CHAPTER 6

METHODS OF FUNCTIONAL ANALYSIS IN ELASTICITY

The purpose of this chapter is to present some basic theorems in elasticity concerning the boundary value problems of elastostatics and elastodynamics. The techniques are primarily those of linear and nonlinear functional analysis. The first five sections are motivated by questions of existence and uniqueness, but the results turn out to bear on basic questions such as: what constitutive inequalities should one impose? Section 6.6 discusses what is currently known (to the authors) concerning a longstanding problem: are minima of the energy stable? This turns out to be a very subtle yet significant point. The final section gives an application of nonlinear analysis to a control problem for a beam as a sample of how the general machinery can be used in a problem arising from nonlinear elasticity.

This chapter is not intended to be comprehensive. Some of the important topics omitted are the methods of variational inequalities (see Duvaut and Lions [1972]) and, except for a few illustrative examples, the existence theory for rods, plates, and shells. The topics omitted include both the approximate models such as the von Karmen equations and the full nonlinear models (see, for example, Ciarlet [1983], Berger [1977], Antman [1978a], [1979b], [1980c], and references therein).

6.1 ELLIPTIC OPERATORS AND LINEAR ELASTOSTATICS

This section discusses existence and uniqueness for linear elastostatics. The methods used are based on elliptic theory. Basic results in this subject are stated without proof; for these proofs, the reader should consult, for example, Agmon

[1965], Friedman [1969], Morrey [1966], or Wells [1980]. These results are then applied to linear elastostatics. The methods and emphasis here differ slightly from those in the important reference work of Fichera [1972a].

To simplify notation and the spaces involved, we shall explicitly assume the body $\mathcal{B}$ is a bounded open set Ω in $\mathbb{R}^n$ with piecewise smooth boundary. The reader is forewarned that standard usage and a number of notational conflicts have resulted in us adopting various notations for the volume element throughout this chapter.

The linearized equations under consideration are as follows (see Chapter 4):

$$\operatorname{div}(\mathbf{a}\cdot\nabla \boldsymbol{u})(x) = \boldsymbol{f}(x), \qquad x \in \Omega, \tag{1}$$

where $\boldsymbol{f} = -\rho \boldsymbol{b} - \operatorname{div} \boldsymbol{\sigma}$, $\boldsymbol{b}$ is the body force, $\boldsymbol{\sigma}$ is the Cauchy stress in the configuration we are linearizing about, and $\mathbf{a} = \boldsymbol{\sigma} \otimes \boldsymbol{\delta} + \mathbf{c}$ is the corresponding elasticity tensor; we assume $\mathbf{a}$ is C^∞. We assume boundary conditions of displacement or traction are used and shall assume them to be homogeneous. [This is no real loss of generality, for if the boundary conditions are not homogeneous, say $\boldsymbol{u}$ equals some displacement $\tilde{\boldsymbol{u}}$ on $\partial\Omega$, replace $\boldsymbol{f}$ by $\boldsymbol{f} + \operatorname{div}(\mathbf{a}\cdot\nabla\tilde{\boldsymbol{u}})$ and $\boldsymbol{u}$ by $\boldsymbol{u} - \tilde{\boldsymbol{u}}$.]

1.1 Definitions Let L^2 denote the Hilbert space of all displacements $\boldsymbol{u}\colon \Omega \longrightarrow \mathbb{R}^n$ that are square integrable:

$$\|\boldsymbol{u}\|_{L^2}^2 = \int_\Omega \|\boldsymbol{u}\|^2\, dv < \infty.$$

Let H^1 denote the Hilbert space of all displacements $\boldsymbol{u}\colon \Omega \longrightarrow \mathbb{R}^n$ that belong to L^2 and such that the gradient $\nabla \boldsymbol{u}$ (initially only a distribution) is also an L^2 tensor. The H^1-norm is defined by

$$\|\boldsymbol{u}\|_{H^1} = \left(\int_\Omega \|\boldsymbol{u}\|^2\, dv + \int_\Omega \|\nabla\boldsymbol{u}\|^2\, dv\right)^{1/2}.$$

Similarly, define H^s for positive integers s, with the convention $H^0 = L^2$. (One can show that H^1 is a Hilbert space: basic facts about these *Sobolev spaces* will be given shortly without proof; we shall not require these detailed proofs, which are contained in the references cited earlier and Adams [1975]. See Box 1.1 for further information and some sample proofs.)

If p is a real number $(1 \le p \le \infty)$, L^p denotes the space of displacements $\boldsymbol{u}$ such that

$$\|\boldsymbol{u}\|_{L^p} = \left(\int_\Omega \|\boldsymbol{u}\|^p\, dv\right)^{1/p} < \infty$$

(with $\|\boldsymbol{u}\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\Omega} \|\boldsymbol{u}(x)\|$). Define $W^{1,p}$ to be the space of $\boldsymbol{u} \in L^p$ such that $\nabla\boldsymbol{u}$ is L^p as well, with norm

$$\|\boldsymbol{u}\|_{W^{1,p}} = (\|\boldsymbol{u}\|_{L^p}^p + \|\nabla\boldsymbol{u}\|_{L^p}^p)^{1/p}.$$

Similarly, define the Sobolev spaces $W^{s,p}$ for s a positive integer.[1] In this

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[1]For s not an integer, these may still be defined by means of the Fourier transform; see Box 1.1.

notation, we have the following coincidences:

$$H^0 = L^2, \qquad W^{s,2} = H^s.$$

Let H^2_∂ denote the set of $\boldsymbol{u} \in H^2$ with homogeneous boundary conditions of displacement or traction (or both) imposed; that is,

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial_d \quad \text{and} \quad (\mathsf{a} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial_\tau$$

(where one of ∂_d or ∂_τ may be empty). (One has to show that H^2_∂ is a well-defined closed subspace of H^2; this is true and relies on the so-called "trace theorems"; cf. Adams [1975], Section 5.22 and p. 330 below.)

Define the linear operator

$$A: H^2_\partial \longrightarrow L^2 \quad \text{by} \quad A(\boldsymbol{u}) = \operatorname{div}(\mathsf{a} \cdot \nabla \boldsymbol{u}). \tag{2}$$

Problem (1) can be phrased as follows: *given* $\boldsymbol{f} \in L^2$, *can we solve* $A\boldsymbol{u} = \boldsymbol{f}$?

Throughout this section, we shall make the following:

1.2 Assumptions (i) The elasticity tensor a is *hyperelastic;* that is,

$$\mathsf{a}^{abcd} = \mathsf{a}^{cdab}.$$

(ii) The elasticity tensor a is *strongly elliptic;* that is, there is an $\epsilon > 0$ such that

$$\mathsf{a}^{abcd}(x)\xi_a\xi_c\eta_b\eta_d \geq \epsilon \,\|\boldsymbol{\xi}\|^2\,\|\boldsymbol{\eta}\|^2$$

for all vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n$ and all $x \in \Omega$.

1.3 Proposition *Assumption* (i) *is equivalent to symmetry of A; that is,*

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, A\boldsymbol{v}\rangle$$

for $\boldsymbol{u}, \boldsymbol{v} \in H^2_\partial$, *where* $\langle\ ,\ \rangle$ *is the* L^2 *inner product.*

Proof Assume (i). Then

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega (\mathsf{a}^{abcd}u_{c|d})_{|b}v_a\, dv.$$

We can integrate by parts noting that the boundary terms are zero, giving the following:

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = -\int_\Omega \mathsf{a}^{abcd}u_{c|d}v_{a|b}\, dv.$$

(This requires some further justification since $\boldsymbol{u}$ and $\boldsymbol{v}$ are not smooth; this is done by an approximation argument which is omitted.) This expression is symmetric in $\boldsymbol{u}$ and $\boldsymbol{v}$. The converse is left as an exercise. ∎

1.4 Definition The mapping B from $H^1 \times H^1$ to $\mathbb{R}$ defined by

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathsf{a}^{abcd}u_{c|d}v_{a|b}\, dv \tag{3}$$

(which is a continuous symmetric bilinear form) is called the *Dirichlet form.*

The next proposition, discussed in Box 1.1, relates the Dirichlet form and strong ellipticity.

1.5 Proposition *Let A be defined as above, and let (i) of 1.2 hold. Then strong ellipticity is equivalent to* Gårding's inequality:

$$\begin{cases} \text{There are constants } c > 0 \text{ and } d > 0 \text{ such that} \\ \qquad B(\boldsymbol{u}, \boldsymbol{u}) \geq c\,\|\boldsymbol{u}\|_{H^1}^2 - d\,\|\boldsymbol{u}\|_{L^2}^2 \\ \text{for all } \boldsymbol{u} \in H^1. \end{cases}$$

Problem 1.1 Use Gårding's inequality and the inequality $2ab \leq \epsilon a^2 + (1/\epsilon)b^2$ to show that $-\langle A\boldsymbol{u}, \boldsymbol{u}\rangle \geq c_1\,\|\boldsymbol{u}\|_{H^1}^2 - d_1\,\|\boldsymbol{u}\|_{L_2}^2$ for all $\boldsymbol{u} \in H_\partial^2$, for suitable positive constants c_1 and d_1.

There are some technical difficulties with the mixed problem that will force us to make hypotheses on ∂_d and ∂_τ that are stronger than one would ideally like. Specifically, for the validity of certain theorems, we shall need to assume that the *closures* of ∂_d and ∂_τ do not intersect. This means, in effect, that ∂_d and ∂_τ do not touch. An example of something that *is* allowed by this is when $\mathcal{B}$ is an annulus, with ∂_d the outer boundary and ∂_τ the inner boundary. For the pure displacement or traction problem, we likewise require $\partial\mathcal{B}$ to be smooth, say C^1. Again this is an unpleasant assumption, for it eliminates bodies with corners.

The assumptions in the previous paragraph may well be just technical and not necessary, but to eliminate them would require nontrivial modifications of what follows. The difficulty lies with the *regularity* of solutions; for example, if one is solving $\Delta\phi = f$ with Dirichlet boundary conditions and $\partial\mathcal{B}$ has corners, the issue of whether or not ϕ is in H^2 when f is in L^2 is delicate. If $\mathcal{B}$ is a square or a cube, this is true and can be seen by using Fourier series. If the angles are not 90°, however, this assertion need not be true in Sobolev spaces, but analogous results may be true in other spaces. As far as we know, this theory has not yet been developed, except for isolated cases. (See also the remarks on p. 371.)

For both the linear and nonlinear theories, these problems do not really affect the existence question if one seeks a generalized solution with less regularity. However, if regularity is desired, or if the inverse function theorem is to be used, the above assumptions must be made, or else the spaces must be modified. The situation calls for further research to see if these difficulties are just technical or are of physical interest. Presumably the modifications in the spaces needed correspond to known asymptotic solutions near corners, cracks, etc.

For the rest of this section we shall make the assumptions of 1.2 and those on $\partial\mathcal{B}$ above without explicit mention.

1.6 Basic Facts about Elliptic Operators

(i) *Elliptic estimates.* If $\boldsymbol{u} \in H_\partial^2$, then for each $s \geq 2$ there is a constant K such that

$$\|\boldsymbol{u}\|_{H^s} \leq K(\|A\boldsymbol{u}\|_{H^{s-2}} + \|\boldsymbol{u}\|_{L^2}),$$

More generally, if $1 < p < \infty$,

$$\|u\|_{W^{s,p}} \leq K(\|Au\|_{W^{s-2,p}} + \|u\|_{L^p}).$$

The proof is omitted. See Box 1.1 for the proof in a simple case.

(ii) *The kernel of A is finite dimensional.* [*Proof:* The elliptic estimates show that $\|u\|_{H^2} \leq C\|u\|_{L^2}$ on Ker A. Rellich's theorem, which states that the inclusion $H^s \longrightarrow H^r$ $(s > r)$ is compact, then implies that the unit ball in Ker A is compact. Hence it is finite dimensional.[2]] We also get $\|u\|_{H^s} \leq C\|u\|_{L^2}$ so $H^\infty = \bigcap_{s\geq 0} H^s \subset$ Ker A. Since $H^s \subset C^k$ if $s > n/2 + k$ (see Box 1.1), $H^\infty = C^\infty$, so *elements of* Ker A *are smooth* in this case. Similarly, *the spectrum of A is discrete* and *each eigenvalue has finite multiplicity.*

(iii) *The range of A is closed in* L^2. The outline of the proof of this for those knowing some functional analysis requires the following facts:

(1) Let T be a closed linear operator in a Banach space $\mathfrak{X}$ with domain $\mathfrak{D}(T)$ (T closed means its graph is closed). Then if T is continuous, $\mathfrak{D}(T)$ is closed.

This follows from the definition.

(2) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, $C\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ a $1-1$ continuous linear map with closed range, and $B\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ a compact linear operator. Suppose $C + B$ is $1 - 1$. Then $C + B$ has closed range.

Proof The operator $(C + B)^{-1}$ defined on Range$(C + B)$ is closed since the inverse of a $1 - 1$ closed operator is closed (consider the graphs). Thus, by (1), it suffices to show that $(C + B)^{-1}$ is continuous. Suppose that $(C + B)^{-1}y_n = x_n$ and $y_n \longrightarrow 0$. Suppose $x_n \not\rightarrow 0$. By passing to a subsequence, we can suppose $\|x_n\| \geq \epsilon > 0$. Let $\bar{x}_n = x_n/\|x_n\|$. Then

$$\|(C + B)(\bar{x}_n)\| = \frac{1}{\|x_n\|}\|(C + B)x_n\| = \frac{\|y_n\|}{\|x_n\|} \leq \frac{1}{\epsilon}\|y_n\|,$$

so $(C + B)(\bar{x}_n) \longrightarrow 0$. Since B is compact and $\|\bar{x}_n\| = 1$, we can suppose $B(\bar{x}_n)$ converges. Thus $C(\bar{x}_n)$ converges too. Since C has closed range, $C(\bar{x}_n) \longrightarrow C\bar{x}$ for some $\bar{x} \in \mathfrak{X}$. By the closed graph theorem, C has a bounded inverse on its range. Thus $\bar{x}_n \longrightarrow \bar{x}$, so $\|\bar{x}\| = 1$. But $(C + B)(\bar{x}_n) \longrightarrow 0$ and so $(C + B)(\bar{x}) = 0$ and thus $\bar{x} = 0$, a contradiction.

(3) Now, let $\tilde{\mathfrak{X}} = H^2_\partial$, $\mathfrak{Y} = L^2 \times L^2$, $Cu = (Au, u)$, and $Bu = (0, -u)$. Let $\mathfrak{X}$ be the Hilbert space orthogonal complement of ker $(C + B)$ in $\tilde{\mathfrak{X}}$, and restrict C and B to $\mathfrak{X}$. By construction, $C + B$ is $1 - 1$. Clearly C is $1 - 1$; moreover, using the sum norm on $\mathfrak{Y}$,

$$\|Cu\|_{\mathfrak{Y}} = \|Au\|_{L^2} + \|u\|_{L^2} \geq \alpha\|u\|_{\mathfrak{X}}$$

by the elliptic estimate (i) with $\alpha = 1/K$. This estimate shows directly that the

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[2]A Banach space is finite dimensional if and only if its unit ball is compact (another theorem of Rellich).

range of C is closed. Finally, B is compact by Rellich's theorem. Thus, by (2), $(C + B)(u) = (Au, 0)$ has closed range. Hence A has closed range. ∎

(iv) *Weak solutions are strong solutions.* Suppose $\boldsymbol{u} \in L^2, \boldsymbol{f} \in L^2$, and $\langle \boldsymbol{u}, A\boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle$ for all $\boldsymbol{v} \in H^2_\partial$. Then $\boldsymbol{u} \in H^2_\partial$ and $A\boldsymbol{u} = \boldsymbol{f}$.

Proof We sketch the proof of (iv) as follows: This is a *regularity* result. The statement $\langle \boldsymbol{u}, A\boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle$ implies that $A\boldsymbol{u} = \boldsymbol{f}$ in the sense of distributions, so $A\boldsymbol{u} \in L^2$. Thus, as in the basic elliptic estimate (i), $\boldsymbol{u} \in H^2$. In fact, the most delicate part here is that $\boldsymbol{u}$ is H^2 near the boundary, a relatively deep fact. That $\boldsymbol{u}$ must satisfy the boundary conditions comes about *formally* as follows. Integrating by parts,

$$\begin{aligned}\langle \boldsymbol{u}, A\boldsymbol{v}\rangle &= \int_\Omega \boldsymbol{u}\cdot\operatorname{div}(\mathbf{a}\cdot\nabla\boldsymbol{v})\,dv \\ &= -\int_\Omega \nabla\boldsymbol{u}\cdot\mathbf{a}\cdot\nabla\boldsymbol{v}\,dv + \int_{\partial\Omega} \boldsymbol{u}\cdot[\mathbf{a}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{n}]\,da \\ &= \int_\Omega \operatorname{div}(\mathbf{a}\cdot\nabla\boldsymbol{u})\boldsymbol{v}\,dv + \int_{\partial\Omega} \{\boldsymbol{u}\cdot[\mathbf{a}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{n}] - \boldsymbol{v}\cdot[\mathbf{a}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{n}]\}\,da.\end{aligned}$$

Suppose the boundary conditions are $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$. If this is to equal $\langle \boldsymbol{f}, \boldsymbol{v}\rangle$ for all $\boldsymbol{v} \in H^2_\partial$, then first take $\boldsymbol{v}$ of compact support in Ω to get $\operatorname{div}(\mathbf{a}\cdot\nabla\boldsymbol{u}) = \boldsymbol{f}$. Thus, we are left with the identity

$$\int_{\partial\Omega} \boldsymbol{u}\cdot[\mathbf{a}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{n}]\,da = 0.$$

Since no condition is imposed on $\mathbf{a}\cdot[\nabla\boldsymbol{v}\cdot\boldsymbol{n}]$ on $\partial\Omega$, $\boldsymbol{u}$ must vanish on $\partial\Omega$. A similar argument holds for the traction or mixed case. ∎

From these facts we can deduce the following crucial result for symmetric elliptic operators:

1.7 Fredholm Alternative Theorem $L^2 = \text{Range } A \oplus \text{Ker } A$, *an* L^2 *orthogonal sum.*

Proof By 1.6(iii), Range A is closed. Let $\mathfrak{N}$ be its L^2 orthogonal complement. We claim that $\mathfrak{N} = \text{Ker } A$. This will prove the result. First of all, if $\boldsymbol{u} \in \text{Ker } A$ and $\boldsymbol{f} = A\boldsymbol{v} \in \text{Range } A$, then as A is symmetric, $\langle \boldsymbol{u}, \boldsymbol{f}\rangle = \langle \boldsymbol{u}, A\boldsymbol{v}\rangle = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 0$. Conversely, if $\boldsymbol{u} \in \mathfrak{N}$, then $\langle \boldsymbol{u}, A\boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in H^2_\partial$. Thus by 1.6(iv) $\boldsymbol{u} \in H^2_\partial$ and $A\boldsymbol{u} = \mathbf{0}$; that is, $\boldsymbol{u} \in \text{Ker } A$. ∎

Remarks From the elliptic estimates, the decomposition in 1.7 has a regularity property:

$$W^{s,p} = \text{Range}(A \mid H^2_\partial \cap W^{s+2,p}) \oplus \text{Ker } A.$$

For non-symmetric elliptic operators, the Fredholm alternative reads

$$L^2 = \text{Range } A \oplus \text{Ker } A^*,$$

where A^* is the adjoint of A.

Problem 1.2 Use the Fredholm alternative applied to $\lambda I - A$ and Gårding's inequality to show that $\lambda I - A \colon H_\partial^2 \longrightarrow L^2$ is an isomorphism, where $\lambda > d_1$, with d_1 given in Problem 1.1.

From 1.7 we obtain the following main result for our boundary value problem.

1.8 Theorem *Let $f \in L^2$. Then there exists a $\boldsymbol{u} \in H_\partial^2$ such that $A\boldsymbol{u} = \boldsymbol{f}$ if and only if*

$$\langle f, \boldsymbol{h} \rangle = 0 \qquad \textit{for all } \boldsymbol{h} \in \operatorname{Ker} A.$$

In this case, (i) $\boldsymbol{u}$ is unique up to addition of elements in Ker A, *and (ii) if f is of class $W^{s,p}$, $\boldsymbol{u}$ is of class $W^{s+2,p}$ (up to and including the boundary).*

Theorem 1.8 gives complete information on when the boundary value problem

$$A\boldsymbol{u} = f, \qquad \boldsymbol{u} \in H_\partial^2$$

is solvable for $\boldsymbol{u}$. To be useful, one must be able to compute Ker A in a specific instance. We show how this may be done for the important case of stable classical elasticity (see Section 4.3). Thus we now deal with the case $A\boldsymbol{u} = \operatorname{div}(\mathbf{c} \cdot \nabla \boldsymbol{u})$ on $\mathcal{B} = \Omega \subset \mathbb{R}^3$.

1.9 Definition Let c^{abcd} be a classical elasticity tensor on Ω. We say $\mathbf{c}$ is *uniformly pointwise stable* if there is an $\eta > 0$ such that

$$\epsilon = \tfrac{1}{2}\boldsymbol{e} \cdot \mathbf{c} \cdot \boldsymbol{e} \geq \eta \| \boldsymbol{e} \|^2$$

for all symmetric $\boldsymbol{e}$ (see 3.5 and 3.6 in Section 4.3).

The argument of 3.9, Section 4.3 shows that *uniform pointwise stability implies strong ellipticity, but not conversely.*

1.10 Lemma *If $\mathbf{c}$ is uniformly pointwise stable and if $\boldsymbol{u} \in$ Ker A, then $\boldsymbol{u}$ is an infinitesimal Euclidean motion; i.e. a rotation or translation.*

Proof If $\boldsymbol{u} \in \operatorname{Ker} A$, then $\operatorname{div}(\mathbf{c} \cdot \nabla \boldsymbol{u}) = \mathbf{0}$. Thus, multiplying by $\boldsymbol{u}$ and integrating,

$$0 = \int_\Omega \boldsymbol{u} \cdot \operatorname{div}(\mathbf{c} \cdot \nabla \boldsymbol{u})\, dv = -\int_\Omega \nabla \boldsymbol{u} \cdot \mathbf{c} \cdot \nabla \boldsymbol{u}\, dv.$$

Here we integrated by parts using the boundary conditions. By $\mathbf{c}$'s symmetries, $\nabla \boldsymbol{u} \cdot \mathbf{c} \cdot \nabla \boldsymbol{u} = \boldsymbol{e} \cdot \mathbf{c} \cdot \boldsymbol{e}$, where $\boldsymbol{e} = \frac{1}{2}\mathcal{L}_{\boldsymbol{u}} \boldsymbol{g}$; that is, $e_{ab} = \frac{1}{2}(u_{a|b} + u_{b|a})$. By 1.9 we conclude that $\boldsymbol{e} = \mathbf{0}$, so $\boldsymbol{u}$ is an infinitesimal rotation or translation. ▮

For the displacement or mixed problem, any infinitesimal rotation must vanish since it vanishes on a portion of $\partial\Omega$ containing three linearly independent points. For the traction problem any infinitesimal rotation or translation lies in Ker A. Thus, we have proved the following:

1.11 Theorem *For classical elasticity, assume* $\mathbf{c}$ *is uniformly pointwise stable, and the hypotheses on the boundary conditions preceding 1.6 hold. Then*

(i) *For displacement or mixed boundary conditions and any* $\boldsymbol{f} \in L^2$, *the problem* $\operatorname{div}(\mathbf{c}\cdot\nabla\boldsymbol{u}) = \boldsymbol{f}$, *has a unique solution* $\boldsymbol{u} \in H^2_\partial$. *If* $\boldsymbol{f} \in W^{s,p}$, *then* $\boldsymbol{u} \in W^{s+2,p}$ *for* $s \geq 0$, $1 < p < \infty$.

(ii) (*Traction Problem*) *The equation*

$$\operatorname{div}(\mathbf{c}\cdot\nabla\boldsymbol{u}) = \boldsymbol{f}, \qquad \boldsymbol{f} \in L^2$$

with $\mathbf{c}\cdot\nabla\boldsymbol{u} = \mathbf{0}$ *on* $\partial\Omega$ *has a solution* $\boldsymbol{u} \in H^2_\partial$ *if and only if*

$$\int_\Omega \boldsymbol{f}(x)\cdot(\boldsymbol{a} + \boldsymbol{b}x)\, dv = 0,$$

where $\boldsymbol{a}$ *is any constant vector and* $\boldsymbol{b}$ *is any skew-symmetric matrix. If this holds,* $\boldsymbol{u}$ *is unique up to the addition of a term of the form* $\boldsymbol{a} + \boldsymbol{b}x$, $\boldsymbol{b}$ *a skew matrix. If* $\boldsymbol{f} \in W^{s,p}$ $(s \geq 0,\ 1 < p < \infty)$, *then* $\boldsymbol{u} \in W^{s+2,p}$.

Sometimes Korn's inequalities are used in studying results like those in Theorem 1.11; however, our presentation did not require or use them. They are very relevant for questions of stability in linear elastodynamics, as we shall see in Section 6.3. We shall state these without detailed proof (see Fichera [1972a], Friedrichs [1947], Payne and Weinberger [1961], the remark below and Box 1.1 for the proofs).

1.12 Korn's Inequalities

(i) *First Inequality.* For $\boldsymbol{u} \in H^2_\partial$ satisfying displacement boundary conditions on $\partial_d \subset \partial\Omega$, we have

$$\int_\Omega \|\boldsymbol{e}\|^2\, dv \geq c\,\|\boldsymbol{u}\|^2_{H^1}$$

for a susitable constant $c > 0$ independent of $\boldsymbol{u}$.

(ii) *Second Inequality.* There is a constant $\bar{c} > 0$ such that

$$\int_\Omega \|\boldsymbol{e}\|^2\, dv + \int_\Omega \|\boldsymbol{u}\|^2\, dv \geq \bar{c}\,\|\boldsymbol{u}\|^2_{H^1}$$

for all $\boldsymbol{u} \in H^1$.

The first inequality is fairly straightforward (see Box 1.1) while the second is more subtle. For the displacement or mixed problem, uniform pointwise stability implies that we have

$$-\langle A\boldsymbol{u}, \boldsymbol{u}\rangle = \int_\Omega \boldsymbol{e}\cdot\mathbf{c}\cdot\boldsymbol{e}\, dv \geq 2\eta \int \|\boldsymbol{e}\|^2\, dv \geq 2\eta c\,\|\boldsymbol{u}\|^2_{H^1}.$$

So it follows that Ker $A = \{0\}$ in this case, reproducing what we found above. As indicated in 2.8, Section 5.2, this inequality will guarantee dynamic stability.

Korn's inequalities are actually special cases of Gårdings inequality for (not necessarily square) elliptic systems. The Lie derivative operator

$$\boldsymbol{u} \mapsto \boldsymbol{e}$$

is in fact elliptic in the sense of *systems* of partial differential equations (see, for example, Berger and Ebin [1969]). Gårdings inequality applied to L is exactly Korn's second inequality. The first inequality comes from the general fact that for elliptic operators with constant coefficients satisfying zero boundary conditions, Gårdings inequality reads $B(\boldsymbol{u}, \boldsymbol{u}) \geq c\|\boldsymbol{u}\|_{H^1}^2$—that is, stability. (This is seen by examining the proof; see Box 1.1.)

Box 1.1 ***Some Useful Inequalities***

This box discusses four topics: (1) Gårdings inequality (see 1.5); (2) Korn's first inequality (see 1.12); (3) a sample elliptic estimate (see 1.6(i)); and (4) some key Sobolev inequalities.

(1) *Garding's inequality*

(a) Let us prove the inequality

$$B(\boldsymbol{u}, \boldsymbol{u}) \geq c\|\boldsymbol{u}\|_{H^1}^2 - d\|\boldsymbol{u}\|_{L^2}^2$$

in case $\mathbf{a}$ is strongly elliptic and is constant (independent of x). We shall also assume $\boldsymbol{u}$ is C^∞ with compact support in $\mathbb{R}^n$.

Let $\hat{\boldsymbol{u}}$ be the Fourier transform of $\boldsymbol{u}$:

$$\hat{\boldsymbol{u}}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot x} \boldsymbol{u}(x)\, dx \quad (i = \sqrt{-1}).$$

The H^1-norm of $\boldsymbol{u}$ is given by

$$\|\boldsymbol{u}\|_{H^1}^2 = \sum_{l,j} \int_{\mathbb{R}^n} \left(\frac{\partial u^l}{\partial x^j}\right)^2 dx + \sum_l \int_{\mathbb{R}^n} (u^l)^2\, dx.$$

Since the Fourier transform preserves the L^2-norm (Plancherel's theorem), and $(\partial u^l/\partial x^j)\hat{} = \xi_j \hat{u}^l/i$, we have

$$\|\boldsymbol{u}\|_{H^1}^2 = \int_{\mathbb{R}^n} |\boldsymbol{\xi} \otimes \hat{\boldsymbol{u}}(\boldsymbol{\xi})|^2\, d\xi + \int_{\mathbb{R}^n} |\hat{\boldsymbol{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}.$$

By strong ellipticity,

$$\begin{aligned} B(\boldsymbol{u}, \boldsymbol{u}) &= \int_{\mathbb{R}^n} a_i{}^j{}_k{}^l \frac{\partial u^i}{\partial x^j} \frac{\partial u^k}{\partial x^l}\, dx \\ &= \int_{\mathbb{R}^n} \xi_j \xi_l a_i{}^j{}_k{}^l \hat{u}^i \hat{u}^k\, d\boldsymbol{\xi} \geq \int_{\mathbb{R}^n} \epsilon |\boldsymbol{\xi}|^2 |\hat{\boldsymbol{u}}|^2\, d\boldsymbol{\xi} \\ &= \epsilon \int_{\mathbb{R}^n} |\boldsymbol{\xi} \otimes \hat{\boldsymbol{u}}|^2\, d\boldsymbol{\xi}. \end{aligned}$$

Thus, we can take $c = \epsilon$ and $d = \epsilon$. The case of variable coefficients and a general domain requires a modification of this basic idea (cf. Yosida [1971] or Morrey [1966]).

(b) Relevant to Gårding's inequality is the *Poincaré inequality*, one version of which states that if $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$, then

$$\|\boldsymbol{u}\|_{L^2} \leq C\|D\boldsymbol{u}\|_{L^2}.$$

In this case, H_0^1 can be normed by $\|D\boldsymbol{u}\|_{L^2}$, and so if $\mathbf{a}$ has constant coefficients, we can choose $d = 0$ in Gårding's inequality.

(c) Next we discuss *Hadamard's theorem* [1902], which states that *Gårding's inequality implies strong ellipticity*. Let us again just prove a simple case. Suppose $\mathbf{a}$ is constant and assume $\boldsymbol{0} \in \Omega$. Choose $\boldsymbol{u}(x) = \boldsymbol{\xi}\phi(\boldsymbol{\lambda}\cdot x)$, where $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$ are constant vectors in $\mathbb{R}^n$ and $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is smooth with compact support. Then

$$\begin{aligned} B(\boldsymbol{u}, \boldsymbol{u}) &= \int_\Omega a^{ijkl}\xi_i\lambda_j\xi_k\lambda_l\,|\phi'(\boldsymbol{\lambda}\cdot x)|^2\,dx \\ &= \left(\int_\Omega |\phi'(\boldsymbol{\lambda}\cdot x)|^2\,dx\right)(a^{ijkl}\xi_i\xi_k\lambda_j\lambda_l). \end{aligned}$$

By assumption, we get

$$\begin{aligned} &\left(\int_\Omega |\phi'(\boldsymbol{\lambda}\cdot x)|^2\,dx\right)(a^{ijkl}\xi_i\xi_k\lambda_j\lambda_l) \\ &\quad\geq c\left(\int_\Omega |\boldsymbol{\xi}|^2|\boldsymbol{\lambda}|^2\,\phi'(\boldsymbol{\lambda}\cdot x)\,dx\right) \\ &\quad\quad + (c - d)\left(\int_\Omega |\boldsymbol{\xi}|^2|\phi(\boldsymbol{\lambda}\cdot x)|^2\,dx\right). \end{aligned}$$

It is easy to see that we can choose a sequence of ϕ_n's such that

$$\left(\int_\Omega |\phi_n(\boldsymbol{\lambda}\cdot x)|^2\,dx\right)\Big/\int_\Omega |\phi_n'(\boldsymbol{\lambda}\cdot x)|^2\,dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Dividing the preceding inequality by $\int_\Omega |\phi_n'(\boldsymbol{\lambda}\cdot x)|^2\,dx$ and letting $n \longrightarrow \infty$, we see that strong ellipticity holds with $\epsilon = c$.

(2) *Korn's first inequality* Let us prove that

$$\int |\boldsymbol{e}|^2\,dx \geq c\|\boldsymbol{u}\|_{H^1}^2$$

for $\boldsymbol{u}$ a C^∞ displacement in $\mathbb{R}^n$ with compact support. As in Gårding's inequality, the general case can be reduced to this one. Let $\hat{\boldsymbol{u}}(\boldsymbol{\xi})$ be the Fourier transform of $\boldsymbol{u}$ as defined above. Thus

$$\hat{e}(\boldsymbol{\xi}) = \tfrac{1}{2}(\boldsymbol{\xi} \otimes \hat{\boldsymbol{u}} + \hat{\boldsymbol{u}} \otimes \boldsymbol{\xi}).$$

Therefore, by Plancherel's theorem for Fourier transforms of tensor fields,

$$\int_{\mathbb{R}^n} |e|^2\, dx = \int_{\mathbb{R}^n} |\hat{e}(\xi)|^2\, d\xi$$
$$= \tfrac{1}{4} \int \sum_{i,j} (\xi_i \hat{u}_j + \xi_j \hat{u}_i)^2\, d\xi.$$

However,

$$\sum_{i,j} (\xi_i \hat{u}_j + \xi_j \hat{u}_i)^2 = \sum_{i,j} (\xi_i^2 \hat{u}_j^2 + \xi_j^2 \hat{u}_i^2 + 2\xi_i \hat{u}_i \xi_j \hat{u}_j)$$
$$= 4 \sum_i (\xi_i^2 \hat{u}_i^2) + \sum_{i \neq j} (\xi_i^2 \hat{u}_j^2 + \xi_j^2 \hat{u}_i^2)$$
$$+ 2(\sum_{i \neq j} \xi_i \hat{u}_i \xi_j \hat{u}_j).$$

Using the inequality $2\xi_i \hat{u}_i \xi_j \hat{u}_j \geq -(\xi_i^2 \hat{u}_i^2 + \xi_j^2 \hat{u}_j^2)$, we get

$$\sum_{i,j} (\xi_i \hat{u}_j + \xi_j \hat{u}_i)^2 \geq 2 \sum_i \xi_i^2 \hat{u}_i^2 + \sum_{i \neq j} (\xi_i^2 \hat{u}_j^2 + \xi_j^2 \hat{u}_i^2)$$
$$= 2 \sum_{i,j} \xi_i^2 \hat{u}_j^2.$$

Thus,
$$\int_{\mathbb{R}^n} |e|^2\, dx \geq 2 \int_{\mathbb{R}^n} |Du|^2\, dx$$

which gives Korn's first inequality (cf. the Poincaré inequality above).

As pointed out in the remark preceding this box, this inequality is, secretly, Gårding's inequality for the elliptic operator $\boldsymbol{u} \mapsto \boldsymbol{e}$.

(3) *A sample elliptic estimate* Suppose **a** is constant and strongly elliptic and $\boldsymbol{u}$ is smooth with compact support on $\mathbb{R}^n$. We will prove the inequality

$$\|\boldsymbol{u}\|_{H^2}^2 \leq C(\|A\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^2}^2)$$

for a constant C. We have

$$\|\boldsymbol{u}\|_{H^2}^2 = \sum_{i,j,k} \int_{\mathbb{R}^n} \xi_i^2 \xi_j^2 \hat{u}_k^2\, d\xi + \sum_{i,j} \int_{\mathbb{R}^n} \xi_i^2 \hat{u}_j^2\, d\xi + \sum_i \int_{\mathbb{R}^n} \hat{u}_i^2\, d\xi.$$

It suffices to deal with the top order terms by Gårding's inequality. Now by strong ellipticity,

$$\|A\boldsymbol{u}\|_{L^2}^2 = \sum_{i,j,k,l} \int_{\mathbb{R}^n} |a^{ijkl} \xi_j \xi_l \hat{u}_k|^2\, d\xi$$
$$\geq \int_{\mathbb{R}^n} \epsilon(\sum_i \xi_i^2)^2 (\sum_k \hat{u}_k^2)\, d\xi \geq \epsilon \int_{\mathbb{R}^n} \sum_{i,k} \xi_i^4 \hat{u}_k^2\, d\xi.$$

Using $\xi_i^2 \xi_j^2 \leq \frac{1}{2}(\xi_i^4 + \xi_j^4)$, we get

$$\sum_{i,j,k} \int_{\mathbb{R}^n} \xi_i^2 \xi_j^2 \hat{u}_k^2\, d\xi \leq \frac{1}{2} \sum_{i,j,k} \int_{\mathbb{R}^n} (\xi_i^4 + \xi_j^4) \hat{u}_k^2\, d\xi$$
$$\leq n \sum_{i,k} \int_{\mathbb{R}^n} \xi_i^4 \hat{u}_k^2\, d\xi \leq \frac{n}{\epsilon} \|A\boldsymbol{u}\|_{L^2}^2.$$

This gives the desired inequality.

(4) *Some key Sobolev inequalities* Let us begin by being a little more precise in our definitions of Sobolev spaces. Let Ω be an open set in $\mathbb{R}^n$ with piecewise smooth boundary. Let $C^\infty(\Omega, \mathbb{R}^l)$ denote the maps $f: \Omega \longrightarrow \mathbb{R}^l$ that have C^∞ extensions to maps $\bar{f}$, which are C^∞ on all of $\mathbb{R}^n$. If Ω is bounded, we set

$$W^{k,p}(\Omega, \mathbb{R}^l) = \begin{cases} \text{completion of } C^\infty(\Omega, \mathbb{R}^l) \text{ in the norm,} \\ \| f \|_{k,p} = \sum_{0 \le i \le k} \| D^i f \|_{L^p}, \end{cases}$$

where $\| \phi \|_{L^p} = \left(\int_\Omega \| \phi(x) \|^p \, dx \right)^{1/p}$ is the L^p-norm on Ω, $D^i f$ is the ith derivative of f, and we take its norm in the usual way. For $p = 2$ we set $H^s(\Omega, \mathbb{R}^l) = W^{s,2}(\Omega, \mathbb{R}^l)$. Thus H^s is a Hilbert space. (One can show that H^s consists of those L^2 functions whose first s derivatives, in the sense of distribution theory, lie in L^2. This is called the *Meyer–Serrin theorem.* A convenient reference for the proof is Friedman [1969].)

For general Ω, set $C_0^\infty(\Omega, \mathbb{R}^l) =$ the C^∞ functions from Ω to $\mathbb{R}^n$ that have compact support in Ω. The completion of this space in the $\| \cdot \|_{k,p}$ norm is denoted $W_0^{k,p}$, and the corresponding H^s space is denoted H_0^s. For $\Omega = \mathbb{R}^n$ we just write $H^s = H_0^s$. Again $H^s(\mathbb{R}^n, \mathbb{R}^l)$ consists of those L^2 functions whose first s derivatives are in L^2.

In order to obtain useful information concerning the Sobolev spaces $W^{k,p}$, we need to establish certain fundamental relationships between these spaces. To do this, one uses the following fundamental inequality of Sobolev, as generalized by Nirenberg and Gagliardo. We give a special case (the more general case deals with Hölder norms as well as $W^{k,p}$ norms).

1.13 Theorem *Let* $1 \le q \le \infty$, $0 \le r \le \infty$, $0 \le j < m$, $j/m \le a \le 1$, $0 < p < \infty$, *with* j, m *integers* ≥ 0; *assume that*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1 - a)\frac{1}{q} \tag{1}$$

(*if* $1 < r < \infty$ *and* $m - j - n/r$ *is an integer* ≥ 0, *assume* $j/m \le a < 1$). *Then there is a constant C such that for any smooth* $\boldsymbol{u}: \mathbb{R}^n \longrightarrow \mathbb{R}^l$, *we have*

$$\| D^j \boldsymbol{u} \|_{L^p} \le C \| D^m \boldsymbol{u} \|_{L^r}^a \| \boldsymbol{u} \|_{L^q}^{1-a}. \tag{2}$$

(*If* $j = 0$, $rm < n$, *and* $q = \infty$, *assume* $\boldsymbol{u} \longrightarrow 0$ *at* ∞ *or* $\boldsymbol{u}$ *lies in* $L^{\tilde{q}}$ *for some finite* $\tilde{q} > 0$.)

Below we shall prove some special cases of this result. (The arguments given by Nirenberg [1959] are geometric in flavor in contrast to the usual Fourier transform proofs and therefore are more suitable for generalization to manifolds; cf. Cantor [1975a] and Aubin [1976].)

The above theorem remains valid for u defined on a region with piecewise smooth boundary, or more generally if the boundary satisfies a certain "cone condition."

Note: If one knows an inequality of the form (2) exists, one can infer that (1) must hold as follows: replace $u(x)$ by $u(tx)$ for a real $t > 0$. Then writing $u_t(x) = u(tx)$, one has

$$\| D^j u_t \|_{L^p} = \| D^j u \|_{L^p} \cdot t^{j-n/p}, \qquad \| D^m u_t \|^a_{L^r} = \| D^m u \|^a_{L^r} \cdot t^{a(m-n/r)}, \quad \text{and}$$
$$\| u_t \|^{1-a}_{L^q} = \| u \|^{1-a}_{L^q} \cdot t^{-n(1-a)/q}.$$

Thus if (2) is to hold for u_t (with the constant independent of t), we must have $j - n/p = a(m - n/r) - n(1 - a)/q$, which is exactly the relation (1).

The following corollary is useful in a number of applications:

1.14 Corollary *With the same relations as in Theorem 1.13, for any $\epsilon > 0$ there is a constant K_ϵ such that*

$$\| D^j u \|_{L^p} \leq \epsilon \| D^m u \|_{L^p} + K_\epsilon \| u \|_{L^q}$$

for all (smooth) functions u.

Proof This follows from 1.13 and Young's inequality: $x^a y^{1-a} \leq ax + (1 - a)y$, which implies that $x^a y^{1-a} = (\epsilon x)^a (K_\epsilon y)^{1-a} \leq a\epsilon x + (1 - a)K_\epsilon y$ where, $K_\epsilon = 1/\epsilon^{a/(1-a)}$. ∎

Let us illustrate how Fourier transform techniques can be used to directly prove the special case of 1.14 in which $n = 3$, $l = 1$, $j = 0$, $p = \infty$, $m = 2$, $r = 2$, and $q = 2$.

1.15 Proposition *There is a constant $c > 0$ such that for any $\epsilon > 0$ and function $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ smooth with compact support, we have*

$$\| f \|_\infty \leq c(\epsilon^{3/2} \| f \|_{L^2} + \epsilon^{-1/2} \| \Delta f \|_{L^2}).$$

(It follows that if $f \in H^2(\mathbb{R}^3)$, then f is uniformly continuous and the above inequality holds.)

Proof Let

$$\hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ik\cdot x} f(x)\, dx$$

denote the Fourier transform. Recall that $(\Delta \hat{f})(k) = -\| k \|^2 \hat{f}(k)$. From

Schwarz' inequality, we have

$$\left(\int |\hat{f}(k)|\,dk\right)^2 \leq \left(\int \frac{dk}{(\epsilon^2 + \|k\|^2)^2}\right)\left(\int (\epsilon^2 + \|k\|^2)^2 |\hat{f}(k)|^2\,dk\right)$$

$$= \frac{c_1}{\epsilon}\|(\epsilon^2 - \Delta)f\|^2_{L^2},$$

where
$$c_1 = \int_{\mathbb{R}^3} \frac{d\xi}{(1 + \|\xi\|^2)^2} < \infty.$$

Here we have used the fact that $h \mapsto \hat{h}$ is an isometry in the L^2-norm (Plancherel's theorem). Thus, from $f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ik\cdot x} \hat{f}(k)\,dk$,

$$(2\pi)^{3/2}\|f\|_\infty \leq \|\hat{f}\|_{L^1} \leq \frac{c_2}{\sqrt{\epsilon}}\|(\epsilon^2 - \Delta)f\|_{L^2}$$
$$\leq c_2(\epsilon^{3/2}\|f\|_{L^2} + \epsilon^{-1/2}\|\Delta f\|_{L^2}). \quad \blacksquare$$

Thus we have shown that $H^2(\mathbb{R}^3) \subset C^0(\mathbb{R}^3)$ and that the inclusion is continuous. More generally one can show by similar arguments that $H^s(\Omega) \subset C^k(\Omega)$ provided $s > n/2 + k$ and $W^{s,p}(\Omega) \subset C^k(\Omega)$ if $s > n/p + k$. This is one of the celebrated *Sobolev embedding theorems.*

> ***Problem 1.3*** Prove that the last assertion is a special case of the result in Theorem 1.13.

For Ω bounded, the inclusion $W^{s,p}(\Omega) \longrightarrow C^k(\Omega)$, $s > n/p + k$ is compact; that is, the unit ball in $W^{s,p}(\Omega)$ is compact in $C^k(\Omega)$. This is proved in a manner similar to the classical Arzela–Ascoli theorem, one version of which states that the inclusion $C^1(\Omega) \subset C^0(\Omega)$ is compact (see Marsden [1974a], for instance). Also, $W^{s,p}(\Omega) \subset W^{s',p'}(\Omega)$ is compact if $s > s'$ and $p = p'$ or if $s = s'$ and $p > p'$. (See Friedman [1969] for the proofs.)

We already saw one application of Rellich's theorem in our proof of the Fredholm alternative. It is often used this way in existence theorems, using compactness to extract convergent sequences.

As we shall see later, compactness comes into existence theory in another crucial way when one seeks weak solutions. This is through the fact that the unit ball in a Banach space is weakly compact—that is, compact in the weak topology. See, for example, Yosida [1971] for the proof (and for refinements, involving weak *sequential* compactness).

We shall give another illustration of Theorem 1.13 through a special case that is useful in the study of nonlinear wave equations. This is the following important inequality in $\mathbb{R}^3$:

$$\|u\|_{L^6} \leq C\|Du\|_{L^2}.$$

1.16 Proposition *Let $u: \mathbb{R}^3 \to \mathbb{R}$ be smooth and have compact support. Then*

$$\int_{\mathbb{R}^3} u^6\, dx \leq 48\left(\int_{\mathbb{R}^3} \|\operatorname{grad} u\|^2\, dx\right)^3 \cdot$$

so $C = \sqrt[6]{48}$.

Proof[3] From

$$u^3(x, y, z) = 3\int_{-\infty}^{x} u^2 \frac{\partial u}{\partial x}\, dx$$

one gets

$$\sup_x |u^3(x, y, z)| \leq 3\int_{-\infty}^{\infty} \left|u^2 \frac{\partial u}{\partial x}\right| dx.$$

Set $I = \int_{\mathbb{R}^3} u^6\, dx$ and write

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \left(\iint |u^3||u^3|\, dy\, dz\right) dx \\
&\leq \int_{-\infty}^{\infty} \left[\left(\sup_y \int_{-\infty}^{\infty} |u^3|\, dz\right)\left(\int_{-\infty}^{\infty} \sup_z |u^3|\, dy\right)\right] dx \\
&\leq 9\int_{-\infty}^{\infty} \left[\left(\iint \left|u^2 \frac{\partial u}{\partial y}\right| dy\, dz\right)\left(\iint \left|u^2 \frac{\partial u}{\partial z}\right| dy\, dz\right)\right] dx \cdot
\end{aligned}$$

Using Schwarz' inequality on this gives

$$\begin{aligned}
I &\leq 9\int_{-\infty}^{\infty} \left[\left(\iint u^4\, dy\, dz\right)\left(\iint_{-\infty}^{\infty} \left(\frac{\partial u}{\partial y}\right)^2 dy\, dz\right)^{1/2}\left(\iint_{-\infty}^{\infty} \left(\frac{\partial u}{\partial z}\right)^2 dy\, dz\right)^{1/2}\right] dx \\
&\leq 9 \max_x \left(\iint_{-\infty}^{\infty} u^4\, dy\, dz\right)\left(\iiint_{-\infty}^{\infty} \left(\frac{\partial u}{\partial y}\right)^2 dx\, dy\, dz\right)^{1/2}\left(\iiint_{-\infty}^{\infty} \left(\frac{\partial u}{\partial z}\right)^2 dx\, dy\, dz\right)^{1/2} \\
&\leq 36\int_{\mathbb{R}^3} \left|u^3 \frac{\partial u}{\partial x}\right| dx\, dy\, dz \left(\int_{\mathbb{R}^3} \left(\frac{\partial u}{\partial y}\right)^2\right)^{1/2}\left(\int_{\mathbb{R}^3} \left(\frac{\partial u}{\partial z}\right)^2\right)^{1/2} \\
&\leq 36\sqrt{I}\left(\int_{\mathbb{R}^3} \left(\frac{\partial u}{\partial x}\right)^2\right)^{1/2}\left(\int_{\mathbb{R}^3} \left(\frac{\partial u}{\partial y}\right)^2\right)^{1/2}\left(\int_{\mathbb{R}^3} \left(\frac{\partial u}{\partial z}\right)^2\right)^{1/2}.
\end{aligned}$$

Now using the arithmetic geometric mean inequality $\sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{c} < (a + b + c)/3$ gives

$$I \leq 36\sqrt{I}\left(\int_{\mathbb{R}^3} \|\operatorname{grad} u\|^2\right)^{3/2} \Big/ 3^{3/2}.$$

---

[3]Following Ladyzhenskaya [1969].

That is,

$$I \leq \frac{(36)^2}{3^3} \cdot \left(\int_{\mathbb{R}^3} \| \operatorname{grad} u \|^2 \right)^3 \cdot \quad \blacksquare$$

One can use the same technique to prove similar inequalities.

There is another important corollary of Theorem 1.13 that we shall prove. The techniques can be used to determine to which $W^{s,p}$ space a product belongs.

1.17 Corollary *For $s > n/2$, $H^s(\mathbb{R}^n)$ is a Banach algebra (under pointwise multiplication). That is, there is a constant $K > 0$ such that for $u, v \in H^s(\mathbb{R}^n)$,*

$$\| u \cdot v \|_{H^s} \leq K \| u \|_{H^s} \| v \|_{H^s}.$$

This is an important property of H^s not satisfied for low s. It certainly is not true *that L^2 forms an algebra under multiplication.*

Proof Choose $a = j/s$, $r = 2$, $q = \infty$, $p = 2s/j$, $m = s$ $(0 \leq j \leq s)$ to obtain

$$\| D^j u \|_{L^{2s/j}} \leq \text{const.} \| D^s u \|_{L^2}^{j/s} \| u \|_{\infty}^{1-j/s} \leq \text{const.} \| u \|_{H^s}.$$

(See 1.15.) Let $j + k = s$. From Holder's inequality we have

$$\| D^j u \cdot D^k v \|_{L^2}^2 \leq \| D^j u \|_{L^{2s/j}}^2 \| D^k v \|_{L^{2s/k}}^2 \leq \text{const.} \| u \|_{H^s}^2 \| v \|_{H^s}^2.$$

Now $D^s(uv)$ consists of terms like $D^j u \cdot D^k v$, so we obtain

$$\| D^s uv \|_{L^2} \leq \text{const.} \| u \|_{H^s} \| v \|_{H^s}.$$

Similarly for the lower-order terms. Summing gives the result. $\blacksquare$

The trace theorems have already been mentioned in 1.1. Generally, they state that the restriction map from Ω to a submanifold $\mathfrak{M} \subset \Omega$ of codimension m induces a bounded operator from $W^{s,p}(\Omega)$ to $W^{s-(1/mp),p}(\mathfrak{M})$. Adams [1975] and Morrey [1966] are good references; the latter contains some useful refinements of this.

There are also some basic extension theorems that are right inverses of restriction maps. Thus, for example, the *Calderon extension theorem* asserts that there is an extension map T: $W^{s,p}(\Omega) \longrightarrow W^{s,p}(\mathbb{R}^n)$ that is a bounded operator and "restriction to Ω" $\circ T =$ Identity. This is related to a classical C^k theorem due to Whitney. See, for example, Abraham and Robbin [1967], Stein [1970], and Marsden [1973a].

Finally, we mention that all these $\mathbb{R}^n$ results carry over to manifolds in a straightforward way. See, for example, Palais [1965] and Cantor [1979].

Problem 1.4 Prove a $W^{s,p}$ version of the ω-lemma given in Box 1.1, Chapter 4, by using the results of this box.

Box 1.2 ***Summary of Important Formulas for Section 6.1***

Sobolev Spaces on $\Omega \subset \mathbb{R}^n$

$$L^2 = \left\{ \boldsymbol{u}\colon \Omega \longrightarrow \mathbb{R}^n \,\middle|\, \int_\Omega \|\boldsymbol{u}^2\|\, dv < \infty \right\}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\, dv = \int_\Omega u^a(x) v_a(x)\, dx$$

$H^s = \{\boldsymbol{u}\colon \Omega \longrightarrow \mathbb{R}^n \,|\, \boldsymbol{u}, D\boldsymbol{u}, \ldots, D^s\boldsymbol{u}$ are in $L^2\}$

norm: $\|\boldsymbol{u}\|^2_{H^s} = \|\boldsymbol{u}\|^2_{L^2} + \cdots + \|D^s\boldsymbol{u}\|^2_{L^2}$.

$H^2_\partial = \{\boldsymbol{u} \in H^2 \,|\, \boldsymbol{u}$ satisfies the boundary conditions (displacement, traction or mixed)$\}$

$W^{s,p} = \{\boldsymbol{u}\colon \Omega \longrightarrow \mathbb{R}^n \,|\, \boldsymbol{u}, D\boldsymbol{u}, \ldots, D^s\boldsymbol{u} \in L^p\}$

Symmetric Elliptic Operator

Form: $A\boldsymbol{u} = \operatorname{div}(\mathsf{a}\cdot\nabla\boldsymbol{u}) \quad (A\boldsymbol{u})^a = (\mathsf{a}^{abcd}u_{c|d})_{|b}$

Symmetry: $\mathsf{a}^{abcd} = \mathsf{a}^{cdab}$, $\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, A\boldsymbol{v}\rangle$ for $\boldsymbol{u}, \boldsymbol{v} \in H^2_\partial$

Strong ellipticity: $\mathsf{a}^{abcd}\xi_a\xi_c\eta_b\eta_d \geq \epsilon\|\boldsymbol{\xi}\|^2\|\boldsymbol{\eta}\|^2$ for some $\epsilon > 0$ and all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^n$

Gårding's Inequality ($\Leftrightarrow$ Strong Ellipticity)

$B(\boldsymbol{u}, \boldsymbol{u}) \geq c\|\boldsymbol{u}\|^2_{H^1} - d\|\boldsymbol{u}\|^2_{L^2}$ for all $\boldsymbol{u} \in H^1$, where

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\mathsf{a}\cdot\nabla\boldsymbol{v}\, dv = \int_\Omega u_{a|b}\mathsf{a}^{abcd}v_{c|d}\, dv$$

$(= -\langle A\boldsymbol{u}, \boldsymbol{v}\rangle$ for $\boldsymbol{u}, \boldsymbol{v} \in H^2_\partial)$.

Fredholm Alternative

(1) $L^2 = \text{Range } A \oplus \text{Ker } A$, an L^2 orthogonal sum.

(2) $A\boldsymbol{u} = \boldsymbol{f}$ is solvable for $\boldsymbol{u} \in H^2_\partial$ if and only if $\boldsymbol{f} \perp \text{Ker } A$.

Classical Elasticity

Uniform pointwise stability: $\frac{1}{2}\,\boldsymbol{e}\cdot\mathsf{c}\cdot\boldsymbol{e} \geq \eta\|\boldsymbol{e}\|^2$, $\frac{1}{2}e_{ab}\mathsf{c}^{abcd}e_{cd} \geq \eta e_{ab}e^{ab}$, implies strong ellipticity. In $\Omega \subset \mathbb{R}^3$,

$\operatorname{div}(\mathsf{c}\cdot\nabla\boldsymbol{u}) = \boldsymbol{f}$ is solvable for $\boldsymbol{u} \in H^2_\partial$ using displacement or mixed boundary conditions for any $\boldsymbol{f}$ and for traction boundary conditions if $\int_\Omega \boldsymbol{f}(x)(\boldsymbol{a} + \boldsymbol{b}x)\, dx = 0$ for $\boldsymbol{a}$ any constant vector and $\boldsymbol{b}$ any 3×3 skew matrix.

Korn's Inequalities

(1) Displacement (or mixed) boundary conditions:

$$\int_\Omega \|\boldsymbol{e}\|^2\, dv \geq c\|\boldsymbol{u}\|^2_{H^1}, \quad e_{ab} = \tfrac{1}{2}(u_{a|b} + u_{b|a})$$

(2) General:

$$\int_\Omega \|\boldsymbol{e}\|^2\, dv + \int_\Omega \|\boldsymbol{u}\|^2\, dv \geq c\|\boldsymbol{u}\|^2_{H^1}$$

6.2 ABSTRACT SEMIGROUP THEORY

This section gives an account of those parts of semigroup theory that are needed in the following section for applications to elastodynamics. Although the account is self contained and gives fairly complete proofs of most of the theorems, it is not exhaustive. For example, we have omitted details about the theory of analytic semigroups, since it will be treated only incidentally in subsequent sections. The standard references for semigroup theory are Hille and Phillips [1957], Yosida [1971], Kato [1966], and Pazy [1974]. This theory also occurs in many books on functional analysis, such as Balakrishnan [1976].

We shall begin with the definition of a semigroup. The purpose is to capture, under the mildest possible assumptions, what we mean by solvability of a linear evolution equation

$$\frac{du}{dt} = Au \quad (t \geq 0), \qquad u(0) = u_0. \tag{1}$$

Here A is a linear operator in a Banach space $\mathfrak{X}$. We are interested in when (1) has unique solutions and when these solutions vary continuously in $\mathfrak{X}$ as the initial data varies in the $\mathfrak{X}$ topology. When this holds, one says that Equation (1) is *well-posed.* If A is a bounded operator in $\mathfrak{X}$, solutions are given by

$$u(t) = e^{tA}u_0 = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} u_0.$$

For partial differential equations, however, A will usually be unbounded, so the problem is to make sense out of e^{tA}. Instead of power series, the operator analogue of the calculus formula $e^x = \lim_{n \to \infty} (1 - x/n)^{-n}$ will turn out to be appropriate.

2.1 Definitions A (C^0) *semigroup* on a Banach space $\mathfrak{X}$ is a family $\{U(t) \mid t \geq 0\}$ of bounded linear operators of $\mathfrak{X}$ to $\mathfrak{X}$ such that the following conditions hold:

(i) $U(t+s) = U(t) \circ U(s)$ $(t, s \geq 0)$ (semigroup property);
(ii) $U(0) =$ Identity; and
(iii) $U(t)x$ is t-continuous at $t = 0$ for each $x \in \mathfrak{X}$; that is, $\lim_{t \downarrow 0} U(t)x = x$. (This pointwise convergence is also expressed by saying strong $\lim_{t \downarrow 0} U(t) = I$.)

The *infinitesimal generator* A of $U(t)$ is the (in general unbounded) linear operator given by

$$Ax = \lim_{t \downarrow 0} \frac{U(t)x - x}{t} \tag{2}$$

on the domain $\mathfrak{D}(A)$ defined to be the set of those $x \in \mathfrak{X}$ such that the limit (2) exists in $\mathfrak{X}$.

We now derive a number of properties of semigroups. (Eventually we will prove an existence and uniqueness theorem for semigroups *given* a generator A.)

For Propositions 2.2–2.12, assume that $U(t)$ is a given C^0 semigroup with infinitesimal generator A.

2.2 Proposition *There are constants $M > 0$, $\beta \geq 0$ such that $\| U(t) \| \leq Me^{t\beta}$ for all $t \geq 0$. In this case we write $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$ and say A is the generator of a semigroup of type (M, β).*

Proof We first show that $\| U(t) \|$ is bounded on some neighborhood of zero. If not, there would be a sequence $t_n \downarrow 0$ such that $\| U(t_n) \| \geq n$. But $U(t_n)x \longrightarrow x$ as $n \longrightarrow \infty$, so $U(t_n)$ is pointwise bounded as $n \longrightarrow \infty$, and therefore by the Uniform Boundedness Theorem[4] $\| U(t_n) \|$ is bounded, which is a contradiction.

Thus for some $\delta > 0$ there is a constant M such that $\| U(t) \| \leq M$ for $0 \leq t \leq \delta$. For $t \geq 0$ arbitrary, let n be the largest integer in t/δ so $t = n\delta + \tau$, where $0 \leq \tau < \delta$. Then by the semigroup property,

$$\| U(t) \| = \| U(n\delta)U(\tau) \| \leq \| U(\tau) \| \| U(\delta) \|^n \leq M \cdot M^n \leq M \cdot M^{t/\delta} \leq Me^{t\beta}.$$

where $\beta = (1/\delta) \log M$. ∎

2.3 Proposition *$U(t)$ is strongly continuous[5] in t; that is, for each fixed $x \in \mathfrak{X}$, $U(t)x$ is continuous in $\mathfrak{X}$ as a function of $t \in [0, \infty)$.*

Proof Let $s > 0$. Since $U(\tau + s)x = U(s)U(\tau)x$, 2.1(iii) gives

$$\begin{aligned}\lim_{t \downarrow s} U(t)x &= \lim_{\tau \downarrow 0} U(\tau + s)x \\ &= U(s) \lim_{\tau \downarrow 0} U(\tau)x = U(s)x,\end{aligned}$$

so we have right continuity in t at $t = s$. For left continuity let $0 \leq \tau \leq s$, and write

$$\| U(s - \tau)x - U(s)x \| = \| U(s - \tau)(x - U(\tau)x) \| \leq Me^{\beta(s-\tau)} \| x - U(\tau)x \|,$$

which tends to zero as $\tau \downarrow 0$. ∎

2.4 Proposition

(i) *$U(t)\mathfrak{D}(A) \subset \mathfrak{D}(A)$;*
(ii) *$U(t)Ax = AU(t)x$ for $x \in \mathfrak{D}(A)$; and*
(iii) *$(d/dt)U(t)x_0 = A(U(t)x_0)$ for all $x_0 \in \mathfrak{D}(A)$ and $t \geq 0$. In other words,*

$$x(t) = U(t)x_0 \quad \textit{satisfies} \quad \frac{dx}{dt} = Ax \textit{ and } x(0) = x_0.$$

[4]This theorem states that if $\{T_\alpha\}$ is a family of bounded linear operators on $\mathfrak{X}$ and if $\{T_\alpha x\}$ is bounded for each $x \in \mathfrak{X}$, then the norms $\| T_\alpha \|$ are bounded. See, for example, Yosida [1971], p. 69.

[5]One can show that strong continuity at $t = 0$ can be replaced by weak continuity at $t = 0$ and strong continuity in $t \in [0, \infty)$ can be replaced by strong measurability in t. See Hille and Phillips [1957] for details.

Proof From $[U(h)U(t)x - U(t)x]/h = U)t)[(U(h)x - x)/h]$ we get (i) and (ii) We get (iii) by using the fact that if $x(t) \in \mathfrak{X}$ has a continuous right derivative, then $x(t)$ is differentiable—from the right at $t = 0$ and two sided if $t > 0$.[6] ▮

From (i) and (ii) we see that if $x \in \mathfrak{D}(A^n)$, then $U(t)x \in \mathfrak{D}(A^n)$. This is often used to derive *regularity* results, because if A is associated with an elliptic operator, $\mathfrak{D}(A^n)$ may consist of smoother functions for larger n. Notice that we have now shown that the concept of semigroup given here and that given in 2.5, Chapter 5, agree.

2.5 Proposition *$\mathfrak{D}(A)$ is dense in $\mathfrak{X}$.*

Proof Let $\phi(t)$ be a C^∞ function with compact support in $[0, \infty)$, let $x \in \mathfrak{X}$ and set

$$x_\phi = \int_0^\infty \phi(t)U(t)x\,dt.$$

Noting that

$$U(s)x_\phi = \int_0^\infty \phi(t)U(t+s)x\,dt = \int_0^\infty \phi(\tau - s)U(\tau)x\,d\tau$$

is differentiable in s, we find that $x_\phi \in \mathfrak{D}(A)$. On the other hand, given any $\epsilon > 0$ we claim that there is a ϕ (close to the "δ function") such that $\|x_\phi - x\| < \epsilon$. Indeed, by continuity, choose $\delta > 0$ such that $\|U(t)x - x\| < \epsilon$ if $0 \leq t \leq \delta$. Let ϕ be C^∞ with compact support in $(0, \delta)$, $\phi \geq 0$ and $\int_0^\infty \phi(t)\,dt = 1$. Then

$$\|x_\phi - x\| = \left\| \int_0^\infty \phi(t)(U(t)x - x)\,dt \right\| \leq \int_0^\delta \phi(t)\|U(t)x - x\|\,dt$$
$$< \epsilon \int_0^\delta \phi(t)\,dt = \epsilon. \quad ▮$$

The same argument in fact shows that $\bigcap_{n=1}^\infty \mathfrak{D}(A^n)$ *is dense in* $\mathfrak{X}$.

2.6 Proposition *A is a closed operator; that is, its graph in $\mathfrak{X} \times \mathfrak{X}$ is closed.*[7]

Proof Let $x_n \in \mathfrak{D}(A)$ and assume that $x_n \longrightarrow x_0$ and $Ax_n \longrightarrow y$. We must show that $x \in \mathfrak{D}(A)$ and $y = Ax$. By 2.4,

$$U(t)x_n = x_n + \int_0^t U(s)Ax_n\,ds.$$

[6] This follows from the corresponding real variables fact by considering $l(u(t))$ for $l \in \mathfrak{X}^*$. See Yosida [1971], p. 235.

[7] We shall prove more than this in Proposition 2.12 below, but the techniques given here are more direct and also apply to certain nonlinear semigroups as well. See Chernoff and Marsden [1974].

Since $U(s)Ax_n \longrightarrow U(s)y$ uniformly for $s \in [0, t]$, we have

$$U(t)x = x + \int_0^t U(s)y\,ds.$$

It follows that $(d/dt+)\,U(t)x\,|_{t=0}$ exists and equals y. ∎

Next we show that integral curves are unique. (Compare 2.15, Chapter 5.)

2.7 Proposition *Suppose $c(t)$ is a differentiable curve in $\mathfrak{X}$ such that $c(t) \in \mathfrak{D}(A)$ and $c'(t) = A(c(t))$ $(t \geq 0)$. Then $c(t) = U(t)c(0)$.*

Proof Fix $t_0 > 0$ and define $h(t) = U(t_0 - t)c(t)$ for $0 \leq t \leq t_0$. Then for τ small,

$$\begin{aligned}\|h(t+\tau) - h(t)\| &= \|U(t_0 - t - \tau)c(t+\tau) - U(t_0 - t - \tau)U(\tau)c(t)\| \\ &\leq Me^{\beta(t_0 - t - \tau)}\|c(t+\tau) - U(\tau)c(t)\|.\end{aligned}$$

However,

$$\frac{1}{\tau}[c(t+\tau) - U(\tau)c(t)] = \frac{1}{\tau}[c(t+\tau) - c(t)] - \frac{1}{\tau}[U(\tau)c(t) - c(t)],$$

which converges to $Ac(t) - Ac(t) = 0$, as $\tau \longrightarrow 0$. Thus, $h(t)$ is differentiable for $0 < t < t_0$ with derivative zero. By continuity, $h(t_0) = \lim_{t \downarrow t_0} h(t) = c(t_0) = \lim_{t \downarrow t_0} h(t) = U(t_0)c(0)$. (The last limit is justified by the fact that $\|U(t)\| \leq Me^{t\beta}$.) This is the result with t replaced by t_0. ∎

One also has uniqueness in the class of weak solutions as is explained in the optional Box 2.1.

Box 2.1 ***Adjoints and Weak Solutions (Balakrishnan*** *[1976]* ***and Ball*** *[1977c])*

Let the adjoint $A^*\colon \mathfrak{D}(A^*) \subset \mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ be defined by $\mathfrak{D}(A^*) = \{v \in \mathfrak{X}^* \mid \text{there exists a } w \in \mathfrak{X}^* \text{ such that } \langle w, x\rangle = \langle v, Ax\rangle \text{ for all } x \in \mathfrak{D}(A)\}$, where $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{X}$ and $\mathfrak{X}^*$. Set $A^*v = w$. If $c(t)$ is a continuous curve in $\mathfrak{X}$ and if, for every $v \in \mathfrak{D}(A^*)$, $\langle c(t), v\rangle$ is absolutely continuous and

$$\frac{d}{dt}\langle c(t), v\rangle = \langle c(t), A^*v\rangle \quad \text{almost everywhere}$$

that is, $\langle c(t), v\rangle = \langle c(0), v\rangle + \int_0^t \langle c(s), A^*v\rangle\,ds$, then $c(t)$ is called a *weak solution* of $dx/dt = Ax$.

2.8 Proposition *Let $\{U(t)\}$ be a C^0 semigroup on $\mathfrak{X}$. If $c(t)$ is a weak solution, then $c(t) = U(t)c(0)$. Conversely, for $x_0 \in \mathfrak{X}$ (not necessarily in the domain of A), then $c(t) = U(t)x_0$ is a weak solution.*

Proof If $x_0 \in \mathfrak{D}(A)$, then $U(t)x_0$ is a solution in $\mathfrak{D}(A)$ and hence a weak solution. Since $U(t)$ is continuous and $\mathfrak{D}(A)$ is dense, the same is true for $x_0 \in \mathfrak{X}$; that is, we can pass to the limit in

$$\langle U(t)x_n, v\rangle = \langle x_n, v\rangle + \int_0^t \langle U(\tau)x_n, v\rangle \, d\tau$$

for $x_n \in \mathfrak{D}(A)$, $x_n \longrightarrow x_0 \in \mathfrak{X}$.

Now suppose $c(t)$ is a weak solution. Let $w(t) = c(t) - U(t)c(0)$. Then $w(0) = 0$ and for $v \in \mathfrak{D}(A^*)$,

$$\langle w(t), v\rangle = \int_0^t \langle w(\tau), A^*v\rangle \, d\tau = \left\langle \int_0^t w(\tau)\, d\tau, A^*v \right\rangle.$$

Thus, $\int_0^t w(\tau)\, d\tau \in \mathfrak{D}(A)$ since A is closed (see 2.6). Here we have used the fact that if A is closed, then $A^{**} \subset A$, where we identify $\mathfrak{X}$ with a subspace of $\mathfrak{X}^{**}$. (If $\mathfrak{X}$ is reflexive, $A^{**} = A$; cf. Kato [1966], p. 168.) It follows that $z(t) = \int_0^t w(\tau)\, d\tau$ satisfies $\dot{z} = Az$, and since $z(0) = 0$, z is identically zero by 2.7. ▮

Ball [1977c] also shows that if the equation $\dot{x} = Ax$ admits unique weak solutions and A is densely defined and closed, then A is a generator.

We continue now to develop properties of a given semigroup of type (M, β). If $\beta = 0$, we say the semigroup is *bounded*, and if $M = 1$, we say it is *quasi-contractive*. If $M = 1$ and $\beta = 0$, it is called *contractive*.

2.9 Proposition *If $U(t)$ is (a C^0 semigroup) of type (M, β) on $\mathfrak{X}$, then: (i) $T(t) = e^{-t\beta}U(t)$ is a bounded semigroup with generator $A - \beta I$; (ii) there is an equivalent norm $|||\cdot|||$ on $\mathfrak{X}$ relative to which $U(t)$ is quasi-contractive.*

We shall leave the proof as an exercise. For (ii), use the norm $|||x||| = \sup_{t\geq 0} \| e^{-t\beta}U(t)x\|$.

2.10 Example Let $\mathfrak{X} = L^2(\mathbb{R})$ with the norm

$$\|f\|^2 = \int_{-1}^1 |f(x)|^2 \, dx + \tfrac{1}{2} \int_{\mathbb{R}\setminus[-1,1]} |f(x)|^2 \, dx$$

and let $(U(t)f)(x) = f(t + x)$. Then $U(t)$ is a C^0 semigroup and $Af = df/dx$

with domain $H^1(\mathbb{R})$ (absolutely continuous functions with derivatives in L^2). Here, $\|U(t)\| \leq 2$. If we form the norm $||| \; |||$, we get the usual L^2-norm and a contraction semigroup.

2.11 Proposition *$U(t)$ is norm continuous at $t = 0$ if and only if A is bounded.*

Proof Choose $\epsilon > 0$ so that $\|U(t) - I\| < \frac{1}{2}$ if $0 \leq t \leq \epsilon$ and pick ϕ to be a C^∞ function with compact support in $[0, \epsilon)$ such that $\phi \geq 0$ and $\int_0^\epsilon \phi(t)\,dt = 1$.

Let $J_\phi(x) = \int_0^\infty \phi(\tau)U(\tau)x\,d\tau$ and note that

$$J_\phi(U(t)x) = \int_0^\infty \phi(\tau)U(\tau + t)x\,d\tau = \int_t^\infty \phi(\tau - t)U(\tau)x\,d\tau.$$

However,

$$\|(J_\phi - I)(x)\| = \left\| \int_0^\infty \phi(\tau)(U(\tau)x - x)\,d\tau \right\| \leq \tfrac{1}{2} \int_0^\infty \phi(\tau)\|x\|\,d\tau = \tfrac{1}{2}\|x\|,$$

so $\|J_\phi - I\| \leq \frac{1}{2}$ and hence J_ϕ is invertible. By construction

$$U(t)x = J_\phi^{-1}\left(\int_t^\infty \phi(\tau - t)U(\tau)x\,d\tau\right),$$

which is therefore differentiable in t for all x and also shows $A \in \mathcal{B}(\mathfrak{X})$ (the set of all bounded linear operators on $\mathfrak{X}$). The converse is done by noting that $e^{tA} = \sum_{n=0}^\infty (tA)^n/(n!)$ is norm continuous in t. ▮

Next we give a proposition that will turn out to be a complete characterization of generators.

2.12 Proposition *Let $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$. Then:*

(i) *$\mathfrak{D}(A)$ is dense in $\mathfrak{X}$;*
(ii) *$(\lambda - A)$ is one-to-one and onto $\mathfrak{X}$ for each $\lambda > \beta$ and the* resolvent *$R_\lambda = (\lambda - A)^{-1}$ is a bounded operator; and*
(iii) *$\|(\lambda - A)^{-n}\| \leq M/(\lambda - \beta)^n$ for $\lambda > \beta$ and $n = 1, 2, \ldots$.*

Note. Here and in what follows, $\lambda - A$ stands for $\lambda I - A$, where I is the identity operator.

Proof Given $x \in \mathfrak{X}$, let

$$y_\lambda = \int_0^\infty e^{-\lambda t}U(t)x\,dt, \qquad \lambda > \beta.$$

Then

$$\begin{aligned}(U(s) - I)y_\lambda &= \int_0^\infty e^{-\lambda t}U(t + s)x\,dt - y_\lambda \\ &= e^{\lambda s}\int_s^\infty e^{-\lambda\tau}U(\tau)x\,d\tau - y_\lambda \\ &= (e^{\lambda s} - 1)y_\lambda - e^{\lambda s}\int_0^s e^{-\lambda t}U(t)x\,dt.\end{aligned}$$

Hence $y_\lambda \in \mathfrak{D}(A)$ and $Ay_\lambda = \lambda y_\lambda - x$. Thus $(\lambda - A)$ is surjective. (Taking $\lambda \to \infty$ shows that $\lambda y_\lambda \longrightarrow x$, which also shows $\mathfrak{D}(A)$ is dense, reproducing 2.5.) The formula

$$u = \int_0^\infty e^{-\lambda t} U(t)(\lambda - A)u\, dt, \qquad u \in \mathfrak{D}(A),$$

which follows from $-(d/dt)e^{-\lambda t}U(t)u = e^{-\lambda t}U(t)(\lambda - A)u$, shows that $(\lambda - A)$ is one-to-one.

Thus we have proved the Laplace transform relation

$$R_\lambda x = (\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}U(t)x\, dt, \quad (\lambda > \beta),$$

from which it follows that

$$\|(\lambda - A)^{-1}\| \leq \int_0^\infty e^{-\lambda t} M e^{\beta t}\, dt = \frac{M}{\lambda - \beta}.$$

The estimate (iii) follows from the formulas

$$(n-1)!\,(\lambda - A)^{-n}x = \int_0^\infty e^{-\lambda t} t^{n-1} U(t)x\, dt, \tag{3}$$

$$\int_0^\infty e^{-\mu t} t^{n-1}\, dt = \frac{(n-1)!}{\mu^n}. \tag{4}$$

Equation (4) is proved by integration by parts and (3) follows from the relation

$$\left(\frac{d}{d\lambda}\right)^{n-1}(\lambda - A)^{-1} = (-1)^{n-1}(n-1)!\,(\lambda - A)^{-n}. \quad ∎$$

Problem 2.1 Show that the *resolvent identity* $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ holds and that $\mu R_\lambda \longrightarrow$ Identity strongly as $\lambda \longrightarrow \infty$.

The following *Hille–Yosida theorem* asserts the converse of 2.12. It is, in effect, an existence and uniqueness theorem. Uniqueness was already proved in 2.7.

2.13 Theorem *Let A be a linear operator in $\mathfrak{X}$ with domain $\mathfrak{D}(A)$. Assume there are positive constants M and β such that:*

(i) *$\mathfrak{D}(A)$ is dense;*
(ii) *$(\lambda - A)$ is one-to-one and onto $\mathfrak{X}$ for $\lambda > \beta$ and $(\lambda - A)^{-1} \in \mathfrak{B}(\mathfrak{X})$; and*
(iii) *$\|(\lambda - A)\|^{-n} \leq M/(\lambda - \beta)^n$ $(\lambda > \beta, n = 1, 2, \ldots)$.*

Then $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$; that is, there exists a C^0 semigroup of type (M, β) whose generator is A. We shall often write e^{tA} for the semigroup generated by A.

Proof If $(A - \beta I)$ generates the semigroup U, then A generates the semigroup $e^{t\beta}U$ (see 2.9). Thus it suffices to prove the theorem for $\beta = 0$.

Rewrite (iii) as $\|(1 - \alpha A)^{-n}\| \leq M$ $(\alpha > 0, n = 1, \ldots)$ by taking $\alpha = 1/\lambda$. Thus, if $x \in \mathfrak{D}(A)$, then $(1 - \alpha A)^{-1}x - x = \alpha(1 - \alpha A)^{-1}Ax$, so $(1 - \alpha A)^{-1} - I$

$\longrightarrow 0$ strongly on $\mathfrak{D}(A)$ as $\alpha \downarrow 0$. Since $(1 - \alpha A)^{-1} - I \in \mathfrak{B}(\mathfrak{X})$, convergence also holds on $\mathfrak{X}$.

Let $U_n(t) = (1 - (t/n)A)^{-n}$, a uniformly bounded sequence of operators. We shall show it converges on a dense set. Write

$$U_n(t)x - U_m(t)x = U_m(t - s)U_n(s)x \,|_{s=0}^{s=t} = \operatorname*{s-lim}_{\epsilon \downarrow 0} \int_\epsilon^t \frac{d}{ds} U_m(t - s)U_n(s)x \, ds$$

$$= \operatorname*{s-lim}_{\epsilon \downarrow 0} \int_\epsilon^t \left(\frac{s}{n} - \frac{t-s}{m}\right) A^2 \left(1 - \frac{t-s}{m} A\right)^{-m-1} \left(1 - \frac{s}{n} A\right)^{-n-1} x \, ds.$$

Thus, if $x \in \mathfrak{D}(A^2)$, we get,

$$\| U_n(t)x - U_m(t)x \| \leq M^2 \| A^2 x \| \tfrac{1}{2} \left(\frac{1}{n} + \frac{1}{m}\right) t^2.$$

Thus $U_n(t)x$ converges for $x \in \mathfrak{D}(A^2)$. But

$$\mathfrak{D}(A^2) = \mathfrak{D}((1 - A)^2) = \operatorname{Range}(1 - A)^{-2} = (1 - A)^{-1}\mathfrak{D}(A).$$

Now $(1 - A)^{-1} : \mathfrak{X} \longrightarrow \mathfrak{D}(A)$ is bounded and surjective. Since $\mathfrak{D}(A) \subset \mathfrak{X}$ is dense, $(1 - A)^{-1}(\mathfrak{D}(A)) \subset \mathfrak{D}(A)$ is dense; that is, $\mathfrak{D}(A^2) \subset \mathfrak{X}$ is dense.

Let $U(t)x = \operatorname{s-lim}_{n \to \infty} U_n(t)x$. Clearly, $\| U(t) \| \leq M$, $U(0)x = x$, and $U(t + s) = U(t) \circ U(s)$. Since $U_n(t)x \longrightarrow U(t)x$ uniformly on compact t-intervals for $x \in \mathfrak{D}(A^2)$ and this is dense, $U(t)x$ is t-continuous. Thus we have a C^0 semigroup.

Let A' be the generator of $U(t)$. We need to show that $A' = A$. For $x \in \mathfrak{D}(A)$,

$$\frac{d}{dt} U_n(t)x = A\left(1 - \frac{t}{n} A\right)^{-1} U_n(t)x.$$

Thus

$$U_n(t)x = x + \int_0^t \left(1 - \frac{s}{n} A\right)^{-1} U_n(s)Ax \, ds,$$

and so

$$U(t)x = x + \int_0^t U(s)Ax \, ds.$$

Therefore $x \in \mathfrak{D}(A')$ and $A' \supset A$. But $(1 - A')^{-1} \in \mathfrak{B}(\mathfrak{X})$ by the previous proposition and $(1 - A)^{-1} \in \mathfrak{B}(\mathfrak{X})$ by assumption, so they must agree. ∎

We shall give some concrete examples of how to check these hypotheses below and in the next section. In this regard, note that if $M = 1$, we have a quasi-contractive semigroup and verification of (iii) for $n = 1$ is sufficient. Also, as the proof shows, it suffices to verify (ii) and (iii) for *some* sufficiently large λ. Finally, we note that if (ii) and (iii) hold for $|\lambda| > \beta$, then $U(t)$ is a *group*—that is, is defined for all $t \in \mathbb{R}$, not just $t \geq 0$.

For applications, there are two special verisions of the Hille–Yosida theorem that are frequently used. These involve the notion of the closure of an operator. Namely, A is called *closable* if the closure of the graph of A in $\mathfrak{X} \times \mathfrak{X}$ is the graph of an operator; this operator is called the *closure* of A and is denoted $\bar{A}$. In practice, this often enlarges the domain of A. (For example, the Laplacian

∇^2 in $\mathfrak{X} = L^2$ may originally be defined on smooth functions satisfying desired boundary conditions; its closure will extend the domain to H^2_∂.)

2.14 First Corollary *A linear operator A has a closure $\bar{A}$ that is the generator of a quasi-contractive semigroup on $\mathfrak{X}$ if and only if (i) $\mathfrak{D}(A)$ is dense and (ii) for λ sufficiently large, $(\lambda - A)$ has dense range and $\|(\lambda - A)x\| \geq (\lambda - \beta)\|x\|$.*

Proof Necessity follows Proposition 2.12. For sufficiency, we use the following:

2.15 Lemma (*a*) *Let B be a closable linear operator with a densely defined bounded inverse B^{-1}. Then $(\overline{B^{-1}})$ is injective, and $(\bar{B})^{-1} = (\overline{B^{-1}})$.*

(*b*) *Suppose that A is a densely defined linear operator such that $(\lambda - A)^{-1}$ exists, is densely defined, with $\|(\lambda - A)^{-1}\| \leq K/\lambda$ for large λ. Then A is closable. (Hence, by part (a), $(\lambda - \bar{A})$ is invertible, with $(\lambda - \bar{A})^{-1} = (\overline{\lambda - A)^{-1}}$.)*

Proof (a) Since B^{-1} is bounded, $\overline{B^{-1}}$ is a bounded, everywhere-defined operator. Suppose that $\overline{B^{-1}}y = 0$. We will show that $y = 0$. Let $y_n \in$ Range of B and $y_n \longrightarrow y$. Then $y_n = Bx_n$ for $x_n \in \mathfrak{D}(B)$, and $\|x_n\| \leq \|B^{-1}\|\|y_n\| \longrightarrow 0$. Since B is closable, we must have $y = 0$. Thus $\overline{B^{-1}}$ is injective and (a) follows.

(b) We shall first show that $\lambda R_\lambda \longrightarrow I$ as $\lambda \longrightarrow \infty$, where $R_\lambda = (\overline{\lambda - A)^{-1}}$ by definition. By assumption, $\|R_\lambda\| \leq K/\lambda$. Now pick any $x \in \mathfrak{D}(A)$. Then $x = R_\lambda(\lambda - A)x$, so $x = \lambda R_\lambda x - R_\lambda Ax$, and $\|R_\lambda Ax\| \leq (K/\lambda)\|Ax\| \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Thus $\lambda R_\lambda \longrightarrow I$ strongly on $\mathfrak{D}(A)$. But $\mathfrak{D}(A)$ is dense and $\|\lambda R_\lambda\| \leq K$ for all large λ, so $\lambda R_\lambda \longrightarrow I$ on the whole of $\mathfrak{X}$.

To prove that A is closable, we suppose $x_n \in \mathfrak{D}(A)$, $x_n \longrightarrow 0$, and $Ax_n \longrightarrow y$. We claim that $y = 0$. Indeed, choose a sequence $\lambda_n \longrightarrow \infty$ with $\lambda_n x_n \longrightarrow 0$. Then $(\lambda_n - A)x_n + y \longrightarrow 0$. Since $\|\lambda_n R_{\lambda_n}\| \leq K$, we have $\lambda_n R_{\lambda_n}[(\lambda_n - A)x_n + y] \longrightarrow 0$. Thus, $\lambda_n x_n + \lambda_n R_{\lambda_n} y \longrightarrow 0$. But $\lambda_n x_n \longrightarrow 0$ and $\lambda_n R_{\lambda_n} y \longrightarrow y$, so $y = 0$. ∎

The rest of 2.14 now follows. Indeed, A satisfies the conditions of part (b) of the lemma, and hence $\bar{A}$ satisfies the hypothesis of the Hille–Yosida theorem with $M = 1$). ∎

Now we give a result in Hilbert space. We will sometimes refer to this result as the *Lumer–Phillips Theorem.*[8] For applications we shall give in the next section, it will be one of the most useful results of this section.

2.16 Second Corollary *Let A be a linear operator in a Hilbert space $\mathfrak{H}$. Then A has a closure $\bar{A}$ that is the generator of a quasi-contractive semigroup on X (that is, $\bar{A} \in \mathcal{G}(\mathfrak{H}, 1, \beta)$) if and only if:*

(i) *$\mathfrak{D}(A)$ is dense in $\mathfrak{H}$;*

[8]See Lumer and Phillips [1961] for the case of Banach spaces. It proceeds in a similar way, using a duality map in place of the inner product.

(ii) $\langle Ax, x\rangle \leq \beta\langle x, x\rangle$ *for all* $x \in \mathfrak{D}(A)$ (*If* β *is zero, we call* A *dissipative*); *and*

(iii) $(\lambda - A)$ *has dense range for sufficiently large* λ.

If in (iii) $(\lambda - A)$ *is onto, then* A *is closed and* $A \in \mathcal{G}(\mathcal{H}, 1, \beta)$.

Proof First suppose (i), (ii), and (iii) hold. Then $\langle(\lambda - A)x, x\rangle \geq (\lambda - \beta)\cdot \|x\|^2$, and so by Schwarz's inequality, $\|(\lambda - A)x\| \geq (\lambda - \beta)\|x\|$. Thus $\bar{A} \in \mathcal{G}(\mathcal{H}, 1, \beta)$ by 2.15.

Conversely, assume $\bar{A} \in \mathcal{G}(\mathcal{H}, 1, \beta)$. We need only show that

$$\langle \bar{A}x, x\rangle \leq \beta\langle x, x\rangle \quad \text{for all } x \in \mathfrak{D}(\bar{A}).$$

By 2.9 we can assume that $\beta = 0$ and $U(t)$ is contractive. Now $\langle x, U(t)x\rangle \leq \|x\|\,\|U(t)x\| \leq \|x\|^2$ and therefore $\langle x, U(t)x - x\rangle \leq 0$. Dividing by t and letting $t \downarrow 0$ gives $\langle x, \bar{A}x\rangle \leq 0$ as desired. ∎

2.17 Further Results Some additional useful results that we just quote are as follows:

1. *Bounded Perturbations* If $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$ and $B \in \mathfrak{B}(\mathfrak{X})$, then $A + B \in \mathcal{G}(\mathfrak{X}, M, \beta + \|B\|M)$ (Kato [1966], p. 495).

2. *Trotter–Kato Theorem* If $A_n \in \mathcal{G}(\mathfrak{X}, M, \beta)$ $(n = 1, 2, 3, \ldots)$ $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$ and for λ sufficiently large, $(\lambda - A_n)^{-1} \longrightarrow (\lambda - A)^{-1}$ strongly, then $e^{tA_n} \longrightarrow e^{tA}$ strongly, uniform on bounded t-intervals (Kato [1966], p. 502).

> ***Problem 2.2*** Show that if $\mathfrak{D}(A_n)$ and $\mathfrak{D}(A)$ all have a common *core* $\mathcal{Y} \subset \mathfrak{X}$—that is, A_n, A are the closures of their restrictions to $\mathcal{Y}$, and $A_n \longrightarrow A$ strongly on $\mathcal{Y}$—then $(\lambda - A_n)^{-1} \longrightarrow (\lambda - A)^{-1}$ strongly, from the resolvent identity (see Problem 2.1).

3. *Lax Equivalence Theorem* If $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$, and $K_\epsilon \in \mathfrak{B}(\mathfrak{X})$ is a family of bounded operators defined for $\epsilon \geq 0$, with $K_0 = I$, we say $\{K_\epsilon\}$ is:

(i) *stable* if $\|K^n_{t/n}\|$ is bounded on bounded t-intervals $(n = 1, 2, \ldots)$;

(ii) *resolvent consistent* if for λ sufficiently large

$$(\lambda - A)^{-1} = \operatorname*{s-lim}_{\epsilon \downarrow 0} \left(\lambda - \frac{1}{\epsilon}(K_\epsilon - I)\right)^{-1} \quad \text{(strong limit)};$$

(iii) *consistent* if $(d/d\epsilon +)K_\epsilon(x)|_{\epsilon=0} = Ax$, $x \in$ a core of A.

The Lax equivalence theorem states that $e^{tA} = \text{s-lim}_{n\to\infty} K^n_{t/n}$ uniformly on bounded t-intervals if and only if $\{K_\epsilon\}$ is stable and resolvent consistent (see Chorin, Hughes, McCracken, and Marsden [1978] for a proof and applications). Assuming stability, consistency implies resolvent consistency.

4. *Trotter Product Formula* If A, B are generators of quasi-contractive semigroups and $C = \overline{A + B}$ is a generator, then

$$e^{tC} = \operatorname*{s-lim}_{n\to\infty} (e^{tA/n}e^{tB/n})^n.$$

(This is a special case of 3.)

5. *Inhomogeneous Equations* Let $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$ and consider the following initial value problem: Let $f(t)$ $(0 \leq t \leq T)$ be a continuous $\mathfrak{X}$-valued function. Find $x(t)$ $(0 \leq t \leq T)$ with $x(0)$ a given member of $\mathfrak{D}(A)$, such that

$$x'(t) = Ax(t) + f(t). \tag{I}$$

If we solve (I) formally by the variation of constants formula, we get

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}f(\tau)\,d\tau \quad (0 \leq t \leq T).$$

However, $x(t)$ need not lie in $\mathfrak{D}(A)$; but it will if f is a C^1 function from $[0, T]$ to $\mathfrak{X}$. Then (I) is satisfied in the classical sense (Kato [1966], p. 486). For uniqueness, suppose $y(t)$ is another solution of (I), with $y(0) = x(0)$. Let $z(t) = x(t) - y(t)$. Then $z'(t) = Az(t)$ and $z(0) = 0$, so $z(t) \equiv 0$ by uniqueness. Thus $x(t) = y(t)$.

6. *Trend to Equilibrium* Let $A \in \mathcal{G}(\mathfrak{X}, 1, \beta)$ and suppose there is a $\delta > 0$ such that the spectrum of e^A lies inside the unit disk a positive distance δ from the unit circle. Then for any $x \in \mathfrak{X}$,

$$e^{tA}x \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty.$$

Indeed, if $0 < \delta' < \delta$, we can, by way of the spectral theorem, find a new norm in which

$$A \in \mathcal{G}(\mathfrak{X}, 1, -\delta'),$$

from which the result follows. (See Marsden and McCracken, [1976], §2A, Slemrod [1976], and Dafermos [1968].) Abstract conditions under which spectrum $e^A = e^{\text{spectrum}\,A}$ are unfortunately rather complex (see Carr [1981] and Roh [1982]).

7. *Analytic Semigroups.* If $\|(\zeta - A)^{-1}\| < M/|\zeta|$ for ζ complex and belonging to a sector $|\arg \zeta| \leq \pi/2 + \omega$ $(\omega > 0)$, then A generates a bounded semigroup $V(t)$ that can be extended to complex t's as an analytic function of t $(t \neq 0)$. For real $t > 0$, $x \in \mathfrak{X}$ one has $V(t)x \in \mathfrak{D}(A)$ and

$$\left\| \frac{d^n V(t)x}{dt^n} \right\| \leq (\text{const.}) \cdot \|x\| \cdot t^{-n}.$$

Consult one of the standard references for details.

2.18 Comments on Operators in Hilbert Space and Semigroups The results here are classical ones due to Stone and von Neumann, which may be found in several of the aforementioned references. A densely defined operator in Hilbert space is called *symmetric* if $A \subset A^*$; that is, $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathfrak{D}(A)$ (see Box 2.1 for the definition of A^*; replace $\mathfrak{X}^*$ by $\mathfrak{X}$ in Hilbert space), *self-adjoint* if $A = A^*$ (i.e., A is symmetric and A and A^* have the the same domain), and *essentially self-adjoint* if $\bar{A}$ is self-adjoint.

For the first two results following, $\mathfrak{X}$ is assumed to be a *complex* Hilbert space.

1. *Let A be closed and symmetric. Then A is self-adjoint if and only if $A + \lambda I$ is surjective when* $\operatorname{Im} \lambda \neq 0$.

> ***Problem 2.3*** (a) Show that a symmetric operator is closable. (b) By consulting a book on functional analysis, prove 1.

2. (*Stone's Theorem*) *A is self-adjoint if and only if iA generates a one-parameter unitary group.*

Proof From the symmetry of A and $\|(A + \lambda)x\|^2 \geq 0$, we get $\|(A + \lambda)x\| \geq |\operatorname{Im} \lambda| \|x\|$, and so for λ real, $\|(\lambda - iA)x\| \geq |\lambda| \|x\|$. Thus 2 results from 1 and the Hille–Yosida theorem. ∎

3. (*Real version of Stone's Theorem*) *Let A be a skew-adjoint operator on a real Hilbert space (i.e., $A = -A^*$). Then A generates a one-parameter group of isometries and conversely.* (This follows by an argument similar to 2.)

4. *Let A be closed, symmetric, and $A \leq 0$; that is, $\langle Ax, x\rangle \leq 0$ for all $x \in \mathfrak{D}(A)$. Then A is self-adjoint if and only if $(\lambda - A)$ is onto $(\lambda \geq 0)$.*

> ***Problem 2.4*** Use this and the Fredholm alternative to show that the symmetric elliptic operator A in Section 6.1 is self-adjoint. See Problem 1.2.

5. *If A dissipative ($\langle Ax, x\rangle \leq 0$) and self-adjoint, then A generates a contraction semigroup.* (This follows from 4 and the Lumer–Phillips theorem.)

2.19 Example (*Heat Equation*) Let $\Omega \subset \mathbb{R}^n$ be an open region with smooth boundary, $\mathfrak{X} = L^2(\Omega)$, $Au = \Delta u$, and $\mathfrak{D}(A) = C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ are the C^∞ functions with compact support in Ω. Then $\bar{A}$ generates a contraction semigroup in $\mathfrak{X}$. ($\bar{A}$ will turn out to be the Laplacian on $\mathfrak{D}(\bar{A}) = H_0^2(\Omega)$, the H^2 functions with 0 boundary conditions.)

Proof Obviously A is symmetric. It follows that it is closable (Problem 2.3). Moreover, for $u \in \mathfrak{D}(A)$,

$$\langle Au, u\rangle = \int_\Omega \Delta u \cdot u \, dx = -\int_\Omega \nabla u \cdot \nabla u \, dx \leq 0,$$

so A is dissipative. By the second corollary of the Hille–Yosida theorem, it suffices to show that for $\lambda > 0$, $(\lambda - A)$ has dense range; that is, A is self-adjoint. Suppose $v \in L_2(\Omega)$ is such that

$$\langle(\lambda - A)u, v\rangle = 0 \quad \text{for all} \quad u \in \mathfrak{D}(A).$$

Then

$$\langle(\lambda - \Delta)u, v\rangle = 0 \quad \text{for all} \quad u \in C_0^\infty(\Omega).$$

By the regularity of weak solutions of elliptic equations, v is in fact C^∞ and $v = 0$ on $\partial\Omega$. (See 1.6(iv).) Thus, setting $u = v$ and integrating by parts,

$$\lambda \int_\Omega |v|^2 \, dx + \int_\Omega |\nabla v|^2 \, dx = 0,$$

and so $v = 0$. ∎

The semigroup produced here is actually analytic.

Problem 2.5 Show directly, using the elliptic theory in Section 6.1 (see, especially, Problem 1.2) that Δ on H_∂^2 generates a semigroup. Conclude that this is $\bar{A}$.

Problem 2.6 Generalize 2.19 to the equation $du/dt = Au$, where A is the operator discussed in Section 6.1.

The above example concerns the *parabolic* equation $\partial u/\partial t = Au$. The *hyperbolic* equation $\partial^2 u/\partial t^2 = Au$ is directly relevant to linear elastodynamics and will be considered in the next section.

Box 2.2 ***Summary of Important Formulas for Section 6.2***

Semigroup in a Banach Space
$U(t)\colon \mathfrak{X} \longrightarrow \mathfrak{X}$ is defined for $t \geq 0$, $U(t+s) = U(t)\circ U(s)$, $U(0) = I$, and $U(t)$ is strongly continuous in t at $t = 0+$ (and hence for all $t \geq 0$).

Infinitesimal Generator

$$Ax = \lim_{t\to 0+} \frac{U(t)x - x}{t}$$

on the domain $\mathfrak{D}(A)$ for which the limit exists ($\mathfrak{D}(A)$ is always dense). We write $U(t) = e^{tA}$.

Class (M, β)
$A \in \mathcal{G}(\mathfrak{X}, M, \beta)$ means A is a generator of a $U(t)$ satisfying $\|U(t)\| \leq Me^{t\beta}$. (Bounded if $\beta = 0$; quasi-contractive if $M = 1$, and contractive if both.)

Evolution Equation
If $x_0 \in \mathfrak{D}(A)$, then $x(t) = U(t)x_0 \in \mathfrak{D}(A)$, $x(0) = x_0$ and $dx/dt = Ax$. Solutions are unique if a semigroup exists.

Hille–Yosida Theorem Necessary and sufficient conditions on an operator A to satisfy $A \in \mathcal{G}(\mathfrak{X}, M, \beta)$ are:

(i) $\mathfrak{D}(A)$ is dense;
(ii) $(\lambda - A)^{-1} \in \mathfrak{B}(\mathfrak{X})$ exists for $\lambda > \beta$; and
(iii) $\|(\lambda - A)^{-n}\| \leq M/(\lambda - \beta)^n$ $(n = 1, 2, \ldots)$.

Useful Special Case (Lumer–Phillips Theorem) On Hilbert space $\mathfrak{H}$, $\bar{A} \in \mathcal{G}(\mathfrak{H}, 1, \beta)$ if and only if $\mathfrak{D}(A)$ is dense, $\langle Ax, x\rangle \leq \beta\langle x, x\rangle$ for all $x \in \mathfrak{D}(A)$, and $(\lambda - A)$ has dense range. (If $(\lambda - A)$ is onto, there is no need to take the closure.)

6.3 LINEAR ELASTODYNAMICS

In this section we discuss various aspects of the initial value problem in linear elastodynamics using the theory of semigroups developed in the previous section. We begin with the main result for hyperelasticity: *strong ellipticity is necessary and sufficient for the equations to generate a quasi-contractive semigroup* in $\mathfrak{X} = H^1 \times L^2$.[9] Following this, we discuss stability and show that definiteness of the energy implies dynamical stability (the linear energy criterion). It is also shown, using a result of Weiss [1967], that if Cauchy elasticity generates stable dynamics, then it is necessarily hyperelastic. Box 3.1 describes some general abstract results for Hamiltonian systems, linear hyperelasticity being a special case. Box 3.2 shows how semigroup techniques can be used in a problem of panel flutter, and in Box 3.3 various linear dissipative mechanisms are considered, again using semigroup techniques. Finally, Box 3.4 considers symmetric hyperbolic systems and how they may be used in linear elasticity.

Table 6.3.1 shows the interrelationships between some of the topics considered in this section.

Table 6.3.1

	A. *Linear Hyperelasticity*	
Strong ellipticity	$\Longleftrightarrow$	The equations of motion generate a quasi-contractive semigroup in $H^1 \times L^2$ (relative to some Hilbert space structure).
$\Uparrow$ (Gårding)		$\Uparrow$
Stability (classical elasticity)	$\Longleftrightarrow$	Generates a *contractive* semigroup on $H^1 \times L^2$ (relative to some Hilbert space norm).
$\Uparrow$ (Korn)		
Elasticity tensor positive-definite		
	B. *Cauchy Elasticity*	
Generation of a contractive semigroup on a Hilbert space of the form $\mathcal{Y} \times L^2$, $\mathcal{Y} \subset L^2$	$\Longrightarrow$	Hyperelasticity
	C. *Logarithmic Convexity*	
Strong ellipticity "strictly" fails	$\Longleftrightarrow$	No semigroup on any space of the form $\mathcal{Y} \times L^2$, $\mathcal{Y} \subset L^2$.

Consider then the equations of linearized elasticity on a region Ω that is (for simplicity) a bounded open set in $\mathbb{R}^n$ with piecewise smooth boundary or else

[9]The fact that the equations generate a semigroup in $\mathfrak{X}$ embodies the idea that we have a continuous linear dynamical system in $\mathfrak{X}$. In particular, the solutions depend continuously in $\mathfrak{X}$ as the initial conditions are varied in the *same* space $\mathfrak{X}$. This is to be compared with other types of continuous data dependence where the solution and initial data vary in *different* spaces. For the latter, strong ellipticity is not required. See Knops and Payne [1971] for an extensive discussion of this point.

$\Omega = \mathbb{R}^n$. These equations are

$$\left.\begin{aligned} \rho\ddot{\boldsymbol{u}} &= \operatorname{div}(\mathsf{a}\cdot\nabla\boldsymbol{u}) \quad \text{on} \quad \Omega, \\ \text{that is,} \qquad \rho\ddot{u}^a &= (\mathsf{a}^{abcd}u_{c|d})_{|b} \quad \text{on} \quad \Omega. \end{aligned}\right\} \tag{1}$$

We have dropped the inhomogeneous terms $\boldsymbol{f} = -\rho\mathring{\boldsymbol{a}} + \rho\boldsymbol{b} + \operatorname{div}(\mathring{\boldsymbol{\sigma}})$, for they cause no added difficulty in questions of existence and uniqueness; see item 5 in 2.17.

The boundary conditions are assumed to be either

$$\left.\begin{aligned} \text{displacement:} \quad \boldsymbol{u} &= \boldsymbol{0} \text{ on } \partial\Omega, \\ \text{traction:} \quad \mathsf{a}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{n} &= \boldsymbol{0} \text{ on } \partial\Omega, \\ \text{or mixed:} \quad \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial_d \text{ and } \mathsf{a}\cdot\nabla\boldsymbol{u} &= \boldsymbol{0} \text{ on } \partial_\tau \end{aligned}\right\} \tag{2}$$

again taken to be homogeneous without loss of generality. We assume the elasticity tensor $\mathsf{a}^{abcd}(x)$ is C^1 in x and $\rho(x) \geq \delta > 0$ is C^0.

We recall that the material in question is *hyperelastic* when $\mathsf{a}^{abcd} = \mathsf{a}^{cdab}$. This is equivalent to *symmetry* of the operator $A\boldsymbol{u} = (1/\rho)\operatorname{div}(\mathsf{a}\cdot\nabla\boldsymbol{u})$ in L^2 as in 1.3, where we put on L^2 a modified inner product corresponding to the $1/\rho$ factor in A, namely, $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \rho(x)u^a(x)v_a(x)\,dv(x)$.

Rewrite the equations of motion as $\ddot{u} = Au$ or

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{u}\\ \dot{\boldsymbol{u}}\end{pmatrix} = A'\begin{pmatrix}\boldsymbol{u}\\ \dot{\boldsymbol{u}}\end{pmatrix}, \qquad \text{where} \quad A' = \begin{pmatrix}0 & I\\ A & 0\end{pmatrix}. \tag{3}$$

The domain of A is taken to be $H^2_\partial(\Omega)$, defined in 1.1.

The main existence theorem for linear elastodynamics is as follows:

3.1 Theorem *Let* a *be hyperelastic and let the symmetric operator* $A\colon \mathfrak{D}(A) = H^2_\partial \longrightarrow L^2$ *be defined by* $A\boldsymbol{u} = [\operatorname{div}(\mathsf{a}\cdot\nabla\boldsymbol{u})]/\rho$. *Let* $\mathfrak{X} = H^1 \times L^2$ *and let* A' *be defined by* (3) *with* $\mathfrak{D}(A') = \mathfrak{D}(A) \times H^1$. *The following assertions are equivalent:*

(i) *There is a Hilbert space norm on* H^1 *such that* A' *generates a quasi-contractive group on* $\mathfrak{X}$.

(ii) a *is strongly elliptic.*

Proof First, assume a is strongly elliptic. For $\boldsymbol{v} \in \mathfrak{D}(A)$ and $\boldsymbol{u} \in H^1(\Omega)$, the Dirichlet from is given by $B(\boldsymbol{u}, \boldsymbol{v}) = -\langle \boldsymbol{u}, A\boldsymbol{v}\rangle$, where the L^2 inner product is weighted with ρ, as above. By Gårding's inequality,

$$B(\boldsymbol{u}, \boldsymbol{u}) + d\,\|\boldsymbol{u}\|^2_{L^2} = |||\boldsymbol{u}|||^2_{H^1}$$

is equivalent to the H^1-norm. We use norm $(|||\boldsymbol{u}|||^2_{H^1} + \|\dot{\boldsymbol{u}}\|^2_{L^2})^{1/2}$ on $\mathfrak{X} = H^1 \times L^2$. If $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{H^1}$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denote the corresponding inner products on H^1 and $\mathfrak{X}$, respectively, then

$$\begin{aligned} \langle\!\langle A'(\boldsymbol{u}, \dot{\boldsymbol{u}}), (\boldsymbol{u}, \dot{\boldsymbol{u}})\rangle\!\rangle &= \langle\!\langle \boldsymbol{u}, \dot{\boldsymbol{u}}\rangle\!\rangle_{H^1} + \langle A\boldsymbol{u}, \dot{\boldsymbol{u}}\rangle_{L^2} \\ &= B(\boldsymbol{u}, \dot{\boldsymbol{u}}) + d\langle \boldsymbol{u}, \dot{\boldsymbol{u}}\rangle_{L^2} + \langle A\boldsymbol{u}, \dot{\boldsymbol{u}}\rangle_{L^2} = d\langle \boldsymbol{u}, \dot{\boldsymbol{u}}\rangle_{L^2} \\ &\leq \tfrac{1}{2}\, d(\|\boldsymbol{u}\|^2_{L^2} + \|\dot{\boldsymbol{u}}\|^2_{L^2}) \leq \tfrac{1}{2}\, d(|||\boldsymbol{u}|||^2_{H^1} + \|\dot{\boldsymbol{u}}\|^2_{L^2}). \end{aligned}$$

The same estimate holds for $-A'$ since $\dot{u}$ can be replaced by $-\dot{u}$. By Problem 1.2 of Section 6.1, $\lambda u - Au = f$ is solvable for u if $\lambda > d_1$. The solution of

$$(\lambda - A')(u, \dot{u}) = (f, \dot{f})$$

is readily checked to be

$$u = (\lambda^2 - A)^{-1}\dot{f} + \lambda f, \qquad \dot{u} = -f + \lambda u,$$

so $(\lambda - A')$ is onto for $|\lambda|$ sufficiently large. Thus by the second corollary of the Hille–Yosida theorem (2.16), A' is the generator of a quasi-contractive semigroup.

Next we prove the converse. If A' generates a quasi-contractive semigroup, then 2.16 gives the estimate

$$\langle\!\langle u, \dot{u}\rangle\!\rangle_{H^1} + \langle Au, \dot{u}\rangle_{L^2} \leq \beta\{\langle\!\langle u, u\rangle\!\rangle_{H^1} + \langle \dot{u}, \dot{u}\rangle_{L^2}\}$$

for all $u \in \mathfrak{D}(A)$ and $\dot{u} \in H^1$. Letting $\dot{u} = \alpha u$ ($\alpha > 0$) and using the equivalence of the $\|\cdot\|_{H^1}$ and $|||\cdot|||_{H^1}$ norms,

$$-\langle Au, u\rangle_{L^2} \geq \frac{1}{\alpha}\{\alpha\langle\!\langle u, u\rangle\!\rangle_{H^1} - \beta\langle\!\langle u, u\rangle\!\rangle_{H^1} - \beta\alpha^2\langle u, u\rangle_{L^2}\}$$

$$\geq \frac{1}{\alpha}\{\alpha\|u\|_{H^1}^2 - \gamma\|u\|_{H^1}^2 - \beta\alpha^2\|u\|_{L^2}^2\}$$

for a constant $\gamma > 0$. Choosing $\alpha > \gamma^2$, we get

$$-\langle Au, u\rangle_{L^2} \geq c\|u\|_{H^1}^2 - d\|u\|_{L^2}^2,$$

where $c = (\alpha - \gamma^2)/\alpha\gamma$, $d = \beta\alpha$. Thus Garding's inequality holds and so by 1.5, we have strong ellipticity. ∎

Remarks

(1) This argument also gives a sharp regularity result. Recall from 1.6(i) that if $u \in H^2(\Omega)$, u satisfies non-mixed boundary conditions, and Ω has a smooth boundary, then for $s \geq 2$,

$$\|u\|_{H^s} < C(\|Au\|_{H^{s-2}} + \|u\|_{L^2}).$$

This shows that $\mathfrak{D}(A^m) \subset H^{2m}$ (which, by the Sobolev embedding theorem, lies in C^k if $k < 2m - n/2$). From the abstract theory of semigroups in Section 6.2, we know that if $(u(0), \dot{u}(0)) \in \mathfrak{D}((A')^m)$, then $(u(t), \dot{u}(t)) \in \mathfrak{D}((A')^m)$ as well. For instance $(u(0), \dot{u}(0)) \in \mathfrak{D}((A')^3)$ means that

$$u(0) \in \mathfrak{D}(A^2) \quad \text{and} \quad \dot{u}(0) \in \mathfrak{D}(A).$$

Note that this automatically means $u(0)$ and $\dot{u}(0)$ must satisfy extra boundary conditions; in general, these extra conditions for $(u, \dot{u})$ to belong to $\mathfrak{D}((A')^m)$ are called the *compatibility conditions.*

In particular, if $u(0)$ and $\dot{u}(0)$ are C^∞ in x and belong to the domain of every power of A, then the solutions are C^∞ in (x, t) in the classical sense.

(2) If we have

$$\mathsf{a}^{abcd}(x)\xi_a\xi_c\eta_b\eta_d \geq \epsilon(x)|\xi|^2|\eta|^2,$$

and $\epsilon(x)$ vanishes at some points, then one can still, under technical condition sufficient to guarantee A is self-adjoint (see, e.g., Reed and Simon [1975], and references therein), get a quasi-contractive semigroup on $\mathcal{Y} \times L^2$, where $\mathcal{Y}$ is the completion of $H^1(\Omega)$ in the energy norm. One can show along the lines of 3.1 that if A' generates a quasi-contractive semigroup on $\mathcal{Y} \times L^2$, then we must have

$$\mathsf{a}^{abcd}(x)\xi_a\xi_c\eta_b\eta_d \geq 0.$$

If $\mathsf{a}^{abcd}\xi_a\xi_c\eta_b\eta_d < 0$ somewhere in Ω, we say strong ellipticity *strictly fails.* In 3.7 it is shown that in this case no semigroup is possible on any space $\mathcal{Y} \times L^2$.

Next we mention the abstract version of 3.1 (see also Box 3.1).

3.2 Theorem (*Weiss* [1967] *and Goldstein* [1969]) *Let $\mathcal{H}$ be a real Hilbert space and A a self-adjoint operator on $\mathcal{H}$ satisfying $\langle Ax, x\rangle \geq c\langle x, x\rangle$ for a constant $c > 0$. Let $A^{1/2}$ be the positive square root of A and let $\mathcal{H}_1$ be the domain of $A^{1/2}$ with the graph norm. Then the operator*

$$A' = \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix}$$

generates a one-parameter group on $\mathcal{H}_1 \times \mathcal{H}$ with domain $\mathfrak{D}(A) \times \mathcal{H}_1$. The semigroup $e^{tA'}$ solves the abstract wave equation $(\partial^2 x)/(\partial t^2) = -Ax$.

Proof Our condition on A means that the graph norm of $A^{1/2}$ is equivalent to the norm $|||x||| = \langle A^{1/2}x, A^{1/2}x\rangle$. Thus on $\mathcal{H}_1 \times \mathcal{H}$ we can take the Hilbert space norm

$$\|(x, y)\|^2 = \langle A^{1/2}x, A^{1/2}x\rangle + \langle y, y\rangle.$$

We will show that A' is skew-adjoint on $\mathcal{H}_1 \times \mathcal{H}$, so the result follows from the real form of Stone's theorem (2.18(3)). Let us first check skew symmetry. Let $x_1, x_2 \in \mathfrak{D}(A)$ and $y_1, y_2 \in \mathcal{H}_1$. Then

$$\begin{aligned}\langle A'(x_1, x_2), (y_1, y_2)\rangle &= \langle (x_2, -Ax_1), (y_1, y_2)\rangle \\ &= \langle A^{1/2}x_2, A^{1/2}y_1\rangle + \langle -Ax_1, y_2\rangle \\ &= \langle Ax_2, y_1\rangle - \langle Ax_1, y_2\rangle\end{aligned}$$

since $x_2 \in \mathfrak{D}(A)$. Similarly, for $x_1, x_2 \in \mathcal{H}$, and $y_1, y_2 \in \mathfrak{D}(A)$,

$$\langle (x_1, x_2), A'(y_1, y_2)\rangle = \langle x_1, Ay_2\rangle - \langle x_2, Ay_1\rangle,$$

so A' is skew-symmetric.

To show A' is skew-adjoint, let $(y_1, y_2) \in \mathfrak{D}(A'^+)$, where A'^+ denotes the skew-adjoint of A'. This means there is $(z_1, z_2) \in \mathcal{H}_1 \times \mathcal{H}$ such that

$$\langle A'(x_1, x_2), (y_1, y_2)\rangle = -\langle (x_1, x_2), (z_1, z_2)\rangle$$

for all $(x_1, x_2) \in \mathfrak{D}(A) \times \mathcal{H}_1$. This assertion is equivalent to

$$\langle A^{1/2}x_2, A^{1/2}y_1\rangle = -\langle x_2, z_2\rangle \quad \text{for all} \quad x_2 \in \mathfrak{D}(A^{1/2})$$

and

$$\langle Ax_1, y_2\rangle = \langle A^{1/2}x_1, A^{1/2}z_1\rangle \quad \text{for all} \quad x_1 \in \mathfrak{D}(A).$$

The first statement implies $A^{1/2}y_1 \in \mathfrak{D}(A^{1/2})$ or $y_1 \in \mathfrak{D}(A)$ and the second implies $y_2 \in \mathfrak{D}(A^{1/2})$. Hence $\mathfrak{D}(A'^{+}) = \mathfrak{D}(A')$ so A' is skew-adjoint. ∎

Remarks

(1) The group generated by A' can be written explicitly in terms of that generated by $C = A^{1/2}$ as

$$e^{tA'} = \cosh(tC)(\text{Identity}) + \frac{\sinh(tC)}{C}A',$$

where for example $\cosh tC = (e^{tC} + e^{-tC})/2$. Division by C is in terms of the operational calculus.

(2) The condition $c > 0$ in the hypothesis can be relaxed to $c = 0$ if the spaces are modified as follows. Let A be self-adjoint and non-negative with trivial kernel and let $\mathcal{H}_A$ be the completion of $\mathcal{H}$ with respect to the norm $\|x\|_A^2 = \langle Ax, x\rangle$. Let $\mathfrak{X} = \mathcal{H}_A \times \mathcal{H}$ and let $A'(x, y) = (y, -Ax)$. Then the closure of A' is a generator in $\mathfrak{X}$. The argument follows the lines above (see Weiss [1967]).

(3) The wave equation $u_{tt} = \nabla^2 u$ does *not* generate a semigroup in $W^{1,p} \times L^p$ if $p \neq 2$ and $n > 1$. See Littman [1973].

Problem 3.1 Show that A' cannot be a generator on $\mathcal{H} \times \mathcal{H}$ unless A is a bounded operator. (*Hint:* $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ is bounded on $\mathcal{H} \times \mathcal{H}$; so if A' is a generator, so is $A_0 = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$. Now compute e^{tA_0}.)

The next abstract theorem will show that stable dynamics implies hyperelasticity. Here, we say A' is *dynamically stable* if it generates a contractive semigroup on $\mathfrak{X}$ (relative to some norm on H^1).

3.3 Theorem (*Weiss* [1967]) *Let A be a linear operator in a Hilbert space $\mathcal{H}$ with domain $\mathfrak{D}(A)$. Let $\mathcal{Y}$ be a Hilbert space, with $\mathfrak{D}(A) \subset \mathcal{Y} \subset \mathcal{H}$. Assume that*

$$A' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \qquad \mathfrak{D}(A') = \mathfrak{D}(A) \times \mathcal{Y}$$

generates a contractive semigroup on $\mathfrak{X} = \mathcal{Y} \times \mathcal{H}$. Then A is a self-adjoint operator, and in particular is symmetric.

Proof By the Lumer–Phillips theorem 2.16, with $\beta = 0$, we have

$$\langle (u, \dot{u}), A'(u, \dot{u})\rangle_{\mathfrak{X}} \leq 0, \qquad \text{that is,} \quad \langle u, \dot{u}\rangle_{\mathcal{Y}} + \langle \dot{u}, Au\rangle_{\mathcal{H}} \leq 0.$$

Since this holds for all $u \in \mathfrak{D}(A)$, $\dot{u} \in \mathcal{Y}$, we can replace $\dot{u}$ by $-\dot{u}$. The left-hand side changes sign, so we must have $\langle u, \dot{u}\rangle_{\mathcal{Y}} + \langle \dot{u}, Au\rangle_{\mathcal{H}} = 0$. Thus, $\langle \dot{u}, Au\rangle_{\mathcal{H}} = -\langle \dot{u}, u\rangle_{\mathcal{Y}}$, and so A is symmetric and non-positive. It is also self-adjoint since $(\lambda - A)$ is surjective for $\lambda > 0$ (see 2.18(4)). ∎

This result shows that *linear Cauchy elasticity cannot give a stable dynamical system in $\mathcal{Y} \times L^2$ unless it is hyperelastic.* This is, presumably, an undesirable situation for Cauchy elasticity. It is a semigroup analogue of the work theorems that are used to cast doubt on Cauchy elasticity (cf. Gurtin [1972b], p. 82).

The proof of the above theorem has another interesting corollary. It shows that, under the hypothesis given, the $\mathcal{Y}$-norm is necessarily the energy norm, $\|u\|_{\mathcal{Y}}^2 = -\langle u, Au\rangle_{\mathcal{K}}$, and that our semigroups are forced to be *groups of isometries.*

Remarks

(1) Weiss [1967] also shows that one is forced into working on Hilbert space as opposed to general Banach spaces (cf. Remark 3 following 3.2).

(2) Related to the contractive assumption is an abstract result of Nagy (Riesz and Nagy [1955], p. 396), namely, that a bounded one-parameter *group* on Hilbert space is actually unitary in an equivalent Hilbert norm.

Next we discuss the *energy criterion* for linear elastodynamics.

3.4 Definition We say that $\mathbf{a}$ is *stable* provided that there is a $c > 0$ such that

$$B(\boldsymbol{u}, \boldsymbol{u}) \geq c\|\boldsymbol{u}\|_{H^1}^2 \quad \text{for all} \quad \boldsymbol{u} \in H^1(\Omega).$$

That is, the elastic potential energy is positive-definite relative to the H^1-norm. We recall from 1.5 that stability implies strong ellipticity.

3.5 Theorem *A' is dynamically stable (in the sense that A' generates a contractive group on $H^1 \times L^2$ relative to some inner product on H^1) if and only if $\mathbf{a}$ is stable.*[10]

Proof In the proof of sufficiency in 3.1 we can take $\beta = 0$ by stability, so we get a contractive group.

Conversely, if we get a contractive group relative to some equivalent inner product $\langle\!\langle\, , \rangle\!\rangle_{H^1}$ on H^1, we saw in the proof of Weiss' theorem 3.3 that we must have $B(\boldsymbol{u}, \boldsymbol{u}) = \langle\!\langle \boldsymbol{u}, \boldsymbol{u}\rangle\!\rangle_{H^1}$, which implies stability. ▮

3.6 Corollary *For the displacement problem in classical elasticity, uniform pointwise stability of the elasticity tensor, that is, there is a $\delta > 0$ such that*

$$\mathbf{e}\cdot\mathbf{c}\cdot\mathbf{e} \geq \delta\|\mathbf{e}\|^2$$

for all symmetric e_{ab}, implies stability.

[10]In 2.7, Chapter 5 we defined a semigroup to be stable when it is bounded. Note that there, stable means contractive relative to *some* Hilbert space structure. There is a slight technical difference.

Proof This follows by virtue of Korn's first inequality

$$\int_\Omega \|\boldsymbol{e}\|^2\,dx \geq c\|\boldsymbol{u}\|^2_{H^1} \quad (c > 0),$$

where $e_{ab} = \frac{1}{2}(u_{a,b} + u_{b,a})$ and $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$. (See 1.12.) ▮

For the traction problem, Korn's first inequality cannot hold since $\boldsymbol{e}$ is invariant under the Euclidean group. Instead we have *Korn's second inequality* (see 1.12),

$$\int_\Omega \|\boldsymbol{e}\|^2\,dx + \int_\Omega \|\boldsymbol{u}\|^2\,dx \geq c\|\boldsymbol{u}\|^2_{H^1}.$$

As it stands, this shows that positive-definiteness of the elasticity tensor implies Gårding's inequality. However, we already know Gårding's inequality is true from strong ellipticity alone. Thus Korn's second inequality is not required for existence. However, there is a deeper reason for Korn's second inequality. Namely, if we view the traction problem as a Hamiltonian system (as in Chapter 5) and move into center of mass and constant angular momentum "coordinates,"[11] then in the appropriate quotient space of $H^1 \times L^2$, we get a new Hamiltonian system and *in this quotient space*, Korn's second inequality can be interpreted as saying that *positive-definiteness of the elasticity tensor implies stability* and hence dynamical stability.

> ***Problem 3.2*** (On the level of a masters thesis.) Carry out the details of the remarks just given.

Finally, we sketch an argument due to Wilkes [1980] based on logarithmic convexity (see Knops and Payne [1971]) to show that dynamics is not possible when strong ellipticity fails, even when H^1 is replaced by some other space $\mathcal{Y}$ in 3.1.

3.7 Theorem *If the strong ellipticity condition strictly fails then A' cannot generate a semigroup on $\mathcal{Y} \times L^2$, where $\mathfrak{D}(A) \subset \mathcal{Y} \subset L^2$ and $\mathcal{Y}$ is a Banach space.*

Proof Suppose A' generates a semigroup $U(t)$ of type (M, β). Since strong ellipticity strictly fails, the argument used to prove Hadamard's theorem (Box 2.2) shows that

$$\inf_{\|\boldsymbol{u}\|_{L^2}=1} -\langle \boldsymbol{u}, A\boldsymbol{u}\rangle = -\infty.$$

(Roughly speaking, one can rescale $\boldsymbol{u}(x)$ keeping its L^2-norm constant, but blowing up its H^1-norm.) We can thus choose $\boldsymbol{u}(0)$, $\dot{\boldsymbol{u}}(0)$ such that

$$2\langle \dot{\boldsymbol{u}}(0), \boldsymbol{u}(0)\rangle > \beta, \qquad \|\boldsymbol{u}(0)\|^2_{L^2} = 1$$

and

$$\tfrac{1}{2}\langle \dot{\boldsymbol{u}}(0), \dot{\boldsymbol{u}}(0)\rangle - \tfrac{1}{2}\langle \boldsymbol{u}(0), A\boldsymbol{u}(0)\rangle = c < 0.$$

---

[11]In Hamiltonian systems language this is a special case of the general procedure of "reduction." See Abraham and Marsden [1978] and Marsden and Weinstein [1974].

Here $\langle\ ,\ \rangle$ is the L^2 inner product. (For example, we can let $\dot{u}(0) = \mu u(0)$ where $\mu > \beta/2$ and $u(0)$ has unit L^2-norm.) Let $(u(t), \dot{u}(t)) = U(t)(u(0), \dot{u}(0))$ and $F(t) = \frac{1}{2}\langle u(t), u(t)\rangle$. Then clearly

$$\dot{F} = \langle \dot{u}, u\rangle \quad \textit{and} \quad \ddot{F} = \langle u, Au\rangle + \langle \dot{u}, \dot{u}\rangle.$$

Note that c is the initial energy, and the energy is constant in time. By Schwarz's inequality,

$$\frac{\dot{F}^2}{F} = \frac{2\langle \dot{u}, u\rangle^2}{\langle u, u\rangle} \leq 2\|\dot{u}\|_{L^2}^2 = 2c + \ddot{F}.$$

Thus, $F\ddot{F} - \dot{F}^2 \geq -2cF \geq 0$. Hence, $(d^2/dt^2)(\log F) \geq 0$, and so

$$F(t) \geq F(0)\exp\left(\frac{\dot{F}(0)}{F(0)}t\right), \qquad \text{that is,} \quad \|u\|_{L^2}^2 \geq \|u(0)\|_{L^2}^2 e^{\gamma t},$$

where $\gamma = 2\langle \dot{u}(0), u(0)\rangle/\langle u(0), u(0)\rangle$. Because $U(t)$ is a semigroup of type (M, β), the $\mathcal{Y}$ topology is stronger than the L^2 topology and $\gamma > \beta$, such an inequality is impossible. ▮

Box 3.1 ***Hamiltonian One-Parameter Groups***

In Section 5.2 we studied some properties of linear Hamiltonian systems. Now we re-examine a few of these topics in the light of semigroup theory. The main result is an abstract existence theorem that applies to hyperelasticity under the assumption of stability.

Let $\mathfrak{X}$ be a Banach space and ω a (weak) symplectic form (see Definition 2.1 in Chapter 5). We recall that a linear operator A on $\mathfrak{X}$ is called *Hamiltonian* when it is ω-skew; that is, $\omega(Ax, y) = -\omega(x, Ay)$ for all $x, y \in \mathfrak{D}(A)$. Let A^+ denote the ω-adjoint of A; that is,

$$\mathfrak{D}(A^+) = \{w \in \mathfrak{X} \mid \text{there is a } z \in \mathfrak{X} \text{ such that } \omega(z, x) = \omega(w, Ax) \text{ for all } x \in \mathfrak{D}(A)\}$$

and $A^+w = z$. We call A *ω-skew-adjoint* when $A = -A^+$. This is a stronger condition than ω-skew symmetry.

In 2.6, Chapter 5, we saw that if A is Hamiltonian and generates a semigroup $U(t)$, then each $U(t)$ is a canonical transformation. The next result shows that A is necessarily ω-skew-*adjoint*. The result, whose proof is based on an idea of E. Nelson, is due to Chernoff and Marsden [1974] (see also Marsden [1968b]).

3.8 Proposition *Let $U(t)$ be a one-parameter group of canonical transformations on $\mathfrak{X}$ with generator A. Then A is ω-skew-adjoint.*

Proof Let A^+ be the ω-adjoint of A. Since $\omega(U(t)x, U(t)y) = \omega(x, y)$, we have

$$\omega(Ax, y) + \omega(x, Ay) = 0 \quad \text{for} \quad x, y \in \mathfrak{D}(A),$$

so $A^+ \supset -A$. Now let $f \in \mathfrak{D}(A^+)$ and $A^+f = g$. For $x \in \mathfrak{D}(A)$, write

$$U(t)x = x + \int_0^t AU(s)x\,ds,$$

so

$$\omega(U(t)x, f) = \omega(x, f) + \int_0^t \omega(AU(s)x, f)\,ds.$$

Thus

$$\omega(x, U(-t)f) = \omega(x, f) + \int_0^t \omega(x, U(-s)A^+f)\,ds.$$

Since $\mathfrak{D}(A)$ is dense,

$$U(-t)f = f + \int_0^t U(-s)A^+f\,ds.$$

It follows that $f \in \mathfrak{D}(A)$ and $-Af = A^+f$. ∎

> ***Problem 3.3*** Deduce that the generator of a one-parameter unitary group on complex Hilbert space is i times a self-adjoint operator (one-half of Stone's theorem).

In 3.8, ω-skew-adjointness is not sufficient for A to be a generator. (This is seen from the ill-posed problem $\ddot{\phi} = -\nabla^2\phi$, for example.) However, it does become sufficient if we add a positivity requirement.

3.9 Theorem (*Chernoff and Marsden* [1974]) *Let $\mathfrak{X}$ be a Banach space and ω a weak symplectic form on $\mathfrak{X}$.*

Let A be an ω-skew-adjoint operator in $\mathfrak{X}$ and set

$$[x, y] = \omega(Ax, y),$$

the energy inner product. Assume the following stability condition:

$$[x, x] \geq c\,\|x\|_{\mathfrak{X}}^2$$

for a constant $c > 0$.

Let $\mathfrak{H}$ be the completion of $\mathfrak{D}(A)$ with respect to [,], *let*

$$\mathfrak{D}(\tilde{A}) = \{x \in \mathfrak{D}(A) \mid Ax \in \mathfrak{H}\} \text{ and set } \tilde{A}x = Ax \text{ for } x \in \mathfrak{D}(\tilde{A}).$$

Then $\tilde{A}$ generates a one-parameter group of canonical transformations in $\mathfrak{H}$ (relative to $\tilde{\omega}$, the restriction of $\tilde{\omega}$ to $\mathfrak{H}$) and these are, moreover, isometries relative to the energy inner product on $\mathfrak{H}$.

Proof Because the energy inner product satisfies $[x, x] \geq c\,\|x\|_{\mathfrak{X}}^2$, we can identify $\mathfrak{H}$ with a subspace of $\mathfrak{X}$. Relative to [,] we note that $\tilde{A}$ is skew-symmetric: for $x, y \in \mathfrak{D}(A)$,

$$[x, \tilde{A}y] = \omega(Ax, Ay) = -\omega(Ay, Ax) = -[y, \tilde{A}x] = -[Ax, y].$$

We next shall show that $\tilde{A}$ is skew-adjoint. To do this, it is enough to show $\tilde{A}: \mathfrak{D}(\tilde{A}) \longrightarrow \mathfrak{K}$ is onto. This will follow if we can show that A: $\mathfrak{D}(A) \longrightarrow \mathfrak{X}$ is onto; see 2.18.

Let $w \in \mathfrak{X}$. By the Riesz representation theorem, there is an $x \in \mathfrak{K}$ such that

$$\omega(w, y) = [x, y] \quad \text{for all} \quad y \in \mathfrak{K}.$$

In particular,

$$\omega(w, y) = \omega(Ay, x) = -\omega(x, Ay) \quad \text{for all} \quad y \in \mathfrak{D}(A).$$

Therefore, $x \in \mathfrak{D}(A^+) = \mathfrak{D}(A)$ and $Ax = w$. Thus A is onto.

It remains to show that $\tilde{\omega}$ is invariant under $U(t) = e^{t\tilde{A}}$. By 2.6, Chapter 5, we need only verify that $\tilde{A}$ is $\tilde{\omega}$-skew. Indeed for $x, y \in \mathfrak{D}(\tilde{A})$,

$$\tilde{\omega}(\tilde{A}x, y) = \omega(Ax, y) = -\omega(x, Ay) = -\tilde{\omega}(x, Ay). \quad \blacksquare$$

3.10 Example (*Abstract Wave Equation*) (See Examples 2.9, Chapter 5 and Theorem 3.2 above.) Let $\mathfrak{K}$ be a real Hilbert space and B a self-adjoint operator satisfying $B \geq c > 0$. Then

$$A = \begin{pmatrix} 0 & I \\ -B & 0 \end{pmatrix}$$

is Hamiltonian on $\mathfrak{X} = \mathfrak{D}(B^{1/2}) \times \mathfrak{K}$ with $\mathfrak{D}(A) = \mathfrak{D}(B) \times \mathfrak{D}(B^{1/2})$, $\omega((x_1, y_1), (x_2, y_2)) = \langle y_2, x_1 \rangle - \langle x_2, y_1 \rangle$, and energy $H(x, y) = \frac{1}{2}\|y\|^2 + \frac{1}{2}\langle Bx, x \rangle$. Theorem 3.9 then reproduces Theorem 3.2, which we proved above concerning the abstract wave equation $\ddot{x} = -Bx$. It follows from 2.6, Chapter 5 that the corresponding one-parameter group consists of canonical transformations that preserve energy. The dynamics is thus stable.

Problem 3.4 Show that if **a** is stable in the sense of 3.4, then Theorem 3.9 applies to A' and reproduces one direction of the result obtained in 3.5.

Problem 3.5 (Converse of 3.9.) Suppose A is Hamiltonian on $\mathfrak{X}$ and generates a one-parameter group $U(t)$. Suppose, relative to some complex Hilbert space structure $\langle\, , \rangle$ on $\mathfrak{X}$, that $U(t)$ is contractive and $\omega(x, y) = -\mathrm{Im}\langle x, y \rangle$. Show that $\omega(Ax, y) = \mathrm{Re}\langle x, y \rangle$ and hence that A is stable (cf. 3.3).

Box 3.2 ***A Semigroup Arising in Panel Flutter***

Consider small vibrations of a panel, as shown in Figure 6.3.1. Neglecting nonlinear and two-dimensional effects, the equations for the panel deflection $v(x, t)$ are

$$\ddot{v} + \alpha\dot{v}'''' + v'''' - \Gamma v'' + \rho v' + \sqrt{\rho}\,\delta\dot{v} = 0, \qquad \text{(PF)}$$

where $\cdot = \partial/\partial t$ and $' = \partial/\partial x$. Here α is a viscoelastic structural damping constant, ρ is an aerodynamic pressure, Γ is an in-plane tensile load, and $\sqrt{\rho}\,\delta$ is aerodynamic damping. We assume $\alpha > 0$, $\delta > 0$, and $\rho > 0$. If the edges of the plate are simply supported, we impose the boundary conditions $v = 0$, $v'' + \alpha\dot{v}'' = 0$ at $x = 0, 1$.

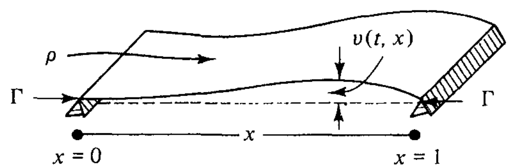


Figure 6.3.1

The equations (PF) are derived in Dowell [1975]. Nonlinear versions of (PF) are discussed in Chapter 7. Let $H_\partial^2 = \{u \in H^2([0, 1]) \,|\, u = 0$ at $x = 0, 1\}$ and $\mathfrak{X} = H_\partial^2 \times L^2$. Define the operator A on $\mathfrak{X}$ by

$$A\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{v} \\ -\alpha\dot{v}'''' - v'''' \end{pmatrix}$$

with $\mathfrak{D}(A) = \{(v, \dot{v}) \in H_\partial^2 \times L^2 \,|\, v + \alpha\dot{v} \in H^4, v'' + \alpha\dot{v}'' = 0$ at $x = 0, 1$, and $\dot{v} \in H_\partial^2\}$. On $\mathfrak{X}$ define the inner product

$$\langle (v, \dot{v}), (w, \dot{w}) \rangle = \langle v'', w'' \rangle_{L^2} + \langle \dot{v}, \dot{w} \rangle_{L^2},$$

where $\langle\ ,\ \rangle_{L^2}$ denotes the L^2 inner product.

Let $B\colon \mathfrak{X} \longrightarrow \mathfrak{X}$ be defined by

$$B\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma v'' - \rho v' - \sqrt{\rho}\,\delta\dot{v} \end{pmatrix}$$

and observe that equations (PF) may be written

$$\frac{d}{dt}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = A\begin{pmatrix} v \\ \dot{v} \end{pmatrix} + B\begin{pmatrix} v \\ \dot{v} \end{pmatrix}. \qquad \text{(PF)}'$$

3.11 Proposition *The equations $(PF)'$ generate a contraction semigroup on $\mathfrak{X}$.*

Proof Since B is a bounded operator, it suffices to show that A is a generator. (See 2.17(1).) To do this, observe that

$$\left\langle A\begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \begin{pmatrix} v \\ \dot{v} \end{pmatrix}\right\rangle = \left\langle \begin{matrix} \dot{v} \\ -\alpha\dot{v}'''' - v'''' \end{matrix}, \begin{matrix} v \\ \dot{v} \end{matrix}\right\rangle = \langle \dot{v}'', v''\rangle_{L^2} - \langle \alpha\dot{v}'''' + v'''', \dot{v}\rangle_{L^2}$$
$$= \langle \dot{v}'', v''\rangle_{L^2} - \langle \alpha\dot{v}'' + v'', \dot{v}''\rangle_{L^2}$$
$$= -\alpha\langle \dot{v}'', \dot{v}''\rangle_{L^2} \leq 0.$$

Thus A is dissipative. Next we show that $(\lambda - A)$ is onto for $\lambda > 0$. We do this in two steps.

First of all, the range of $\lambda - A$ is closed; let $x_n = (v_n, \dot{v}_n) \in \mathfrak{D}(A)$ and suppose $y_n = (\lambda - A)(v_n, \dot{v}_n) \longrightarrow y \in \mathfrak{X}$. From the above dissipative estimate, and Schwarz's inequality, we get

$$\|\lambda x_n - Ax_n\|_x \geq \lambda \|x_n\|,$$

from which it follows that x_n converges to say x in $\mathfrak{X}$. Since $y_n \longrightarrow y$, Ax_n converges as well. Thus $\alpha\dot{v}_n + v_n$ converges in H^4, so $x \in \mathfrak{D}(A)$ and $Ax = y$.

Secondly, the range of $\lambda - A$ is dense. Suppose that there is a $y \in \mathfrak{X}$ such that $\langle(\lambda - A)x, y\rangle = 0$ for all $x \in \mathfrak{D}(A)$. Thus if $x = (v, \dot{v})$, $y = (w, \dot{w})$, then

$$\langle \lambda v'' - \dot{v}'', w''\rangle_{L^2} = 0 \quad \text{and} \quad \langle \lambda\dot{v} + \alpha\dot{v}'''' + v'''', \dot{w}\rangle_{L^2} = 0.$$

Set $\dot{v} = 0$; the first equation gives $w'' = 0$, so $w = 0$. The second equation with $\dot{v} = 0$ shows $\dot{w}$ satisfies $\dot{w}'''' = 0$ in the weak sense, so is smooth. Setting $v = 0$ and $\dot{v} = \dot{w}$ then shows $\dot{w} = 0$ since $\lambda > 0$ and $\alpha > 0$.

The result now follows from 2.16. ∎

Problem 3.6

(a) Show that the operator

$$f_0\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{v} \\ -v'''' + \Gamma v'' \end{pmatrix}$$

is Hamiltonian on $\mathfrak{X}$ and generates a group. Show that the energy is

$$H(v, \dot{v}) = \tfrac{1}{2}\|\dot{v}\|_{L^2}^2 - \frac{\Gamma}{2}\|v'\|_{L^2}^2 + \tfrac{1}{2}\|v''\|_{L^2}^2.$$

(b) Prove the inequality $\|v'\|_{L^2}^2 \leq \|v''\|_{L^2}^2/\pi^2$ using Fourier series. (A Wirtinger inequality.)

(c) Show that f_0 is stable if $0 \leq \Gamma \leq \pi^2$. (Buckling occurs for $\Gamma > \pi^2$.)

We have shown that e^{tA} is a contractive semigroup. Actually, the origin is globally attracting in the sense that in a suitable equivalent norm

$$\| e^{tA} \|_{\mathfrak{X}} \leq e^{\epsilon t} \qquad (\epsilon > 0).$$

This is, roughly speaking, because the spectrum of A (computed by separation of variables) is discrete and consists of eigenvalues

$$\lambda_j = -\frac{j^4\pi^4\alpha}{2}\left(1 \pm \sqrt{1 - \frac{4}{\alpha^2\pi^4 j^4}}\right) \quad (j = 1, 2, \ldots)$$

with $\operatorname{Re} \lambda_j \leq \max\{-\alpha\pi^4/2, -1/\alpha\}$. (See 2.17(vi).)

For additional information, see Chapter 7, Parks [1966], Holmes and Marsden [1978a], and Walker [1980].

Box 3.3 ***Linear Elastodynamics with Dissipative Mechanisms***

We give three examples of how semigroup theorems can be applied to modifications of the equations of linear elastodynamics. These are: (1) *viscoelasticity with dissipation of rate type;* (2) *thermal dissipation;* and (3) *viscoelasticity of memory (or Boltzmann) type.* References for these topics, where related topics may be found, are Weiss [1967], Dafermos [1976], and Navarro [1978].

1. *Dissipation of Rate Type* The form of the abstract equations is $\ddot{u} = Au + B\dot{u}$, and the relevant abstract theorem is as follows.

3.12 Proposition *Suppose A and B generate (quasi)-contractive semigroups on Hilbert space $\mathcal{H}$ and $\mathfrak{D}(B) \subset \mathfrak{D}(A)$. Suppose $A' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ generates a (quasi)-contractive semigroup on $\mathcal{Y} \times \mathcal{H}$ with domain $\mathfrak{D}(A) \times \mathcal{Y}$, where $\mathfrak{D}(A) \subset \mathcal{Y}$. Let $C = \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}$ with $\mathfrak{D}(C) = \mathfrak{D}(A) \times \mathfrak{D}(B)$. Then $\bar{C}$, the closure of C, generates a (quasi)-contractive semigroup on $\mathcal{Y} \times \mathcal{H}$.*

The proof will use the following result.

3.13 Lemma *Let A and B generate (quasi)-contractive semigroups on a Banach space $\mathfrak{X}$ and let $\mathfrak{D}(B) \subset \mathfrak{D}(A)$. Then there is a $\delta > 0$ such that $cA + B$ generates a quasi-contractive semigroup if $0 \leq c \leq \delta$.*

This result is due to Trotter [1959]. The interested reader should look up the original article for the proof, or else deduce it from 2.17(4).

Proof of 3.12 From 2.16, there is a $\beta > 0$ such that

$$\langle (u, \dot{u}), A'(u, \dot{u}) \rangle_{\mathcal{Y} \times \mathcal{X}} \leq \beta \| (u, \dot{u}) \|^2_{\mathcal{Y} \times \mathcal{X}}$$

and a $\gamma > 0$ be such that $\langle B\dot{u}, \dot{u} \rangle_{\mathcal{X}} \leq \gamma \| \dot{u} \|^2_{\mathcal{X}}$. Setting $B' = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$, we get

$$\langle (u, \dot{u}), (A' + B')(u, \dot{u}) \rangle_{\mathcal{Y} \times \mathcal{X}}$$
$$\leq \beta(\| u \|^2_{\mathcal{Y}} + \| \dot{u} \|^2_{\mathcal{X}}) + \gamma \| \dot{u} \|^2_{\mathcal{X}} \leq \rho \| (u, \dot{u}) \|^2_{\mathcal{Y} \times \mathcal{X}},$$

where $\rho = \beta + \gamma$. By 2.16 it is sufficient to show that $\lambda - C = \lambda - A' - B'$ has dense range for λ sufficiently large. Suppose $(v, \dot{v})$ is orthogonal to the range of $\lambda - C$. Then

$$\langle \lambda u - \dot{u}, v \rangle_{\mathcal{Y}} + \langle \lambda \dot{u} - Au - B\dot{u}, \dot{v} \rangle_{\mathcal{X}} = 0$$
$$\text{for} \quad u \in \mathfrak{D}(A) \quad \text{and} \quad \dot{u} \in \mathfrak{D}(A).$$

Setting $\dot{u} = \lambda u$, we get

$$\langle \lambda^2 u - Au - \lambda Bu, \dot{v} \rangle = 0, \qquad \text{that is,} \quad \left\langle \lambda u - \frac{1}{\lambda} Au - Bu, \dot{v} \right\rangle = 0.$$

If $\lambda > \delta^{-1}$, where δ is given in the lemma, we conclude that $\lambda - A/\lambda - B$ is onto, so $\dot{v} = 0$. From the original orthogonality condition, we get $v = 0$. ∎

If A is symmetric, then $\ddot{u} = Au$ is Hamiltonian. Furthermore, if $B \leq 0$, then the energy is decreasing:

$$\tfrac{1}{2} \frac{d}{dt}(\langle \dot{u}, \dot{u} \rangle - \langle u, Au \rangle) = \langle \dot{u}, B\dot{u} \rangle \leq 0.$$

This is the usual situation for rate-type dissipation.

3.14 Example If a^{abcd} is strongly elliptic, then

$$\rho \ddot{u}^a = (\mathrm{a}^{abcd} u_{c|d})_{|b} + \dot{u}^a{}_{|b|b},$$

that is,

$$\rho \ddot{u} = \operatorname{div}(\mathbf{a} \cdot \nabla u) + \nabla^2 \dot{\boldsymbol{u}}$$

with, say, displacement boundary conditions, generates a quasi-contractice semigroup on $\mathcal{X} = H^1 \times L^2$. If the elastic energy is positive-definite—that is, stability holds—then the semigroup is contractive. One can establish trend to equilibrium results either by spectral methods (see 2.17(vi)) or by Liapunov techniques (see Dafermos [1976] and Potier–Ferry [1978a], Ch. 11).

Problem 3.6 Show that 3.12 applies to the panel flutter equations (PF) in the preceding box, with A' Hamiltonian. For what parameter values is the energy decreasing?

2. *Dissipation of Thermal Type* Now the equations take the form

$$\ddot{u} = Au + B\theta, \qquad \dot{\theta} = C\theta + D\dot{u}. \tag{TE}$$

We make these assumptions:

(i) $A' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ generates a quasi-contractive semigroup on $\mathcal{Y} \times \mathcal{H}$.

(ii) C is a non-positive self-adjoint operator on a Hilbert space $\mathcal{H}_\theta$.

(iii) B is an operator from $\mathcal{H}_\theta$ to $\mathcal{H}$, is densely defined; $D = -B^*$, and is densely defined.

(iv) $\mathfrak{D}(A) \subset \mathfrak{D}(D) \subset \mathcal{Y}$.

(v) $\mathfrak{D}(C) \subset \mathfrak{D}(B)$.

(vi) $B(1 - C)^{-1}D$, a non-positive symmetric operator, has self-adjoint closure (i.e., is essentially self-adjoint). (In Example 3.16 below $B(1 - C)^{-1}D$ is bounded.)

Let $G = \begin{pmatrix} 0 & I & 0 \\ A & 0 & B \\ 0 & D & C \end{pmatrix}$ with domain $\mathfrak{D}(A) \times \mathfrak{D}(D) \times D(C) \subset \mathcal{Y} \times \mathcal{H} \times \mathcal{H}_\theta = \mathfrak{X}$, so that $\dfrac{d}{dt}\begin{pmatrix} u \\ \dot{u} \\ \theta \end{pmatrix} = G\begin{pmatrix} u \\ \dot{u} \\ \theta \end{pmatrix}$ represents the thermoelastic system (TE).

3.15 Proposition *Under assumptions* (i)–(vi), $\bar{G}$ *generates a quasi-contractive semigroup on* $\mathfrak{X}$. *If* A' *generates a contractive semigroup, so does* $\bar{G}$.

Proof In the inner product on $\mathcal{Y} \times \mathcal{H} \times \mathcal{H}_\theta$, we have

$$\begin{aligned}\langle (u, \dot{u}, \theta), G(u, \dot{u}, \theta)\rangle &= \langle (u, \dot{u}, \theta), (\dot{u} + Au + B\theta, C\theta + D\dot{u})\rangle \\ &= \langle u, \dot{u}\rangle_{\mathcal{Y}} + \langle \dot{u}, Au + B\theta\rangle_{\mathcal{H}} + \langle \theta, C\theta + D\dot{u}\rangle_{\mathcal{H}_\theta}\end{aligned}$$

$$\begin{aligned}&\leq \beta\|(u, \dot{u})\|^2_{\mathcal{Y}\times\mathcal{H}} + \langle \dot{u}, B\theta\rangle + \langle \theta, D\dot{u}\rangle + \langle \theta, C\theta\rangle\\&= \beta\|(u, \dot{u})\|^2_{\mathcal{Y}\times\mathcal{H}} + \langle \theta, C\theta\rangle\\&\leq \beta\|(u, \dot{u})\|^2_{\mathcal{Y}\times\mathcal{H}} \leq \beta\|(u, \dot{u}, \theta)\|^2_{\mathcal{X}},\end{aligned}$$

so $(G - \beta)$ is dissipative. By 2.16, it remains to show that for λ sufficiently large, $(\lambda - G)$ has dense range. Let $(v, \dot{v}, g)$ be orthogonal to the range:

$$\langle \lambda u - \dot{u}, v\rangle + \langle \lambda\dot{u} - Au - B\theta, \dot{v}\rangle + \langle \lambda\theta - C\theta - D\dot{u}, g\rangle = 0.$$

For $u \in \mathfrak{D}(A)$, let $\dot{u} = \lambda u$ and $\theta = \lambda(\lambda - C)^{-1}Du$. Then

$$\langle \lambda^2 u - Au - \lambda B(\lambda - C)^{-1}Du, \dot{v}\rangle = 0.$$

Using 3.13 and the same argument as in the preceding proposition, we see that if λ is sufficiently large, $\lambda^2 - A - \lambda B(\lambda - C)^{-1}D$ has dense range. Thus $\dot{v} = 0$. Taking $\dot{u}$ and $\theta = 0$, one sees that $v = 0$ and taking $u = 0 = \dot{u}$, one gets $\theta = 0$. ∎

3.16 Example If $\mathbf{a}$ is strongly elliptic, then the system

$$\begin{cases}\rho\ddot{u} = \operatorname{div}(\mathbf{a}\cdot\nabla u) + m\nabla\theta,\\ c\dot{\theta} = k\nabla^2\theta + \dfrac{m}{\rho}\nabla\cdot\dot{u} \quad (u, \theta = 0 \quad \text{on} \quad \partial\Omega),\end{cases}$$

generates a quasi-contractive semigroup on $H^1 \times L^2 \times L^2$ (with the L^2 spaces appropriately weighted), where $c, k > 0, m > 0$.[12]

3. *Viscoelasticity of Memory (or Boltzmann) Type* The equations now have the form

$$\ddot{u} = Au + Bw, \qquad \dot{w} = Cw. \tag{BE}$$

We make these assumptions:

(i) $A' = \begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$ generates a quasi-contractive semigroup on $\mathcal{Y} \times \mathcal{H}$.

(ii) C generates a contractive semigroup on $\tilde{\mathcal{H}}$.

(iii) There is an injection $i: \mathcal{H} \longrightarrow \tilde{\mathcal{H}}$ (corresponding to constant histories) such that $C \circ i = 0$.

(iv) B is a densely defined operator of $\tilde{\mathcal{H}}$ to $\mathcal{H}$ such that $i \circ B$ is symemtric and non-negative, $\mathfrak{D}(B) \subset \mathfrak{D}(C)$, $\mathfrak{D}(B)$ is a core for C, and B is one-to-one.

[12]In Theorem 3.1 we saw that "well-posedness" of the elastic part implies $\mathbf{a}$ is strongly elliptic. The Clausius–Duhem inequality implies $k \geq 0$. Well-posedness of the heat part then implies $c > 0$, and then of the whole implies $m \geq 0$.

(v) Let $\mathfrak{A}(w_1, w_2) = \langle i \cdot Bw_1, w_2 \rangle_{\tilde{\mathcal{H}}}$ for $w_1, w_2 \in \mathfrak{D}(B)$. Suppose that on $\mathfrak{X} = \mathcal{Y} \times \mathcal{H} \times \tilde{\mathcal{H}}$,

$$\|(u, \dot{u}, w)\|_{\mathfrak{X}}^2 = \|(u, \dot{u})\|_{\mathcal{Y}\times\mathcal{H}}^2 + \mathfrak{A}(iu - w, iu - w) - \mathfrak{A}(iu, iu)$$

is an inner product equivalent to the original one. Let

$$G = \begin{pmatrix} 0 & I & 0 \\ A & 0 & B \\ 0 & 0 & C \end{pmatrix},$$

which is the operator on $\mathfrak{X}$ corresponding to the equations (BE), with domain $\mathfrak{D}(A) \times \mathcal{Y} \times \mathfrak{D}(B)$.

(vi) $\langle iBw, Cw \rangle \geq 0$ for all $w \in \mathfrak{D}(B)$.

3.17 Proposition *Under assumptions* (i)–(vi), $\bar{G}$ *generates a quasi-contractive semigroup on* $\mathfrak{X}$. *Moreover,* $\exp(t\bar{G})$ *is contractive if* A' *generates a contractive semigroup.*

Proof Using the $\mathfrak{X}$ inner product of (*v*),

$$\begin{aligned}
&\langle (u, \dot{u}, w), (\dot{u}, Au + Bw, Cw) \rangle \\
&\quad = \langle u, \dot{u} \rangle + \langle \dot{u}, Au \rangle + \langle \dot{u}, Bw \rangle + \langle iB(iu - w), i\dot{u} - Cw \rangle \\
&\qquad\qquad - \langle iBiu, iu \rangle \\
&\quad = \langle u, \dot{u} \rangle + \langle \dot{u}, Au \rangle + \langle iB(iu - w), -Cw \rangle \\
&\quad = \langle u, \dot{u} \rangle + \langle \dot{u}, Au \rangle - \langle iB(iu - w), C(iu - w) \rangle \\
&\quad \leq \langle u, \dot{u} \rangle + \langle \dot{u}, , Au \rangle \leq \beta \| u, \dot{u} \|_{\mathcal{Y}\times\mathcal{H}}^2 \leq \beta \|(u, \dot{u}, w)\|_{\mathfrak{X}}^2
\end{aligned}$$

for a constant β.

It remains to show that $(\lambda - G)$ has dense range for λ sufficiently large. If $(v, \dot{v}, h)$ is orthogonal to the range in the original inner product,

$$\langle \lambda u - \dot{u}, v \rangle_{\mathcal{Y}} + \langle \lambda u - Au - Bw, \dot{v})_{\mathcal{H}} + \langle \lambda w - Cw, h \rangle_{\tilde{\mathcal{H}}} = 0,$$

for all $u \in \mathfrak{D}(A)$, $\dot{u} \in \mathcal{Y}$ and $w \in \mathfrak{D}(B)$. Taking $\dot{u} = \lambda u$ and $w = 0$, and using the fact that $(\lambda - A)$ is surjective, we get $\dot{v} = 0$. Choosing $u, \dot{u} = 0$, and using the fact that $\mathfrak{D}(B)$ is a core for C, we find that $h = 0$, and finally choosing $\dot{u} = 0$ and $w = 0$ gives $v = 0$. ▮

3.18 Example (See Coleman and Mizel [1966] and Navarro [1978].) Suppose that the body occupies a bounded region $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, and that the reference configuration is a natural state in which stress is zero and base temperature θ_0 is a strictly positive constant. Let $x \in \Omega$ be the position of a material point, $\mathbf{u}(x, t)$ the displacement, and $\theta(x, t)$ the temperature difference from θ_0.

We assume that the Cauchy stress $\boldsymbol{\sigma}$ and specific entropy difference

η are given by functionals depending upon both displacement and temperature difference history in the following form:

$$\boldsymbol{\sigma}(x, t) = \mathbf{g}(x, 0)\cdot\nabla\boldsymbol{u}(x, t) - \theta(x, t)\boldsymbol{l}(x, 0) + \int_0^\infty \{\mathbf{g}'(x, s)\cdot\nabla\boldsymbol{u}(x, t - s) - \boldsymbol{l}'(x, s)\theta(x, t - s)\}\, ds$$

$$\rho(x)\eta(x, t) = \boldsymbol{l}(x, 0)\cdot\nabla\boldsymbol{u}(x, t) + \rho(x)c(x, 0)\theta(x, t)/\theta_0 + \int_0^\infty \{\boldsymbol{l}'(x, s)\cdot\nabla\boldsymbol{u}(x, t - s) + \rho(x)c'(x, s)\theta(x, t - s)/\theta_0\}\, ds,$$

where $\rho(x)$ is the mass density in the natural state and $\partial a/\partial s$ is denoted by a'. The material functions $\mathbf{g}(x, s)$, $\boldsymbol{l}(x, s)$, and $c(x, s)$ $(s \geq 0)$ are the relaxation tensors of fourth, second, and zero order, respectively ($\mathbf{g}$ is assumed to have the symmetry $\mathrm{g}^{abcd} = \mathrm{g}^{cdab}$). We call $\mathbf{g}(x, 0)$, $\boldsymbol{l}(x, 0)$, and $c(x, 0)$ the *instantaneous elastic modulus*, *instantaneous stress-temperature tensor*, and *instantaneous specific heat*, respectively. We also assume Fourier's law for the heat flux vector $\boldsymbol{q}(x, t)$;

$$\boldsymbol{q}(x, t) = -\boldsymbol{\kappa}(x)\cdot\nabla\theta(x, t), \qquad \text{that is,} \quad q^a = -k^{ab}\theta_{/b},$$

where $\boldsymbol{\kappa}(x)$ is the *thermal conductivity* in the reference configuration. The local equation for balance of momentum is then

$$\rho(x)\dot{\boldsymbol{v}}(x, t) = \operatorname{div}\boldsymbol{\sigma}(x, t), \quad \text{where} \quad \boldsymbol{v} = \dot{\boldsymbol{u}},$$

while the linearized energy balance equation becomes

$$\theta_0\rho(x)\dot{\eta}(x, t) + \operatorname{div}\boldsymbol{q}(x, t) = 0.$$

Substitution of $\boldsymbol{\sigma}$, η, and $\boldsymbol{q}$ into these equations then yields the system of coupled equations for the linear theory of thermoeleastic materials with memory (with no body forces or heat supply):

$$\left.\begin{aligned}
\dot{\boldsymbol{v}}(x, t) &= \frac{1}{\rho(x)}\operatorname{div}(\mathbf{g}(x, 0)\cdot\nabla\boldsymbol{u}(x, t)) - \theta(x, t)\boldsymbol{l}(x, 0) \\
&\quad + \int_0^\infty \{\mathbf{g}'(x, s)\cdot\nabla\boldsymbol{u}(x, t - s) - \boldsymbol{l}'(x, s)\theta(x, t - s)\}\, ds) \\
\dot{\theta}(x, t) &= \theta_0(\operatorname{div}\{\boldsymbol{\kappa}(x)\cdot\nabla\theta(x, t)\}/\theta_0 - \boldsymbol{l}(x, 0)\cdot\nabla\boldsymbol{v}(x, t) \\
&\quad + \int_0^\infty \{\boldsymbol{l}'(x, s)\cdot\nabla\dot{\boldsymbol{u}}(x, t - s) \\
&\qquad + (\rho(x)\theta_0)c'(x, s)\dot{\theta}(x, t - s)\}\, ds)/(\rho(x)c(x, 0))
\end{aligned}\right\} \text{(TEM)}$$

The boundary conditions are assumed to be

$$\boldsymbol{u}(x, t) = \mathbf{0}, \qquad \theta(x, t) = 0 \quad \text{on} \quad \partial\Omega \times [0, \infty),$$

while the prescribed initial histories for the displacement and temperature difference are given by

$$\boldsymbol{u}(x, -s) = \boldsymbol{w}^0(x, -s), \qquad \theta(x, -s) = \alpha^0(x, -s) \quad (0 \leq s < \infty,\ x \in \bar{\Omega}.)$$

We shall now state the main hypothesis on the material properties.

First, we assume that $\boldsymbol{\kappa}(x)$, and $\mathbf{g}(x, s)$, $\boldsymbol{l}(x, s)$, and $c(x, s)$ for fixed $s \geq 0$, are Lebesgue measurable and essentially bounded on Ω.

In addition, we also postulate the following conditions:

(i) $0 < \rho_0 \leq \operatorname{ess.inf}_{x \in \Omega} \rho(x)$.

(ii) $0 < c_0 < \operatorname{ess.inf}_{x \in \Omega} c(x, 0)$.

(iii) $\mathbf{g}$ is stable:
$$\int_\Omega \nabla \boldsymbol{y}(x) \cdot \mathbf{g}(x, \infty) \cdot \nabla \boldsymbol{y}(x)\, dV \geq g \int_\Omega |\nabla \boldsymbol{y}|^2\, dV,$$
where g is a positive constant and $\mathbf{g}(x, \infty) = \lim_{s \to \infty} \mathbf{g}(x, s)$ is called the *equilibrium elastic modulus*.

(iv) $\boldsymbol{\kappa}$ is positive-definite:
$$\int_\Omega \nabla \beta(x) \cdot \boldsymbol{\kappa}(x) \cdot \nabla \beta(x)\, dV \geq k \int_\Omega |\nabla \beta|^2,$$
where k is a positive constant.

(v) $\mathbf{g}''$ is stable:
$$\int_\Omega \nabla \boldsymbol{y}(x) \cdot \mathbf{g}''(x, s) \cdot \nabla \boldsymbol{y}(x)\, dV \geq g_2(s) \int_\Omega |\nabla \boldsymbol{y}|^2\, dV,$$
where $0 \leq s < \infty$ and $g_2(s) > 0$.

(vi) $$-\int_\Omega c''(x, s) \beta^2(x)\, dV \geq c_2(s) \int_\Omega |\beta(x)|^2\, dV,$$
where $0 \leq s < \infty$ and $c_2(s) > 0$.

(vii) For all $s \geq 0$,
$$\|\boldsymbol{l}'(s)\| = \operatorname{ess.sup}_{x \in \Omega} \|\boldsymbol{l}'(x, s)\| \in L^1(0, \infty)$$
$$\|\boldsymbol{l}''(s)\| \leq \left(\frac{\rho_0}{\theta_0}\right)^{1/2} [c_2(s)]^{1/2} [g_2(s)]^{1/2}.$$

Let $\boldsymbol{w}(x, t, s)$ and $\alpha(x, t, s)$ denote the displacement and temperature difference histories; that is, $\boldsymbol{w}(x, t, s) = \boldsymbol{u}(x, t - s)$ and $\alpha(x, t, s) = \theta(x, t - s) (0 \leq s \leq \infty)$. Denote by $\mathfrak{X}$ the Hilbert space obtained as the completion of the space
$$\begin{aligned}(\boldsymbol{u}, \boldsymbol{v}, \theta, \boldsymbol{w}, \alpha) \in C_0^\infty(\Omega; \mathbb{R}^3) &\times C_0^\infty(\Omega; \mathbb{R}^3) \times C_0^\infty(\Omega) \\ &\times C^\infty([0, \infty); H_\partial^1(\Omega; \mathbb{R}^3)) \\ &\times C^\infty([0, \infty); H_\partial^1(\Omega))\end{aligned}$$
under the norm induced by the inner product
$$\begin{aligned}&\langle (\boldsymbol{u}, \boldsymbol{v}, \theta, \boldsymbol{w}, \alpha), (\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\theta}, \bar{\boldsymbol{w}}, \bar{\alpha}) \rangle \\ &= \int_\Omega \left\{ \nabla \boldsymbol{u} \cdot \mathbf{g}(\infty) \cdot \nabla \bar{\boldsymbol{u}} + \rho \boldsymbol{v} \cdot \bar{\boldsymbol{v}} + \frac{\rho c(0)}{\theta_0} \theta \bar{\theta} \right\} dV \\ &\quad - \iint_{\Omega}\int_0^\infty \Big\{ [\nabla \boldsymbol{u} - \nabla \boldsymbol{w}(s)] \cdot \mathbf{g}'(s) \cdot [\nabla \bar{\boldsymbol{u}} - \nabla \bar{\boldsymbol{w}}(s)] + \bar{\alpha}(s) \boldsymbol{l}'(s) \cdot [\nabla \boldsymbol{u} \\ &\quad - \nabla \boldsymbol{w}(s)] + \alpha(s) \boldsymbol{l}'(s) \cdot [\nabla \bar{\boldsymbol{u}} - \nabla \bar{\boldsymbol{w}}(s)] - \frac{\rho}{\theta_0} c'(s) \alpha(s) \bar{\alpha}(s) \Big\}\, ds\, dV.\end{aligned}$$

Define the operator

$$G\begin{pmatrix} u \\ v \\ \theta \\ \\ w \\ \alpha \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{\rho}\operatorname{div}\left\{\mathbf{g}(0)\cdot\nabla u - \theta \boldsymbol{l}(0) + \int_0^\infty [\mathbf{g}'(s)\cdot\nabla w(s) - \boldsymbol{l}'(s)\alpha(s)]\,ds\right\} \\ \frac{\theta_0}{\rho c(0)}\left\{-\boldsymbol{l}'(0)\cdot\nabla u - \boldsymbol{l}(0)\cdot\nabla v + \frac{\operatorname{div}(\boldsymbol{\kappa}\cdot\nabla\theta)}{\theta_0} - \frac{\rho c'(0)}{\theta_0}\right. \\ \left. - \int_0^\infty [\boldsymbol{l}''(s)\cdot\nabla w(s) + (\rho/\theta_0)c''(s)\alpha(s)]\,ds\right\} \\ -w'(s) \\ -\alpha'(s) \end{pmatrix}$$

with domain $\mathfrak{D}(G)$ given by those $(u, v, \theta, w, \alpha)$ such that the right-hand side of the above equation lies in $\mathfrak{X}$. Thus, we obtain the abstract evolutionary equation

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \\ \theta(t) \\ w(t) \\ \alpha(t) \end{pmatrix} = G\begin{pmatrix} u(t) \\ v(t) \\ \theta(t) \\ w(t) \\ \alpha(t) \end{pmatrix}, \quad \begin{pmatrix} u(0) \\ v(0) \\ \theta(0) \\ w(0) \\ \alpha(0) \end{pmatrix} = \begin{pmatrix} w^0(0) \\ v^0 \\ \alpha^0(0) \\ w^0 \\ \alpha^0 \end{pmatrix}. \qquad \text{(TEM)}'$$

The existence and uniqueness result for equation (TEM) is as follows:

The operator G is the infinitesimal generator of a C^0 contractive semigroup on $\mathfrak{X}$.

In fact, 3.17 shows that $\bar{G}$ generates a contractive semigroup. A slightly more careful analysis shows $G = \bar{G}$; that is, G is already closed. (See Navarro [1978].)

Box 3.4 ***Symmetric Hyperbolic Systems***

Here we study the symmetric hyperbolic systems of Friedrichs [1954], [1958]. This type of system occurs in many problems of mathematical physics—for example, Maxwell's equations, as is shown in Courant and Hilbert [1962]. As we shall see below, this includes the equations of linear elasticity. As Friedrichs has shown, many nonlinear equations are also covered by systems of this type. For elasticity, see Section 6.5 below, John [1977], Hughes and Marsden [1978], and, for general relativity, see Fischer and Marsden [1972b]. For time-dependent and non-linear cases, see Section 6.5 below, Dunford and Schwartz [1963], and Kato [1975a]. We consider the linear equations in all of space ($\Omega = \mathbb{R}^n$) for simplicity. For general Ω, see Rauch and Massey [1974].

Let $U(x, t) \in \mathbb{R}^N$ for $x \in \mathbb{R}^m$, $t \geq 0$ and consider the following evolution equation

$$\boldsymbol{a}_0(x)\frac{\partial U}{\partial t} = \sum_{j=1}^{m} \boldsymbol{a}_j(x)\frac{\partial U}{\partial x^j} + \boldsymbol{b}(x)U + \boldsymbol{f}(x), \tag{SH}$$

where $\boldsymbol{a}_0$, $\boldsymbol{a}_j$ and $\boldsymbol{b}$ are $N \times N$ matrix functions. We assume $\boldsymbol{a}_0$ and $\boldsymbol{a}_j$ are *symmetric* and $\boldsymbol{a}_0$ is uniformly positive-definite; that is, $\boldsymbol{a}_0(x) \geq \epsilon$ for some $\epsilon > 0$. (This is a matrix inequality; it means $\langle \boldsymbol{a}_0(x)\xi, \xi\rangle \geq \epsilon \|\xi\|^2$ for all $\xi \in \mathbb{R}^N$.) To simplify what follows we shall take $\boldsymbol{a}_0 =$ Identity. The general case is dealt with in the same way by weighting the L^2-norm by $\boldsymbol{a}_0$. We can also assume $\boldsymbol{f} = \boldsymbol{0}$ by the remarks 2.17(v) on inhomogeneous equations.

We make the following technical assumptions. The functions $\boldsymbol{a}_j$ and $\boldsymbol{b}$ are to be of class C^1, uniformly bounded and with uniformly bounded first derivatives.

3.19 Theorem *Let the assumptions just stated hold and let $A_{\min}$: $C_0^\infty \longrightarrow L^2(\mathbb{R}^m, \mathbb{R}^N)$ (C_0^∞ denotes the C^∞ functions U: $\mathbb{R}^m \longrightarrow \mathbb{R}^N$ with compact support) be defined by*

$$A_{\min}U = \sum_{j=1}^{m} \boldsymbol{a}_j(x)\frac{\partial U}{\partial x^j} + \boldsymbol{b}(x)U(x).$$

Let A be a closure of $A_{\min}$. Then A generates a quasi-contractive one-parameter group in $L^2(\mathbb{R}^m, \mathbb{R}^N)$.

Proof Define $B_{\min}$ on C_0^∞ by

$$B_{\min}U = -\sum \frac{\partial}{\partial x^j}(\boldsymbol{a}_j(x)U) + \boldsymbol{b}(x)U.$$

Integration by parts shows that $B_{\min}$ is a restriction of the adjoint of $A_{\min}$ on C_0^∞: $A_{\min}^* \supset B_{\min}$. Let $A_{\max} = B_{\min}^*$. (In distribution language, $A_{\max}$ is just $A_{\min}$ defined on all U for which $A_{\min}\, U$ lies in L^2 with derivatives in the sense of distributions.)

We shall need the following:

3.20 Lemma *$A_{\max}$ is the closure of $A_{\min}$.*

Proof We shall sketch out the main steps. The method is often called that of the "Friedrichs Mollifier."

Let $U \in \mathfrak{D}(A_{\max})$. We have to show there is $U_n \in C_0^\infty$ such that $U_n \longrightarrow U$ and $A_{\min} U_n \longrightarrow A_{\max} U \in L^2$. Let $\rho: \mathbb{R}^m \longrightarrow \mathbb{R}$ be C^∞ with support in the unit ball, $\rho \geq 0$ and $\int \rho \, dx = 1$. Set

$$\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho\left(\frac{x}{\epsilon}\right) \quad \text{for} \quad \epsilon > 0.$$

Let $U_\epsilon = \rho_\epsilon * U$ (componentwise convolution). We assert that $U_\epsilon \longrightarrow U$ as $\epsilon \longrightarrow 0$ (in L^2). Indeed, $\| U_\epsilon \| \leq \| U \|$, so it is not enough to check this for $U \in C_0^\infty$. Then it is a standard (and easy) argument; one actually obtains uniform convergence.

Now each U_ϵ is C^∞. Let L denote the differential operator

$$L = \sum a_j(x) \frac{\partial}{\partial x_j} + b(x).$$

Then one computes that

$$\begin{aligned} L(U_\epsilon) = \int & \left\{ -\sum_j \frac{\partial}{\partial y^j} [a_j(y) \rho_\epsilon(x-y)] + b(y) \rho_\epsilon(x-y) \right\} U(y) \, dy \\ & + \int \left\{ \sum \frac{\partial}{\partial y^j} ([a_j(y) - a_j(x)] \rho_\epsilon(x-y)) \right. \\ & \left. - [b(y) - b(x)] \rho_\epsilon(x-y) \right\} U(y) \, dy. \end{aligned}$$

The first term is just $\rho_\epsilon * (A_{\max} U)$ and thus we have proved that $L(\rho_\epsilon * U) - \rho_\epsilon * A_{\max} U \longrightarrow 0$ as $\epsilon \longrightarrow 0$. It follows that $C_0^\infty \cap L^2$ is a core of $A_{\max}$. That is, $A_{\max}$ restricted to $C_0^\infty \cap L^2 \cap \mathfrak{D}(A)_{\max}$ has closure $A_{\max}$.

Let $\omega \in C_0^\infty(\mathbb{R}^m)$, ω with support in a ball of radius 2, and $\omega \equiv 1$ on a ball of radius 1, and let $\omega_n(x) = \omega(x/n)$. Then $\omega_n U_\epsilon \in C_0^\infty$ and

$$L(\omega_n U_\epsilon) = \omega_n L U_\epsilon + \sum a_j(x) \frac{\partial \omega_n}{\partial x^j} U_\epsilon.$$

As $n \longrightarrow \infty$, this converges to $L(U_\epsilon)$, which proves the lemma. ▼

Now we shall complete the proof of 3.19. Let $A = A_{\max}$. For $U \in C_0^\infty$,

$$\begin{aligned} \langle AU, U \rangle &= \int_\Omega \sum_j \left\langle a_j \frac{\partial U}{\partial x^j}, U \right\rangle + \langle bU, U \rangle \, dx \\ &= \int_\Omega \left\{ \sum \tfrac{1}{2} \frac{\partial}{\partial x_j} \langle a_j U, U \rangle - \tfrac{1}{2} \left\langle \frac{\partial a_j}{\partial x^j} U, U \right\rangle + \langle bU, U \rangle \right\} dx \\ & \qquad \qquad \qquad \qquad \text{(by symmetry of } a_j) \\ &= \int_\Omega -\tfrac{1}{2} \left\langle \frac{\partial a_j}{\partial x^j} U, U \right\rangle + \langle bU, U \rangle \, dx. \end{aligned}$$

Thus,

$$\langle AU, U \rangle \leq \beta_1 \int_\Omega \langle U, U \rangle \, dx, \quad \text{where} \quad \beta_1 = \sup\left(\tfrac{1}{2} \left| \frac{\partial a_j}{\partial x^j} \right| + |b| \right).$$

By the lemma, this same inequality holds for all $U \in \mathfrak{D}(A)$. Thus, $\langle(\lambda - A)U, U\rangle \geq (\lambda - \beta_1)\langle U, U\rangle$ from which it follows that $\|(\lambda - A)U\| \geq (\lambda - \beta_1)\|U\|$. Thus $(\lambda - A)$ has closed range if $\lambda \geq \beta_1$, and is one-to-one. To show the range is the whole space we must show that $(\lambda - A)^*\boldsymbol{\omega} = \mathbf{0}$ implies $\boldsymbol{\omega} = \mathbf{0}$. ($(\lambda - A)^*\boldsymbol{\omega} = \mathbf{0}$ means $\boldsymbol{\omega}$ is orthogonal to the range.) But $B =$ closure of $B_{\min}$ (defined in the proof of the lemma) equals A^*. Thus $(\lambda - B)\boldsymbol{\omega} = \mathbf{0}$. As above, we have $\|(\lambda - B)\boldsymbol{\omega}\| \geq (\lambda - \beta_2)\|\boldsymbol{\omega}\|$ so $(\lambda - B)\boldsymbol{\omega} = \mathbf{0}$ implies $\boldsymbol{\omega} = \mathbf{0}$ for $\lambda > \beta_2$. For $\beta = \sup(\beta_1, \beta_2)$ and $\lambda > \beta$, we have

$$\|(\lambda - A)^{-1}\| < 1/(\lambda - \beta).$$

Since conditions on A are unaffected by replacing A with $-A$, we see that

$$\|(A + \lambda)^{-1}\| \leq 1/(|\lambda| - \beta) \quad (|\lambda| > \beta).$$

Hence A generates a quasi-contractive group. ∎

Provided the coefficients are smooth enough, one can also show that A generates a semigroup on H^s as well as on $H^0 = L^2$. (This follows by using Gronwall's inequality to show that the H^s norm remains bounded under the flow on L^2.)

3.21 Example (The Wave Equation) Consider the system:

$$\left.\begin{aligned}
\frac{\partial U^0}{\partial t} &= U^{n+1}, \\
\frac{\partial U^1}{\partial t} &= \frac{\partial U^{n+1}}{\partial x^1}, \\
&\ \vdots \\
\frac{\partial U^n}{\partial t} &= \frac{\partial U^{n+1}}{\partial x^n}, \\
\frac{\partial U^{n+1}}{\partial t} &= \frac{\partial U^1}{\partial x^1} + \cdots + \frac{\partial U^n}{\partial x^n}.
\end{aligned}\right\} \tag{W}$$

Here

$$U = (U^0, U^1, \ldots, U^{n+1}), \qquad \boldsymbol{a}_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & 1 & \cdots & 0 \end{pmatrix}, \quad \text{and so on}$$

so our system is symmetric. By Theorem 3.19, it generates a group. Let $u, \dot{u} \in H^1 \times L^2$ and consider the initial data

$$U^0 = u, \qquad U^1 = \frac{\partial u}{\partial x^1}, \ldots, U^n = \frac{\partial u}{\partial x^n}, \qquad U^{n+1} = \dot{u}.$$

Then the equations for U reduce exactly to the wave equation for u, so $\partial^2 u/\partial t^2 = \Delta u$ generates a group on $H^1 \times L^2$, reproducing the result we found by second-order methods in 3.2.

3.22 Example (Linear Hyperelasticity) We will use symmetric hyperbolic methods to reproduce the implication (ii) $\Longrightarrow$ (i) in Theorem 3.1 for $\Omega = \mathbb{R}^n$. Consider, then, the system $\rho\ddot{u} = \operatorname{div}(\mathbf{a}\cdot\nabla u)$, where $\mathbf{a}$ is symmetric (hyperelastic) and strongly elliptic.

This is easiest to carry out in the case of stable classical elasticity, so we consider it first. Thus, we begin by dealing with

$$\rho\frac{\partial^2 u^a}{\partial t^2} = (\mathrm{c}^{abcd}u_{c|d})_{|b} = \mathrm{c}^{abcd}u_{c|d|b} + \mathrm{c}^{abcd}{}_{|b}u_{c|d}.$$

Following John [1977], let U be defined by $U_{ab} = u_{a|b}$ and $U_{a0} = \dot{u}_a$. The system under consideration is thus (in Euclidean coordinates)

$$\begin{cases} \rho\dfrac{\partial}{\partial t}U_{i0} = \mathrm{c}_{ikjm}\dfrac{\partial}{\partial x_k}U_{jm} + c_{ikjm,k}U_{jm}, \\ c_{ikjm}\dfrac{\partial}{\partial t}U_{jm} = \mathrm{c}_{ikjm}\dfrac{\partial}{\partial x_m}U_{j0}. \end{cases}$$

This has the form (SH) and a_0 is positive-definite if $\mathbf{c}$ is uniformly pointwise stable. Thus Theorem 3.19 applies.

In the general strongly elliptic case, one can replace a_{ikjm} by

$$\mathfrak{A}_{ikjm} = \mathrm{a}_{ikjm} + \gamma(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk})$$

for a suitable constant γ, so $\mathfrak{A}_{ikjm}$ becomes positive-definite: $\mathfrak{A}_{ikjm}\zeta^{ik}\zeta^{jm} \geq \epsilon\,|\boldsymbol{\zeta}|^2$ for all 3×3 matrices $\boldsymbol{\zeta}$ (not necessarily symmetric). This may be proved by the arguments in 3.10, Chapter 4.*

Problem 3.7 Carry out the details of this, and for isotropic classical elasticity, show that one can choose $\gamma = c_2^2$, where c_2 is the wave velocity defined in Problem 3.4, Chapter 4.

A calculation shows that the added term, miraculously, does not affect the equations of motion.[13] Therefore, the preceding reduction applies and we get a symmetric hyperbolic system.

[13]As was pointed out by J. Ball, this is because it is the elasticity tensor of a *null Lagrangian* —that is, a Lagrangian whose Euler–Lagrange operator vanishes identically. Apparently this trick of adding on γ was already known to Hadamard (cf. Hill [1957]). See Ball [1977a] for a general discussion of the role of null-Lagrangians in elasticity. They will appear again in our discussion of the energy criterion in Section 6.6.

*As pointed out by S. Spector, this trick does not universally work. See also *Arch. Rat. Mech. An.* 98 (1987):1-30.

Box 3.5 ***Summary of Important Formulas for Section 6.3***

Linear Elastodynamics

$$
\text{(LE)} \qquad \begin{cases} \rho\ddot{\boldsymbol{u}} = \operatorname{div}(\mathbf{a}\cdot\nabla \boldsymbol{u}) \\ \boldsymbol{u} = \mathbf{0} \text{ on } \partial_d, \\ \quad \mathbf{a}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial_\tau \end{cases} \qquad \begin{aligned} \rho\ddot{u}^a &= (\mathrm{a}^{abcd}u_{c|d})_{|b} \\ u^a &= 0 \text{ on } \partial_d, \\ \mathrm{a}^{abcd}u_{c|d}n_b &= 0 \text{ on } \partial_\tau \end{aligned}
$$

Initial Value Problem (Hyperelasticity)

The equations (LE) are well posed (give a quasi-contractive semigroup in $H^1 \times L^2$) if and only if $\mathbf{a}$ is strongly elliptic.

Cauchy Elasticity

If (LE) generates a contraction semigroup in $H^1 \times L^2$, then $\mathbf{a}$ is hyperelastic.

Energy Criterion

If $\mathbf{a}$ is hyperelastic and stable:

$$
\begin{aligned}
\int_\Omega \nabla\boldsymbol{u}\cdot\mathbf{a}\cdot\nabla\boldsymbol{u}\, dv &= \int_\Omega u_{a|b}\mathrm{a}^{abcd}u_{c|d}\, dv \\
&\geq c\int_\Omega u_{a|b}u^{a|b}\, dv \quad (c = \text{const.} > 0),
\end{aligned}
$$

then $\mathbf{a}$ is strongly elliptic and (LE) generates stable dynamics (a contractive semigroup in a suitable norm) in $H^1 \times L^2$.

Korn's Lemma (Classical Elasticity)

If $\mathbf{c}$ is uniformly pointwise stable, then $\mathbf{c}$ is stable.

Hamiltonian Existence Theorem

If $(\mathfrak{X}, \omega)$ is a Banach space with symplectic form ω and A is ω-skew-adjoint and if the energy $H(x) = \omega(Ax, x)$ satisfies the stability condition $H(x) \geq c\|x\|^2$, then in the completion of $\mathfrak{D}(A)$ in the energy norm, A generates a one-parameter group.

Panel Flutter Equations

The equations

$$
\begin{gathered}
\ddot{v} + \alpha\dot{v}'''' + v'''' + \Gamma'' + \rho v' + \sqrt{\rho}\,\delta\dot{v} = 0, \\
v = v'' + \alpha\dot{v}'' = 0 \quad \text{at} \quad x = 0, 1.
\end{gathered}
$$

generate a semigroup in $\mathfrak{X} = H_0^2 \times L^2$.

Dissipative Mechanisms

(1) Rate type:

$$
\ddot{u} = Au + B\dot{u} \quad \text{or} \quad \rho\ddot{\boldsymbol{u}} = \operatorname{div}(\mathbf{a}\cdot\nabla\boldsymbol{u}) + \nabla^2\dot{\boldsymbol{u}}
$$

(2) Thermal type:

$$
\begin{aligned}
\ddot{u} &= Au + B\theta \quad \text{or} \quad \rho\ddot{\boldsymbol{u}} = \operatorname{div}(\mathbf{a}\cdot\nabla\boldsymbol{u}) + m\nabla\theta \\
\dot{\theta} &= C\theta + D\dot{u}, \qquad c\dot{\theta} = k\nabla^2\theta + \frac{m}{\rho}\nabla\cdot\dot{\boldsymbol{u}}
\end{aligned}
$$

(3) Memory:

$$\ddot{u} = Au + Bw$$

(In models, $w(x, t, s) = u(x, t - s)$ is the retarded value of u.)

$$\dot{w} = Cw$$

Symmetric Hyperbolic Systems for $U(x, t) \in \mathbb{R}^N$, $x \in \mathbb{R}^m$

$$a_0 \frac{\partial U}{\partial t} = a_j \frac{\partial U}{\partial x^j} + bU + f, \qquad \text{(SH)}$$

where a_0, a_j, b are $N \times N$ matrices with a_0, a_j symmetric and a_0 uniformly positive-definite. (SH) generates a quasi-contractive group in $L^2(\mathbb{R}^m, \mathbb{R}^N)$.

For linear elasticity use the vector U given by $U_{ij} = u_{i,j}$, $U_{i0} = \dot{u}_i$ to write (LE) in the form (SH).

6.4 NONLINEAR ELASTOSTATICS

This section begins by giving the perturbation theory for nonlinear elastostatics in cases where solutions of the linear problem correspond faithfully to those of the nonlinear problem—that is, when there is no bifurcation. Bifurcation problems are studied in Chapter 7. In particular, we show that if the linearized problem has unique solutions, then so does the nonlinear one, nearby. This is done using the linear theory of Section 6.1 and the implicit function theorem (see Section 4.1). These results are due essentially to Stoppelli [1954]. This procedure fails in the important case of the pure traction problem because of its rotational invariance; this case is treated in Section 7.3. We also give an example from Ball, Knops, and Marsden [1978] showing that care must be taken with the function spaces.

Following this, we briefly describe some aspects of the global problem for three-dimensional elasticity following Ball [1977a, b] and state some of the open problems.

For the perturbation theorem (4.2 below), we make the following assumptions;

4.1 Assumptions

(i) The material is hyperelastic; $P = \partial W/\partial F$ and P is a smooth function of x and F (C^1 will do).

(ii) The boundary of $\mathcal{B}$ is smooth (C^1 will do). Here we use $\mathcal{B} \subset \mathbb{R}^3$ to simplify the function spaces involved, but this is not an essential assumption.

(iii) ∂_d; the portion of $\partial\mathcal{B}$ on which the displacement is prescribed is a connected component of $\partial\mathcal{B}$ (or a union of components), and ∂_τ the portion of $\partial\mathcal{B}$ on which the traction is prescribed is a (union of) component(s) of $\partial\mathcal{B}$. Assume $\partial_d \neq \varnothing$.

(iv) $\boldsymbol{\phi}_0\colon \mathcal{B} \to \mathbb{R}^3$ is a given regular deformation (C^3 will do) and the elasticity tensor at $\boldsymbol{\phi}_0$ is strongly elliptic.

(v) The equations of the linearized theory have unique solutions; that is,

$$\mathrm{DIV}[\mathbf{A}\cdot\nabla \boldsymbol{U}] = \mathbf{0}$$

$\boldsymbol{U} = \mathbf{0}$ on ∂_d and $\langle \mathbf{A}\cdot\nabla \boldsymbol{U}, \boldsymbol{N}\rangle = 0$ on ∂_τ implies $\boldsymbol{U} = \mathbf{0}$ ($\mathbf{A}$ is the elasticity tensor evaluated at $\boldsymbol{\phi}_0$).

There are several remarks to be made on these assumptions.

(a) Condition (ii) is unpleasant but is necessitated by the function spaces we use. Bodies with corners are excluded by these methods.[14] The example given below in 4.4 shows that in function spaces such as $W^{1,p}$ (suitable for regions with corners), the results predicted by a formal application of the inverse function theorem simply are not true. It is possible that the source of the difficulties is stress concentrations in corners and that this may be a real problem. (See also p. 318.)

(b) Similar remarks hold for (iii). We do *not* allow genuine mixed problems where ∂_d and ∂_τ have common boundaries.

(c) These assumptions imply existence and uniqueness for the linearized problem by the basic existence theorem for linear elasticity, 1.8. Indeed, assumption (v) states exactly that Ker $A = \{0\}$.

Next define the Banach space $\mathfrak{X}$ to be all $W^{s,p}$ maps $\boldsymbol{\phi}\colon \mathcal{B} \to \mathbb{R}^3$ and let $\mathcal{C} \subset \mathfrak{X}$ be the regular $\boldsymbol{\phi}$'s. If $s > 3/p + 1$, then $\mathcal{C}$ is open in $\mathfrak{X}$ because $W^{s,p} \subset C^1$. Let $\mathcal{Y}$ consist of all triples $(\boldsymbol{B}, \boldsymbol{\psi}, \boldsymbol{\tau})$, where $\boldsymbol{B}$ is a $W^{s-2,p}$ vector function $\boldsymbol{B}$: $\mathcal{B} \to \mathbb{R}^3$, $\boldsymbol{\psi}$ is a $W^{s-1/p,p}$ map of ∂_d to $\mathbb{R}^3$, and $\boldsymbol{\tau}$ is a $W^{s-1-1/p,p}$ map of ∂_τ to $\mathbb{R}^3$.

Define $F\colon \mathcal{C} \subset \mathfrak{X} \to \mathcal{Y}$ by

$$F(\boldsymbol{\phi}) = (-\mathrm{DIV}\, \boldsymbol{P}(\boldsymbol{F}), \boldsymbol{\phi}\,|\,\partial_d, \boldsymbol{P}\cdot\boldsymbol{N}\,|\,\partial_\tau).$$

4.2 Theorem (*Stoppelli* [1954])[15] *Make the assumptions (i)–(v) above and that $s > 3/p + 1$ $(1 < p < \infty)$ (e.g., for $p = 2$, $s > \frac{5}{2}$). Then there are neighborhoods $\mathcal{U}$ of $\boldsymbol{\phi}_0$ in $\mathfrak{X}$ and $\mathcal{V}$ of $(\rho_{\mathrm{Ref}}\mathring{\boldsymbol{B}} = -\mathrm{DIV}\,\mathbf{P}(\mathring{\boldsymbol{F}}), \boldsymbol{\phi}_0\,|\,\partial_d, \mathring{\boldsymbol{P}}\cdot\boldsymbol{N}\,|\,\partial_\tau)$ such that $F\colon \mathcal{U} \to \mathcal{V}$ is one-to-one and onto.*

Proof By the ω-lemma for composition (see Box 1.1, Chapter 3 and Box 1.1, this chapter), F is a C^1 mapping. By Theorem 1.8, and elliptic regularity,

---

[14]The situations in Figure 4.2, page 13 are examples where the results do not apply unless the corners are "smoothed out" a bit or meet at 90°. There is a clear call for better techniques here. See Kellogg and Osborn [1976] and references therein for related work.

[15]Related references are Van Buren [1968], John [1972], and Wang and Truesdell [1973]. The work of John [1972] is especially interesting since he is able to prove uniqueness under a hypothesis of small *stress*.

the derivative of F at ϕ_0 is a linear isomorphism. Therefore the result follows by the implicit function theorem (Box 1.1, Chapter 4). ▮

The assumption $s > 3/p + 1$ (or $n/p + 1$ in n dimensions) is crucial for F to be smooth. Indeed, Example 4.4 below shows that the conclusion is not true without this assumption.

Rephrased, 4.2 states that if the linearized problem is strongly elliptic, hyperelastic,[16] and has unique solutions, then for slight perturbations of the load or boundary conditions from their values at the given displacement, the nonlinear problem

$$\mathrm{DIV}\, \boldsymbol{P} + \rho_{\mathrm{Ref}}\boldsymbol{B} = \boldsymbol{0}, \qquad \phi = \phi_d \quad \text{on} \quad \partial_d, \qquad \boldsymbol{P}\cdot\boldsymbol{N} = \boldsymbol{\tau} \quad \text{on} \quad \partial_\tau$$

has a unique solution ϕ near ϕ_0. Moreover, ϕ is a smooth function of $\boldsymbol{B}$, ϕ_d and $\boldsymbol{\tau}$ (in the topologies above).

4.3 Corollary *Assume (i), (ii), (iii) with $\partial_d \neq \varnothing$ in 4.1, take $\phi_0 = Identity$ and assume that ϕ_0 is a natural state; that is $\boldsymbol{P} = \boldsymbol{0}$. Assume that the elasticity tensor* $\mathbf{c}$ *is uniformly pointwise stable. Then conditions (iv) and (v) are satisfied and so Theorem 4.2 applies.*

The proof of 4.2 shows that the existence of a unique solution remains valid if the constitutive function $\boldsymbol{P}(\boldsymbol{F})$ is also perturbed. This is important philosophically since in any given situation $\boldsymbol{P}(\boldsymbol{F})$ is only known approximately and the "real" material will have various imperfections. This apparently obvious remark takes on deeper significance in the context of bifurcation theory in Chapter 7.

It is clear that in most situations, the neighborhoods $\mathfrak{U}$ and $\mathfrak{V}$ in Theorem 4.2 really are small. Indeed, a bar in a natural state has unique solutions if the boundary is given a small prescribed displacement. If, however, the prescribed displacements are large, buckling and non-uniqueness can occur. (The fact that Theorem 4.2 cannot handle the traction or mixed problems makes examples of this a bit artificial.) One example, due to Fritz John, is shown in Figure 6.4.1(a) and another for a nonhomogeneous material pointed out by John Ball is shown in Figure 6.4.1(b).

Problem 4.1 Let ϕ_0 = Identity be a stress-free (natural) configuration for a strongly elliptic hyperelastic material, satisfying material frame indifference. Consider the problem DIV $\boldsymbol{P} = \boldsymbol{0}$ with $\boldsymbol{P}\cdot\boldsymbol{N} = \boldsymbol{0}$ on $\partial\mathcal{B}$. We have the solution $\phi = \phi_0$. We wish to show that all solutions near ϕ_0 are obtained by translating and rotating ϕ_0. The case $\boldsymbol{B} \neq \boldsymbol{0}$ and $\boldsymbol{P}\cdot\boldsymbol{N} \neq \boldsymbol{0}$ is the most interesting and is the one discussed in Section 7.3.

(a) Let $\mathcal{C}_{\mathrm{tr}} = \{\phi \in \mathcal{C} \mid \boldsymbol{P}\cdot\boldsymbol{N} = \boldsymbol{0} \text{ on } \partial\mathcal{B}\}$. Show that $\mathcal{C}_{\mathrm{tr}} \subset \mathcal{C}$ is a smooth submanifold of $W^{s,p}$, $s < n/p + 1$. (*Hint:* Show

[16] If the problem is not hyperelastic, one must also assume the adjoint A^* has trivial kernel.

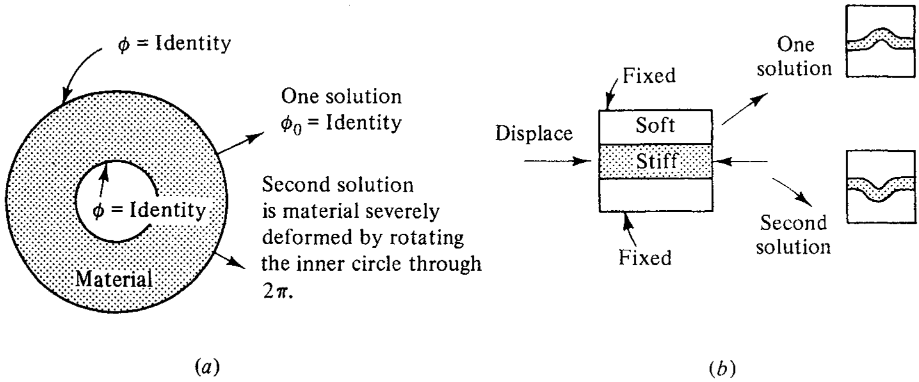


Figure 6.4.1 Global non-uniqueness of solutions for the displacement problem.

that $\boldsymbol{\phi} \mapsto \boldsymbol{P}\cdot\boldsymbol{N}|\,\partial\boldsymbol{B}$ has a surjective derivative by using strong ellipticity.)

(b) Let $\mathfrak{X}$ be the space of $W^{s-2,p}$ vector fields on $\mathcal{B}$ and let $\mathfrak{X}_0 \subset \mathfrak{X}$ be the six-dimensional subspace of vector fields of the form $\boldsymbol{a} + \boldsymbol{b}\boldsymbol{x}$, as in 1.11 (at $\boldsymbol{\phi} = \boldsymbol{\phi}_0$, identify material and spatial coordinates). Define $F\colon \mathcal{C}_{\mathrm{tr}} \to \mathfrak{X}$ by $F(\boldsymbol{\phi}) = \mathrm{DIV}\,\boldsymbol{P}$. Use 1.11(ii) to show that the range of $DF(\boldsymbol{\phi}_0)$ complements $\mathfrak{X}_0$. (We say F is *transversal* to $\mathfrak{X}_0$.)

(c) Deduce from (b) and the implicit function theorem that $F^{-1}(\mathfrak{X}_0)$ is, near $\boldsymbol{\phi}_0$, a smooth manifold of dimension six. Compute its tangent space at $\boldsymbol{\phi}_0$.

(d) Let $\mathcal{O}_{\phi_0}$ denote all rigid motions of $\boldsymbol{\phi}_0$. Show that $\mathcal{O}_{\phi_0} \subset \mathcal{C}_{\mathrm{tr}}$ is six dimensional and, by material frame indifference, show that $F(\mathcal{O}_{\phi_0}) = 0$. Deduce that every zero of F near $\boldsymbol{\phi}_0$ lies in $\mathcal{O}_{\phi_0}$.

The problems with mixed boundary conditions discussed above may be just technicalities that could be overcome by means of a more powerful differential calculus. For example, it seems intuitive that the inverse function theorem and bifurcation theory should work for situations like that in Figure 6.4.2. (Formally, the operator A remains strongly elliptic but picks up a kernel when buckling occurs; this kernel is associated with the axial symmetry of the problem; see Chapter 7.) One way to handle such situations to allow, for example, large stress concentrations in the corners, is to use function spaces like $W^{1,p}$ for $\boldsymbol{\phi}$. Then the stress can be (pointwise) unbounded. These function spaces are used in the global theory of Ball, described below. However, we shall now give an example which shows that in $W^{1,p}$, the conclusions of 4.2 are not true. In fact, we do it just in one dimension ($n = 1$) and show that while the formal linearization is an isomorphism and so by 4.2 the solution is isolated in $W^{2,p}$, it

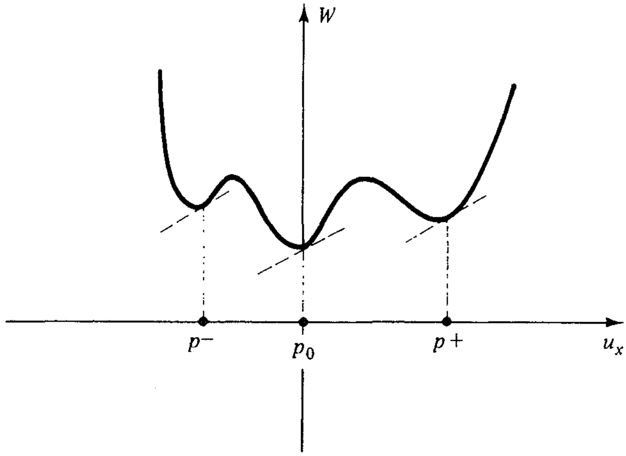


Figure 6.4.2

is *not isolated* in $W^{1,p}$. The difficulty is precisely that the nonlinear map is *not* C^1. (See Morrey [1966] and Martini [1979] for related results.)

4.4 Example (Ball, Knops, and Marsden [1978]). We consider the displacement problem in one dimension, writing u for the nonlinear displacement: $u = \phi$ — Identity. On [0, 1] we consider a stored energy function $W(u_x)$, suppose there are no external forces, and assume the boundary conditions $u(0) = u(1) = 0$. Assume W is smooth and let $p_- < 0 < p_+$ be such that

$$W'(p_-) = W'(0) = W'(p_+) \quad \text{and} \quad W''(0) > 0.$$

(See Figure 6.4.2.)

In $W^{2,p}$ (with the boundary conditions $u(0) = 0$, $u(1) = 0$), the trivial solution $u_0 \equiv 0$ is isolated because the map $u \mapsto W(u_x)_x$ from $W^{2,p}$ to L^p is smooth and its derivative at u_0 is the linear isomorphism $v \mapsto W''(0)v_{xx}$. Therefore, by the inverse function theorem, zeros of $W(u_x)_x$ are isolated in $W^{2,p}$, as above. Note that the second variation of the energy $V(u) = \int_0^1 W(u_x)\,dx$ is positive-definite (relative to the $H^1 = W^{1,2}$ topology) at u_0 because if v is in $W^{1,2}$ and vanishes at $x = 0, 1$, then

$$\frac{d^2}{d\epsilon^2} V(u_0 + \epsilon v)_{|\epsilon=0} = W''(0) \int_0^1 v_x^2\,dx \geq c\,\|v\|^2_{W^{1,2}}.$$

Now we show that u_0 *is not isolated in* $W^{1,p}$. Given $\epsilon > 0$, let

$$u_\epsilon(x) = \begin{cases} p_+ x & \text{for} \quad 0 \leq x \leq \epsilon, \\ p_+\epsilon + p_-(x - \epsilon) & \text{for} \quad \epsilon \leq x \leq \dfrac{p_- - p_+}{p_-}\epsilon, \\ 0 & \text{for} \quad \dfrac{p_- - p_+}{p_-}\epsilon \leq x \leq 1. \end{cases}$$

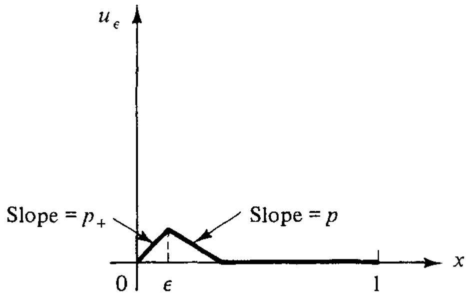


Figure 6.4.3

(See Figure 6.4.3.) Since $W'(u_{\epsilon x})$ is constant, each u_ϵ is an extremal. Also

$$\int_0^1 |u_{\epsilon x} - u_{0x}|^p \, dx = \epsilon |p_+|^p + \epsilon |p_-|^p |p_+|^p,$$

which tends to zero as $\epsilon \longrightarrow 0$. Thus u_0 is not isolated in $W^{1,p}$.

Remarks If $W(p_-) = W(p_+) = W(0)$ and if $W(p) \geq W(0)$ for all p, the same argument shows that there are absolute minima of V arbitrarily close to u_0 in $W^{1,p}$.

Phenomena like this seem to have first been noticed by Weierstrass. See Bolza [1904], footnote 1, p. 40. The "pathology" could possibly be eliminated by making good constitutive assumptions on W.

Now we turn to a discussion of global existence of solutions for the elastotatics problem. Thus, we wish to remove the restriction of locality in Theorem 4.2 and prove the existence of solutions for arbitrary loads. This is, then, a *global* nonlinear elliptic boundary value problem for a system. There have been at least four (overlapping) general methods used for dealing with such problems:

1. Convexity methods and monotone operators
2. Continuity methods
3. Topological methods and degree theory
4. Calculus of variations and minimizers

Method 1 considers an abstract operator $F\colon \mathcal{Y} \longrightarrow \mathcal{X}$, where $\mathcal{X}$ is a Hilbert space and $\mathcal{Y} \subset \mathcal{X}$ is a compactly embedded Banach space. We seek to solve $F(u) = f$ assuming:

(i) F is C^1; $F(0) = 0$.

(ii) $DF(u)$ is symmetric, so $F(u) = f$ is the Euler–Lagrange operator of $J\colon \mathcal{Y} \longrightarrow \mathbb{R}$,

$$J(u) = \int_0^1 \langle F(tu), u \rangle \, dt - \langle f, u \rangle \quad \text{(see Box 7.2, Chapter 1).}$$

(iii) $\langle DF(u)\cdot h, h \rangle > c \, \|h\|_{\mathcal{Y}}^2$ for all $h \in \mathcal{Y}$.

Condition (iii) implies that J is strictly convex: if $u \neq v$ and $0 < t < 1$, then $J(tu + (1 - t)v) < tJ(u) + (1 - t)J(v)$, that J is bounded below and that F is strictly monotone:

$$\langle F(u) - F(v), u - v \rangle > 0 \quad \text{if} \quad u \neq v.$$

From this it is easy to see that solutions of $F(u) = f$ are unique. Also, by selecting a minimizing sequence, one shows that it converges to a unique (generalized) solution of $F(u) = f$. This idea is developed by Langenbach [1961] and is applied to elasticity by, for example, Beju [1971]. However, the convexity assumption (iii) is unsatisfactory for three good reasons: (1) it implies uniqueness of solutions and so precludes buckling; (2) convexity is incompatible with material frame indifference (Coleman and Noll [1959]); and (3) even in ranges where solutions may be unique, convexity need not hold, as is seen from the stretching experiment discussed in Ball [1977a].

Thus, while method 1 is mathematically very powerful and is appropriate for a number of important situations, it is not appropriate in its present form for three-dimensional nonlinear elasticity.

Gurtin and Spector [1979] (see also Gurtin [1981a]) have looked for regions in the space of deformations where a convexity argument might be useful, and thereby have attempted to find sets of deformations in which uniqueness does hold.

Method 2 is based on the following "principle": to solve $F(u) = f$, we look at the range of F. Using the implicit function theorem as in 4.2, one can sometimes show that the range of F is open. Some additional estimates could be used to show the range is closed. By connectedness, this would imply that Range(F) = $\mathfrak{X}$, so $F(u) = f$ would be solvable. The following proposition illustrates the idea, although what is important is the method of proof. As far as we know, this technique has not yet been successfully applied in elasticity.[17]

4.5 Proposition *Suppose $\mathfrak{X}$, $\mathfrak{Y}$ are Banach spaces and $F: \mathfrak{Y} \to \mathfrak{X}$ is a C^1 map and that at each point $x \in \mathfrak{Y}$, $DF(x)$ is an isomorphism of $\mathfrak{Y}$ onto $\mathfrak{X}$. Assume that F is* proper; *that is, if $C \subset \mathfrak{X}$ is compact, then $F^{-1}(C) \subset \mathfrak{Y}$ is compact. Then $F: \mathfrak{Y} \to \mathfrak{X}$ is onto.*

Proof We can suppose without loss of generality that $F(0) = 0$. Let $x \in \mathfrak{X}$ and consider the curve $\sigma(t) = tx$. By the inverse function theorem, there is a unique C^1 curve $\rho(t)$ defined for $0 \leq t \leq \epsilon$ such that $F(\rho(t)) = \sigma(t)$. As in ordinary differential equations, extend ρ uniquely to its maximum domain of existence; say $0 \leq t < T \leq 1$. Let $t_n \to T$; then $F(\rho(t_n)) = \sigma(t_n) \to \sigma(T)$, so by the properness of F, $\rho(t_n)$ has a convergent subsequence, converging, say, to y_0. But as F is a local diffeomorphism in a neighborhood of y_0, the curve $\rho(t)$ can

[17]For another proof and references, see Wu and Desoer [1972].

be defined up to and including T and beyond T if $T < 1$. Thus p is defined for $0 \leq t \leq 1$ and $F(p(1)) = x$. ∎

Problem 4.2 (a) Show that F is proper if one has the estimate $\| DF(x)^{-1} \| \geq m > 0$ for a constant $m > 0$. (b) Show that F in 4.5 is, in fact, one-to-one by using the above proof and the fact that $\mathfrak{X}$ is simply connected.

This proposition may be regarded as a primitive version of method 3, as well as an illustration of method 2. However, in method 3 the most common device used is not so much the use of curves in the domains and ranges of F, but rather to join the map F to another one F_0 that can be understood; for example, F_0 can be a linear map and the curve could be a straight line: $F(t) = tF + (1 - t)F_0$. The idea is to now invoke topological tools of degree theory to show that questions of solvability of $F(u) = f$ can be continued back to those for solvability of $F_0(u) = f$. The details are not appropriate for us to go into at this point; however, the method is powerful in a number of contexts. It allows multiple solutions (indeed it is very useful in bifurcation theory) and does not require any convexity assumptions. For general background and some applications, see Choquet-Bruhat, DeWitte, and Morette [1977]. For applications to rod and shell theory in elasticity, see Antman [1976a, b]. A global uniqueness theorem in the same spirit is given by Meisters and Olech [1963].

Now we turn to method 4, which we shall discuss a bit more extensively, following parts of Ball [1977a, b]. Let $\mathcal{B} = \Omega$ be a region in $\mathbb{R}^3$ with piecewise C^1 boundary, and $\phi\colon \Omega \longrightarrow \mathbb{R}^3$ a typical deformation. Let $W(\boldsymbol{F})$ be a given smooth stored energy function, $\mathcal{V}_B$ a potential for the body forces, and $\mathcal{V}_\tau$ a potential for the tractions. As usual, the displacement will be prescribed $\phi = \phi_d$ on $\partial_d \subset \partial\Omega$ and the traction will be prescribed on ∂_τ: $\mathbf{P} \cdot N = \boldsymbol{\tau}$. The energy functional whose critical points we seek is (see Section 5.1)

$$I(\phi) = \int_\Omega W(\boldsymbol{F})\, dV + \int_\Omega \mathcal{V}_B\, dV + \int_{\partial_\tau} \mathcal{V}_\tau\, dA.$$

In fact, we seek to minimize $I(\phi)$ over all ϕ satisfying $\phi = \phi_d$ on ∂_d. For dead loads we can take $\mathcal{V}_B = -\boldsymbol{B} \cdot \boldsymbol{\phi}$ and $\mathcal{V}_\tau = -\boldsymbol{\tau} \cdot \boldsymbol{\phi}$, which are linear functions of ϕ. The essential part of $I(\phi)$ is the stored energy function, so we shall assume that $I(\phi) = \int_\Omega W(\boldsymbol{F})\, dV$ for simplicity.

The method for minimizing $I(\phi)$ proceeds according to the following outline. On a suitable function space $\mathcal{C}$ of ϕ's:

(1) Show I is bounded below; then $m = \inf_{\psi \in \mathcal{C}} I(\psi)$ exists as a real number. Let ϕ_n be such that $I(\phi_n) \longrightarrow m$ as $n \longrightarrow \infty$; that is, select a minimizing sequence (this is possible as $m > -\infty$).

(2) Find a subsequence of ϕ_n that converges weakly; that is, $\phi_n \rightharpoonup \phi$ (this notion is explained in Box 4.1).

(3) Show that I is weakly sequentially lower semicontinuous; that is, $\phi_n \rightharpoonup \phi$ implies that $I(\phi) \leq \liminf_{n\to\infty} I(\phi_n)$.

If each of these steps can be effected, then ϕ is the minimizer of I. [*Proof:* Clearly $I(\phi) \geq m$ by definition of m. Also, if $I(\phi_n) \longrightarrow m$, then $\liminf_{n\to\infty} I(\phi_n) = m$, so by 3, $I(\phi) \leq m$. Thus $I(\phi) = m$.]

In one dimension, this method can only work if W is convex. Indeed for this case, a theorem of Tonelli states that if $I(\phi)$ is weakly sequentially semicontinuous, then W is convex, and conversely. This is actually not difficult and is proved in Box 4.1. The related "relaxation theorem" is discussed in this box as well.

In elasticity, even in one dimension, two of the properties of minimizers ϕ that have to be carefully considered are smoothness and invertibility of ϕ. These are not simple; in one dimension, however, the situation is fairly well understood (mostly due to Antman [1976a, b]) and is discussed in Box 4.1.

In three dimensions, convexity of W is *not* necessary for this method to work. Indeed, it can be made to work under hypotheses that are reasonable for elasticity. The analog of Tonelli's result is due to Morrey [1952]; assume throughout that $|W(\boldsymbol{F})| \leq C_1 + C_2|\boldsymbol{F}|^p$ for constants C_1 and C_2, so $I(\phi)$ is defined for $\phi \in W^{1,p}(\Omega)$.

4.6 Proposition *If $I(\phi) = \int_\Omega W(\boldsymbol{F})\,dV$ is weakly sequentially lower semicontinuous on $W^{1,p}(\Omega)$, then W is* quasi-convex; *that is, for all (constant)* 3×3 *matrices $\boldsymbol{F}$ with* $\det \boldsymbol{F} > 0$ *and all $\psi\colon \Omega \longrightarrow \mathbb{R}^3$ that are C^∞ with compact support in Ω,*

$$\int_\Omega W(\boldsymbol{F} + \nabla\psi(X))\,dV(X) \geq W(\boldsymbol{F}) \times \textit{volume}(\Omega). \tag{QC}$$

This also implies strong ellipticity.

The proof is actually similar to Tonelli's theorem proved in Box 4.1, and reduces to it in one dimension, so is omitted. The inequality (QC) says essentially that if ϕ_h is a homogeneous deformation, $I(\phi_h)$ is a minimum among deformations ϕ with the same boundary conditions. Morrey also shows that (QC) and growth conditions imply sequential weak lower semicontinuity. However, this does not apply to elasticity because of the condition $\det \boldsymbol{F} > 0$ that we must be aware of.

To understand which stored energy functions give sequentially weakly lower semicontinuous (s.w.l.s) I's, first consider the question of which ones give sequential weak continuity (s.w.c.). We state the following without proof:

4.6′ Proposition (*Ericksen, Edelen, Reshetnyak, Ball*) *Let $W\colon M^{m\times n}$ (the $m \times n$ matrices) $\longrightarrow \mathbb{R}$ be continuous, $|W(\boldsymbol{F})| \leq C_1 + C_2|\boldsymbol{F}|^p$ $(1 < p < \infty)$, and let $L(\phi) = W(\nabla\phi)$; $L\colon W^{1,p} \longrightarrow L^1$. The following are equivalent:*

(1) *L is s.w.c.*
(2) *L is a null Lagrangian; that is, $L(\phi + \psi) = L(\phi)$ for all ψ smooth with*

compact support in Ω and all C^1 maps $\phi\colon \Omega \longrightarrow \mathbb{R}^3$. (If W is smooth, this means the Euler–Lagrange equations for W are identically satisfied for any ϕ.)

(3) *$W(F) =$ Constant $+$ linear combination of $r \times r$ minors of F for $1 \le r \le \min(m, n)$.*

For example, in three dimensions the null Lagrangians are $W(\boldsymbol{F}) = C + \boldsymbol{A}\cdot\boldsymbol{F} + \boldsymbol{B}\cdot(\operatorname{adj}\boldsymbol{F}) + D(\det\boldsymbol{F})$, where (adj F) is the matrix of cofactors of $\boldsymbol{F}$ and C, $\boldsymbol{A}$, $\boldsymbol{B}$, and D are constant. Some feeling for what is involved in 4.6 can be gained by working the following problem. (See Ball [1977a] for the complete proof.)

> ***Problem 4.3*** (Ball) Prove that $\det\boldsymbol{F}$ is s.w.c. for 2×2 matrix functions by using the identity
>
> $$\det \nabla\phi = \phi^1_{,1}\phi^2_{,2} - \phi^1_{,2}\phi^2_{,1} = (\phi^1\phi^2_{,2})_{,1} - (\phi^1\phi^2_{,1})_{,2}.$$
>
> (This sort of trick is used in "compensated compactness"; cf. Tarter [1979].)

A good hypothesis on constitutive functions must be invariant under various transformations of coordinates. The following is one that is invariant under adding on null Lagrangians and the inversion transformation of fields $\phi \mapsto \phi^{-1}$.

4.7 Definition W is called (strictly) *polyconvex* if there is a (strictly) convex function $g\colon M^{3\times3}_+ \times M^{3\times3} \times (0, \infty) \longrightarrow \mathbb{R}$ such that $W(\boldsymbol{F}) = g(\boldsymbol{F}, \operatorname{adj}\boldsymbol{F}, \det\boldsymbol{F})$ for all $\boldsymbol{F} \in M^{3\times3}_+$.

One has the following chain of implications

$$
\begin{tikzcd}
\text{Strict Convexity} \arrow[r, Rightarrow] & \text{Strict Polyconvexity} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Quasi-Convexity} \arrow[dr, Rightarrow] & \\
 & & \text{s.w.l.s.} \arrow[r, Rightarrow] & \text{Strong Ellipticity}
\end{tikzcd}
$$

As remarked before, convexity is not a useful assumption. However, polyconvexity is valid for many specific materials, such as the Mooney–Rivlin and Ogden materials.

The following is a sample of one of Ball's results.

4.8 Theorem (*Ball* [1977b]) *Suppose:*

(H1) *W is polyconvex.*

(H2) *$g(\boldsymbol{F}, \boldsymbol{H}, \delta) \le C + K(|\boldsymbol{F}|^p + |\boldsymbol{H}|^q + \delta^r)$ for constants $K > 0$, $p \ge 2$, $q \ge p/(p-1)$, and $r > 1$.*

(H3) *if $(\boldsymbol{F}_n, \boldsymbol{H}_n, \delta_n) \longrightarrow (\boldsymbol{F}, \boldsymbol{H}, 0)$, then $g(\boldsymbol{F}_n, \boldsymbol{H}_n, \delta_n) \longrightarrow \infty$.*

Let $\mathfrak{a} = \{\phi \in W^{1,p}(\Omega)\,|\,\operatorname{adj}(\nabla\phi) \in L^q, \det\nabla\phi \in L^r, I(\phi) < \infty$, and $\phi = \phi_d$ on $\partial_d\}$ and suppose $\mathfrak{a} \ne \varnothing$. Then there exists a $\phi \in \mathfrak{a}$ that minimizes I in $\mathfrak{a}$.

Sketch of Proof Take a minimizing sequence ϕ_n. By (H2), we see that $\nabla\phi_n$ is bounded in L^p, $\operatorname{adj}(\nabla\phi_n)$ is bounded in L^q, and $\det(\nabla\phi_n)$ is bounded in L^r. By the Poincaré inequality and weak compactness of the unit ball we get a subsequence ϕ_μ such that

$$\phi_\mu \rightharpoonup \phi_0 \quad \text{in} \quad W^{1,p}, \qquad \operatorname{adj} \nabla\phi_\mu \rightharpoonup H \quad \text{in} \quad L^q, \quad \text{and} \quad \det \nabla\phi_\mu \rightharpoonup \delta \quad \text{in} \quad L^r$$

By 4.6′, adj and det are weakly continuous, so $H = \operatorname{adj}(\nabla\phi_0)$ and $\delta = \det(\nabla\phi_0)$ Since g is convex, $\phi \mapsto I(\phi) = \int_\Omega g(F, \operatorname{adj} F, \det F)\, dV$ is s.w.l.s. so ϕ_0 is the required minimizer. ▮

This sort of argument works well for the incompressible case, too, by weak continuity of det.

Open Problems

(a) Are minimizers weak solutions of the Euler–Lagrange equations?
(b) Are minimizers C^1? (Ball [1980] has shown that strong ellipticity is *necessary* for regularity; however it may be sufficient if $W(F)/|F|^3 \longrightarrow \infty$ as $|F| \longrightarrow \infty$. Without such a condition, Ball [1982] has shown by means of a very important example involving *cavitation* that minimizers need not be smooth.)
(c) Are minimizers 1–1 deformations? (Using a result of Meisters and Olech [1963], Ball [1981] shows that this is true in the incompressible case.)

Box 4.1 ***Some Facts About Weak Convergence***[18]

In this box we shall state a few basic properties and examples of weak convergence; prove Tonelli's theorem and discuss the related relaxation theorem of L.C. Young; and discuss the proof of existence and regularity for one-dimensional problems.

These results only hint at the extensive literature on uses of the weak topology. Besides the work of Ball already cited, the articles of Tartar [1979] and DiPerna [1982] are indicative of current research in this area.

If $\mathfrak{X}$ is a Banach space and x_n is a sequence in $\mathfrak{X}$, we write $x_n \rightharpoonup x$ and say x_n *converges weakly* to x if for all $l \in \mathfrak{X}^*$ (i.e., $l\colon \mathfrak{X} \longrightarrow \mathbb{R}$ is continuous and linear), $l(x_n) \longrightarrow l(x)$ in $\mathbb{R}$. If $\mathfrak{X} = L^p([0, 1])$, $(1 < p < \infty)$, then $u_n \rightharpoonup u$ means that

$$\int_0^1 u_n v\, dx \longrightarrow \int_0^1 uv\, dx$$

for all $v \in L^{p'}[0, 1]$, where $(1/p) + (1/p') = 1$. This is because $(L^p)^* =$

[18]We thank J. Ball for help with this box.

$L^{p'}$ (Riesz representation theorem). For $\mathfrak{X} = L^\infty$ and we choose $v \in L^1$, we speak of *weak* convergence.* Clearly ordinary convergence implies weak convergence.

Problem 4.4 Let $0 < \lambda < 1$, $a, b > 0$, and $u_n = a$ on $[0, \lambda/n]$, $u_n = b$ on $[\lambda/n, 1/n]$, and repeat on every subinterval $[i/n, (i+1)/n]$ $(i = 0, 1, \ldots, n-1)$. Prove that $u_n \rightharpoonup u = \lambda a + (1-\lambda)b$ weak* in L^∞ [and hence in $L^p[0, 1]$ $(1 < p < \infty)$], but $u_n \not\to u$.

In a reflexive separable Banach space $\mathfrak{X}$ (such as $W^{s,p}(\Omega)$, $1 < p < \infty$) the unit ball is weakly sequentially compact; that is, if $u_n \in \mathfrak{X}$ and $\|u_n\| = 1$, then there is a subsequence $u_{n_k} \rightharpoonup u \in \mathfrak{X}$. The unit ball (or in fact any closed convex set) is weakly closed, so $\|u\| \le 1$. (This result may be found in Yosida [1971], p. 125.)

Problem 4.5 In $L^2[0, 2\pi]$ show that $(1/\sqrt{\pi}) \sin nx = u_n(x)$ satisfies $u_n \rightharpoonup 0$, yet $\|u_n\| = 1$. (*Hint:* Use the Riemann–Lebesgue lemma from Fourier series.)

Suppose $W\colon \mathbb{R}^+ \to \mathbb{R}$ is a given smooth function and for $\phi\colon [0, 1] \to \mathbb{R}$, $\phi' \ge 0$ almost everywhere, define

$$I(\phi) = \int_0^1 W(\phi'(X))\, dX.$$

Assume $|W(F)| \le C_1 + C_2 |F|^p$ so I maps $\{\phi \in W^{1,p} \mid \phi' \ge 0 \text{ a.e.}\}$ to $\mathbb{R}$.

4.9 Proposition (*Tonelli*) *I is weakly sequentially lower semicontinuous (w.s.l.s.) if and only if W is convex.*

Proof First of all, assume W is convex and let $\phi_n \rightharpoonup \phi$ in $W^{1,p}$. By Mazur's theorem (see Yosida [1971]) there is a sequence of finite convex combinations, say $\psi_n = \sum_j \lambda_n^j \phi_j$, $\sum_j \lambda_n^j = 1$ $(0 \le \lambda_j \le 1)$ such that $\psi_n \to \phi$ (strongly) in $W^{1,p}$. By going to a further subsequence we can suppose $\psi_n' \to \psi'$ a.e. By Fatou's lemma,

$$\int_0^1 W(\phi'(X))\, dX \le \liminf_{n\to\infty} \int_0^1 W(\psi_n'(X))\, dX.$$

By convexity of W, the right-hand side does not exceed

$$\liminf_{n\to\infty} \sum_j \lambda_n^j \int_0^1 W(\phi_j'(X))\, dX \le \liminf_{n\to\infty} \int_0^1 W(\phi_n'(X))\, dX.$$

Thus I is w.s.l.s.

Conversely, assume I is w.s.l.s. Let $\phi_n(0) = 0$ and let $\phi_n'(X) = u_n(X)$, given by Problem 4.4. By that problem, $\phi_n \rightharpoonup \phi(X) = (\lambda a + (1-\lambda)b)X$ and $W(\phi_n') \rightharpoonup \lambda W(a) + (1-\lambda) W(b)$. Thus by w.s.l.s. $I(\phi) \le \liminf_{n\to\infty}$

$I(\phi_n)$ becomes $W(\lambda a + (1 - \lambda)b) \leq \lambda W(a) + (1 - \lambda)W(b)$; that is, W is convex. ▮

Problem 4.6 Use this argument to prove that I is w.s.c. if and only if $\phi(X) = aX + b$ for constants a and b. (This is the special case of Proposition 4.6 when $n = 1 = m$.)

In one dimension there is evidence that non-convex W's may be useful for describing phase transitions (Ericksen [1975]).[19] However, in static experiments the non-convexity of W may not be observable. Indeed, the relaxation theorem of L.C. Young states that the minimum of $\int_0^1 W(\phi'(X))\, dX$ is the same as that of $\int_0^1 W^*(\phi'(X)\, dX$, where W^* is the convex lower envelope of W (Figure 6.4.4). A convenient reference for this is Ekeland and Temam [1974]. In three dimensions the situation is far from settled.

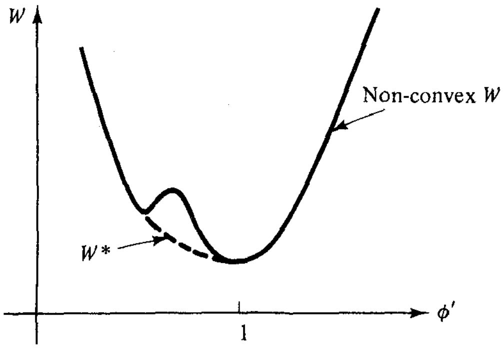


Figure 6.4.4

Remaining in one dimension, consider the following hypotheses on $W(X, F)$:

(H1) $W: [0, 1] \times (0, \infty) \longrightarrow \mathbb{R}$ is C^1.
(H2) $W(X, F) \longrightarrow +\infty$ as $F \longrightarrow 0+$.
(H3) $W(X, F)/F \longrightarrow \infty$ as $F \longrightarrow +\infty$ uniformly in X.
(H4) $W(X, F)$ is convex in F.

[19]Phase transitions contemplated here may be seen if the polyethelene used in beer can packaging is stretched with your hands. It turns white, changing phase. Gentle heat will restore the original phase.

(H5) $V: [0, 1] \times [0, \infty) \longrightarrow \mathbb{R}$ is C^1. Let

$$I(\phi) = \int_0^1 W(X, \phi'(X))\, dX + \int_0^1 V(X, \phi(X))\, dX.$$

The following refines 4.8 slightly in one dimension.

4.10 Proposition *If (H1)–(H4) hold and α is given, then there is a member of*

$$\mathcal{A} = \{\phi \in W^{1,1} \mid I(\phi) < \infty, \phi(0) = 0, \phi(1) = \alpha\}$$

that minimizes I. (Note that $W^{1,1} \subset C^0$, so if $\phi \in \mathcal{A}$, then by (H2), $\phi' > 0$ a.e., so ϕ is one-to-one.)

Proof Note that $\mathcal{A} \neq \varnothing$ since $\phi(X) = \alpha X$ lies in $\mathcal{A}$. We first prove I is bounded below on $\mathcal{A}$.

By (H2) and (H3), W is bounded below, so $\int_0^1 W(X, \phi'(X)\, dX$ is bounded below. Since members of $\mathcal{A}$ are continuous and one-to-one, they are bounded between 0 and α. Thus $V(X, \phi(X))$ is uniformly bounded. Hence $I(\phi)$ is bounded below.

Let $\phi_n \in \mathcal{A}$ be such that $I(\phi_n) \searrow \inf\{I(\phi) \mid \phi \in \mathcal{A}\}$. (H3) implies ϕ_n are bounded in $W^{1,1}$. However, as $p = 1$, the space is not reflexive, so it is not obvious we can extract a weakly convergent subsequence. However, a result of de la Valleé Poussin shows that we can (see Morrey [1966]). The proof is then completed using w.s.l.s. of I from 4.9. ▮

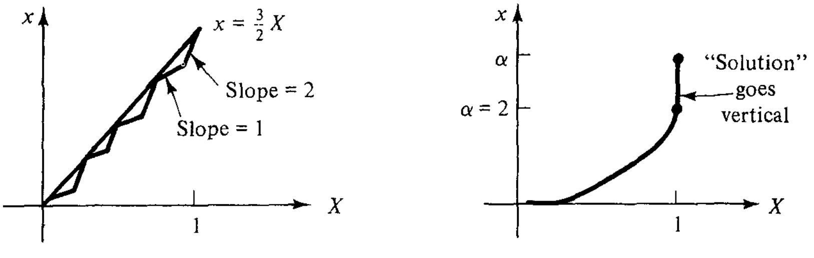
The condition $W(X, F)/F \longrightarrow +\infty$ as $F \longrightarrow +\infty$ has direct physical meaning. Namely, consider a small piece of material that undergoes the homogeneous deformation $\phi(X) = FX$. This stretches a length l/F to the length l. The energy required to do this is $W(F)l/F$. Thus, (H3) means it takes more and more energy to stretch small lengths out to a prescribed length. The analogue in dimension three was mentioned in open problem (b) above.

Without the conditions of convexity or $W(X, F)/F \longrightarrow \infty$ as $F \longrightarrow \infty$, one runs into difficulties. The following examples (motivated by examples of L.C. Young) show this.

4.11 Examples (J. Ball) (a) Consider the (non-convex) problem of minimizing

$$\int_0^1 \left(\frac{(\phi' - 1)^2(\phi' - 2)^2}{\phi'} + (\phi - \tfrac{3}{2}X)^2\right) dX$$

for $x = \phi(X)$ with $\phi' \geq 0$, $\phi(0) = 0$, and $\phi(1) = \frac{3}{2}$. One sees by taking small broken segments as in Figure 6.4.5(a) that the minimum is zero.



(*a*) (*b*)

Figure 6.4.5

However, the minimum can never be attained. Roughly, it is a line with infinitely many zig-zags. If W does not depend explicitly on X, then examples like this are not possible (cf. Aubert and Tahraoui [1979]).

(b) An example violating $W(X, F)/F \longrightarrow \infty$ is

$$\int_0^1 \left(\frac{1}{\phi'} + \phi' + \phi\right) dX, \qquad \phi(0) = 0, \qquad \phi(1) = \alpha.$$

Direct calculation shows that the solution does not exist for $\alpha > 2$. See Figure 6.4.5(b). (This problem can be dealt with by making the transformation $\phi \mapsto \phi^{-1}$—that is, by interchanging the roles of x and X.)

In one dimension there is more known about regularity than in three dimensions. In fact, we have the following (Antman [1976b]):

4.12 Proposition *Suppose (H1), (H2), and (H5) hold and ϕ minimizes I in $\mathfrak{A}$. Then the Euler–Lagrange equations*

$$\frac{d}{dX} W_F(X, \phi'(X)) = V_\phi(X, \phi(X))$$

hold a.e. on $[0, 1]$. *If, moreover, W is strictly convex in F and (H3) holds, then ϕ is C^1 and ϕ' is bounded away from zero.*

Proof The usual derivation of the Euler–Lagrange equations is not valid, so some care is needed; indeed, on $\mathfrak{A}$, I is not differentiable. Let

$$\Omega_n = \{X \in [0, 1] \mid 1/n \leq \phi'(X) \leq n\}.$$

Thus $\Omega_n \subset \Omega_{n+1}$ and $\mu([0, 1]\backslash \cup \Omega_n) = 0$. Let $v \in L^\infty$, let χ_n be the characteristic function of Ω_n, and define $\psi(t, X)$ by

$$\psi'(t, X) = \phi'(X) + t\chi_n(X)v(X), \qquad \psi(0) = 0.$$

We will show that $(d/dt)I(\psi(t, \cdot))|_{t=0}$ exists. Since ϕ is a minimum, this is then zero. We have

$$\frac{1}{t}[I(\psi(t, \cdot)) - I(\psi(0, \cdot))] =$$

$$\frac{1}{t}\int_0^1 \{W(X, \phi'(X) + t\chi_n(X)v(X)) - W(X, \phi'(X))\}\, dX$$
$$+ \frac{1}{t}\int_0^1 \left\{V\left(X, \int_0^X [\phi'(\bar{X}) + t\chi_n(\bar{X})v(\bar{X})]\, d\bar{X}\right) - V(X, \phi(X)\right\} dX$$

From the mean value theorem and the definition of Ω_n we see that the integrands are bounded, so we can pass to the limit $t \longrightarrow 0$ by the dominated convergence theorem to get

$$0 = \frac{d}{dt} I(\psi(t, \cdot))\bigg|_{t=0} = \int_{\Omega_n} W_F(X, \phi'(X))v(X)\, dX$$
$$+ \int_{\Omega_n} V_\phi(X, \phi(X))\left(\int_0^X \chi_n(\bar{X})v(\bar{X})\, d\bar{X}\right) dX.$$

Integration by parts gives

$$0 = \int_{\Omega_n} W_F(X, \phi'(X))v(X)\, dX - \int_{\Omega_n}\left(\int_0^X V_\phi(\bar{X}, \phi(\bar{X}))\, d\bar{X}\right) v(X)\, dX.$$

Since v and n are arbitrary, we get

$$W_F(X, \phi'(X)) = \int_0^X V_\phi(\bar{X}, \phi(\bar{X}))\, d\bar{X} \text{ a.e.}$$

But this implies, by continuity of the integrand,

$$\frac{d}{dX} W_F(X, \phi'(X)) = V_\phi(X, \phi(X)).$$

The second part follows by implicitly solving

$$W_F(X, \phi'(X)) = \int_0^X V_\phi(\bar{X}, \phi(\bar{X}))\, d\bar{X}$$

for ϕ'. ∎

Box 4.2 ***Summary of Important Formulas for Section 6.4***

Local Existence Theory for Elastostatics

If about a given configuration ϕ_0 the linearized problem has unique solutions (see Section 6.1, so the problem is not pure traction and no bifurcation occurs), and the boundaries of $\mathcal{B}$ are smooth and no mixed boundary conditions with contiguous parts occur, then a small change in any of the data (boundary conditions, forces, or constitutive functions) produces a corresponding unique configuration ϕ in $W^{s,p}$, $s > 3/p + 1$ depending smoothly on the data.

Non-Applicability of the Inverse Function Theorem in $W^{1,p}$

In $W^{1,p}$ solutions to the elastostatics equations need not be isolated, even though the formal linearization of the equations is an isomorphism.

Convexity

The assumptions of convexity and monotonicity are not appropriate for the operators in three-dimensional elasticity.

Topological Methods

Degree theory, Morse theory, and so on may be useful in three-dimensional elasticity, but so far have not been successfully applied because of technical difficulties.

Minimizers

Global solutions in $W^{1,p}$ for elastostatics can be found by using weak convergence and minimizers. Under the assumption of polyconvexity and growth conditions on the stored energy function, minimizers exist. Their regularity is not known, except in one dimension.

6.5 NONLINEAR ELASTODYNAMICS

This section surveys some results that are relevant to elastodynamics. The only part of this theory that is well understood is that dealing with *semilinear equations*—that is, equations that are linear plus lower-order nonlinear terms. This theory, due to Jorgens [1961] and Segal [1962], will be presented and applied to an example—the equations of a vibrating panel. The theory for *quasi-linear* equations—equations whose leading terms are nonlinear but depend linearly on the highest derivative—is appropriate for three-dimensional nonlinear elasticity. This will be briefly sketched, but much less is known. The primary difficulty is the problem of shock waves. The recent literature will be briefly discussed concerning this problem.

Elastostatics is imbedded in elastodynamics; each solution of the elastostatic equations is a fixed point for the equations of elastodynamics. Eventually, the dynamical context provides important additional information and intuition. For example, we may wish to know if the fixed points are stable, unstable, or are saddle points. We may also wish to find periodic orbits and examine their stability. For ordinary differential equations this leads to the large subject of dynamical systems (cf. Abraham and Marsden [1978] for more information and references). For partial differential equations a good deal is known for semilinear equations and we shall give some examples in Section 6.6 and in Chapter 7. However, for the quasi-linear equations of three-dimensional elasticity, much less is known about qualitative dynamics.

Let us begin by recalling some general terminology (see Definition 3.3, Chapter 5).

A *continuous local semiflow* on a Banach space $\mathcal{Y}$ is a continuous map $F: \mathfrak{D} \subset \mathcal{Y} \times \mathbb{R}^+ \longrightarrow \mathcal{Y}$, where $\mathfrak{D}$ is open, such that: (i) $\mathcal{Y} \times \{0\} \subset \mathfrak{D}$ and $F(x, 0) = x$; and (ii) if $(x, t) \in \mathfrak{D}$, then $(x, t + s) \in \mathfrak{D}$ if and only if $(F(x, t), s) \in \mathfrak{D}$ and in this case $F(x, t + s) = F(F(x, t), s)$.

Suppose $\mathcal{Y}$ is continuously included in another Banach space $\mathfrak{X}$ and G maps $\mathcal{Y}$ (or an open subset $\mathfrak{D}$ of $\mathcal{Y}$) to $\mathfrak{X}$. We say G *generates* the semiflow if for $t \geq 0$ and for each $x \in \mathcal{Y}$, $F(x, t)$ is t-differentiable and

$$\frac{d}{dt} F(x, t) = G(F(x, t)). \tag{1}$$

As usual, if $F(x, t)$ is defined for all $t \in \mathbb{R}$, we call F a *flow*.

There is a slight philosophical difference with the linear case. For the latter, we started with $F_t: \mathfrak{X} \longrightarrow \mathfrak{X}$ and constructed the generator by looking at where F_t is t-differentiable at $t = 0+$. The domain of the generator is a Banach space $\mathcal{Y}$ with the graph norm, and F_t maps it to itself. This is compatible with the above definition. In the nonlinear case, it is better to start right off with F_t defined on the smaller space $\mathcal{Y}$. Then F_t may or may not extend to all of $\mathfrak{X}$. As we shall see, it does in the semilinear case. In this case we can use the phrase "G is the generator of F_t."

Often we are given G and want to construct $F(x, t)$ such that (1) holds. If an F exists, satisfying 5.1, we say the equations $dx/dt = G(x)$ are *well-posed.* If solutions exist for all time $t \geq 0$ (or all t for flows), we say the equations generate *global solutions.*

A crucial part of well-posedness is the continuous dependence of the solution $F(x, t)$ on the initial data x—that is, continuity of the map $x \mapsto F(x, t)$ from (an open subset of) $\mathcal{Y}$ to $\mathcal{Y}$ for each $t \geq 0$. The satisfactory answer to such problems can depend on the choice of $\mathcal{Y}$ made.

Sometimes it is necessary to study the case in which G depends explicitly on time. Then the flow is replaced by evolution operators $F_{t,s}: \mathcal{Y} \longrightarrow \mathcal{Y}$ satisfying $F_{s,s} =$ Identity and $F_{t,s} \circ F_{s,r} = F_{t,r}$. This is just as in Definition 6.5, Chapter 1. We replace (1) by

$$\frac{d}{dt} F_{t,s}(x) = G(F_{t,s}(x), t) \tag{1'}$$

with initial condition $F_{s,s}(x) = x$ (so "s" is the *starting time*).

We begin now with a discussion of the semilinear case. The main method is based on the Duhamel, or variation of constants, formula; namely, the fact that the solution of

$$\frac{du}{dt} = Au + f, \qquad u(0) = u_0$$

is

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)\,ds.$$

(See 2.17(5) in Section 6.2.) If B depends on t and u, we conclude that the solution $u(t)$ of

$$\frac{du}{dt} = Au + B(t, u), \qquad u(0) = u_0 \tag{2}$$

satisfies the implicit equation

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}B(s, u(s))\,ds. \tag{3}$$

The point now is that if B is a Lipschitz operator on the Banach space $\mathfrak{X}$ (on which e^{tA} defines a semigroup), then the Picard iteration technique from ordinary differential equations applies to (3). If B is actually a C^1 map of $\mathfrak{X}$ to $\mathfrak{X}$, then we will show that solutions of (3) are in fact in $\mathfrak{D}(A)$ if $u_0 \in \mathfrak{D}(A)$ and satisfy (2) in the strict sense.

The part of the analysis of (3) that is the same as that in ordinary differential equations is outlined in the following problem.

Problem 5.1 (*Existence*) Let A generate a C^0 semigroup on $\mathfrak{X}$ of type (M, β) and let $\mathfrak{F} = \{u \in C([0, t_0], \mathfrak{X}) \mid \|u(t) - u_0\| \leq R\}$, where $t_0 > 0$ and $R > 0$ are constants and $C([0, t_0], \mathfrak{X})$ is the set of continuous maps of $[0, t_0]$ to $\mathfrak{X}$. Suppose $B(t, u)$ is continuous and $\|B(t, u)\| \leq Cp(t)$ if $\|u - u_0\| \leq R$ and $0 \leq t \leq t_0$. Define

$$T\colon \mathfrak{F} \to C([0, t_0], \mathfrak{X});\ (Tu)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}B(s, u(s))\,ds.$$

(i) Show that if

$$\|e^{tA}u_0 - u_0\| + Me^{\beta t_0}C\int_0^{t_0} |p(s)|\,ds \leq R \quad (0 \leq t \leq t_0),$$

then T maps $\mathfrak{F}$ to $\mathfrak{F}$.

(ii) If B satisfies $\|B(t, u_1) - B(t, u_2)\| \leq Kp(t)\|u_1 - u_2\|$ for u_1 and u_2 in the ball $\|u - u_0\| \leq R$ and $0 \leq t \leq t_0$, where K is a (Lipschitz) constant (depending on R), then T satisfies

$$\|Tu_1 - Tu_2\| \leq \alpha\|u_1 - u_2\|,$$

where the distance function on $\mathfrak{F} \subset C([0, t_0], \mathfrak{X})$ is

$$\|u_1 - u_2\| = \sup_{0 \leq t \leq t_0} \|u_1(t) - u_2(t)\|$$

and α is defined by

$$\alpha = KMe^{\beta t_0}\int_0^{t_0} |p(s)|\,ds.$$

(iii) Deduce that if $\alpha < 1$, then T has a unique fixed point by the contraction mapping principle (see Lemma 1.3 of Section 4.1).

(iv) Show that instead of continuity of B, it is sufficient for $B(t, u)$ to be L^1 in t and Lipschitz in u.

Problem 5.2 (Uniqueness and Continuous Dependence on Initial Data)

(i) Prove *Gronwall's inequality:* if p is integrable on $[a, b]$ and v is non-negative, bounded, and measurable on $[a, b]$ and there is a constant $C \geq 0$ such that for all $t \in [a, b]$,

$$v(t) \leq C + \int_a^t |p(s)|\, v(s)\, ds,$$

then
$$v(t) \leq C \exp\left(\int_a^t |p(s)|\right) ds.$$

(See any book on ordinary differential equations; the solution is on p. 124 of Carroll [1969].)

(ii) In the setting of Problem 5.1, show that solutions of (3) satisfy

$$\| u(t) \| \leq Me^{\beta t_0} \| u_0 \| \exp\left(CMe^{\beta t_0} \int_0^t |p(s)|\, ds\right)$$

and $$\| u(t) - v(t) \| \leq Me^{\beta t_0} \| u_0 - v_0 \| \exp\left(KMe^{\beta t_0} \int_0^t |p(s)|\, ds\right).$$

(iii) Use (ii) to show that solutions of (3) are Lipschitz functions of the initial data.

(iv) Use (ii) to give another proof of local uniqueness of solutions.

(v) Prove that any two solutions of (3) are globally unique on their common domain of definition if B is locally Lipschitz. (*Hint:* Show that the set of t where the two solutions coincide is both open and closed.)

Now we shall state the main theorem for semilinear equations in a version due to Segal [1962]. (Further important information is contained in subsequent corollaries.) To simplify the exposition, we shall assume B is independent of s. (The reader should do the general case.)

5.1 Theorem *Let $\mathfrak{X}$ be a Banach space and U_t a linear semigroup on $\mathfrak{X}$ with generator A having domain $\mathfrak{D}(A)$. Suppose $B\colon \mathfrak{X} \longrightarrow \mathfrak{X}$ is a C^k* map *($k \geq 1$). Let $G(u) = Au + B(u)$ on $\mathfrak{D}(A)$. Then (3) defines a unique local semiflow $u(t) = F_t(u_0)$ on $\mathfrak{X}$ ($t \geq 0$); F_t is a local flow if A generates a group. If $u_0 \in \mathfrak{D}(A)$, then $F_t(u_0) \in \mathfrak{D}(A)$ and (1) holds. Moreover, for each fixed t, F_t is a C^k mapping of an open set in $\mathfrak{X}$ to $\mathfrak{X}$.*

Proof Problems 5.1 and 5.2 show that (3) defines a local flow and that F_t is locally Lipschitz. We next show that $F_t(x)$ is differentiable in x. To do so one can appeal to a general theorem on ordinary differential equations in Hale [1969] (Theorem 3.2, p. 7). We can also show that F_t is C^1 by a direct calculation as follows. For $x \in \mathfrak{X}$, let $\theta_t(x) \in \mathfrak{B}(\mathfrak{X})$ (the bounded linear operators from $\mathfrak{X}$ to $\mathfrak{X}$) satisfy the linearized equations:

$$\theta_t(x) = U_t + \int_0^t U_{t-s} DB(F_s(x)) \cdot \theta_s(x) \, ds.$$

$\theta_t(x)$ is defined as long as $F_t(x)$ is defined. It is easy to check that $t \mapsto \theta_t(x)$ is continuous in the strong operator topology and that (for fixed t), $x \mapsto \theta_t(x)$ is norm continuous. We claim that $DF_t(x) = \theta_t(x)$, which will thus prove F_t is C^1. Let

$$\lambda_t(x, h) = \| F_t(x + h) - F_t(x) - \theta_t(x) \cdot h \|.$$

Then

$$\lambda_t(x,h) = \left\| \int_0^t U_{t-s}\{B(F_s(x + h)) - B(F_s(x)) - DB(F_s(x)) \cdot \theta_s(x) \cdot h\} \, ds \right\|$$

$$\leqq M \exp(\beta |t|) \left\{ \int_0^t \| B(F_s(x+h)) - B(F_s(x)) - DB(F_s(x)) \cdot [F_s(x+h) - F_s(x) \| d \right.$$

$$\left. + \int_0^t \| DB(F_s(x)) \cdot [F_s(x + h) - F_s(x) - \theta_s(x) \cdot h] \| \, ds \right\}.$$

Thus, given $\epsilon > 0$, there is a $\delta > 0$ such that $\| h \| < \delta$ implies

$$\lambda_t(x, h) \leqq (\text{const.}) \cdot \left\{ \| h \| \epsilon + \int_0^t \lambda_s(x, h) \, ds \right\}.$$

Hence (by Gronwall's inequality), $\lambda_t(x, h) \leqq C(t) \, \| h \| \, \epsilon$. Hence, by definition of the derivative, $DF_t(x) = \theta_t(x)$. Thus F_t is C^1. An induction argument can be used to show F_t is C^k.

Now we prove that F_t maps $\mathfrak{D}(A)$ to $\mathfrak{D}(A)$ and

$$\frac{d}{dt} F_t(u_0) = G(F_t(u_0))$$

is continuous in t. Let $u_0 \in \mathfrak{D}(A)$. Then, setting $u(t) = F_t(u_0)$, (3) gives

$$\frac{1}{h}[u(t + h) - u(t)] = \frac{1}{h}(U_{t+h}u_0 - U_t u_0) + \frac{1}{h}\int_0^t (U_{t+h-s} - U_{t-s}) B(u(s)) \, ds$$

$$+ \frac{1}{h}\int_0^{t+h} U_{t+h-s} B(u(s)) \, ds$$

$$= \frac{1}{h}[U_h(u(t)) - u(t)] + \frac{1}{h}\int_t^{t+h} U_{t+h-s} B(u(s)) \, ds. \tag{4}$$

The second term approaches $B(u(t))$ as $h \longrightarrow 0$. Indeed,

$$\left\| \frac{1}{h}\int_t^{t+h} U_{t+h-s} B(u(s)) \, ds - B(u(t)) \right\| \leqq \frac{1}{h}\int_t^{t+h} \| U_{t+h-s} B(u(s)) - B(u(t)) \| \, ds$$

$$\leqq \frac{1}{h}\int_t^{t+h} \| U_{t+h-s} B(u(s)) - U_{t+h-s} B(u(t)) \| \, ds$$

$$+ \frac{1}{h}\int_t^{t+h} \| U_{t+h-s} B(u(t)) - B(u(t)) \| \, ds$$

$$\leqq \frac{1}{h} \cdot C(t) \cdot \int_t^{t+h} \| B(u(s)) - B(u(t)) \| \, ds$$
$$+ \frac{1}{h} \int_t^{t+h} \| U_{t+h-s} B(u(t)) - B(u(t)) \| \, ds$$

and each term $\longrightarrow 0$ as $h \longrightarrow 0$.

It follows that $F_t(u_0)$ is right differentiable at $t = 0$ and has derivative $G(u_0)$. To establish the formula at $t \neq 0$ we first prove that $u(t) \in \mathfrak{D}(A)$. But

$$\frac{1}{h}(F_{t+h}u_0 - F_t u_0) = \frac{1}{h}(F_t F_h u_0 - F_t u_0)$$

has a limit as $h \longrightarrow 0$ since F_t is of class C^1. Hence, from (4),

$$\frac{1}{h}[U_h(u(t)) - u(t)]$$

has a limit as $h \longrightarrow 0$. Thus, $u(t) \in \mathfrak{D}(A)$. It follows that

$$\frac{d}{dt} F_t(u_0) = G(F_t(u_0)) = DF_t(u_0) \cdot G(u_0).$$

Since the right derivative is continuous, the ordinary (two-sided) derivative exists as well. ∎

Next we give a criterion for global existence.

5.2 Proposition *Let the hypotheses of 5.1 hold. Suppose $u(t)$ is a solution of (3) defined for $0 \leq t < T$ and that $\| B(u(t)) \|$ is an integrable function of t on $[0, T]$. Then $u(t)$ can be continued to a solution for $0 \leq t \leq T + \epsilon$ for $\epsilon > 0$.*

Proof For $0 \leq t, s < T$, we have from (3),

$$\| u(t) - u(s) \| \leq \| e^{tA} u_0 - e^{sA} u_0 \| + \left\| \int_s^t e^{(t-\tau)A} B(u(\tau)) \right\| d\tau$$
$$\leq \| e^{tA} u_0 - e^{sA} u_0 \| + M e^{\beta T} \int_s^t \| B(u(\tau)) \| \, d\tau$$

Thus $u(t)$ is a Cauchy sequence as $t \longrightarrow T$, so converges, to, say u_T. The local existence theory applied in a neighborhood of u_T shows the time of existence is uniformly bounded away from zero as $t \longrightarrow T$; so the result follows. ∎

Here is an example of how this works.

5.3 Corollary *Suppose B satisfies $\| B(u) \| \leq C + K \| u \|$ for constants C and K. Then the semiflow F_t is global—that is, is defined for all $t \geq 0$.*

Proof Fix $T > 0$ and let $u(t)$ solve (3) for $0 \leq t < T$. Then

$$\| u(t) \| \leq M e^{\beta T} \| u_0 \| + \int_0^t M e^{\beta T} (C + K \| u(\tau) \|) \, d\tau.$$

From Grownwall's inequality (Problem 5.2),

$$\|u(t)\| \leq [Me^{\beta T}\|u_0\| + CT]\exp(KMe^{\beta T})$$

so $\|u(t)\|$ and hence $\|B(u(t))\|$ are bounded for $0 \leq t < T$. Thus u can be extended beyond T by 5.2. Hence u is defined for all $t \geq 0$. ▮

Problem 5.3 Prove a global existence theorem if

$$\|B(t, u)\| \leq p(t)[C + K\|u\|],$$

where $p(t)$ is L^1 on every finite interval.

Another way of obtaining global existence is through energy estimates. That is, if the equations are Hamiltonian (or are related to a Hamiltonian system), there may be a conserved (or decreasing) energy function that can be used to obtain the needed estimate in 5.2. We give an example in the following:

5.4 Corollary *Suppose the conditions of Theorem 5.1 hold. Suppose, moreover, that there is a C^1 function $H: \mathfrak{X} \longrightarrow \mathbb{R}$ such that:*

(i) *there is a monotone increasing function $\phi: [a, \infty) \longrightarrow [0, \infty)$, where $[a, \infty) \supset$ Range H, satisfying $\|x\| \leq \phi(H(x))$;*
(ii) *there is a constant $K \geqq 0$ such that if $u(t)$ satisfies (3), then*

$$\frac{d}{dt} H(u(t)) \leq KH(u(t)).$$

Then $F_t(u_0)$ is defined for all $t \geq 0$ and $u_0 \in \mathfrak{X}$; that is, the semiflow is global.

If, in addition, H is bounded on bounded sets and

(iii)
$$\frac{d}{dt} H(u(t)) \leq 0 \quad \text{if} \quad \|u(t)\| \geq B,$$

then any solution of (3) remains uniformly bounded in $\mathfrak{X}$ for all time; that is, given $u_0 \in \mathfrak{D}(A)$, there is a constant $C = C(u_0)$ such that $\|u(t)\| \leqq C$ for all $t \geqq 0$.

Proof By (ii), $H(u(t)) \leq H(u_0)\exp(Kt)$ so by (i), $\|u(t)\| \leq \phi(H(u_0)\exp(Kt))$. Thus, global existence follows by 5.2. Let $H_B = \sup\{H(u) \mid \|u\| \leq B\}$, so, by (iii), $H(u(t)) \leq \max\{H(u_0), H_B\}$. Hence, by (i), we can take $C = \phi(\max\{H(u_0), H_B\})$. ▮

Next we have a criterion for asymptotic stability of a fixed point x_0; that is, $F_t(x_0) = x_0$ and $F_t(x) \longrightarrow x_0$ as $t \longrightarrow +\infty$ for all $x \in \mathfrak{X}$. (See Theorem 4.1 of Chapter 7 for the use of spectral methods to obtain a related result.)

5.5 Proposition *Let (i) and (ii) of 5.4 hold, and suppose $G(x_0) = 0$; that is, x_0 is a fixed point, and:*

(iii′) *There is a continuous monotone function $f: [0, \infty) \longrightarrow [0, \infty)$, locally Lipschitz on $(0, \infty)$ such that*

(a) $\frac{dH}{dt}(F_t(x)) \le -f(H(F_t(x))$ *for* $x \in U$, *a neighborhood of* x_0,

and

(b) *solutions of* $\dot{r} = -f(r)$ *tend to zero as* $t \longrightarrow +\infty$.

(iv) $H(x) \ge 0$ *and there is a strictly monotone continuous function* $\psi\colon [0, \infty) \longrightarrow [0, \infty)$ *such that*

$$\| x - x_0 \| \le \psi(H(x)).$$

Then x_0 *is asymptotically stable.*

Proof Let $r(t)$ be the solution of $\dot{r} = -f(r)$ with $r(0) = H(x)$, $x \in U, x \ne x_0$. Then, by (iii′), $H(F_t(x)) \le r(t)$. Hence $H(F_t(x)) \longrightarrow 0$ as $t \longrightarrow +\infty$. Thus by (iv), $F_t(x)$ converges to x_0 as $t \longrightarrow +\infty$. ∎

> ***Problem 5.4*** Consider $\ddot{u} + \dot{u} + u^3 = 0$. Show that solutions decay to zero like $C/\sqrt{t}$ as $t \longrightarrow +\infty$, by considering $H(u, \dot{u}) = (u + \dot{u})^2 + \dot{u}^2 + u^4$. (See Ball and Carr [1976] for more information.)

The following is a specific situation relevant for some semilinear wave equations. See 3.10 in Box 3.1, Section 6.3.

5.6 Proposition *Let* $\mathcal{H}$ *be a real Hilbert space and* B *a self-adjoint operator in* $\mathcal{H}$ *with* $B \ge c > 0$. *On* $\mathfrak{X} = \mathfrak{D}(B^{1/2}) \times \mathcal{H}$, *let* $A = \begin{bmatrix} 0 & I \\ -B & 0 \end{bmatrix}$ *with domain* $\mathfrak{D}(A) = \mathfrak{D}(B) \times \mathfrak{D}(B^{1/2})$. *Let* $V\colon \mathfrak{D}(B^{1/2}) \longrightarrow \mathbb{R}$ *be a smooth function and suppose* V *has a smooth gradient; that is,* $\nabla V\colon \mathfrak{D}(B^{1/2}) \longrightarrow \mathcal{H}$ *is smooth and satisfies*

$$\langle \nabla V(x), y \rangle = dV(x) \cdot y$$

for all $x, y \in \mathfrak{D}(B^{1/2})$. *Suppose* $V \ge 0$ *and* V *is bounded on bounded sets in* $\mathfrak{D}(B^{1/2})$. *Let*

$$G\colon \mathfrak{D}(A) \longrightarrow \mathfrak{X}, \textit{ defined by } G(u, \dot{u}) = A(u, \dot{u}) + (0, \nabla V(u)).$$

Then the flow of G *is globally defined and solutions are uniformly bounded for all time.*

Proof Clearly G satisfies the conditions of 5.1. Also, G is Hamiltonian with energy

$$H(u, u) = \tfrac{1}{2}\| u \|^2 + \tfrac{1}{2}\langle B^{1/2}u, B^{1/2}u \rangle + V(u)$$

and H is conserved along solutions. Since $V \ge 0$,

$$H(u, u) \ge \tfrac{1}{2}(\| u \|^2 + \| B^{1/2}u \|^2).$$

The result therefore follows by 5.4. ∎

> ***Problem 5.5*** Instead of $V \ge 0$, assume $V(0) = 0$, $DV(0) = 0$ and $D^2V(0) = 0$. Show that in a neighborhood of (0, 0), solutions are globally defined and remain in a ball about (0, 0). (*Hint*: Apply Taylor's theorem to

H to show that in a neighborhood of $(0, 0)$, $C_1\|(u, \dot{u})\|^2 \leq H(u, \dot{u}) \leq C_2\|(u, \dot{u})\|^2$ for constants C_1 and C_2.)

Problem 5.6 (*Regularity*) Let 5.1 hold and suppose that $B\colon \mathfrak{D}(A^{k-l}) \to \mathfrak{D}(A^{k-l})$ is C^{l-1}, $l = 1, \ldots, k-1$, where $\mathfrak{D}(A^{k-l})$ has the graph norm. Prove that F_t maps $\mathfrak{D}(A^{k-l})$ to itself and is C^{k-l}.

Problem 5.7 (*Singular Case*) Let A generate a C^0 semigroup on $\mathfrak{X}$ and let $\mathfrak{D}(A)$ have the graph norm. Suppose $B\colon \mathfrak{D}(A) \to \mathfrak{X}$ is C^1 and the tangent of B, $TB\colon (u, v) \mapsto (u, DB(u)\cdot v)$ extends to a C^1 map of $\mathfrak{D}(A) \times \mathfrak{X}$ to $\mathfrak{D}(A) \times \mathfrak{X}$. Then $G = A + B$ generates a unique local C^1 flow. (*Hint*: Apply 5.1 to the operator $TG(u, v) = (0, Av) + TB(u, v)$; the first component of this flow gives the flow of G; cf. Segal [1962].)

We will now give two applications of this theory. The first is to semilinear non-linear wave equations and the second is to a problem of panel flutter. The first is perhaps not directly relevant to elasticity, but it is a topic of current interest and illustrates the methods well. (See Reed [1976] for similar results from a different point of view. For a recent spectacular application of the semilinear theory, see Eardley and Moncrief [1982].)

5.7 Example (Semilinear Wave Equation) We consider the following equation for $\phi(x, t) \in \mathbb{R}$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$\left.\begin{aligned} &\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi - m^2\phi - g\phi^p, \\ &\phi(x, 0) \text{ and } \dot{\phi}(x, 0) \text{ given} \end{aligned}\right\} \tag{5}$$

where $p \geq 2$ is an integer and $g \in \mathbb{R}$ is a constant. (One can also consider this problem on complete Riemannian manifolds, but we stick to $\mathbb{R}^n$ for simplicity.) To obtain the results, essential use is made of the Sobolev spaces $H^s(\mathbb{R}^n)$ and the Sobolev inequalities. To obtain global solutions, these inequalities must be applied with care. The relevant facts we need were given in Box 1.1, Section 6.1.

The results for Equation (5) are summarized as follows:

(i) *For $n = 1, 2$, $m \geq 0$, $g \geq 0$, p an odd integer, and $s \geq 1$, Equation* (5) *has unique global solutions in $H^{s+1}(\mathbb{R}^n, \mathbb{R}) \times H^s(\mathbb{R}^n, \mathbb{R})$ for any initial data in this space.*

(ii) *For $n = 3$, $m \geq 0$, $g \geq 0$, and $p = 3$, the conclusion* (*i*) *remains valid.*

(iii) *For $n = 1, 2$, $m \geq 0$, any g, and any p, there are unique global solutions as in* (*i*) *if the initial data is small enough in $H^1 \times L^2$ norm. The same holds for $n = 3$ if $p \leq 4$.*

(iv) *For any n, m, g, p, there are unique solutions local in time in $H^{s+1}(\mathbb{R}^n, \mathbb{R}) \times H^s(\mathbb{R}^n, \mathbb{R})$ if $s + 1 > n/2$.*

Note: From hyperbolicity arguments (Courant and Hilbert [1962]) it follows that if we start with C^∞ data with compact support, the solution will be C^∞ with compact support as well.

To establish (i) and (ii), we use Proposition 5.6 (with a slight modification if $m = 0$), and to prove (iii) we use Problem 5.5, and to prove (iv) we use Theorem 5.1. We shall give the details for case (ii).

First, suppose $m > 0$. Then as was discussed in Section 6.3 the Klein–Gordon operator

$$A = \begin{bmatrix} 0 & I \\ \Delta - m^2 & 0 \end{bmatrix}$$

is skew-adjoint on $H^1 \times L^2$ with the energy norm, which is equivalent to the usual Sobolev norm. We set

$$V: H^1 \longrightarrow \mathbb{R}, \qquad V(\phi) = \frac{g}{4} \int_{\mathbb{R}^3} \phi^4 \, dx.$$

Here is the key fact:

Lemma 1 *V is a well-defined smooth map with derivative*

$$DV(\phi)\cdot\psi = g \int_{\mathbb{R}^3} \phi^3 \psi \, dx.$$

Proof From the Sobolev–Nirenberg–Gagliardo inequality (see Box 1.1), there is a positive constant C such that for $\phi \in L^2(\mathbb{R}^3)$,

$$\|\phi\|_{L^p} \le C \|\nabla\phi\|_{L^2}^{a} \|\phi\|_{L^2}^{1-a}$$

for $2 \le p \le 6$ and $a = 3\cdot(\frac{1}{2} - 1/p)$. In particular, we have the estimate

$$\|\phi\|_{L^p} \le C \|\phi\|_{H^1}$$

for $2 \le p \le 6$. The main case, $p = 6$, was proved in Proposition 1.16. Taking $p = 4$ we see that V is well defined. Consider

$$\mathcal{V}: H^1 \times H^1 \times H^1 \times H^1 \longrightarrow \mathbb{R}, \qquad \mathcal{V}(\phi_1, \phi_2, \phi_3, \phi_4) = \int_{\mathbb{R}^3} \left(\prod_{i=1}^{4} \phi_i\right) dx.$$

From the Schwartz inequality,

$$\|\mathcal{V}(\phi_1, \phi_2, \phi_3, \phi_4)\| \le \prod_{i=1}^{4} \|\phi_i\|_{L^4} \le C \prod_{i=1}^{4} \|\phi_i\|_{H^1}.$$

Hence $\mathcal{V}$ is a continuous multilinear map; so is C^∞. Hence V is as well, and so the lemma follows. ▮

Lemma 2 *Set $Y: H^1 \longrightarrow L^2$; $Y(\phi) = (-g\phi^3)$. Then Y is a C^∞ map and the conditions of Proposition 5.6 hold, with A as above, $\mathcal{H} = L^2$, and $\nabla V = Y$.*

Proof Since $L^6 \subset H^1$, Y is well defined. Consider $\mathfrak{Y}: H^1 \times H^1 \times H^1 \longrightarrow L^2$, $\mathfrak{Y}(\phi_1, \phi_2, \phi_3) = \prod_{i=1}^{3} \phi_i$. Then by Hölder's inequality,

$$\|\mathfrak{Y}(\phi_1, \phi_2, \phi_3)\|_{L^2} \le \prod_{i=1}^{3} \|\phi_i\|_{L^6} \le C \prod_{i=1}^{3} \|\phi_i\|_{H^1}.$$

Thus $\mathfrak{Y}$ and hence Y is smooth.

A straightforward calculation shows that $\nabla V = Y$. ▮

That solutions which start in $H^{s+1} \times H^s$ for $s \geq 2$ stay in that space follows by regularity (Problem 5.6). We ask the reader to supply the details.

The case $m = 0$ may be dealt with as follows. Using the usual norm on $H^1 \times L^2$, conservation of energy and $V \geq 0$ implies that $\|\nabla\phi\|_{L^2} + \|\dot{\phi}\|_{L^2}$ is bounded, say, by M. But this implies that $\|\nabla\phi\|_{L^2} + \|\phi\|_{L^2} + \|\dot{\phi}\|_{L^2} \leq M + tM\|\phi_0\|_{L^2}$, so the $(H^1 \times L^2)$-norm is bounded on finite t-intervals. Thus we again get our result.

The result (iv) is perhaps the easiest of all. Indeed, for $s > n/2$, $H^s(\mathbb{R}^n, \mathbb{R})$ is a ring: $\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}$, so multiplication is smooth. (See 1.17 in Box 6.1.1.) Therefore, we can apply Theorem 5.1 directly because any polynomial Y on H^s will be smooth.

5.8 Example (Panel Flutter) The linear problem was considered in Box 3.2. (See Figure 6.3.1.) Here we consider the nonlinear problem. The equations are

$$\alpha\dot{v}'''' + v'''' - \left\{\Gamma + \kappa\int_0^1 (v'(t,\xi))^2\,d\xi + \sigma\int_0^1 (v'(t,\xi)\dot{v}'(t,\xi))\,d\xi\right\}v''$$
$$+ \rho v' + \sqrt{\rho}\,\delta\dot{v} + \ddot{v} = 0. \tag{6}$$

(See Dowell [1975] and Holmes [1977a].) As in Box 3.1, $\dot{} = \partial/\partial t$, $' = \partial/\partial x$, and we have included viscoelastic structural damping terms α, σ as well as aerodynamic damping $\sqrt{\rho}\,\delta$; κ represents nonlinear (membrane) stiffness, ρ the dynamic pressure, and Γ an in-plane tensile load. We have boundary conditions at $x = 0, 1$ that might typically be simply supported ($v = (\dot{v} + \alpha v)'' = 0$) or clamped ($v = v' = 0$). To be specific, let us choose the simply supported condition.

To proceed with the methods above, we first write (6) in the form (2), choosing as our basic space $\mathfrak{X} = H^2_\partial([0, 1]) \times L^2([0, 1])$, where H^2_∂ denotes H^2 functions in $[0, 1]$ that vanish at 0, 1. Set $\|(v, \dot{v})\|_{\mathfrak{X}} = (\|\dot{v}\|^2 + \|v''\|^2)^{1/2}$, where $\|\cdot\|$ denotes the usual L^2-norm. This is equivalent to the usual norm because of the boundary conditions. In fact, the two Poincaré-type inequalities $\|v'\|^2 \geq \pi^2\|v\|^2$ and $\|v''\| \geq \pi^4\|v\|^2$ may be checked using Fourier series. Define

$$A = \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}, \quad \text{where } \begin{cases} Cv = -v'''' + \Gamma v'' - \rho v', \\ D\dot{v} = -\alpha\dot{v}'''' - \sqrt{\rho}\,\delta\dot{v}. \end{cases} \tag{7}$$

The domain $\mathfrak{D}(A)$ of A consists of all pairs $(v, \dot{v}) \in \mathfrak{X}$ such that $\dot{v} \in H^2_\partial$, $v + \alpha\dot{v} \in H^4$, and $v'' + \alpha\dot{v}'' = 0$ at $x = 0, 1$. Define the nonlinear operator $B(v, \dot{v}) = (0, [\kappa\|v'\|^2 + \sigma\langle v', \dot{v}'\rangle]v'')$, where $\langle\cdot\rangle$ denotes the L^2 inner product; so (6) can be rewritten as

$$\frac{dx}{dt} = Ax + B(x) \equiv G(x), \quad \text{where } x = (v, \dot{v}) \text{ and } x(t) \in \mathfrak{D}(A). \tag{8}$$

By Proposition 3.11 in Box 3.2, Section 6.2, A generates a semigroup on $\mathfrak{X}$. In one dimension H^1 forms a ring; so as in the previous example, $B\colon \mathfrak{X} \longrightarrow \mathfrak{X}$ is a C^∞ mapping bounded on bounded sets. Thus, by Theorem 5.1, Equations (6) generate a local semiflow on $\mathfrak{X}$. We claim that this semiflow is global. To see this, we temporarily omit the dissipative terms in (6), namely, we consider

$$v'''' - \left\{\Gamma + \kappa \int_0^1 (v'(t, \xi))^2 \, d\xi\right\} v'' + \ddot{v} = 0.$$

As in Problem 3.6, Section 6.3, this is Hamiltonian with

$$H(v, \dot{v}) = \tfrac{1}{2}\|\dot{v}\|^2 + \tfrac{1}{2}\|v''\|^2 + \frac{\Gamma}{2}\|v'\|^2 + \frac{\kappa}{4}\|v'\|^4.$$

For the full equation (6) we find by a simple calculation that

$$\frac{dH}{dt} = -\rho\langle v', \dot{v}\rangle - \sqrt{\rho}\,\delta\|\dot{v}\|^2 - \alpha\|\dot{v}''\|^2 - \sigma\|(v, \dot{v})\|^2.$$
$$\leq -\rho\langle v', \dot{v}\rangle \leq \rho\|v'\|\|\dot{v}\| \leq \tfrac{1}{2}\rho(\|v'\|^2 + \|\dot{v}\|^2).$$

For $\|v'\|$ large enough,

$$H(v, \dot{v}) \geq C\|(v, \dot{v})\|_X^2.$$

[Note that Γ can be ≤ 0; write

$$\frac{\Gamma}{2}\|v'\|^2 + \frac{\kappa}{4}\|v'\|^4 = \frac{\kappa}{4}\|v'\|^2\left\{\|v'\|^2 + \frac{2\Gamma}{\kappa}\right\}$$

to see this.] Thus, for $\|v'\|$ large, hypotheses (i) and (ii) of 5.4 are satisfied, and so we have global solutions. We shall return to this example again in Chapter 7.

Problem 5.8 (Parks [1966], Holmes and Marsden [1978a]) Let $\tilde{H}(v, \dot{v}) = \frac{1}{2}\{\sqrt{\rho}\,\delta\|v\|^2 + \alpha\|v''\|^2 + 2\langle v, \dot{v}\rangle + (\sigma/2)\|v'\|^4\}$ and let $\mathcal{H}(v, \dot{v}) = H(v, \dot{v}) + \nu\tilde{H}(v, \dot{v})$, where $\nu = (\sqrt{\rho}\,\delta + \alpha\pi^4)/2$. If $\rho^2 < (\sqrt{\rho}\,\delta + \alpha\pi^4)^2 \times (\Gamma + \rho^2)$, show that $d\mathcal{H}/dt < 0$ along non-zero solutions. Use 5.5 to show that in these circumstances solutions tend to $(0, 0)$ as $t \to +\infty$.

While the above theory does apply to a number of special situations in elasticity involving rod and plate approximations, it does not apply to the "full" theory of nonlinear elasticity, even in one dimension. Here the equations have the form

$$\dot{u} = A(t, u)u + f(t, u) \qquad (0 \leq t \leq T, \quad u(0) = u_0), \tag{9}$$

the point being that the linear operator A depends on u. For example, one-dimensional elasticity has this form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} 0 & I \\ \sigma'(\phi_x)\dfrac{\partial^2}{\partial x^2} & 0\end{pmatrix}\begin{pmatrix}\phi \\ \dot{\phi}\end{pmatrix}.$$

Such equations in which the highest derivatives occur linearly, but possibly multiplied by functions of lower derivatives, are called *quasi-linear*. The remainder of this section discusses the theory for these equations. This theory began with Schauder [1935] with contributions by Petrovskii [1937], Sobolev [1939], Choquet-Bruhat [1952], and many others. However, most of these treatments had a few loose ends, and none proved the continuous dependence on initial data (in the same space Y). We shall follow the formulations of Kato [1975b], [1977], and of Hughes, Kato, and Marsden [1977].

The abstract theory for (9) divides into the local theory and the global theory. We begin with a discussion of the local theory. Proofs will be omitted as they are rather technical.

We start from four (real) Banach spaces

$$\mathcal{Y} \subset \mathfrak{X} \subset \mathcal{Z}' \subset \mathcal{Z},$$

with all the spaces reflexive and separable and the inclusions continuous and dense. We assume that

(Z′) $\mathcal{Z}'$ is an interpolation space between $\mathcal{Y}$ and $\mathcal{Z}$; thus if $U \in \mathcal{B}(\mathcal{Y}) \cap \mathcal{B}(\mathcal{Z})$, then $U \in B(\mathcal{Z}')$ with $\|U\|_{\mathcal{Z}'} \leq c \max\{\|U\|_{\mathcal{Y}}, \|U\|_{\mathcal{Z}}\}$; $\mathcal{B}(\mathcal{Y})$ denotes the set of bounded operators on $\mathcal{Y}$.

Let $\mathfrak{N}(\mathcal{Z})$ be the set of all norms in $\mathcal{Z}$ equivalent to the given one $\|\ \|_{\mathcal{Z}}$. Then $\mathfrak{N}(\mathcal{Z})$ is a metric space with the distance function

$$d(\|\ \|_\mu, \|\ \|_\nu) = \log \max\{\sup_{0 \neq z \in \mathcal{Z}} \|z\|_\mu/\|z\|_\nu, \sup_{0 \neq z \in \mathcal{Z}} \|z\|_\nu/\|z\|_\mu\}.$$

We now introduce four functions, A, N, S, and f on $[0, T] \times \mathcal{W}$, where $T > 0$ and $\mathcal{W}$ is an open set in $\mathcal{Y}$, with the following properties:

For all $t, t', \ldots, \in [0, T]$ and all $w, w', \ldots \in \mathcal{W}$, there is a real number β and there are positive numbers $\lambda_N, \mu_N, \ldots$ such that the following conditions hold:

(N) $N(t, w) \in \mathfrak{N}(\mathcal{Z})$, with

$$d(N(t, w), \|\ \|_{\mathcal{Z}}) \leq \lambda_N,$$
$$d(N(t', w'), N(t, w)) \leq \mu_N(|t' - t| + \|w' - w\|_{\mathfrak{X}}).$$

(S) $S(t, w)$ is an isomorphism of $\mathcal{Y}$ onto $\mathcal{Z}$, with

$$\|S(t, w)\|_{\mathcal{Y},\mathcal{Z}} \leq \lambda_S, \quad \|S(t, w)^{-1}\|_{\mathcal{Z},\mathcal{Y}} \leq \lambda_s',$$
$$\|S(t', w') - S(t, w)\|_{\mathcal{Y},\mathcal{Z}} \leq \mu_S(|t' - t| + \|w' - w\|_{\mathfrak{X}}).$$

(A1) $A(t, w) \in G(\mathcal{Z}_{N(t,w)}\ 1, \beta)$, where $\mathcal{Z}_{N(t,w)}$ denotes the Banach space $\mathcal{Z}$ with norm $N(t, w)$. This means that $A(t, w)$ is a C_0-generator in $\mathcal{Z}$ such that $\|e^{\tau A(t,w)} z\| \leq e^{\beta\tau}\|z\|$ for all $\tau \geq 0$ and $z \in \mathcal{Z}$.

(A2) $S(t, w)A(t, w)S(t, w)^{-1} = A(t, w) + B(t, w)$, where $B(t, w) \in \mathcal{B}(\mathcal{Z})$, $\|B(t, w)\|_{\mathcal{Z}} \leq \lambda_B$.

(A3) $A(t, w) \in \mathcal{B}(\mathcal{Y}, \mathfrak{X})$, with $\|A(t, w)\|_{\mathcal{Y},\mathfrak{X}} \leq \lambda_A$ and $\|A(t, w') - A(t, w)\|_{\mathcal{Y},\mathcal{Z}'} \leq \mu_A\|w' - w\|_{\mathcal{Z}'}$ and with $t \mapsto A(t, w) \in \mathcal{B}(\mathcal{Y}, \mathcal{Z})$ continuous in norm.

(f1) $f(t, w) \in \mathcal{Y}$, $\|f(t, w)\|_{\mathcal{Y}} \leq \lambda_f$, $\|f(t, w') - f(t, w)\|_{\mathcal{Z}'} \leq \mu_f\|w' - w\|_{\mathcal{Z}}$, and $t \mapsto f(t, w) \in \mathcal{Z}$ is continuous.

Remarks (i) If $N(t, w) = \text{const.} = \| \ \|_{\mathfrak{Z}}$, condition (N) is redundant. If $S(t, w) = \text{const.} = S$, condition (S) is trivial. If both are assumed, and $\mathfrak{X} = \mathfrak{Z}' = \mathfrak{Z}$, we have the case of Kato [1975b].

(ii) In most applications we can choose $\mathfrak{Z}' = \mathfrak{Z}$ and/or $\mathfrak{Z}' = \mathfrak{X}$.

(iii) The paper of Hughes, Kato, and Marsden [1977] had an additional condition (A4) that was then shown to be redundant in Kato [1977].

(iv) The possibility of dropping (S) and related refinements are discussed in Graff [1981] and Altman [1982].

5.9 Theorem *Let conditions* (Z′), (N), (S), (A1) *to* (A3), *and* (f1) *be satified. Then there are positive constants* ρ' *and* $T' \leq T$ *such that if* $u_0 \in \mathcal{Y}$ *with* $\| u_0 - y_0 \|_{\mathcal{Y}} \leq \rho'$, *then* (9) *has a unique solution* u *on* $[0, T']$ *with*

$$u \in C^0([0, T']; \mathcal{W}) \cap C^1([0, T']; \mathfrak{X}).$$

Here ρ' depends only on λ_N, λ_S, λ'_S, and $R = \operatorname{dist}(y_0, \mathcal{Y}\backslash\mathcal{W})$, while T' may depend on all the constants β, λ_N, μ_N, . . . and R. When u_0 varies in $\mathcal{Y}$ subject to $\| u_0 - y_0 \|_{\mathcal{Y}} \leq \rho'$, the map $u_0 \mapsto u(t)$ is Lipschitz continuous in the $\mathfrak{Z}'$-norm, uniformly in $t \in [0, T']$.

To establish well-posedness, we have to strengthen some of the assumptions. We assume the following conditions:

(B) $\| B(t, w') - B(t, w) \|_{\mathfrak{Z}} \leq \mu_B \| w' - w \|_{\mathcal{Y}}$.
(f2) $\| f(t, w') - f(t, w) \|_{\mathcal{Y}} \leq \mu'_f \| w' - w \|_{\mathcal{Y}}$.

5.10 Theorem *Let conditions* (Z′), (N), (S), (A1) *to* (A3), (B), (f1), *and* (f2) *be satisfied, where* $S(t, w)$ *is assumed to be independent of* w. *Then there is a positive constant* $T'' \leq T'$ *such that when* u_0 *varies in* $\mathcal{Y}$ *subject to* $\| u_0 - y_0 \|_{\mathcal{Y}} \leq \rho'$, *the map* $u_0 \mapsto u(t)$ *given by Theorem* 5.9 *is continuous in the* $\mathcal{Y}$*-norm, uniformly in* $t \in [0, T'']$.

As in Kato [1975b], one can prove a similar continuity theorem when not only the initial value u_0 but also the functions N, A, and f are varied; that is, the solution is "stable" when the equations themselves are varied. Variation of S is discussed in Graff [1981]. In summary, Theorem 5.10 guarantees the existence of (locally defined) evolution operators $F_{t,s}\colon \mathcal{Y} \longrightarrow \mathcal{Y}$ that are continuous in all variables.

The idea behind the proofs of the Theorems 5.9 and 5.10 is to fix a curve $v(t)$, in $\mathcal{Y}$ satisfing $v(0) = u_0$ and let $u(t)$ be the solution of the "frozen coefficient problem"

$$\dot{u} = A(t, v)u + f(t, v), \qquad u(0) = u_0,$$

which is guaranteed by linear theory (Kato [1970], [1973]). This defines a map $\Phi\colon v \mapsto u$ and we look for a fixed point of Φ. In a suitable function space and for T' sufficiently small, Φ is in fact a contraction, so has a unique fixed point.

However, it is not so simple to prove that u depends continuously on u_0 and detailed estimates from the linear theory are needed. The proof more or less has to be delicate since the dependence on u_0 is not locally Lipschitz in general. For details of these proofs, we refer to Kato [1975b], [1977] and Hughes, Kato, and Marsden [1977]. The continuous dependence of the solution on u_0 leads us to investigate if it is smooth in any sense. This is important in Hamiltonian systems, as we saw in Proposition 3.4, Chapter 5. This is explored in Box 5.1 below.

Next we give some specific systems to which 5.10 applies. These will be nonlinear versions of first-order symmetric hyperbolic systems and second-order hyperbolic systems. Each of these applies to nonlinear elasticity; to verify the hypotheses (details of which are omitted) one uses the linear theory presented in Theorems 3.1 (for the second-order version) and 3.19 of Section 6.3. We shall treat these equations in all of $\mathbb{R}^m$ as the machinery needed for boundary value problems is yet more complex (see Kato [1977] and Box 5.2 below).

5.11 Example (Quasi-linear Symmetric Hyperbolic Systems) Consider the equation

$$\boldsymbol{a}_0(t, x, \boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial t} = \sum_{j=1}^{m} \boldsymbol{a}_j(t, x, \boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x^j} + \boldsymbol{a}(t, x, \boldsymbol{u}). \tag{10}$$

for $x \in \mathbb{R}^m$, $t \in \mathbb{R}$, $\boldsymbol{u}(t, x) \in \mathbb{R}^N$, and $\boldsymbol{a}_j$, $\boldsymbol{a}$ real $N \times N$ matrices. We assume:

(i) $s > \frac{1}{2}m + 1$ and $\boldsymbol{a}_0$, $\boldsymbol{a}$ are of class C^{s+1} in the variables $t, x, \boldsymbol{u}$ (possibly locally defined in $\boldsymbol{u}$);

(ii) there are constant matrices $\boldsymbol{a}_j^\infty$, $\boldsymbol{a}^\infty$ such that

$$\boldsymbol{a}_j - \boldsymbol{a}_j^\infty,\ \boldsymbol{a} - \boldsymbol{a}^\infty \in C([0, T], H^0(\mathbb{R}^m) \cap L^\infty([0, T], H^s(\mathbb{R}^m)) \qquad (j = 0, 1, \ldots, m),$$

$$\boldsymbol{a}_0 - \boldsymbol{a}_0^\infty \in \mathrm{Lip}([0, T], H^{s-1}(\mathbb{R}^m)),$$
locally uniformly in $\boldsymbol{u}$;

(iii) $\boldsymbol{a}_j$, $(j = 0, \ldots, m)$ are symmetric;

(iv) $\boldsymbol{a}_0(t_1 x, \boldsymbol{u}) \geq cI$ for some $c > 0$ for all x and locally in t, $\boldsymbol{u}$.

Under these conditions, the hypotheses of Theorem 5.10 hold with

$$\mathfrak{X} = H^{s-1}(\mathbb{R}^m), \qquad \mathcal{Y} = H^s(\mathbb{R}^m), \qquad \mathcal{Z} = \mathcal{Z}' = L^2(\mathbb{R}^m),$$

$$S = (1 - \Delta)^{s/2}, \qquad A = \boldsymbol{a}_0^{-1}\left(\sum_{j=1}^{m} \boldsymbol{a}_j \frac{\partial}{\partial x^j} + \boldsymbol{a}\right).$$

Thus (10) generates a unique continuous evolution system $F_{t,s}$ on $\mathcal{Y}$.

5.12 Example (Second-order Quasi-linear Hyperbolic Systems) Consider the equation

$$\boldsymbol{a}_{00}(t, s, \boldsymbol{u}, \nabla\boldsymbol{u})\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \sum_{i,j=1}^{m} \boldsymbol{a}_{ij}(t, x, \boldsymbol{u}, \nabla\boldsymbol{u})\frac{\partial^2 \boldsymbol{u}}{\partial x^i\, \partial x^j}$$
$$+ 2\sum_{i=1}^{m} \boldsymbol{a}_{0i}(t, x, \boldsymbol{u}, \nabla\boldsymbol{u})\frac{\partial^2 \boldsymbol{u}}{\partial t\, \partial x^i} + \boldsymbol{a}(t, x, \boldsymbol{u}, \nabla\boldsymbol{u}). \tag{11}$$

Here
$$\nabla \boldsymbol{u} = \left(\frac{\partial \boldsymbol{u}}{\partial x^1}, \ldots, \frac{\partial \boldsymbol{u}}{\partial x^m}, \frac{\partial \boldsymbol{u}}{\partial t}\right).$$

We assume:

(i) $s > (m/2) + 1$ and $\boldsymbol{a}_{\alpha\beta}$, $\boldsymbol{a}$ are of class C^{s+1} in all variables (possibly locally defined in $\boldsymbol{u}$);

(ii) there are constant matrices $\boldsymbol{a}^{\infty}_{\alpha\beta}$, $\boldsymbol{a}^{\infty}$ such that
$$\boldsymbol{a}_{\alpha\beta} - \boldsymbol{a}^{\infty}_{\alpha\beta},\ \boldsymbol{a} - \boldsymbol{a}^{\infty} \in \mathrm{Lip}([0, T];\ H^{s-1}(\mathbb{R}^m)) \subset L^{\infty}([0, T];\ H^{s}(\mathbb{R}^m)),$$
locally uniformly in $\boldsymbol{u}$;

(iii) $\boldsymbol{a}_{\alpha\beta}$ is symmetric;

(iv) $\boldsymbol{a}_{00}(t, x, \boldsymbol{u}) \geq c\boldsymbol{I}$ for some $c > 0$;

(v) strong ellipticity. There is an $\epsilon > 0$ such that
$$\sum_{i,j=1}^{m} \boldsymbol{a}_{ij}(t, x, \boldsymbol{u})\xi^i\xi^j \geq \epsilon\left(\sum_{j=1}^{m} [\xi^j]^2\right)$$
(a matrix inequality) for all $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$, $x \in \mathbb{R}^m$, and locally in t, $\boldsymbol{u}$.

Under these conditions, Theorem 5.10 holds with
$$\mathfrak{X} = H^s(\mathbb{R}^m) \times H^{s-1}(\mathbb{R}^m),$$
$$\mathfrak{Z} = Z' = H^1(\mathbb{R}^m) \times H^0(\mathbb{R}^m),$$
$$\mathfrak{Y} = H^{s+1}(\mathbb{R}^m) \times H^s(\mathbb{R}^m),$$
$$S = (1 - \Delta)^{s/2} \times (1 - \Delta)^{s/2},$$
$$A(t) = \left(\begin{array}{c|c} 0 & I \\ \hline \boldsymbol{a}_{00}^{-1}\left[\sum \boldsymbol{a}_{ij}\dfrac{\partial^2}{\partial x^i\, \partial x^j}\right] & \boldsymbol{a}_{00}^{-1}\left[2\sum \boldsymbol{a}_{0j}\dfrac{\partial}{\partial x^j}\right]\end{array}\right).$$
Thus (11) generates a unique continuous evolution system on $\mathfrak{Y}$.

From either 5.11 or 5.12 we conclude that the equations of nonlinear elastodynamics generate a unique continuous local evolution system on the space of ϕ, $\dot{\phi}$, which are sufficiently smooth; ϕ is at least C^2 and $\dot{\phi}$ is C^1.

There are several difficulties with theorems of this type: (A). The existence is only local in time. (B). The function spaces are too restrictive to allow shocks and other discontinuities.

With regard to (A), some global results for (11) have been proved by Klainermann [1978] (and simplified by Shatah [1982]) in four or higher dimensions for small initial data. However, these global Properties are not true in three dimensions (John [1979]). Much work has been done on (B), but the success is very limited. In fact, the problem in one dimension is not settled. Simple dissipative mechanisms sometimes relieve the situation, as is discussed below.

A few selected topics of current interest related to difficulties (A) and (B) are as follows:

1. Lax [1964] proved the non-existence of global smooth solutions to

$$u_{tt} = \sigma(u_x)_x \tag{12}$$

assuming that σ'' never vanishes. Equation (12) is studied by writing it as a system of conservation laws:

$$\begin{cases} w_t = v_x \\ v_t = \sigma(w)_x \end{cases}$$

where $w = u_x$ and $v = u_t$.

2. The assumption $\sigma'' \neq 0$ is unrealistic for nonlinear elasticity. Indeed it is reasonable that a stress-strain function $\sigma(u_x)$ should satisfy $\sigma(p) \longrightarrow +\infty$ as $p \longrightarrow +\infty$ and $\sigma(p) \longrightarrow -\infty$ as $p \longrightarrow 0+$. Strong ellipticity is the assertion that $\sigma'(p) \geq \epsilon > 0$. This does not imply that σ'' never vanishes. This hypothesis was overcome by MacCamy and Mizel [1967] under realistic conditions on σ for the displacement problem on [0, 1] (representing longitudinal displacements of a bar). These results have been improved by Klainermann and Majda [1980] who show that singularities develop in finite time for arbitrarily small initial data under rather general hypotheses. The results are still, however, one dimensional.

3. A general existence theorem for weak solutions of (12) was proved by Glimm [1965] and Dafermos [1973]. The crucial difficulty is in selecting out the "correct" solution by imposing an entropy condition. This problem has been recast into the framework of nonlinear contractive semigroups by several authors, such as Quinn [1971] and Crandall [1972].

Glimm constructed solutions in the class of functions of bounded variation by exploiting the classical solution to the Riemann problem for shocks. The proof that the scheme converges involves probabilistic considerations. DiPerna [1982], using ideas of Tartar [1979], obtains solutions in L^∞ as limits of solutions of a viscoelastic problem as the viscosity tends to zero. DiPerna's results involve acceptable hypotheses on the stress σ and impose correct entropy conditions on the solutions. (The solutions are not known to be of bounded variation).

4. Much has been done on equations of the form (12) with a dissipative mechanism added on. For example, viscoelastic-type dissipation is considered in MacCamy [1977], Matsumura and Nishida [1980] and Potier–Ferry [1980] and thermoeleastic dissipation is considered in Coleman and Gurtin [1965] and Slemrod [1981]. It is proved that smooth initial data with small norm has a global solution. Dafermos [1982], using the ideas in Andrews [1980], shows smooth, global existence for arbitrarily large initial data for thermovisco-elasticity.

5. The viscoelastic equations $u_{tt} = \sigma(u_x)_x + u_{xxt}$ *without* the assumption of strong ellipticity are shown to have unique global weak solutions in Andrews [1980]. This is of interest since, as we mentioned in the previous section, a σ without the restriction $\sigma' > 0$ may be relevant to phase transitions.

Box 5.1 ***Differentiability of Evolution Operators***

This box gives an abstract theorem which shows that in some sense the evolution operators $F_{t,s}$ for a system of the type (1)′ are differentiable. This is of interest for quasi-linear systems and applies, in particular, to the situation of 5.10. For semilinear systems, $F_{t,s}$ is smooth in the usual sense, as we already proved in Theorem 5.1. However, the example $u_t + uu_x = 0$ shows that $F_t\colon H^s \longrightarrow H^s$ is not even locally Lipschitz, although $F_t\colon H^s \longrightarrow H^{s-1}$ is differentiable (see Kato [1975a]), so the situation is more subtle for quasi-linear *hyperbolic* systems. Despite this, there is a notion of differentiability that is adequate for Proposition 3.4, Chapter 5, for example. For quasi-linear *parabolic* systems, the evolution operator will be smooth in the usual sense. For more details, see Marsden and McCracken [1976], Dorroh and Graff [1979], and Graff [1981].

First, we give the notion of differentiability appropriate for the generator. Let $\mathfrak{X}$ and $\mathcal{Y}$ be Banach spaces with $\mathcal{Y} \subset \mathfrak{X}$ continuously and densely included. Let $\mathcal{U} \subset \mathcal{Y}$ be open and $f\colon \mathcal{U} \longrightarrow \mathfrak{X}$ be a given mapping. We say f is *generator-differentiable* if for each $x \in \mathcal{U}$ there is a bounded linear operator $Df(x)\colon \mathcal{Y} \longrightarrow \mathfrak{X}$ such that

$$\frac{\|f(x+h) - f(x) - Df(x)\cdot h\|_{\mathfrak{X}}}{\|h\|_{\mathfrak{X}}} \longrightarrow 0$$

as $\|h\|_{\mathcal{Y}} \longrightarrow 0$. If f is generator-differentiable and $x \mapsto Df(x) \in \mathcal{B}(\mathcal{Y}, \mathfrak{X})$ is norm continuous, we call f C^1 *generator-differentiable.* Notice that this is *stronger* than C^1 in the Fréchet sense. If f is generator-differentiable and

$$\|f(x+h) - f(x) - Df(x)\cdot h\|_{\mathfrak{X}}/\|h\|_{\mathfrak{X}}$$

is uniformly bounded for x and $x + h$ in some $\mathcal{Y}$ neighborhood of each point, we say that f is *locally uniformly generator-differentiable.*

Most concrete examples can be checked using the following proposition:

5.13 Proposition *Suppose $f\colon \mathcal{U} \subset \mathcal{Y} \longrightarrow \mathfrak{X}$ is of class C^2, and locally in the $\mathcal{Y}$ topology*

$$x \mapsto \frac{\|D^2f(x)(h, h)\|_{\mathfrak{X}}}{\|h\|_{\mathcal{Y}}\|h\|_{\mathfrak{X}}}$$

is bounded. Then f is locally uniformly C^1 generator-differentiable.

This follows easily from the identity

$$f(x+h) - f(x) - Df(x)\cdot h = \int_0^1 \int_0^1 D^2f(x + sth)(h, h)\, ds\, dt.$$

Next, we turn to the appropriate notion for the evolution operators. A map $g: \mathcal{U} \subset \mathcal{Y} \longrightarrow \mathfrak{X}$ is called *flow-differentiable* if it is generator-differentiable and $Dg(x)$, for each $x \in \mathcal{U}$, extends to a bounded operator $\mathfrak{X}$ to $\mathfrak{X}$. Flow-differentiable maps obey a chain rule. For example, if $g_1: \mathcal{Y} \longrightarrow \mathcal{Y}$, $g_2: \mathcal{Y} \longrightarrow \mathcal{Y}$ and each is flow-differentiable (as maps of $\mathcal{Y}$ to $\mathfrak{X}$) and are continuous from $\mathcal{Y}$ to $\mathcal{Y}$, then $g_2 \circ g_1$ is flow-differentiable with, of course,

$$D(g_2 \circ g_1)(x) = Dg_2(g_1(x)) \cdot Dg_1(x).$$

The proof of this fact is routine. In particular, one can apply the chain rule to $F_{t,s} \circ F_{s,r} = F_{t,s}$ if each $F_{t,s}$ is flow-differentiable. Differentiating this in s at $s = r$ gives the backwards equation:

$$\frac{\partial}{\partial s} F_{t,s}(x) = -DF_{t,s}(x) \cdot G(x).$$

Differentiation in r at $r = s$ gives

$$DF_{t,s}(x) \cdot G(x) = G(F_{t,s}(x)),$$

the flow invariance of the generator.

Problem 5.9 Use the method of Proposition 2.7, Section 6.2 to show that integral curves are unique if (1)′ has an evolution operator that is flow-differentiable.

For the following theorem we assume these hypotheses: $\mathcal{Y} \subset \mathfrak{X}$ is continuously and densely included and $F_{t,s}$ is a continuous evolution system on an open subset $\mathfrak{D} \subset \mathcal{Y}$ and is generated by a map $G(t): \mathfrak{D} \longrightarrow \mathfrak{X}$. Also, we assume:

(H_1) $G(t): \mathfrak{D} \subset \mathcal{Y} \longrightarrow \mathfrak{X}$ is locally uniformly C^1 generator-differentiable. Its derivative is denoted $D_xG(t, x)$ and is assumed strongly continuous in t.

(H_2) For $x \in \mathfrak{D}$, $s \geq 0$, let $T_{x,s}$ be the lifetime of x beyond s: that is, $T_{x,s} = \sup\{t \geq s \mid F_{t,s}(x) \text{ is defined}\}$. Assume there is a strongly continuous linear evolution system $\{U^{x,s}(\tau, \sigma) \mid 0 \leq \sigma \leq \tau \leq T_{x,s}\}$ in $\mathfrak{X}$ whose $\mathfrak{X}$-infinitesimal generator is an extension of $\{D_xG(t, F_{t,s}x) \in B(\mathcal{Y}, \mathfrak{X}) \mid 0 \leq t \leq T_{x,s}\}$; that is, if $y \in \mathcal{Y}$, then

$$\frac{\partial}{\partial \tau_+} U^{x,s}(\tau, \sigma) \cdot y \bigg|_{\sigma=\tau} = D_xG(\tau, F_{\tau,s}(x)) \cdot y.$$

(Sufficient conditions for (H2) are given in Kato [1973].)

5.14 Theorem *Under the hypotheses above, $F_{t,s}$ is flow-differentiable at x and, in fact,*

$$DF_{t,s}(x) = U^{x,s}(t, s).$$

Proof Define $\varphi_t(x, y) = \varphi(t, x, y)$ by

$$G(t, x) - G(t, y) = D_x G(t, y)\cdot(x - y) + \| x - y \|_{\mathfrak{X}} \varphi_t(x, y)$$

(or zero if $x = y$) and notice that by local uniformity, $\| \varphi(t, x, y) \|_{\mathfrak{X}}$ is uniformly bounded if x and y are $\mathfrak{Y}$-close. By joint continutity of $F_{t,s}(x)$, for $0 < t < T_{x,s}$, $\| \varphi(t, F_{t,s}(y), F_{t,s}(x)) \|_{\mathfrak{X}}$ is bounded for $0 \le s \le T$ if $\| x - y \|_{\mathfrak{Y}}$ is sufficiently small.

By construction, we have the equation

$$\frac{d}{dt} F_{t,s}(x) = G[F_{t,s}(x)] \quad (0 \le s \le t \le T_{x,s}, x \in \mathfrak{D}).$$

Let $\qquad w(t, s) = F_{t,s}(y) - F_{t,s}(x)$

so that
$$\frac{\partial w(t, s)}{\partial t} = G(t, F_{t,s}(y)) - G(t, F_{t,s}(x))$$
$$= D_x G(t, F_{t,s}(x)) w(t, s) + \| w(t, s) \|_{\mathfrak{X}} \varphi(t, F_{t,s} y, F_{t,s} x).$$

Since $D_x G(t, F_{t,s} x)\cdot w(t, s)$ is continuous in t, s with values in $\mathfrak{X}$, and writing $U = U^{x,s}$, we have the backwards differential equation:

$$\frac{\partial}{\partial \sigma} U(t, \sigma) w(\sigma, s) = U(t, \sigma) \frac{\partial w(\sigma, s)}{\partial \sigma} - U(t, \sigma) D_x G(\sigma, F_{\sigma,s}(x))\cdot w(\sigma, s)$$
$$= U(t, \sigma)\cdot \| w(\sigma, s) \|_{\mathfrak{X}} \varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x)).$$

Hence, integrating from $\sigma = s$ to $\sigma = t$,

$$w(t, s) = U(t, s)(y - x) + \int_s^t U(t, \sigma) \| w(\sigma, s) \|_{\mathfrak{X}} \varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x))\, d\sigma.$$

Let $\| U(\tau, \sigma) \|_{\mathfrak{X},\mathfrak{X}} \le M$, and $\| \varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x)) \|_{\mathfrak{X}} \le M_2$, $(0 \le s \le \sigma \le \tau \le T)$. Thus, by Gronwall's inequality,

$$\| w(t, s) \|_{\mathfrak{X}} \le M_1 e^{M_1 M_2 T} \| y - x \|_{\mathfrak{X}} = M_3 \| y - x \|_{\mathfrak{X}}.$$

In other words,

$$\frac{\| F_{t,s}(y) - F_{t,s}(x) - U(t, s)(y - x) \|_{\mathfrak{X}}}{\| y - x \|_{\mathfrak{X}}}$$
$$\le M_1 M_3 \int_s^t \| \varphi(\sigma, F_{\sigma,s}(y), F_{\sigma,s}(x)) \|_{\mathfrak{X}}\, d\sigma.$$

From the Lebesgue bounded convergence theorem, we conclude that $F_{t,s}$ is flow-differentiable at x and $DF_{t,s}(x) = U(t, s)$; $(\varphi(t, F_{t,s}(y), F_{t,s}(x))$ is strongly measurable in s since $\varphi(x, y)$ is continuous for $x \ne y$.) ∎

Box 5.2 ***Remarks on Continuity in the Initial Data for the Initial Boundary Value Problem for Quasi-linear Systems***

As stated, Example 5.12 applies only to an elastic body filling all of space. For the initial boundary value problem, Kato [1977] has shown local existence and uniqueness, but the continuous dependence on the initial data is not yet known. The method is to replace $\mathcal{Y}$ by the domain of a power of A, such as A^3. The major complication that needs to be dealt with is the compatability conditions and the possibility that the domains of powers of A will be time dependent, even though that of A is not. All of this is caused by the degree of smoothness required in the methods and the dependence of A on u.

We shall now sketch a method that may be used in proving the continuous dependence in certain cases.[20] Write the equations this way:

$$\dot{u} + \mathfrak{A}(u) = 0, \qquad u_0 = \phi,$$

where $\mathfrak{A}(u) = A(u)\cdot u$ and we have dropped the term $f(u)$ for simplicity. Suppose that the boundary conditions are written

$$B(u) = 0, \qquad u \in \mathfrak{D}(\mathfrak{A}).$$

If we seek solutions in the domain of $[A(u)]^3$, then the compatability conditions for the initial data are obtained by differentiating $B(u) = 0$ twice:

(i) $B(\phi) = 0$;
(ii) $B'(\phi)\cdot\mathfrak{A}(\phi) = 0$;
(iii) $B''(\phi)\cdot(\mathfrak{A}(\phi), \mathfrak{A}(\phi)) + B'(\phi)\cdot\mathfrak{A}'(\phi)\cdot\mathfrak{A}(\phi) = 0.$

The difficulty is that even if $B(\phi) = 0$ are linear boundary conditions, this is a nonlinear space of functions in which we seek the solution. Let C denote the space of functions satisfying the compatability conditions. It seems natural to try to show C is a smooth manifold. We assume B itself is *linear* for simplicity and assume our function spaces are Hilbert spaces, so closed subspaces will split.

5.15 Proposition *Let $\phi \in C$ and assume:*

(a) *B is surjective;*
(b) *the linear boundary value problem*

$$\mathfrak{A}'(\phi)\cdot\psi = \rho, \qquad B\cdot\psi = 0,$$

has a solution ψ for any ρ; and

[20] M. Ortiz has pointed out to us that this result may also be provable using product formula methods. C. Dafermos has noted that for the second order hyperbolic case, which includes elasticity, classical energy and elliptic estimates may give the result.

(c) *the linear boundary value problem*

$$\mathfrak{A}''(\phi)(\mathfrak{A}(\phi), \psi) + (\mathfrak{A}'(\phi))^2\psi = \rho \qquad B\cdot\psi = 0,$$
$$B\cdot\mathfrak{A}'(\phi)\cdot\psi = 0$$

has a solution ψ *for any* ρ.

Then C *is a smooth manifold in the neighborhood of* ϕ.

Proof Let $C_1 = \operatorname{Ker} B$, the "first" boundary conditions. Map

$$\phi \in C_1 \mapsto B(\phi)\cdot\mathfrak{A}(\phi) = B\cdot\mathfrak{A}(\phi).$$

This has derivative at ϕ given by $\psi \mapsto B\cdot\mathfrak{A}'(\phi)\cdot\psi$. Since B is surjective and $\mathfrak{A}'(\phi)\colon \operatorname{Ker} B \longrightarrow$ (range space) is surjective by (b), this map has a surjective derivative, so $C_2 = \{\phi \in C_1 \mid B\cdot\mathfrak{A}(\phi) = 0\}$ is a submanifold of C_1 with tangent space $T_\phi C_2 = \{\psi\colon B\psi = 0 \text{ and } B\cdot\mathfrak{A}'(\phi)\cdot\psi = 0\}$, by the implicit function theorem.

Finally, map

$$C_2 \longrightarrow \text{(range space)}, \qquad \phi \mapsto B\cdot\mathfrak{A}'(\phi)\cdot\mathfrak{A}(\phi),$$

which has the derivative $\psi \mapsto B\cdot[\mathfrak{A}''(\phi)(\mathfrak{A}(\phi), \psi) + (\mathfrak{A}'(\phi))^2\cdot\psi]$. Thus, by assumption (c), this is surjective on $T_\phi C_2$. Thus,

$$C_3 = \{\phi \in C_2 \mid B\cdot\mathfrak{A}'(\phi)\cdot\mathfrak{A}(\phi) = 0\} = C$$

is a submanifold by the implicit function theorem. ∎

We want to solve

$$\frac{du}{dt} + \mathfrak{A}(u) = 0$$

for $u(t) \in C$ with a given initial condition $\phi \in C$. To do so, we can use the local diffeomorphism

$$\Phi\colon C \longrightarrow \text{(linear space)} = \gamma,$$

with $\Phi(\phi) = \phi$ mapping a neighborhood of ϕ in C to a ball about ϕ in a linear space obtained from the proof above. (So Φ is only defined implicitly.) Let $v = \Phi(u)$. Thus

$$\frac{dv}{dt} = \Phi'(u)\frac{du}{dt} = \Phi'(\Phi^{-1}(v))\cdot(-\mathfrak{A}(\Phi^{-1}(v))),$$

so v satisfies the modified equation

$$\frac{dv}{dt} + \tilde{\mathfrak{A}}(v) = 0,$$

where $\tilde{\mathfrak{A}}(v) = \Phi'(\Phi^{-1}(v))\cdot\mathfrak{A}(\Phi^{-1}(v))$. (In geometry notation, $\tilde{\mathfrak{A}} = \Phi_*\mathfrak{A}$.) If the modified problem is well-posed, then clearly the original one is as well. We can choose $Y = \{\psi \mid B\psi = 0,\ B\cdot\mathfrak{A}'(\phi)\cdot\psi = 0$ and

$B \cdot [\mathfrak{A}''(\phi)(\mathfrak{A}(\phi), \psi) + (\mathfrak{A}'(\phi))^2\psi] = 0\}$ and let Φ be the projection of C onto Y; see Figure 6.5.1.

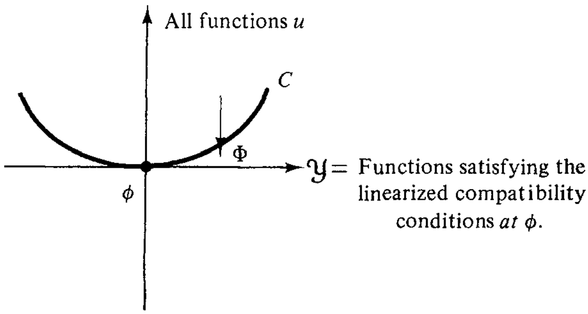


Figure 6.5.1

Now let $\mathfrak{A}$ have the form $\mathfrak{A}(u) = A(u) \cdot u$. Since Φ is a *linear* projection, the equation

$$\tilde{\mathfrak{A}}(v) = \Phi \cdot A(\Phi^{-1}(v)) \cdot \Phi^{-1}(v)$$

is still quasi-linear. It seems reasonable to ask that the abstract quasi-linear Theorem 5.11 applies to the new system $\tilde{\mathfrak{A}}$. If it does, then continuous dependence for the initial boundary value problem follows.

Box 5.3 ***Remarks on Incompressible Elasticity***

In Example 4.6 Section 5.4 we remarked that the configuration space for incompressible elasticity is $\mathcal{C}_{\text{vol}}$, the deformations ϕ with $J = 1$. (This requires careful interpretation if ϕ is not C^1.) The equations of motion are modified by replacing the first Piola–Kirchhoff stress $P_a{}^A = \rho_{\text{Ref}}(\partial W/\partial F^a{}_A)$ by $\rho_{\text{Ref}}(\partial W/\partial F^a{}_A) + Jp(F^{-1})_a{}^A$, where p is a scalar function to be determined by the condition of incompressibility. A fact discovered by Ebin and Marsden [1970] is that this extra term $P_{\text{presure}}(\phi, \dot{\phi}) = Jp(F^{-1})$ in material coordinates is a C^∞ function of $(\phi, \dot{\phi})$ in the $H^s \times H^s$ (and hence $H^{s+1} \times H^s$) topology. (In fact, there is one order of smoothing in the $H^{s+1} \times H^s$ topology; see also Cantor [1975b].) This remark enables one to deduce that in all of space the equations of incompressible elasticity are well-posed as a consequence of that for the compressible equations. (One can use, for example, Marsden

[1981] for this, or the abstract Theorem 5.10.) Again the boundary value problem requires further technical requirements.

The smoothness (as a function of ϕ) of the pressure term P_{pressure} in the stress seems not to have been exploited fully. For example, it may help with studies of weak (Hopf) solutions in fluid mechanics. In particular, it may help to prove additional regularity or uniqueness of these solutions. In general (as has been emphasized by J. Ball) it is a good idea in fluid mechanics not to forget that material coordinates could be very useful and natural in studies of existence and uniqueness.

We conclude with a few remarks on the incompressible limit, based on Rubin and Ungar [1957], Ebin [1977], and Klainerman and Majda [1918]. If the "incompressible pressure" p is replaced by a constitutive law $p_k(\rho)$, where $(dp_k/d\rho) = k$, so $1/k$ is the *compressibility*, then a potential V_k is added to our Hamiltonian which has the property:

$$V_k(\phi) = 0 \quad \text{if} \quad \phi \in \mathcal{C}_{\text{vol}} \quad \text{and} \quad V_k(\phi) \longrightarrow \infty \text{ as } k \longrightarrow +\infty$$
$$\text{if } \phi \notin \mathcal{C}_{\text{vol}}.$$

In such a case, it is intuitively clear from conservation of energy that this ought to force compressible solutions with initial data in $\mathcal{C}_{\text{vol}}$ to converge to the incompressible solutions as $k \longrightarrow \infty$.

Such convergence in the *linear* case can be proved by the Trotter–Kato theorem (see Section 6.2). Kato [1977] has proved analogues of this for non-linear equations that, following Theorem 5.10, are applicable to nonlinear elastodynamics. These approximation theorems may now be used in the proofs given by Ebin [1977]. Although all the details have not been checked, it seems fairly clear that this method will enable one to prove the convergence of solutions in the incompressible limit, at least for a short time. (See Klainerman and Kohn [1982].)

We also mention that the smoothness of the operator P_{pressure} and the convergence of the constraining forces as $k \longrightarrow \infty$ should enable one to give a simple proof of convergence of solutions of the stationary problem merely by employing the implicit function theorem. Rostamian [1978] gives some related results.

Problem 5.10 Formulate the notion of the "rigid limit" by considering $SO(3) \subset \mathcal{C}$ and letting the rigidity $\longrightarrow \infty$.

Problem 5.11 A compressible fluid may be regarded as a special case of elasticity, with stored energy function $W(\boldsymbol{F}) = h(\det \boldsymbol{F})$, where h is a strictly convex function.

(a) Show that the first Piola–Kirchhoff stress is $\boldsymbol{P} = h'(\det \boldsymbol{F}) \operatorname{adj} \boldsymbol{F}$ (or $\boldsymbol{\sigma} = h'(\det \boldsymbol{F})\boldsymbol{I}$) and the elasticity tensor is

$$\mathbf{A} = h''(\det F)\operatorname{adj} F \otimes \operatorname{adj} F + h'(\det F) \cdot \frac{\partial(\operatorname{adj} F)}{\partial F}.$$

(b) Show explicitly in two dimensions using suitable coordinates, that **A** is *not* strongly elliptic.

(c) Despite (b), show that the equations are locally well-posed by using Ebin [1977] or a direct argument to check the abstract hypotheses of 5.9 and 5.10.

Box 5.4 ***Summary of Important Formulas for Section 6.5***

Semiflow on $\mathcal{Y}$

Autonomous: $F_t\colon \mathcal{Y} \to \mathcal{Y}$; $F_0 = \text{Identity}$, $F_{t+s} = F_t \circ F_s$

Time-dependent: $F_{t,s}\colon \mathcal{Y} \to \mathcal{Y}$; $F_{s,s} = \text{Identity}$, $F_{t,s} = F_{t,r} \circ F_{r,s}$

Generator

$$\frac{d}{dt}F_t(x) = G(F_t(x)) \quad \text{(autonomous)}$$

$$\frac{d}{dt}F_{t,s}(x) = G(F_{t,s}(x), t) \quad \text{(time-dependent)}$$

Variation of Constants Formula

If $du/dt = Au + f(t)$, then $u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)\, ds.$

Semilinear Equations

If A is a linear generator on $\mathfrak{X}$ and $B\colon \mathfrak{X} \to \mathfrak{X}$ is C^∞, then $G(x) = Ax + B(x)$ generates a smooth local semiflow on $\mathfrak{X}$. If an a priori bound for $\| B(x) \|$ can be found, the semiflow is global.

Panel Flutter

The equations

$$\alpha \dot{v}'''' + v'''' - \{\Gamma + \kappa \| v' \|_{L^2}^2 + \sigma \langle v', \dot{v}' \rangle_{L^2}\} v'' + \rho v' + \sqrt{\rho}\, \delta \dot{v} + \ddot{v} = 0 \quad \text{on } [0, 1]$$

with boundary conditions

$$v = (\dot{v} + \alpha v)'' = 0 \quad \text{at} \quad x = 0, 1$$

generate a unique smooth global semiflow on $H_\partial^2 \times L^2$.

Quasilinear Equations

Hyperbolic (symmetric first-order or strongly elliptic second-order) systems define local flows on spaces of sufficiently smooth functions.

The evolution operators are differentiable in the technical sense of "flow differentiability."

Little is known about global solutions for quasi-linear equations, although some results that necessarily involve entropy conditions are known in one dimension.

6.6 THE ENERGY CRITERION

The energy criterion states that *minima of the potential energy are dynamically stable.* A major problem in elasticity (that is not yet settled) is to find conditions appropriate to nonlinear elasticity under which this criterion can be proved as a theorem. The purpose of this section is to discuss some of what is known about the problem for nonlinear elasticity. The linearized case has been discussed in Theorem 3.5. There are both positive and negative results concerning the criterion for nonlinear elasticity; examples discussed below show that the criterion is probably false as a sweeping general criterion; but there are theorems to indicate that these counterexamples can be eliminated with reasonable hypotheses.

The energy criterion has been extensively discussed in the literature. It seems to have first been explicitly recognized as a genuine difficulty by Koiter [1945], [1965b], [1976a], although it has been very successfully used in engineering for much longer. Relevant mathematical theory is closely related to bifurcation theory and goes back at least to Poincaré [1885]. A detailed account of the history and basic results available up until 1973 are contained in Knops and Wilkes [1973], which should be consulted by any readers who have a serious interest in pursuing this subject. *Some* of the other more recent references that are important are (chronologically): Ericksen [1966a, b], Coleman [1973], Naghdi and Trapp [1973], Knops, Levine, and Payne [1974], Gurtin [1975], Payne and Sattinger [1975], Mejlbro [1976], Ball, Knops, and Marsden [1978], Knops and Payne [1978], Ball and Marsden [1980], and Ball [1982]. There are many references that deal with stability in the context of bifurcation theory such as Ziegler [1968] and Thompson and Hunt [1973]; these aspects will be discussed in Chapter 7.

The difficulties with the energy criterion for conservative infinite-dimensional quasi-linear systems may be genuine ones. Some possible alternatives are:

(i) Use some kind of averaging technique to mask the higher frequency motions, thereby making semilinear techniques applicable (see 6.10 below). As far as we know this possibility has not been explored.

(ii) Employ a dissipative mechanism in addition to a conservative minimum of the potential. It is possible that a suitable mechanism will move the spectrum into the left half-plane so that Liapunov's theorem (4.1 of Chapter 7) becomes applicable. In Potier–Ferry [1980] this is established for viscoelasticity of Kelvin-Voigt type, as in Proposition 3.12, Section 6.3, generalizing the work of Dafermos [1969], [1976]. For a dissipative mechanism of thermal type, see Slemrod [1981]. For both thermal and viscous dissipation, see Dafermos [1982].

The contents of this section are as follows.

1. First we give (in 6.3) sufficient conditions for validity of the energy criterion. This involves the notion of a *potential well.*
2. The applicability of this theorem to elasticity is then criticized and examples are presented to show that it is at best difficult to satisfy the hypotheses.
3. A positive result is proved showing that potential wells can be obtained in $W^{1,p}$ (spaces in which the existence of dynamics is questionable).
4. Semilinear equations are discussed and the validity of the energy criterion is proved under acceptable hypotheses in this case.
5. Some discussion of the role of dissipative mechanisms is given.

Let us begin with sufficient conditions for the validity of the energy criterion. This version goes back at least to Lagrange and Dirichlet and is satisfactory for finite degree of freedom systems and for some infinite degree of freedom systems (the semilinear ones), as we shall see.

The general context is that of Hamiltonian systems with energy of the form kinetic-plus-potential. To be a bit more concrete than the context of Section 5.3, let us adopt the following set-up:

(a) $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces.
(b) $K\colon \mathfrak{Y} \longrightarrow \mathbb{R}$, called a kinetic energy function, is a given continuous function that satisfies $K(y) \geq 0$, with $K(y) = 0$ only if $y = 0$.
(c) $V\colon \mathfrak{X} \longrightarrow \mathbb{R}$ is a given continuous potential energy function. Let H: $\mathfrak{X} \times \mathfrak{Y} \longrightarrow \mathbb{R}$ be defined by $H(x, y) = K(y) + V(x)$.

We are contemplating the dynamical system associated with Hamilton's equations, so let us write $\dot{x}$ for y, although it need not be a time derivative. We consider the following two assumptions.

6.1 Assumptions Let $x_0 \in \mathfrak{X}$ be fixed. For each $\eta > 0$ and $\delta > 0$, let the *K-neighborhood* about $(x_0, 0)$ of radius (η, δ) be defined by

$$\mathcal{S}_{\eta,\delta} = \{(x, \dot{x}) \in \mathfrak{X} \times \mathfrak{Y} \mid \|x - x_0\| \leq \eta \quad \text{and} \quad K(\dot{x}) \leq \delta\}.$$

Assume:

(S1) There is an $\eta_0 > 0$, $\delta_0 > 0$, and $\tau > 0$ and a continuous local semiflow F_t defined on $\mathcal{S}_{\eta_0,\delta_0}$ for $0 \leq t \leq \tau$ such that (i) $(x_0, 0)$ is a fixed point for F_t—that is, $F_t(x_0, 0) = (x_0, 0)$—and (ii) $H(x(t), \dot{x}(t)) \leq H(x(0)\ \dot{x}(0))$ for all $(x(0), \dot{x}(0)) \in \mathcal{S}_{\eta_0,\delta_0}$, where $(x(t), \dot{x}(t)) =$

$F_t(x(0), \dot{x}(0))$. [This implies that there is a uniform time for which $(x(t), \dot{x}(t))$ is defined if $(x(0), \dot{x}(0)) \in \mathcal{S}_{\eta_0, \delta_0}$ and that H decreases along orbits for as long as they are defined.]

(S2) x_0 lies in a *potential well*; that is, there is an $\epsilon_0 > 0$ such that (a) $V(x_0) < V(x)$ if $0 < \|x - x_0\| < \epsilon_0$, and (b) for each $0 < \epsilon \leq \epsilon_0$, $\rho(\epsilon) > 0$, where $\rho(\epsilon) = \inf_{\|x-x_0\|=\epsilon} V(x) - V(x_0)$.

(If $\mathfrak{X}$ is finite dimensional, (b) follows from (a), but this need not be true in infinite dimensions.)

6.2 Definition We say $(x_0, 0)$ is (Liapunov) *K-stable* if for any $\eta, \delta > 0$ there is an (η_1, δ_1) such that if $(x(0), \dot{x}(0)) \in \mathcal{S}_{\eta_1, \delta_1}$, then $(x(t), \dot{x}(t))$ is defined for all $t \geq 0$ and lies in $\mathcal{S}_{\eta, \delta}$. If $K(\dot{x}) = \|\dot{x}\|^2/2$, we will just say "stable" for "K-stable."

6.3 Theorem *If* (S1) *and* (S2) *hold, then* $(x_0, 0)$ *is K-stable.*

Proof Let $\eta > 0$ and $\delta > 0$ be given. Choose $\bar{\delta} = \min(\delta, \epsilon_0)$ and $\bar{\eta} = \min(\eta, \rho(\bar{\delta}))$. By (S2), $\bar{\eta} > 0$. Next, choose $\delta_1 < \bar{\delta}$ such that $V(x(0)) - V(x_0) < \bar{\eta}/2$ if $\|x(0) - x_0\| < \delta_1$ and choose $\eta_1 = \bar{\eta}/2$.

We will show that if $(x(0), \dot{x}(0)), \in \mathcal{S}_{(\eta_1, \delta_1)}$, then $(x(t), \dot{x}(t))$ is defined for all $t \geq 0$ and lies in $\mathcal{S}_{\bar{\eta}, (\bar{\delta})}$. Suppose $(x(t), \dot{x}(t))$ lies in $\mathcal{S}_{\bar{\eta}, \bar{\delta}}$ for a maximal interval $[0, T)$. By (S1), $(x(t), \dot{x}(t))$ is defined on $[0, T + \tau)$, and so $(x(T), \dot{x}(T))$ is defined and lies on the boundary of $\mathcal{S}_{\bar{\eta}, \bar{\delta}}$. Thus, $\|x(T) - x_0\| = \bar{\delta}$ or $K(\dot{x}(T)) = \bar{\eta}$. We will derive a contradiction that will prove the proposition. We have

$$K(\dot{x}(T)) + V(x(T)) \leq K(\dot{x}(0)) + V(x(0)),$$

so
$$K(\dot{x}(T)) + V(x(T)) - V(x_0) \leq K(\dot{x}(0)) + V(x(0)) - V(x_0)$$
$$< \bar{\eta}/2 + \bar{\eta}/2 = \bar{\eta}.$$

But the left-hand side is at least $\bar{\eta}$. ∎

Now we need to discuss whether or not this "potential well approach" is applicable to nonlinear elasticity. (One should bear in mind that a totally different approach based on more than simple energy estimates may be ultimately required.)

In discussing this, the choice of topologies is crucial. Let us first suppose that fairly strong topologies are chosen so that results from the previous section guarantee that a local semiflow F_t exists. We shall show in the example below (6.4) by means of a one-dimensional example that the choice of a strong topology implies that while (S2)(a) is satisfied, (S2)(b) *cannot* be satisfied. (See Knops and Payne [1978] for some related three-dimensional examples.)

Conclusion: Theorem 6.3 *is not applicable in function spaces for which the elastodynamic equations are known to be* (*locally*) *well-posed.* (In such spaces, the deformations are at least C^1.)

6.4 Example (Ball, Knops, and Marsden [1978]) Let $\mathcal{B} = [0, 1]$ and consider displacements $\phi: [0, 1] \to \mathbb{R}$ with $\phi(0) = 0$ and $\phi(1) = \lambda > 0$ prescribed. The potential is $V(\phi) = \int_0^1 W(\phi_X)\, dX$. Suppose that W is C^2, $W'(\lambda) = 0$, and $W''(\lambda) > 0$. Then $\phi_0(X) = \lambda X$ is an extremal for V. Let $\mathfrak{X}$ be a Banach space continuously included in $W^{1,\infty}$.

1. *Under these conditions,* (S2)(a) *holds; that is, there is an* $\epsilon > 0$ *such that* if $0 < \|\phi - \phi_0\|_{\mathfrak{X}} < \epsilon$, then $V(\phi) > V(\phi_0)$. That is, ϕ_0 *is a strict local minimum for* V.

Proof This follows from the fact that λ is a local minimum of W and that the topology on $\mathfrak{X}$ is as strong as that of $W^{1,\infty}$. ∎

2. *We necessarily have failure of* (S2)(b) *in* $\mathfrak{X}$; that is,

$$\inf_{\|\phi-\phi_0\|_{\mathfrak{X}}=\epsilon} V(\phi) = V(\phi_0).$$

Proof By Taylor's theorem,

$$\begin{aligned} V(\phi) - V(\phi_0) &= \int_0^1 (W(\phi_X) - W(\lambda))\, dX \\ &= \int_0^1 \int_0^1 (1 - s)W''(s\phi_X + (1 - s)\lambda) \;\; ds\, dX. \end{aligned}$$

Thus, as $s\phi_X$ is uniformly bounded ($\mathfrak{X} \subset W^{1,\infty}$) and W'' is continuous, there is $C > 0$ such that

$$V(\phi) - V(\phi_0) \leq C \int_0^1 (\phi_X - \lambda)^2\, dX.$$

However, the topology on $\mathfrak{X}$ is *strictly* stronger than the $W^{1,2}$ topology, and so

$$\inf_{\|\phi-\phi_0\|_{\mathfrak{X}}=\epsilon} \int_0^1 (\phi_X - \lambda)^2\, dX = 0,$$

which proves our claims. ∎

3. *In* $W^{1,p}$ *one cannot necessarily conclude that* ϕ_0 *is a local minimum even though the second variation of* V *at* ϕ_0 *is positive-definite.*

Proof The example $W(\phi_X) = \frac{1}{2}(\phi_X^2 - \phi_X^4)$ shows that in any $W^{1,p}$ neighborhood of ϕ_0, $V(\phi)$ can be unbounded below. ∎

There is a Morse lemma available which enables one to verify S2(a) and to bring V into a normal form. See Tromba [1976], Golubitsky and Marsden [1983] and Buchner, Marsden and Schecter [1983]. The hypotheses of this theorem are verified in item 1 above but fail in item 3.

A more important example of the failure of the energy criterion has been constructed by Ball [1982]. He shows that a sphere undergoing a radial tension will eventually rupture due to cavitation. This is done within the framework of minimizers in $W^{1,p}$ discussed in Section 6.4. The "rupture" solution corresponds to a change in topology from $\mathcal{B}$, a sphere, to $\phi(\mathcal{B})$, a hollow sphere; clearly ϕ cannot be continuous at the origin, but one can find such a ϕ in $W^{1,p}$ for a suitable $p < 3$. Formally, the energy criterion would say that the trivial unruptured radial solution is stable, but in fact it is unstable to rupturing, a phenomenon not "detected" by the criterion.

Since the hypothesis (S2)(b) fails if the topology is too strong, it is natural to ask if it is true in weaker topologies, especially $W^{1,p}$. In fact, we shall prove that *with reasonable hypotheses in* $W^{1,p}$, (S2)(b) *follows from* (S2)(a). *However, as there is no existence theorem for elastodynamics in* $W^{1,p}$, (S1) *is unknown* (and presumably is a difficult issue).

The heart of the argument already occurs in one dimension, so we consider it first.

Let $W\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a *strictly convex* C^1 function satisfying the growth condition

$$c_0 + \alpha_0 |s|^p \leq W(s)$$

for constants $c_0 \geq 0, \alpha_0 > 0, p > 1$. Fix $\lambda > 0$ and let $W_\lambda^{1,p}$ denote those $\phi \in W^{1,p}([0, 1])$ such that $\phi(0) = 0$, and $\phi(1) = \lambda$. Define $V\colon W_\lambda^{1,p} \longrightarrow [0, \infty]$ by

$$V(\phi) = \int_0^1 W(\phi_X)\, dX$$

and let

$$\|\phi\|_{1,p} = \left(\int |\phi_X|^p\, dX\right)^{1/p}$$

Let $\phi_0(X) = \lambda X$, so $\phi_0 \in W_\lambda^{1,p}$ and $V(\phi_0) = W(\lambda)$.

6.5 Proposition (*Ball and Marsden* [1980]) *V has a potential well at* ϕ_0; *that is, (a)* $V(\phi) > V(\phi_0)$ *for* $\phi \neq \phi_0$, $\phi \in W_\lambda^{1,p}$, *and (b) for* $\epsilon > 0$,

$$\inf_{\substack{\|\phi-\phi_0\|_{1,p}=\epsilon \\ \phi\in W_\lambda^{1,p}}} V(\phi) > V(\phi_0).$$

Proof

(a) Integration by parts and the boundary conditions give

$$0 = \int_0^1 W'(\lambda)(\phi_X - (\phi_0)_X)\, dX$$

and so

$$\int_0^1 [W(\phi_X) - W(\lambda)]\, dX$$

$$= \int_0^1 \{W(\phi_X) - W(\lambda) - W'(\lambda)(\phi_X - \lambda)\}\, dX > 0$$

since W lies strictly above its tangent line. This proves (a).

(b) We prove (b) by contradiction. By (a),

$$\inf_{\|\phi-\phi_0\|_{1,p}=\epsilon} V(\phi) \geq V(\phi_0).$$

Suppose equality held. Then there would be a squence $\phi_n \in W_\lambda^{1,p}$ satisfying

$$\|\phi_n - \phi_0\|_{1,p} = \epsilon \tag{1}$$

and

$$\int_0^1 W((\phi_n)_X)\, dX \longrightarrow W(\lambda). \tag{2}$$

From (1) we can extract a weakly convergent subsequence of the ϕ_n (See Box 4.1, Section 6.4). Thus we may suppose that $\phi_n \rightharpoonup \bar{\phi}$ in $W^{1,p}$. ▮

6.6 Lemma $\bar{\phi} = \phi_0$.

Proof By weak lower sequential continuity of V (Tonelli's theorem; see Box 4.1),

$$\int_0^1 W(\bar{\phi}_X)\, dX \leq \underline{\lim} \int_0^1 W(\phi_{nX})\, dX.$$

By (2) we get

$$\int_0^1 W(\bar{\phi}_X)\, dX \leq W(\lambda),$$

so by part (a), $\bar{\phi} = \phi_0$. ▮

6.7 Lemma *If $v_n \longrightarrow v$ in L^p,*

$$\int_0^1 W(v)\, dX < \infty \quad \text{and} \quad \int_0^1 W(v_n)\, dX \longrightarrow \int_0^1 W(v)\, dX,$$

then there is a subsequence $v_{n_k} \longrightarrow v$ a.e.

Proof Fix some $\theta \in (0, 1)$ and let

$$f_n = \theta W(v_n) + (1 - \theta) W(v) - W(\theta v_n + (1 - \theta) v).$$

Then $f_n(X) \geq 0$ by convexity of W. Notice that

$$\int_0^1 W(\theta v_n + (1 - \theta) v)\, dX < \infty$$

from $f_n(X) \geq 0$ and the finiteness of $\int_0^1 W(v_n)\, dX$ and $\int_0^1 W(v)\, dX$. Now

$$\begin{aligned}
0 &\leq \overline{\lim} \int_0^1 f_n(X)\, dX \\
&= \theta \int_0^1 W(v)\, dX + (1 - \theta) \int_0^1 W(v)\, dX - \underline{\lim} \int_0^1 W(v_n + (1 - \theta) v)\, dX \\
&= \int_0^1 W(v)\, dX - \underline{\lim} \int_0^1 W(\theta v_n + (1 - \theta) v)\, dX.
\end{aligned}$$

By weak lower sequential continuity of $\int_0^1 W(v)\, dX$, $v_n \rightharpoonup v$ implies

$$\begin{aligned}
\underline{\lim} \int_0^1 W(\theta v_n + (1 - \theta) v)\, dX &\geq \int_0^1 W(\theta v + (1 - \theta) v)\, dX \\
&= \int_0^1 W(v)\, dX.
\end{aligned}$$

Thus $0 \leq \underline{\lim} \int_0^1 f_n(X)\,dX \leq 0$ and so $\lim \int_0^1 f_n(X)\,dX = 0$. There is a subsequence such that $f_{n_k}(X) \longrightarrow 0$ a.e. Since W is strictly convex, this implies $v_{n_k} \longrightarrow v$ a.e. ∎

From Lemmas 6.6 and 6.7 we can pass to a subsequence and obtain $\phi_{nX} \longrightarrow \phi_{0X} = \lambda$ a.e. By the growth condition, $W(s) - c_0 - \alpha|s|^p \geq 0$, and so by Fatou's lemma

$$\int_0^1 \underline{\lim}\,[W(\phi_{nX}) - c_0 - \alpha|\phi_{nX}|^p]\,dX \leq \underline{\lim} \int_0^1 [W(\phi_{nX}) - c_0 - \alpha|\phi_{nX}|^p]\,dX,$$

that is,

$$\int_0^1 [W(\lambda) - c_0 - \alpha|\lambda|^p]\,dX \leq W(\lambda) - c_0 + \underline{\lim} \int_0^1 \alpha|\phi_{nX}|^p\,dX$$

by $\phi_{nX} \longrightarrow \lambda$ a.e. and (2). Thus

$$-\alpha|\lambda|^p \leq \underline{\lim} \int_0^1 -\alpha_0|\phi_{nX}|^p\,dX = -\alpha\,\overline{\lim} \int_0^1 |\phi_{nX}|^p\,dX,$$

and so

$$\overline{\lim} \int_0^1 |\phi_{nX}|^p\,dX \leq \int_0^1 |\phi_{0X}|^p\,dX$$

$$\leq \underline{\lim} \int_0^1 |\phi_{nX}|^p\,dX$$

(again using Fatou's lemma).

Thus $\|\phi_{nX}\|_p \longrightarrow \|\phi_{0X}\|_p$. But in L^p, convergence of the norms and weak convergence implies strong convergence (see Riesz and Nagy [1955], p. 78), so $\phi_n \longrightarrow \phi_0$ strongly in $W_\lambda^{1,p}$. This contradicts our assumption (1) that $\|\phi_n - \phi_0\|_{1,p} = \epsilon > 0$. (We do not know if the infimum in (b) of the theorem is actually attained.) ∎

Now we discuss the three-dimensional case. Following the results of Section 6.4 we assume that W is strictly polyconvex; that is,

$$W(\boldsymbol{F}) = g(\boldsymbol{F}, \operatorname{adj}\boldsymbol{F}, \det\boldsymbol{F}), \tag{3}$$

where

$$g\colon M_+^{3\times 3} \times M^{3\times 3} \times (0, \infty) \longrightarrow \mathbb{R}$$

is strictly convex; $M^{3\times 3}$ denotes the space of 3×3 matrices and $\boldsymbol{F}$ denotes the deformation gradient, $\boldsymbol{F} = D\phi$. Assume g satisfies the growth conditions

$$g(\boldsymbol{F}, \boldsymbol{H}, \delta) \geq c_0 + k(|\boldsymbol{F}|^p + |\boldsymbol{H}|^q + \delta^r), \tag{4}$$

where $k > 0$ and, say, $c_0 > 0$. Assume that p, q, r satisfy

$$p \geq 2, \qquad q \geq \frac{p}{p-1}, \qquad r > 1. \tag{5}$$

Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain with, say, piecewise C^1 boundary, and let displacement (and/or traction boundary conditions) be fixed on $\partial\Omega$. Denote by $\mathfrak{X}$ the space of $W^{1,p}$ maps $\phi\colon \Omega \longrightarrow \mathbb{R}^3$ subject to the given boundary conditions and satisfying

$$\boldsymbol{F} \in L^p, \operatorname{adj}\boldsymbol{F} \in L^q \quad \text{and} \quad \det\boldsymbol{F} \in L^r$$

with the metric induced from $L^p \times L^q \times L^r$. (Note that $\mathfrak{X} \cong W_{\partial}^{1,p}$ if $q \leq p/2$, $1/p + 1/q < 1/r$). Let $d_{\mathfrak{X}}(\phi, \psi)$ denote the distance between ϕ and ψ in $\mathfrak{X}$.

6.8 Theorem (*Ball and Marsden* [1980]) *Suppose conditions* (3), (4), *and* (5) *above hold and that* $\phi_0 \in \mathfrak{X}$ *is a strict local minimizer; that is, for some* $\epsilon > 0$,

$$\int_\Omega W(D\phi)\, dX > \int_\Omega W(D\phi_0)\, dX.$$

if $0 < d_{\mathfrak{X}}(\phi, \phi_0) \leq \epsilon$. *Then there is a potential well at* ϕ_0; *that is,*

$$\inf_{\substack{d_{\mathfrak{X}}(\phi,\phi_0)=\epsilon \\ \phi \in \mathfrak{X}}} \int_\Omega W(D\phi)\, dX > \int_\Omega W(D\phi_0)\, dX.$$

The one-dimensional proof readily generalizes to this case, so we can omit it.

Remarks

1. There is a similar two-dimensional theorem for $W(\boldsymbol{F}) = g(\boldsymbol{F}, \det \boldsymbol{F})$.
2. Examples of stored energy functions W appropriate for natural rubber satisfying (3)–(5) and having a unique natural state $\boldsymbol{F} = \boldsymbol{I}$ (up to rotations) are the Ogden materials:

$$W(\boldsymbol{F}) = \lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha + (\lambda_2\lambda_3)^\beta + (\lambda_3\lambda_1)^\beta + (\lambda_1\lambda_2)^\beta + h(\lambda_1\lambda_2\lambda_3),$$

where $\alpha \geq 3$, $\beta \geq \frac{3}{2}$, $h'' > 0$, $\alpha + 2\beta + h'(1) = 0$, and where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches (eigenvalues of $(\boldsymbol{F}^T\boldsymbol{F})^{1/2}$).

3. Homogeneous deformations provide basic examples of strict local minimizers.
4. The method shows that minimizing sequences actually converge strongly.
5. An obvious question concerns when (S2)(a) holds in three dimensions—that is, with $V(\phi) = \int_\Omega W(D\phi)\, dX$, if $V'(\phi_0) = 0$ and $V''(\phi_0) > 0$ under conditions of polyconvexity—is ϕ_0 a strict local minimum of V in $W^{1,p}$? (Example 6.4 shows that some condition like polyconvexity is needed.) Also, Ball's example on cavitation shows that the answer is generally "no" without additional growth conditions on W. However, Weierstrass' classical work in the calculus of variations (see Bolza [1904]) indicates that this problem is tractable. (Weierstrass made the big leap from C^1 to C^0 for the validity of (S2)(a) in one-dimensional variational problems.)

In practice the energy criterion has great success, according to Koiter [1976a]. However, this is consistent with the possibility that the energy criterion may fail for nonlinear elastodynamics. Indeed, "in practice" one usually does not observe the very high frequency motions. Masking these high frequencies may amount to an averaging process in which the quasi-linear equations of elastodynamics are replaced by finite-dimensional or semilinear approximations. For the latter, the energy criterion is valid under reasonable conditions. Indeed, for semilinear equations there usually are function spaces in which (S1) holds by using the semilinear existence theorem 5.1 and in which (S2) can be checked by just using differential calculus. The method was already applied to nonlinear wave equations in the previous section.[21] We isolate the calculus lemma as follows.

6.9 Proposition *Let $\mathfrak{X}$ be a Banach space and $V: \mathfrak{X} \to \mathbb{R}$ be C^2 in a neighborhood of $x_0 \in \mathfrak{X}$. Suppose that (i) $DV(x_0) = 0$, and (ii) there is a $c > 0$ such that $D^2V(x_0)\cdot(v, v) \geq c\|v\|^2$. Then x_0 lies in a potential well for V.*

Proof *By Taylor's theorem and (i),*

$$\begin{aligned} V(x) - V(x_0) &= \int_0^1 (1-s)D^2V(sx + (1-s)x_0)(x - x_0, x - x_0)\,ds \\ &= \tfrac{1}{2}D^2V(x_0)(x - x_0, x - x_0) \\ &\quad + \int_0^1 (1-s)[D^2V(sx + (1-s)x_0) - D^2V(x_0)](x - x_0, x - x_0)\,ds \\ &= \tfrac{1}{2}D^2V(x_0)(x - x_0, x - x_0) + R(x, x_0). \end{aligned}$$

By continuity of D^2V, there is a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $|R(x, x_0)| \leq (c/4)\|x - x_0\|^2$. Since the first term is $\geq c\|x - x_0\|^2/2$, we get

$$V(x) - V(x_0) \geq \frac{c}{4}\|x - x_0\|^2 \quad \text{if} \quad \|x - x_0\| < \delta.$$

This inequality gives (S2). ∎

Note that condition (ii) states that $\langle v, w\rangle = D^2V(x_0)\cdot(v, w)$ is an inner product on $\mathfrak{X}$ whose topology is the same as the $\mathfrak{X}$-topology.

The following theorem shows how the semilinear theory works. We take $x_0 = 0$ for simplicity.

6.10 Theorem *Let $\mathcal{H}$ be a Hilbert space and let A be a positive-definite ($A \geq c > 0$) self-adjoint operator and let $\mathcal{H}_1$ be the domain of $\sqrt{A}$ with the graph norm. Let $V: \mathcal{H}_1 \to \mathbb{R}$ be given by $V(x) = \frac{1}{2}\langle\sqrt{A}x, \sqrt{A}x\rangle + \tilde{V}(x)$, where $\tilde{V}$ is C^2 and $\tilde{V}(0) = 0$, $D\tilde{V}(0) = 0$, and $D^2\tilde{V}(0) \geq 0$; that is, $D^2\tilde{V}(0)(v, v) \geq 0$. Suppose that $\nabla\tilde{V}(x)$ exists and is a C^1 map of $\mathcal{H}_1$ to $\mathcal{H}$* (cf. 5.6). *Then the hypotheses* (S1) *and* (S2) *hold for the Hamiltonian equations*

$$\frac{\partial}{\partial t}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ -\nabla\tilde{V}(x) \end{pmatrix} \tag{6}$$

with $H(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + \frac{1}{2}\|\sqrt{A}x\|^2 + \tilde{V}(x)$ and so (0, 0) *is dynamically stable.*

Proof This follows directly from 3.2, 5.1, 5.6, and 6.9, with $V(x) = \langle\sqrt{A}x, \sqrt{A}x\rangle + \tilde{V}(x)$. ∎

Problem 6.1 Use this theorem to re-derive the results of Example 5.7(iii).

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[21] These methods appear in a number of references, such as Marsden [1973b] and Payne and Sattinger [1975].

6.11 Example (R. Knops) Consider the following equations:

$$u_{tt} = (P(u_x))_x - u_{xxxx}, \tag{7}$$

where $P(u_x) = W'(u_x)$, $W(0) = W'(0) = 0$, $W''(0) \geq 0$, and W is smooth. We work in one dimension on an interval, say $[0, 1]$, with boundary conditions, say $u = 0$ and $u_{xx} = 0$, at the ends. We claim that Theorem 6.10 applies to this example with $\mathcal{H} = L^2[0, 1]$,

$$A(u) = -u_{xxxx}, \quad \sqrt{A}u = u_{xx}$$

$$\mathcal{H}_1 = \mathfrak{D}(\sqrt{A}) = H^2_\partial = \{u \in H^2 \mid u = 0 \text{ at } x = 0, 1\}$$

and

$$\mathfrak{D}(A) = H^4_\partial = \{u \in H^4 \mid u = u_{xx} = 0 \text{ at } x = 0, 1\}.$$

We have $\tilde{V}(u) = \int W(u_x)\,dx$. Then since $H^2 \subset C^1$ in one dimension, $\tilde{V}$ is smooth, with

$$D\tilde{V}(u)\cdot v = \int W'(u_x)\cdot v_x\,dx = -\int (W'(u_x))_x\cdot v\,dx \tag{8a}$$

and

$$D^2\tilde{V}(u)\cdot(u, w) = \int W''(u_x)\cdot u_x\cdot w_x\,dx. \tag{8b}$$

From (8a) $-\nabla\tilde{V}(u) = (W'(u_x))_x$, so (6) and (7) coincide and $-\nabla\tilde{V}$ is a smooth function from H^2_∂ to L^2. Finally, it is clear from (8a) that $D\tilde{V}(0) = 0$ and as $W''(0) > 0$, (8b) gives $D^2\tilde{V}(0)(v, v) \geq 0$. Thus, the trivial solution $(0, 0)$ is (dynamically) stable in $H^2_\partial \times L^2$.

> ***Problem 6.2*** Modify (7) to $u_{tt} = P(u_x)_x + u_{xxxx}$ where boundary conditions $u(0) = 0$, $u(1) = \lambda$, $u_{xx}(0) = 0$, $u_{xx}(1) = 0$ are imposed and we insist $u'(x) > 0$.

Box 6.1 ***Summary of Important Formulas for Section 6.6***

Energy Criterion

"Minima of the potential energy are stable." This assertion "works," but a satisfactory theorem justifying it for nonlinear elasticity is not known.

Potential Well Conditions

(a) $V(x) > V(x_0)$; $x \neq x_0$, x near x_0.

(b) $\left(\inf_{\|x-x_0\|=\epsilon} V(x)\right) > V(x_0)$.

Stability Theorem

Well-posed dynamics and a potential well implies stability.

Applicability

1. The potential well condition (b) cannot hold for nonlinear elas-

ticity in topologies as strong as C^1, but in $W^{1,p}$ (b) follows from (a) and the assumption of polyconvexity. Dynamics however is not known to exist in $W^{1,p}$.

2. The stability theorem does apply to situations in which the basic equations are semilinear (rather than the quasi-linear equations of nonlinear elasticity).

6.7 A CONTROL PROBLEM FOR A BEAM EQUATION[22]

This section discusses the abstract problem of controlling a semilinear evolution equation and applies the formalism to the case of a vibrating beam. The beam equation in question is

$$w_{tt} + w_{xxxx} + p(t)w_{xx} = 0 \quad (0 \le x \le 1)$$

with boundary conditions $w = w_{xx} = 0$ at $x = 0, 1$; see Figure 6.7.1. Here w represents the transverse deflection of a beam with hinged ends, and $p(t)$ is an axial force. The control question is this: given initial conditions $w(x, 0)$, $\dot{w}(x, 0)$, can we find $p(t)$ that controls the solution to a prescribed w, $\dot{w}$ after a prescribed time T?

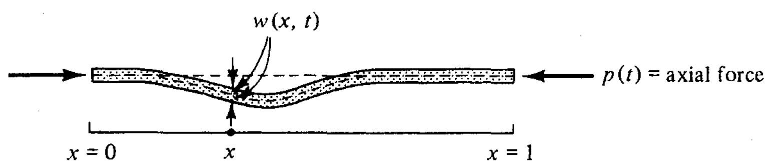

Figure 6.7.1

The beam equation is just one illustration of many in control theory, but it has a number of peculiarities that point out the caution needed when setting up a general theory for the control of partial, rather than ordinary, differential equations. In particular, we shall see that it is easy to prove controllability of any finite number of modes at once, but it is very delicate to prove controllability of *all* the modes simultaneously. The full theory needed to prove the latter is omitted here, but the points where the "naive" method breaks down will be discussed. For the more sophisticated theory needed to deal with all the modes

[22]*This section was done in collaboration with* J. M. Ball *and* M. Slemrod *and is based on* Ball, Marsden, and Slemrod [1982].

of the beam equation, see Ball, Marsden, and Slemrod [1982]. The paper of Ball and Slemrod [1979] considers some related stabilization problems. For background in control theory, see, for example, Brockett [1982] and Russell [1979]. References related to the work in this section are Sussman [1977], Jurdjevic and Quinn [1978] and Hermes [1979]. For control in the Hamiltonian context, see for example, Van der Schaft [1981].

We begin by treating the situation for abstract semilinear equations. Thus, we consider an evolution equation of the form

$$\dot{u}(t) = \mathfrak{A}u(t) + p(t)\mathfrak{B}(u(t)), \tag{1}$$

where $\mathfrak{A}$ generates a C^0 semigroup on a Banach space $\mathfrak{X}$, $p(t)$ is a real valued function of t that is locally L^1 and $\mathfrak{B}\colon \mathfrak{X} \longrightarrow \mathfrak{X}$ is C^k ($k \geq 1$). Let u_0 be given initial data for u and let $T > 0$ be given. For $p = 0$, the solution of (1) after time T is just $e^{T\mathfrak{A}}u_0$, which we call the *free solution.*

7.1 Definition If there exists a neighborhood $\mathcal{V}$ of $e^{\mathfrak{A}T}u_0$ in $\mathfrak{X}$ with the property that for any $v \in V$, there exists a p such that the solution of (1) with initial data u_0 reaches v after time T, we say (1) is *locally controllable* about the free solution $e^{\mathfrak{A}T}u_0$.

One way to tackle this problem of local controllability is to use the implicit function theorem. Write (1) in integrated form:

$$u(t) = e^{\mathfrak{A}t}u_0 + \int_0^t e^{(t-s)\mathfrak{A}}p(s)\mathfrak{B}(u(s))\,ds. \tag{2}$$

Let p belong to a specified Banach space $\mathcal{Z} \subset L^1([0, T], \mathbb{R})$. The techniques used to prove Theorem 5.1 show that for short time, (2) has a unique solution $u(t, p, u_0)$ that is C^k in p and u_0. By Corollary 5.3, if $\|\mathfrak{B}(x)\| \leq C + K\|x\|$ (for example, $\mathfrak{B}$ linear will be of interest to us), then solutions are globally defined, so we do not need to worry about taking short time intervals.

Let $L\colon \mathcal{Z} \longrightarrow \mathfrak{X}$ denote the derivative of $u(T, p, u_0)$ with respect to p at $p = 0$. This may be found by implicitly differentiating (2). One gets

$$Lp = \int_0^T e^{(T-s)\mathfrak{A}}p(s)\mathfrak{B}(e^{s\mathfrak{A}}u_0)\,ds. \tag{3}$$

The implicit function theorem then gives:

7.2 Theorem *If $L\colon \mathcal{Z} \longrightarrow \mathfrak{X}$ is a surjective linear map, then* (1) *is locally controllable around the free solution.*

If $\mathfrak{A}$ generates a group and $\mathfrak{B}$ is linear, then surjectivity of L is clearly equivalent to surjectivity of $\hat{L}p = \int_0^T p(s)\, e^{-s\mathfrak{A}}\mathfrak{B}e^{s\mathfrak{A}}u_0\,ds.$

7.3 Example If $\mathfrak{X} = \mathbb{R}^n$ and $\mathfrak{B}$ is linear, we can expand

$$e^{-s\mathfrak{A}}\mathfrak{B}e^{s\mathfrak{A}} = \mathfrak{B} + s[\mathfrak{A}, \mathfrak{B}] + \frac{s^2}{2}[\mathfrak{A}, [\mathfrak{A}, \mathfrak{B}]] + \cdots$$

to recover a standard controllability criterion: if

$$\dim \operatorname{span}\{\mathfrak{B}u_0, [\mathfrak{A}, \mathfrak{B}]u_0, [\mathfrak{A}, [\mathfrak{A}, \mathfrak{B}]]u_0, \ldots\} = n,$$

then (1) is locally controllable. These ideas lead naturally to differential geometric aspects of control theory using Lie brackets of vector fields and the Frobenius theorem. See, for example, Sussman [1977] and Brockett [1982].

Next, suppose we wish only to control a finite-dimensional piece of u or a finite number of modes of u. To do so, we should "observe" only a finite-dimensional projection of u. This idea leads to the following:

7.4 Definition We say that (1) is *locally controllable about the free solution* $e^{T\mathfrak{A}}u_0$ *for finite-dimensional observations* if for any surjective continuous linear map $G\colon \mathfrak{X} \longrightarrow \mathbb{R}^n$ there exists a neighborhood $\mathcal{U}$ of $G(e^{\mathfrak{A}T}u_0)$ in $\mathbb{R}^n$ with the property that for any $v \in \mathcal{U}$ there exists a $p \in \mathfrak{Z}$ such that

$$G(u(T, p, u_0)) = v.$$

As above, we have:

7.5 Theorem *Suppose* L, *defined by* (3), *has dense range in* $\mathfrak{X}$. *Then* (1) *is locally controllable about the free solution for finite-dimensional observations.*

Proof The map $p \mapsto G(u(T, p, u_0))$ has derivative $G \circ L\colon \mathfrak{Z} \longrightarrow \mathbb{R}^n$ at $p = 0$. Since L has dense range, $G \circ L$ is surjective, so the implicit function theorem applies. ∎

As above, if $\mathfrak{A}$ generates a group, it is enough to show that $\hat{L}$ has dense range to get the conclusion of 7.5. *For the beam equation we shall see that* $\hat{L}$ *does have dense range but in the space* $\mathfrak{X}$ *corresponding to the energy norm and with* $\mathfrak{Z} = L^2$, *is not surjective.* To see this, we shall need some more detailed computations concerning hyperbolic systems in general.

Let A be a positive self-adjoint operator on a real Hilbert space $\mathfrak{H}$ with inner product $\langle\, , \rangle_{\mathfrak{H}}$. Let A have spectrum consisting of eigenvalues λ_n^2 $(0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots)$ with corresponding orthonormalized eigenfunctions ϕ_n. Let $B\colon \mathfrak{D}(A^{1/2}) \longrightarrow \mathfrak{H}$ be bounded. We consider the equation

$$\ddot{w} + Aw + pBw = 0. \tag{4}$$

This in the form (1) with

$$u = \begin{pmatrix} w \\ \dot{w} \end{pmatrix}$$

and

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \qquad \mathfrak{B} = \begin{pmatrix} 0 & 0 \\ -B & 0 \end{pmatrix}.$$

Here $\mathfrak{X} = \mathfrak{D}(A^{1/2}) \times \mathfrak{H}$ and, by 3.2, $\mathfrak{A}$ generates a C^0 group of isometries on $\mathfrak{X}$. The inner product on $\mathfrak{X}$ is given by the "energy inner product":

$$\langle (y_1, z_1), (y_2, z_2) \rangle_{\mathfrak{X}} = \langle A^{1/2}y_1, A^{1/2}y_2 \rangle_{\mathfrak{H}} + \langle z_1, z_2 \rangle_{\mathfrak{H}}.$$

Write
$$u_0 = \begin{pmatrix} \sum_{m=1}^{\infty} b_m \phi_m \\ \sum_{m=1}^{\infty} - \lambda_m c_m \phi_m \end{pmatrix} \in \mathfrak{X},$$
where
$$\sum_{m=1}^{\infty} \lambda_m^2 (b_m^2 + c_m^2) < \infty.$$
If we set $a_m = \frac{1}{2}(b_m + ic_m)$, we have
$$e^{\mathfrak{A}s} u_0 = \begin{pmatrix} \sum_{m=1}^{\infty} [a_m \exp(i\lambda_m s) + \bar{a}_m \exp(-i\lambda_m s)]\phi_m \\ \sum_{m=1}^{\infty} i\lambda_m [a_m \exp(i\lambda_m s) - \bar{a}_m \exp(-i\lambda_m s)]\phi_m \end{pmatrix}.$$
In particular, this gives an explicit formula for the free solution. Applying $\mathfrak{B}$, we get
$$\mathfrak{B} e^{\mathfrak{A}s} u_0 = \begin{pmatrix} 0 \\ \sum_{m=1}^{\infty} [a_m \exp(i\lambda_m s) + \bar{a}_m \exp(-i\lambda_m s)] B\phi_m \end{pmatrix}.$$

To simplify matters, let us *assume that* B is diagonal; that is, $\langle B\phi_m, \phi_n \rangle_{\mathfrak{X}} = d_m \delta_{mn}$. Then a little computation gives
$$e^{-s\mathfrak{A}} \mathfrak{B} e^{s\mathfrak{A}} u_0 = \begin{bmatrix} \sum_{n=1}^{\infty} \frac{-id_n}{2\lambda_n} \{a_n \exp(2i\lambda_n s) - \bar{a}_n \exp(-2i\lambda_n s) - (a_n - \bar{a}_n)\}\phi_n \\ \sum_{n=1}^{\infty} - \frac{d_n}{2} \{a_n \exp(2i\lambda_n s) + \bar{a}_n \exp(-2i\lambda_n s) + (a_n + \bar{a}_n)\}\phi_n \end{bmatrix} \tag{5}$$
This formula can now be inserted into (3) to give Lp in terms of the basis ϕ_n. Since $\mathfrak{A}$ generates a group, surjectivity of L comes down to solvability of
$$\hat{L}p = \int_0^T p(s) e^{-s\mathfrak{A}} \mathfrak{B}(e^{s\mathfrak{A}} u_0)\, ds = h \tag{6}$$
for $p(s)$ given $h \in \mathfrak{X}$. Also, (4) is locally controllable for finite-dimensional observers if $\hat{L}$ has dense range.

Let us now turn to our vibrating beam with hinged ends and an axial load $p(t)$ as a control:
$$\begin{aligned} w_{tt} + w_{xxxx} + p(t) w_{xx} &= 0 \quad (0 \le x \le 1), \\ w = w_{xx} &= 0 \quad \text{at} \quad x = 0, 1. \end{aligned} \tag{7}$$
Here, $Aw = w_{xxxx}$, $Bw = w_{xx}$, $\lambda_n = n^2\pi^2$, $\phi_n = (1/\sqrt{2}) \sin(n\pi x)$, $d_n = -n^2\pi^2$, and $\mathfrak{X} = H_\partial^2 \times L^2$ (see Box 3.2).

7.6 Theorem *If $T \ge 1/\pi$, $\mathfrak{X} = H_\partial^2 \times L^2$, $\mathfrak{Z}$ is L^2 or more generally is dense in L^1 and the Fourier coefficients of the initial data u_0 satisfy $a_n \neq 0$ ($n = 1, 2, 3, \ldots$) (i.e., u_0 is active in all modes), then (7) is locally controllable about the free solution $e^{T\mathfrak{A}} u_0$ for finite-dimensional observations.*

Proof We have to show that $\hat{L}$ defined by (5) and (6) has dense range. To do this we show that any $h \in \mathfrak{X}$ orthogonal to the range of $\hat{L}$ must vanish.

If $h \in \mathfrak{X}$ is orthogonal to the range of $\hat{L}$, then

$$\int_0^T p(s)\langle e^{-s\mathfrak{A}}\mathfrak{B}e^{s\mathfrak{A}}u_0, h\rangle\, ds = 0$$

for all $p \in L^2$. This implies that

$$\langle e^{-s\mathfrak{A}}\mathfrak{B}e^{s\mathfrak{A}}u_0, h\rangle = 0, \qquad s \in [0, T]. \tag{8}$$

Write

$$h = \begin{pmatrix} \sum \alpha_m \phi_m \\ \sum - \lambda_m \beta_m \phi_m \end{pmatrix} \tag{9}$$

so $h \in \mathfrak{X}$ means $\sum n^4\pi^4(\alpha_m^2 + \beta_m^2) < \infty$. Using (5) and (9), (7) simplifies to

$$\begin{aligned}\sum_{n=1}^{\infty} i\frac{n^4\pi^4}{2}\{[a_n e^{2in^2\pi^2 s} - \bar{a}_n e^{-2in^2\pi^2 s} - (a_n - \bar{a}_n)]\,\alpha_n \\ + i[a_n e^{2in^2\pi^2 s} + \bar{a}_n e^{-2in^2\pi^2 s} + (a_n + \bar{a}_n)]\beta_n\} = 0.\end{aligned} \tag{10}$$

However, this represents some of the terms of a convergent Fourier series on $[0, 1/\pi]$, so all the coefficients must vanish. Since $a_n \neq 0$, this implies α_n and $\beta_n = 0$. ∎

Problem 7.1 Redo the above computations for general hyperbolic equations using complex notation by writing $z = \sqrt{A}w + i\dot{w}$. Show that (4) becomes

$i\dot{z} = \sqrt{A}z + pBA^{-1/2}\operatorname{Re} z$ so $\mathfrak{A} = -i\sqrt{A}$, $\mathfrak{B} = -iB\circ A^{-1/2}\circ\operatorname{Re}$

and (5) becomes

$$e^{-s\mathfrak{A}}\mathfrak{B}e^{s\mathfrak{A}}z = \sum_{n=1}^{\infty} \frac{-b_n}{2\lambda_n}(\bar{z}_n e^{2i\lambda_n s} + z_n)\,\phi_n.$$

Use this to give a cleaner looking expression for (10).

One can now ask if $\hat{L}$ is surjective. For $\mathrm{Z} = L^2$, we can see that it is *not*, as follows. To solve $\hat{L}p = h$ we loose no generality by seeking p's on $[0, 1/\pi]$ in the form

$$p(s) = p_0 + \sum \{p_{n^2}\exp(2in^2\pi^2 s) + \bar{p}_{n^2}\exp(-2in^2\pi^2 s)\} \tag{11}$$

and suppressing the remaining coefficients. Inserting (5) and (11) into (6), we can determine the Fourier coefficients of h. As in (9), the condition for h to be in $\mathfrak{X}$ is $\sum \lambda_m^2(\alpha_m^2 + \beta_m^2) < \infty$. But the condition for h to be in the range of $\hat{L}$ with an L^2 function p is that $\{a_n d_n p_{n^2}\} \in \ell_2$. This is, however, a stronger condition than $h \in \mathfrak{X}$. Thus, we conclude that $\hat{L}$ and hence L has range that is not equal to $\mathfrak{X}$.

In fact, one can show that not only is L not surjective, but that (7) is *not* locally controllable in the energy norm. To overcome this difficulty one can contemplate more sophisticated inverse function theorems and indeed these may be necessary in general. However, for a certain class of equations another trick works. Namely, instead of the $\mathfrak{X}$-norm, make up a new space related to the range of $\hat{L}$ and the graph norm. Miraculously, the solution $u(t, p, u_0)$ stays in this space and is still a smooth function of p in the new topology. Consequently, in this stronger norm the implicit function theorem can, in effect still be used. For the beam example, this trick (combined with a certain averaging method) actually works. In the end one finds a new space that differs from the space of the finite energy solution in that a more severe condition on the asymptotic decay rate of the modal amplitudes is required. In this space, control *is* possible. See Ball, Marsden, and Slemrod [1982] for details.

Problem 7.2 Show that the problem

$$u_{tt} - u_{xx} + pu = 0 \quad (0 < x < 1),$$
$$u = 0 \quad \text{at} \quad x = 0, 1,$$

is controllable for finite-dimensional observers provided the initial data is active in all modes.

Box 7.1 ***Summary of Important Formulas for Section 6-7***

Local controllability of $\dot{u} = \mathfrak{A}u + p\mathfrak{B}u$ about the free solution $e^{T\mathfrak{A}}u_0$ means that we can hit a whole neighborhood of $e^{T\mathfrak{A}}u_0$ by starting at u_0 and varying the control p in a neighborhood of 0.

If we ask only for Gu to hit a neighborhood of $G(e^{T\mathfrak{A}}u_0)$, where G is any surjective linear map to $\mathbb{R}^n$, we says the equation is locally controllable for *finite-dimensional observations.*

Criteria

1. If $Lp = \int_0^T e^{(T-s)\mathfrak{A}}p(s)\mathfrak{B}e^{s\mathfrak{A}}u_0\, ds$ is surjective, then the equation is locally controllable.

2. If L has dense range, then the equation is controllable for finite-dimensional observations.

3. If the equation is on $\mathbb{R}^n$, $\mathfrak{B}$ is linear, and

$$\dim \operatorname{span}\{\mathfrak{B}u_0, [\mathfrak{A}, \mathfrak{B}]u_0, [\mathfrak{A}, [\mathfrak{A}, \mathfrak{B}]]u_0, \ldots\} = n,$$

then the equation is locally controllable.

The Beam Equation

Equation (7) is controllable for finite-dimensional observations provided the initial data are active in all modes. It is never controllable to a full neighborhood of the free solution in the energy norm; however, it is controllable to a dense subset.
