta e^{-i\omega_d t}}{2\sqrt{\zeta^2-1}} + \dfrac{e^{i\omega_d t} + e^{-i\omega_d t}}{2} & \dfrac{e^{i\omega_d t} - e^{-i\omega_d t}}{2\sqrt{\zeta^2-1}} \\ \dfrac{e^{-i\omega_d t} - e^{i\omega_d t}}{2\sqrt{\zeta^2-1}} & \dfrac{\zeta e^{-i\omega_d t} - \zeta e^{i\omega_d t}}{2\sqrt{\zeta^2-1}} + \dfrac{e^{i\omega_d t} + e^{-i\omega_d t}}{2} \end{bmatrix},$$

where $\omega_d = \omega_0\sqrt{\zeta^2-1}$. Note that ω_d and $\sqrt{\zeta^2-1}$ can be either real or complex, but the combinations of terms will always yield a real value for the entries in the matrix exponential. ∇

An important class of linear systems are those that can be converted into diagonal form. Suppose that we are given a system

$$\frac{dx}{dt} = Ax$$

such that all the eigenvalues of A are distinct. It can be shown (Exercise 4.14) that we can find an invertible matrix T such that TAT^{-1} is diagonal. If we choose a set of coordinates $z = Tx$, then in the new coordinates the dynamics become

$$\frac{dz}{dt} = T\frac{dx}{dt} = TAx = TAT^{-1}z.$$

By construction of T, this system will be diagonal.

Now consider a diagonal matrix A and the corresponding kth power of At, which is also diagonal:

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \qquad (At)^k = \begin{bmatrix} \lambda_1^k t^k & & & 0 \\ & \lambda_2^k t^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k t^k \end{bmatrix},$$

It follows from the series expansion that the matrix exponential is given by

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}.$$

A similar expansion can be done in the case where the eigenvalues are complex, using a block diagonal matrix, similar to what was done in Section 4.3.

Jordan Form

Some matrices with equal eigenvalues cannot be transformed to diagonal form. They can, however, be transformed to a closely related form, called the *Jordan form*, in which the dynamics matrix has the eigenvalues along the diagonal. When there are equal eigenvalues, there may be 1's appearing in the superdiagonal indicating that there is coupling between the states.

More specifically, we define a matrix to be in Jordan form if it can be written as

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & J_{k-1} & 0 \\ 0 & 0 & \dots & 0 & J_k \end{bmatrix}, \quad \text{where} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}. \tag{5.9}$$

Each matrix J_i is called a *Jordan block*, and λ_i for that block corresponds to an eigenvalue of J. A first-order Jordan block can be represented as a system consisting of an integrator with feedback λ. A Jordan block of higher order can be represented as series connections of such systems, as illustrated in Figure 5.3.

Theorem 5.2 (Jordan decomposition)**.** *Any matrix $A \in \mathbb{R}^{n \times n}$ can be transformed into Jordan form with the eigenvalues of A determining λ_i in the Jordan form.*

Proof. See any standard text on linear algebra, such as Strang [187]. The special case where the eigenvalues are distinct is examined in Exercise 4.14. □

Converting a matrix into Jordan form can be complicated, although MATLAB can do this conversion for numerical matrices using the `jordan` function. The structure of the resulting Jordan form is particularly interesting since there is no

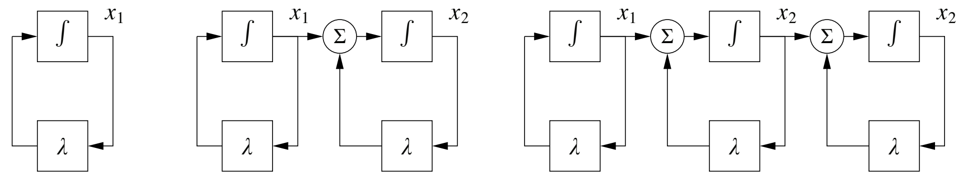


Figure 5.3: Representations of linear systems where the dynamics matrices are Jordan blocks. A first-order Jordan block can be represented as an integrator with feedback λ, as shown on the left. Second- and third-order Jordan blocks can be represented as series connections of integrators with feedback, as shown on the right.

requirement that the individual λ_i's be unique, and hence for a given eigenvalue we can have one or more Jordan blocks of different sizes.

Once a matrix is in Jordan form, the exponential of the matrix can be computed in terms of the Jordan blocks:

$$e^{J} = \begin{bmatrix} e^{J_1} & 0 & \dots & 0 \\ 0 & e^{J_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & e^{J_k}. \end{bmatrix}. \tag{5.10}$$

This follows from the block diagonal form of J. The exponentials of the Jordan blocks can in turn be written as

$$e^{J_i t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & & 1 & \ddots & \vdots \\ & & & \ddots & t \\ 0 & \dots & & 0 & 1 \end{bmatrix} e^{\lambda_i t}. \tag{5.11}$$

When there are multiple eigenvalues, the invariant subspaces associated with each eigenvalue correspond to the Jordan blocks of the matrix A. Note that λ may be complex, in which case the transformation T that converts a matrix into Jordan form will also be complex. When λ has a nonzero imaginary component, the solutions will have oscillatory components since

$$e^{\sigma + i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t).$$

We can now use these results to prove Theorem 4.1, which states that the equilibrium point $x_e = 0$ of a linear system is asymptotically stable if and only if $\operatorname{Re} \lambda_i < 0$.

Proof of Theorem 4.1. Let $T \in \mathbb{C}^{n \times n}$ be an invertible matrix that transforms A into Jordan form, $J = TAT^{-1}$. Using coordinates $z = Tx$, we can write the solution $z(t)$ as

$$z(t) = e^{Jt} z(0).$$

Since any solution $x(t)$ can be written in terms of a solution $z(t)$ with $z(0) = Tx(0)$, it follows that it is sufficient to prove the theorem in the transformed coordinates.

The solution $z(t)$ can be written in terms of the elements of the matrix exponential. From equation (5.11) these elements all decay to zero for arbitrary $z(0)$ if and only if $\operatorname{Re}\lambda_i < 0$. Furthermore, if any λ_i has positive real part, then there exists an initial condition $z(0)$ such that the corresponding solution increases without bound. Since we can scale this initial condition to be arbitrarily small, it follows that the equilibrium point is unstable if any eigenvalue has positive real part. □

The existence of a canonical form allows us to prove many properties of linear systems by changing to a set of coordinates in which the A matrix is in Jordan form. We illustrate this in the following proposition, which follows along the same lines as the proof of Theorem 4.1.

Proposition 5.3. *Suppose that the system*

$$\frac{dx}{dt} = Ax$$

has no eigenvalues with strictly positive real part and one or more eigenvalues with zero real part. Then the system is stable if and only if the Jordan blocks corresponding to each eigenvalue with zero real part are scalar (1×1*) blocks.*

Proof. See Exercise 5.6b. □

The following example illustrates the use of the Jordan form.

Example 5.4 Linear model of a vectored thrust aircraft

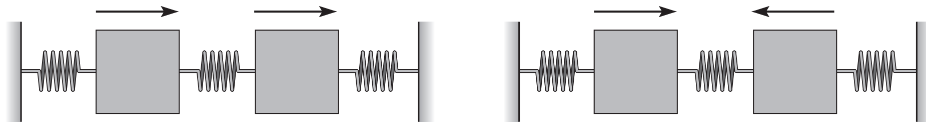
Consider the dynamics of a vectored thrust aircraft such as that described in Example 2.9. Suppose that we choose $u_1 = u_2 = 0$ so that the dynamics of the system become

$$\frac{dz}{dt} = \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ -g\sin z_3 - \frac{c}{m} z_4 \\ -g(\cos z_3 - 1) - \frac{c}{m} z_5 \\ 0 \end{bmatrix}, \qquad (5.12)$$

where $z = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$. The equilibrium points for the system are given by setting the velocities $\dot{x}$, $\dot{y}$ and $\dot{\theta}$ to zero and choosing the remaining variables to satisfy

$$\begin{aligned} -g\sin z_{3,e} &= 0 \\ -g(\cos z_{3,e} - 1) &= 0 \end{aligned} \qquad \Longrightarrow \qquad z_{3,e} = \theta_e = 0.$$

This corresponds to the upright orientation for the aircraft. Note that x_e and y_e are not specified. This is because we can translate the system to a new (upright) position and still obtain an equilibrium point.



(a) Mode 1 (b) Mode 2

Figure 5.4: Modes of vibration for a system consisting of two masses connected by springs. In (a) the masses move left and right in synchronization in (b) they move toward or against each other.

To compute the stability of the equilibrium point, we compute the linearization using equation (4.11):

$$A = \left.\frac{\partial F}{\partial z}\right|_{z_e} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -c/m & 0 & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the system can be computed as

$$\lambda(A) = \{0, 0, 0, 0, -c/m, -c/m\}.$$

We see that the linearized system is not asymptotically stable since not all of the eigenvalues have strictly negative real part.

To determine whether the system is stable in the sense of Lyapunov, we must make use of the Jordan form. It can be shown that the Jordan form of A is given by

$$J = \left[\begin{array}{c|ccc|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -c/m & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -c/m \end{array}\right].$$

Since the second Jordan block has eigenvalue 0 and is not a simple eigenvalue, the linearization is unstable. ∇

Eigenvalues and Modes

The eigenvalues and eigenvectors of a system provide a description of the types of behavior the system can exhibit. For oscillatory systems, the term *mode* is often used to describe the vibration patterns that can occur. Figure 5.4 illustrates the modes for a system consisting of two masses connected by springs. One pattern is when both masses oscillate left and right in unison, and another is when the masses move toward and away from each other.

The initial condition response of a linear system can be written in terms of a matrix exponential involving the dynamics matrix A. The properties of the matrix A

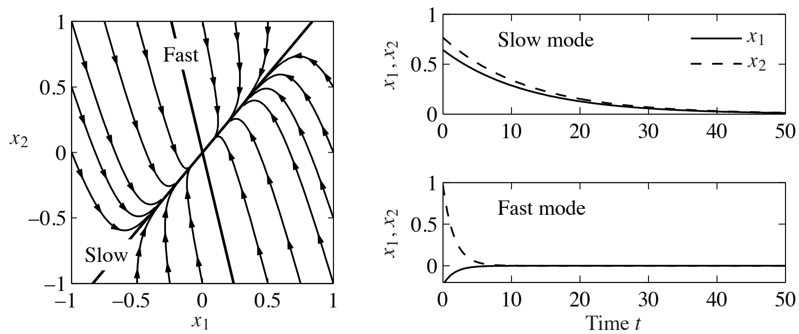


Figure 5.5: The notion of modes for a second-order system with real eigenvalues. The left figure shows the phase portrait and the modes corresponding to solutions that start on the eigenvectors (bold lines). The corresponding time functions are shown on the right.

therefore determine the resulting behavior of the system. Given a matrix $A \in \mathbb{R}^{n\times n}$, recall that v is an eigenvector of A with eigenvalue λ if

$$Av = \lambda v.$$

In general λ and v may be complex-valued, although if A is real-valued, then for any eigenvalue λ its complex conjugate λ^* will also be an eigenvalue (with v^* as the corresponding eigenvector).

Suppose first that λ and v are a real-valued eigenvalue/eigenvector pair for A. If we look at the solution of the differential equation for $x(0) = v$, it follows from the definition of the matrix exponential that

$$e^{At}v = \big(I + At + \frac{1}{2}A^2t^2 + \cdots\big)v = v + \lambda t v + \frac{\lambda^2 t^2}{2}v + \cdots = e^{\lambda t}v.$$

The solution thus lies in the subspace spanned by the eigenvector. The eigenvalue λ describes how the solution varies in time, and this solution is often called a *mode* of the system. (In the literature, the term "mode" is also often used to refer to the eigenvalue rather than the solution.)

If we look at the individual elements of the vectors x and v, it follows that

$$\frac{x_i(t)}{x_j(t)} = \frac{e^{\lambda t}v_i}{e^{\lambda t}v_j} = \frac{v_i}{v_j},$$

and hence the ratios of the components of the state x are constants for a (real) mode. The eigenvector thus gives the "shape" of the solution and is also called a *mode shape* of the system. Figure 5.5 illustrates the modes for a second-order system consisting of a fast mode and a slow mode. Notice that the state variables have the same sign for the slow mode and different signs for the fast mode.

The situation is more complicated when the eigenvalues of A are complex. Since A has real elements, the eigenvalues and the eigenvectors are complex conjugates

$\lambda = \sigma \pm i\omega$ and $v = u \pm iw$, which implies that

$$u = \frac{v + v^*}{2}, \qquad w = \frac{v - v^*}{2i}.$$

Making use of the matrix exponential, we have

$$e^{At}v = e^{\lambda t}(u + iw) = e^{\sigma t}\big((u\cos\omega t - w\sin\omega t) + i(u\sin\omega t + w\cos\omega t)\big),$$

from which it follows that

$$e^{At}u = \frac{1}{2}\Big(e^{At}v + e^{At}v^*\Big) = ue^{\sigma t}\cos\omega t - we^{\sigma t}\sin\omega t,$$
$$e^{At}w = \frac{1}{2i}\Big(e^{At}v - e^{At}v^*\Big) = ue^{\sigma t}\sin\omega t + we^{\sigma t}\cos\omega t.$$

A solution with initial conditions in the subspace spanned by the real part u and imaginary part w of the eigenvector will thus remain in that subspace. The solution will be a logarithmic spiral characterized by σ and ω. We again call the solution corresponding to λ a mode of the system, and v the mode shape.

If a matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then the initial condition response can be written as a linear combination of the modes. To see this, suppose for simplicity that we have all real eigenvalues with corresponding unit eigenvectors $v_1, \dots, v_n$. From linear algebra, these eigenvectors are linearly independent, and we can write the initial condition $x(0)$ as

$$x(0) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Using linearity, the initial condition response can be written as

$$x(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n.$$

Thus, the response is a linear combination of the modes of the system, with the amplitude of the individual modes growing or decaying as $e^{\lambda_i t}$. The case for distinct complex eigenvalues follows similarly (the case for nondistinct eigenvalues is more subtle and requires making use of the Jordan form discussed in the previous section).

Example 5.5 Coupled spring–mass system

Consider the spring–mass system shown in Figure 5.4. The equations of motion of the system are

$$m_1\ddot{q}_1 = -2kq_1 - c\dot{q}_1 + kq_2, \qquad m_2\ddot{q}_2 = kq_1 - 2kq_2 - c\dot{q}_2.$$

In state space form, we define the state to be $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$, and we can rewrite the equations as

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{2k}{m} & \dfrac{k}{m} & -\dfrac{c}{m} & 0 \\ \dfrac{k}{m} & -\dfrac{2k}{m} & 0 & -\dfrac{c}{m} \end{pmatrix} x.$$

We now define a transformation $z = Tx$ that puts this system into a simpler form. Let $z_1 = \frac{1}{2}(q_1 + q_2)$, $z_2 = \dot{z}_1$, $z_3 = \frac{1}{2}(q_1 - q_2)$ and $z_4 = \dot{z}_3$, so that

$$z = Tx = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} x.$$

In the new coordinates, the dynamics become

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k}{m} & -\dfrac{c}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{3k}{m} & -\dfrac{c}{m} \end{bmatrix} x,$$

and we see that the system is in block diagonal (or *modal*) form.

In the z coordinates, the states z_1 and z_2 parameterize one mode with eigenvalues $\lambda \approx c/(2\sqrt{km}) \pm i\sqrt{k/m}$, and the states z_3 and z_4 another mode with $\lambda \approx c/(2\sqrt{3km}) \pm i\sqrt{3k/m}$. From the form of the transformation T we see that these modes correspond exactly to the modes in Figure 5.4, in which q_1 and q_2 move either toward or against each other. The real and imaginary parts of the eigenvalues give the decay rates σ and frequencies ω for each mode. ∇

5.3 Input/Output Response

In the previous section we saw how to compute the initial condition response using the matrix exponential. In this section we derive the convolution equation, which includes the inputs and outputs as well.

The Convolution Equation

We return to the general input/output case in equation (5.3), repeated here:

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du. \tag{5.13}$$

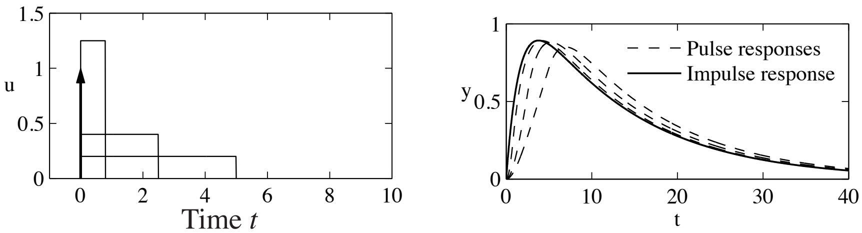
Using the matrix exponential, the solution to equation (5.13) can be written as follows.

Theorem 5.4. *The solution to the linear differential equation* (5.13) *is given by*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{5.14}$$

Proof. To prove this, we differentiate both sides and use the property (5.8) of the matrix exponential. This gives

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t) = Ax + Bu,$$



(a) Pulse and impulse functions (b) Pulse and impulse responses

Figure 5.6: Pulse response and impulse response. (a) The rectangles show pulses of width 5, 2.5 and 0.8, each with total area equal to 1. The arrow denotes an impulse $\delta(t)$ defined by equation (5.17). The corresponding pulse responses for a linear system with eigenvalues $\lambda = \{-0.08, -0.62\}$ are shown in (b) as dashed lines. The solid line is the true impulse response, which is well approximated by a pulse of duration 0.8.

which proves the result. Notice that the calculation is essentially the same as for proving the result for a first-order equation. □

It follows from equations (5.13) and (5.14) that the input/output relation for a linear system is given by

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{5.15}$$

It is easy to see from this equation that the output is jointly linear in both the initial conditions and the state, which follows from the linearity of matrix/vector multiplication and integration.

Equation (5.15) is called the *convolution equation*, and it represents the general form of the solution of a system of coupled linear differential equations. We see immediately that the dynamics of the system, as characterized by the matrix A, play a critical role in both the stability and performance of the system. Indeed, the matrix exponential describes *both* what happens when we perturb the initial condition and how the system responds to inputs.

Another interpretation of the convolution equation can be given using the concept of the *impulse response* of a system. Consider the application of an input signal $u(t)$ given by the following equation:

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & t < 0 \\ 1/\epsilon & 0 \le t < \epsilon \\ 0 & t \ge \epsilon. \end{cases} \tag{5.16}$$

This signal is a *pulse* of duration ϵ and amplitude $1/\epsilon$, as illustrated in Figure 5.6a. We define an *impulse* $\delta(t)$ to be the limit of this signal as $\epsilon \to 0$:

$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t). \tag{5.17}$$

This signal, sometimes called a *delta function,* is not physically achievable but provides a convenient abstraction in understanding the response of a system. Note that the integral of an impulse is 1:

$$\begin{aligned}\int_0^t \delta(\tau)\,d\tau &= \int_0^t \lim_{\epsilon\to 0} p_\epsilon(t)\,d\tau = \lim_{\epsilon\to 0}\int_0^t p_\epsilon(t)\,d\tau\\ &= \lim_{\epsilon\to 0}\int_0^\epsilon 1/\epsilon\,d\tau = 1 \qquad t > 0.\end{aligned}$$

In particular, the integral of an impulse over an arbitrarily short period of time is identically 1.

We define the *impulse response* of a system $h(t)$ to be the output corresponding to having an impulse as its input:

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)\,d\tau = Ce^{At}B, \tag{5.18}$$

where the second equality follows from the fact that $\delta(t)$ is zero everywhere except the origin and its integral is identically 1. We can now write the convolution equation in terms of the initial condition response, the convolution of the impulse response and the input signal, and the direct term:

$$y(t) = Ce^{At}x(0) + \int_0^t h(t-\tau)u(\tau)\,d\tau + Du(t). \tag{5.19}$$

One interpretation of this equation, explored in Exercise 5.2, is that the response of the linear system is the superposition of the response to an infinite set of shifted impulses whose magnitudes are given by the input $u(t)$. This is essentially the argument used in analyzing Figure 5.2 and deriving equation (5.5). Note that the second term in equation (5.19) is identical to equation (5.5), and it can be shown that the impulse response is formally equivalent to the derivative of the step response.

The use of pulses as approximations of the impulse function also provides a mechanism for identifying the dynamics of a system from data. Figure 5.6b shows the pulse responses of a system for different pulse widths. Notice that the pulse responses approach the impulse response as the pulse width goes to zero. As a general rule, if the fastest eigenvalue of a stable system has real part $-\sigma_{\max}$, then a pulse of length ϵ will provide a good estimate of the impulse response if $\epsilon\sigma_{\max} \ll 1$. Note that for Figure 5.6, a pulse width of $\epsilon = 1$ s gives $\epsilon\sigma_{\max} = 0.62$ and the pulse response is already close to the impulse response.

Coordinate Invariance

The components of the input vector u and the output vector y are given by the chosen inputs and outputs of a model, but the state variables depend on the coordinate frame chosen to represent the state. This choice of coordinates affects the values of the matrices A, B and C that are used in the model. (The direct term D is not affected since it maps inputs to outputs.) We now investigate some of the consequences of changing coordinate systems.

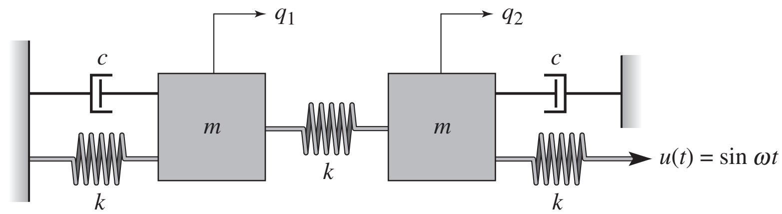


Figure 5.7: Coupled spring mass system. Each mass is connected to two springs with stiffness k and a viscous damper with damping coefficient c. The mass on the right is drive through a spring connected to a sinusoidally varying attachment.

Introduce new coordinates z by the transformation $z = Tx$, where T is an invertible matrix. It follows from equation (5.3) that

$$\frac{dz}{dt} = T(Ax + Bu) = TAT^{-1}z + TBu =: \tilde{A}z + \tilde{B}u,$$
$$y = Cx + Du = CT^{-1}z + Du =: \tilde{C}z + Du.$$

The transformed system has the same form as equation (5.3), but the matrices A, B and C are different:

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}. \tag{5.20}$$

There are often special choices of coordinate systems that allow us to see a particular property of the system, hence coordinate transformations can be used to gain new insight into the dynamics.

We can also compare the solution of the system in transformed coordinates to that in the original state coordinates. We make use of an important property of the exponential map,

$$e^{TST^{-1}} = Te^{S}T^{-1},$$

which can be verified by substitution in the definition of the matrix exponential. Using this property, it is easy to show that

$$x(t) = T^{-1}z(t) = T^{-1}e^{\tilde{A}t}Tx(0) + T^{-1}\int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)\,d\tau.$$

From this form of the equation, we see that if it is possible to transform A into a form $\tilde{A}$ for which the matrix exponential is easy to compute, we can use that computation to solve the general convolution equation for the untransformed state x by simple matrix multiplications. This technique is illustrated in the following example.

Example 5.6 Coupled spring–mass system

Consider the coupled spring–mass system shown in Figure 5.7. The input to this system is the sinusoidal motion of the end of the rightmost spring, and the output is the position of each mass, q_1 and q_2. The equations of motion are given by

$$m_1\ddot{q}_1 = -2kq_1 - c\dot{q}_1 + kq_2, \qquad m_2\ddot{q}_2 = kq_1 - 2kq_2 - c\dot{q}_2 + ku.$$

In state space form, we define the state to be $x = (q_1, q_2, \dot q_1, \dot q_2)$, and we can rewrite the equations as

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{bmatrix} u.$$

This is a coupled set of four differential equations and is quite complicated to solve in analytical form.

The dynamics matrix is the same as in Example 5.5, and we can use the coordinate transformation defined there to put the system in modal form:

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3k}{m} & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{2m} \\ 0 \\ -\frac{k}{2m} \end{bmatrix} u.$$

Note that the resulting matrix equations are block diagonal and hence decoupled. We can solve for the solutions by computing the solutions of two sets of second-order systems represented by the states (z_1, z_2) and (z_3, z_4). Indeed, the functional form of each set of equations is identical to that of a single spring–mass system. (The explicit solution is derived in Section 6.3.)

Once we have solved the two sets of independent second-order equations, we can recover the dynamics in the original coordinates by inverting the state transformation and writing $x = T^{-1}z$. We can also determine the stability of the system by looking at the stability of the independent second-order systems. ∇

Steady-State Response

Given a linear input/output system

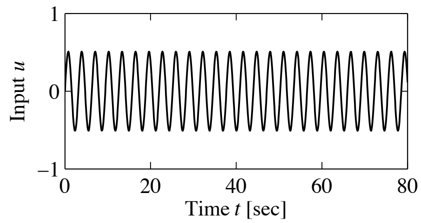
$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{5.21}$$

the general form of the solution to equation (5.21) is given by the convolution equation:

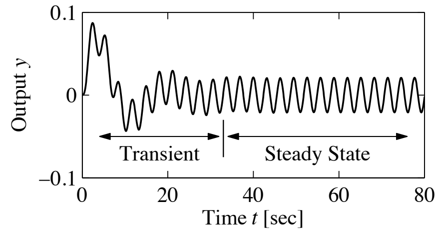
$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

We see from the form of this equation that the solution consists of an initial condition response and an input response.

The input response, corresponding to the last two terms in the equation above, itself consists of two components—the *transient response* and the *steady-state*



(a) Input



(b) Output

Figure 5.8: Transient versus steady-state response. The input to a linear system is shown in (a), and the corresponding output with $x(0) = 0$ is shown in (b). The output signal initially undergoes a transient before settling into its steady-state behavior.

response. The transient response occurs in the first period of time after the input is applied and reflects the mismatch between the initial condition and the steady-state solution. The steady-state response is the portion of the output response that reflects the long-term behavior of the system under the given inputs. For inputs that are periodic the steady-state response will often be periodic, and for constant inputs the response will often be constant. An example of the transient and the steady-state response for a periodic input is shown in Figure 5.8.

A particularly common form of input is a *step input*, which represents an abrupt change in input from one value to another. A *unit step* (sometimes called the Heaviside step function) is defined as

$$u = S(t) = \begin{cases} 0 & t = 0 \\ 1 & t > 0. \end{cases}$$

The *step response* of the system (5.21) is defined as the output $y(t)$ starting from zero initial condition (or the appropriate equilibrium point) and given a step input. We note that the step input is discontinuous and hence is not practically implementable. However, it is a convenient abstraction that is widely used in studying input/output systems.

We can compute the step response to a linear system using the convolution equation. Setting $x(0) = 0$ and using the definition of the step input above, we have

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = C\int_0^t e^{A(t-\tau)}Bd\tau + D \\ &= C\int_0^t e^{A\sigma}Bd\sigma + D = C\left(A^{-1}e^{A\sigma}B\right)\Big|_{\sigma=0}^{\sigma=t} + D \\ &= CA^{-1}e^{At}B - CA^{-1}B + D. \end{aligned}$$

If A has eigenvalues with negative real part (implying that the origin is a stable

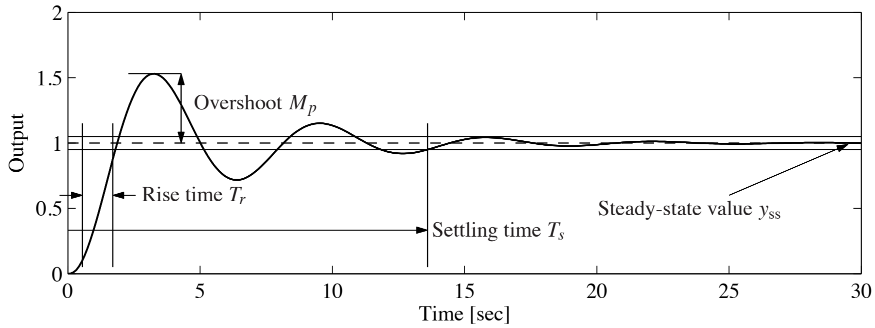


Figure 5.9: Sample step response. The rise time, overshoot, settling time and steady-state value give the key performance properties of the signal.

equilibrium point in the absence of any input), then we can rewrite the solution as

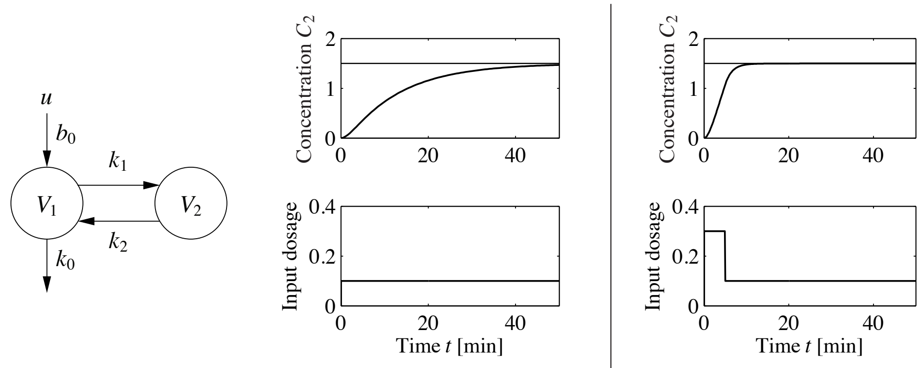
$$y(t) = \underbrace{CA^{-1}e^{At}B}_{\text{transient}} + \underbrace{D - CA^{-1}B}_{\text{steady-state}}, \qquad t > 0. \tag{5.22}$$

The first term is the transient response and decays to zero as $t \to \infty$. The second term is the steady-state response and represents the value of the output for large time.

A sample step response is shown in Figure 5.9. Several terms are used when referring to a step response. The *steady-state value* y_{ss} of a step response is the final level of the output, assuming it converges. The *rise time* T_r is the amount of time required for the signal to go from 10% of its final value to 90% of its final value. It is possible to define other limits as well, but in this book we shall use these percentages unless otherwise indicated. The *overshoot* M_p is the percentage of the final value by which the signal initially rises above the final value. This usually assumes that future values of the signal do not overshoot the final value by more than this initial transient, otherwise the term can be ambiguous. Finally, the *settling time* T_s is the amount of time required for the signal to stay within 2% of its final value for all future times. The settling time is also sometimes defined as reaching 1% or 5% of the final value (see Exercise 5.7). In general these performance measures can depend on the amplitude of the input step, but for linear systems the last three quantities defined above are independent of the size of the step.

Example 5.7 Compartment model

Consider the compartment model illustrated in Figure 5.10 and described in more detail in Section 3.6. Assume that a drug is administered by constant infusion in compartment V_1 and that the drug has its effect in compartment V_2. To assess how quickly the concentration in the compartment reaches steady state we compute the step response, which is shown in Figure 5.10b. The step response is quite slow, with a settling time of 39 min. It is possible to obtain the steady-state concentration much faster by having a faster injection rate initially, as shown in Figure 5.10c. The response of the system in this case can be computed by combining two step



(a) Schematic (b) Step input (c) Pulse input

Figure 5.10: Response of a compartment model to a constant drug infusion. A simple diagram of the system is shown in (a). The step response (b) shows the rate of concentration buildup in compartment 2. In (c) a pulse of initial concentration is used to speed up the response.

responses (Exercise 5.3). ∇

Another common input signal to a linear system is a sinusoid (or a combination of sinusoids). The *frequency response* of an input/output system measures the way in which the system responds to a sinusoidal excitation on one of its inputs. As we have already seen for scalar systems, the particular solution associated with a sinusoidal excitation is itself a sinusoid at the same frequency. Hence we can compare the magnitude and phase of the output sinusoid to the input. More generally, if a system has a sinusoidal output response at the same frequency as the input forcing, we can speak of the frequency response of the system.

To see this in more detail, we must evaluate the convolution equation (5.15) for $u = \cos \omega t$. This turns out to be a very messy calculation, but we can make use of the fact that the system is linear to simplify the derivation. In particular, we note that

$$\cos \omega t = \frac{1}{2}\Big(e^{i\omega t} + e^{-i\omega t}\Big).$$

Since the system is linear, it suffices to compute the response of the system to the complex input $u(t) = e^{st}$ and we can then reconstruct the input to a sinusoid by averaging the responses corresponding to $s = i\omega t$ and $s = -i\omega t$.

Applying the convolution equation to the input $u = e^{st}$ we have

$$\begin{aligned} y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Be^{s\tau}\,d\tau + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}B\,d\tau + De^{st}. \end{aligned}$$

If we assume that none of the eigenvalues of A are equal to $s = \pm i\omega$, then the matrix $sI - A$ is invertible, and we can write

$$\begin{aligned} y(t) &= Ce^{At}x(0) + Ce^{At}\left((sI - A)^{-1}e^{(sI-A)\tau}B\right)\Big|_0^t + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}(sI - A)^{-1}\left(e^{(sI-A)t} - I\right)B + De^{st} \\ &= Ce^{At}x(0) + C(sI - A)^{-1}e^{st}B - Ce^{At}(sI - A)^{-1}B + De^{st}, \end{aligned}$$

and we obtain

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}. \tag{5.23}$$

Notice that once again the solution consists of both a transient component and a steady-state component. The transient component decays to zero if the system is asymptotically stable and the steady-state component is proportional to the (complex) input $u = e^{st}$.

We can simplify the form of the solution slightly further by rewriting the steady-state response as

$$y_{ss}(t) = Me^{i\theta}e^{st} = Me^{(st+i\theta)},$$

where

$$Me^{i\theta} = C(sI - A)^{-1}B + D \tag{5.24}$$

and M and θ represent the magnitude and phase of the complex number $C(sI - A)^{-1}B + D$. When $s = i\omega$, we say that M is the *gain* and θ is the *phase* of the system at a given forcing frequency ω. Using linearity and combining the solutions for $s = +i\omega$ and $s = -i\omega$, we can show that if we have an input $u = A_u \sin(\omega t + \psi)$ and an output $y = A_y \sin(\omega t + \varphi)$, then

$$\text{gain}(\omega) = \frac{A_y}{A_u} = M, \qquad \text{phase}(\omega) = \varphi - \psi = \theta.$$

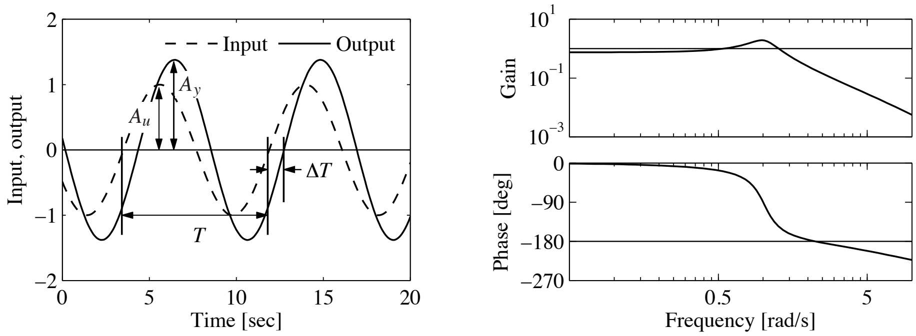
The steady-state solution for a sinusoid $u = \cos\omega t$ is now given by

$$y_{ss}(t) = M\cos(\omega t + \theta).$$

If the phase θ is positive, we say that the output *leads* the input, otherwise we say it *lags* the input.

A sample sinusoidal response is illustrated in Figure 5.11a. The dashed line shows the input sinusoid, which has amplitude 1. The output sinusoid is shown as a solid line and has a different amplitude plus a shifted phase. The gain is the ratio of the amplitudes of the sinusoids, which can be determined by measuring the height of the peaks. The phase is determined by comparing the ratio of the time between zero crossings of the input and output to the overall period of the sinusoid:

$$\theta = -2\pi \cdot \frac{\Delta T}{T}.$$



(a) Input/output response (b) Frequency response

Figure 5.11: Response of a linear system to a sinusoid. (a) A sinusoidal input of magnitude A_u (dashed) gives a sinusoidal output of magnitude A_y (solid), delayed by ΔT seconds. (b) Frequency response, showing gain and phase. The gain is given by the ratio of the output amplitude to the input amplitude, $M = A_y/A_u$. The phase lag is given by $\theta = -2\pi\,\Delta T/T$; it is negative for the case shown because the output lags the input.

A convenient way to view the frequency response is to plot how the gain and phase in equation (5.24) depend on ω (through $s = i\omega$). Figure 5.11b shows an example of this type of representation.

Example 5.8 Active band-pass filter
Consider the op amp circuit shown in Figure 5.12a. We can derive the dynamics of the system by writing the *nodal equations*, which state that the sum of the currents at any node must be zero. Assuming that $v_- = v_+ = 0$, as we did in Section 3.3, we have

$$0 = \frac{v_1 - v_2}{R_1} - C_1\frac{dv_2}{dt}, \quad 0 = C_1\frac{dv_2}{dt} + \frac{v_3}{R_2} + C_2\frac{dv_3}{dt}, \quad 0 = C_2\frac{dv_3}{dt} + \frac{v_3}{R_2} - C_1\frac{dv_2}{dt}.$$

Choosing v_2 and v_3 as our states and using the first and last equations, we obtain

$$\frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1C_1}, \qquad \frac{dv_3}{dt} = \frac{-v_3}{R_2C_2} - \frac{v_1 - v_2}{R_1C_2}.$$

Rewriting these in linear state space form, we obtain

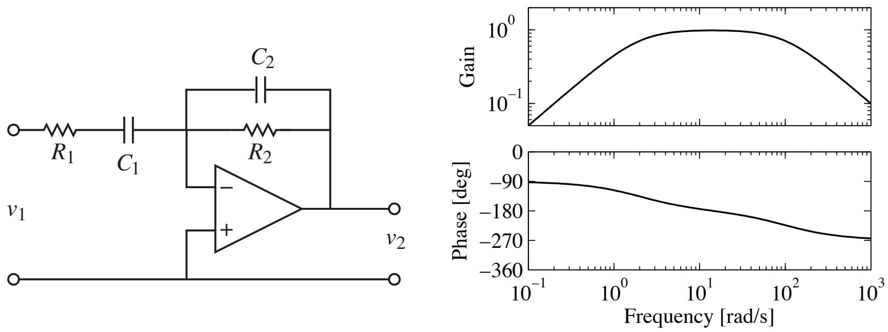
$$\frac{dx}{dt} = \begin{pmatrix} -\dfrac{1}{R_1C_1} & 0 \\ \dfrac{1}{R_1C_2} & -\dfrac{1}{R_2C_2} \end{pmatrix} x + \begin{pmatrix} \dfrac{1}{R_1C_1} \\ \dfrac{-1}{R_1C_2} \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} x, \tag{5.25}$$

where $x = (v_2, v_3)$, $u = v_1$ and $y = v_3$.

The frequency response for the system can be computed using equation (5.24):

$$Me^{j\theta} = C(sI - A)^{-1}B + D = -\frac{R_2}{R_1}\frac{R_1C_1s}{(1 + R_1C_1s)(1 + R_2C_2s)}, \qquad s = i\omega.$$

The magnitude and phase are plotted in Figure 5.12b for $R_1 = 100\ \Omega$, $R_2 = 5\ \mathrm{k}\Omega$ and $C_1 = C_2 = 100\ \mu\mathrm{F}$. We see that the circuit passes through signals with



(a) Circuit diagram (b) Frequency response

Figure 5.12: Active band-pass filter. The circuit diagram (a) shows an op amp with two *RC* filters arranged to provide a band-pass filter. The plot in (b) shows the gain and phase of the filter as a function of frequency. Note that the phase starts at -90° due to the negative gain of the operational amplifier.

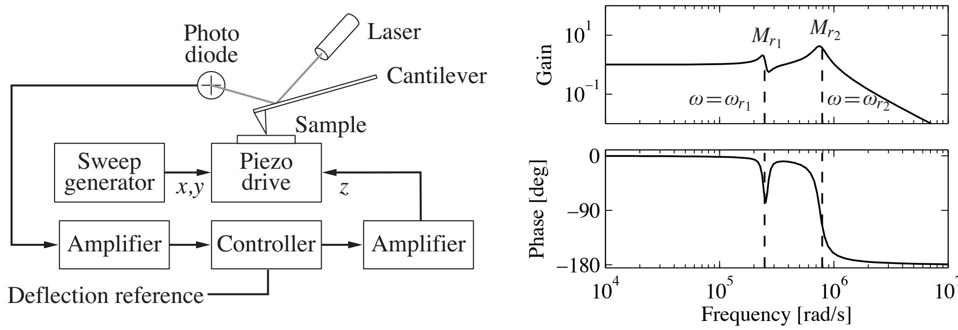
frequencies at about 10 rad/s, but attenuates frequencies below 5 rad/s and above 50 rad/s. At 0.1 rad/s the input signal is attenuated by 20× (0.05). This type of circuit is called a *band-pass filter* since it passes through signals in the band of frequencies between 5 and 50 rad/s. ∇

As in the case of the step response, a number of standard properties are defined for frequency responses. The gain of a system at $\omega = 0$ is called the *zero frequency gain* and corresponds to the ratio between a constant input and the steady output:

$$M_0 = -CA^{-1}B + D.$$

The zero frequency gain is well defined only if A is invertible (and, in particular, if it does not have eigenvalues at 0). It is also important to note that the zero frequency gain is a relevant quantity only when a system is stable about the corresponding equilibrium point. So, if we apply a constant input $u = r$, then the corresponding equilibrium point $x_e = -A^{-1}Br$ must be stable in order to talk about the zero frequency gain. (In electrical engineering, the zero frequency gain is often called the *DC gain*. DC stands for direct current and reflects the common separation of signals in electrical engineering into a direct current (zero frequency) term and an alternating current (AC) term.)

The *bandwidth* ω_b of a system is the frequency range over which the gain has decreased by no more than a factor of $1/\sqrt{2}$ from its reference value. For systems with nonzero, finite zero frequency gain, the bandwidth is the frequency where the gain has decreased by $1/\sqrt{2}$ from the zero frequency gain. For systems that attenuate low frequencies but pass through high frequencies, the reference gain is taken as the high-frequency gain. For a system such as the band-pass filter in Example 5.8, bandwidth is defined as the range of frequencies where the gain is larger than $1/\sqrt{2}$ of the gain at the center of the band. (For Example 5.8 this would give a bandwidth of approximately 50 rad/s.)



(a) AFM block diagram

(b) Frequency response

Figure 5.13: AFM frequency response. (a) A block diagram for the vertical dynamics of an atomic force microscope in contact mode. The plot in (b) shows the gain and phase for the piezo stack. The response contains two frequency peaks at resonances of the system, along with an antiresonance at $\omega = 268$ krad/s. The combination of a resonant peak followed by an antiresonance is common for systems with multiple lightly damped modes.

Another important property of the frequency response is the *resonant peak* M_r, the largest value of the frequency response, and the *peak frequency* ω_{mr}, the frequency where the maximum occurs. These two properties describe the frequency of the sinusoidal input that produces the largest possible output and the gain at the frequency.

Example 5.9 Atomic force microscope in contact mode

Consider the model for the vertical dynamics of the atomic force microscope in contact mode, discussed in Section 3.5. The basic dynamics are given by equation (3.23). The piezo stack can be modeled by a second-order system with undamped natural frequency ω_3 and damping ratio ζ_3. The dynamics are then described by the linear system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k/(m_1+m_2) & -c/(m_1+m_2) & 1/m_2 & 0 \\ 0 & 0 & 0 & \omega_3 \\ 0 & 0 & -\omega_3 & -2\zeta_3\omega_3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_3 \end{pmatrix} u,$$

$$y = \frac{m_2}{m_1+m_2} \begin{pmatrix} \dfrac{m_1 k}{m_1+m_2} & \dfrac{m_1 c}{m_1+m_2} & 1 & 0 \end{pmatrix} x,$$

where the input signal is the drive signal to the amplifier and the output is the elongation of the piezo. The frequency response of the system is shown in Figure 5.13b. The zero frequency gain of the system is $M_0 = 1$. There are two resonant poles with peaks $M_{r1} = 2.12$ at $\omega_{mr1} = 238$ krad/s and $M_{r2} = 4.29$ at $\omega_{mr2} = 746$ krad/s. The bandwidth of the system, defined as the lowest frequency where the gain is $\sqrt{2}$ less than the zero frequency gain, is $\omega_b = 292$ krad/s. There is also a dip in the gain $M_d = 0.556$ for $\omega_{md} = 268$ krad/s. This dip, called an *antiresonance*, is associated with a dip in the phase and limits the performance when the system is controlled by simple controllers, as we will see in Chapter 10. ∇

Sampling

It is often convenient to use both differential and difference equations in modeling and control. For linear systems it is straightforward to transform from one to the other. Consider the general linear system described by equation (5.13) and assume that the control signal is constant over a sampling interval of constant length h. It follows from equation (5.14) of Theorem 5.4 that

$$x(t+h) = e^{Ah}x(t) + \int_t^{t+h} e^{t+h-\tau} Bu(k)\, d\tau = \Phi x(t) + \Gamma u(t), \tag{5.26}$$

where we have assumed that the discontinuous control signal is continuous from the right. The behavior of the system at the sampling times $t = kh$ is described by the difference equation

$$x[k+1] = \Phi x[k] + \Gamma u[k], \qquad y[k] = Cx[k] + Du[k]. \tag{5.27}$$

Notice that the difference equation (5.27) is an exact representation of the behavior of the system at the sampling instants. Similar expressions can also be obtained if the control signal is linear over the sampling interval.

The transformation from (5.26) to (5.27) is called *sampling*. The relations between the system matrices in the continuous and sampled representations are as follows:

$$\Phi = e^{Ah}, \quad \Gamma = \Bigl(\int_0^h e^{As}\, ds\Bigr)B; \qquad A = \frac{1}{h}\log\Phi, \quad B = \Bigl(\int_0^h e^{At}\, dt\Bigr)^{-1}\Gamma. \tag{5.28}$$

Notice that if A is invertible, we have

$$\Gamma = A^{-1}\bigl(e^{Ah} - I\bigr).$$

All continuous-time systems can be sampled to obtain a discrete-time version, but there are discrete-time systems that do not have a continuous-time equivalent. The precise condition is that the matrix Φ cannot have real eigenvalues on the negative real axis.

Example 5.10 IBM Lotus server

In Example 2.4 we described how the dynamics of an IBM Lotus server were obtained as the discrete-time system

$$y[k+1] = ay[k] + bu[k],$$

where $a = 0.43$, $b = 0.47$ and the sampling period is $h = 60$ s. A differential equation model is needed if we would like to design control systems based on continuous-time theory. Such a model is obtained by applying equation (5.28); hence

$$A = \frac{\log a}{h} = -0.0141, \qquad B = \Bigl(\int_0^h e^{At}\, dt\Bigr)^{-1} b = 0.0116,$$

and we find that the difference equation can be interpreted as a sampled version of the ordinary differential equation

$$\frac{dx}{dt} = -0.0141x + 0.0116u.$$

∇

5.4 Linearization

As described at the beginning of the chapter, a common source of linear system models is through the approximation of a nonlinear system by a linear one. These approximations are aimed at studying the local behavior of a system, where the nonlinear effects are expected to be small. In this section we discuss how to locally approximate a system by its linearization and what can be said about the approximation in terms of stability. We begin with an illustration of the basic concept using the cruise control example from Chapter 3.

Example 5.11 Cruise control
The dynamics for the cruise control system were derived in Section 3.1 and have the form

$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mgC_r \operatorname{sgn}(v) - \frac{1}{2}\rho C_v A v^2 - mg\sin\theta, \tag{5.29}$$

where the first term on the right-hand side of the equation is the force generated by the engine and the remaining three terms are the rolling friction, aerodynamic drag and gravitational disturbance force. There is an equilibrium (v_e, u_e) when the force applied by the engine balances the disturbance forces.

To explore the behavior of the system near the equilibrium we will linearize the system. A Taylor series expansion of equation (5.29) around the equilibrium gives

$$\frac{d(v-v_e)}{dt} = a(v-v_e) - b_g(\theta-\theta_e) + b(u-u_e) + \text{higher order terms}, \tag{5.30}$$

where

$$a = \frac{u_e\alpha_n^2 T'(\alpha_n v_e) - \rho C_v A v_e}{m}, \qquad b_g = g\cos\theta_e, \qquad b = \frac{\alpha_n T(\alpha_n v_e)}{m}. \tag{5.31}$$

Notice that the term corresponding to rolling friction disappears if $v \neq 0$. For a car in fourth gear with $v_e = 25$ m/s, $\theta_e = 0$ and the numerical values for the car from Section 3.1, the equilibrium value for the throttle is $u_e = 0.1687$ and the parameters are $a = -0.0101$, $b = 1.32$ and $c = 9.8$. This linear model describes how small perturbations in the velocity about the nominal speed evolve in time.

Figure 5.14 shows a simulation of a cruise controller with linear and nonlinear models; the differences between the linear and nonlinear models are small, and hence the linearized model provides a reasonable approximation. ∇

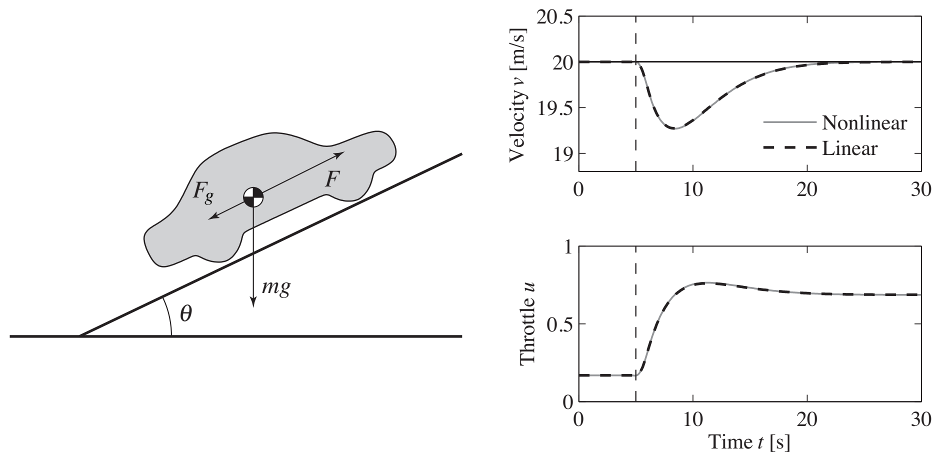


Figure 5.14: Simulated response of a vehicle with PI cruise control as it climbs a hill with a slope of 4°. The solid line is the simulation based on a nonlinear model, and the dashed line shows the corresponding simulation using a linear model. The controller gains are $k_p = 0.5$ and $k_i = 0.1$.

Jacobian Linearization Around an Equilibrium Point

To proceed more formally, consider a single-input, single-output nonlinear system

$$\frac{dx}{dt} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R},$$
$$y = h(x,u), \qquad y \in \mathbb{R}, \tag{5.32}$$

with an equilibrium point at $x = x_e$, $u = u_e$. Without loss of generality we can assume that $x_e = 0$ and $u_e = 0$, although initially we will consider the general case to make the shift of coordinates explicit.

To study the *local* behavior of the system around the equilibrium point (x_e, u_e), we suppose that $x - x_e$ and $u - u_e$ are both small, so that nonlinear perturbations around this equilibrium point can be ignored compared with the (lower-order) linear terms. This is roughly the same type of argument that is used when we do small-angle approximations, replacing $\sin\theta$ with θ and $\cos\theta$ with 1 for θ near zero.

As we did in Chapter 4, we define a new set of state variables z, as well as inputs v and outputs w:

$$z = x - x_e, \qquad v = u - u_e, \qquad w = y - h(x_e, u_e).$$

These variables are all close to zero when we are near the equilibrium point, and so in these variables the nonlinear terms can be thought of as the higher-order terms in a Taylor series expansion of the relevant vector fields (assuming for now that these exist).

Formally, the *Jacobian linearization* of the nonlinear system (5.32) is

$$\frac{dz}{dt} = Az + Bv, \qquad w = Cz + Dv, \tag{5.33}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}. \tag{5.34}$$

The system (5.33) approximates the original system (5.32) when we are near the equilibrium point about which the system was linearized. Using Theorem 4.3, if the linearization is asymptotically stable, then the equilibrium point x_e is locally asymptotically stable for the full nonlinear system.

It is important to note that we can define the linearization of a system only near an equilibrium point. To see this, consider a polynomial system

$$\frac{dx}{dt} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + u,$$

where $a_0 \neq 0$. A set of equilibrium points for this system is given by $(x_e, u_e) = (x_e, -a_0 - a_1 x_e - a_2 x_e^2 - a_3 x_e^3)$, and we can linearize around any of them. Suppose that we try to linearize around the origin of the system $x = 0, u = 0$. If we drop the higher-order terms in x, then we get

$$\frac{dx}{dt} = a_0 + a_1 x + u,$$

which is *not* the Jacobian linearization if $a_0 \neq 0$. The constant term must be kept, and it is not present in (5.33). Furthermore, even if we kept the constant term in the approximate model, the system would quickly move away from this point (since it is "driven" by the constant term a_0), and hence the approximation could soon fail to hold.

Software for modeling and simulation frequently has facilities for performing linearization symbolically or numerically. The MATLAB command `trim` finds the equilibrium, and `linmod` extracts linear state space models from a SIMULINK system around an operating point.

Example 5.12 Vehicle steering

Consider the vehicle steering system introduced in Example 2.8. The nonlinear equations of motion for the system are given by equations (2.23)–(2.25) and can be written as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} v\cos(\alpha(\delta)+\theta) \\ v\sin(\alpha(\delta)+\theta) \\ \dfrac{v_0}{b}\tan\delta \end{bmatrix}, \qquad \alpha(\delta) = \arctan\Bigl(\frac{a\tan\delta}{b}\Bigr),$$

where x, y and θ are the position and orientation of the center of mass of the vehicle, v_0 is the velocity of the rear wheel, b is the distance between the front and rear wheels and δ is the angle of the front wheel. The function $\alpha(\delta)$ is the angle between the velocity vector and the vehicle's length axis.

We are interested in the motion of the vehicle about a straight-line path ($\theta = \theta_0$) with fixed velocity $v_0 \neq 0$. To find the relevant equilibrium point, we first set $\dot\theta = 0$ and we see that we must have $\delta = 0$, corresponding to the steering wheel being straight. This also yields $\alpha = 0$. Looking at the first two equations in the dynamics, we see that the motion in the xy direction is by definition *not* at equilibrium since $\dot{\xi}^2 + \dot{\eta}^2 = v_0^2 \neq 0$. Therefore we cannot formally linearize the full model.

Suppose instead that we are concerned with the lateral deviation of the vehicle from a straight line. For simplicity, we let $\theta_e = 0$, which corresponds to driving along the x axis. We can then focus on the equations of motion in the y and θ directions. With some abuse of notation we introduce the state $x = (y, \theta)$ and $u = \delta$. The system is then in standard form with

$$f(x,u) = \begin{Bmatrix} v \sin(\alpha(u) + x_2) \\ \dfrac{v_0}{b} \tan u \end{Bmatrix}, \quad \alpha(u) = \arctan\Bigl(\frac{a \tan u}{b}\Bigr), \quad h(x,u) = x_1.$$

The equilibrium point of interest is given by $x = (0, 0)$ and $u = 0$. To compute the linearized model around this equilibrium point, we make use of the formulas (5.34). A straightforward calculation yields

$$A = \frac{\partial f}{\partial x}\bigg|_{\substack{x=0\\u=0}} = \begin{Bmatrix} 0 & v_0 \\ 0 & 0 \end{Bmatrix}, \qquad B = \frac{\partial f}{\partial u}\bigg|_{\substack{x=0\\u=0}} = \begin{Bmatrix} a v_0/b \\ v_0/b \end{Bmatrix},$$
$$C = \frac{\partial h}{\partial x}\bigg|_{\substack{x=0\\u=0}} = \begin{Bmatrix} 1 & 0 \end{Bmatrix}, \qquad D = \frac{\partial h}{\partial u}\bigg|_{\substack{x=0\\u=0}} = 0,$$

and the linearized system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du \tag{5.35}$$

thus provides an approximation to the original nonlinear dynamics.

The linearized model can be simplified further by introducing normalized variables, as discussed in Section 2.3. For this system, we choose the wheel base b as the length unit and the unit as the time required to travel a wheel base. The normalized state is thus $z = (x_1/b, x_2)$, and the new time variable is $\tau = v_0 t/b$. The model (5.35) then becomes

$$\frac{dz}{d\tau} = \begin{Bmatrix} z_2 + \gamma u \\ u \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 0 & 0 \end{Bmatrix} z + \begin{Bmatrix} \gamma \\ 1 \end{Bmatrix} u, \qquad y = \begin{Bmatrix} 1 & 0 \end{Bmatrix} z, \tag{5.36}$$

where $\gamma = a/b$. The normalized linear model for vehicle steering with nonslipping wheels is thus a linear system with only one parameter. ∇

Feedback Linearization

Another type of linearization is the use of feedback to convert the dynamics of a nonlinear system into those of a linear one. We illustrate the basic idea with an example.

Example 5.13 Cruise control

Consider again the cruise control system from Example 5.11, whose dynamics are

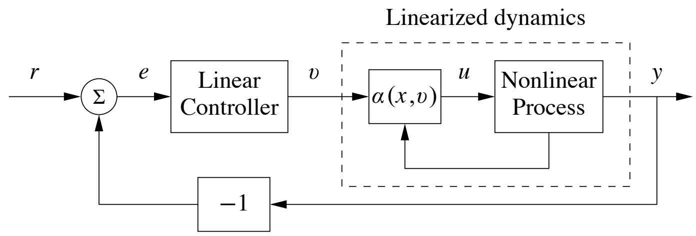


Figure 5.15: Feedback linearization. A nonlinear feedback of the form $u = \alpha(x, v)$ is used to modify the dynamics of a nonlinear process so that the response from the input v to the output y is linear. A linear controller can then be used to regulate the system's dynamics.

given in equation (5.29):

$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mgC_r \,\mathrm{sgn}(v) - \frac{1}{2}\rho C_d A v^2 - mg\sin\theta.$$

If we choose u as a feedback law of the form

$$u = \frac{1}{\alpha_n T(\alpha_n v)}\left(u' + mgC_r \,\mathrm{sgn}(v) + \frac{1}{2}\rho C_v A v^2\right), \tag{5.37}$$

then the resulting dynamics become

$$m\frac{dv}{dt} = u' + d, \tag{5.38}$$

where $d = -mg\sin\theta$ is the disturbance force due the slope of the road. If we now define a feedback law for u' (such as a proportional-integral-derivative [PID] controller), we can use equation (5.37) to compute the final input that should be commanded.

Equation (5.38) is a linear differential equation. We have essentially "inverted" the nonlinearity through the use of the feedback law (5.37). This requires that we have an accurate measurement of the vehicle velocity v as well as an accurate model of the torque characteristics of the engine, gear ratios, drag and friction characteristics and mass of the car. While such a model is not generally available (remembering that the parameter values can change), if we design a good feedback law for u', then we can achieve robustness to these uncertainties. ∇

More generally, we say that a system of the form

$$\frac{dx}{dt} = f(x, u), \qquad y = h(x),$$

is *feedback linearizable* if we can find a control law $u = \alpha(x, v)$ such that the resulting closed loop system is input/output linear with input v and output y, as shown in Figure 5.15. To fully characterize such systems is beyond the scope of this text, but we note that in addition to changes in the input, the general theory also allows for (nonlinear) changes in the states that are used to describe the system, keeping only the input and output variables fixed. More details of this process can be found in the textbooks by Isidori [106] and Khalil [123].

One case that comes up relatively frequently, and is hence worth special mention, is the set of mechanical systems of the form

$$M(q)\ddot{q} + C(q, \dot{q}) = B(q)u.$$

Here $q \in \mathbb{R}^n$ is the configuration of the mechanical system, $M(q) \in \mathbb{R}^{n\times n}$ is the configuration-dependent inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^n$ represents the Coriolis forces and additional nonlinear forces (such as stiffness and friction) and $B(q) \in \mathbb{R}^{n\times p}$ is the input matrix. If $p = n$, then we have the same number of inputs and configuration variables, and if we further have that $B(q)$ is an invertible matrix for all configurations q, then we can choose

$$u = B^{-1}(q)\big(M(q)v - C(q, \dot{q})\big). \tag{5.39}$$

The resulting dynamics become

$$M(q)\ddot{q} = M(q)v \qquad \Longrightarrow \qquad \ddot{q} = v,$$

which is a linear system. We can now use the tools of linear system theory to analyze and design control laws for the linearized system, remembering to apply equation (5.39) to obtain the actual input that will be applied to the system.

This type of control is common in robotics, where it goes by the name of *computed torque*, and in aircraft flight control, where it is called *dynamic inversion*. Some modeling tools like Modelica can generate the code for the inverse model automatically. One caution is that feedback linearization can often cancel out beneficial terms in the natural dynamics, and hence it must be used with care. Extensions that do not require complete cancellation of nonlinearities are discussed in Khalil [123] and Krstić et al. [129].

5.5 Further Reading

The majority of the material in this chapter is classical and can be found in most books on dynamics and control theory, including early works on control such as James, Nichols and Phillips [110] and more recent textbooks such as Dorf and Bishop [61], Franklin, Powell and Emami-Naeini [79] and Ogata [162]. An excellent presentation of linear systems based on the matrix exponential is given in the book by Brockett [44], a more comprehensive treatment is given by Rugh [171] and an elegant mathematical treatment is given in Sontag [182]. Material on feedback linearization can be found in books on nonlinear control theory such as Isidori [106] and Khalil [123]. The idea of characterizing dynamics by considering the responses to step inputs is due to Heaviside, he also introduced an operator calculus to analyze linear systems. The unit step is therefore also called the *Heaviside step function*. Analysis of linear systems was simplified significantly, but Heaviside's work was heavily criticized because of lack of mathematical rigor, as described in the biography by Nahin [157]. The difficulties were cleared up later by the mathematician Laurent Schwartz who developed *distribution theory* in the late 1940s. In engineering, linear systems have traditionally been analyzed using Laplace transforms as described in Gardner and Barnes [81]. Use of the matrix exponential started with developments of control theory in the 1960s, strongly stimulated by a textbook by Zadeh and Desoer [207]. Use of matrix techniques expanded rapidly when the powerful methods of numeric linear algebra were packaged in programs like LabVIEW, MATLAB and Mathematica.

Exercises

5.1 (Response to the derivative of a signal) Show that if $y(t)$ is the output of a linear system corresponding to input $u(t)$, then the output corresponding to an input $\dot u(t)$ is given by $\dot y(t)$. (Hint: Use the definition of the derivative: $\dot y(t) = \lim_{\epsilon\to 0}\big(y(t+\epsilon) - y(t)\big)/\epsilon$.)

5.2 (Impulse response and convolution) Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear system to an input $u(t)$ (assuming a zero initial condition) can be written as

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system.

5.3 (Pulse response for a compartment model) Consider the compartment model given in Example 5.7. Compute the step response for the system and compare it with Figure 5.10b. Use the principle of superposition to compute the response to the pulse input shown in Figure 5.10c.

5.4 (Matrix exponential for second-order system) Assume that $\zeta < 1$ and let $\omega_d = \omega_0\sqrt{1-\zeta^2}$. Show that

$$\exp\begin{pmatrix} -\zeta\omega_0 & \omega_d \\ -\omega_d & -\zeta\omega_0 \end{pmatrix} t = \begin{pmatrix} e^{-\zeta\omega_0 t}\cos\omega_d t & e^{-\zeta\omega_0 t}\sin\omega_d t \\ -e^{-\zeta\omega_0 t}\sin\omega_d t & e^{-\zeta\omega_0 t}\cos\omega_d t \end{pmatrix}.$$

5.5 (Lyapunov function for a linear system) Consider a linear system $\dot x = Ax$ with $\operatorname{Re}\lambda_j < 0$ for all eigenvalues λ_j of the matrix A. Show that the matrix

$$P = \int_0^\infty e^{A^T\tau} Q e^{A\tau}\,d\tau$$

defines a Lyapunov function of the form $V(x) = x^T P x$.

5.6 (Nondiagonal Jordan form) Consider a linear system with a Jordan form that is non-diagonal.

(a) Show that there exists a periodic input that does not produce a periodic output.

(b) Prove Proposition 5.3 by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time. Extend this argument to the case of complex eigenvalues with Re $\lambda = 0$ by using the block Jordan form

$$J_i = \begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}.$$

5.7 (Rise time for a first-order system) Consider a first-order system of the form

$$\tau \frac{dx}{dt} = -x + u, \qquad y = x.$$

We say that the parameter τ is the *time constant* for the system since the zero input system approaches the origin as $e^{-t/\tau}$. For a first-order system of this form, show that the rise time for a step response of the system is approximately 2τ, and that 1%, 2%, and 5% settling times approximately corresponds to 4.6τ, 4τ and 2τ.

5.8 (Discrete-time systems) Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x_0 + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

(c) Let $u[k] = A\sin(\omega k)$ represent an oscillatory input with frequency $\omega < \pi$ (to avoid "aliasing"). Show that the steady-state component of the response has gain M and phase θ, where

$$Me^{j\theta} = C(e^{i\omega} I - A)^{-1} B + D.$$

(d) Show that if we have a nonlinear discrete-time system

$$\begin{aligned} x[k] &= f(x[k], u[k]), \qquad x[k] \in \mathbb{R}^n,\ u \in \mathbb{R}, \\ y[k] &= h(x[k], u[k]), \qquad y \in \mathbb{R}, \end{aligned}$$

then we can linearize the system around an equilibrium point (x_e, u_e) by defining the matrices A, B, C and D as in equation (5.34).

5.9 (Keynesian economics) Consider the following simple Keynesian macroeconomic model in the form of a linear discrete-time system discussed in Exercise 5.8:

$$\begin{Bmatrix} C[t+1] \\ I[t+1] \end{Bmatrix} = \begin{Bmatrix} a & a \\ ab-a & ab \end{Bmatrix} \begin{Bmatrix} C[t] \\ I[t] \end{Bmatrix} + \begin{Bmatrix} a \\ ab \end{Bmatrix} G[t],$$
$$Y[t] = C[t] + I[t] + G[t].$$

Determine the eigenvalues of the dynamics matrix. When are the magnitudes of the eigenvalues less than 1? Assume that the system is in equilibrium with constant values capital spending C, investment I and government expenditure G. Explore what happens when government expenditure increases by 10%. Use the values $a = 0.25$ and $b = 0.5$.

5.10 Consider a scalar system

$$\frac{dx}{dt} = 1 - x^3 + u.$$

Compute the equilibrium points for the unforced system ($u = 0$) and use a Taylor series expansion around the equilibrium point to compute the linearization. Verify that this agrees with the linearization in equation (5.33).

5.11 (Transcriptional regulation) Consider the dynamics of a genetic circuit that implements *self-repression*: the protein produced by a gene is a repressor for that gene, thus restricting its own production. Using the models presented in Example 2.13, the dynamics for the system can be written as

$$\frac{dm}{dt} = \frac{\alpha}{1+kp^2} + \alpha_0 - \gamma m - u, \qquad \frac{dp}{dt} = \beta m - \delta p, \tag{5.40}$$

where u is a disturbance term that affects RNA transcription and $m, p \geq 0$. Find the equilibrium points for the system and use the linearized dynamics around each equilibrium point to determine the local stability of the equilibrium point and the step response of the system to a disturbance.