ON THE WEIGHT OF HEAT AND THERMAL EQUILIBRIUM IN GENERAL RELATIVITY

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ABSTRACT

In accordance with the special theory of relativity all forms of energy, including heat, have inertia and hence in accordance with the equivalence principle also have weight. The purpose of the present article is to investigate the thermodynamic implications of the idea that heat has weight. In particular an investigation is made to see if a temperature gradient is a necessary accompaniment of thermal equilibrium in a gravitational field, in order to prevent the flow of heat from regions of higher to those of lower gravitational potential.

A preliminary non-rigorous treatment of this problem is first given by attempting to modify the classical thermodynamics only to the extent of associating with each intrinsic quantity of energy an additional amount of potential gravitational energy. In this way an expression is obtained for increase in equilibrium temperature with decrease in gravitational potential which, however, could in any case only be correct as a first approximation in a weak gravitational field. A discussion of the uncertainties and lack of rigor of this preliminary treatment is then given and the necessity pointed out for a rigorous treatment based on the principles of general relativity.

A rigorous relativistic treatment is then undertaken using the extension of thermodynamics to general relativity previously presented by the author. The system to be treated is taken as a static spherical distribution of perfect fluid which has come to gravitational and thermodynamic equilibrium. The principles of relativistic mechanics are first applied to such a system in order to obtain results needed in the later work. And it is then shown that these *mechanical* principles themselves are sufficient to determine the temperature distribution as a function of potential in the simple case of black-body radiation. The principles of relativistic thermodynamics are then applied to this same case of pure black-body radiation and the same expression for temperature as a function of potential obtained by the *thermodynamic* as by the mechanical treatment. This may be regarded as giving some measure of check on the validity of the proposed relativistic thermodynamics.

Following this, a thermodynamic treatment is given for the temperature distribution in the more general case of matter and radiation and a result found which harmonizes with that for radiation alone. A treatment is then given to the distribution of a perfect monatomic gas in a gravitational field both on the assumption that the total number of atoms must remain constant and on the assumption of the ready interconvertibility of matter and radiation. In the latter case the same dependence of concentration on temperature is obtained as was found by Stern and by the author for the case of flat space-time.

Using a system of coordinates such that the line element for the sphere of fluid takes the form

$$ds^2 = -e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + e^{\nu} dt^2$$

the general result for the relation between gravitational potential and equilibrium temperature T_0 as measured by a local observer in proper coordinates can be given by the equation

$$\frac{d\,\ln T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr}$$

This equation reduces in the case of a weak field to that obtained by the preliminary non-rigorous treatment, and gives a very small change of temperature with position in fields of ordinary intensity. The result, however, is one of great theoretical interest, since constant temperature throughout any system which has come to thermal equilibrium has hitherto been regarded as an inescapable thermodynamic conclusion. It is also not out of the question that the effect might sometime be of experimental or observational importance.

§1. Introduction

ONE of the most important results of the special theory of relativity was the relation between the total energy of a system U and its inertial mass m, given by Einstein's equation

$$U = mc^2 \tag{1}$$

where c is the velocity of light. In accordance with this equation we must ascribe the property of inertia to energy. For ordinary matter, however, we are in the possession of very exact experimental results showing a proportionality between inertia and weight, and hence the question at once arises whether we must also ascribe to energy the property of weight.

As early as 1911, in developing those ideas which finally led to the general theory of relativity, Einstein[1] considered this question, as to the weight of energy, and showed by a simple application of the equivalence hypothesis that the property of weight must indeed also be ascribed to energy if we are to maintain the postulated equivalence between behavior in a homogeneous gravitational field and behavior with reference to a set of uniformly accelerated axes. The later more complete developments of the general theory of relativity have shown that the Newtonian concept of the weight of a body as a force acting on it when placed in a gravitational field, is an idea which is only suitable for the treatment of slow-moving particles in weak gravitational fields. Nevertheless, these more complete developments of the general theory of relativity have completely confirmed the fundamental nature of the idea that weight must be ascribed to energy, since they have shown in any case that all forms of energy will be subject to the same gravitational action when placed under the same conditions in a gravitational field. Furthermore, in the case of weak enough fields and slow enough motions so that the Newtonian concepts may still be taken as approximately valid, the general theory of relativity has shown that the gravitational force F acting on any quantity of energy U will be given as would be expected by the simple equation

[1] Einstein, Ann. d. Physik **35**, 898 (1911).

$$F = (U/c^2)g \tag{2}$$

where g is the acceleration due to gravity.

If we accept the conclusion that energy has weight, it is evident that we must also ascribe weight to energy in the form of heat, and hence must expect to find thermodynamic consequences of the new idea. Thus considering for example cases where the Newtonian approximation is valid, we might expect the flow of heat from a place of higher to a place of lower gravitational potential to be accompanied by a decrease in potential energy, and this in turn would lead us to suspect that the condition of thermal equilibrium in a gravitational field might involve higher temperatures at the lower gravitational levels in order to prevent any thermal flow. It is the purpose of the present article to investigate the thermodynamic consequences of this idea that heat has weight, making use of principles obtained in an extension of thermodynamics to general relativity which I have already given.[2]

In the immediately following section, §2, we shall first consider a preliminary treatment of temperature equilibrium in the case of a system in such a weak gravitational field that we shall feel warranted in trying to apply the Newtonian concept of gravitation as a first approximation. And in §3, we shall examine the inadequacies of this preliminary treatment and show the necessity for the more rigorous treatments to follow. In §4, we shall then prepare for the rigorous general relativity treatment by applying the principles of relativistic mechanics to a spherical distribution of perfect fluid to obtain results which will be needed in the later developments. And in §5, we shall show that these purely mechanical results are alone sufficient to determine the temperature distribution in the simple case of a spherical distribution of pure radiation. In §6, we shall then briefly restate that result of the earlier extension of thermodynamics to general relativity which is necessary for our present considerations and then apply it in §7 to this same special case of pure radiation, and show that the same results are also obtained from the thermodynamic as from the purely mechanical treatment; this may be regarded as furnishing a partial confirmation of the correctness of the new system of relativistic thermodynamics. Following this, in §8, we shall use our relativistic thermodynamics to obtain a general equation for the distribution of temperature in any spherical mass of gravitating fluid, which has come to equilibrium. In §9, we shall then consider the equilibrium distribution of matter in a gravitational sphere of fluid; in particular the cases of a perfect monatomic gas, both when in equilibrium with radiation and when the total number of atoms is constant, will be treated. Finally in §10, we shall make some concluding remarks.

§2. Preliminary Treatment of Temperature Equilibrium in a Weak Gravitational Field

To obtain a preliminary non-rigorous treatment of the temperature equilibrium in a weak gravitational field, we shall endeavor to apply the

[2] Tolman, Proc. Nat. Acad. **14,** 268 (1928); ibid. **14,** 701 (1928); This Journal, 896, ibid.

principles of the classical theomodynamics, modified only by assuming that the force of gravity will act on any quantity of energy U as though it had a weight, in the Newtonian sense, corresponding to its inertial mass U/c^2. Under these circumstances a quantity of energy which has the intrinsic magnitude U, when measured at the zero of gravitational potential at a great distance from bodies producing a gravitational field, will be assumed to have associated with it the potential energy $U\Psi/c^2$ when brought to a point where the gravitational potential has the value Ψ.

Consider now an isolated sphere of material which is held together by its own gravitational attraction, but has a small enough mass so that the gravitational field is everywhere weak. In accordance with the classical thermodynamics, this system should be in equilibrium provided it has the maximum entropy S consistent with its total energy U. To determine the temperature distribution which corresponds to these conditions, let us then consider a small variation in temperature distribution, leaving unaltered, however, the amount of substance of each component in each element of volume dV. For the variation in total entropy we shall write

$$\delta S = \int \frac{\delta u}{T} dV \tag{3}$$

where T is the temperature at the point where the element of volume dV is located, δu is the change in the intrinsic energy density at that point produced by the variation in temperature, and the integration is to be taken over the whole volume of the system. On the other hand for the variation in the total energy of the system, we shall write

$$\delta U = \int \left(1 + \frac{\Psi}{c^2}\right) \delta u \, dV \tag{4}$$

where Ψ is the gravitational potential at the point in question.

Setting equation (3) equal to zero as the condition that the entropy be a maximum, and equation (4) equal to zero as the subsidiary condition of constant energy, and combining by the method of Lagrange, we obtain

$$\int \left[\frac{1}{T} - \lambda\left(1 + \frac{\Psi}{c^2}\right)\right] \delta u \, dV = 0 \tag{5}$$

where λ is a constant undetermined multiplier, and this equation can only be true for arbitrary variations δu, if we have

$$\frac{1}{T} = \lambda\left(1 + \frac{\Psi}{c^2}\right) \tag{6}$$

which is the desired expression giving the distribution of the temperature T throughout the system as a function of the gravitational potential Ψ.

Differentiating this equation with respect to the radius r, to obtain an expression for the rate of change of the temperature with distance from the center of the gravitating sphere, we have

$$-\frac{1}{T^2}\frac{dT}{dr}=\frac{\lambda}{c^2}\frac{d\Psi}{dr}$$

and substituting the expression for λ which can be obtained from (6), we obtain

$$\frac{d\ln T}{dr}=-\frac{(1/c^2)(d\Psi/dr)}{1+\Psi/c^2}=-\frac{d}{dr}\ln\left(1+\frac{\Psi}{c^2}\right) \tag{7}$$

or since by hypothesis the field is weak enough so that Ψ/c^2 is small compared with unity we have approximately

$$\frac{d\ln T}{dr}=-\frac{1}{c^2}\frac{d\Psi}{dr}=-\frac{g}{c^2} \tag{8}$$

where g is the acceleration due to gravity.

For the case of a field of vanishing gravitational intensity with g negligible, the result degenerates into the usual condition for thermal equilibrium, $T=$constant. Furthermore, owing to the magnitude of the velocity of light the percentage change in temperature with height would be very small in fields of any ordinary intensity. Thus in a gravitational field having the strength of that at the surface of the earth we should have approximately

$$\frac{d\ln T}{dr}=-10^{-17}\text{ cm}^{-1}.$$

§3. Inadequacy of the Preliminary Treatment

The foregoing treatment contains of course many inadequacies and uncertainties, which can only be removed by a rigorous treatment of the problem from the point of view of general relativity. In the first place, in accordance with the classical thermodynamics, we have assumed maximum entropy and constant energy as the criteria of equilibrium for an isolated system, without any certain knowledge as to the form or even the validity of such principles in the presence of a gravitational field. Equation (3) for the entropy assumes it to depend on what we have called the intrinsic energy but the justification for this is by no means clear. Equation (4) for the energy, on the other hand, even if satisfactory for the case of weak gravitational fields,[3] can certainly not be regarded as correct in general when we recall the great modifications in the nature of the energy principle which have to be introduced in the theory of general relativity. Furthermore, it should be noted that we have given no very clear definition of the quantities temperature T and intrinsic energy density u which occur in the equations, although we might perhaps guess that we should take the values of these quantities as measured by local observers using proper coordinates. In addition, even the interpretation of the quantity dV appears dubious when we recall the differences between pro-

[3] See §8 in the article already referred to, This Journal, 888, for a satisfactory approximate treatment of the energy of a sphere of perfect fluid in a weak gravitational field.

per volume and coordinate volume which occur in the general theory of relativity.

It is hence abundantly evident that we cannot regard the results of the preceding section as having any certainty of validity, unless indeed we can show them to be an approximation in weak fields for conclusions which can be obtained from a more rigorous general relativity treatment. In what follows we shall now turn our attention to the general relativity treatment of the distribution of temperature and matter in a sphere of gravitating fluid which has come to thermodynamic equilibrium.

§4. The Relativity Mechanics of a Spherical Distribution of Perfect Fluid

As a preparation for our later work we must first consider the application of the mechanics of general relativity to a static distribution of perfect fluid having spherical symmetry. For a sufficiently general line element for such a system we may evidently write

$$ds^2 = -e^{\mu}\,(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) + e^{\nu}dt^2 \tag{9}$$

where the conditions that the system is to be static and spherically symmetrical are fulfilled if μ and ν can be taken as functions of r alone.

The components of the metrical tensor corresponding to this line element are evidently

$$\begin{aligned}
&g_{11}=-e^{\mu}, \qquad g_{22}=-e^{\mu}r^2, \qquad g_{33}=-e^{\mu}r^2\sin^2\theta, \qquad g_{44}=e^{\nu}\\
&g^{11}=-e^{-\mu}, \qquad g^{22}=-\frac{e^{-\mu}}{r^2}, \qquad g^{33}=-\frac{e^{-\mu}}{r^2\sin^2\theta}, \qquad g^{44}=e^{-\nu}\\
&g_{\rho\sigma}=g^{\rho\sigma}=0 \quad (\rho\neq\sigma) \qquad \sqrt{-g}=e^{\frac{3\mu+\nu}{2}}\,r^2\sin\theta
\end{aligned} \tag{10}$$

and the components of the contracted Riemann-Christoffel tensor corresponding to this metrical tensor have already been worked out and are known to have the values[4]

$$\begin{aligned}
&G_{11}=\mu''+\frac{\nu''}{2}+\frac{\mu'}{r}-\frac{\mu'\nu'}{4}+\frac{\nu'^2}{4}\\
&G_{22}=\frac{3}{2}r\mu'+\frac{r\nu'}{2}+\frac{r^2\mu''}{2}+\frac{r^2\mu'^2}{4}+\frac{r^2\mu'\nu'}{4}\\
&G_{33}=G_{22}\sin^2\theta\\
&G_{44}=-e^{\nu-\mu}\left[\frac{\nu''}{2}+\frac{\nu'}{r}+\frac{\mu'\nu'}{4}+\frac{\nu'^2}{4}\right]\\
&G_{\rho\sigma}=0 \qquad (\rho\neq\sigma)
\end{aligned} \tag{11}$$

[4] See Eddington, "The Mathematical Theory of Relativity," Cambridge 1923. The results in question can be obtained by setting $\lambda=\mu$ in Eddington's equations (43.5).

where the primes indicate differentiation with respect to r. Raising suffixes with the help of the values of the metrical tensor (10), and substituting into the fundamental equation

$$-8\pi T_\sigma^\rho = G_\sigma^\rho - \tfrac{1}{2} g_\sigma^\rho G \tag{12}$$

which connects the energy-momentum tensor T_σ^ρ with the metric we can then obtain, after a simple calculation, for the separate components of T_σ^ρ

$$\begin{aligned} -8\pi T_1^1 &= e^{-\mu}\left[\frac{\mu'+\nu'}{r} + \frac{\mu'\nu'}{2} + \frac{\mu'^2}{4}\right] \\ -8\pi T_2^2 = -8\pi T_3^3 &= e^{-\mu}\left[\frac{\mu''+\nu''}{2} + \frac{\mu'+\nu'}{2r} + \frac{\nu'^2}{4}\right] \\ -8\pi T_4^4 &= e^{-\mu}\left[\mu'' + \frac{2\mu'}{r} + \frac{\mu'^2}{4}\right]. \end{aligned} \tag{13}$$

On the other hand in the case of a perfect fluid the energy tensor is known to depend on the properties of the fluid in accordance with the equation[5]

$$T_{00} = (\rho_{00} + p_0)\frac{dx_\rho}{ds}\frac{dx_\sigma}{ds} - g^{\rho\sigma} p_0 \tag{14}$$

where ρ_{00} is the proper *macroscopic* density of energy in the fluid, p_0 its proper pressure and the quantities dx_ρ/ds are *macroscopic* "velocities." Since we assume a static condition in our fluid, these "velocities" will all be zero except for the case ρ or $\sigma = 4$, and in accordance with the line element (9) will then have the value $dx_4/ds = e^{-\nu/2}$. Hence noting the values for $g^{\rho\sigma}$ given by equations (10) we obtain for the separate components of $T^{\rho\sigma}$

$$T^{11} = e^{-\mu} p_0, \qquad T^{22} = \frac{e^{-\mu}}{r^2} p_0, \qquad T^{33} = \frac{e^{-\mu}}{r^2 \sin^2\theta} p_0, \qquad T^{44} = e^{-\nu}\rho_{00} \tag{15}$$

or lowering suffixes with the help of the metrical tensor (10) we have

$$T_1^1 = T_2^2 = T_3^3 = -p_0 \qquad T_4^4 = \rho_{00}. \tag{16}$$

Comparing these results with those given by equations (13) we can now write

$$p_0 = \frac{e^{-\mu}}{8\pi}\left[\frac{\mu'+\nu'}{r} + \frac{\mu'\nu'}{2} + \frac{\mu'^2}{4}\right] \tag{17}$$

$$p_0 = \frac{e^{-\mu}}{8\pi}\left[\frac{\mu''+\nu''}{2} + \frac{\mu'+\nu'}{2r} + \frac{\nu'^2}{4}\right] \tag{18}$$

$$\rho_{00} = -\frac{e^{-\mu}}{8\pi}\left[\mu'' + \frac{2\mu'}{r} + \frac{\mu'^2}{4}\right] \tag{19}$$

[5] See Eddington, reference 4, equations (54.81) and (54.82).

as expressions connecting the proper pressure p_0 and proper macroscopic density ρ_{00} with the metric. The reason that we obtain two separate expressions (17) and (18) connecting the pressure with the metric lies in the fact that the line element (9) with which we have started is sufficiently general to apply to any static distribution having spherical symmetry, while our later assumption of a perfect fluid involves at each point of the fluid an equality between the stresses in the radial and tangential directions.

Adding equations (17), (19), and (18) taken twice we obtain

$$(\rho_{00}+3p_0)=\frac{e^{-\mu}}{8\pi}\left(\nu''+\frac{2\nu'}{r}+\frac{\mu'\nu}{2}+\frac{\nu'^2}{2}\right)$$

and this can be rewritten in a form which will later prove useful

$$8\pi(\rho_{00}+3p_0)e^{\frac{3\mu+\nu}{2}}r^2=e^{\frac{\mu+\nu}{2}}\left(\nu''+\frac{2\nu'}{r}+\frac{\mu'\nu'}{2}+\frac{\nu'^2}{2}\right)r^2=\frac{d}{dr}(e^{\frac{\mu+\nu}{2}}\nu' r^2). \tag{20}$$

We are now ready to apply the principles of mechanics to our system, in accordance with the fundamental equation of relativity mechanics which can be written in the form.[6]

$$\frac{\partial \mathfrak{T}_\sigma^\rho}{\partial x_\rho}-\frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\sigma}=0 \tag{21}$$

Considering first the case $\sigma=1$, substituting values for the tensor *densities* of energy and momentum obtained from (15) and (16) through multiplication by $\sqrt{-g}$, and substituting for $g_{\alpha\beta}$ from (10), we can write

$$\frac{\partial}{\partial r}(-p_0\sqrt{-g})\ -\frac{1}{2}(e^{-\mu}p_0\sqrt{-g})\frac{\partial}{\partial r}(-e^\mu)\ -\frac{1}{2}\left(\frac{e^{-\mu}}{r^2}p_0\sqrt{-g}\right)\frac{\partial}{\partial r}(-e^\mu r^2)$$
$$-\frac{1}{2}\left(\frac{e^{-\mu}}{r^2\sin^2\theta}p_0\sqrt{-g}\right)\frac{\partial}{\partial r}(-e^\mu r^2\sin^2\theta)\ -\frac{1}{2}(e^{-\nu}\rho_{00}\sqrt{-g})\frac{\partial}{\partial r}(e^\nu)=0$$

and substituting the value $\sqrt{-g}=r^2\sin\theta\, e^{\frac{3\mu+\nu}{2}}$ given in (10), this is easily found to reduce to the form

$$\frac{\partial p_0}{\partial r}=-\frac{\rho_{00}+p_0}{2}\frac{\partial\nu}{\partial r} \tag{22}$$

which furnishes us with a very simple and useful expression for the change in pressure with radius. The equation may be regarded as the general relativity analogue of the Newtonian equation

$$\frac{dp}{dr}=-\rho\frac{d\Psi}{dr}$$

where ρ is now expressed in terms of mass per unit volume, and indeed can be shown to reduce to this in case the gravitational field is weak and the sys-

[6] See Eddington, reference 4, equation (55.6).

tem consists of ordinary matter with the pressure p_0 negligible compared with the density ρ_{00}.

It is also of interest to note that equation (22) can alternatively be derived by combining equations (17), (18) and (19), without the use of the mechanical principle given by equation (21). The mechanical principle and the equality of the two different expressions for pressure thus imply the same restrictions on the pressure distribution within the sphere.

Returning now to the mechanical equation (21) and considering the cases $\sigma=2$ and $\sigma=3$, it is easy to show by a similar method to the one just used that we obtain the results

$$\frac{\partial p_0}{\partial \theta}=\frac{\partial p_0}{\partial \phi}=0 \tag{23}$$

as was to be expected.

Finally, considering the case $\sigma=4$, we obtain by substitution into (21)

$$\frac{\partial}{\partial t}(\rho_{00}\sqrt{-g})-\frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t}=0$$

and since by our hypothesis of a static system the $g_{\alpha\beta}$ are independent of the time t, we have the result

$$\frac{\partial \rho_{00}}{\partial t}=0 \tag{24}$$

the proper density remaining constant with time at each point in the distribution, in complete agreement with our original assumption of a static system

§5. Mechanical Treatment of Temperature Distribution in the Case of Radiation

In the case of a spherical distribution of pure black-body radiation such as might surround a gravitating sphere of denser matter, relativistic mechanics without a resort to relativistic thermodynamics will be sufficient to determine the temperature distribution. This arises because of the simple relations in the case of black-body radiation directly connecting the thermodynamic quantity temperature with the mechanical quantities energy density and pressure.

For the energy density and pressure of black-body radiation we can evidently write in accordance with well known equations

$$\rho_{00}=aT_0^4 \tag{25}$$

and

$$p_0=\tfrac{1}{3}aT_0^4 \tag{26}$$

where a is the Stefan-Boltzmann constant and T_0 is the *proper* temperature, as measured by a local observer, located at a point where ρ_{00} and p_0 are the proper macroscopic energy density and proper pressure of the radiation.

Substituting these expressions into our mechanical equation (22), we at once obtain as exact relativistic expressions for the temperature distribution within a field of radiation

$$\frac{d\ ln\ T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr} \tag{27}$$

or

$$T_0 = Ce^{-\nu/2} \tag{28}$$

where C is a constant of integration. Furthermore, in the case of weak enough fields so that the Newtonian theory of gravitation is a sufficient approximation, equation (27) can be shown to reduce to

$$\frac{d\ ln\ T_0}{dr} = -\frac{1}{c^2}\frac{d\Psi}{dr} = -\frac{g}{c^2} \tag{29}$$

in agreement with equation (8) obtained by our preliminary treatment. The qualitative nature of this result is entirely reasonable since it is evident that the pressure of radiation must increase as we go to lower gravitational levels in order to support the increasing amount of the radiation above, and in the case of pure radiation such an increase in pressure can only be the result of an increase in temperature.

It is a matter of great interest that we have thus been able to determine, by a straightforward application of relativity mechanics without any resort to the new relativistic thermodynamics, the effect of gravity on temperature distribution in the particular case of a field of pure radiation. We have thus obtained in a rather unimpeachable manner a justification in at least one case for our original general idea, as expressed in §1, that the condition of thermodynamic equilibrium under the action of gravitation would involve higher temperatures at lower gravitational levels in order to prevent the downward flow of heat, and shall be ready to expect similar effects of gravitational action on temperature in more complicated cases whose solution will involve our new system of relativistic thermodynamics.

§6. The Thermodynamic Conditions for Equilibrium

To prepare for our thermodynamic treatment, we may now restate the conditions for equilibrium in a static system obtained in the previous development of relativistic thermodynamics.[7] Using the polar coordinates adopted in §4 of this article, these conditions took the form that the general relativity expression for the entropy of the system lying within the spherical shell between the constant radii r_1 to r_2 should be a maximum in accordance with the variational equation

$$\delta\left[4\pi\int_{r_1}^{r_2}\phi_0\, e^{3\mu/2}\, r^2 dr\right] = 0 \tag{30}$$

under the subsidiary condition coming from consideration of the energy-momentum principle

[7] See in particular §5 in the article in This Journal.

$$\delta\mu=\delta\mu'=\delta\nu=\delta\nu'=0 \quad (\text{at } r_1 \text{ and } r_2) \tag{31}$$

where μ and ν are the exponents occurring in our expression for the line element, and the quantity ϕ_0 occurring in equation (30) is the proper density of entropy.

§7. Thermodynamic Treatment of Temperature Distribution in the Case of Radiation

Let us now make use of the thermodynamic method to obtain a treatment of temperature distribution in the case of radiation. To do this we can re-express the entropy density ϕ_0 occurring in equation (30) in terms of energy density ρ_{00}, and by means of our previous relation connecting energy density with the metric can obtain the condition for maximum entropy in a form in which it depends explicitly on the metrical variable μ and its differential coefficients. The variation indicated in equation (30) can then be performed and the subsidiary conditions imposed by equation (31) on the variable μ easily introduced.

For the proper density of entropy ϕ_0 in terms of the proper temperature T_0 we can evidently write in accordance with the well known properties of black-body radiation

$$\phi_0=\frac{4}{3}aT_0{}^3 \tag{32}$$

where a is the Stefan-Boltzmann constant. Combining this with equation (25), connecting the energy density with temperature, we obtain

$$\phi_0=\frac{4}{3}a^{1/4}\rho_{00}{}^{3/4} \tag{33}$$

and using equation (19) which connects the energy density with the metric we can write

$$\phi_0=-\frac{4a^{1/4}}{24\pi}e^{-3\mu/4}\left[\mu''+\frac{2\mu'}{r}+\frac{\mu'^2}{4}\right]^{3/4}. \tag{34}$$

Substituting this into equation (30) and dividing out the constant factors, we finally obtain as the condition of maximum entropy in terms of the metrical variable μ

$$\delta\int_{r_1}^{r_2}\left(\mu''+\frac{2\mu'}{r}+\frac{\mu'^2}{4}\right)^{3/4}e^{3\mu/4}\,r^2dr=0 \tag{35}$$

under the condition

$$\delta\mu=\delta\mu'=0 \quad (\text{at } r_1 \text{ and } r_2). \tag{36}$$

Performing the variation indicated in equation (35) we obtain

$$\int_{r_1}^{r_2}\left[\frac{3}{4}\left(\mu''+\frac{2\mu'}{r}+\frac{\mu'^2}{4}\right)^{-1/4}\left(\delta\mu''+\frac{2}{r}\delta\mu'+\frac{\mu'}{2}\delta\mu'\right)e^{3\mu/4}\right.$$
$$\left.+\frac{3}{4}\left(\mu''+\frac{2\mu'}{r}+\frac{\mu'^2}{4}\right)^{3/4}e^{3\mu/4}\delta\mu\right]r^2\,dr=0 \tag{37}$$

and resubstituting the expression for energy density given by equation (19) this can be rewritten in the simpler form

$$\int_{r_1}^{r_2}\left[\frac{e^{(\mu/2)}r^2}{\rho_{00}{}^{1/4}}\left(\delta\mu''+\frac{2}{r}\delta\mu'+\frac{\mu'}{2}\delta\mu'\right)-8\pi\rho_{00}{}^{3/4}\,e^{3\mu/2}\,r^2\,\delta\mu\right]dr=0$$

Noting, however, in accordance with equation (36) that $\delta\mu'$ and $\delta\mu$ are equal to zero at the limits r_1 and r_2 this can evidently be transformed in the usual manner with the help of partial integrations, dropping terms that become zero at the limits, to the following form containing $\delta\mu$ alone.

$$\int_{r_1}^{r_2}\left[\frac{d^2}{dr^2}\left(\frac{e^{\mu/2}r^2}{\rho_{00}{}^{1/4}}\right)-2\frac{d}{dr}\left(\frac{e^{\mu/2}r}{\rho_{00}{}^{1/4}}\right)-\frac{d}{dr}\left(\frac{e^{\mu/2}r^2}{\rho_{00}{}^{1/4}}\,\frac{\mu'}{2}\right)-8\pi\rho_{00}{}^{3/4}\,e^{3\mu/2}\,r^2\right]\delta\mu dr=0. \quad (38)$$

This equation, however, can be true for arbitrary variations $\delta\mu$ only if the quantity in the square brackets is equal to zero, and this can evidently be rewritten to give us

$$\frac{d}{dr}\left\{e^{\mu/2}\,r^2\,\frac{d}{dr}\left(\frac{1}{\rho_{00}{}^{1/4}}\right)\right\}=8\pi\rho_{00}{}^{3/4}\,e^{3\mu/2}\,r^2. \quad (39)$$

And substituting the relation betwen energy density ρ_{00} and temperature T_0 given by equation (25), this becomes

$$\frac{d}{dr}\left\{e^{\mu/2}\,r^2\,\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\}=8\pi\,\frac{\rho_{00}}{T_0}e^{3\mu/2}\,r^2. \quad (40)$$

To solve this equation for T_0 as a function of r, we note that in the case of radiation we have $8\pi\rho_{00}=4\pi(\rho_{00}+3p_0)$ and substituting equation (20) we can then rewrite (40) in the form

$$\frac{d}{dr}\left\{e^{\mu/2}\,r^2\,\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\}=\frac{e^{-\nu/2}}{T_0}\,\frac{d}{dr}\left\{e^{\mu/2}\,r^2\,\frac{d}{dr}(e^{\nu/2})\right\}$$

or

$$\left(\frac{1}{T_0}\right)^{-1}\frac{d}{dr}\left\{e^{\mu/2}\,r^2\,\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\}=(e^{\nu/2})^{-1}\,\frac{d}{dr}\left\{e^{\mu/2}\,r^2\,\frac{d}{dr}(e^{\nu/2})\right\}. \quad (41)$$

Postponing a more general discussion of the solution of this equation until the next section, §8, where the equation again occurs, we note at once that a particular solution of the equation is given by

$$T_0 = Ce^{-\nu/2} \quad (42)$$

where C is a constant of integration and this equation for the temperature distribution in a field of radiation which has come to equilibrium under the action of gravity is exactly the same as equation (28) which we obtained in §5. Hence in this simple case of radiation, where a mechanical treatment can be given, the thermodynamic and the mechanical treatment of temperature

distribution under the action of gravity lead to the same result. This can be regarded as some confirmation of the validity of the system of relativistic thermodynamics which I have proposed.

§8. Temperature Distribution in the General Case of Any Perfect Fluid

Let us now consider the application of our condition for a maximum of the relativistic expression for entropy

$$\delta\left[4\pi\int_{r_1}^{r_2}\phi_0\, e^{3\mu/2}r^2dr\right]=0 \tag{43}$$

to a determination of the temperature distribution in the general case of any perfect fluid. To apply this equation it will be convenient to note that $4\pi e^{3\mu/2}r^2\,dr$ is evidently the *proper* spatial volume lying in the spherical shell between r and $r+dr$. This can be seen from the general relation for the element of proper spatial volume

$$dV_0\,ds = \sqrt{-g}dx_1dx_2dx_3dx_4$$

which, in the case of our system and line element (9), gives

$$dV_0=e^{\frac{3\mu+\nu}{2}}r^2\sin\theta drd\theta d\phi\frac{dt}{ds} = e^{3\mu/2}\,r^2\,\sin\theta drd\theta d\phi.$$

And if we integrate this over all values of θ and ϕ, we obtain

$$V_0 = 4\pi e^{3\mu/2}\,r^2\,dr \tag{44}$$

for the proper spatial volume lying between r and $r+dr$, where V_0 is of course an infinitesimal quantity.

Under these circumstances, since ϕ_0 is the proper density of entropy, we may evidently write

$$S_0 = 4\pi\,\phi_0\,e^{3\mu/2}\,r^2\,dr \tag{45}$$

as an expression for the entropy, as measured in proper coordinates, associated with the material lying between r and $r+dr$. As this quantity of entropy is measured in proper coordinates, it will evidently obey the classical laws of thermodynamics and may be taken as a function of temperature T_0, volume V_0 and number of molecules $N_1, N_2 \cdots N_n$ of the n different components which are necessary to specify the composition, in the way ordinarily employed in thermodynamics.

We are now ready to investigate the temperature distribution in our fluid. To do this let us consider the effect of a variation in the temperature T_0 and volume V_0 associated with the shell r to $r+dr$, keeping the composition of the material in this shell unaltered, that is holding $N_1 \cdots N_n$ constant. Under these conditions, we shall have in accordance with the principles of ordinary thermodynamics

$$\delta S_0 = \frac{\delta U_0}{T_0} + \frac{p_0}{T_0}\delta V_0 \tag{46}$$

and in accordance with equations (44) and (45) this can evidently be written in the form

$$\delta(4\pi\, \phi_0\, e^{3\mu/2}\, r^2\, dr) = \frac{\delta(4\pi\, \rho_{00}\, e^{3\mu/2}\, r^2\, dr)}{T_0} + \frac{p_0}{T_0}\,\delta(4\pi\, e^{3\mu/2}\, r^2\, dr). \tag{47}$$

Substituting this in equation (43) for the maximum of the relativistic expression for entropy we now obtain

$$4\pi\int_{r_1}^{r_2}\left[\frac{\delta(\rho_{00}\, e^{3\mu/2})}{T_0} + \frac{p_0}{T_0}\,\delta(e^{3\mu/2})\right] r^2 dr = 0 \tag{48}$$

which gives the condition for a maximum of the relativistic entropy in a form which can be made to depend solely on the variation of the metrical variable μ and its derivatives, by substituting for the energy density ρ_{00} its values as given by equation (19). Doing so we obtain

$$\int_{r_1}^{r_2}\left[-\frac{1}{2T_0}\,\delta\left\{e^{\mu/2}\left(\mu'' + \frac{2\mu'}{r} + \frac{\mu'^2}{4}\right)\right\} + \frac{4\pi p_0}{T_0}\,\delta(e^{3\mu/2})\right] r^2 dr = 0 \tag{49}$$

Performing the indicated variations, we have

$$\int_{r_1}^{r_2}\left[-\frac{e^{\mu/2}}{2T_0}\left(\delta\mu'' + \frac{2\delta\mu'}{r} + \frac{\mu'}{2}\delta\mu'\right) - \frac{e^{\mu/2}}{4T_0}\left(\mu'' + \frac{2\mu'}{r} + \frac{\mu'^2}{4}\right)\delta\mu \right.$$
$$\left. + \frac{12\pi p_0}{2T_0}\, e^{3\mu/2}\,\delta\mu\right] r^2 dr = 0.$$

Noting, however, that the variations are to be carried out in accordance with equation (31) which gives $\delta\mu' = \delta\mu = 0$ at r_1 and r_2, the first term in the integrand can evidently be transformed in the usual manner with the help of partial integrations, dropping terms that become zero at the limits, to a form depending on $\delta\mu$ alone; and the second term in the integrand can be simplified by resubstituting equation (19) for the energy density ρ_{00}. We thus obtain

$$\int_{r_1}^{r_2}\left[\frac{d^2}{dr^2}\left(\frac{e^{\mu/2}\, r^2}{T_0}\right) - 2\frac{d}{dr}\left(\frac{e^{\mu/2}\, r}{T_0}\right) - \frac{d}{dr}\left(\frac{e^{\mu/2}\, r^2}{T_0}\,\frac{\mu'}{2}\right) - \frac{4\pi(\rho_{00} + 3p_0)}{T_0} e^{3\mu/2}\, r^2\right]\delta\mu\, dr = 0.$$

This equation can be true, however, for arbitrary variations $\delta\mu$ only if the term in square brackets is equal to zero, and this can evidently be rewritten to give us

$$\frac{d}{dr}\left\{e^{\mu/2}\, r^2\,\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\} = \frac{4\pi(\rho_{00} + 3p_0)}{T_0}\, e^{3\mu/2}\, r^2 \tag{50}$$

and substituting equation (20), we finally obtain as an expression connecting T_0 with the coordinate r and the metrical variables μ and ν

$$\frac{d}{dr}\left\{e^{\mu/2}r^2\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\} = \frac{e^{-\nu/2}}{T_0}\frac{d}{dr}\left\{e^{\mu/2}r^2\frac{d}{dr}(e^{\nu/2})\right\}. \tag{51}$$

This differential equation for T_0 is the same as equation (41) which we got when we applied the thermodynamic method to pure radiation. A first integral for this equation can be written in the form

$$\frac{d\ ln\ T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr} + \frac{Be^{-\frac{\mu+\nu}{2}}}{r^2}T_0 \tag{52}$$

where B is a constant of integration, and substituting equation (22) this can be written as

$$\frac{d\ ln\ T_0}{dr} = \frac{1}{\rho_{00}+p_0}\frac{dp_0}{dr} + \frac{Be^{-\frac{\mu+\nu}{2}}}{r^2}T_0.$$

If, however, we now assume on physical grounds that at the center of the sphere $r=0$, we have dT_0/dr and dp_0/dr equal to zero, T_0 not equal to zero and the other functions of r finite, it is evident that the constant of integration B must be equal to zero.[8] Under these circumstances the second integral of our equation is then easily seen to be

$$T_0 = Ce^{-\nu/2} \tag{53}$$

where C is a second constant of integration.

This result is exactly of the same form as that obtained for radiation alone and that is a satisfactory outcome since our method of derivation did not involve the assumption that any matter at all was necessarily present. The equation gives a definite relation connecting the equilibrium temperature in a gravitational field with the variable ν which is itself determined in a known way by the gravitational field. The general nature of the result can perhaps be more easily appreciated by redifferentiating equation (53) with respect to r and substituting equation (22). We then obtain

$$\frac{d\ ln\ T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr} = \frac{1}{\rho_{00}+p_0}\frac{dp_0}{dr} \tag{54}$$

and can at once see that temperature and pressure will increase together as we go towards the center of the sphere. It is also of interest to note once more the approximation

$$\frac{d\ ln\ T_0}{dr} = -\frac{1}{c^2}\frac{d\Psi}{dr} = -\frac{g}{c^2}$$

[8] It would be interesting to investigate the possibility of solutions of physical interest with $B\neq 0$.

valid in weak enough fields so that the Newtonian method is applicable, and this is the same equation as was obtained in our preliminary non-rigorous treatment.[9]

§9. The Distribution of Matter in a Sphere of Perfect Fluid

The principles of relativistic thermodynamics which we have used in the preceding section should be sufficient to give us information not only as to the temperature distribution in a sphere of fluid but also as to the equilibrium distribution of matter. To obtain such information let us return once more to our condition for a maximum of the relativistic expression for entropy

$$\delta\left[4\pi\int_{r_1}^{r_2}\phi_0\, e^{3\mu/2}\, r^2\, dr\right]=0. \tag{55}$$

In using this equation to determine the distribution of temperature, we considered the result of varying the temperature as a function of r, holding constant the number of molecules $N_1 \cdots N_n$ of each of the different components of the system in each spherical shell lying between r and $r+dr$. To determine the distribution of matter, on the other hand, we may hold the temperature constant and consider the result of varying the composition of the layers. As simple illustrations, we shall apply this method to a system consisting of a mixture of a perfect monatomic gas and radiation, both on the assumption that radiation and matter are interconvertible, and on the assumption that the total number of molecules in the system cannot be varied.

For the proper density of entropy of a mixture of perfect monatomic gas and radiation, we may evidently write in accordance with well known equations

$$\phi_0=N_0k\ ln\frac{CT_0^{3/2}}{N_0}+\frac{4}{3}aT_0^3 \tag{56}$$

where N_0 is the number of molecules in unit volume as measured in proper coordinates by a local observer, T_0 is the proper temperature, k and a are respectively the Boltzmann constant and Stefan constant, and C is a constant so chosen that the starting points for the measurement of the entropy of matter and radiation will be in agreement.

If now we vary the proper concentration of molecules N_0 holding the temperature T_0 constant, we obtain

$$\delta\phi_0=\left(k\ ln\frac{CT_0^{3/2}}{N_0}-k\right)\delta N_0. \tag{57}$$

On the other hand for the proper density of energy we evidently have

$$\rho_{00}=N_0mc^2+\tfrac{3}{2}N_0kT_0+aT_0^4 \tag{58}$$

[9] As a justification for the results of this section, my colleague Dr. J. Robert Oppenheimer has kindly pointed out to me that the relation which I have obtained between temperature and gravitational potential is a necessary one if the Planck radiation law is to hold at different gravitational levels.

where m is the rest mass of one molecule and c the velocity of light,—mc^2 thus being the internal energy and $(3/2)kT_0$ the average kinetic energy per molecule. Hence, holding the temperature constant as before, and varying N_0 we obtain

$$\delta\rho_{00}=(mc^2+\tfrac{3}{2}kT_0)\delta N_0 \tag{59}$$

and by combining equations (57) and (59) we can express the variation in the density of entropy in terms of the variation in density of energy in the form

$$\delta\phi_0=\frac{k\ ln\ CT_0{}^{3/2}/N_0-k}{mc^2+3/2kT_0}\,\delta\rho_{00}$$

or

$$\delta\phi_0=f\,(N_0,T_0)\ \delta\rho_{00} \tag{60}$$

where for simplification we have written $f(N_0,\ T_0)$ as an abbreviation for the coefficient of $\delta\rho_{00}$.

Since $\delta\rho_{00}$, however, can be expressed as a function of the metrical variable μ, we can now return to our equation (55) for maximum entropy and write as a condition for equilibrium

$$4\pi\int_{r1}^{r_2}\left[\delta\phi_0\,e^{3\mu/2}+\tfrac{3}{2}\phi_0\,e^{3\mu/2}\,\delta\mu\right]r^2dr=0$$

or substituting equation (60)

$$4\pi\int_{r^1}^{r_2}\left[f(N_0,T_0)e^{3\mu/2}\,\delta\rho_{00}+\tfrac{3}{2}\phi_0\,e^{3\mu/2}\,\delta\mu\right]r^2dr=0 \tag{61}$$

and substituting in turn the value of ρ_{00} in terms of the metrical variable give by equation (19) we obtain

$$\int_{r1}^{r_2}\left[-\frac{1}{2}f(\,N_0,T_0)e^{\mu/2}\left(\delta\mu''+\frac{2}{r}\delta\mu'+\frac{\mu'}{2}\delta\mu'\right)\right.$$
$$\left.+\frac{1}{2}f(\,N_0,T_0)e^{\mu/2}\left(\mu''+\frac{2\mu'}{r}+\frac{\mu'^2}{4}\right)\delta\mu+4\pi\frac{3}{2}\phi_0\,e^{3\mu/2}\,\delta\mu\right]r^2dr=0.$$

Noting as in previous sections, however, that the variations are to be carried out in accordance with equation (31) which gives $\delta\mu'=\delta\mu=0$ at r_1 and r_2, the first term in the integrand can evidently be transformed in the usual manner with the help of partial integrations, dropping terms that become zero at the limits, to a form depending on $\delta\mu$ alone; and the second term in the integrand can be simplified by resubstituting equation (19) for the energy density ρ_{00}. We thus obtain

$$\int_{r_1}^{r_2}\left[\frac{d^2}{dr^2}\{f(N_0,T_0)e^{\mu/2}r^2\}-2\frac{d}{dr}\{f(N_0,T_0)e^{\mu/2}\,r\}\right.$$
$$\left.-\frac{d}{dr}\left\{f(N_0,T_0)e^{\mu/2}\,r^2\,\frac{\mu'}{2}\right\}+4\pi\left\{2f(N_0,T_0)\rho_{00}-3\phi_0\right\}e^{3\mu/2}\,r^2\right]\delta\mu dr=0$$

and combining the first three terms this can be rewritten in the form

$$\int_{r_1}^{r_2}\left[\frac{d}{dr}\left\{e^{\mu/2}r^2\frac{d}{dr}f(N_0,T_0)\right\}+4\pi\left\{2f(N_0,T_0)\rho_{00}-3\phi_0\right\}e^{3\mu/2}r^2\right]\delta\mu dr=0. \quad (62)$$

This equation has resulted from an application of the condition for maximum relativistic entropy under the subsidiary condition, given by equation (31), which satisfies the requirements of the energy-momentum principle. In case we assume the ready interconvertibility of matter and radiation no further subsidiary conditions need be introduced. On the other hand, if we are interested in what might be only a temporary condition of equilibrium reached in a length of time such that the total number of molecules could not change, we must evidently add as a further subsidiary condition

$$\delta\left[4\pi\int_{r1}^{r_2}N_0\,e^{3\mu/2}\,r^2\,dr\right]=0 \quad (63)$$

where N_0 is the proper concentration of molecules and as we saw above $4\pi\,e^{3\mu/2}\,r^2dr$ is the proper volume in the spherical shell lying between r and $r+dr$.

Performing the indicated variation, keeping of course the temperature constant as in the previous variation, and substituting the value for δN_0 given by equation (58), we obtain

$$4\pi\int_{r_1}^{r^2}\left[\frac{e^{3\mu/2}}{mc^2+(3/2)kT_0}\delta\rho_{00}+\frac{3}{2}N_0\,e^{3\mu/2}\,\delta\mu\right]r^2dr=0.$$

This equation, however, is of the same general form as (61) above, and can be treated by the same methods which led from (61) to (62), and will then evidently reduce to

$$\int_{r_1}^{r_2}\left[\frac{d}{dr}\left\{e^{\mu/2}r^2\frac{d}{dr}\left(\frac{1}{mc^2+(3/2)kT_0}\right)\right\}\right.$$
$$\left.+4\pi\left\{\frac{2\rho_{00}}{mc^2+(3/2)kT_0}-3N_0\right\}e^{3\mu/2}r^2\right]\delta\mu dr=0. \quad (64)$$

Equations (64) and (62) may now be combined by the method of Lagrange to give us a single condition

$$\int_{r_1}^{r_2}\left[\frac{d}{dr}\left\{e^{\mu/2}r^2\frac{d}{dr}\left(f(N_0,T_0)+\frac{\lambda}{mc^2+(3/2)kT_0}\right)\right\}\right.$$
$$\left.+4\pi\left\{2\left(f(N_0,T_0)+\frac{\lambda}{mc^2+(3/2)kT_0}\right)\rho_{00}-3\phi_0-3\lambda N_0\right\}e^{3\mu/2}r^2\right]\delta\mu dr=0$$

where λ is a constant undetermined multiplier, whose value will be zero in case equilibrium between matter and radiation is established, since the additional condition (63) will then not be needed. This equation, however, can only be true for arbitrary variations $\delta\mu$, if the quantity in square brackets

is equal to zero. Hence, resubstituting the value of $f(N_0, T_0)$ given by equation (60) we have as our final differential equation

$$\frac{d}{dr}\left\{e^{\mu/2}\,r^2\frac{d}{dr}\,\frac{k\ ln\ CT_0^{3/2}/N_0-k+\lambda}{mc^2+(3/2)kT_0}\right\}$$
$$=4\pi\left\{3\phi_0+3\lambda N_0-2\frac{k\ ln\ CT_0^{3/2}/N_0-k+\lambda}{mc^2+(3/2)kT_0}\rho_{00}\right\}e^{3\mu/2}\,r^2 \quad (65)$$

where $\lambda=0$ in case the equilibrium between matter and radiation is established.

We have thus obtained a second order differential equation connecting the thermodynamic quantities concentration N_0 and temperature T_0 with the coordinate r. We have already found in the previous section, §8, however, equations for T_0 as a function of r, and hence the present equation requires that N_0 be such a function of T_0 as not to disagree with the previous results. We shall then suggest as an expression for the dependence of N_0 on T_0 and hence implicitly on r

$$k\ ln\frac{CT_0^{3/2}}{N_0}-k+\lambda=\frac{mc^2}{T_0}+\frac{3}{2}k \quad (66)$$

or

$$N_0=Ce^{\lambda/k-5/2}T_0^{3/2}\,e^{-mc^2/kT_0}$$

and test the suggestion by substituting into (65). Doing so, we obtain

$$\frac{d}{dr}\left\{e^{\mu/2}\,r^2\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\}=4\pi\left\{3\phi_0+3\lambda N_0-\frac{2\rho_{00}}{T_0}\right\}e^{3\mu/2}\,r^2.$$

Comparing the first form of equation (66), however, with the expression for ϕ_0 given by (56), and introducing the expression for ρ_{00} given by (58), we obtain

$$\frac{d}{dr}\left\{e^{\mu/2}\,r^2\frac{d}{dr}\left(\frac{1}{T_0}\right)\right\}=4\pi\left\{\frac{3N_0mc^2}{T_0}+\frac{15}{2}N_0k-3\lambda N_0+4aT_0^3\right.$$
$$\left.+3\lambda N_0-2\frac{N_0mc^2}{T_0}-3N_0k-2aT_0^3\right\}e^{3\mu/2}\,r^2$$
$$=4\pi\left\{\frac{N_0mc^2+(3/2)N_0kT_0+aT_0^4+3(N_0kT_0+(a/3)T_0^4)}{T_0}\right\}e^{3\mu/2}\,r^2$$
$$=4\pi\frac{(\rho_{00}+3p_0)}{T_0}e^{3\mu/2}\,r^2 \quad (67)$$

since the gas pressure is evidently N_0kT_0 and the radiation pressure $\frac{1}{3}aT_0^4$. And this equation is exactly the same as our general equation (50) for the dependence of temperature on the coordinate and metric as obtained in the previous section §8. Hence our suggested expression equation (66) for the dependence of the concentration on temperature can be accepted as

satisfying the information previously obtained as to the dependence of temperature on the metric, and thus as a satisfactory solution of the problem of the distribution of a perfect monatomic gas in a gravitational field.

It is also significant to point out that equation (66) can be subjected to another test, since this expression for the proper concentration of the gas makes it possible to calculate the partial pressure of the gas and by combining with the partial pressure of radiation, then examine the dependence of total pressure on the radius r to see if it agrees with the purely mechanical equation $dp_0/dr = -(\rho_{00}+p_0)\nu'/2$, obtained in §4. And as a matter of fact equation (66) does lead to this result.[10]

Finally, it is interesting to compare the dependence of proper concentration on proper temperature given by equation (66) with the result obtained for the equilibrium concentration in flat space-time by Stern[11] and myself.[12] Taking the case where equilibrium between matter and radiation has been established, we have $\lambda = 0$ and equation (66) can be written

$$N_0 = (Ce^{5/2})\, T_0{}^{3/2}\, e^{-mc^2/kT_0}$$

which agrees with my previous result as it stands, and agrees with the Stern result if we assign to the constant term $(Ce^{5/2})$ the not necessarily correct value which Stern obtained for it. Hence the conclusion can again be stated, owing to the great magnitude of the negative exponent for reasonable values of m and T, that the equilibrium concentration would be extremely small unless the constant term could be shown to have an enormous magnitude.

§10. Conclusion

In accordance with the special theory of relativity and the equivalence principle all forms of energy, including heat, must be regarded as having both inertia and weight, and the purpose of the present article has been to investigate, in as consequent a manner as may be, the thermodynamic implications of the notion that heat has weight. The most striking result of the investigation has been the discovery of a definite relation connecting gravitational potential with the distribution of temperature throughout a system which has come to thermodynamic equilibrium.

Qualitatively, the increase in equilibrium temperature which was found to accompany decrease in gravitational potential, may be regarded as due to the necessity of having a temperature gradient to prevent the flow of heat from places of higher to those of lower potential energy; and quantitatively, a first approximation to the magnitude of this temperature gradient was

[10] It may seem strange that this purely mechanical equation holding within the interior of the system should be derivable from the application of thermodynamics to the system as a whole. The result, however, is the relativity analogue to the equation for change in pressure with height obtained by Gibbs ("Scientific Papers," Longmans, Green 1906, equation 230, p. 145) in his thermodynamic treatment of the conditions of equilibrium under the influence of gravity. Indeed the whole treatment of this article may be regarded as the relativistic extension of this part of Gibbs' work.

[11] Stern, Zeits. f. Elektrochem. **31,** 448 (1925); Trans. Farad. Soc. **21,** 477 (1925–26).

[12] Tolman, Proc. Nat. Acad. **12,** 670 (1926).

obtained by modifying the classical thermodynamics merely by ascribing to each given intrinsic quantity of energy the right additional quantity of potential gravitational energy. For a rigorous treatment, however, it was obviously necessary to investigate the whole problem from the standpoint of general relativity, and this was done using the extension of thermodynamics to general relativity which I have previously given. In this way it was possible to obtain what appears to be a rigorous equation connecting equilibrium temperature with gravitational potential. In addition in the case of black-body radiation it was possible to test the thermodynamic method, since the same temperature distribution in the case of this simple system was also obtained by the use of relativistic mechanics, without the necessity for the use of the new relativistic thermodynamics.

This discovery of a dependence of equilibrium temperature on gravitational potential must be regarded as something essentially new in thermodynamics, since uniform temperature throughout any system which has come to equilibrium has hitherto been taken as an inescapable part of thermodynamic theory. The new result hence has a very considerable theoretical interest, and even though the effect of gravitational potential on temperature may usually be extremely small the result may sometime be of experimental or observational interest.