TEMPERATURE EQUILIBRIUM IN A STATIC GRAVITATIONAL FIELD

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(Received October 27, 1930)

Abstract

In the case of a gravitating mass of perfect fluid which has come to thermodynamic equilibrium, it has previously been shown that the proper temperature T_0 as measured by a local observer would depend in a definite manner on the gravitational potential at the point where the measurement is made. In the present article the conditions of thermal equilibrium are investigated in the case of a general static gravitational field which could correspond to a system containing solid as well as fluid parts. Writing the line element for the general static field in the form

$$ds^2 = g_{ij}dx_i dx_j + g_{44}dt^2 \qquad i, j=1, 2, 3,$$

where the g_{ij} and g_{44} are independent of the time t, it is shown that the dependence of proper temperature on position at thermal equilibrium is such as to make the quantity $T_0\sqrt{g_{44}}$ a constant throughout the system.

§ I. Introduction

In several previous articles the principles of relativistic thermodynamics have been discussed[1] and then applied[2] to determine the conditions for thermodynamic equilibrium in the presence of gravitational fields. In the case of a gravitating mass of perfect fluid which has come to thermodynamic equilibrium, it was shown in the course of the work[3] that the proper temperature as measured by a local observer would depend in a definite manner on the gravitational potential at the point where the measurement is made. Thus if we write the line element, for a spherical mass of fluid which has come to equilibrium, in the form

$$ds^2 = -e^{\mu}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) + e^{\nu}dt^2 \tag{1}$$

where μ and ν are functions of the coordinate r, the dependence of the proper temperature T_0 on position was found to be given by the equation

$$\frac{d\log T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr}. \tag{2}$$

This result is a very interesting one since uniform temperature throughout a system which has come to thermodynamic equilibrium has hitherto been regarded as an inescapable part of thermodynamic theory. In accordance with this equation, however, the proper temperature is found to

[1] Tolman, Proc. Nat. Acad. **14**, 268 (1928); ibid. **14**, 701 (1928); Phys. Rev. **35**, 875 (1930); ibid. **35**, 896 (1930).

[2] Tolman, Proc. Nat. Acad. **14**, 348 (1928); ibid. **14**, 353 (1928); Phys. Rev. **35**, 904(1930).

[3] See reference 2, last article.

increase as we move inwards towards the center of the sphere, and this can be qualitatively interpreted by ascribing to heat the property of weight and regarding the increase in temperature as we move inwards as necessary in order to prevent the flow of heat from higher to lower gravitational levels.

Integrating equation (2) we obtain the result

$$\log T_0 + \frac{\nu}{2} = \text{const.}$$

or

$$T_0 e^{\nu/2} = T_0\sqrt{g_{44}} = \text{const.} \tag{3}$$

and have thus obtained a quantity which has a constant value throughout the fluid even though the proper temperature itself does not. This result, however has so far only been proved for the equilibrium condition of a perfect fluid, and the question naturally arises whether the quantity $T_0\sqrt{g_{44}}$ would also have a constant value in the case of a more complicated system containing solid parts. In the present article we shall investigate temperature equilibrium in the case of a general static gravitational field by considering that the parts of the system whose temperatures are to be compared are in thermal contact with a small connecting tube containing radiation. Such a tube may be called a *radiation thermometer*, and by calculating the change in pressure as we go from one portion of the tube to another it will be easy to show that the radiation has the same value of $T_0\sqrt{g_{44}}$ throughout the thermometer.

§ 2. The Energy-Momentum Tensor for Black-Body Radiation

In order to solve our problem by the method suggested, we shall need an expression for the energy-momentum tensor for black-body radiation and shall take this as given by the equation

$$T^{\mu\nu} = (\rho_{00} + p_0)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} - g^{\mu\nu}p_0 \tag{4}$$

with

$$\rho_{00} = 3p_0 \tag{5}$$

where ρ_{00} is the proper macroscopic density of the radiation at the point of interest, p_0 its proper pressure, and the velocities dx_μ/ds correspond to the macroscopic motion of the radiation; i.e., to the motion, in the coordinate system which is being used, of an observer who finds on the average no net flow of energy in the radiation field.

Equation (4) is well known in general relativity as being an expression for the energy-momentum tensor of a perfect fluid,[4] and the primary justification for adopting it as applying to black body radiation resides in its

[4] See for example Eddington, "The Mathematical Theory of Relativity," Cambridge1923, equations (54.81) and (54.82).

general applicability in studying the relativistic mechanics of any system whose local properties can be specified by the two scalars, proper density ρ_{00} and pressure p_0. And such is evidently the case for black-body radiation, with the additional simplification $\rho_{00}=3p_0$.

In addition to this primary justification, however, it will also be of interest to start with the usual relativistic generalization of the Maxwell-Lorentz electromagnetic equations and derive the above expression for the energy-momentum tensor by treating black-body radiation as an electromagnetic phenomenon.

In accordance with this relativistic generalization, the electromagnetic field at any point can be specified by a certain antisymmetric tensor $F_{\mu\nu}$ whose components are directly related in Galilean coordinates to the classical components of electric field strength X, Y, Z and magnetic field strength α, β, γ. And the energy-momentum tensor is given in terms of the $F_{\mu\nu}$ by the equation[5]

$$T^{\mu\nu} = - g^{\mu\gamma}F^{\nu\alpha}F_{\gamma\alpha} + \tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{6}$$

Using Galilean coordinates at the point of interest and substituting for the components of $F_{\mu\nu}$ the values which they then have, it is found that the components of $T^{\mu\nu}$ assume the values indicated by the following typical examples[6]

$$\begin{aligned} T^{11} &= - \tfrac{1}{2}(X^2 - Y^2 - Z^2) - \tfrac{1}{2}(\alpha^2 - \beta^2 - \gamma^2) \\ T^{12} &= - \alpha\beta - XY \\ T^{14} &= - \beta Z + \gamma Y \\ T^{44} &= \tfrac{1}{2}(X^2 + Y^2 + Z^2) + \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) \end{aligned} \tag{7}$$

Thus in Galilean coordinates T^{11}, T^{22}, T^{33}, T^{12}, T^{13}, and T^{23} become the classical components of the stress in the field, T^{14}, T^{24}, and T^{34} become the classical components of the Poynting vector, and T^{44} becomes the classical density of energy.

If now we consider that the electromagnetic field in question corresponds to black-body radiation, and take a proper system of Galilean coordinates in which the radiation as a whole is at rest, it is evident that we shall have *on the average*

$$X^2 = Y^2 = Z^2 \quad \text{and} \quad \alpha^2 = \beta^2 = \gamma^2 \tag{8}$$

since the average field strength will be independent of direction,

$$XY = YZ = ZX = 0 \quad \text{and} \quad \alpha\beta = \beta\gamma = \gamma\alpha = 0 \tag{9}$$

since the lack of phase relations between waves will make it equally probable that the instantaneous values of these products will have positive or negative magnitude, and

[5] See Eddington, reference 4, equation (77.2).

[6] See Eddington, reference 4, equations (77.41–2–3–4).

$$-\beta Z + \gamma Y = -\gamma X + \alpha Z = -\alpha Y + \beta X = 0 \tag{10}$$

since there will be no net flow of energy in the proper system of coordinates that we are now using. Hence, introducing these results into equations 7, it is evident that in proper coordinates the averaged energy-momentum tensor for black-body radiation will have as its only surviving components

$$T^{11} = T^{22} = T^{33} = p_0 \quad \text{and} \quad T^{44} = \rho_{00} \tag{11}$$

with

$$\rho_{00} = 3p_0 \tag{12}$$

where ρ_{00} is the proper macroscopic density at the point of interest and the three surviving components of the stress are each equal to the pressure p_0.

This expression for the energy-momentum tensor of black-body radiation holds, however, only for a set of proper coordinates x, y, z, t, in which there is no net flow of energy, and to pass to a general set of coordinates x_1', x_2', x_3', x_4' we shall have to substitute the above values into the general transformation equation

$$T'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x_\nu'}{\partial x_\beta}T^{\alpha\beta}.$$

Doing so we at once obtain

$$T'^{\mu\nu} = (\rho_{00} + p_0)\frac{\partial x_\mu'}{\partial t}\frac{\partial x_\nu'}{\partial t} + p_0\left(\frac{\partial x_\mu'}{\partial x}\frac{\partial x_\nu'}{\partial x} + \frac{\partial x_\mu'}{\partial y}\frac{\partial x_\nu}{\partial y} + \frac{\partial x_\mu'}{\partial z}\frac{\partial x_\nu'}{\partial z} - \frac{\partial x_\mu'}{\partial t}\frac{\partial x_\nu'}{\partial t}\right) \tag{13}$$

For the macroscopic velocity of the radiation in our new set of coordinates, however, we can evidently write

$$\frac{dx_\mu'}{ds} = \frac{\partial x_\mu'}{\partial x}\frac{dx}{ds} + \frac{\partial x_\mu'}{\partial y}\frac{dy}{ds} + \frac{\partial x_\mu'}{\partial z}\frac{dz}{ds} + \frac{\partial x_\mu'}{\partial t}\frac{dt}{ds} = \frac{\partial x_\mu'}{\partial t} \tag{14}$$

owing to the null value for the components dx/ds, dy/ds and dz/ds and the equality of dt and ds in proper coordinates. And from the transformation equation

$$g'^{\mu\nu} = \frac{\partial x_\mu'}{\partial x_\alpha}\frac{\partial x_\nu'}{\partial x_\beta}g^{\alpha\beta}$$

we obtain

$$g'^{\mu\nu} = -\frac{dx_\mu'}{\partial x}\frac{\partial x_\nu'}{\partial x} - \frac{\partial x_\mu'}{\partial y}\frac{\partial x_\nu'}{\partial y} - \frac{\partial x_\mu'}{\partial z}\frac{\partial x_\nu'}{\partial z} + \frac{\partial x_u'}{\partial t}\frac{\partial x_\nu'}{\partial t}. \tag{15}$$

Hence, substituting (14) and (15) into (13) and dropping primes we at once obtain the result which was to be proved

$$T^{\mu\nu} = (\rho_{00} + p_0)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} - g^{\mu\nu}p_0$$

with

$$\rho_{00} = 3p_0.$$

The second justification which we have thus presented for our expression for the energy-momentum tensor for black-body radiation suffers from the fact that it makes our final expression, which contains none but macroscopically measurable quantities, a consequence of the relativistic generalization of the Maxwell-Lorentz equations, which assume possibilities of microscopic measurement in conflict with the modern ideas of quantum mechanics. Nevertheless, in view of the primary justification which we presented, it is evident that our expression for the energy-momentum tensor for black-body radiation may be used with reasonable confidence at the present time, and may actually be regarded as more certain than the basis of the second justification. At some future time, the expression might have to be modified in the light of a successful unified field theory if that should ever be achieved.

§ 3. The Line Element

Having justified the expression which we shall use for the energy-momentum tensor of radiation, we are now ready to proceed to our problem of determining the distribution of radiation in a static gravitational field.

For the case of a general static system which may contain solid parts we can not assume spherical symmetry as a necessary accompaniment of equilibrium, and must take the line element in the *general static* form

$$ds^2 = g_{ij}dx_i dx_j + g_{44}dt^2 \qquad i, j = 1, 2, 3 \tag{16}$$

where $g_{ij}dx_i dx_j$ is a negative quadratic form.

We adopt the convention of using Latin indices i, j, etc. to correspond with the spatial coordinates x_1, x_2 and x_3, and shall reserve Greek indices α, β etc. to correspond with all four coordinates x_1, x_2, x_3 and t. In accordance with the usual definition of a static system we take the potentials g_{14}, g_{24} and g_{34} equal to zero, and take the other potentials g_{ij} and g_{44} as independent of the time t, although depending in any arbitrary way desired on the spatial coordinates x_1, x_2 and x_3.

For the potential g_{44}, we note from the form of the line element that we have the simple relation

$$g^{44} = \frac{1}{g_{44}}\,. \tag{17}$$

§ 4. Energy-Momentum Tensor for Black-Body Radiation in a Static Field

We must now consider the form taken by the energy-momentum tensor for black-body radiation in the field defined by the above line element. Returning to our general Eq. (4) for the energy-momentum tensor

$$T^{\mu\nu} = (\rho_{00} + p_0)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} - g^{\mu\nu}p_0 \tag{18}$$

we note that in a static system the macroscopic velocities dx_μ/ds will evidently be zero for $\mu = 1, 2, 3$

$$\frac{dx_i}{ds} = 0 \tag{19}$$

and taking account of Eq. (17) will reduce for the case $\mu = 4$ to

$$\frac{dx_4}{ds} = \frac{dt}{ds} = \frac{1}{\sqrt{g_{44}}} = \sqrt{g^{44}}. \tag{20}$$

Substituting in (18), the energy-momentum tensor degenerates into

$$\begin{aligned} T^{ij} &= -\, g^{ij} p_0 \\ T^{44} &= g^{44} \rho_{00} \end{aligned} \tag{21}$$

And on lowering suffixes we have

$$T^i_j = g_{j\alpha} T^{i\alpha} = -\, g_{j\alpha} g^{i\alpha} p_0 = -\, g^i_j p_0$$

$$T^4_4 = g_{44} g^{44} \rho_{00} = \rho_{00}$$

so that the only surviving components become

$$T^1_1 = T^2_2 = T^3_3 = -\, p_0 \qquad T^4_4 = \rho_{00}. \tag{22}$$

§ 5. Application of the Principles of Mechanics

We can now investigate the pressure of radiation in our thermometer by applying the principles of relativistic mechanics in the form of the well-known equation

$$\frac{\partial \mathfrak{T}^\nu_\mu}{\partial x_\nu} - \frac{1}{2} \mathfrak{T}^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_\mu} = 0. \tag{23}$$

Taking the case $\mu = 1$ and substituting Eqs. (21) and (22), we obtain

$$\frac{\partial}{\partial x_1}(-\, p_0\sqrt{-\,g}) - \frac{1}{2}(-\, g^{ij} p_0 \sqrt{-\,g}) \frac{\partial g_{ij}}{\partial x_1} - \frac{1}{2}(g^{44}\rho_{00}\sqrt{-\,g}) \frac{\partial g_{44}}{\partial x_1} = 0$$

which can evidently be rewritten in the form

$$\sqrt{-g}\,\frac{\partial p_0}{\partial x_1} + p_0 \frac{\partial \sqrt{-\,g}}{\partial x_1} - \frac{1}{2} p_0 \sqrt{-\,g}\left(g^{ij}\frac{\partial g_{ij}}{\partial x_1} + g^{44}\frac{\partial g_{44}}{\partial x_1}\right)$$
$$+ \frac{1}{2}(\rho_{00} + p_0)\sqrt{-\,g}\, g^{44}\frac{\partial g_{44}}{\partial x_1} = 0 \tag{24}$$

This equation can easily be simplified, however, since we are permitted to write in accordance with a well-known result of tensor analysis

$$g^{ij}\frac{\partial g_{ij}}{\partial x_1} + g^{44}\frac{\partial g_{44}}{\partial x_1} = g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_1} = \frac{1}{g}\,\frac{\partial g}{\partial x_1} \tag{25}$$

and if this is substituted in Eq. (24), it is found that the second and third terms will cancel. Making other obvious simplifications, we then obtain for the dependence of pressure on position the simple expression

$$\frac{\partial p_0}{\partial x_1} + \frac{\rho_{00} + p_0}{2} \frac{\partial \log g_{44}}{\partial x_1} = 0 \tag{26}$$

and similar relations will hold with respect to the other spatial coordinates x_1 and x_2.

For the case of radiation, moreover, we have the additional simplification

$$\rho_{00} = 3p_0$$

so that equation (26) further reduces to

$$\frac{\partial \log p_0}{\partial x_1} + 2\frac{\partial \log g_{44}}{\partial x_1} = 0 \tag{27}$$

and since similar relations hold for the other spatial coordinates, we can express the dependence of radiation pressure on position in our gravitational field by the remarkably simple equation

$$p_0(g_{44})^2 = \text{const.} \tag{28}$$

§ 6. Dependence of Temperature on Position

Finally, however, in the case of radiation we can connect proper pressure and proper temperature by the well-known result of Boltzmann

$$p_0 = \frac{a}{3} T_0{}^4 \tag{29}$$

where a is the Stefan Boltzmann constant, and substituting this into Eq. (28) we at once obtain the desired conclusion for the dependence of proper temperature on position in the case of a general static gravitational field

$$T_0\sqrt{g_{44}} = \text{const.} \tag{30}$$

§ 7. Discussion

In conclusion several remarks may be made with respect to the above result which will be of interest.

In the first place it should be noted from the method of derivation that the constancy of $T_0\sqrt{g_{44}}$ has been proven in the first instance sole'y for points inside the radiation thermometer. Nevertheless since we shall expect T_0 and g_{44} to be continuous functions of position, we shall feel justified in concluding that $T_0\sqrt{g_{44}}$ is also constant in the system itself where it comes in thermal contact with the thermometer.

In the second place it should be noted that the derivation was carried out on the assumption that the system had already been provided with a radiation thermometer connecting the parts whose temperatures were to be compared. Hence in the case of a given system of interest the question arises whether a thermometer can be inserted to connect the desired points without thereby seriously altering the system itself. Thus if we had a gravitating system containing solid parts it would be necessary to make a hole into the solid and insert a radiation thermometer if we wished to obtain information as to the temperature of the interior by the method that we have suggested. This procedure, however, would certainly affect the gravitational potentials $g_{\mu\nu}$ which are themselves completely determined by the distribution of the matter and energy in the system. Nevertheless since the equation of connection

$$-8\pi\rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = -8\pi T^{\mu\nu} = G^{\mu\nu} - \frac{1}{2}Gg^{\mu\nu} + \Lambda g^{\mu\nu} \tag{31}$$

is a differential one giving the distribution of matter and energy in terms of the $g_{\mu\nu}$ ***and*** their first and second differential coefficients, it seems correct to assume that the insertion of a thermometer of small dimensions can be made without appreciably affecting the values of the $g_{\mu\nu}$ themselves. This question might bear further investigation, however, since singular cases of interest might be found.

Finally, it might be emphasized that although the proper temperature itself, T_0, varies from point to point in a gravitational system which has come to equilibrium, nevertheless the constancy of the combined quantity $T_0\sqrt{g_{44}}$ provides many of the advantages of the older principle of constant temperature throughout as necessary for equilibrium. Indeed it would be possible to label $T_0\sqrt{g_{44}}$ as *the* temperature of a system, except for the undesirability of multiplying the different things that are signified by that word. In this connection it is also interesting to recall that Einstein himself was led in his early speculations on the nature of gravitation[7] to distinguish between a quantity, called "wahre Temperatur," which would be constant throughout a system in thermal equilibrium and a second quantity, called at the suggestion of Ehrenfest "Taschentemperatur," which would vary with gravitational potential. The considerations were of only a limited applicability since this was at a time before Einstein's complete development of the general theory of relativity; the quantities in question, however, were quite analogous to our present $T_0\sqrt{g_{44}}$ and T_0.

[7] Einstein, Ann. d. Physik **38,** 443 (1912).