CALIFORNIA INSTITUTE OF TECHNOLOGY

DYNAMICS LABORATORY

RESPONSE OF LINEAR, VISCOUS DAMPED SYSTEMS TO EXCITATIONS HAVING TIME-VARYING FREQUENCY

by

Donald Leslie Cronin

A report on research conducted for the National Aeronautics and Space Administration and the National Science Foundation

Pasadena, California

1965

RESPONSE OF LINEAR, VISCOUS DAMPED SYSTEMS TO EXCITATIONS HAVING TIME-VARYING FREQUENCY

Thesis by

Donald Leslie Cronin

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1966

(Submitted August 5, 1965)

ACKNOWLEDGMENTS

The author wishes to acknowledge the contribution of his former advisor, the late Professor C. E. Crede, who suggested the present study, and who provided guidance in germane experimental matters.

Thanks in large measure are due the author's present advisor, Professor T. K. Caughey for the unflagging interest and invaluable advice which made possible the completion of this investigation.

Thanks, too, belong to those who provided help during the course of investigation and during preparation of this thesis, as well as to those who, by indicating their willingness to be of assistance, created an environment which made work a pleasure.

The author is indebted to the California Institute of Technology for generous financial assistance and to the Marshall Space Flight Center, National Aeronautics and Space Administration for subsidizing this research under contract number NAS 8-2451.

ABSTRACT

The response of linear, viscous damped systems to excitations having time-varying frequency is the subject of exact and approximate analyses, which are supplemented by an analog computer study of single degree of freedom system response to excitations having frequencies depending linearly and exponentially on time.

The technique of small perturbations and the methods of stationary phase and saddle-point integration, as well as a novel bounding procedure, are utilized to derive approximate expressions characterizing the system response envelope--particularly near resonances--for the general time-varying excitation frequency.

Descriptive measurements of system resonant behavior recorded during the course of the analog study--maximum response, excitation frequency at which maximum response occurs, and the width of the response peak at the half-power level--are investigated to determine dependence upon natural frequency, damping, and the functional form of the excitation frequency.

The laboratory problem of determining the properties of a physical system from records of its response to excitations of this class is considered, and the transient phenomenon known as "ringing" is treated briefly.

It is shown that system resonant behavior, as portrayed by the above measurements and expressions, is relatively insensitive to the specifics of the excitation frequency-time relation and may be

described to good order in terms of parameters combining system properties with the time derivative of excitation frequency evaluated at resonance.

One of these parameters is shown useful for predicting whether or not a given excitation having a time-varying frequency will produce strong or subtle changes in the response envelope of a given system relative to the steady-state response envelope. The parameter is shown, additionally, to be useful for predicting whether or not a particular response record will exhibit the "ringing" phenomenon.

TABLE OF CONTENTS

NOTATION

B	decay constant used with exponentially sweeping excitation (τ^{-1})
$\bar{B}$	decay constant used with exponentially sweeping excitation (t^{-1})
$G(t)$	excitation phase angle expressed as a function of t
$g(\tau)$	excitation phase angle expressed as a function of τ
h =	$\frac{1}{2\pi}\ddot{G}$, the rate of change of excitation frequency for the linearly sweeping case (cps/s, a constant)
j =	$\sqrt{-1}$
K	spring constant
M	mass (except where noted in Chapter V)
m =	$\sqrt{1-\zeta^2}$
m_e =	$\sqrt{1-\zeta_e^2}$
N	natural frequency in cps
N_i	i^{th} natural frequency in cps
q =	$\frac{N^2}{h}$
$P(t)$	excitation expressed as a function of t
P_o	modulus of excitation
$p(\tau)$	excitation expressed as a function of τ
p	Laplace transform variable
t	time
t_i, t_{r_i}	time at which the excitation frequency equals the i^{th} resonant frequency
W	width of the response peak at the half-power level (dimensionless, or in cps where noted)
W_e	measured W

x	displacement
x_{mod}	modulus of displacement
y =	$\dfrac{x}{(\frac{P_o}{K})} = \dfrac{M\omega^2 x}{P_o}$ (except where noted in Chapter VI)
y_{mod}	modulus of y
y_{max}	maximum of y
$y_{max, ss}$	maximum steady-state value of y
$y_{max, 2}$	first secondary maximum of y
$\alpha(\tau)$	instantaneous frequency of excitation, $g'(\tau)$
α_{max}	value of α for which $y = y_{max}$
α_0 =	$\sqrt{1 - 2\zeta^2}$
α_{oe} =	$\sqrt{1 - 2\zeta_e^2}$
β	damping constant
γ =	$\dfrac{\overline{B}}{N} = 2\pi B$
ζ =	$\beta / 2\sqrt{KM}$
ζ_e	measured ζ
$\Theta(t)$	angular displacement expressed as a function of t
$\theta(\tau)$	angular displacement expressed as a function of τ
τ	ωt
τ_r	value of τ for which excitation frequency equals resonant frequency
τ_{max}	value of τ for which $y = y_{max}$

$\tau_{max, 2}$		value of τ for which $y = y_{max, 2}$
φ	=	$(1-\alpha^2)^2 + 4\zeta^2\alpha^2$
$\Omega(t)$		instantaneous excitation frequency, $\dot{G}(t)$
ω	=	$\sqrt{K/M}$
ω_i		i^{th} natural frequency in radians/second
ω_e		measured natural frequency

Dots over variables denote differentiation with respect to t; primes denote differentiation with respect to τ.

I. INTRODUCTION

The first analysis of the response of a vibrating system influenced by an excitation having a time dependent frequency was performed by Lewis[1] in 1932. In order to obtain a solution in terms of known functions, Lewis imposed the restrictive requirement that the excitation frequency be linearly related to time. Succeeding papers treating the analytical aspect of the problem adhere, in the main, to Lewis' restriction. Authors of these more recent papers include: Barber and Ursell[2], Hok[3], and Baker[4].

The excitation used most often in experiment, on the other hand, has been one in which the frequency depends exponentially on time. Its use is due primarily to the fact that devices for its synthesis are simply related to the ubiquitous audio oscillator and have long been in service. Papers dealing with the exponentially sweeping excitation have, in general, been experimentally orientated. Authors of such works include Crede[5], and Hartenstein[6].

Direct answers to questions concerning the response of a given system to a sweeping excitation other than the two mentioned would not be forthcoming from the above references, nor from others uncited. The wealth of published data for the linearly sweeping excitation fails to make clear the quantitative effect of damping upon system response. Little, in addition, may be inferred regarding the effects of differences in sweeping excitations, since results for the linearly sweeping and the exponentially sweeping excitations--traditionally associated with the analytical and experimental aspects of the problem respectively--are not written to permit ready comparison.

The following chapters deal with the response of linear, viscous damped systems influenced primarily by specific sweeping excitations. An effort has been made, however, to present results in such a way as to permit insight into the nature of system response to any sweeping excitation. Response envelope behavior near system resonances has received particular attention, and descriptive measurements of this behavior have been analyzed to determine the character of their dependence on damping, natural frequency, and parameters associated with the sweeping excitation.

II. A PHYSICAL AND MATHEMATICAL DESCRIPTION OF SYSTEM RESPONSE

When a simple system is forced by a constant amplitude excitation having a time-varying, or sweeping, frequency, and this frequency is permitted to pass through an important natural frequency of the system, the response will manifest characteristics seen in Figs. 2.1 and 2.2.

The oscillograph trace shown in Fig. 2.1 represents the end acceleration of a Lucite cantilever influenced by an excitation with frequency sweeping exponentially in time through the system's lowest natural frequency. The record shown in Fig. 2.2 represents the mid-point displacement of a simply-supported glass beam influenced again by an excitation with frequency sweeping exponentially in time through the lowest natural frequency of the beam.

The most striking feature of the response envelope is the beat pattern, or "ringing", succeeding the anticipated response peak in time. Crede[5] provided the following physical explanation of the phenomenon:

> "A short time interval after the excitation frequency coincides with the natural frequency, the response amplitude is a maximum. If the excitation were to cease suddenly at this moment of maximum response amplitude, the subsequent vibration of the responding system would be free vibration with a continuously decreasing amplitude as controlled by the damping of the system. However, the excitation continues with a gradually increasing frequency and the forced vibration of the system thus has the same frequency as the excitation. The response of the system thus embodies two frequencies of nearly equal value, the free vibration continuing from the maximum response and the forced vibration at the excitation frequency.

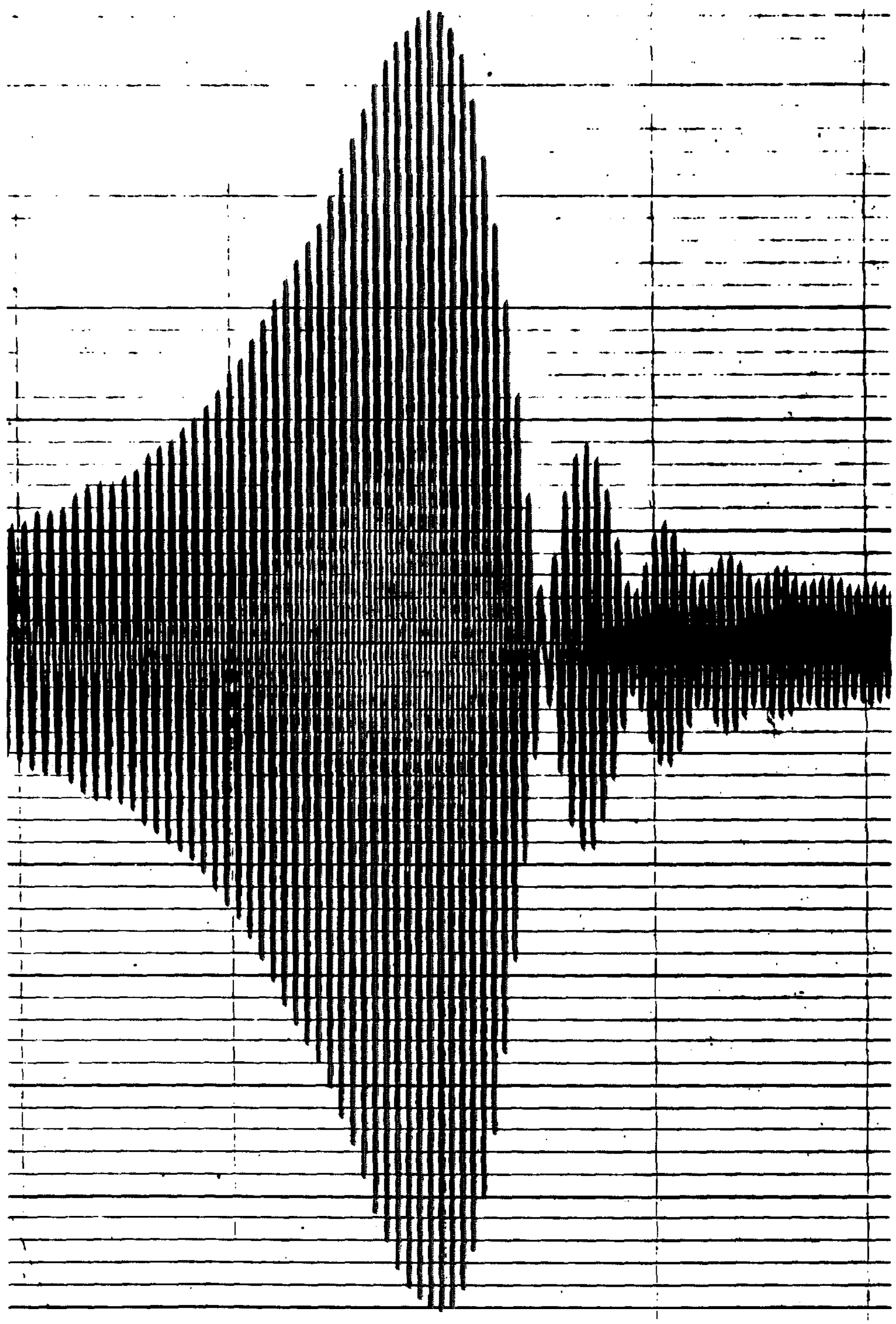


FIG. 2.1

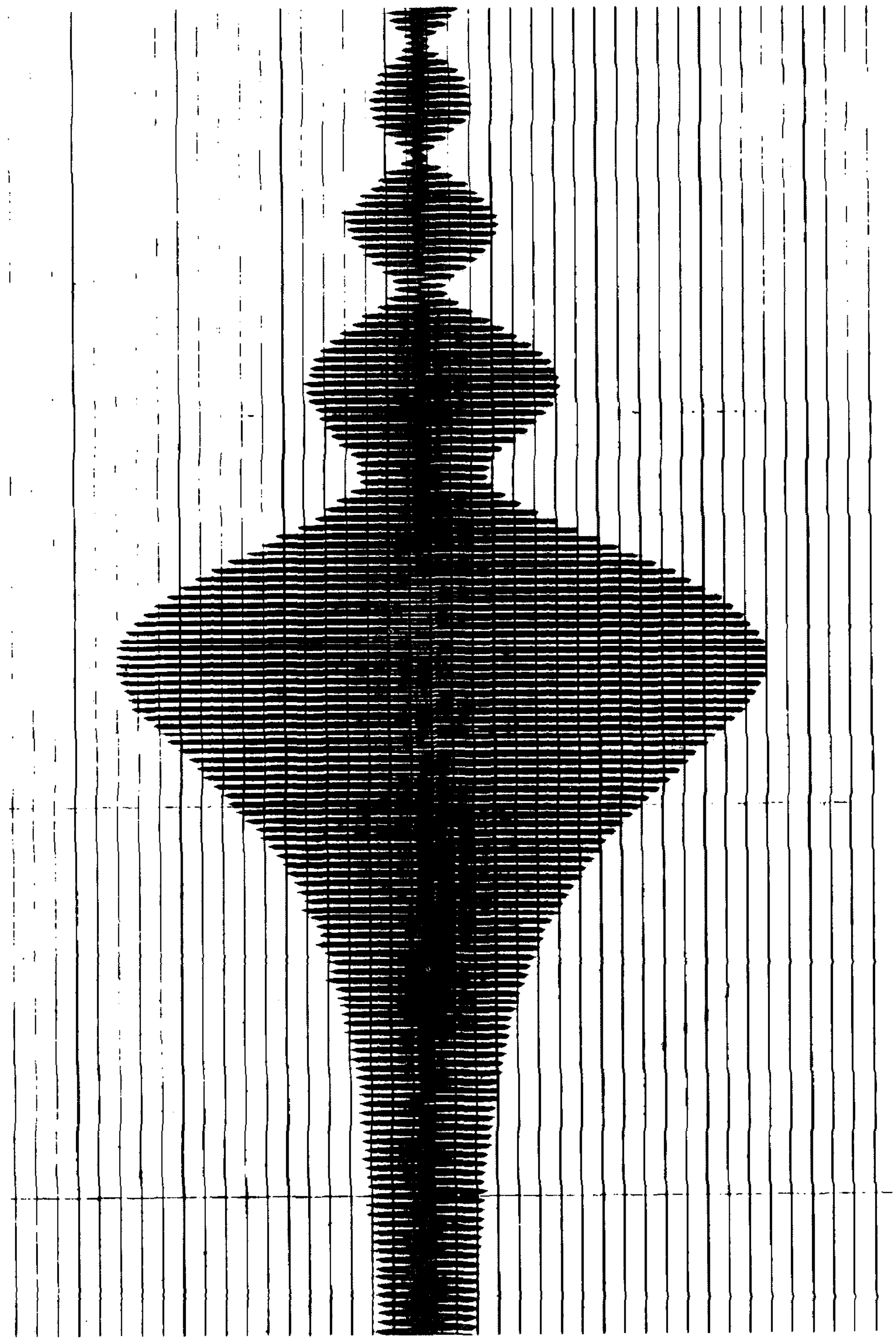


FIG. 2.2

. . . . It may be noted that the beat period decreases continuously with time as the difference between the natural frequency and the excitation frequency increases during the sweep pattern. "

Less striking, but of great importance, nevertheless, is the pronounced effect upon the response peak produced by a sweeping excitation. A comparison of the curve of response amplitude versus excitation frequency for the sweeping case to the curve obtained by plotting the system's steady-state response amplitude as a function of excitation frequency will show that sweeping generally produces an attenuated, broadened response peak.

As indicated above, the response maximum for a system influenced by a sweeping excitation will occur shortly after the excitation frequency equals the system natural frequency. The response peak center frequency will then be higher or lower than the steady-state value depending upon whether the excitation frequency passes up or down through system resonance.

Figures 2. 4 through 2. 8 illustrate the response of an electrical analog model of a viscous damped single degree of freedom system influenced by an excitation in which the frequency is a linear function of time. Figure 2. 3 is the steady-state response amplitude versus excitation frequency curve for the same system provided for the purpose of comparison. Figure 2. 4 represents the slowest sweep rate of excitation frequency in this series of illustrations, and it is seen that the resulting response envelope has a form very similar to the steady-state curve. For a slightly higher sweep rate, Fig. 2. 5, the

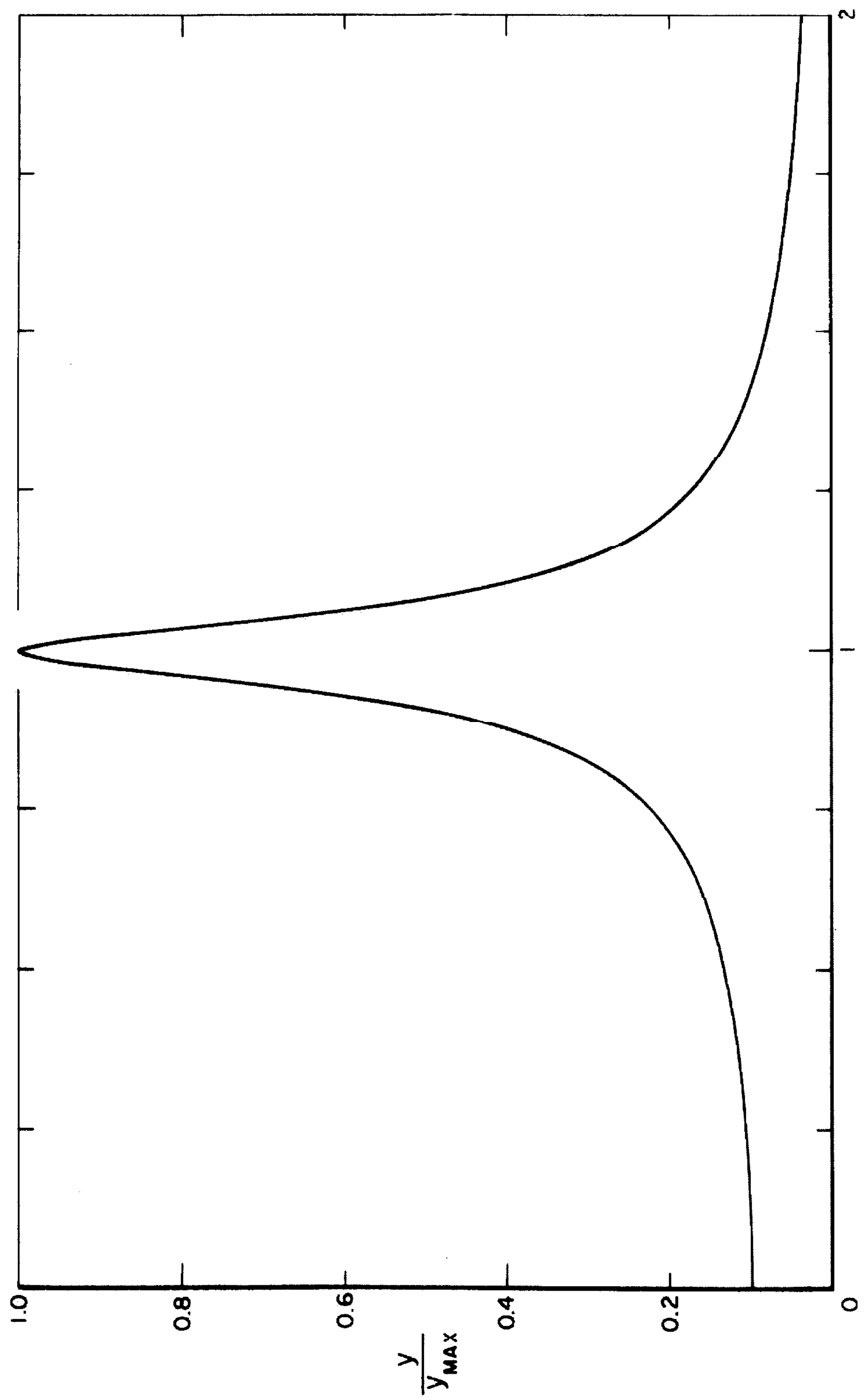


FIG. 2.3

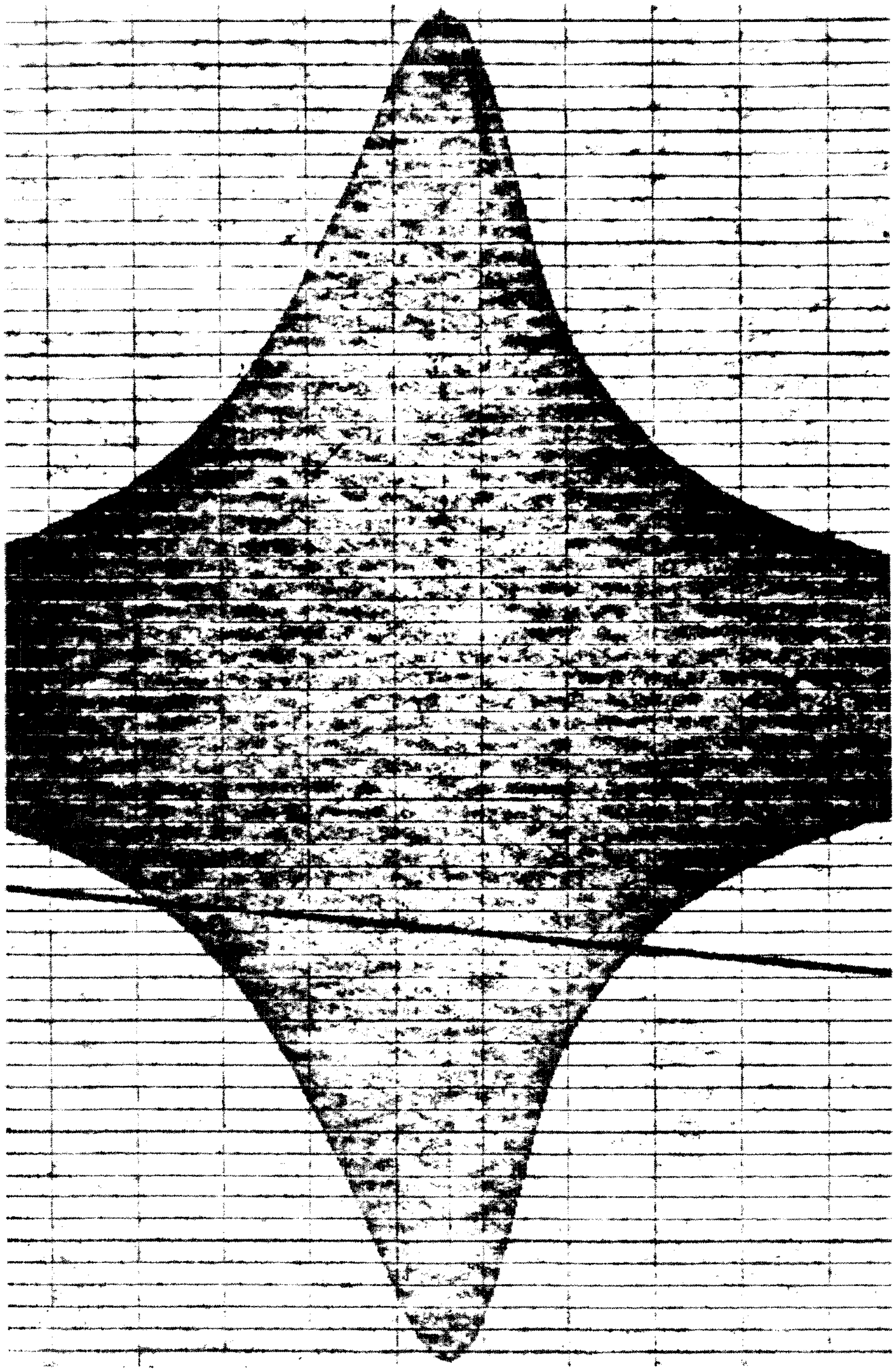


FIG. 2.4

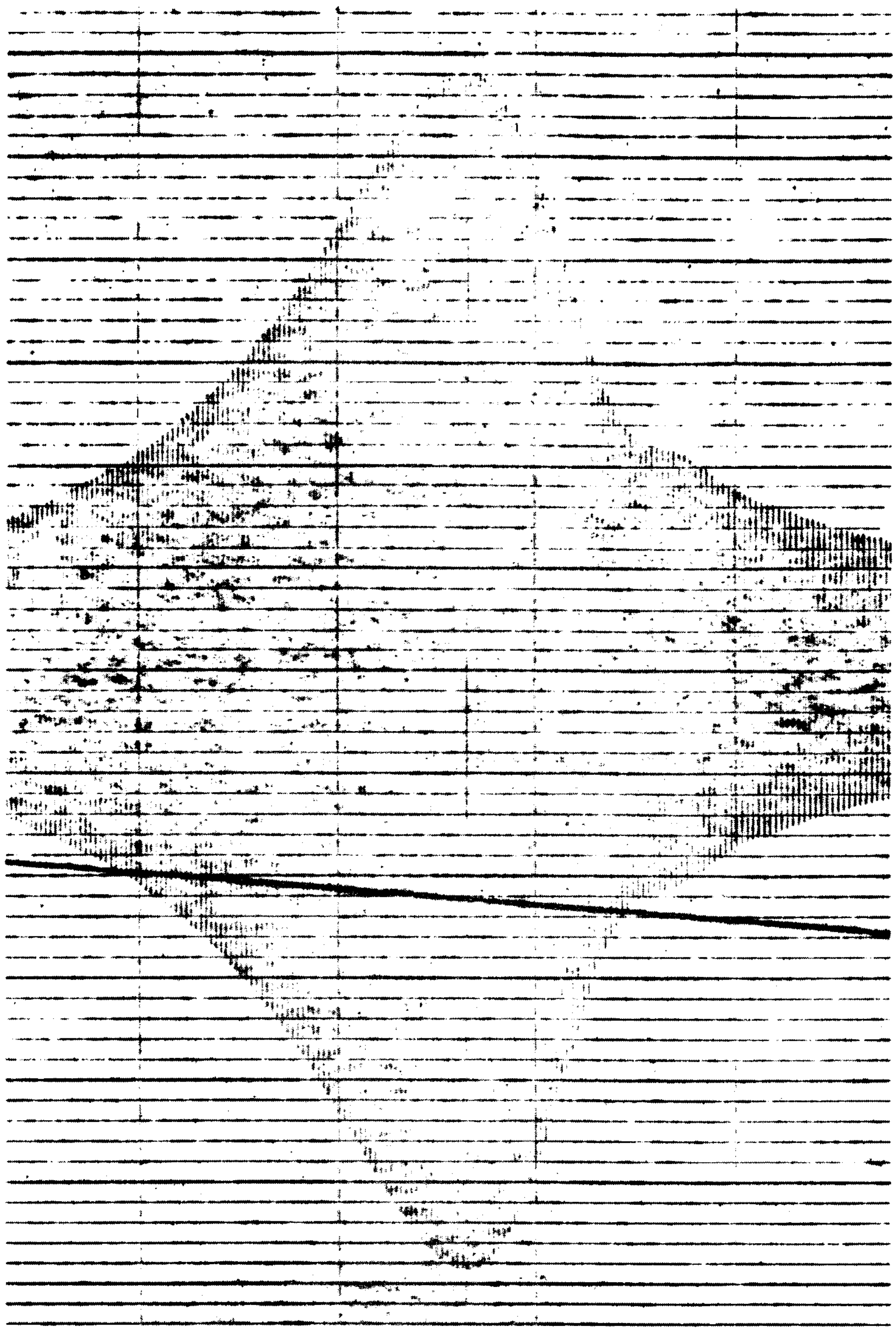


FIG. 2.5

maximum amplitude is attenuated somewhat, and the response peak is broadened. In addition, a secondary hump appears, marking the onset of the "ringing" phenomenon. The response illustrated in Fig. 2.6 represents a still faster sweep rate. The secondary peaks are highly developed, and the primary peak has been further attenuated and broadened. When the sweep rate is increased further, the response assumes the form indicated in Fig. 2.7a. An interesting change has taken place here, for the response envelope which manifests a concavity both before and after the maximum has been attained in the preceding illustrations now appears convex on the side corresponding to times later than the time that the excitation frequency equals the system natural frequency. This convexity suggests more the envelope of a beat pattern produced, for instance, when a system is excited by two closely spaced frequencies, Fig. 2.7b, than the steady-state response curve, Fig. 2.3. The last figure, Fig. 2.8, represents the fastest sweep rate of the series. It illustrates a gross distortion of the response curve, and a degradation of the response envelope for times corresponding to times after the excitation frequency coincides with the system natural frequency. Response curves such as the one shown in Fig. 2.8 are rarely seen in practice, since exciters of physical systems generally do not sweep in this very rapid fashion.

The notion of a characteristic time--defined as that time measuring a significant change in the dependent variable under observation--may help to explain the form of the response curves discussed above.

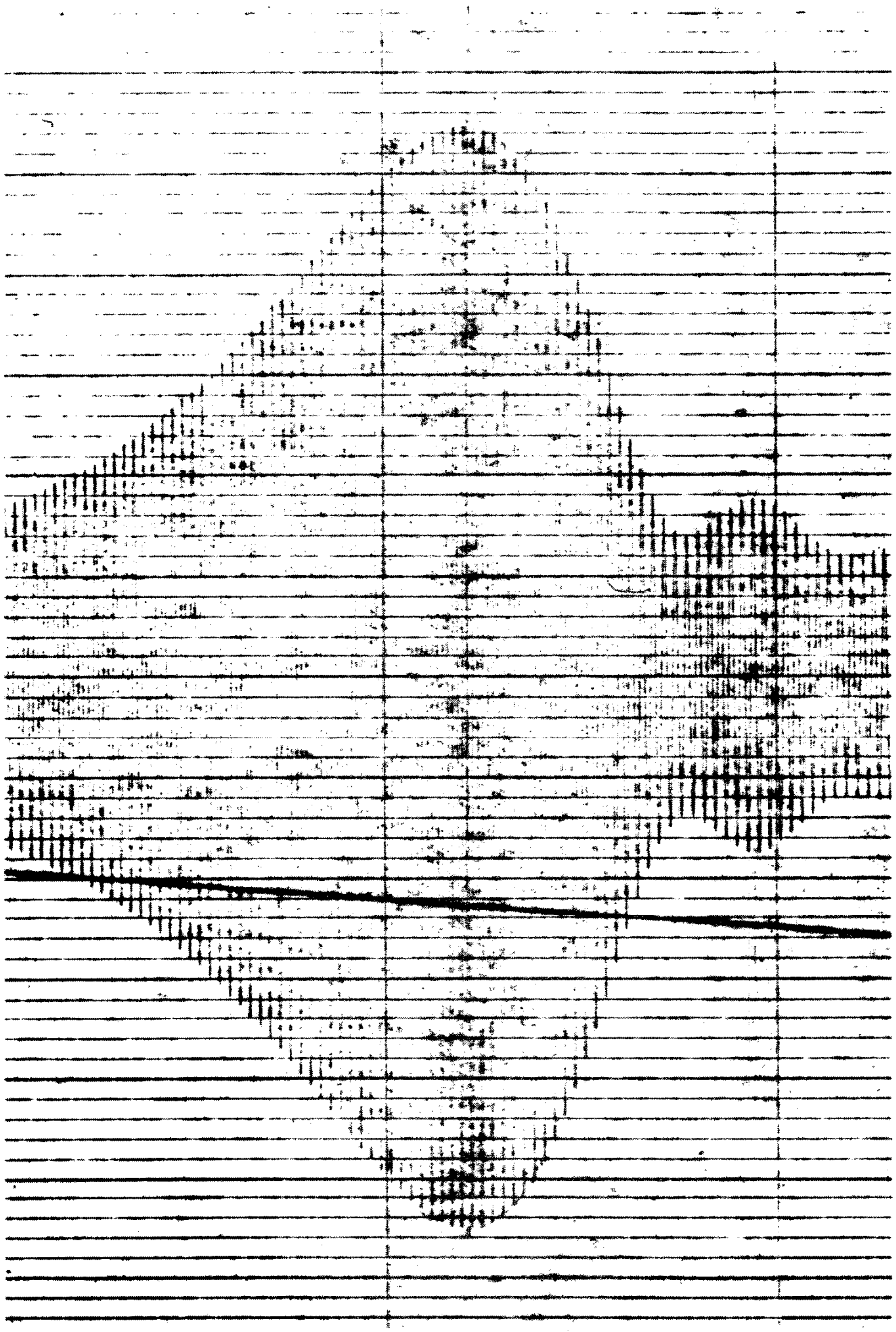


FIG. 2.6

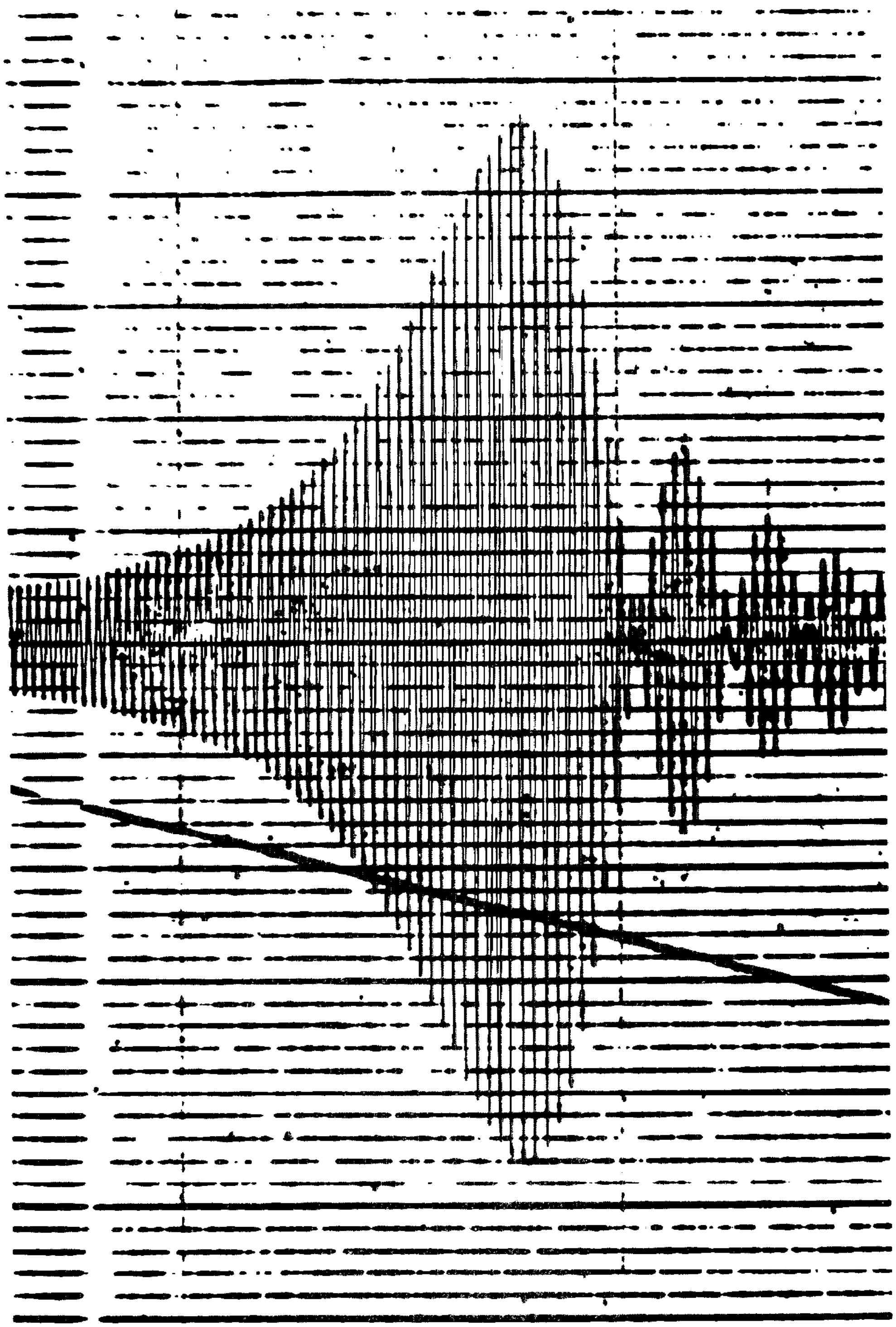


FIG. 2.7a

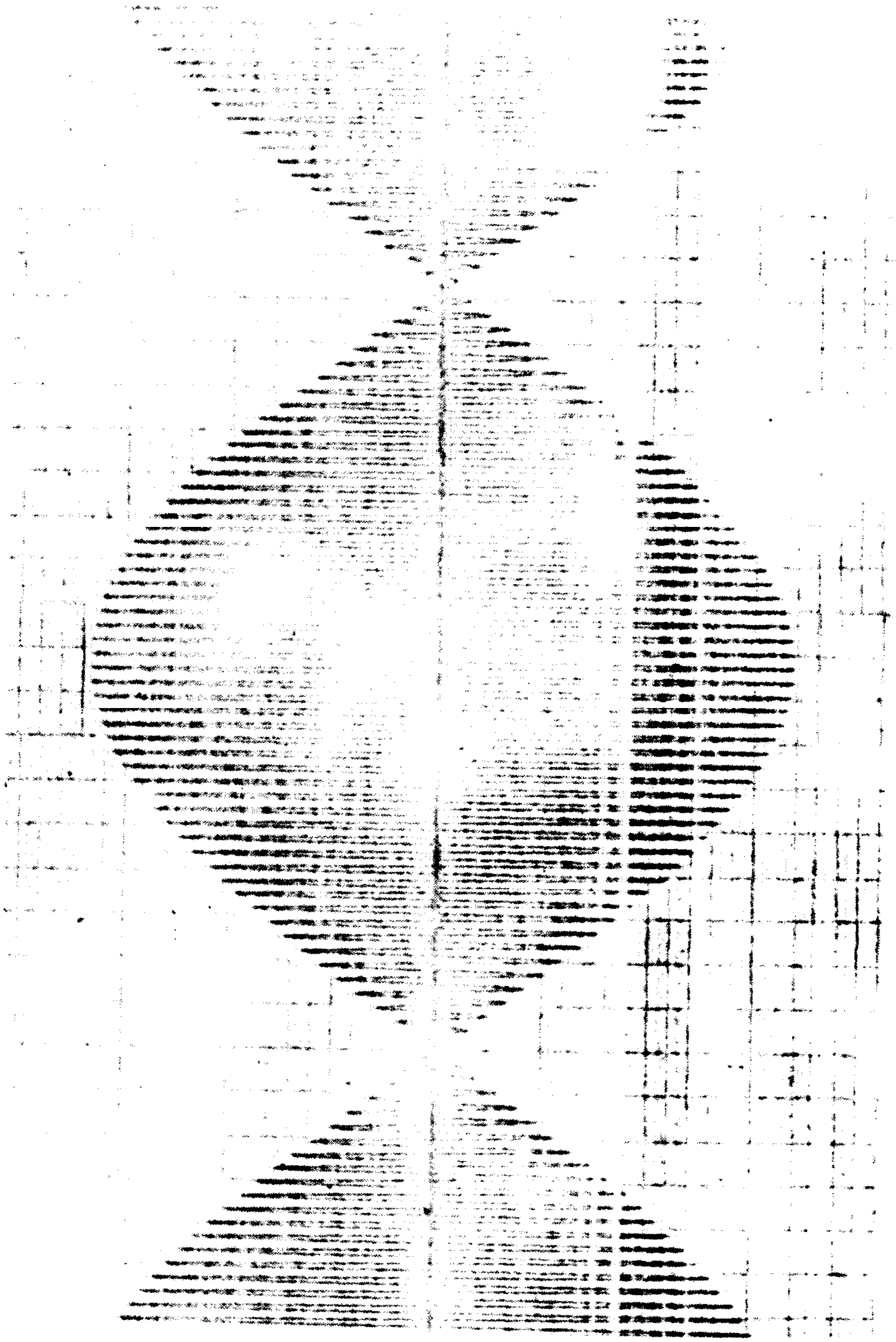


FIG. 2.7b

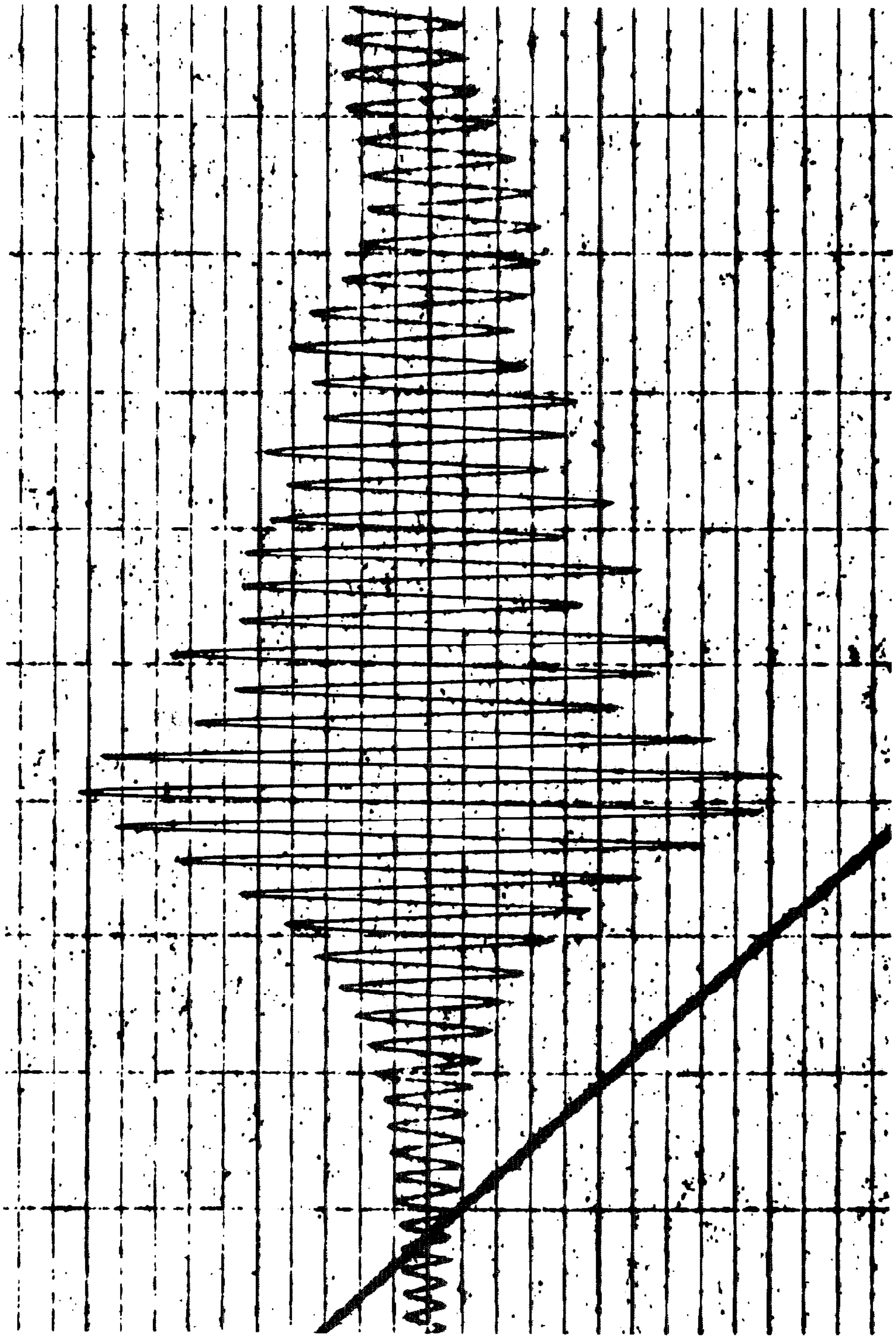


FIG. 2.8

There will be two characteristic times associated with the response of a freely oscillating damped system--one related to the period of oscillation, and the other related to the exponential decay. If the latter is large with respect to the period of oscillation, it will define a long term, or envelope, behavior of the response.

A harmonic excitation provides another characteristic time (associated with its period), and the resulting system response will contain behavior described by all three times. The system times will appear in starting, or transient behavior.

If the excitation frequency were allowed to vary, a fourth characteristic time, related to the time derivative of excitation frequency, would be introduced into the response.

When the excitation frequency sweeps very slowly in time, the characteristic time of sweeping will be large with respect to the system's characteristic times. Transient behavior will contribute very little to the response, which will assume the form of Fig. 2.4. For faster sweep rates, the characteristic time of sweeping will be of the same order as the time of system decay, and transient behavior becomes important, as seen in Figs. 2.5 through 2.8.

In these instances, the system has had insufficient time to build to its steady-state response maximum; the free decay from the peak value interferes with the forced oscillation to form the beat pattern following the primary peak. Further interference and degradation of the envelope appear, as seen in Fig. 2.8, when the dependent characteristic time of beating--proportional to the difference between the period of excitation and natural period of the system--is of the same order as the characteristic time of sweep.

The significance of the effect of sweeping excitation upon the form of the response envelope as compared to the steady-state curve is a function of system damping as well as the sweep rate. If two systems differing only in damping are influenced by an identical sweeping excitation, the response envelope of the more heavily damped system will bear a stronger resemblance to its steady-state curve than will that of the system with lighter damping.

The effect of increased damping upon the sweeping response envelope holds with the notion of characteristic time presented above, since the characteristic time of sweeping even for rather fast sweep rates will be large with respect to the characteristic time of decay for a heavily damped system. Significant changes in the sweeping response envelope of a heavily damped system will then require very fast sweep rates.

In the remainder of this chapter, and in the succeeding three chapters, the response of a viscous damped single degree of freedom system influenced by sweeping excitations will be analyzed. Discrete systems having more than one degree of freedom, and continuous systems will be discussed briefly in Chapter VI.

The sweeping excitations to be discussed will have the general form

$$P(t) = P_o \sin G(t) \quad . \tag{2.1}$$

The instantaneous frequency of excitation will be defined herein as

$$\Omega(t) = \frac{dG(t)}{dt} \quad .$$

The equation of motion for the system illustrated in Fig. 2.9 influenced by the excitation (2.1) is

$$M \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + Kx = P_o \sin G(t) \; , \tag{2.2}$$

where M, β, and K are respectively the mass, the viscous damping constant, and the spring constant.

Equation (2.2) is generally written

$$\frac{d^2x}{dt^2} + 2\zeta\omega \frac{dx}{dt} + \omega^2 x = \frac{P_o}{M} \sin G(t) \; , \tag{2.3}$$

where $\omega = \sqrt{\frac{K}{M}}$ is the undamped natural frequency, and where $\zeta = \frac{\beta}{2\sqrt{KM}}$ is the fraction of critical damping.

It will simplify results if the independent variable is replaced by a dimensionless time, $\tau = \omega t$, and if the dependent variable is replaced by $y = \frac{x}{\left(\frac{P_o}{K}\right)} = \frac{M\omega^2 x}{P_o}$. The new dependent variable may be thought of as the ratio of dynamic response to static displacement resulting from application of the force P_o.

Equation (2.3) upon substitution of the new dependent and independent variable becomes

$$y'' + 2\zeta y' + y = \sin g(\tau) \; , \tag{2.4}$$

where $G(t) = g(\tau)$, and where primes denote differentiation with respect to τ.

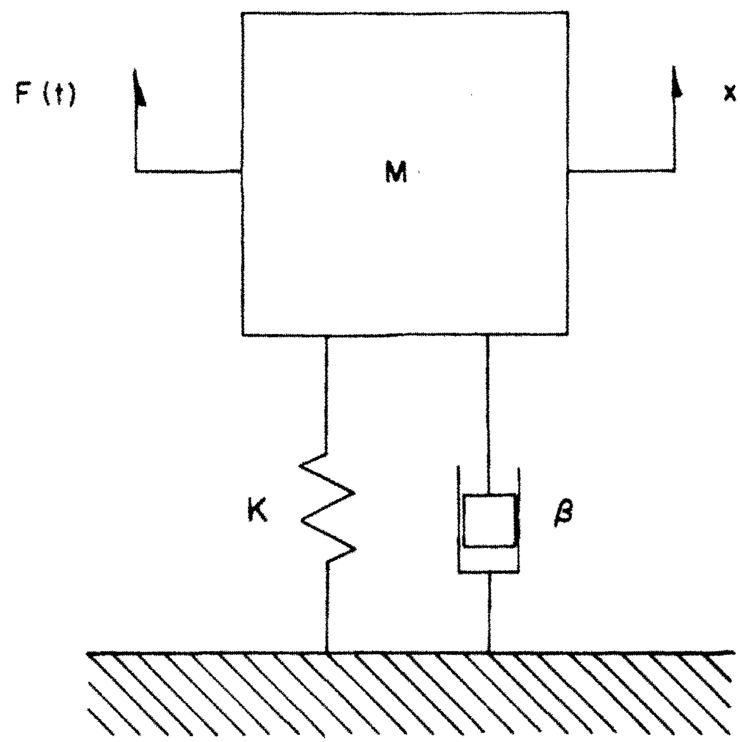


FIG. 2.9

The instantaneous frequency ratio of the excitation will be defined as

$$\alpha(\tau) = g'(\tau) \quad .$$

Equation (2.4) has the complementary solution:

$$y_c(\tau) = Ae^{-\zeta\tau} \sin m\tau + Be^{-\zeta\tau} \cos m\tau \quad , \tag{2.5}$$

where A and B are arbitrary constants of integration and where $m = \sqrt{1 - \zeta^2}$.

The particular solution of (2.4) may be written in terms of Duhamel's integral:

$$y_p(\tau) = \int_{\tau_o}^{\tau} h(\tau - \tau') \sin g(\tau') d\tau' \quad , \tag{2.6}$$

in which

$$h(\tau - \tau') = \frac{e^{-\zeta(\tau - \tau')} \sin m(\tau - \tau')}{m} \quad .$$

The lower limit of the integral in (2.6) represents the time of initial application of the excitation.

The complete solution to (2.4) may be written using the results (2.5) and (2.6):

$$y(\tau) = y(\tau_o) \cdot u(\tau - \tau_o) + y'(\tau_o) \cdot v(\tau - \tau_o) + \int_{\tau_o}^{\tau} h(\tau - \tau') \sin g(\tau') d\tau' . \tag{2.7}$$

The functions $u(\tau - \tau_o)$ and $v(\tau - \tau_o)$ are formed from the two homogeneous solutions. These functions have the properties:

$$u(0) = v'(0) = 1 \quad ,$$
$$u'(0) = v(0) = 0 \quad .$$

Closed-form evaluation of the integral portion of (2. 7) for the general trigonometric argument, $g(\tau')$, is not possible, although the nature of its behavior may be inferred from the physical discussion at the start of this chapter. Further knowledge may be gained concerning the general problem from the following analysis.

It will be helpful to deal in the following with the infinite time operating system, i. e., the system to which the excitation

$$p(\tau) = \sin g(\tau)$$

is first applied at time equals $-\infty$.

Starting conditions, $y(-\infty)$ and $y'(-\infty)$ will, in addition, be taken equal to zero in the undamped case. The complete response may then be written

$$y(\tau) = \int_{-\infty}^{\tau} h(\tau-\tau') \cdot p(\tau')\, d\tau' \quad . \tag{2. 8}$$

If the change of variable

$$u = \tau - \tau'$$

is made in (2. 8), the solution may be written:

$$y(\tau) = \int_{0}^{\infty} h(u) \cdot p(\tau-u)\, du \quad . \tag{2. 9}$$

Expressions (2. 8) and (2. 9) will be used interchangeably.

The response of the undamped system to harmonic excitation having frequency equal to system natural frequency can not be bounded, since, given this circumstance, response grows linearly with time.

For the case in which the excitation frequency is constrained to sweep once through system resonance at a finite rate, it appears reasonable to assume that the response of the undamped system will be bounded, since the excitation frequency will equal system natural frequency for only a finite period, providing insufficient time for the build-up of an unbounded response.

This argument may be extended to more general cases where the excitation frequency is cycled a finite number of times through the system resonant frequency. For if each pass through resonance increases the response by a finite amount, then a finite number of passes cannot produce an infinite response.

It is concluded that the response of an undamped system subject to any sweeping excitation should be bounded provided only that the frequency of excitation equals the system natural frequency for a finite total time. This conclusion eliminates sweeps in which the frequency of excitation approaches the system natural frequency asymptotically or in which the frequency of excitation is cycled through the system natural frequency an infinite number of times (more will be said about this case later).

Granting the above restrictions on the sweeping excitation, it should be possible to derive an upper bound for the response in the undamped case.

It could be surmised that one of the simplest types of sweeping excitations would be one in which the frequency, everywhere positive, is a strictly decreasing function of time. Such an excitation will have a frequency equal to the system natural frequency at one time during the interval (taken to be $\tau = 0$, without loss of generality). An excitation of this type will be assumed in the following demonstration.

The response of the undamped system to the above excitation will be

$$y = \int_{-\infty}^{\tau} \sin(\tau - \tau') \sin g(\tau') d\tau' \quad . \tag{2.10}$$

The instantaneous excitation frequency:

$$\alpha(\tau) = g'(\tau)$$

is such that

$$\alpha(0) = 1 \quad ,$$

and

$$\alpha'(\tau) = g''(\tau) < 0 \; .$$

Expression (2.10) may be rewritten by factoring the time dependence from the integrand, and applying well-known trigonometric identities:

$$y = \frac{1}{2}\sqrt{(I_1 - I_2)^2 + (I_3 - I_4)^2} \cdot \sin\left[\tau + \tan^{-1}\frac{(I_3 - I_4)}{(I_1 - I_2)}\right] , \tag{2.11}$$

where

$$I_1 = \int_{-\infty}^{\tau} \sin\left[\tau' + g(\tau')\right] d\tau' \quad ,$$

$$I_2 = \int_{-\infty}^{\tau} \sin\left[\tau' - g(\tau')\right] d\tau' \quad ,$$

$$I_3 = \int_{-\infty}^{\tau} \cos\left[\tau' + g(\tau')\right] d\tau' \quad ,$$

$$I_4 = \int_{-\infty}^{\tau} \cos\left[\tau' - g(\tau')\right] d\tau' \quad .$$

The frequency of the integrands of I_1 and I_3 is

$$\phi_{1,3}(\tau') = 1 + \alpha(\tau') \quad .$$

The frequency of the integrands of I_2 and I_4 is

$$\phi_{2,4}(\tau') = 1 - \alpha(\tau') \quad .$$

The restriction on $\alpha(\tau')$ requires that $\phi_{1,3}$ and $\phi_{2,4}$ will be as indicated in Fig. 2.10. It is seen that $\phi_{1,3}$ will never be zero and, in fact, will never be less than one. The function $\phi_{2,4}$, on the other hand, will have a single zero in the region of interest.

The integrals $I_1, \ldots, I_4$ represent members of a class of integrals discussed by G. N. Watson[7] in his treatise on the method of stationary phase. Watson notes that if the frequency of the integrand is zero somewhere in the interval of integration, then the prime contribution to the integral comes from the neighborhood of the point corresponding to zero frequency. A corollary contention would be that if no such point exists in the interval of integration then the value of the

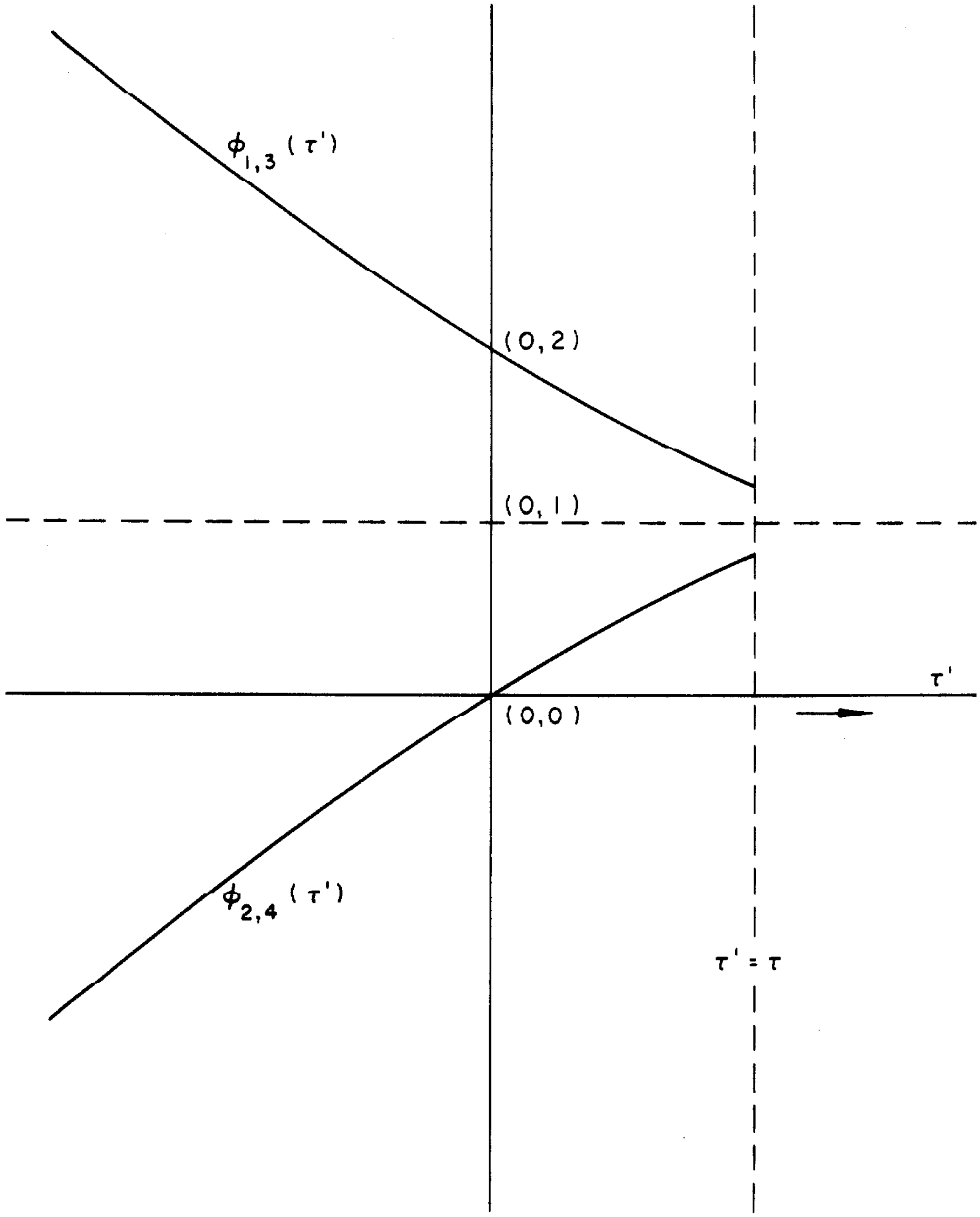


FIG. 2.10

integral is comparatively small.

Of the four integrals making up the expression for the amplitude of the response, y, only two satisfy Watson's condition. These two integrals should, and do, as will be seen in the following, make the prime contribution to the response.

Each of the integrals in (2.11) may be thought of as an infinite sequence of integrals taken over sub-intervals defined by the zeros of the oscillating integrand (plus a residual contribution owing to the fact that the upper limit of integration, τ, need not correspond to a root of the integrand). The signs of successive terms in the sequence alternate. Convergence of the sequence is thus assured if the N^{th} term exceeds the $(N+1)^{th}$ term in absolute value, and if the N^{th} term goes to zero as N goes to ∞.

The requirement that $\alpha(\tau')$ and, consequently, that the frequency of each integrand increases to ∞ as time regresses from $\tau = 0$, to $\tau = -\infty$ will constrain the separation between the zeros of the integrand to diminish to zero as time goes to $-\infty$.

The interval of integration of the $(N+1)^{th}$ element in the sequence will, therefore, be smaller than that of the N^{th} element (which succeeds the former in time), and it may be shown that the absolute value of the $(N+1)^{th}$ contribution to the sequence will be smaller than that of the N^{th}. Furthermore, the N^{th} contribution will go to zero as $N \to \infty$.

The alternating sequences thus satisfy the requirements for convergence.

In order to bound the contribution of each of the integrals forming the response, (2.11), it should be noted that the sum of an infinite alternating sequence may be predicted by the sum of the first N terms. The error in such a prediction will have an absolute value less than that of the first neglected term and will carry the sign of the first neglected term. Further, the sum of a truncated alternating series is bounded by the absolute value of the first term provided that the first term exceeds the second in absolute value, and so on.

The integrals I_1 and I_3 may now be bounded after first noting that the frequency of the integrands strictly increases as time regresses from $\tau' = \tau$.

The integral I_1 over the range $(-\infty, \tau)$ is then bounded by

$$I_1 < \left| \int_{\tau_{12}}^{\tau_{11}} \sin[\tau'+g(\tau')]\,d\tau' \right| + \left| \int_{\tau_{11}}^{\tau} \sin[\tau'+g(\tau')]\,d\tau' \right| , \qquad (2.12a)$$

and the integral I_3 over the range $(-\infty, \tau)$ is bounded by

$$I_3 < \left| \int_{\tau_{32}}^{\tau_{31}} \cos[\tau'+g(\tau')]\,d\tau' \right| + \left| \int_{\tau_{31}}^{\tau} \cos[\tau'+g(\tau')]\,d\tau' \right| . \qquad (2.12b)$$

The τ_{i1} and τ_{i2} equal respectively the first and the second zero preceding $\tau' = \tau$ for the relevant integrand, and the second integral in each of the above expressions represents the residual contribution deriving from the fact that τ need not correspond to a zero of the integrand.

The integrals I_2 and I_4 over the interval $(-\infty, 0)$ are bounded in the same fashion. The integrals I_2 and I_4 over the interval $(0, \tau)$ may be bounded after noting that the frequency of the integrands increases as time progresses from $\tau' = 0$ to $\tau' = \tau$. Each of these integrals may then be thought of as the sum of a truncated alternating sequence in which the first element exceeds the second, and so on.

The integral I_2 over $(0, \tau)$ is then bounded by

$$I_2 < \left| \int_{\tau_{21}}^{\tau_{22}} \sin[\tau' - g(\tau')]\, d\tau' \right| + \left| \int_{0}^{\tau_{21}} \sin[\tau' - g(\tau')]\, d\tau' \right| \quad , \qquad (2.13a)$$

and the integral I_4 over $(0, \tau)$ is bounded by

$$I_4 < \left| \int_{\tau_{41}}^{\tau_{42}} \cos[\tau' - g(\tau')]\, d\tau' \right| + \left| \int_{0}^{\tau_{41}} \cos[\tau' - g(\tau')\, d\tau' \right| \quad . \qquad (2.13b)$$

The τ_{i1} and τ_{i2} equal the first and second zero of the relevant integrand occurring after $\tau = 0$. The second integrals again represent residual terms.

These bounding expressions--integrals taken over specific small segments of the interval of integration--are easier to work with than the original integrals, since they may be either estimated or bounded by very simple functions. Bounds may be further refined by including additional elements of the sequences described above.

The technique described will now be applied to a particular problem.

Consider the excitation

$$p(\tau) = \sin(\tau - \tau^2/4\pi q) \quad , \quad \tau \le 2\pi q \quad . \qquad (2.14)$$

The frequency of excitation is

$$\alpha(\tau) = 1 - \tau/2\pi q \quad .$$

The excitation will sweep in frequency from $+\infty$ at $\tau = -\infty$, through system resonance at $\tau = 0$, to zero at $\tau = 2\pi q$.

It is desired to bound the response amplitude maximum of the undamped system subject to this excitation. Response will be written in the form of expression (2. 11). The integrals appearing in this formulation will be

$$\begin{aligned} I_1 &= \int_{-\infty}^{\tau} \sin\left(2\tau' - \frac{\tau'^2}{4\pi q}\right) d\tau' \quad , \\ I_2 &= \int_{-\infty}^{\tau} \sin\left(\frac{\tau'^2}{4\pi q}\right) d\tau' \quad , \\ I_3 &= \int_{-\infty}^{\tau} \cos\left(2\tau' - \frac{\tau'^2}{4\pi q}\right) d\tau' \quad , \\ I_4 &= \int_{-\infty}^{\tau} \cos\left(\frac{\tau'^2}{4\pi q}\right) d\tau' \quad . \end{aligned} \tag{2. 15}$$

In the range of interest, $\tau \le 2\pi q$, the frequency of the integrands of I_1, and I_3 will always be greater than one. It may be shown by application of (2. 12a) and (2. 12b) that these integrals attain their maximum possible value for $\tau \cong 2\pi q$, and that the size of these maxima will be $0[1]$.

Analysis based on (2. 12a), (2. 12b), (2. 13a), and (2. 13b) shows that $I_2(\tau)$ attains its maximum value for

$$\tau = 2\pi\sqrt{q} \quad ,$$

corresponding to the first root of the integrand occurring after the stationary phase point, i. e. , the point at which excitation frequency equals system natural frequency.

The integral I_2 over $(-\infty, 0)$ may be bounded as outlined, but its value is well known and may be written directly:

$$\int_{-\infty}^{0} \sin\left(\frac{\tau'^2}{4\pi q}\right) d\tau' = \pi\sqrt{q/2} \quad .$$

Over the interval $(0, \tau)$, $I_2(\tau)$ is bounded by

$$\int_{0}^{2\pi\sqrt{q}} \sin\left(\frac{\tau'^2}{4\pi q}\right) d\tau' \quad ,$$

which may in turn be bounded by

$$\int_{0}^{2\sqrt{\pi q}} \left(\frac{\tau'^2}{4\pi q}\right) d\tau' + \int_{2\sqrt{\pi q}}^{2\pi\sqrt{q}} d\tau' \quad ,$$

since both 1 and $\left(\frac{\tau'^2}{4\pi q}\right)$ bound $\sin\left(\frac{\tau'^2}{4\pi q}\right)$.

The integral I_2 is thus bounded by

$$I_2 < 6.14\sqrt{q} \ .$$

The above analysis applied to $I_4(\tau)$ shows that it attains its maximum value for

$$\tau = \pi\sqrt{2q} \quad .$$

The portion of the integral over $(-\infty, 0)$ may again be written directly:

$$\int_{-\infty}^{0} \cos\left(\frac{\tau'^2}{4\pi q}\right) d\tau' = \pi\sqrt{q/2} \quad ,$$

and the bounding integral for the interval $(0, \tau)$

$$\int_{0}^{\pi\sqrt{2q}} \cos\left(\frac{\tau'^2}{4\pi q}\right) d\tau' \quad ,$$

will be bounded by

$$\int_{0}^{\pi\sqrt{2q}} d\tau' = \pi\sqrt{2q} \quad .$$

The integral I_4 over $(-\infty, \tau)$ is then bounded by

$$I_4 < 6.52\sqrt{q} \quad .$$

The value of q will be quite large for cases of interest. The integrals I_2 and I_4 will then dominate the expression for the response, and the contributions of I_1 and I_3 may be neglected. An upper bound for the response amplitude may then be written

$$y_{max} < 4.48\sqrt{q} \quad .$$

Lewis[1] computed the value of these integrals and suggested that the approximate equality

$$y_{max} \cong 3.67\sqrt{q} \tag{2.16}$$

is good for large q. The derived bound differs from this result by 22%. The quality of this bound was, of course, influenced by the fact that in the case treated several of the integrals were replaced by their exact values. In the general case, the infinite integrals would have to be over-estimated, as in the finite case above.

In addition to predicting a maximum bound for the undamped response of a system, the method outlined also predicts the approximate location in time of the response maximum for, considering previous remarks, this maximum should lie in the range

$$\pi\sqrt{2q} < \tau_{max} < 2\pi\sqrt{q} \quad ,$$

or

$$4.45 \sqrt{q} < \tau_{max} < 6.28\sqrt{q} \quad .$$

Lewis' suggested approximation for the location of this response maximum, when written in the present notation, is

$$\tau_{max} \cong 5.36\sqrt{q} \quad . \qquad (2.17)$$

If the expression (2.11) is assumed truly valid for times displaced still further from $\tau = 0$, the time of resonance, then the strong oscillatory behavior of I_2 and I_4 must determine the form of the "ringing" pattern succeeding the response maximum. In the example under study, the time of occurrence of the first secondary peak would have as a lower limit the time defining the end of the second positive swing of the integrand of I_4, and as an upper limit the time defining the end of the second positive swing of the integrand of I_2. The time of occurrence of the first secondary peak would then lie in the range

$$2\pi\sqrt{2q} < \tau_{max,\,2} < 2\pi\sqrt{3q} \quad ,$$

or

$$8.9\sqrt{q} < \tau_{max,\,2} < 11.0\sqrt{q} \quad .$$

A comparison to experimental results indicates fair agreement.

The response of the undamped system may still be overestimated by the above technique when the requirement that the excitation frequency is everywhere positive, and strictly decreasing is relaxed. The intervals of integration will be divided into subintervals in which the frequency of the integrand either increases, decreases, or remains constant. Since the truncated alternating sequence is bounded by the contribution of the first term provided, as before, that the first term exceeds the second in absolute value, and so on, the integrals over the intervals where the frequency increases, or decreases may be bounded. In intervals in which the frequency remains constant, the integrals may be computed or may be bounded by the smaller of the following expressions:

$$\frac{2}{\alpha} \geq \int_a^b \begin{matrix}\sin\\ \cos\end{matrix}\left[\alpha \tau'\right] d\tau' \quad ,$$

or

$$(b-a) \geq \int_a^b \begin{matrix}\sin\\ \cos\end{matrix}\left[\alpha \tau'\right] d\tau' \quad .$$

For the case $\alpha = 0$, b and a will both be finite as a consequence of the condition excluding sweep functions with frequency equaling the system natural frequency for an infinite period.

In the general case permitting multiple passes through the system natural frequency, the maximum bound will obtain shortly after the last pass through resonance. The actual response maximum may occur after some other pass through resonance, but the method, dealing as it does only with response amplitude, takes no account of possible phase cancellation that may occur and requires the "safe" choice, from the bounding standpoint, that peak behavior will always add.

Bounding the response for the damped system subjected to sweeping excitation presents none of the previous problems, since the presence of the decaying exponential term in the integral expression for the response, (2. 9), ensures absolute boundedness provided that

$$| f(\tau - u) | \leq Q \quad ,$$

where Q is some constant less than infinity. This condition is satisfied for the present class of excitations, since the modulus of the excitation is one. Expression (2. 9) may then be bounded by

$$y \leq 1/m\zeta \tag{2. 18}$$

The bound (2. 18) utilizes none of the convergence properties of the sweeping excitation, and it might be hoped that the method discussed in reference to bounding the undamped response could be applied to give a harder bound for the damped response.

Given non-zero damping, the integrals of (2. 11) become

$$I_1 = \int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')} \sin\left[\tau'+g(\tau')\right] d\tau' \quad ,$$

$$I_2 = \int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')} \sin\left[\tau'-g(\tau')\right] d\tau' \quad ,$$

$$I_3 = \int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')} \cos\left[\tau'+g(\tau')\right] d\tau' \quad , \tag{2.19}$$

$$I_4 = \int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')} \cos\left[\tau'-g(\tau')\right] d\tau' \quad .$$

Each of these integrals may be again thought of as the sum of an infinite alternating sequence. Convergence of these sequences is assured for any $g(\tau)$ by the presence of the decaying exponential term, which attenuates the maximum amplitude of each preceding half-cycle oscillation of the integrands as a function of the time of its occurrence measured from the time corresponding to the upper limit of integration.

If an excitation with strictly decreasing frequency is applied to the system, the integrand frequencies, $\varphi_{1,3}(\tau')$ and $\varphi_{2,4}(\tau')$, take the form indicated in Fig. 2.10. The integrals I_1 and I_3, as well as the portions of I_2 and I_4 over $(-\infty, 0)$ will be overestimated by expressions similar to (2.12a) and (2.12b):

$$I_i < \left| \int_{\tau_{i2}}^{\tau_{i1}} e^{-\zeta(\tau-\tau')} \begin{matrix}\sin\\ \cos\end{matrix}\left[\tau'\pm g(\tau')\right] d\tau' \right| + \left| \int_{\tau_{i1}}^{\tau} e^{-\zeta(\tau-\tau')} \begin{matrix}\sin\\ \cos\end{matrix}\left[\tau'\pm g(\tau')\right] d\tau' \right| , \tag{2.20}$$

in which τ_{i1} and τ_{i2} equal the first and second zeros of the relevant integrand occurring before τ. The second integrals again represent residual contributions deriving from the fact that τ need not correspond to a zero of the integrands.

These bounds should be better than those written for the undamped case, since the exponential terms ensure more rapid convergence.

Trouble arises, however, in the attempt to bound the integrals I_2 and I_4 over the interval $(0, \tau)$ for it is no longer clear that these integrals are bounded for any $\tau > 0$ by expressions similar to (2.13a) and (2.13b). This is true since the importance of each half-cycle of the integrand to the sum representing the integral over $(0, \tau)$ is a function not only of its period, but also of its time of occurrence measured from the upper limit of integration.

It could be demanded that the excitation frequency be arranged such that the contribution over the first half-cycle bound the integral over the interval $(0, \tau)$ for any τ. It would only be necessary then to choose τ to maximize the size of this contribution. Such a condition, it may be easily shown, would require that the absolute value of the integrand frequency of I_2 and I_4 take the form

$$\left|\varphi_{2,4}\right| \sim e^{\Delta\tau} \quad , \qquad \Delta > \zeta \quad .$$

When this is not the case, the importance of the initial contributions to I_2 and I_4 will be diminished as the upper limit of the integrals is raised, and the importance of succeeding terms, although representing shorter periods, will be amplified. A great deal of work would be required to bound these integrals by the method used for the undamped case, and even more work would be required to prove that the result obtained was actually a bound and not just an estimate.

It is certainly true that for the damped case the response maximum obtains a short time after the excitation frequency equals the system natural frequency, and an estimate could be developed for this maximum by considering the contributions of I_2 and I_4 near $\tau = 0$.

Consider the response of the damped system subject to the excitation (2.14). The integrals I_2 and I_4 of (2.19) will be approximated by (2.20) in which the limits of integration will be taken to be equal to the zeros of the relevant integrand occurring on either side of $\tau = 0$, the point of stationary phase. The integrands will then be approximated by convenient polynomials.

For I_2 the above work leads to an expression:

$$I_{2\,max} = \int_{-\infty}^{\tau_{max}} e^{-\zeta(\tau_{max} - \tau')} \sin\left(\frac{\tau'^2}{4\pi q}\right) d\tau'$$

$$\cong \frac{2e^{-2\pi\zeta\sqrt{q}}}{\zeta^2} \left\{ \begin{array}{l} \frac{1}{\sqrt{q}}\left[\cosh(2\pi\zeta\sqrt{q}) - \cosh(2\pi\zeta\sqrt{q} - \zeta\sqrt{q})\right] \\ - \frac{1}{\sqrt{\pi q}} \cosh(2\zeta\sqrt{\pi q}) + \frac{1}{2\pi\zeta q} \sinh(2\zeta\sqrt{\pi q}) \end{array} \right\} \qquad (2.21a)$$

and for I_4:

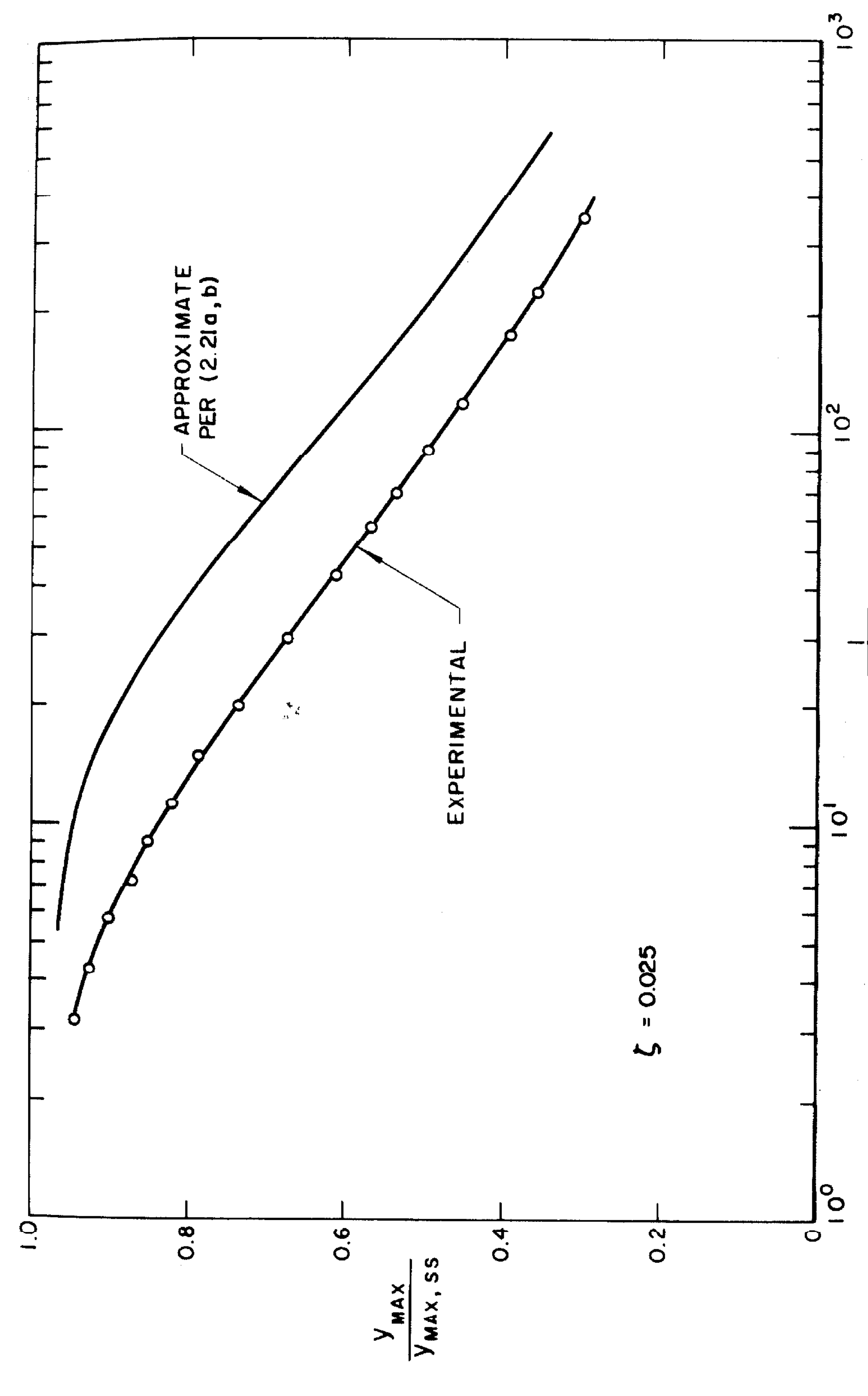
$$I_{4\,max} = \int_{-\infty}^{\tau_{max}} e^{-\zeta(\tau_{max}-\tau')} \cos\left(\frac{\tau'^2}{4\pi q}\right) d\tau'$$

$$\cong \sqrt{\frac{2}{q}}\,\frac{e^{-\pi\sqrt{2q}}}{\zeta^2}\left\{\begin{array}{l}\cosh(\pi\zeta\sqrt{2q}) \\ \\ -\cosh[(\pi-1)\zeta\sqrt{2q}\,]\end{array}\right\} \qquad (2.21b)$$

These expressions may be combined to form the modulus of the system response, y, per (2.11) giving an estimate for y_{max}. This estimate is plotted in Fig. 2.11 expressed as a ratio of the maximum response under sweeping conditions to the maximum response attained by a similarly damped system subjected to a fixed frequency excitation, $y_{max,\,ss}$. The shape of the curve predicts the shape of experimentally determined curves and constitutes a fair prediction of experimental results.

The preceding estimating method may, of course, again be applied to excitations other than the linearly sweeping frequency excitation discussed above.

It should be pointed out that no effort has been made to bound the response maximum under sweeping conditions by the steady-state response maximum. It will be shown in Chapter III that such is not the case. The steady-state response maximum may estimate the response maximum for slowly varying excitation frequency, but it will not bound this maximum.



APPROXIMATE
PER (2.21a,b)
EXPERIMENTAL
$\zeta = 0.025$
$\frac{y_{MAX}}{y_{MAX,\,SS}}$
1.0
0.8
0.6
0.4
0.2
0
10^0
10^1
10^2
10^3
$\frac{1}{|q|\zeta^2}$

FIG. 2.11

III. RESPONSE OF SYSTEMS TO SLOWLY SWEEPING EXCITATIONS

For cases in which the frequency of excitation is a slowly varying function of time, it is possible to develop a more orderly approach to the estimation of system response by using the technique of small perturbations.

The small perturbation quantity--related to the rate of change of the excitation frequency--does not appear in the equation of motion, (2.4). Its introduction may be accomplished, however, in the following straightforward way.

If the excitation in equation (2.4) is replaced by the complex expression

$$p(\tau) = e^{jg(\tau)} \quad , \qquad (3.1)$$

the response may be written

$$y(\tau) = A(\tau)\, e^{jg(\tau)} \quad . \qquad (3.2)$$

The function $A(\tau)$ is the complex modulus. Its amplitude represents the amplitude of the response, and its phase represents the phase difference of the response.

Substitution of (3.1) and (3.2) into equation (2.4) leads to the following equation for the complex modulus $A(\tau)$:

$$A'' + (2\zeta + 2j\alpha)A' + (1 - \alpha^2 + 2j\zeta\alpha + j\alpha')\,A = 1 \quad , \qquad (3.3)$$

where $\alpha = g'$, and where primes denote differentiation with respect to the dimensionless time, τ.

It is assumed that the initial application of the excitation occurred sufficiently far in the past, and that the frequency of excitation has behaved in such a way as to make the contribution of transient terms to the solution of (3.3) near $\tau = 0$ very small. The latter assumption requires that the excitation frequency last equaled system natural frequency a long time in the past. It is assumed further that near $\tau = 0$ the excitation frequency varies slowly and has the form

$$\alpha = \alpha_0 + \epsilon\tau + \epsilon^2 C\tau^2 + \ldots \quad . \tag{3.4}$$

The leading term in (3.4), $\alpha_0 = \sqrt{1 - 2\zeta^2}$, is the frequency corresponding to the steady-state response maximum.

Let the complex modulus take the form

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \ldots \quad . \tag{3.5}$$

Inserting (3.4) and (3.5) into (3.3) leads to the following set of equations for the powers of ϵ:

$$\begin{aligned}
A_0'' + \chi A_0' + \zeta\chi A_0 &= 1 \ , \\
A_1'' + \chi A_1' + \zeta\chi A_1 &= -2j\tau A_0' - j(\chi\tau+1)A_0 \ , \\
A_2'' + \chi A_2' + \zeta\chi A_2 &= -2j\tau A_1' - j(\chi\tau+1)A_1 \\
&\quad -2jC\tau^2 A_0' - jC\tau(\chi\tau+2)A_0 \\
&\quad + \tau^2 A_0 \ , \\
&\ldots
\end{aligned} \tag{3.6}$$

In these equations

$$\chi = 2\zeta + 2j\alpha_0 \quad .$$

The zeroth order solution may be written directly:

$$A_0 = \frac{1}{2\zeta^2 + 2j\zeta\alpha_0} \quad . \tag{3.7}$$

This expression has the modulus

$$|A_0| = \frac{1}{2\zeta m} \quad , \qquad m = \sqrt{1-\zeta^2} \quad ,$$

corresponding to the response maximum under steady-state conditions.

The second of the set (3.6) and the value of A_0 are used to obtain A_1:

$$A_1 = \frac{-j}{\zeta^2\chi} \cdot \left(\tau + \frac{1}{\chi} - \frac{1}{\zeta}\right) . \tag{3.8}$$

The third equation of the set (3.6), and the results (3.7) and (3.8) may be used to obtain the coefficient of ϵ^2:

$$\begin{aligned} A_2 = & \left(\frac{-1}{\zeta^3\chi} + \frac{1}{\zeta^2\chi^2} - \frac{jC}{\zeta^2\chi}\right)\tau^2 \\ & + \left[\left(\frac{3}{\zeta^4\chi} - \frac{6}{\zeta^3\chi^2}\right) + 2jC\left(\frac{1}{\zeta^3\chi} - \frac{1}{\zeta^2\chi^2}\right)\right]\tau \\ & + \left(-\frac{3}{\zeta^5\chi} + \frac{9}{\zeta^4\chi^2} - \frac{3}{\zeta^3\chi^3}\right) \\ & + 2jC\left(-\frac{1}{\zeta^4\chi} + \frac{2}{\zeta^3\chi^2}\right) \quad . \end{aligned}$$

The modulus of A to second order:

$$|A| \cong |A_0| \cdot \left[\left|1 + \varepsilon \operatorname{Re}\left(\frac{A_1}{A_0}\right) + \varepsilon^2 \left\{\operatorname{Re}\left(\frac{A_2}{A_0}\right) + \frac{1}{2}\left[\operatorname{Im}\left(\frac{A_1}{A_0}\right)\right]^2\right\}\right|\right] , \tag{3.9}$$

may be written

$$|A| \cong \frac{1}{2\zeta m} \cdot \left\{1 - \frac{\alpha_0 \varepsilon}{2\zeta m^2} + \varepsilon^2 \left[-\frac{\alpha_0^2 \tau^2}{2\zeta^2 m^2} + \frac{(4-9\zeta^2)}{2\zeta^3 m^2}\tau + \frac{(-20+78\zeta^2-69\zeta^4)}{8\zeta^4 m^4}\right] + C\varepsilon^2 \left(-\frac{\alpha_0 \tau}{\zeta m^2} + \frac{2\alpha_0}{\zeta^2 m^2}\right)\right\} . \tag{3.10}$$

Expression (3.10) has a single maximum occurring at

$$\tau_{max} = \frac{\zeta}{\alpha_0} \cdot \left[\frac{(4-9\zeta^2)}{2\alpha_0 \zeta^2} - C\right] ,$$

corresponding to an excitation frequency of

$$\alpha_{max} = \alpha_0 + \frac{\varepsilon\zeta}{\alpha_0}\left[\frac{(4-9\zeta^2)}{2\alpha_0\zeta^2} - C\right] .$$

The maximum value of (3.10) is

$$|A|_{max} \cong \frac{1}{2\zeta m}\left[1 - \frac{\alpha_0 \varepsilon}{2\zeta m^2} + \frac{\varepsilon^2}{8\alpha_0^2 \zeta^4 m^4} \cdot (-4+30\zeta^2-72\zeta^4+57\zeta^6) + \frac{C\varepsilon^2}{2\alpha_0 m^2}(1+\alpha_0 C)\right] . \tag{3.11}$$

The modulus of the response, $|A|$, may be approximated usefully by the perturbation series to first order in ε, provided that ε is chosen sufficiently small. When the excitation frequency decreases through the system natural frequency, ε--the rate of change of excitation frequency to first order--will be negative, and the perturbed modulus will be larger than the unperturbed expression. It is, therefore, not possible to conclude that the response of a system to sweeping excitation is bounded above by the maximum of the steady-state response.

Observations of systems with damping less than ten per cent of critical indicate that the response peak for even small negative values of ε is less than the steady-state response maximum. The range of applicability of the first order approximation is thus too small to make it generally useful for normally encountered damping ratios.

Given moderate values of C and τ, and small values of ζ, the coefficient of a particular power of ε in the series expression for the perturbed solution will be dominated by terms multiplying $1/\zeta$ raised to the highest power. A study of the complex response expression indicates that the highest power of $1/\zeta$ associated with ε^n will be 2n. For expressions (3.10) and (3.11) to constitute useful approximations of the response amplitude, it will be necessary to demand that $\varepsilon/\zeta^2 << 1$.

It may be seen that for a given value of ζ there exists a range of ε satisfying the given condition and also causing the absolute value of the second order correction of (3.11) to exceed the absolute value of the first order correction. In this range, the response maximum under sweeping conditions will be less than the steady-state response even if ε takes a minus sign.

It should be noted in addition that the contribution to the second order modulus corrector associated with the nonlinear term in the frequency expression, (3.4), is quite small. The factor C appearing in the expression for the perturbed response maximum, (3.11), will multiply a number much smaller than ζ^4 if ε lies in the prescribed range.

The foregoing analysis provides potentially useful information concerning system response near resonance, but provides small insight into the nature of the response for large times and, consequently, for sweeping excitation frequencies much displaced from system resonance. One is led to seek a more general perturbation technique.

Such a technique proceeds from the observation that the steady-state response amplitude and phase are functions of the frequency of excitation, and the oscillatory behavior of the response is a function of the time integral of the excitation frequency. An appropriate conclusion would be that, for sufficiently slow sweep rates, time will enter the response expression in two indirect, distinct ways--through frequency, and through the time integral of the frequency. In the steady-state case, frequency and its time integral are independent variables. In the sweeping case, these variables are linked due to their mutual time dependence through the perturbation quantity--again related to the time rate of change of the excitation frequency.

In equation (2.4):

$$y'' + 2\zeta y' + y = \sin g(\tau) \quad , \qquad (2.4)$$

the solution, y, may be written in terms of the variables described above:

$$y(\tau) \;=\; y[\alpha(\tau)\;,\;g(\tau)]\;\;.$$

The derivatives of $y(\tau)$ may then be written

$$\frac{dy}{d\tau} \;=\; \alpha' \frac{\partial y}{\partial \alpha} + \alpha \frac{\partial y}{\partial g}\;\;, \tag{3.12}$$

and

$$\frac{d^2y}{d\tau^2} \;=\; (\alpha')^2 \,\frac{\partial^2 y}{\partial \alpha^2} + 2\alpha\alpha' \frac{\partial^2 y}{\partial\alpha\partial g} + (\alpha)^2 \,\frac{\partial^2 y}{\partial g^2}$$

$$+\,\alpha'' \frac{\partial y}{\partial \alpha} + \alpha' \frac{\partial y}{\partial g}\;\;.$$

It will be assumed that the excitation was initially applied a long time in the past, and that the frequency of excitation has varied slowly throughout its history. The time derivatives of the excitation will be written

$$\begin{aligned} \alpha' &= \epsilon\,\kappa(\alpha)\;\;, \\ \alpha'' &= \epsilon^2\kappa(\alpha)\,\frac{d\kappa(\alpha)}{d\alpha}\;\;. \end{aligned} \tag{3.13}$$

The right-hand side of (2.4) will be replaced by a more general complex expression permitting variations in the excitation amplitude as a function of frequency, and the perturbation quantity ϵ:

$$f(\tau) = [1 + \epsilon\eta(\alpha)]\;e^{jg}\;\;. \tag{3.14}$$

It will be assumed that the solution, y, may be written as the product of a function of the instantaneous excitation frequency embodying the long-time behavior, and a function of the integral of frequency embodying the oscillatory character of the solution:

$$y(\alpha, g) \;=\; B(\alpha)\cdot e^{jg} \tag{3.15}$$

Substitution of expressions (3.12) through (3.15) in equation (2.4) leads to the following equation for the complex modulus, $B(\alpha)$:

$$\begin{aligned}\epsilon^2\kappa^2\frac{d^2B}{d\alpha^2} &+ \left(2j\epsilon\alpha\kappa + 2\zeta\epsilon\kappa + \epsilon^2\kappa\frac{d\kappa}{d\alpha}\right)\cdot\frac{dB}{d\alpha} \\ &+ (1-\alpha^2 + 2j\zeta\alpha + j\epsilon\kappa)\,B = 1 + \epsilon\eta(\alpha) \;.\end{aligned} \tag{3.16}$$

If $B(\alpha)$ takes the form

$$B(\alpha) \;=\; B_0(\alpha) + \epsilon B_1(\alpha) + \epsilon^2 B_2(\alpha) + \ldots \quad ,$$

its substitution in (3.16) leads to the following set of equations for the coefficients of the various powers of ϵ:

$$\begin{aligned}\sigma B_0 &= 1 \quad , \\ \sigma B_1 &= -\tilde{\chi}\kappa\frac{dB_0}{d\alpha} - j\kappa B_0 + \eta \;, \\ \sigma B_2 &= -\tilde{\chi}\kappa\frac{dB_1}{d\alpha} - j\kappa B_1 \\ &\quad -\kappa^2\frac{d^2B_0}{d\alpha^2} - \kappa\frac{d\kappa}{d\alpha}\cdot\frac{dB_0}{d\alpha} \quad , \\ &\cdots\end{aligned} \tag{3.17}$$

In the above equations

$$\sigma = 1 - \alpha^2 + 2j\zeta\alpha \quad ,$$
$$\tilde{\chi} = 2\zeta + 2j\alpha \quad .$$

The zeroth order solution may be written directly:

$$B_0 = \frac{1}{\sigma} \ . \tag{3.18}$$

The modulus of B_0 is

$$|B_0| = \frac{1}{\sqrt{\varphi}} \quad ,$$

in which

$$\varphi = (1-\alpha^2)^2 + 4\zeta^2\alpha^2 \quad .$$

The second of the set (3.17) may be solved for B_1 using the result (3.18):

$$B_1 = \frac{j\tilde{\chi}^2\kappa}{\sigma^3} - \frac{j\kappa}{\sigma^2} + \frac{\eta}{\sigma} \quad , \tag{3.19}$$

and the third of the set (3.17) may be solved for B_2 using the results (3.18) and (3.19):

$$\begin{aligned} B_2 = & \left(- \frac{3\tilde{\chi}^4}{\sigma^5} + \frac{9\tilde{\chi}^2}{\sigma^4} - \frac{3}{\sigma^3}\right) \kappa^2 \\ & + j \cdot \left(\frac{2\tilde{\chi}}{\sigma^3} - \frac{\tilde{\chi}^3}{\sigma^4}\right) \cdot \kappa \cdot \frac{d\kappa}{d\alpha} \\ & + j \cdot \left(\frac{\tilde{\chi}^2}{\sigma^3} - \frac{1}{\sigma^2}\right) \cdot \kappa \cdot \eta \\ & - \frac{\tilde{\chi}\kappa \frac{d\eta}{d\alpha}}{\sigma^2} \quad . \end{aligned} \tag{3.20}$$

The modulus of B may now be written to second order in ε by inserting the results (3.18), (3.19), and (3.20) into expression (3.9):

$$
\begin{aligned}
|B| \cong \frac{1}{\sqrt{\varphi}} \Bigg[\Bigg[1 &- \frac{2\zeta\alpha\varepsilon}{\varphi^2}\left[4(1+\alpha^2)(\alpha_0^2-\alpha^2)+\varphi\right]\cdot\kappa \\
&+ \frac{\varepsilon^2}{\varphi^4}\Big[-40\zeta^4(1+\alpha^2)^4 - 40\alpha^4(\alpha_0^4-\alpha^4) \\
&\qquad + 272\,\alpha^2\zeta^2(1+\alpha^2)^2(\alpha_0^2-\alpha^2)^2\Big]\cdot\kappa^2 \\
&+ \frac{\varepsilon^2}{\varphi^3}\Big[32\zeta^2(1-\alpha^2)(1+\alpha^2)^2 - 32\alpha^2(1-\alpha^2)(\alpha_0^2-\alpha^2)^2 \\
&\qquad + 144\,\alpha^2\zeta^2(1+\alpha^2)(\alpha_0^2-\alpha^2)\Big]\cdot\kappa^2 \\
&+ \frac{\varepsilon^2}{\varphi^2}\Big[-\frac{5}{2}(1-\alpha^2)^2+12\zeta^2\alpha^2\Big]\cdot\kappa^2 \\
&+ \underbrace{\varepsilon\eta - \varepsilon^2\kappa\left[\frac{2\zeta\alpha\eta}{\varphi}+\frac{8\zeta\alpha\eta(1+\alpha^2)(\alpha_0^2-\alpha^2)}{\varphi^2}+\frac{2\zeta\frac{d\eta}{d\alpha}(1+\alpha^2)}{\varphi}\right]}_{\text{contribution due to varying excitation amplitude}} \\
&+ \underbrace{\varepsilon^2\kappa\frac{d\kappa}{d\alpha}\left\{\frac{8\alpha(\alpha_0^2-\alpha^2)}{\varphi^3}\left[3\zeta^2(1+\alpha^2)^2-\alpha^2(\alpha_0^2-\alpha^2)^2\right] - 4\alpha\frac{[(1-\alpha^2)^2-4\zeta^2]}{\varphi^2}\right\}\Bigg]}_{\text{contribution duc to varying rate of change of frequency.}}
\end{aligned}
\tag{3.21}
$$

The maximum of the perturbed response may be obtained by assuming that this maximum occurs at a frequency slightly displaced from the frequency of the steady-state maximum:

$$\alpha_{10} = \alpha_0 + \epsilon\,\delta_1 + \epsilon^2\delta_2 + \ldots \quad .$$

The expression (3.21) is differentiated, its derivative set equal to zero for $\alpha = \alpha_{10}$, and the component functions are expanded about α_0. The resulting frequency of the maximum to first order is

$$\alpha_{10} = \alpha_0 + \epsilon\left[\frac{(4-9\zeta^2)\cdot\kappa(\alpha_0)}{2\zeta\alpha_0^2} - \frac{\zeta\left.\frac{d\kappa}{d\alpha}\right|_{\alpha_0}}{2\alpha_0} + \frac{\zeta^2 m^2}{\alpha_0^2}\cdot\left.\frac{d\eta}{d\alpha}\right|_{\alpha_0}\right]. \quad (3.22)$$

The value of the maximum of (3.21) to second order is

$$\begin{aligned}
|B|_{max} \cong \frac{1}{2\zeta m}\Bigg\{ & 1 - \frac{\epsilon\alpha_0\kappa(\alpha_0)}{2\zeta m^2} + \frac{\epsilon^2[\kappa(\alpha_0)]^2(-4+30\zeta^2-72\zeta^4+57\zeta^6)}{8\alpha_0^2\zeta^4 m^4} \\
& + \frac{\epsilon^2\kappa(\alpha_0)}{2\zeta}\left[\frac{(2-5\zeta^2)}{\alpha_0^2}\left.\frac{d\eta}{d\alpha}\right|_{\alpha_0} - \frac{\alpha_0\eta(\alpha_0)}{m^2}\right] \\
& + \frac{\epsilon^2\kappa(\alpha_0)}{4m^2\alpha_0}\cdot\left.\frac{d\kappa}{d\alpha}\right|_{\alpha_0} + \frac{\epsilon^2\left(\left.\frac{d\kappa}{d\alpha}\right|_{\alpha_0}\right)^2}{8m^2} - \frac{\epsilon^2\zeta}{2\alpha_0}\left(\left.\frac{d\kappa}{d\alpha}\right|_{\alpha_0}\cdot\left.\frac{d\eta}{d\alpha}\right|_{\alpha_0}\right) \\
& + \frac{\zeta^2 m^2}{2\alpha_0^2}\left(\left.\frac{d\eta}{d\alpha}\right|_{\alpha_0}\right)^2\Bigg\} . \qquad (3.23)
\end{aligned}$$

Constant amplitude sweeping excitations having frequencies varying linearly, and exponentially in time are of particular interest.

Expressions (3.21), (3.22), and (3.23) may be written for these specific cases.

For the linear sweep

$$\alpha' = \text{constant} = \frac{1}{2\pi q} \quad .$$

The response amplitude of a system influenced by a slowly varying, linearly sweeping excitation is approximated by

$$y_{mod} = \frac{1}{\sqrt{\varphi}} \left[1 - \frac{\zeta\alpha}{\pi q} \left[\frac{4(1+\alpha^2)(\alpha_0^2-\alpha^2) + \varphi}{\varphi^2} \right] \right.$$

$$+ \frac{1}{4\pi^2 q^2} \left\{ \frac{\left[-40\zeta^4(1+\alpha^2)^4 - 40\alpha^4(\alpha_0^2-\alpha^2)^4 + 272\zeta^2\alpha^2(1+\alpha^2)^2(\alpha_0^2-\alpha^2)^2\right]}{\varphi^4} \right.$$

$$+ \frac{\left[32\zeta^2(1-\alpha^2)(1+\alpha^2)^2 - 32\alpha^2(1-\alpha^2)(\alpha_0^2-\alpha^2)^2 + 144\alpha^2\zeta^2(1+\alpha^2)(\alpha_0^2-\alpha^2)\right]}{\varphi^3}$$

$$\left.\left. + \frac{\left[-\frac{5}{2}(1-\alpha^2)^2 + 12\zeta^2\alpha^2\right]}{\varphi^2} \right\} \right] \quad . \tag{3.24}$$

The maximum of (3.24) occurs at

$$\alpha_{max} = \alpha_0 + \frac{(4-9\zeta^2)}{4\zeta\pi q\alpha_0^2} \quad ,$$

and the value of this maximum is

$$y_{max} = \frac{1}{2\zeta m} \left[1 - \frac{\alpha_0}{4\pi q\zeta m^2} + \frac{(-4+30\zeta^2-72\zeta^4+57\zeta^6)}{32\pi^2 q^2\alpha_0^2\zeta^4 m^4} \right] \quad . \tag{3.25}$$

For the exponential sweep

$$\alpha'/\alpha = \text{constant} = \gamma/2\pi \quad ,$$

and the response amplitude of a system influenced by a slowly varying, exponentially sweeping excitation is approximated by

$$y_{mod} = \frac{1}{\sqrt{\varphi}} \Bigg[1 - \frac{\zeta\alpha^2\gamma}{\pi}\left[\frac{4(1+\alpha^2)(\alpha_0^2-\alpha^2)+\varphi}{\varphi^2}\right]$$

$$+ \frac{\gamma^2\alpha^2}{4\pi^2}\Bigg\{ \frac{\left[-40\zeta^4(1+\alpha^2)^4 - 40\alpha^4(\alpha_0^2-\alpha^2)^4 + 272\alpha^2\zeta^2(1+\alpha^2)^2(\alpha_0^2-\alpha^2)^2\right]}{\varphi^4}$$

$$+ \frac{\left[56\zeta^2(1-\alpha^2)(1+\alpha^2)^2 - 40\alpha^2(1-\alpha^2)(\alpha_0^2-\alpha^2)^2 - 48\zeta^4(1+\alpha^2)^2 + 16\alpha^2\zeta^2(\alpha_0^2-\alpha^2)^2 + 144\alpha^2\zeta^2(1+\alpha^2)(\alpha_0^2-\alpha^2)\right]}{\varphi^3}$$

$$+ \frac{\left[-\frac{13}{2}(1-\alpha^2)^2 + 12\zeta^2\alpha^2 + 16\zeta^2\right]}{\varphi^2}\Bigg\}\Bigg] \quad . \qquad (3.26)$$

The maximum of (3.26) obtains for

$$\alpha_{max} = \alpha_0 + \frac{(4-10\zeta^2)\cdot\gamma}{4\zeta\pi\alpha_0} \quad ,$$

and the value of this maximum to second order is

$$y_{max} = \frac{1}{2\zeta m}\left[1 - \frac{\alpha_0^2\gamma}{4\pi\zeta m^2} + \frac{\gamma^2(-4+30\zeta^2-69\zeta^4+54\zeta^6)}{32\pi^2\zeta^4 m^4}\right] \quad ; \qquad (3.27)$$

In Fig. 3.1 typical first order correction functions for the linear response, (3.24), and for the exponential response, (3.26), are plotted versus the frequency of excitation. It is seen that both functions are quite similar, and indeed, are nearly equal at system resonance.

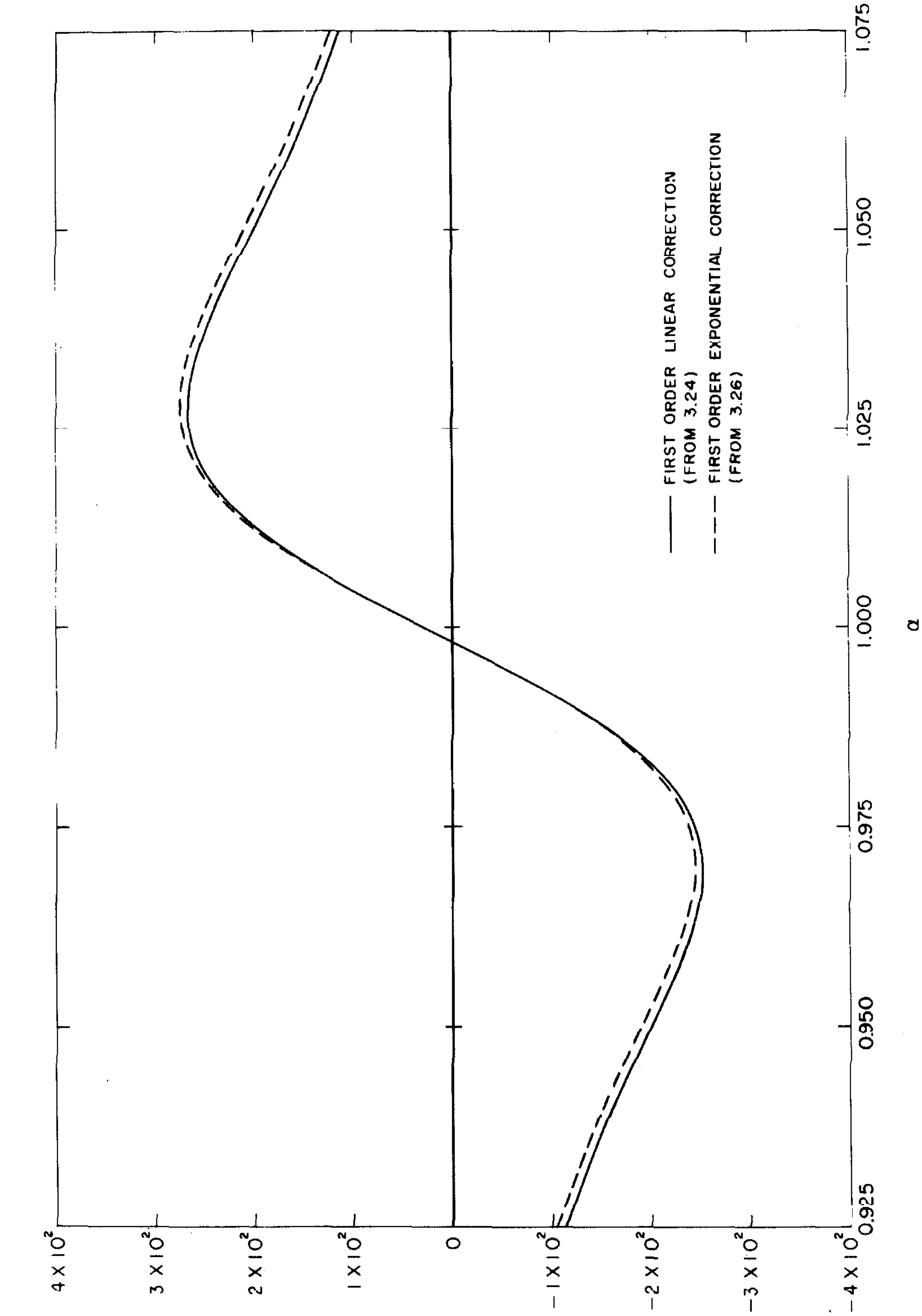

4 X 10²
3 X 10²
2 X 10²
1 X 10²
0
−1 X 10²
−2 X 10²
−3 X 10²
−4 X 10²
0.925
0.950
0.975
1.000
1.025
1.050
1.075
α
FIRST ORDER LINEAR CORRECTION
(FROM 3.24)
FIRST ORDER EXPONENTIAL CORRECTION
(FROM 3.26)

FIG. 3.1

For positive rates of change of excitation frequency, both linear and exponential sweeping reduce the amplitude of the response for frequencies less than system natural frequency, and increase the amplitude of the response for frequencies greater than system natural frequency. Given equal values of the perturbation parameters, exponential sweeping produces the more significant amplitude increase in the post-resonance era, and linear sweeping produces the more significant amplitude reduction in the pre-resonance era.

In Fig. 3.2 typical second order correction functions associated with (3.24) and (3.26) are presented, and again a pronounced similarity, especially near system resonance, is remarked. As might have been predicted, the second order correction functions are approximately symmetric about the system natural frequency and make their largest contribution there.

The first order correction terms make a very small contribution to system response at resonance; they serve primarily to shape the response curve for frequencies displaced from resonance and provide maximum shaping near resonance. The large negative contribution of the second order correction functions at resonance provides, in part, the observed response peak reduction under sweeping conditions.

Since the value of the coefficients multiplying the perturbation parameter raised to the first and second power in the expressions derived above depends upon the frequency of the excitation at which the response amplitude is desired, the limiting size of ϵ, beyond which these expressions cease to represent useful approximations, must also

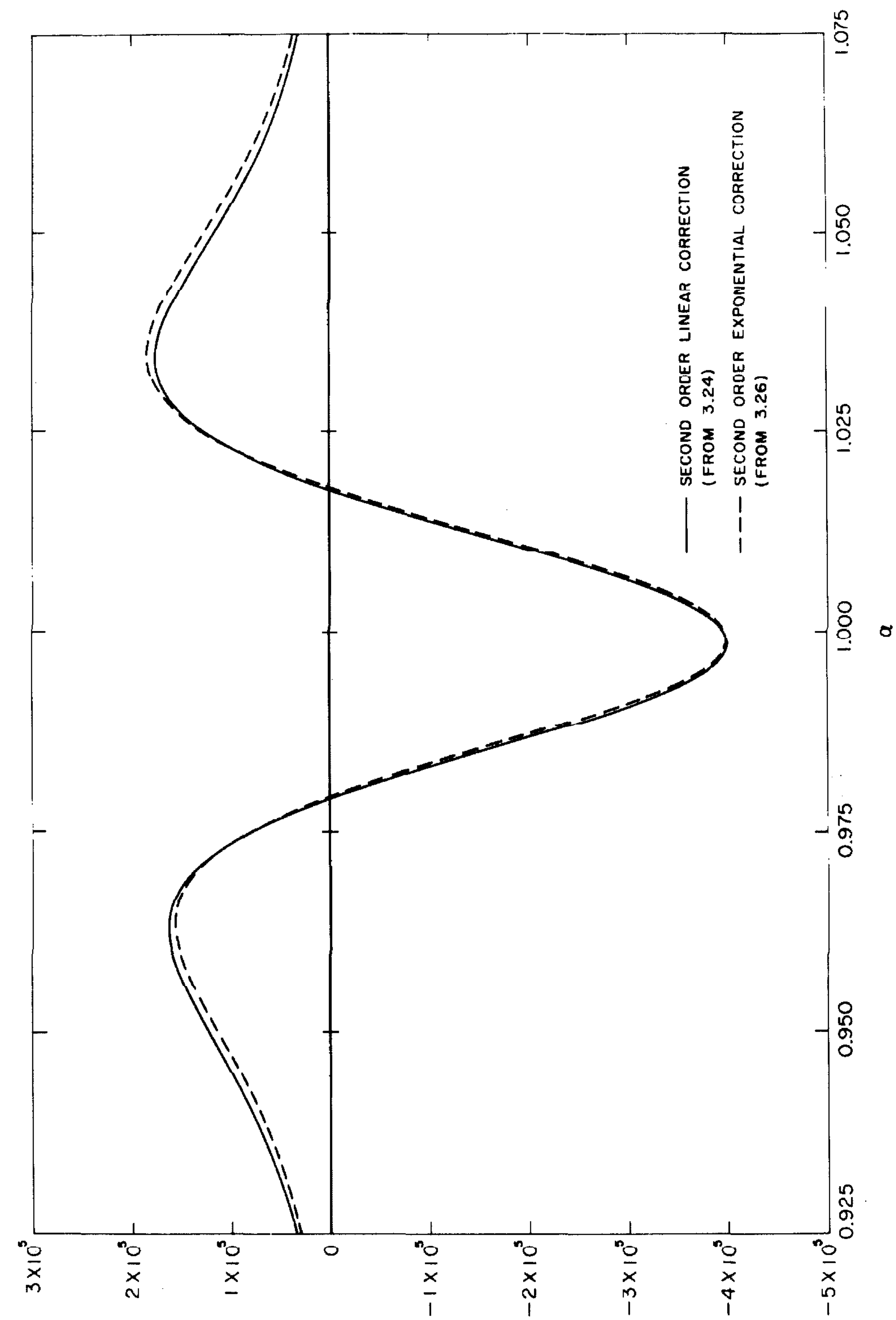
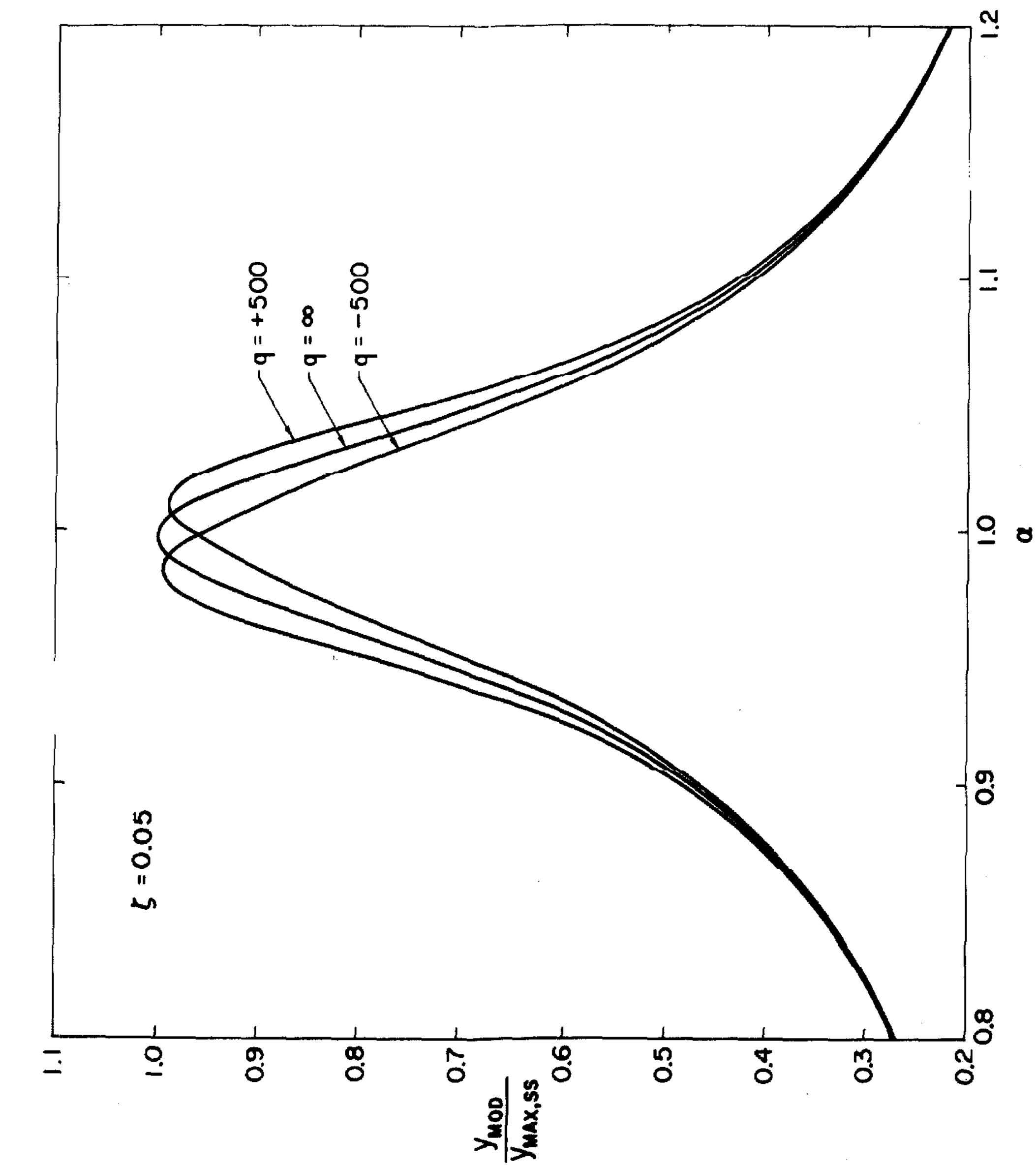

3 X 10^5
2 X 10^5
1 X 10^5
0
-1 X 10^5
-2 X 10^5
-3 X 10^5
-4 X 10^5
-5 X 10^5
0.925
0.950
0.975
1.000
1.025
1.050
1.075
α
SECOND ORDER LINEAR CORRECTION
(FROM 3.24)
SECOND ORDER EXPONENTIAL CORRECTION
(FROM 3.26)

FIG. 3.2

be thought of as a function of the frequency of interest. For frequencies displaced from system resonance, it should be required that the ratio of the absolute values of the first and second order coefficients be much greater than ε. For frequencies near system resonance, the correction to the peak response is primarily second order, and the demand that $\varepsilon/\zeta^2 << 1$ may be substituted for the above. A caveat should be added here. Expressions (3.25) and (3.27) provide estimates for the response maximum agreeing well with experimental results for values of $\varepsilon/\zeta^2 \sim .2$. If, however, an attempt is made to generate the perturbed response curve near resonance for such values of ε using (3.24) or (3.26), the two maxima of the second corrector, Fig. 3.2, appear in the response. The larger of these peaks represents the perturbed response maximum of (3.25), and (3.27); the smaller of these peaks has no physical significance. When ε assumes the size indicated above, the expression (3.10), adapted to the case of interest, provides a better prediction of behavior near resonance.

The similarity of the results obtained for the response maximum by the first and second perturbation techniques suggests a correspondence between the two methods. It is not difficult to derive the method one expression for the response envelope near responance, (3.10), from the more general result (3.21), by expanding the frequency, α, in (3.21) about the frequency of the unperturbed maximum per (3.4).



$\zeta = 0.05$
q = +500
q = ∞
q = −500
$\frac{y_{MOD}}{y_{MAX,SS}}$
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.8
0.9
1.0
1.1
1.2
α

FIG. 3.3

Some of the more important results of the present section are given in Figs. 3.3 through 3.7.

Figure 3.3 illustrates amplitude histories for a system subjected to linearly sweeping excitations. On a plot of this type, differences between exponentially and linearly sweeping excitations sharing a common value of ε would not appear. The steady-state amplitude-excitation frequency curve ($q = \infty$) has been included for the purpose of comparison.

Figures 3.4a, b indicate the change in system amplitude peak response as a function of damping and sweep rate for an exponentially sweeping excitation. Differences in peak response for linearly and exponentially sweeping excitation are again very small.

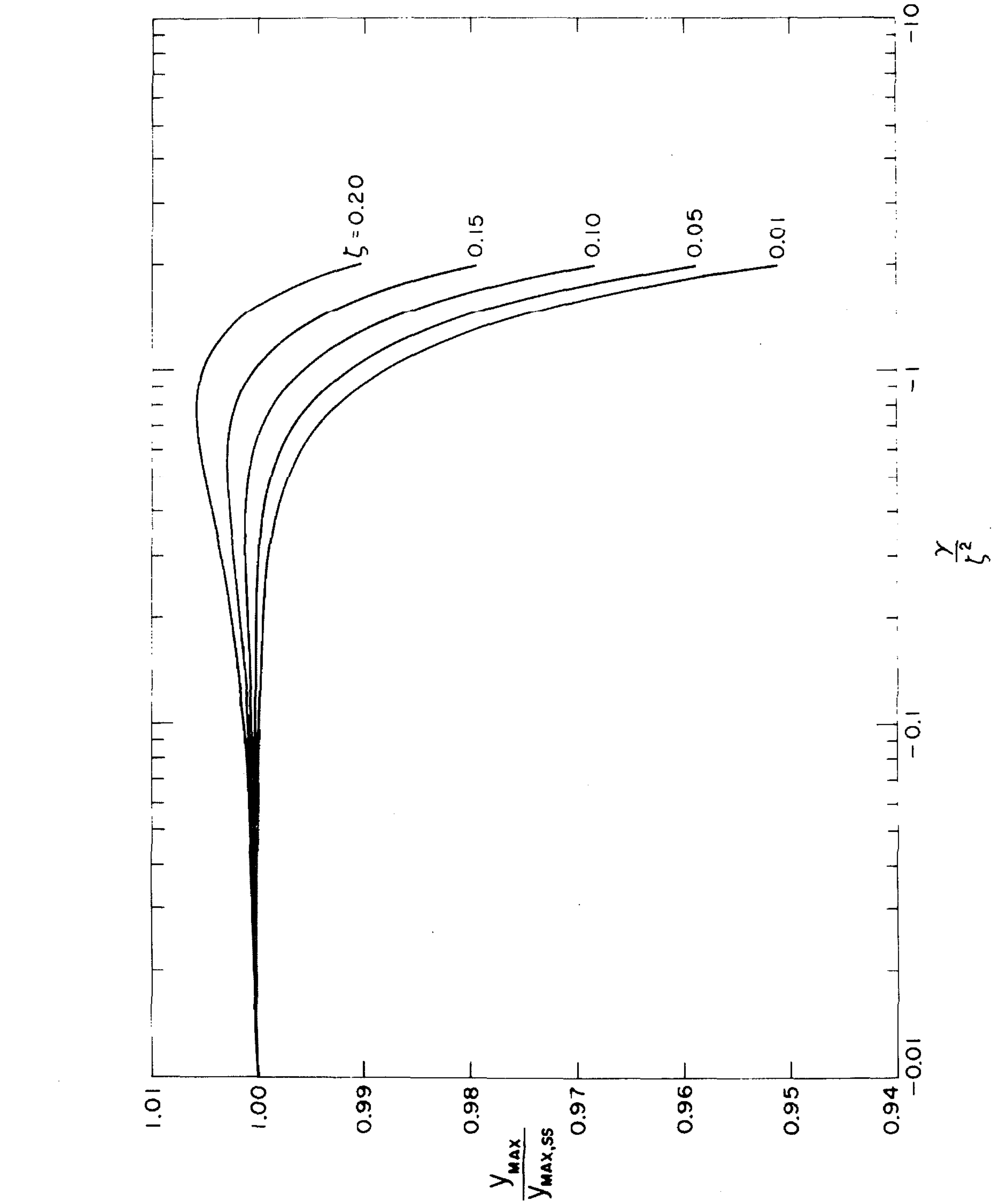
The shift of the frequency corresponding to the response amplitude maximum is presented in Figs. 3.5a, b for the linear sweep case.

When a system approximating the viscous damped single degree of freedom model is influenced by a slowly sweeping excitation having known characteristics, it is possible to correct the usual estimates of system damping and natural frequency by using the results of the second perturbation technique.

If damping is to be determined by measurement of the level of the response peak, and solution of

$$1/2\ \zeta_e m_e = y_{max}\ , \qquad m_e = \sqrt{1 - \zeta_e^2}\ , \tag{3.28}$$

it will be possible to correct the estimated damping, ζ_e, to second order in the sweep parameter by the following method:



ζ = 0.20
0.15
0.10
0.05
0.01
1.01
1.00
0.99
0.98
0.97
0.96
0.95
0.94
$\frac{y_{MAX}}{y_{MAX,SS}}$
−0.01
−0.1
−1
−10
$\frac{\gamma}{\zeta^2}$

FIG. 3.4a

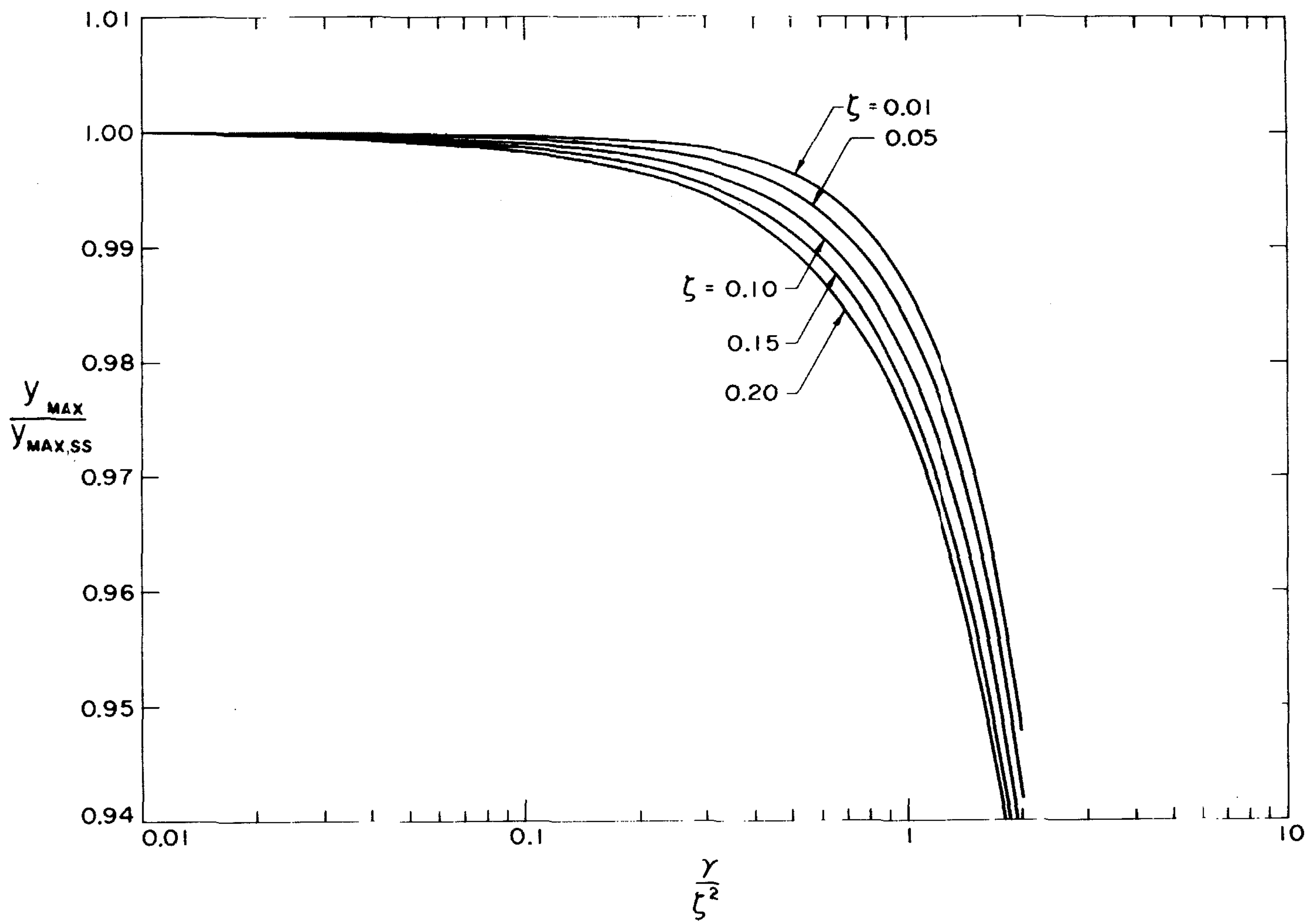

1.01
1.00
0.99
0.98
0.97
0.96
0.95
0.94
$\frac{y_{MAX}}{y_{MAX,SS}}$
0.01
0.1
1
10
$\frac{\gamma}{\zeta^2}$
$\zeta = 0.01$
0.05
$\zeta = 0.10$
0.15
0.20

FIG. 3.4b

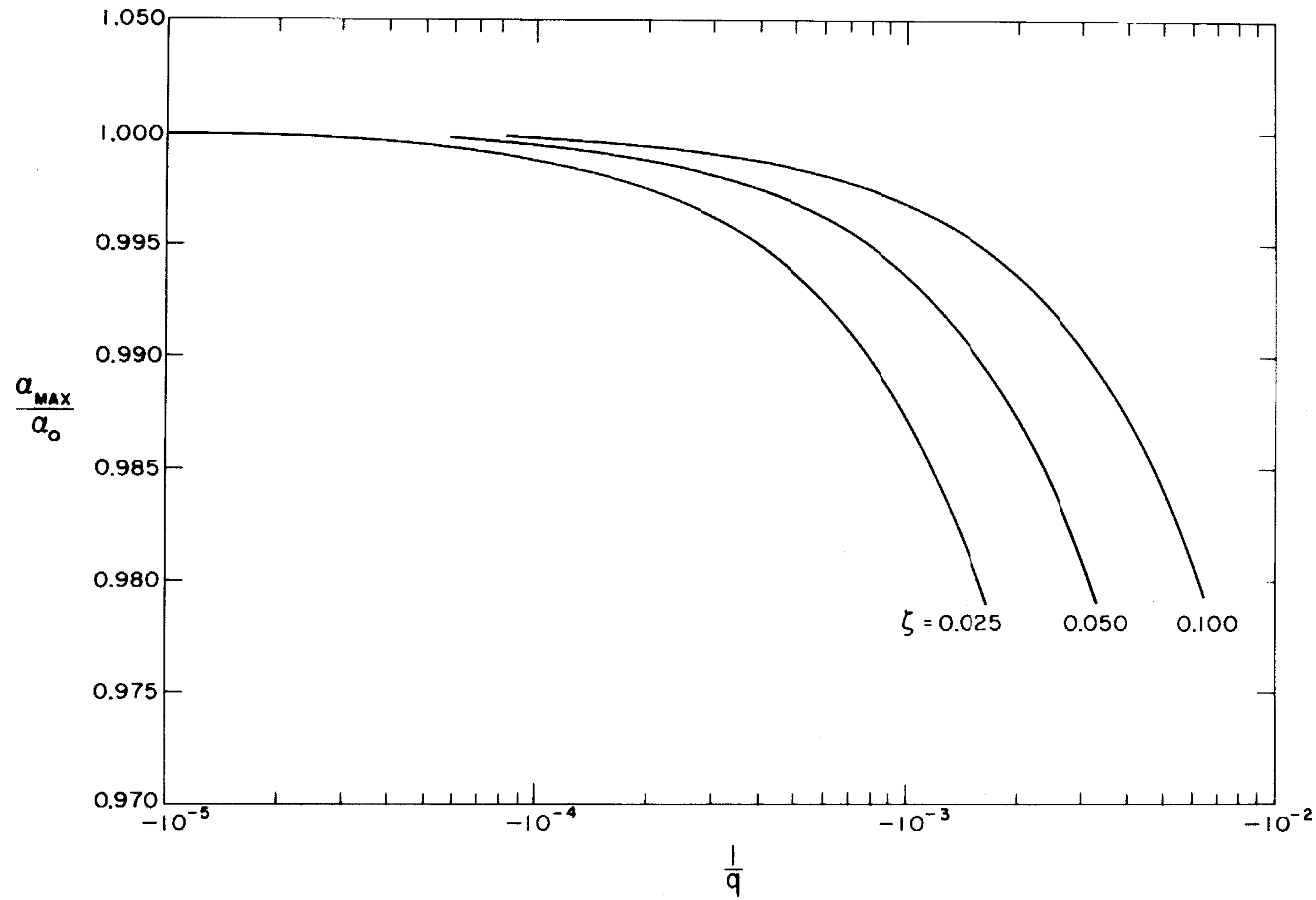
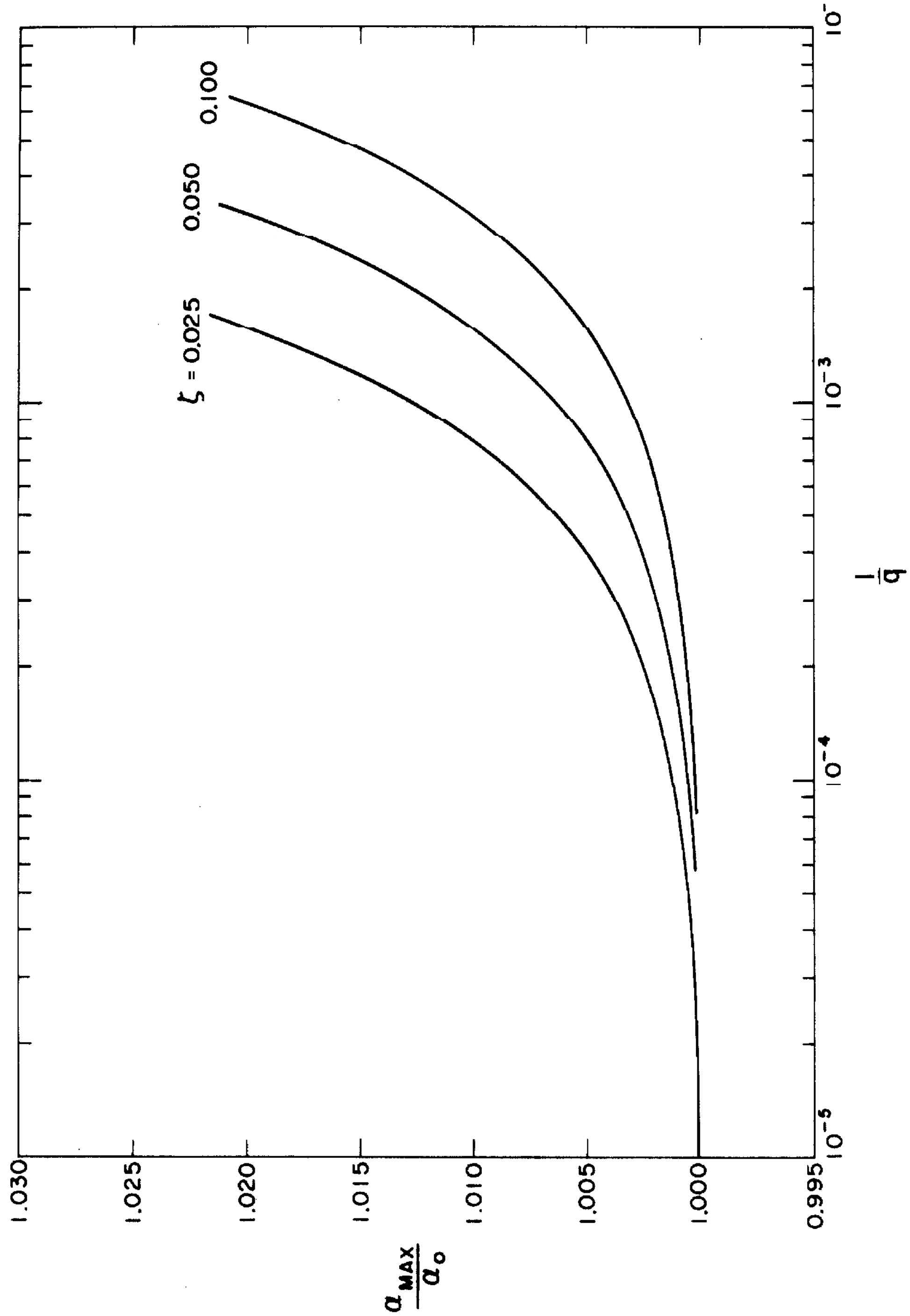

1.050
1.000
0.995
0.990
$\frac{a_{MAX}}{a_o}$
0.985
0.980
0.975
0.970
$\zeta = 0.025$
0.050
0.100
-10^{-5}
-10^{-4}
-10^{-3}
-10^{-2}
$\frac{1}{q}$

FIG. 3.5a



0.100
0.050
ζ = 0.025
1.030
1.025
1.020
1.015
1.010
1.005
1.000
0.995
10^-5
10^-4
10^-3
10^-2
1/q
a_MAX/a_o

FIG. 3.5b

Let the actual damping, ζ, take the form

$$\zeta = \zeta_e + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots \qquad . \tag{3.29}$$

An equation which may be solved for λ_1 and λ_2 is derived by substituting the right hand side of (3.29) for ζ in (3.23), reexpanding (3.23) in terms of ε, and replacing the left hand side of (3.23) by the left hand side of (3.28).

For the linearly sweeping case, the damping to second order will be

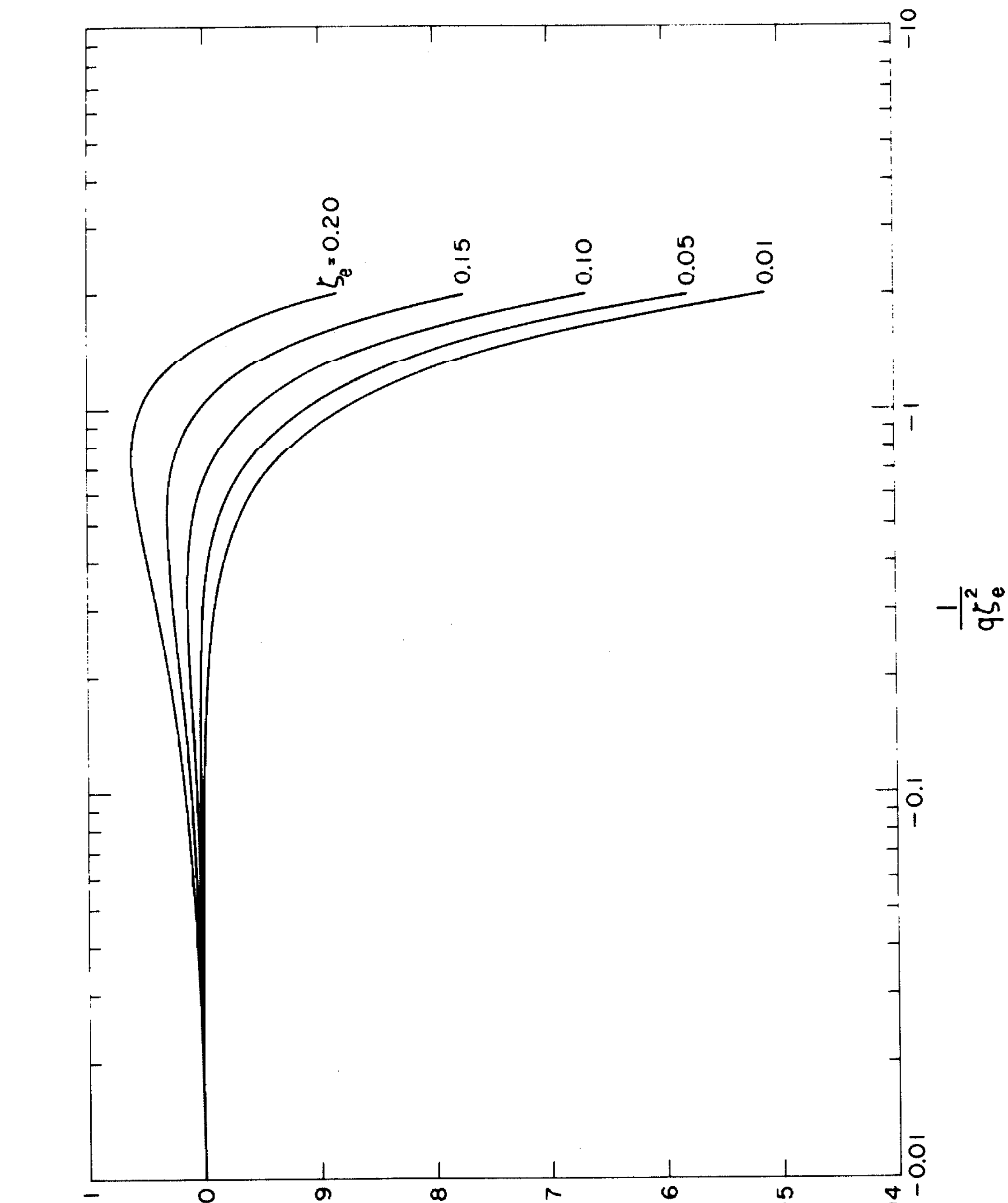
$$\zeta \cong \zeta_e - \frac{1}{4\pi q\alpha_{0e}} + \frac{(-4 + 28\zeta_e^2 - 63\zeta_e^4 + 47\zeta_e^6)}{32\pi^2 q^2 \alpha_{0e}^4 \zeta_e^3 m_e^2} , \tag{3.30}$$

in which the quantities carrying the subscript e are computed using the estimated damping.

For the case of exponentially sweeping excitation, the damping to second order will be

$$\zeta \cong \zeta_e - \frac{\gamma}{4\pi} + \gamma^2 \frac{(-4 + 28\zeta_e^2 - 64\zeta_e^4 + 48\zeta_e^6)}{32\pi^2 \alpha_{0e}^2 \zeta_e^3 m_e^2} \tag{3.31}$$

The above expressions are related to peak behavior, hence they should be useful for $\varepsilon/\zeta^2 << 1$. The ratio ζ/ζ_e given by (3.30) is plotted as a function of sweep rate for various values of ζ_e in Figs. 3.6a, b. Virtually identical curves will result for the exponential case, (3.31), if the abscissae in Figs. 3.6a, b are taken as $\dfrac{\alpha_{0e}\gamma}{\zeta_e^2}$.



$\frac{\zeta}{\zeta_e}$
1.01
1.00
0.99
0.98
0.97
0.96
0.95
0.94
$\zeta_e = 0.20$
0.15
0.10
0.05
0.01
−0.01
−0.1
−1
−10
$\frac{1}{q\zeta_e^2}$

FIG. 3.6a

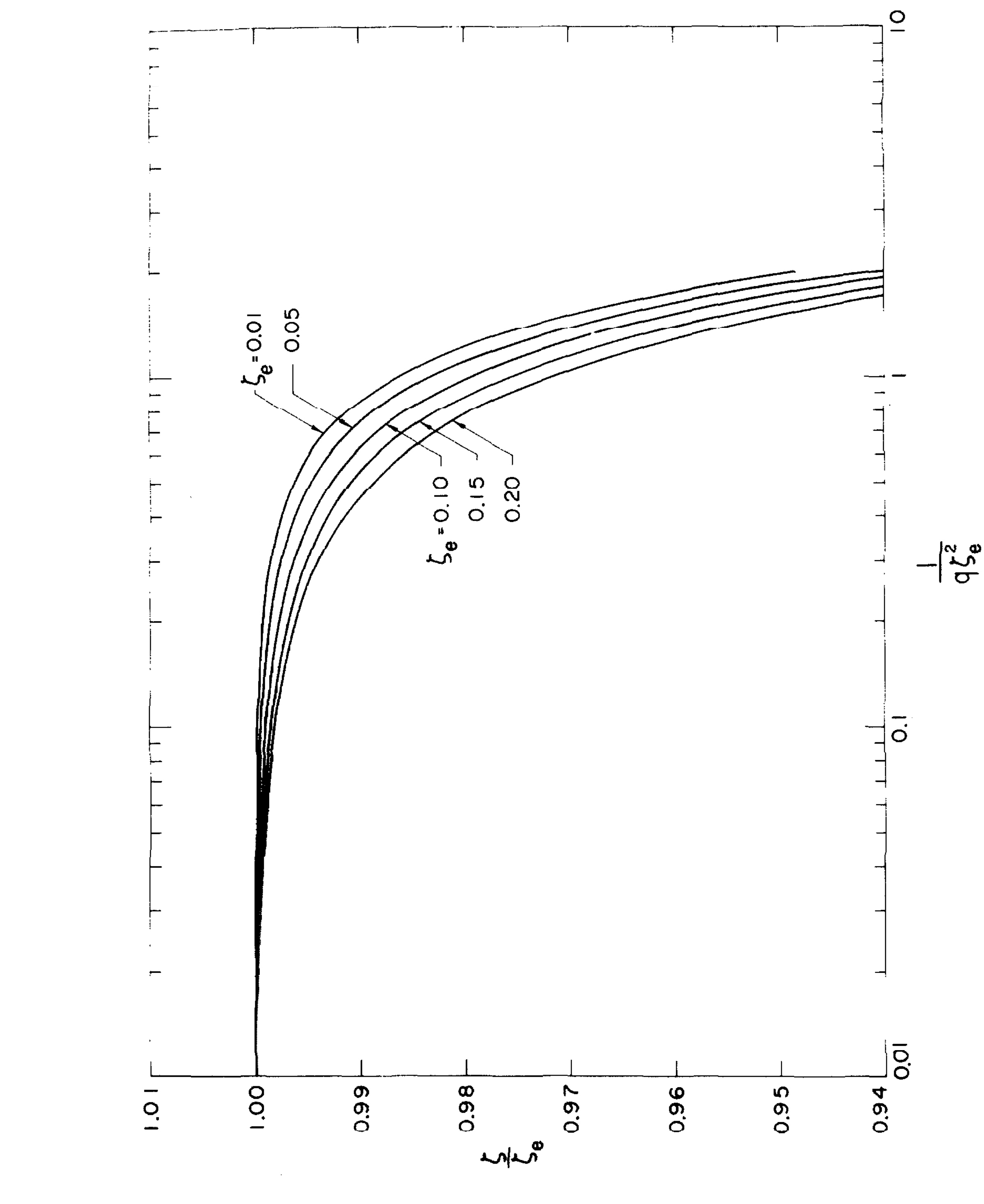

ζ/ζ_e
1.01
1.00
0.99
0.98
0.97
0.96
0.95
0.94
0.01
0.1
1
10
$\zeta_e = 0.01$
0.05
$\zeta_e = 0.10$
0.15
0.20
$\frac{1}{q\zeta_e^2}$

FIG. 3.6b

Damping is also determined in practice by measuring the width of the response peak at the half-power points, that is, the width between the points at which $y = \frac{y_{max}}{\sqrt{2}}$, since, for the steady-state response

$$W_e = 2\zeta_e + O\left[\zeta_e^3\right] .$$

It is again possible to derive a corrected expression for damping for the case of slowly sweeping excitation.

The response maximum to second order will be given by (3. 23). The equation for the half-power points of the perturbed curve may be written by equating (3. 23) divided by $\sqrt{2}$ to the perturbed expression for the response amplitude, (3. 21). This equation may be solved by assuming that the roots of interest take the form

$$\alpha_1' = \alpha_1 + \epsilon\nu_1 + \epsilon^2\nu_2 + \ldots \quad ,$$

$$\alpha_2' = \alpha_2 + \epsilon\mu_1 + \epsilon^2\mu_2 + \ldots \quad ,$$

where α_1 and α_2 correspond to the half-power points of the unperturbed curve.

The equation is easily solved for the first order correction to the measured width, $W_e = \alpha_2' - \alpha_1'$, in terms of the actual width, $W = \alpha_2 - \alpha_1$:

$$W_e = W + \epsilon\left\{\begin{array}{c}\dfrac{\alpha_0\kappa(\alpha_0)}{m}\left[\dfrac{1}{\alpha_1} - \dfrac{1}{\alpha_2}\right] + \dfrac{1}{\zeta}\left[\kappa(\alpha_2) - \kappa(\alpha_1)\right] \\ + \dfrac{1}{2m}\left[\kappa(\alpha_2) + \kappa(\alpha_1)\right]\end{array}\right\} ,$$

or in terms of the measured damping, ζ_e, and the actual damping, ζ:

$$\zeta_e = \zeta + \frac{\epsilon}{2}\left\{\begin{aligned} &\frac{\alpha_0 \kappa(\alpha_0)}{m}\left[\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right] + \frac{1}{\zeta}\left[\kappa(\alpha_2) - \kappa(\alpha_1)\right] \\ &+ \frac{1}{2m}\left[\kappa(\alpha_2) + \kappa(\alpha_1)\right] \end{aligned}\right\} + O\left[\zeta^3\right] .$$

It is clear that the quantity ϵ must be of $O\left[\zeta^2\right]$ multiplying quantities of $O[1]$, since the original expression for the damping is only good to $O\left[\zeta^3\right]$.

For the linear sweep:

$$\zeta \cong \zeta_e - \frac{1}{4\pi q} \quad , \tag{3.32a}$$

and for the exponential sweep:

$$\zeta \cong \zeta_e - \frac{3\nu}{4\pi} \quad . \tag{3.32b}$$

The corrector for the linear sweep in (3.32a) agrees closely with the first order corrector of (3.30). The exponential corrector in (3.32b), however, exceeds the first order correction of (3.31) by a factor of three.

Once the actual system damping has been determined, it will be possible to determine the system natural frequency from the measured frequency of the response peak. A first order expression for the natural frequency, ω, in terms of this measured frequency, ω_e, is developed from (3.22):

$$\omega = \frac{\omega_e}{\alpha_0}\left\{1 - \epsilon\left[\frac{\kappa(\alpha_0)\cdot(4-9\zeta^2)}{2\zeta\alpha_0^3} - \frac{\zeta}{2\alpha_0^2}\cdot\frac{d\kappa}{d\alpha}\bigg|_{\alpha_0}\right]\right\} .$$

The factor associated with varying excitation amplitude has been omitted for the sake of simplicity.

IV. EXACT SOLUTIONS

The response of a system influenced by linearly sweeping excitation has been, as noted in Chapter I, the prime subject of analysis. A major reason for the popularity of this particular case is that undamped response may be written in closed-form in terms of Fresnel integrals:

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2} t^2\right) dt \qquad ,$$

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2} t^2\right) dt \qquad .$$

Recent tabulations of error integrals of complex argument (8, 9, 10) also make possible the writing of undamped system response for linearly sweeping excitation in closed-form.

In equation (2. 4) let the excitation take the form

$$p(t) = \sin\left(m\tau - \frac{\tau^2}{4\pi q}\right) \quad , \qquad \tau \leq 2\pi q m \quad ,$$

$$p(t) = 0 \quad , \qquad \tau > 2\pi q m \quad .$$

Excitation frequency will be

$$\alpha(\tau) = m - \frac{\tau}{2\pi q} \quad , \qquad \tau \leq 2\pi q m \quad ,$$

$$\alpha(\tau) = 0 \quad , \qquad \tau \geq 2\pi q m \quad .$$

The frequency of excitation will thus sweep down from an infinite value at $\tau = -\infty$, through system resonance at $\tau = 0$, to zero at $\tau = 2\pi qm$.

System response for $\tau > 2\pi qm$ will be

$$y = \left[\frac{y'(2\pi qm) + \zeta y(2\pi qm)}{m} \sin m(\tau - 2\pi qm) \right.$$
$$\left. + \, y(2\pi qm) \cos m(\tau - 2\pi qm) \right] e^{-\zeta(\tau - 2\pi qm)} \quad .$$

For $\tau \leq 2\pi qm$, the steady-state response may be written directly

$$y = \frac{1}{m} \int_0^\infty e^{-\zeta u} \sin mu \sin\left[m(\tau - u) - \frac{(\tau - u)^2}{4\pi q} \right] du \quad . \tag{4.1}$$

Equation (4.1) may be rewritten

$$y = \frac{1}{2m} \int_0^\infty e^{-\zeta u} \cos\left[mu - m(\tau - u) + \frac{(\tau - u)^2}{4\pi q} \right] du$$
$$- \frac{1}{2m} \int_0^\infty e^{-\zeta u} \cos\left[mu + m(\tau - u) - \frac{(\tau - u)^2}{4\pi q} \right] du \quad ,$$

or in complex notation:

$$\begin{aligned} y = {} & \frac{\mathrm{Re}}{2m} \int_0^\infty \exp\left\{ -\zeta u + j\left[mu - m(\tau - u) + \frac{(\tau - u)^2}{4\pi q} \right]\right\} du \\ & - \frac{\mathrm{Re}}{2m} \int_0^\infty \exp\left\{ -\zeta u + j\left[mu + m(\tau - u) - \frac{(\tau - u)^2}{4\pi q} \right]\right\} du \quad . \end{aligned} \tag{4.2}$$

Completing the square on the exponents in (4.2), and making a suitable change of variables, leads to the result

$$y = \frac{\pi\sqrt{q}}{2m} \exp\left[-\zeta(\tau - 4\pi q m)\right] \cdot \operatorname{Re}\left[\!\left[\exp j(m\tau + \pi q \zeta^2 - 4\pi q m^2 + \frac{\pi}{4}) \cdot \right.\right.$$

$$\left.\left. \left\{ 1 - \operatorname{erf}\left[\frac{-\tau e^{-j\frac{\pi}{4}}}{2\sqrt{\pi q}} + \sqrt{\pi q}(2m + j\zeta) e^{-j\frac{\pi}{4}} \right] \right\} \right]\!\right]$$

$$- \frac{\pi\sqrt{q}}{2m} \exp\left[-\zeta\tau\right] \cdot \operatorname{Re}\left[\!\left[\exp j(m\tau - \pi q \zeta^2 - \frac{\pi}{4}) \cdot \right.\right.$$

$$\left.\left. \left\{ 1 - \operatorname{erf}\left[\frac{-\tau e^{-j\frac{\pi}{4}}}{2\sqrt{\pi q}} - j\sqrt{\pi q}\, \zeta\, e^{j\frac{\pi}{4}} \right] \right\} \right]\!\right] .$$

When the frequency of excitation equals zero at some finite time in the past and sweeps linearly <u>up</u> through resonance at $\tau = 0$ to an infinite value in the future, the instantaneous frequency of excitation may be written

$$\alpha(\tau) = m + \frac{\tau}{2\pi q} \quad , \quad \tau \geq -2\pi q m \quad ,$$

$$\alpha(\tau) = 0 \quad , \quad \tau < -2\pi q m \quad .$$

The excitation in this case will be

$$p(\tau) = \sin(m\tau + \frac{\tau^2}{4\pi q} + \pi q m^2) \quad , \quad \tau \geq -2\pi q m \quad ,$$

$$p(\tau) = 0 \quad , \quad \tau < -2\pi q m \quad .$$

Prior to the application of the excitation the system will be at the state of rest:

$$y(\tau) = y'(\tau) = 0 \quad , \quad \tau \leq -2\pi q m \quad .$$

For $\tau > -2\pi q m$, the response may be written

$$y = \frac{1}{m}\int_0^{(\tau+2\pi qm)} e^{-\zeta u}\sin mu \cdot \sin\left[m(\tau-u)+\frac{(\tau-u)^2}{4\pi q}+\pi qm^2\right]du \ ,$$

which may be rewritten

$$y = \frac{1}{2m}\int_0^{(\tau+2\pi qm)} e^{-\zeta u}\cos\left[mu - m(\tau-u) - \frac{(\tau-u)^2}{4\pi q} - \pi qm^2\right]du$$

$$- \frac{1}{2m}\int_0^{(\tau+2\pi qm)} e^{-\zeta u}\cos\left[mu + m(\tau-u) + \frac{(\tau-u)^2}{4\pi q} + \pi qm^2\right]du \ ,$$

or in complex notation:

$$\begin{aligned} y = \frac{\mathrm{Re}}{2m}\int_0^{(\tau+2\pi qm)} &\exp\left\{-\zeta u + j\left[mu - m(\tau-u) - \frac{(\tau-u)^2}{4\pi q} - \pi qm^2\right]\right\} du \\ - \frac{\mathrm{Re}}{2m}\int_0^{(\tau+2\pi qm)} &\exp\left\{-\zeta u + j\left[mu + m(\tau-u) + \frac{(\tau-u)^2}{4\pi q} + \pi qm^2\right]\right\} du \ . \end{aligned} \qquad (4.4)$$

Completing the square on the exponents, and making a suitable change of variables in (4.4) leads to the result

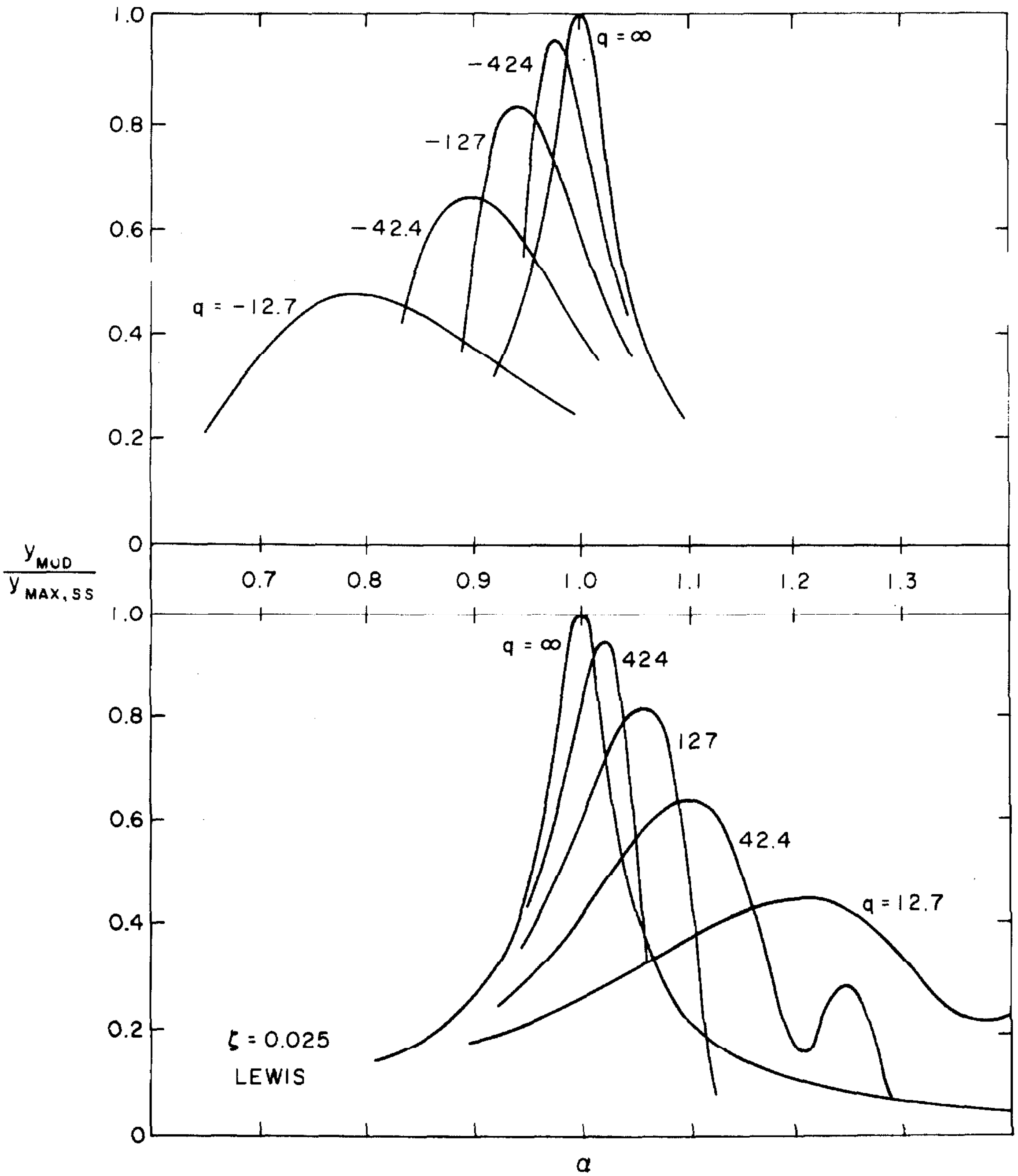
$$y = \frac{\pi\sqrt{q}}{2m} \exp\left[-\zeta(\tau + 4\pi q m)\right] \cdot \mathrm{Re}\left[\!\!\left[\exp j(m\tau + 3\pi q m^2 - \pi q \zeta^2 - \frac{\pi}{4}) \cdot \left\{\mathrm{erf}\left[-\sqrt{\pi q}(m+j\zeta)\, e^{j\frac{\pi}{4}}\right] - \mathrm{erf}\left[-\frac{\tau e^{j\frac{\pi}{4}}}{2\sqrt{\pi q}} - \sqrt{\pi q}\,(2m+j\zeta)\, e^{j\frac{\pi}{4}}\right]\right\}\right]\!\!\right]$$
$$- \frac{\pi\sqrt{q}}{2m} \exp(-\zeta\tau) \cdot \mathrm{Re}\left[\!\!\left[\exp j(m\tau + \pi q m^2 + \pi q \zeta^2 + \frac{\pi}{4}) \cdot \left\{\mathrm{erf}\left[\sqrt{\pi q}(m+j\zeta) e^{-j\frac{\pi}{4}}\right] - \mathrm{erf}\left[\left(-\frac{\tau}{2\sqrt{\pi q}} + j\sqrt{\pi q}\,\zeta\right) e^{-j\frac{\pi}{4}}\right]\right\}\right]\!\!\right] \quad (4.5)$$

Illustrated in Figs. 4.1a, b are examples of graphically computed solutions obtained by Lewis[1] for the response amplitude of a damped system influenced by excitations sweeping linearly up through system resonance and down through system resonance.

When the damping equals zero, equation (4.3) becomes

$$y = \pi\sqrt{q/2}\left\{\left[\frac{1}{2} - C\left(2\sqrt{2q} - \frac{\tau}{\pi\sqrt{2q}}\right)\right]\cos(\tau - 4\pi q) - \left[\frac{1}{2} - S\left(2\sqrt{2q} - \frac{\tau}{\pi\sqrt{2q}}\right)\right]\sin(\tau - 4\pi q)\right\}$$
$$-\pi\sqrt{q/2}\left\{\left[\frac{1}{2} + C\left(\frac{\tau}{\pi\sqrt{2q}}\right)\right]\cos\tau + \left[\frac{1}{2} + S\left(\frac{\tau}{\pi\sqrt{2q}}\right)\right]\sin\tau\right\} .$$

Under the same circumstances, equation (4.5) becomes



1.0
0.8
0.6
0.4
0.2
0
q = ∞
−424
−127
−42.4
q = −12.7
0.7
0.8
0.9
1.0
1.1
1.2
1.3
$\frac{y_{MOD}}{y_{MAX,SS}}$
424
127
42.4
q = 12.7
ζ = 0.025
LEWIS
α

FIG. 4.1a

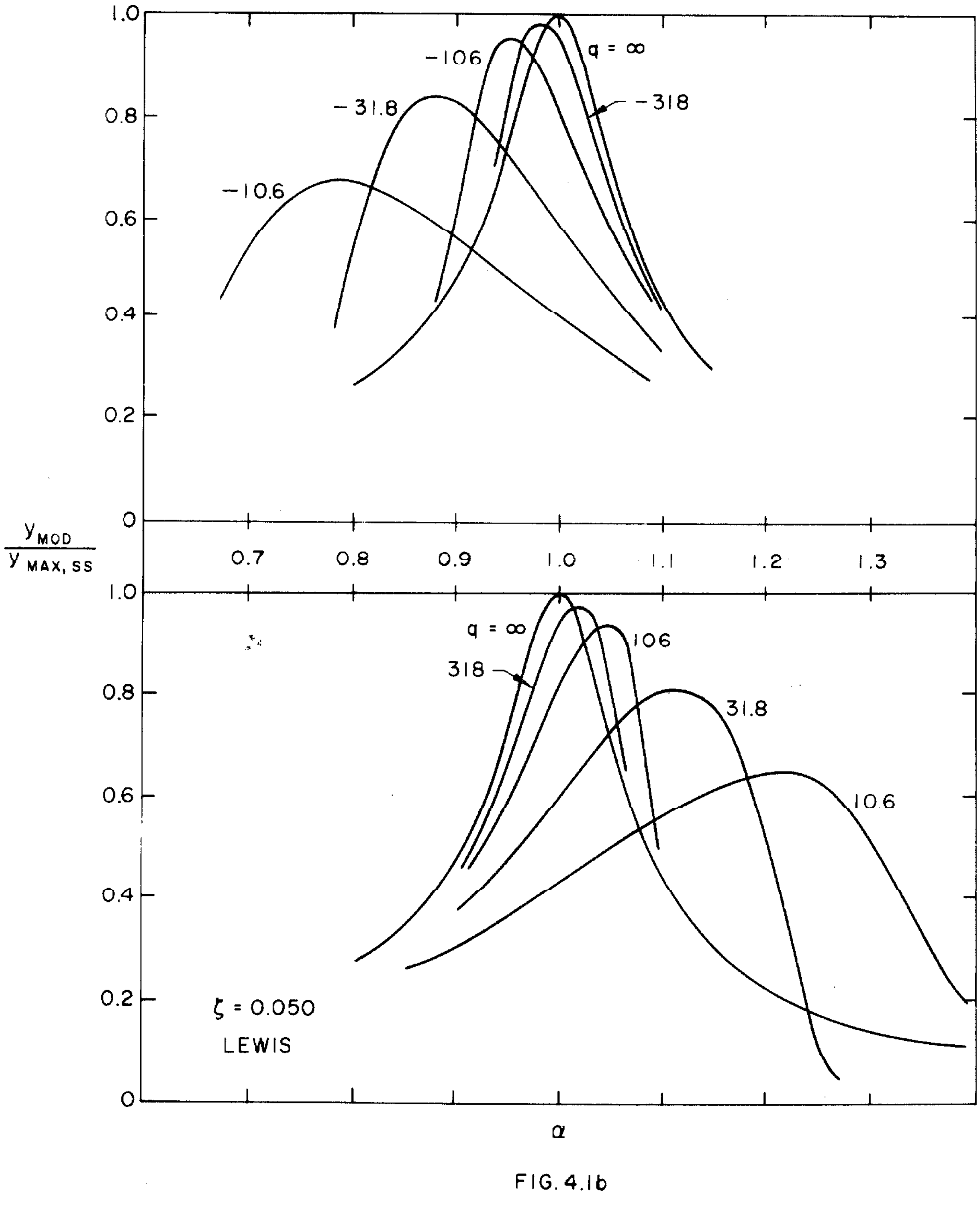

1.0
0.8
0.6
0.4
0.2
0
q = ∞
-106
-318
-31.8
-10.6
$\frac{y_{MOD}}{y_{MAX, SS}}$
0.7
0.8
0.9
1.0
1.1
1.2
1.3
q = ∞
318
106
31.8
10.6
ζ = 0.050
LEWIS
α

FIG. 4.1b

$$y = \pi\sqrt{q/2}\left\{\left[C\left(\frac{\tau}{\pi\sqrt{2q}} + 2\sqrt{2q}\right) - C(\sqrt{2q})\right]\cos(\tau+3\pi q) + \left[S\left(\frac{\tau}{\pi\sqrt{2q}} + 2\sqrt{2q}\right) - S(\sqrt{2q})\right]\sin(\tau+3\pi q)\right\}$$

$$- \pi\sqrt{q/2}\left\{\left[C\left(\frac{\tau}{\pi\sqrt{2q}}\right) + C(\sqrt{2q})\right]\cos(\tau+\pi q) - \left[S\left(\frac{\tau}{\pi\sqrt{2q}}\right) + S(\sqrt{2q})\right]\sin(\tau+\pi q)\right\} .$$

When a viscous damped single degree of freedom system is excited by a rotating mass whose angle of rotation obeys the law

$$\Theta(t) = \omega m t - \pi h t^2 , \qquad t \le \frac{\omega m}{\pi h} ,$$

$$\Theta = 0 , \qquad t > \frac{\omega m}{\pi h} ,$$

an expression for the response may be developed in terms of error integrals of complex argument. This problem is of physical interest in that its solution may be used to predict the behavior of structures influenced by decelerating rotating mass exciters. The quality of such a prediction will depend upon the separation of natural frequencies of the structure, and the actual frequency vs. time relation for the decelerating rotating mass exciter, which is determined not only by the physical parameters of the exciter, but in addition, by the influence of the structure upon the exciter. Nielsen(11) determined the frequency-time relation for such a mass exciter and found that it could be approximated roughly by a decaying exponential function. Since the decay constant was rather large, it was possible to further approximate the frequency-time relation in the region of interest, i.e., the region of system resonance, by a linear relation. Nielsen then used

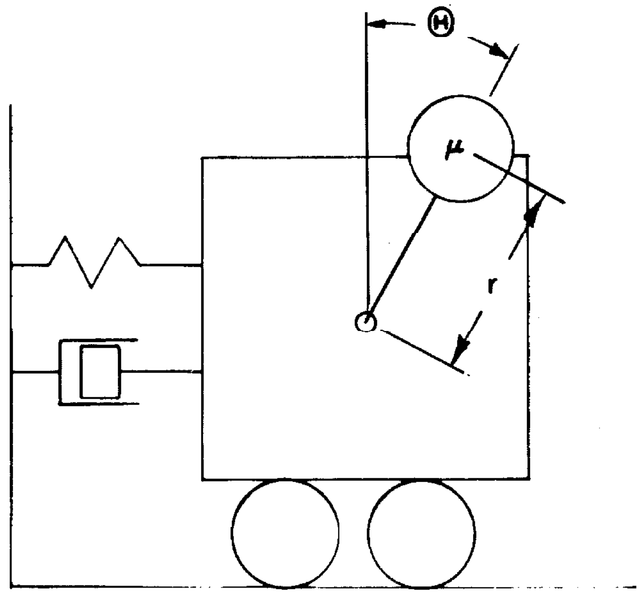


FIG. 4.2 a TOP VIEW

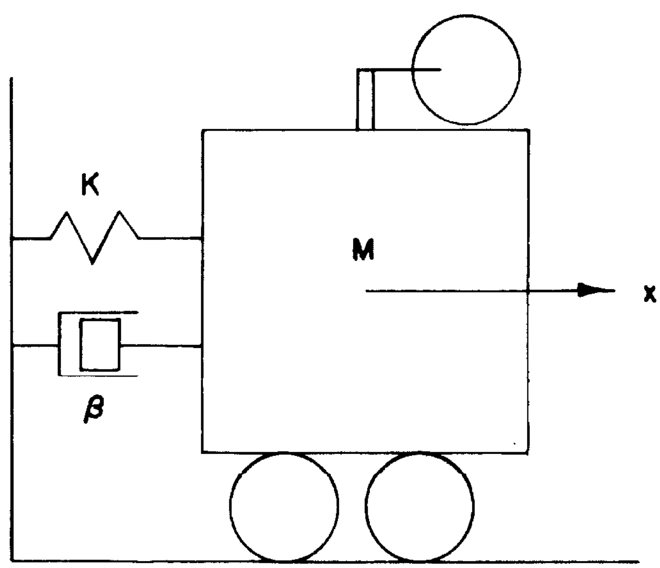


FIG. 4.2 b SIDE VIEW

the results of Lewis[1] to predict the experimentally measured response with fair accuracy. Since Lewis' results were derived for the case of constant amplitude excitation, the solution for the response of a single degree of freedom system under the influence of a linearly sweeping excitation with amplitude proportional to frequency squared should provide a better estimate for the response of real structures.

The equation of motion for the system described in Figs. 4.2a, b is

$$(M+\mu)\ddot{x}+\beta\dot{x}+Kx = -\mu r \frac{d}{dt}(\dot{\Theta}\cos\Theta) \quad .$$

The equation may be rewritten in terms of the dimensionless time, $\tau = \omega t$, and the new dependent variable $z = -\dfrac{(M+\mu)x}{\mu r}$, after noting that $\dfrac{d}{dt}(\dot{\Theta}\cos\Theta)$ is equal to $\dfrac{d^2}{dt^2}(\sin\Theta)$:

$$z'' + 2\zeta z' + z = \frac{d^2}{d\tau^2}\sin\theta(\tau) \quad , \tag{4.6}$$

where

$$\theta(\tau) = m\tau - \frac{\tau^2}{4\pi q} \quad .$$

The solution to (4.6) will be

$$z = \frac{1}{m}\int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')}\sin m(\tau-\tau') \cdot \frac{d^2}{d\tau'^2}\sin\theta(\tau')\,d\tau' \quad .$$

Two integrations by parts lead to the result

$$z = \sin\theta(\tau) - \frac{\alpha_0^2}{m}\int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')}\sin m(\tau-\tau')\sin\theta(\tau')d\tau'$$

$$-2\zeta\int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')}\cos m(\tau-\tau')\sin\theta(\tau')d\tau' \quad .$$

The first integral in (4.7) has already been solved and is represented by $-\alpha_0^2$ multiplying expression (4.3). The second integral may be solved using the procedure previously employed, and the response factor z may be written

$$z = \sin(m\tau - \frac{\tau^2}{4\pi q})$$

$$- \frac{\alpha_0^2\pi\sqrt{q}}{2m}\exp[-\zeta(\tau-4\pi q)] \cdot$$

$$\mathrm{Re}\left[\left[\exp j\left[m\tau+\pi q(\zeta^2-4m^2)+\frac{\pi}{4}\right]\cdot\left\{1-\mathrm{erf}\left[\left(-\frac{\tau}{2\sqrt{\pi q}}+2m\sqrt{\pi q}+j\zeta\sqrt{\pi q}\right)e^{-j\frac{\pi}{4}}\right]\right\}\right]\right]$$

$$+ \frac{\alpha_0^2\pi\sqrt{q}}{2m}\exp[-\zeta\tau]\cdot$$

$$\mathrm{Re}\left[\left[\exp j\left[m\tau-\pi q\zeta^2-\frac{\pi}{4}\right]\cdot\left\{1-\mathrm{erf}\left[\left(-\frac{\tau}{2\sqrt{\pi q}}-j\zeta\sqrt{\pi q}\right)e^{j\frac{\pi}{4}}\right]\right\}\right]\right]$$

$$- \zeta\pi\sqrt{q}\exp[-\zeta\tau]\cdot$$

$$\mathrm{Im}\left[\left[\exp j\left[m\tau-\pi q\zeta^2-\frac{\pi}{4}\right]\cdot\left\{1-\mathrm{erf}\left[\left(-\frac{\tau}{2\sqrt{\pi q}}-j\zeta\sqrt{\pi q}\right)e^{j\frac{\pi}{4}}\right]\right\}\right]\right]$$

$$+ \zeta\pi\sqrt{q}\exp\left[-\zeta(\tau-4\pi qm)\right]\cdot$$

$$\mathrm{Im}\left[\left[\exp j\left[m\tau+\pi q(\zeta^2-4m^2)+\frac{\pi}{4}\right]\cdot\left\{1-\mathrm{erf}\left[\left(-\frac{\tau}{2\sqrt{\pi q}}+2m\sqrt{\pi q}+j\zeta\sqrt{\pi q}\right)e^{-j\frac{\pi}{4}}\right]\right\}\right]\right] .$$

It should be noted in the above expression that, if q is large and ζ small, the result will be dominated by the terms substituted from expression (4.3), that is, the solution for the response factor z in this case will be much like the solution for the response factor y in the case of constant amplitude excitation. Recall, however, that

$$y \sim M\omega^2 x \quad ,$$

and

$$z \sim Mx \quad .$$

In contrast to the linear sweep treated previously, the exponential sweep:

$$p(\tau) = \sin\left(\frac{m}{B}\, e^{\pm B\tau}\right) \quad ,$$
$$|\alpha(\tau)| = m e^{\pm B\tau} \quad , \tag{4.9}$$

has been the subject of very little analytical work. Parker[12] and Hawkes[13] developed digital computer programs to handle the problem. Parker's method consisted of expanding the Duhamel integral representation for the response in terms of several infinite series, and integrating these term by term. Hawkes formed two second order simultaneous differential equations for the real, and the imaginary parts of the complex modulus expression for the response and solved these.

A potentially useful power series solution to equation (2.4) may be developed for the case of exponentially sweeping excitation, (4.9).

Consider

$$y'' + 2\zeta y' + y = \sin\left(\frac{m}{B}\, e^{B\tau}\right) \quad . \tag{4.10}$$

Define the new independent variable

$$g = \frac{m}{B} e^{B\tau} \quad .$$

Then

$$\frac{d}{d\tau} = \frac{dg}{d\tau} \cdot \frac{d}{dg} = Bg \frac{d}{dg} \quad ,$$

and

$$\frac{d^2}{d\tau^2} = \frac{dg}{d\tau} \cdot \frac{d}{dg} \left(\frac{dg}{d\tau} \cdot \frac{d}{dg} \right) = B^2 g^2 \frac{d^2}{dg^2} + B^2 g \frac{d}{dg} \quad .$$

Equation (4. 10) may be written in terms of the new independent variable:

$$B^2 g^2 \frac{d^2 y}{dg^2} + \left(B^2 g + 2\zeta Bg\right) \frac{dy}{dg} + y = \sin(g) \quad . \tag{4. 11}$$

Assume a solution

$$y = \sum_{n=0}^{\infty} a_{2n+1}\, g^{2n+1} \quad , \tag{4. 12}$$

and replace sin(g) by its series representation

$$\sin(g) = \sum_{n=0}^{\infty} \frac{(-)^n g^{2n+1}}{(2n+1)!} \quad .$$

Substitution of (4. 12) into (4. 11) leads to an expression for a_{2n+1}:

$$a_{2n+1} = \frac{(-)^n \cdot 1}{(2n+1)! \left[B^2(2n+1)(2n) + (B^2+2\zeta B)(2n+1) + 1 \right]} \quad ,$$

and the solution to (4. 11) may be written

$$y = \sum_{n=0}^{\infty} \frac{(-)^n g^{2n+1}}{(2n+1)!\left[B^2(2n+1)(2n) + (B^2+2\zeta B)(2n+1) + 1\right]} , \tag{4.13a}$$

or

$$y = \sum_{n=0}^{\infty} \frac{(-)^n \left(\frac{m}{B} e^{B\tau}\right)^{2n+1}}{(2n+1)!\left[B^2(2n+1)(2n) + (B^2+2\zeta B)(2n+1) + 1\right]} . \tag{4.13b}$$

The factorial in the denominator of (4.13a, b) ensures absolute convergence of the series for finite values of the numerator. The series expression is, however, an expression for the complete response. A more useful expression would be one in which the oscillatory character of the solution could be factored out, leaving information concerning the response envelope.

It is noted that the Laplace transform for

$$\sin(ae^{-t}) ,$$

is equal to

$$a^{-p}\Gamma(p)\left[U_p(2a, 0) \sin a - U_{p+1}(2a, 0) \cos a\right], \quad \mathrm{Re}\ p > 0$$

The function $U_\nu(\omega, z)$ is the Lommel function of order ν. It is defined[14] as

$$U_\nu(\omega, z) = \sum_{n=0}^{\infty} (-)^n \left(\frac{\omega}{z}\right)^{\nu+2n} J_{\nu+2n}(z) . \tag{4.14}$$

In the special case, $z = 0$, (4.14) reduces to

$$U_\nu(\omega, 0) = \sum_{n=0}^{\infty} \frac{(-)^n (\omega/2)^{\nu+2n}}{\Gamma(\nu + 2n+1)}$$

Thus the Laplace transform of sin (ae^{-t}) in terms of the transform variable p is

$$a^{-p}\Gamma(p)\sum_{n=0}^{\infty}\frac{(-)^n a^{p+2n}}{\Gamma(p+2n+1)}\sin(a) - a^{-p}\Gamma(p)\sum_{n=0}^{\infty}\frac{(-)^n a^{p+2n+1}}{\Gamma(p+2n+2)}\cos(a)\,. \tag{4.15}$$

The Duhamel integral form for the response of a system subject to excitation (4. 9) may be written

$$y = \frac{1}{m}\,\mathrm{Im}\int_0^{\infty} e^{-(\zeta-jm)u'}\cdot \sin\left\{\frac{m}{B}e^{B\tau}e^{-Bu'}\right\}du' \;, \tag{4.16}$$

but (4. 16) has as a solution (4. 15) when $p = \frac{\zeta-jm}{B}$, and $a = \frac{m}{B}e^{B\tau}$. Then

$$\begin{aligned} y = {} & \frac{1}{m}\,\mathrm{Im}\sum_{n=0}^{\infty}\frac{(-)^n(me^{B\tau})^{2n}\sin\left(\frac{m}{B}e^{B\tau}\right)}{(\zeta-jm)(\zeta+B-jm)\ldots(\zeta+2nB-jm)} \\ & - \frac{1}{m}\,\mathrm{Im}\sum_{n=0}^{\infty}\frac{(-)^n(me^{B\tau})^{2n+1}\cos\left(\frac{m}{B}e^{B\tau}\right)}{(\zeta-jm)(\zeta+B-jm)\ldots(\zeta+(2n+1)B-jm)}\,. \end{aligned} \tag{4.17}$$

The oscillating character of the response may be factored out in the expression (4. 17) and an envelope expression may be written

$$\begin{aligned} y_{mod} = \frac{1}{m}\Bigg\{ & \left[\mathrm{Im}\sum_0 \frac{(-)^n(me^{B\tau})^{2n}}{(\zeta-jm)(\zeta+B-jm)\ldots(\zeta+2nB-jm)}\right]^2 \\ & + \left[\mathrm{Im}\sum_0 \frac{(-)^n(me^{B\tau})^{2n+1}}{(\zeta-jm)(\zeta+B-jm)\ldots(\zeta+(2n+1)B-jm)}\right]^2\Bigg\}^{1/2} \end{aligned} \tag{4.18}$$

Comments concerning the boundedness of the undamped response in the case where the excitation frequency is cycled an infinite number of times through system resonance lead one to inquire into the possibility of obtaining an exact solution for such a case to permit a more thorough examination of the response.

Consider the excitation

$$p(\tau) = \sin\left(\nu\tau + \frac{\nu}{C}\cos C\tau\right) \quad .$$

The frequency of excitation is

$$\alpha(\tau) = \nu(1 - \sin C\tau) \quad .$$

System resonance will be included in the range of excitation frequency if $2\nu \geq 1$.

The response

$$y(\tau) = \int_{-\infty}^{\tau} \sin(\tau-\tau') \cdot \sin\left(\nu\tau' + \frac{\nu}{C}\cos C\,\tau'\right)d\tau' \quad ,$$

may be rewritten in the form

$$\begin{aligned} y(\tau) = \sin\tau \cdot &\int_{-\infty}^{\tau} \cos\tau' \sin\left(\nu\tau' + \frac{\nu}{C}\cos C\,\tau'\right)d\tau' \\ -\cos\tau \cdot &\int_{-\infty}^{\tau} \sin\tau' \sin\left(\nu\tau' + \frac{\nu}{C}\cos C\,\tau'\right)d\tau' \quad . \end{aligned} \tag{4. 19}$$

The integrals of (4. 19) may be decomposed further by trigonometric substitution, and the trigonometric functions of trigonometric argument may be replaced by their well-known Fourier series representations:

$$\cos(A\cos B) = J_o(A) + 2\sum_{n=1}^{\infty} (-)^n J_{2n}(A)\cos 2nB \quad ,$$

and

$$\sin(A\cos B) = 2\sum_{n=0}^{\infty} (-)^n J_{2n+1}(A)\cos(2n+1)B \quad .$$

A term-by-term integration of the series expressions is possible, and the final result is

$$y = J_o\left(\frac{\nu}{C}\right)\frac{\sin\nu\tau}{1-\nu^2}$$

$$+\sum_{n=1}^{\infty}(-)^n J_{2n}\left(\frac{\nu}{C}\right)\left\{\frac{\sin(\nu+2nC)\tau}{1-(\nu+2nC)^2}+\frac{\sin(\nu-2nC)\tau}{1-(\nu-2nC)^2}\right\} \tag{4.20}$$

$$+\sum_{n=0}^{\infty}(-)^n J_{2n+1}\left(\frac{\nu}{C}\right)\left\{\frac{\cos[\nu+(2n+1)C]\tau}{1-[\nu+(2n+1)C]^2}+\frac{\cos[\nu-(2n+1)C]\tau}{1-[\nu-(2n+1)C]^2}\right\}$$

It is seen that (4.20) diverges if one of an infinite number of possible conditions exists. Divergence occurs if the center frequency of the sweep, ν, equals the resonant frequency of the system, or if ν and C, the sweep parameters, interact in such a way as to cause the vanishing of any denominator in the series.

This discussion concerning exact solutions indicates that there are many particular excitation frequency-time relations for which the exact solution to the problem at hand may be presented in some potentially useful form or other.

The quadratic sweep

$$\alpha(\tau) = b\tau^2 + c\tau + d$$

produces a response containing Airy integrals (for particular ratios of the coefficients), and even more complicated polynomial sweeps may be treated using the more general Airy-Hardy integrals[15].

A good table of Laplace transforms suggests further candidates for analysis.

V. ANALOG STUDY OF SYSTEM RESPONSE TO LINEARLY AND EXPONENTIALLY SWEEPING EXCITATIONS

The approximate analytical method of Chapter III may be utilized to provide a fairly complete description of system response for the general slowly sweeping excitation. For faster sweep rates, these methods fail to give useful results. The techniques outlined in Chapter II may still be used, but more detailed knowledge requires analysis by digital or analog means. In the present problem, the choice of either digital or analog computation may be justified for a variety of reasons. The latter was chosen for the current investigation primarily because of the availability of a good analog computer, and quality recording gear.

The choice of analog computation presented the immediate problem of generating the required sweeping excitations. To be sure, the laboratory had available a Brüel and Kjaer Automatic Vibration Exciter Control Type 1018 capable of providing excitations with exponentially varying frequency, but the need for a wider range of sweep rate, and for a selection of frequency-time relationships necessitated the design and construction of the electronic function generator detailed in Appendix A.

In the experiment, the function generator was programmed to excite an analog model of a viscous damped single degree of freedom system with forcing voltages having frequency sweeping both linearly and exponentially in time.

The time histories of excitation frequency and system response were recorded on an oscillograph for various sweep rates and damping ratios. A typical response history for the linearly sweeping case is given in Fig. 5. 1a; Fig. 5. 1b illustrates a similar record for the exponentially sweeping case. Records of this type were analyzed to give the results presented below.

The experimental set-up (Fig. 5. 2) was designed to permit determination of significant parameters--system damping, system natural frequency, and the excitation frequency sweep rate--to within plus or minus one percent.

The error in measurement of response characteristics depended on the excitation frequency sweep rate, for envelope behavior became more difficult to determine at faster sweep rates, since fewer response maxima defined the envelope in the range where significant changes took place. Comparison to published results indicated that the ratio of the maximum response amplitude under sweeping conditions to the steady-state response maximum, for instance, was determined to within plus or minus three percent. For all but the fastest sweeps, the same accuracy was assumed to apply to the measurement of the other response characteristics.

The laboratory model represents a finite time operating system, and the assumption that the present experimental results are generally applicable, i. e., that they closely approximate the response

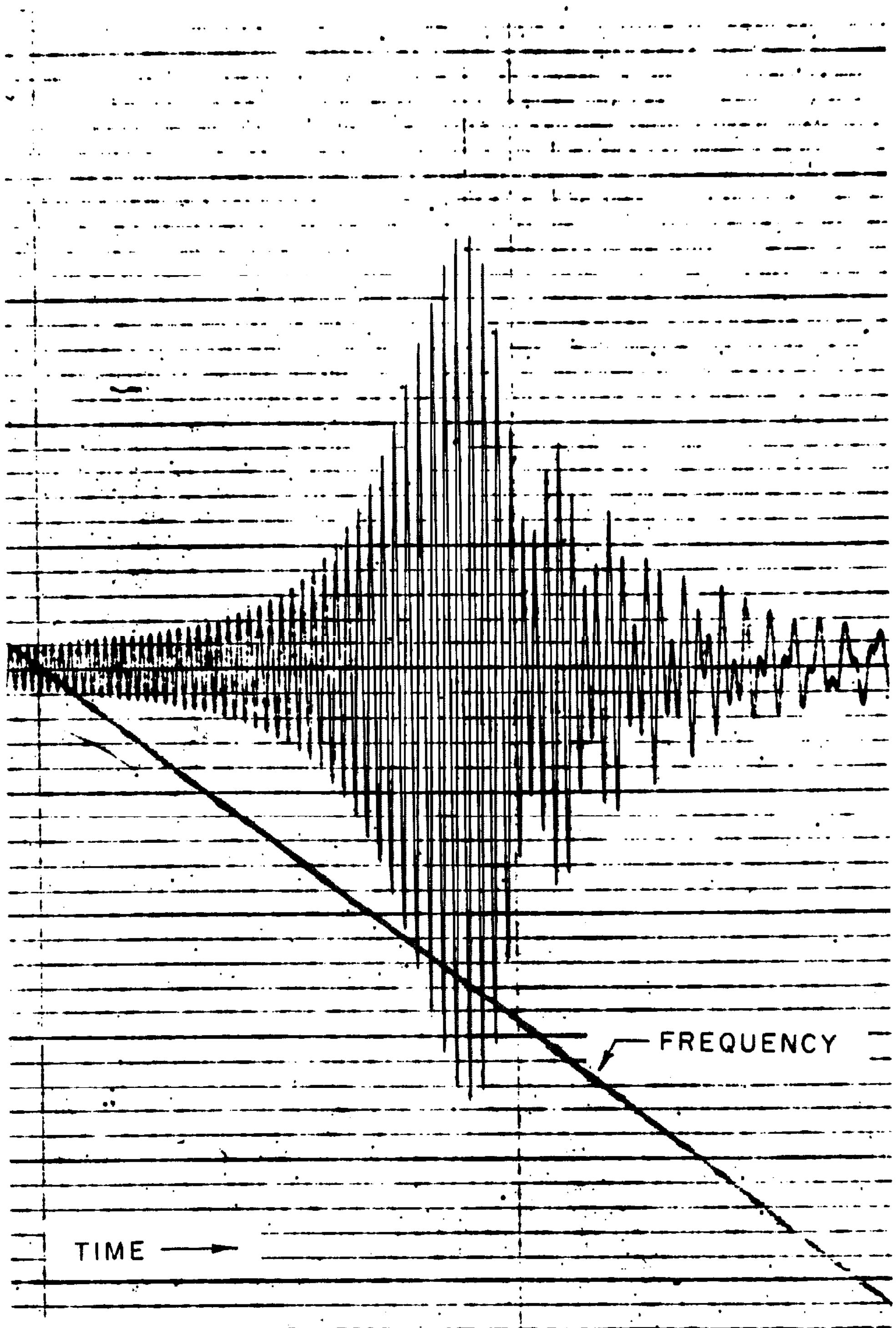


FIG. 5.1a

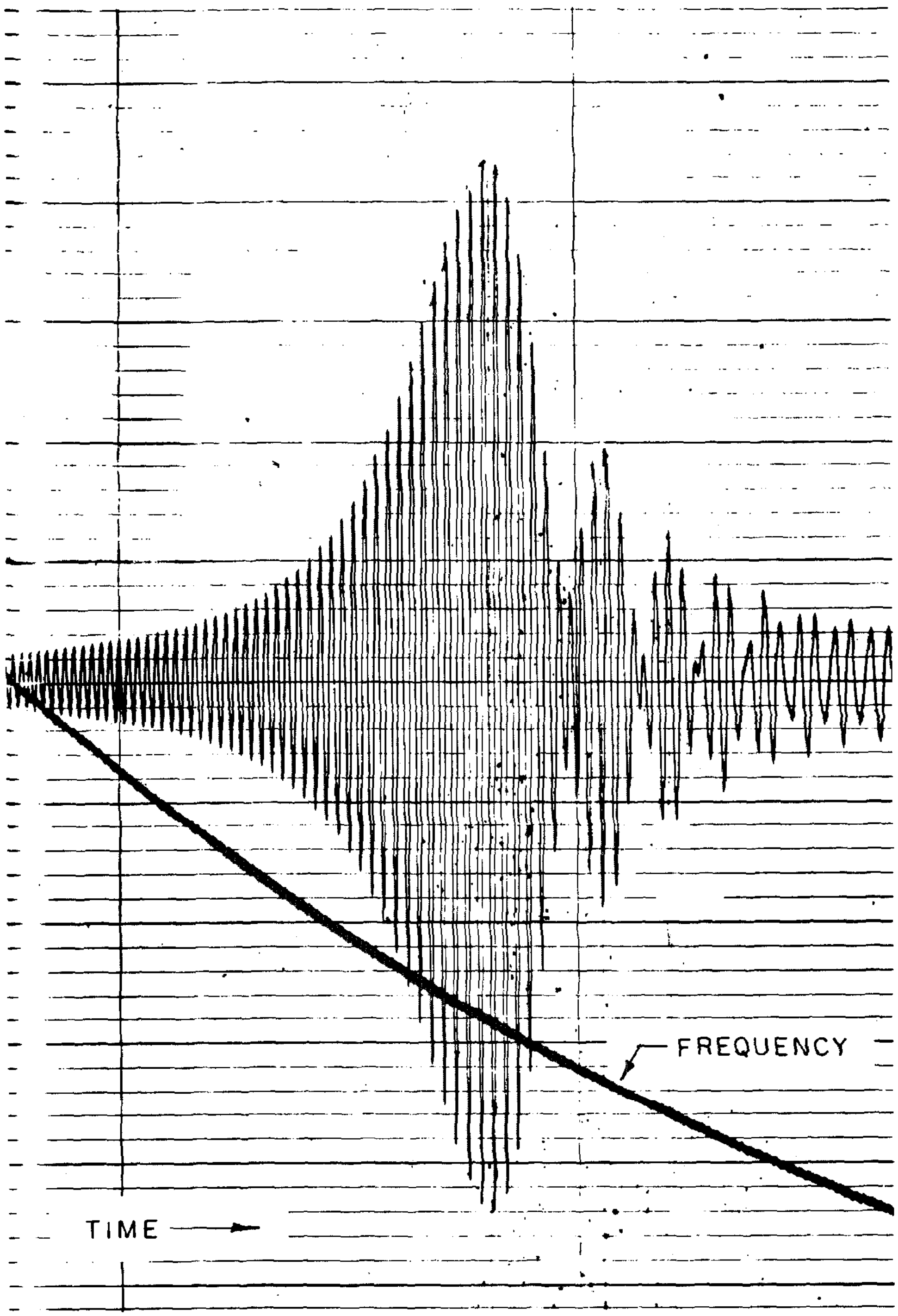


FIG. 5.1b

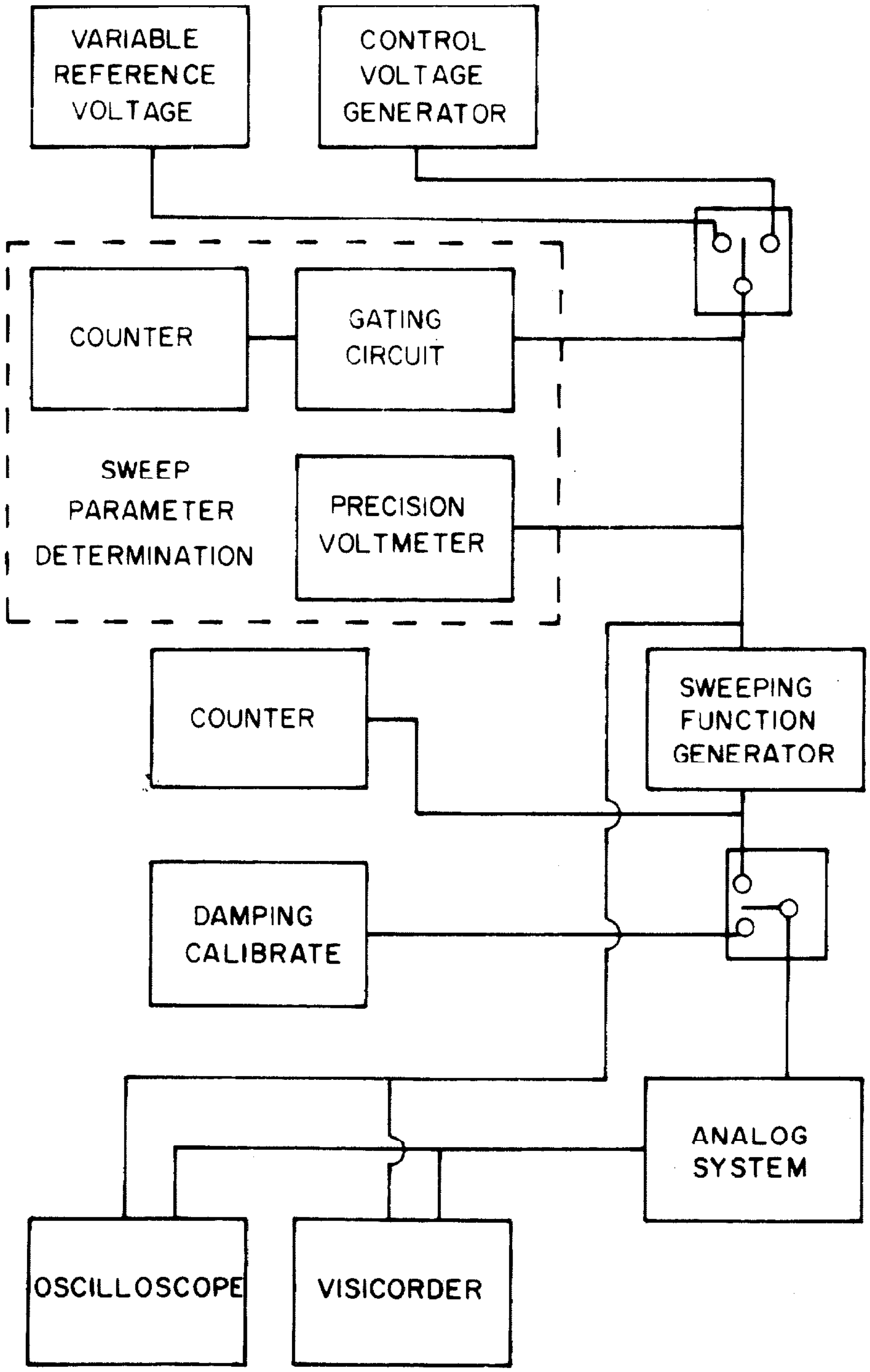

VARIABLE REFERENCE VOLTAGE
CONTROL VOLTAGE GENERATOR
COUNTER
GATING CIRCUIT
SWEEP PARAMETER DETERMINATION
PRECISION VOLTMETER
COUNTER
SWEEPING FUNCTION GENERATOR
DAMPING CALIBRATE
ANALOG SYSTEM
OSCILLOSCOPE
VISICORDER

FIG. 5.2

characteristics of the infinite time operating system, will require a demonstration that the contribution of the starting conditions to system response near resonance is small.

Most experimental models had non-zero damping and were excited for a long period of time prior to each run by a forcing voltage of constant frequency corresponding to either the lower or upper limit of operation of the function generator depending upon whether it was desired during the run to sweep up or down in frequency through system resonance.

If $\bar{\alpha}$ is taken as the initial frequency of the excitation, the system response before time equals zero may be written

$$y = \frac{1}{\sqrt{(1-\bar{\alpha}^2)^2 + 4\zeta^2\bar{\alpha}^2}} \cdot \sin\left[\bar{\alpha}\tau - \tan^{-1}\left(\frac{2\zeta\bar{\alpha}}{1-\bar{\alpha}^2}\right)\right] .$$

For times after the start of the run, $\tau > 0$, system response may then be written

$$y = \int_0^\tau h(\tau-\tau')\sin g(\tau')d\tau' + \frac{\bar{\alpha} \cdot e^{-\zeta\tau} \cdot \sin\left[m\tau - \tan^{-1}\left(\frac{2\zeta m}{\alpha_0^2-\bar{\alpha}^2}\right)\right]}{m\sqrt{(1-\bar{\alpha}^2)^2 + 4\zeta^2\bar{\alpha}^2}} . \qquad (5.1)$$

For the infinite time operating system, an analogous expression will be

$$y_\infty = \int_0^\tau h(\tau-\tau')\sin g(\tau')d\tau' + C e^{-\zeta\tau}\sin(m\tau + \psi) \quad , \qquad (5.2)$$

where

$$C e^{-\zeta\tau} \sin(m\tau + \psi) = \int_{-\infty}^{0} h(\tau - \tau') \sin g(\tau') d\tau' \quad .$$

The factors C and ψ may be estimated by the method of Chapter II, and C will be small if the frequency of the excitation on the interval $(-\infty, 0)$ is far removed from system resonance.

Measured systems response at resonance will be written

$$y_{max} = \frac{\varkappa}{2\zeta m} \quad , \tag{5.3}$$

in which $\varkappa$ is a peak reduction factor, $O[1]$, and the results (5.1), (5.2), and (5.3) may be combined to form an estimating expression for the fractional change in the maximum response amplitude due to the contribution of the present starting conditions:

$$\left| \frac{y_{max} - y_{\infty\, max}}{y_{max}} \right| \lesssim \frac{2\zeta m}{\varkappa} \left\{ |C| + \frac{|\bar{\alpha}|}{m\sqrt{(1-\bar{\alpha}^2)^2 + 4\zeta^2\bar{\alpha}^2}} \right\} e^{-\zeta\tau_r} \quad .$$

The time τ_r represents the time of resonance and is a function of the sweep rate, initial excitation frequency, and system natural frequency.

If the excitation frequency is sufficiently displaced from system resonance, the sweeping response amplitude will not differ greatly from the steady-state response for the same instantaneous frequency, and

$$|C| \cong \frac{|\bar{\alpha}|}{m\sqrt{(1-\bar{\alpha}^2)^2 + 4\zeta^2\bar{\alpha}^2}} \quad .$$

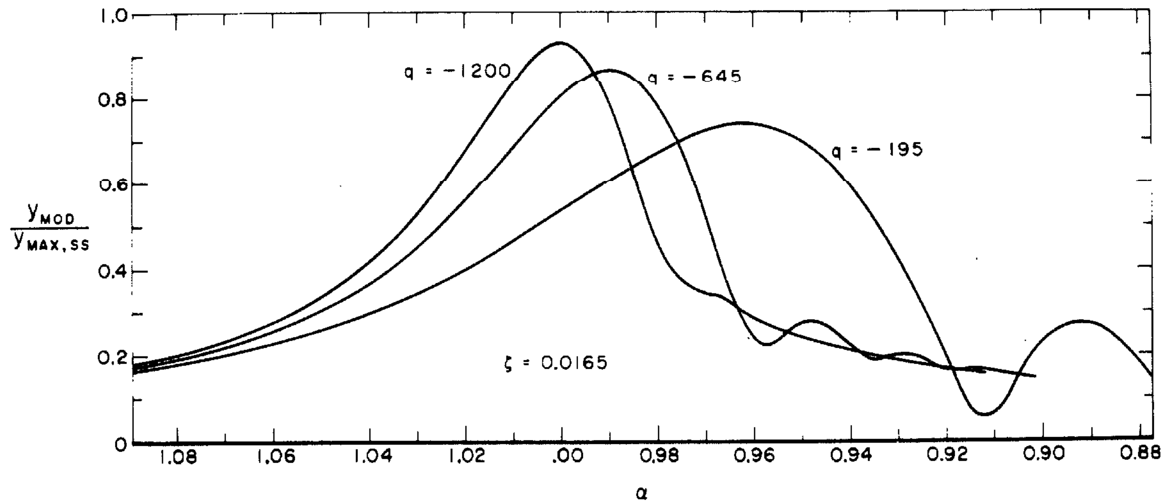


FIG. 5.3a

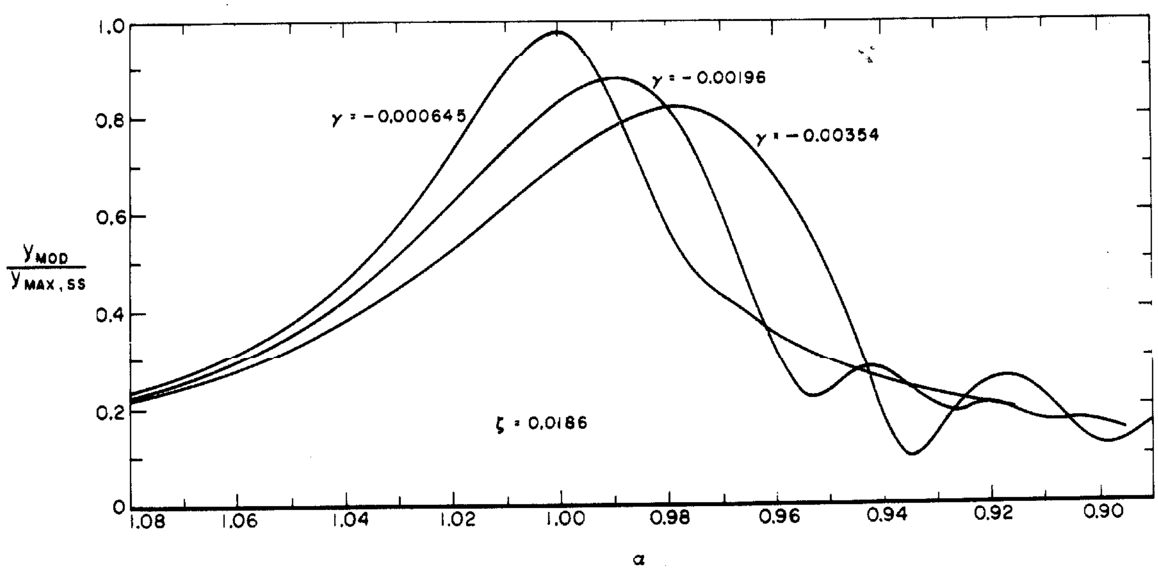


FIG. 5.3b

A more tractable estimate for the fractional change in the maximum response amplitude may then be written

$$\left| \frac{y_{max} - y_{\infty\,max}}{y_{max}} \right| \lesssim \frac{4\zeta}{\varkappa} \frac{|\bar{\alpha}| \cdot e^{-\zeta\tau_r}}{\sqrt{(1-\bar{\alpha}^2)^2 + 4\zeta^2\bar{\alpha}^2}} \quad . \tag{5.4}$$

The largest value of (5.4) obtained during the experiment was about two tenths of one percent. The assumption that the experimentally derived response characteristics closely approximate those of the infinite time operating system is a good one.

Response versus excitation frequency curves for the sweeping excitation case may be constructed from the time histories exemplified in Figs. 5.1a, b. Such curves have been presented by Lewis[1], Parker[12], and Barber and Ursell[2]. Additional examples are provided in Figs. 5.3a, b.

Figure 5.4 illustrates the fractional reduction of the maximum response amplitude of a single degree of freedom system influenced by linearly and exponentially sweeping excitations as a function of sweep rate and system damping.

It is observed that, within the limits of accuracy of the present investigation, a system influenced by a linearly sweeping excitation will experience the same peak attenuation as will the identically damped system influenced by the exponentially sweeping excitation provided that the linear sweep factor introduced in Chapter II:

$$1/q = \text{constant} = 2\pi\alpha'$$

is equal to the exponential sweep factor defined in Chapter III:

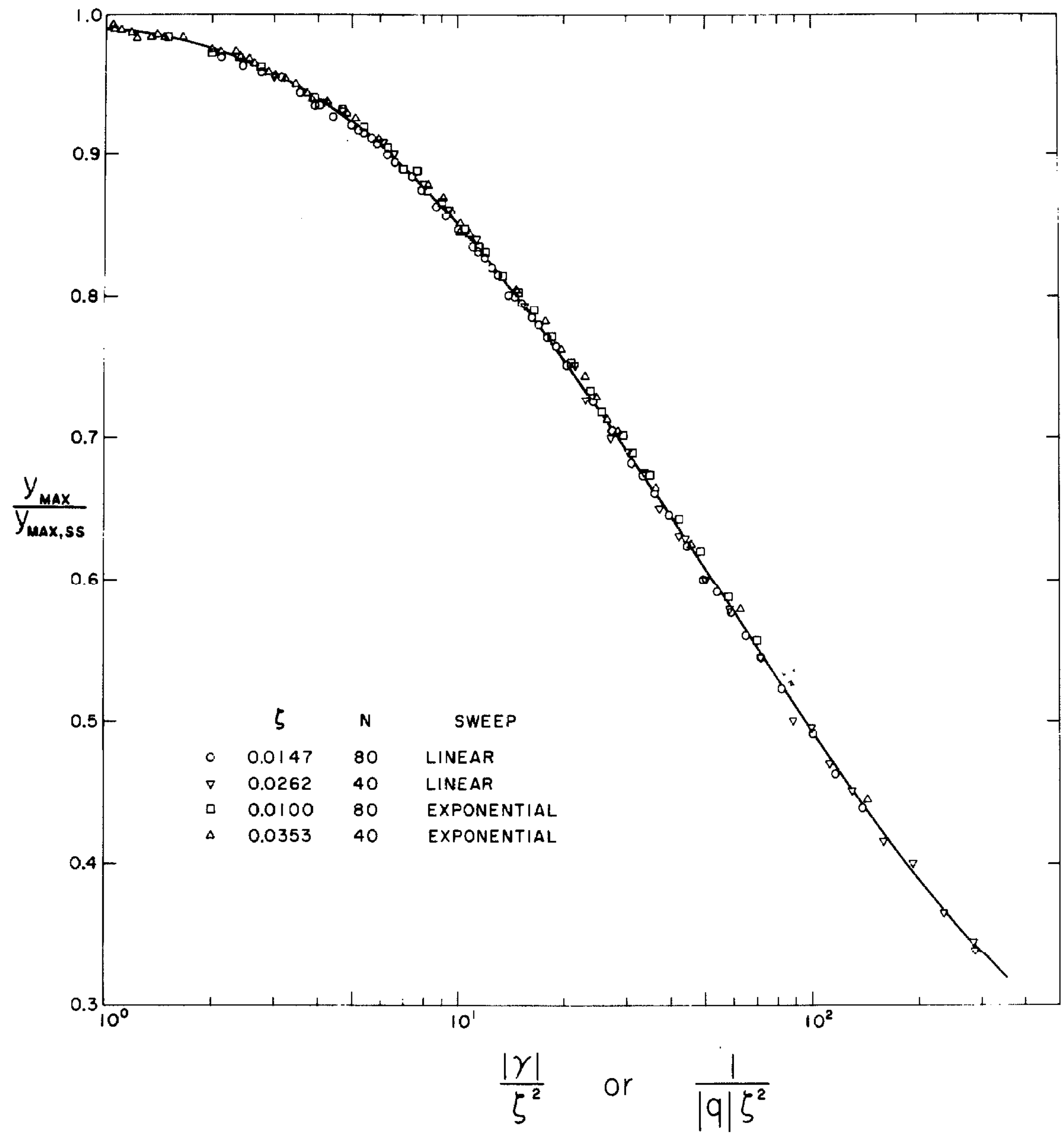

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
$\frac{y_{MAX}}{y_{MAX,SS}}$
10^0
10^1
10^2
$\frac{|\gamma|}{\zeta^2}$ or $\frac{1}{|q|\zeta^2}$
ζ N SWEEP
○ 0.0147 80 LINEAR
▽ 0.0262 40 LINEAR
□ 0.0100 80 EXPONENTIAL
△ 0.0353 40 EXPONENTIAL

FIG. 5.4

$$\gamma = \text{constant} = 2\pi(\alpha'/\alpha) \quad .$$

It is observed, in addition, that the level of the response peak depends only upon the relevant sweep factor divided by the square of the system damping.

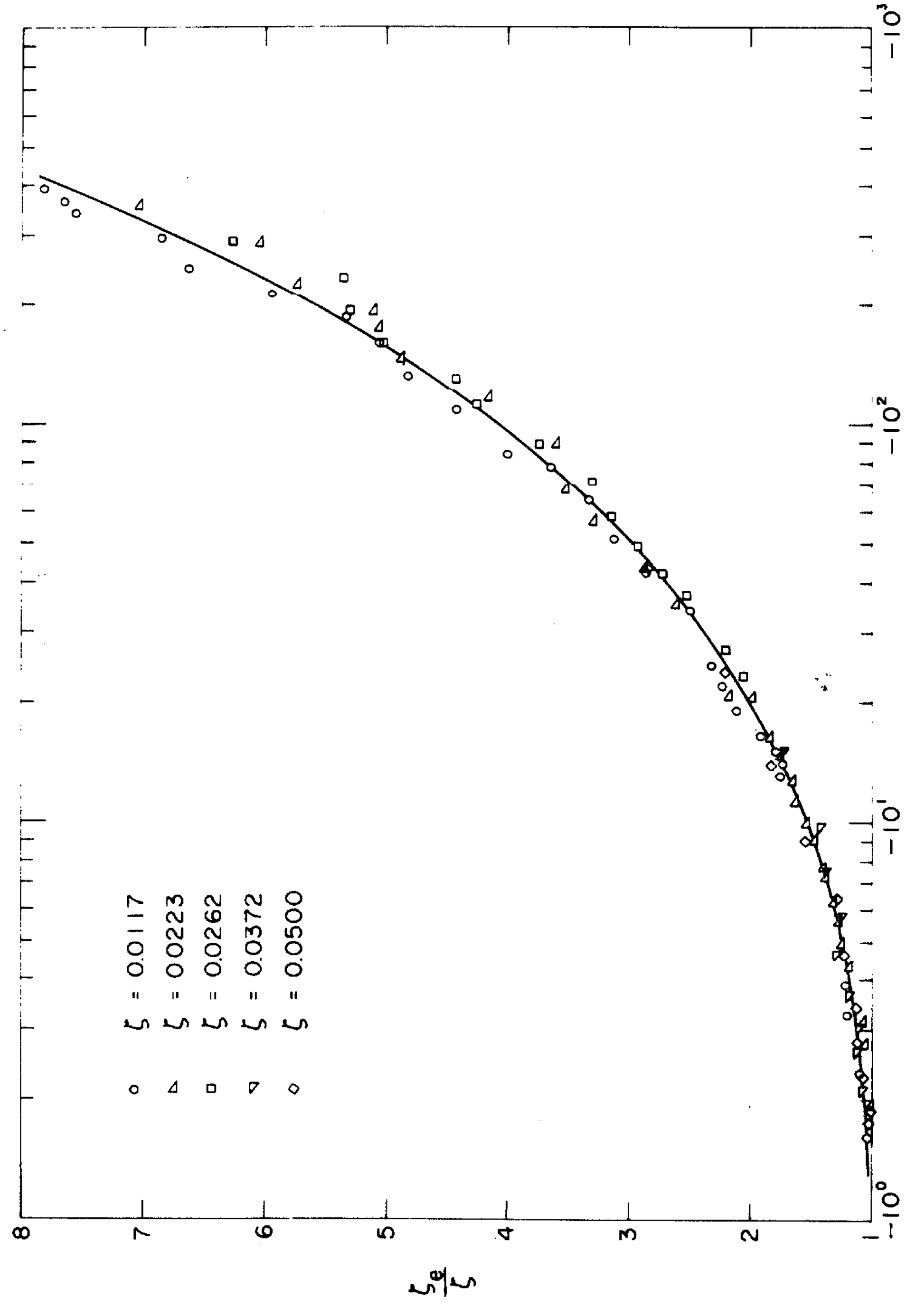
Experimental results illustrated in Fig. 5.4 indicate no measurable difference between the peak response of a system influenced by an excitation sweeping up through resonance, and that of a system influenced by an excitation sweeping at the same rate down through resonance. (The 80 cps system was influenced by the former excitation; the 40 cps system was influenced by the latter.) The results of Chapter III predict a difference, but it is apparently small for the range of damping and sweep rate covered in the present study.

Figure 5.4 shows that peak reduction as a consequence of sweeping is a rather strong effect. The broadening of the response peak under sweeping conditions is, however, an even stronger effect as seen in Figs. 5.5a, b. These figures illustrate the change in the width of the response peak measured at the half-power points as a function of damping and sweep rate expressed in terms of the ratio of measured damping to actual damping.

Measured damping is defined by an approximate expression, truly applicable only for the steady-state case, as

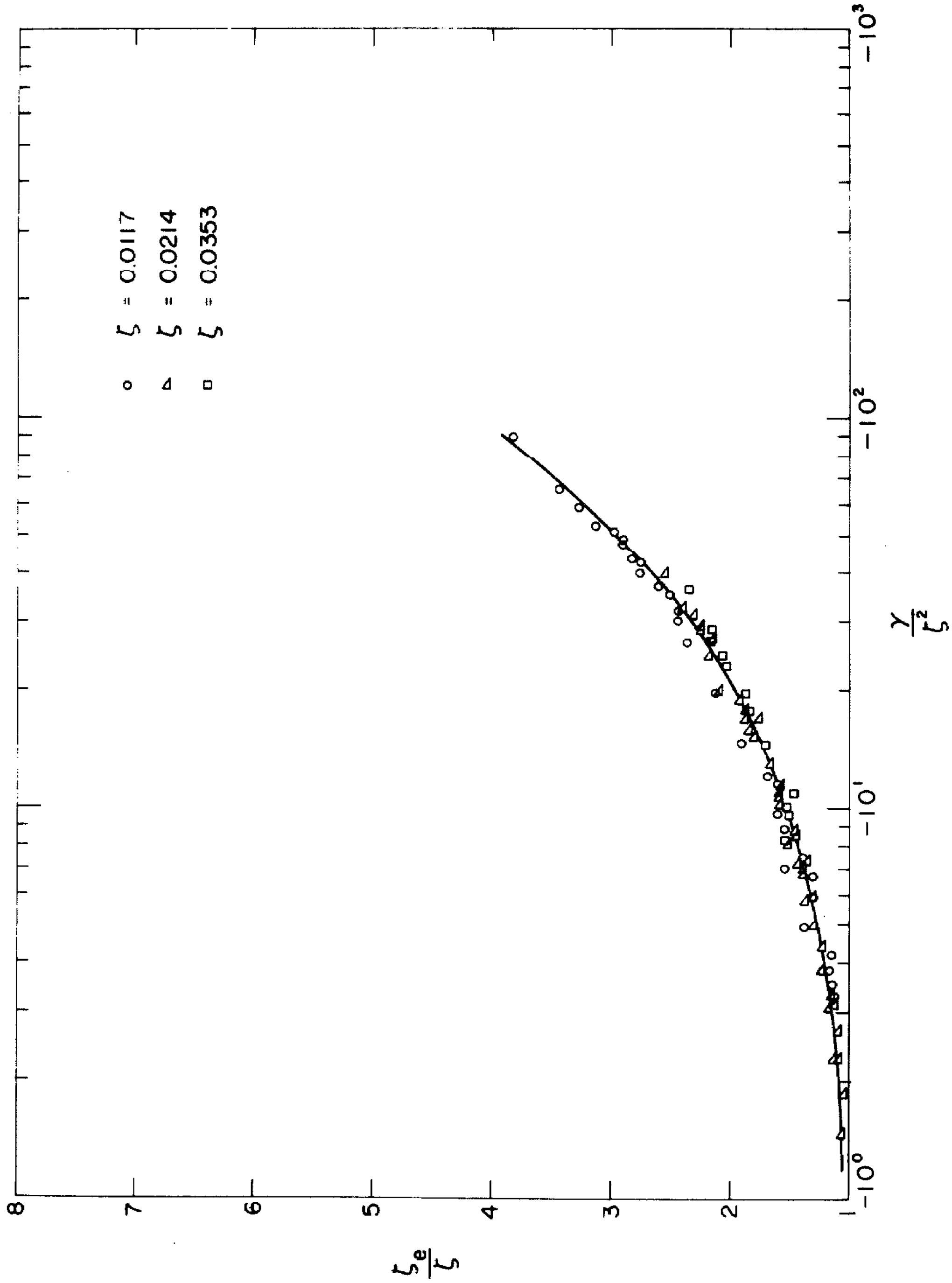
$$\zeta_e \cong \frac{\alpha_2 - \alpha_1}{2} \quad , \tag{5.5}$$

in which α_2 and α_1 represent the frequency ratios where



ζ = 0.0117
ζ = 0.0223
ζ = 0.0262
ζ = 0.0372
ζ = 0.0500
$\frac{\zeta_e}{\zeta}$
$\frac{1}{q\zeta^2}$
-10^0
-10^1
-10^2
-10^3
1
2
3
4
5
6
7
8

FIG. 5.5a



ζ = 0.0117
ζ = 0.0214
ζ = 0.0353
ζ_e / ζ
γ / ζ^2
-10^0
-10^1
-10^2
-10^3
1
2
3
4
5
6
7
8

FIG. 5.5b

$$y = \frac{1}{\sqrt{2}} \cdot y_{max} \quad .$$

Experimental results indicate that the width of the response peak measured at the half-power points is a function only of the relevant sweep factor divided by the square of the actual system damping. Comparison of Fig. 5.5a to Fig. 5.5b indicates further that for

$$1/q\zeta^2 \ , \ \gamma/\zeta^2 < 10^2 \quad ,$$

there will be no measurable difference between the width of the response peak for a system forced by a linearly sweeping excitation, and that of a similarly damped system forced by an exponentially sweeping excitation provided that the sweep factors are equal.

Figure 5.5b may also be used to estimate $\hat{M}$, the number of response maxima exceeding $y_{max}/\sqrt{2}$, for the exponential sweep case, since the number of response maxima occurring in the interval (t_1, t_2) will be approximately twice the number of cycles of the excitation occurring in the interval:

$$M \cong \frac{1}{\pi} \int_{t_1}^{t_2} \Omega(t)\, dt \quad ,$$

or

$$M \cong \frac{1}{\pi} \Big[G(t_2) - G(t_1) \Big] \quad . \tag{5.6}$$

For the exponential sweep case

$$G(t) = \frac{\omega m}{\overline{B}} e^{\overline{B}t} \quad ,$$

and

$$\Omega(t) = \omega m e^{\overline{B}t} = \overline{B}G(t) \ .$$

Then (5.6) becomes

$$M \cong \frac{1}{\pi \overline{B}} \left[\Omega(t_2) - \Omega(t_1) \right] \ ,$$

or in the presently used notation:

$$M \cong \frac{2}{\gamma} (\alpha_2 - \alpha_1) \quad . \tag{5.7}$$

Insertion of the definition (5.5) in (5.7) leads to the result

$$M \cong \frac{4\zeta}{\gamma} (\zeta_e / \zeta) \quad .$$

Examination of Figs. 5.4 and 5.5a, b reveals that the two common methods for the experimental determination of the damping of a linear system--measurement of the peak response and solution of the approximate expression

$$\frac{1}{2\zeta_e} \cong y_{max} \quad ,$$

and measurement of the width of the response peak at the half-power points and solution of (5.5)--will lead to quite different results if these measurements are made from a record of system response under sweeping conditions.

The families of curves given in Figs. 5.6a, b, and Figs. 5.7a, b illustrate the shift in frequency of the resonance peak as a function of system damping and sweep rate expressed as a ratio of the dimensionless peak frequency, α_{max}, to the steady-state dimensionless peak frequency, $\alpha_0 = \sqrt{1-2\zeta^2}$. As stated in Chapter II, the maximum response amplitude for a system influenced by sweeping excitation occurs after the excitation frequency equals the system natural frequency. For negative sweep rates, the frequency corresponding to peak response will thus be smaller than the steady-state value, and for positive sweep rates, the frequency corresponding to peak response will be larger than the steady-state value.

In Figs. 5.6a, b the prediction derived from (2.17) for the peak shift in the undamped system:

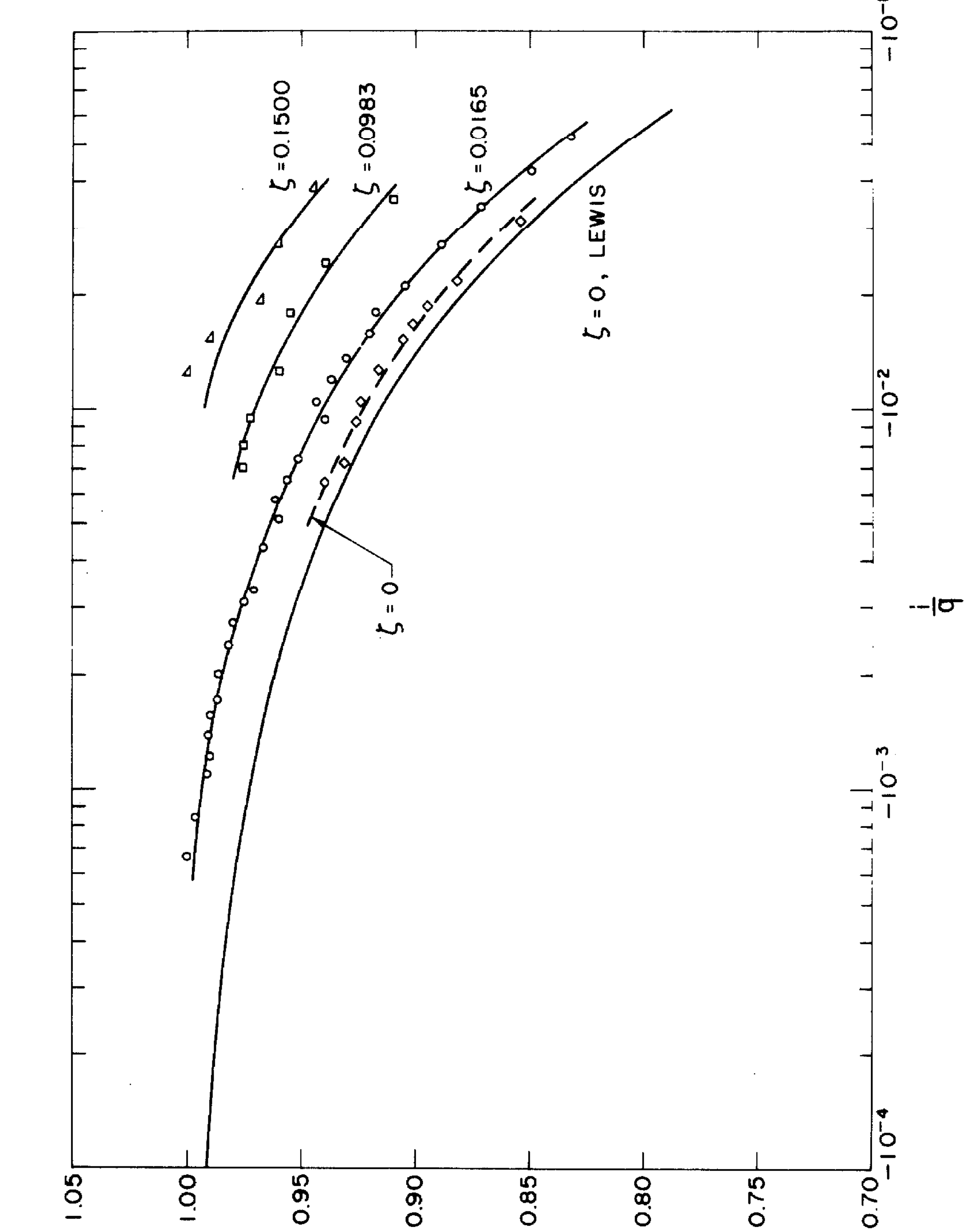
$$\frac{\alpha_{max}}{\alpha_0} \cong 1 + \mathrm{sgn}(q) \cdot \frac{(.852)}{\sqrt{|q|}}$$

is included for comparison to data taken for the undamped system. The difference between this prediction and experimental results is within one percent.

When the method derived in Chapter II for predicting the upper and lower limit of the time of resonance for the undamped system is applied to the exponential sweep case, two transcendental equations result. If the sweep factor, γ, is small, these equations may be simply approximated and solved. The frequency of excitation evaluated at the average of the resulting limiting times is given by

$$\frac{\alpha_{max}}{\alpha_0} \cong 1 + \mathrm{sgn}\,(\gamma) \cdot (.854) \sqrt{|\gamma|} \quad . \qquad (5.8)$$

Expression (5.8) is plotted in Figs. 5.7a, b for comparison to data taken for the undamped system. The difference is again within one percent.



$\frac{\alpha_{MAX}}{\alpha_o}$
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
-10^{-4}
-10^{-3}
-10^{-2}
-10^{-1}
$\frac{1}{q}$
$\zeta = 0.1500$
$\zeta = 0.0983$
$\zeta = 0.0165$
$\zeta = 0$
$\zeta = 0$, LEWIS

FIG. 5.6a

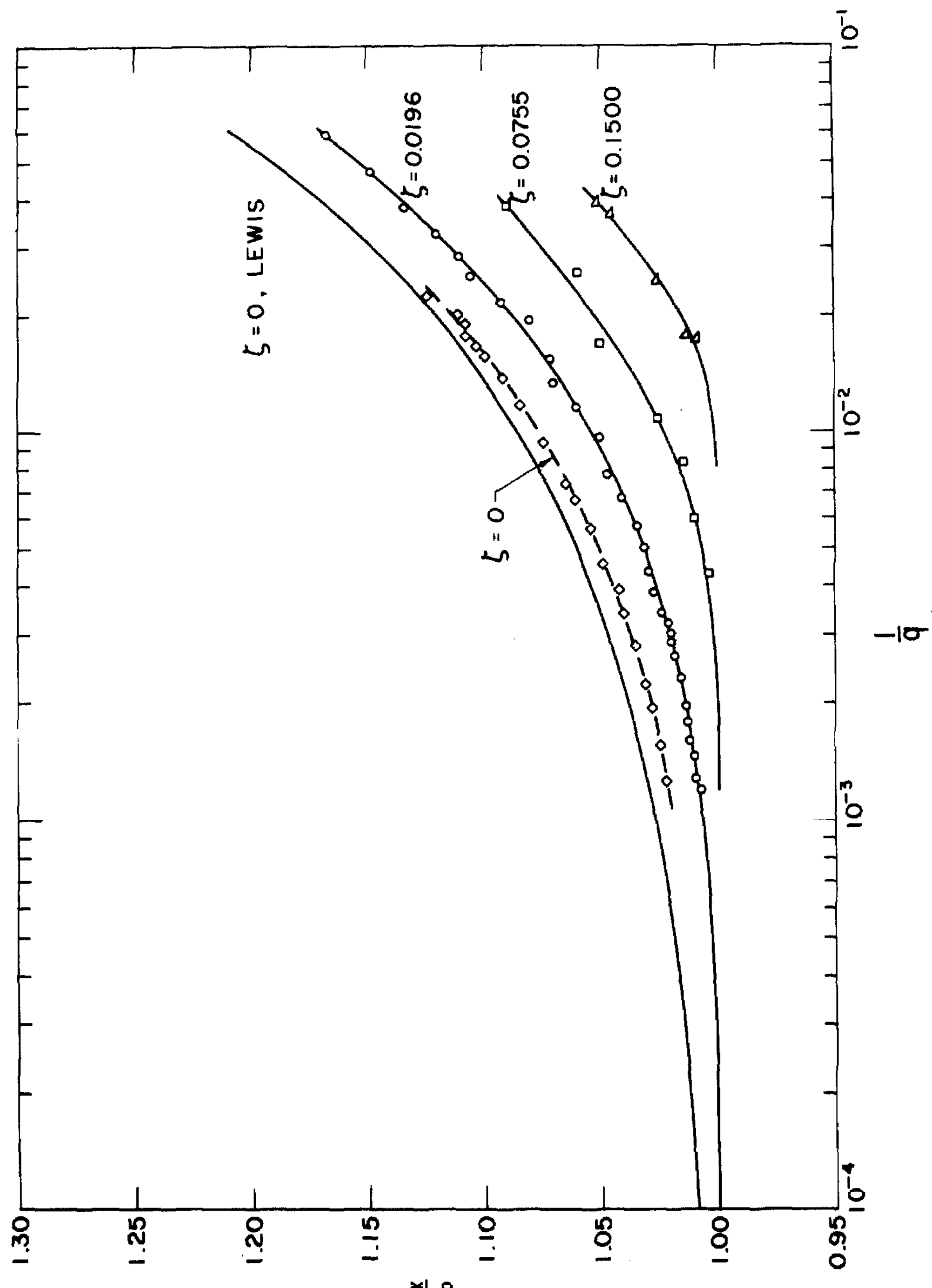

$\frac{\alpha_{MAX}}{\alpha_o}$
1.30
1.25
1.20
1.15
1.10
1.05
1.00
0.95
10^{-4}
10^{-3}
10^{-2}
10^{-1}
$\frac{1}{q}$
$\zeta = 0$, LEWIS
$\zeta = 0.0196$
$\zeta = 0.0755$
$\zeta = 0.1500$
$\zeta = 0$

FIG. 5.6b

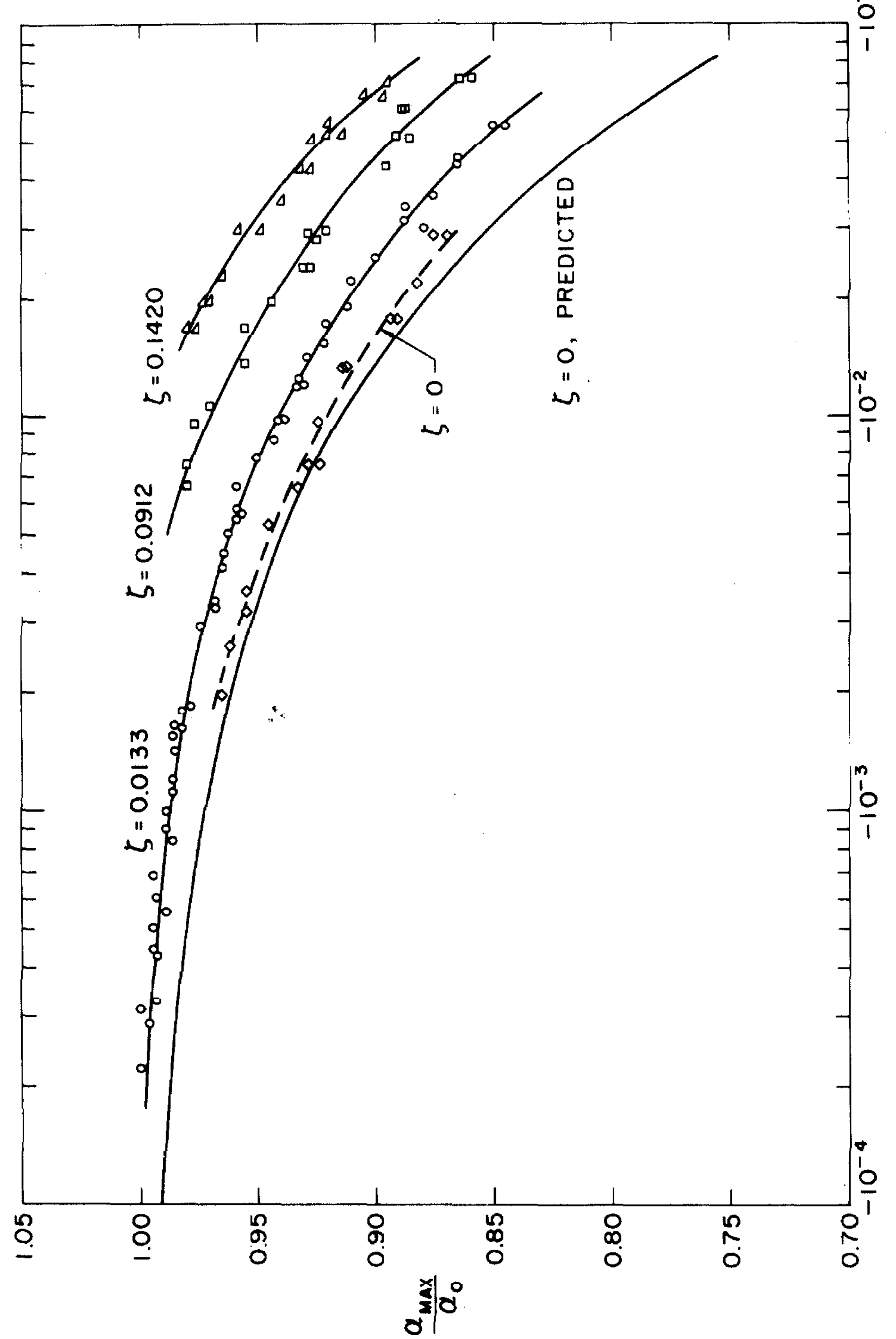
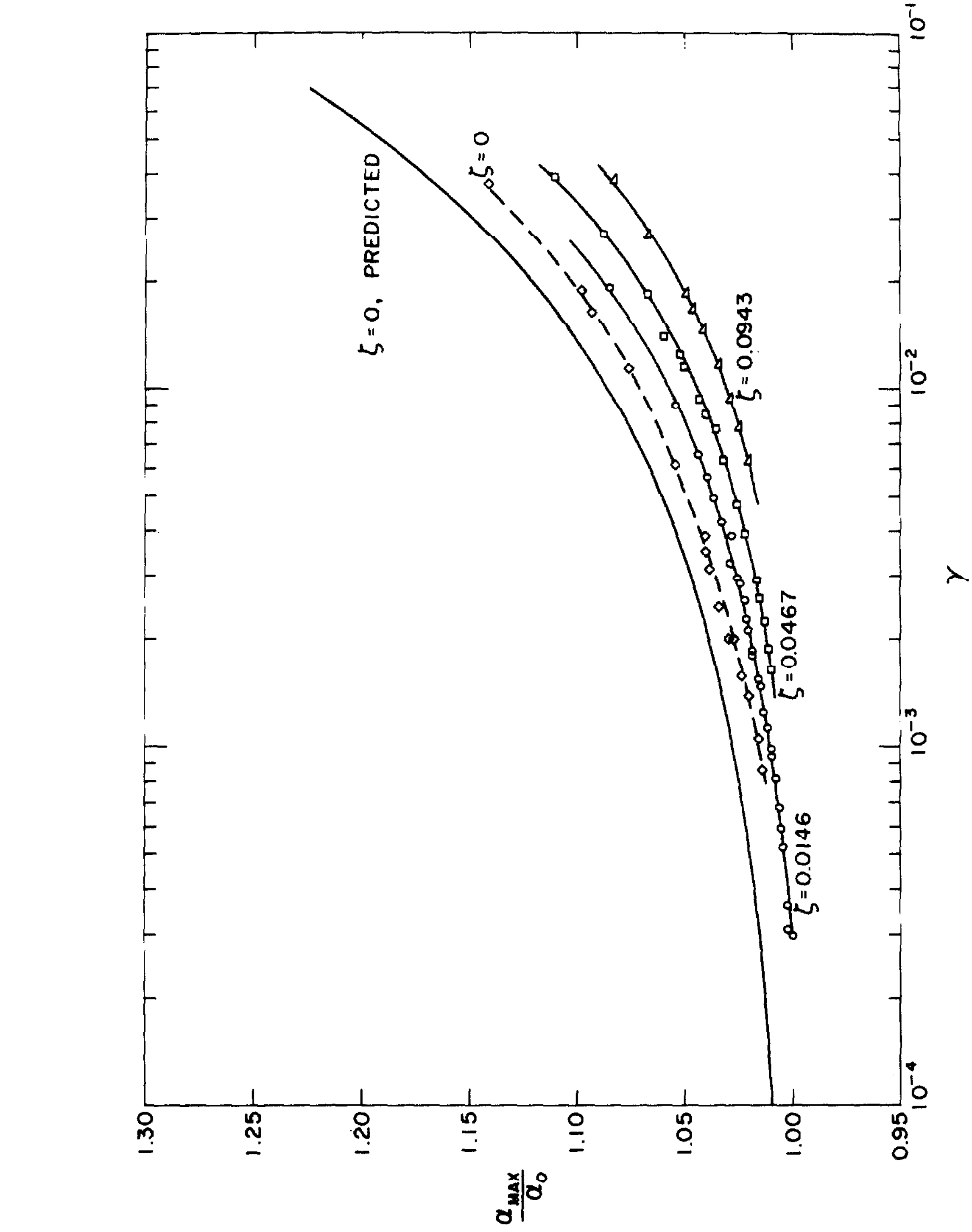

$\zeta = 0.1420$
$\zeta = 0.0912$
$\zeta = 0.0133$
$\zeta = 0$
$\zeta = 0$, PREDICTED
$\frac{a_{MAX}}{a_o}$
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
-10^{-4}
-10^{-3}
-10^{-2}
-10^{-1}
γ

FIG. 5.7a



$\frac{a_{MAX}}{a_o}$
γ
$\zeta = 0$, PREDICTED
$\zeta = 0$
$\zeta = 0.0943$
$\zeta = 0.0467$
$\zeta = 0.0146$
1.30
1.25
1.20
1.15
1.10
1.05
1.00
0.95
10^{-4}
10^{-3}
10^{-2}
10^{-1}

FIG. 5.7b

A discussion of the results presented here would be incomplete without a few words concerning the similarity between the response characteristics of a system influenced by linearly sweeping excitation, and those of a system influenced by exponentially sweeping excitation. The interesting dependence of both the peak attenuation, and the response peak width upon the sweep factor divided by the square of system damping also deserves attention.

It will be helpful to the discussion to analyze the sweep factors $1/q$ or q, and γ more carefully.

The linear sweep factor, q, is defined as

$$q = \frac{N^2}{h} \quad ,$$

in which N is the system natural frequency in cps, and h is the real time derivative of the excitation frequency (cps/s).

Hawkes[13] first defined a parameter related to γ and stressed its lack of connection to q. This is unfortunate, for such a connection may be easily established. If a system having the natural frequency N is subjected to an excitation having the instantaneous frequency

$$\Omega = \Omega_o e^{\overline{B}t} \quad ,$$

the ratio defining q will be

$$q(t) = N^2 \Big/ \frac{\Omega_o}{2\pi} \overline{B} e^{\overline{B}t} \quad .$$

The value of this expression evaluated at steady-state resonance will then be

$$q(t_r) = N/\bar{B} \quad ,$$

but

$$1/\gamma = \frac{1}{2\pi\left(\frac{\alpha'}{\alpha}\right)} = N/\bar{B} \quad ,$$

and

$$q(t_r) = 1/\gamma \quad .$$

Both q and γ are similarly defined parameters and are related in a non-dimensional way to the rate of change of excitation frequency evaluated at system resonance.

The results of the experiment then indicate that, within the limits of experimental accuracy, the measured response characteristics depend only upon the system damping, and the non-dimensionalized rate of change of excitation frequency evaluated at system resonance for sweep rates up to and including the fastest generally attained in practice, and for normally encountered damping ratios ($\zeta < .1$).

It will be recalled that $1/q\zeta^2$ and γ/ζ^2 first appear as important quantities in the Chapter III perturbation series expressions for the system maximum response amplitude.

In the envelope expression of Chapter II, (2.11), the two integrals of (2.19) making the primary contribution to system response near resonance, for the linear sweep case, will be

$$I_2 = \int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')} \sin\left(\frac{\tau'^2}{4\pi q}\right) d\tau' \quad , \qquad (5.9a)$$

and

$$I_4 = \int_{-\infty}^{\tau} e^{-\zeta(\tau-\tau')} \cos\left(\frac{\tau'^2}{4\pi q}\right) d\tau' \quad . \tag{5.9b}$$

The change of variables in (5.9a, b):

$$u' = \frac{\tau'}{2\sqrt{\pi q}} \quad ,$$

leads to the expression:

$$\frac{y_{mod}}{y_{max,\, ss}} \cong 2\sqrt{\pi} \cdot \zeta\sqrt{q} \cdot \left[\left(\int_{-\infty}^{u} e^{-2\sqrt{\pi} \cdot \zeta\sqrt{q}\,(u-u')} \sin u'^2 \, du'\right)^2 + \left(\int_{-\infty}^{u} e^{-2\sqrt{\pi} \cdot \zeta\sqrt{q}(u-u')} \cos u'^2 du'\right)^2\right]^{1/2} \quad , \tag{5.10}$$

and the grouping of factors $\zeta\sqrt{q}$ again appears important to system response near resonance.

The grouping of factors $\zeta\sqrt{\gamma}$ will obtain when (5.10) is rewritten for the exponential sweep case, provided the experimentally justified assumption is made that the excitation frequency may be replaced by the first two terms of its Taylor series expansion about the time of resonance.

The above assumption may only be used to predict behavior associated with the response peak. For excitation frequencies displaced by a moderate amount from system resonance, differences between the response of a system excited by a linearly sweeping excitation, and that of a system excited by an exponentially sweeping excitation may be observed.

Figure 5. 8 illustrates, for instance, that the level of the first secondary peak, $\frac{y_{max, 2}}{y_{max, ss}}$, for the value of the characteristic parameter associated with the first appearance of the "ringing" phenomenon, is different. The level of the primary response peak at the onset of "ringing", it may be concluded with less certainty, is also different.

	Log Sweep	Linear Sweep
Value of Characteristic Parameter	$\gamma/\zeta^2 =$ $-3.46 \pm .13$	$1/q\zeta^2 =$ $-3.73 \pm .14$
$\frac{y_{max}}{y_{max, ss}}$	$.949 \pm .002$	$.937 \pm .005$
$\frac{y_{max, 2}}{y_{max, ss}}$	$.352 \pm .003$	$.314 \pm .009$

Fig. 5. 8

It is seen in Fig. 5. 8 that for a value of the characteristic parameter between three and four, the beating phenomenon makes its first appearance for both types of sweeping excitation. At the onset of "ringing", as mentioned in Chapter II, the first beat appears as a secondary hump on a response curve that looks otherwise much like the steady-state response curve.

The beat amplitude at first appearance represents the largest attained for any sweep rate--about thirty-five percent of the maximum steady-state response. As the sweep rate is increased, the amplitude of the first beat decreases to a minimum as seen in Fig. 5.9. For faster sweeping, the amplitude of the first beat then increases to a maximum, and "ringing" may be said to be fully developed. Still faster sweep rates bring about post-resonance degradation of the response curve, and the beat pattern is gradually destroyed.

In Fig. 5.9, the amplitude of the first beat relative to the amplitude of the maximum steady-state response, $\frac{y_{max,\,2}}{y_{max,\,ss}}$, and the amplitude relative to amplitude of the sweeping response maximum, $\frac{y_{max,\,2}}{y_{max}}$, are presented for a typical system as a function of sweep rate.

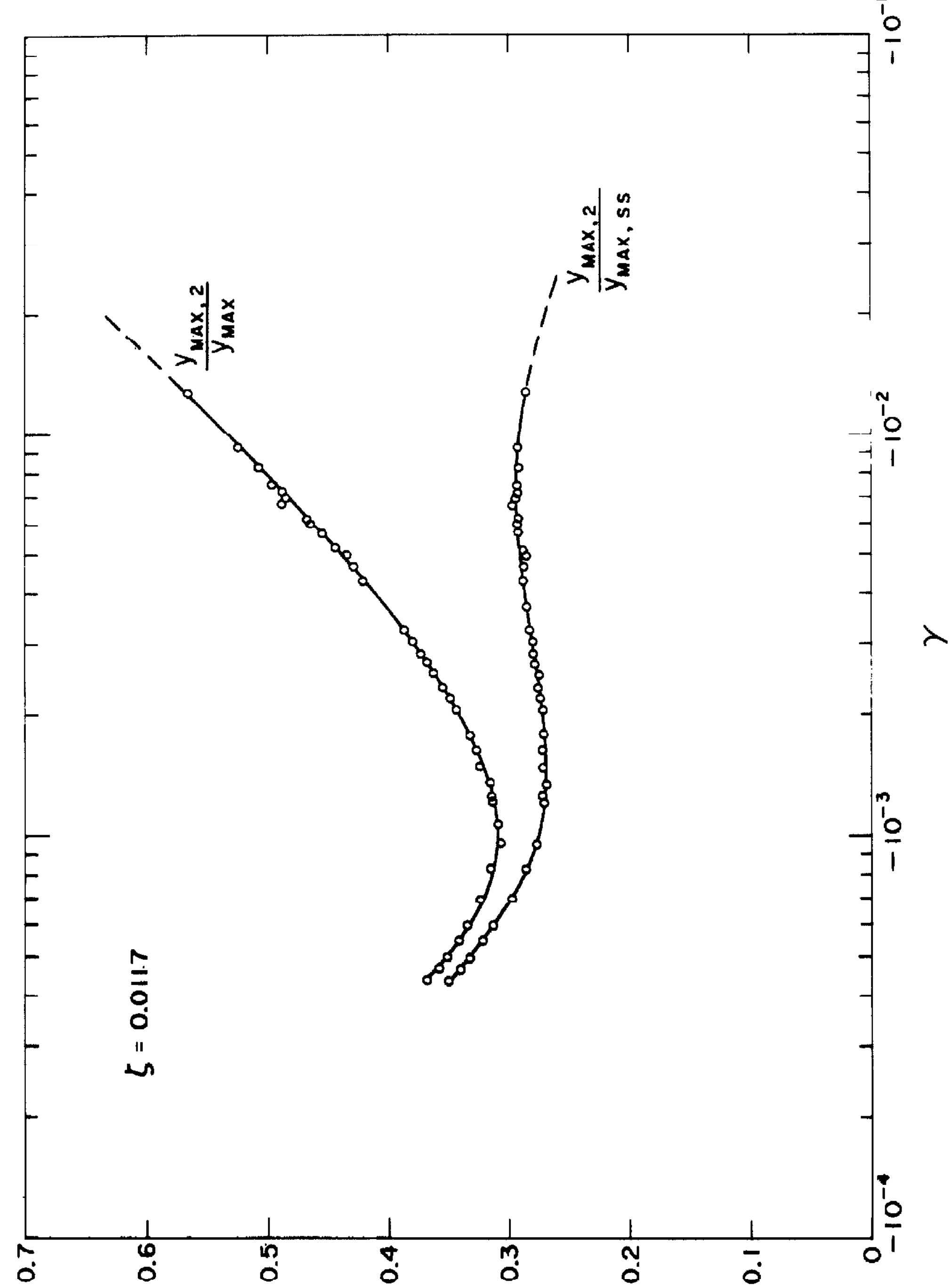

$\zeta = 0.0117$
$\frac{y_{MAX,2}}{y_{MAX}}$
$\frac{y_{MAX,2}}{y_{MAX,SS}}$
γ
-10^{-4}
-10^{-3}
-10^{-2}
-10^{-1}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

FIG. 5.9

VI. RESPONSE OF MULTI-DEGREE OF FREEDOM AND CONTINUOUS SYSTEMS TO SWEEPING EXCITATIONS

When a continuous or discrete system is excited by a set of forces sharing a mutual harmonic, or slowly sweeping quasi-harmonic time dependence, the system response for an excitation frequency equal to an important natural frequency will consist primarily of the mode of vibration associated with that natural frequency provided that system resonances are widely spaced, and that the set of forces has a spatial description involving approximately equal amounts of all system eigenvectors. Since such a system acts essentially as though it had a single degree of freedom near these important resonances, it should be possible to apply previously discussed methods to a simplified mathematical model to predict amplitude peak behavior at points in the system for the case of sweeping excitation. It should be further possible to predict steady-state response amplitude peak characteristics from the response history of the system subjected to sweeping excitation by application of corrections suggested in Chapter III, and in Chapter V.

Many systems combine light modal damping with closely spaced resonances. Steady-state response measured at a representative point in such a system influenced by an excitation of frequency equal to one of the resonant frequencies will, in general, contain sizeable contributions from the modes of vibration associated with nearby resonances. When the excitation sweeps in frequency, the response for an instantaneous excitation frequency equal to one of the system resonances will, in addition, contain transient contributions from the modes of vibration corresponding to resonances occurring earlier in time. These transients may be quite large and may lead to sweeping excitation response histories bearing small resemblance to steady-state response versus excitation frequency curves.

Consider, for example, a lightly damped two degree of freedom system, having closely spaced resonances of approximately the same strength at ω and $\omega + \Delta$, influenced by an excitation sweeping rapidly from $\Omega_1 > \omega + \Delta > \omega$ to $\Omega_2 < \omega$. The first response peak occurring in time--corresponding to $\omega + \Delta$--will be attenuated, broadened, and shifted in center frequency in accord with previous statements. The beat pattern, or transient behavior, associated with this peak may interact, however, with forced response corresponding to ω, producing amplification, attenuation, or complete destruction of the second response peak. McCann and Bennett[16] noted this behavior when performing an analog study of a two degree of freedom torsional pendulum.

It will still be possible to predict the response of a general system to sweeping excitation by the method of Chapter II if the system can be approximated by a classically damped N degree of freedom model, since the integrals solving the decoupled equations of motion will be similar to those previously treated. It will also be possible to predict the influence of the transient behavior associated with one resonance peak on other resonance peaks.

This estimating method may be applied, for instance, to analyze response characteristics of the rotating system shown in Fig. 6.1, influenced by a prescribed angular acceleration.

It will be assumed that the system may be approximated by a two mass model in which the masses, M_1 and M_2, having centers of gravity displaced from the shaft axis by e_1 and e_2, respectively, are firmly affixed to a weightless, non-twisting shaft, and are constrained to move in planes perpendicular to the line defining the rest position of the shaft axis. The motion of each mass will then be completely described by the coordinants of Fig. 6.2.

The shaft segments I, II, and I in Fig. 6.1 will be replaced by equivalent spring-dashpot pairs acting in the x, and in the y directions.

The equations of motion will be written

$$
\begin{aligned}
M_1 \frac{d^2}{dt^2}\left\{x_1+e_1\cos[\Theta(t)+\sigma_1]\right\} &= -K_I x_1 - K_{II}(x_1-x_2) - \beta_I\dot{x}_1 - \beta_{II}(\dot{x}_1-\dot{x}_2)\,, \\
M_1 \frac{d^2}{dt^2}\left\{y_1+e_1\sin[\Theta(t)+\sigma_1]\right\} &= -K_I y_1 - K_{II}(y_1-y_2) - \beta_I\dot{y}_1 - \beta_{II}(\dot{y}_1-\dot{y}_2)\,, \\
M_2 \frac{d^2}{dt^2}\left\{x_2+e_2\cos[\Theta(t)+\sigma_2]\right\} &= -K_I x_2 - K_{II}(x_2-x_1) - \beta_I\dot{x}_2 - \beta_{II}(\dot{x}_2-\dot{x}_1)\,, \\
M_2 \frac{d^2}{dt^2}\left\{y_2+e_2\sin[\Theta(t)+\sigma_2]\right\} &= -K_I y_2 - K_{II}(y_2-y_1) - \beta_I\dot{y}_2 - \beta_{II}(\dot{y}_2-\dot{y}_1)\,.
\end{aligned}
\tag{6.1}
$$

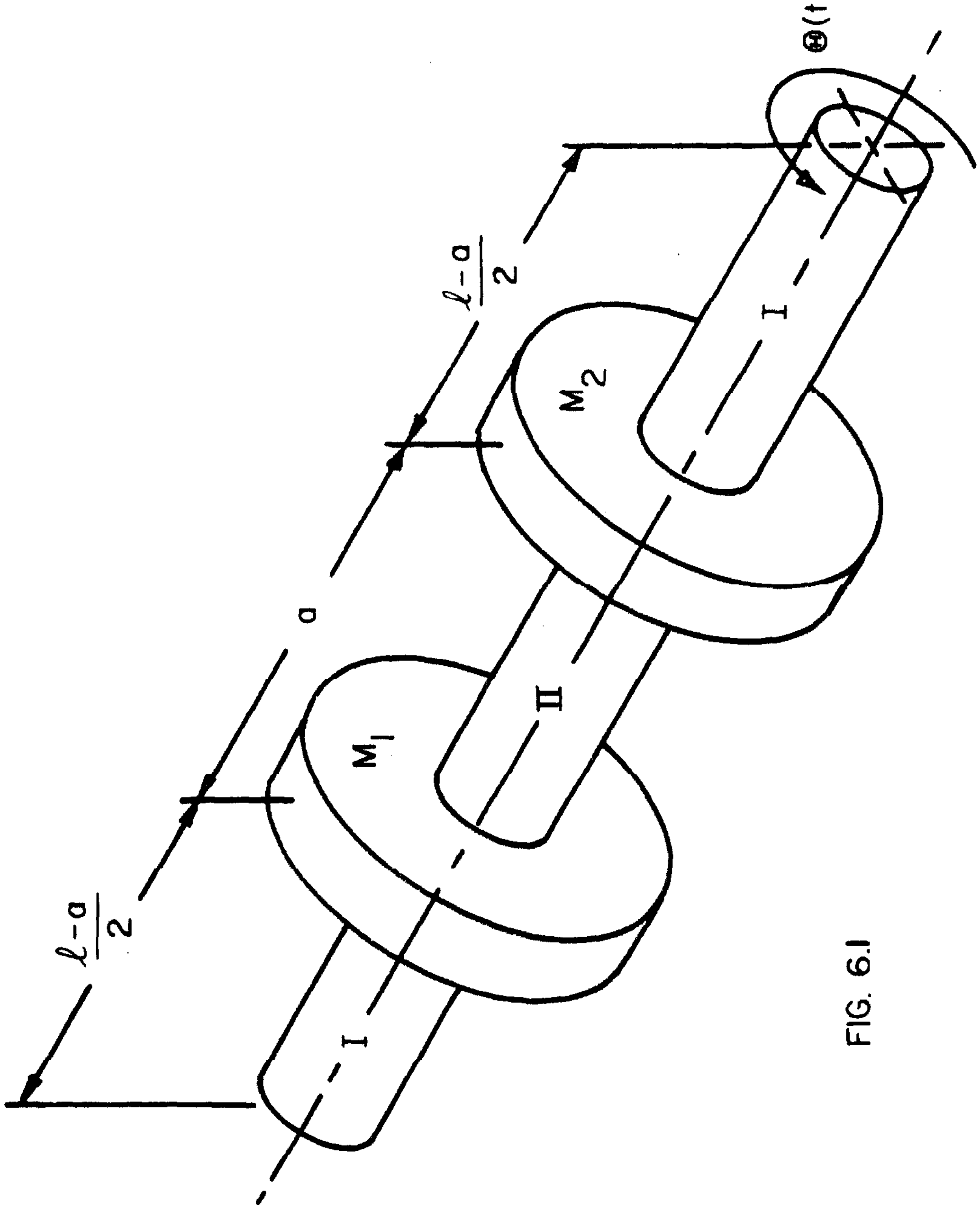


FIG. 6.1

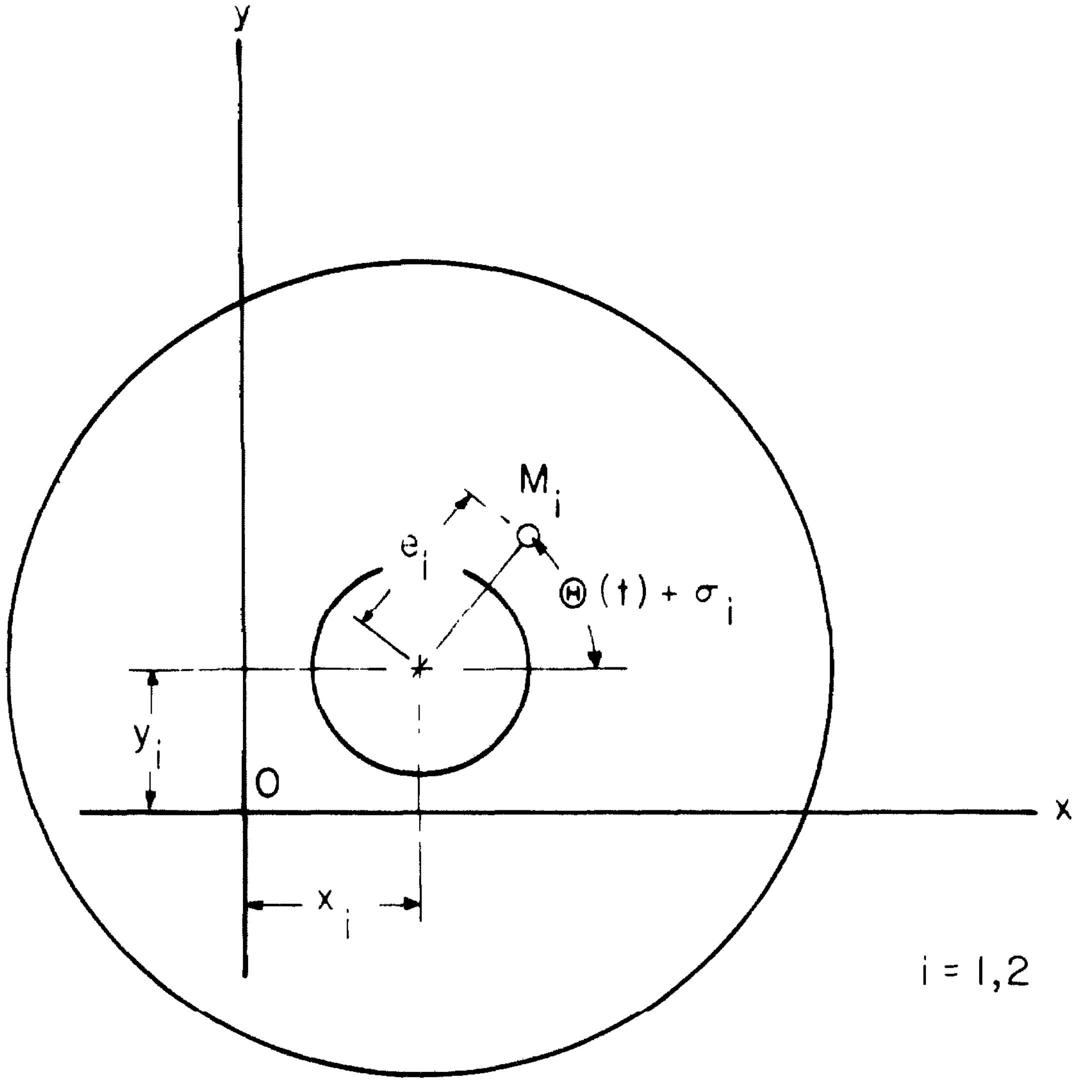


FIG. 6.2

The set (6. 1) in matrix form is

$$[M]\{\ddot{X}\} + [B]\{\dot{X}\} + [K]\{X\} = \{F\} \quad ; \tag{6.2}$$

$$[M] = \left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right] \; , \quad [B] = \left[\begin{array}{c|c} \beta_I + \beta_{II} & -\beta_{II} \\ \hline -\beta_{II} & \beta_I + \beta_{II} \end{array}\right] \; ,$$

$$[K] = \left[\begin{array}{c|c} K_I + K_{II} & -K_{II} \\ \hline -K_{II} & K_I + K_{II} \end{array}\right] \; ,$$

$$\{X\} = \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{Bmatrix} \; , \qquad \{F\} = \begin{Bmatrix} -M_1 e_1 \dfrac{d^2}{dt^2} \cos\left[\Theta(t) + \sigma_1\right] \\ -M_1 e_1 \dfrac{d^2}{dt^2} \sin\left[\Theta(t) + \sigma_1\right] \\ -M_2 e_2 \dfrac{d^2}{dt^2} \cos\left[\Theta(t) + \sigma_2\right] \\ -M_2 e_2 \dfrac{d^2}{dt^2} \sin\left[\Theta(t) + \sigma_2\right] \end{Bmatrix} \; .$$

Expressions to follow will be simplified if it is assumed that

$$M_1 = M_2 = M \; .$$

Premultiplication of (6.2) by $[M]^{-1}$ gives

$$[I]\{\ddot{X}\} + [\bar{B}]\{\dot{X}\} + [\bar{K}]\{X\} = [M]^{-1}\{F\} \quad ; \qquad (6.3)$$

$$[\bar{B}] = [M]^{-1}[B] \quad ,$$

$$[\bar{K}] = [M]^{-1}[K] \quad .$$

It may be shown that the necessary and sufficient condition for classical normal modes(17) prevails, i.e., that

$$[\bar{K}][\bar{B}] = [\bar{B}][\bar{K}] \quad .$$

Equation (6.3) is thus diagonalized by the transformation diagonalizing $[\bar{K}]$.

The eigenvalues of $[\bar{K}]$ are

$$\lambda_1 \; , \; \lambda_2 = K_I / M$$

$$\lambda_3 \; ; \; \lambda_4 = (K_I + 2K_{II}) / M \quad ,$$

corresponding to system natural frequencies:

$$\omega_1 = \sqrt{K_I / M} \quad ,$$

$$\omega_2 = \sqrt{(K_I + 2K_{II}) / M} \quad .$$

System eigenvectors may be obtained directly:

$$\eta^1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} , \qquad \eta^2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} ,$$

$$\eta^3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} , \qquad \eta^4 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} , \qquad (6.4)$$

and the transformation

$$[H] = [\eta^1, \eta^2, \eta^3, \eta^4] \quad ,$$

formed from (6.4), will diagonalize $[\bar{K}]$ and $[\bar{B}]$.

In (6.3) let

$$\{X\} = [H]\{Z\} \quad ,$$

and premultiply by $[H]^T$. The following decoupled set of equations obtains:

$$[I]\{\ddot{Z}\} + \left[{}^{\diagdown}2\zeta\omega_{\diagdown}\right]\left\{\dot{Z}\right\} + \left[{}^{\diagdown}\omega^2_{\diagdown}\right]\left\{Z\right\} = [H]^T[M]^{-1}\{F\} \; . \qquad (6.5)$$

Solutions to (6.5) may be written in the form

$$\{Z\} = \int_{-\infty}^{t} [h(t-t')][H]^T[M]^{-1}\{F(t')\}\,dt' \qquad (6.6)$$

$$[h(t-t')] = \left[\begin{array}{c|c} h_1(t-t') & 0 \\ \hline 0 & h_2(t-t') \end{array}\right] \quad ,$$

$$h_i(t-t') = \frac{e^{-\zeta_i\omega_i(t-t')}\sin m_i\omega_i(t-t')}{m_i\omega_i} \quad , \qquad i = 1, 2 \; .$$

The response, $\{X\}$, may now be written after premultiplying (6.6) by $[H]$:

$$\{X\} = \int_{-\infty}^{t} [H][h(t-t')][H]^T[M]^{-1}\{F(t')\}\,dt' \quad .$$

The elements of the response, $\{X\}$, are:

$$x_1 = \frac{1}{2}\int_{-\infty}^{t} e_1\left[h_1(t-t') + h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\cos\left[\Theta(t') + \sigma_1\right]\right\}$$

$$+ e_2\left[h_1(t-t') - h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\cos\left[\Theta(t') + \sigma_2\right]\right\} dt' ,$$

$$y_1 = \frac{1}{2}\int_{-\infty}^{t} e_1\left[h_1(t-t') + h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\sin\left[\Theta(t') + \sigma_1\right]\right\}$$

$$+ e_2\left[h_1(t-t') - h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\sin\left[\Theta(t') + \sigma_2\right]\right\} dt' ,$$

$$x_2 = \frac{1}{2}\int_{-\infty}^{t} e_1\left[h_1(t-t') - h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\cos\left[\Theta(t') + \sigma_1\right]\right\}$$

$$+ e_2\left[h_1(t-t') + h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\cos\left[\Theta(t') + \sigma_2\right]\right\} dt' ,$$

$$y_2 = \frac{1}{2}\int_{-\infty}^{t} e_1\left[h_1(t-t') - h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\sin\left[\Theta(t') + \sigma_1\right]\right\}$$

$$+ e_2\left[h_1(t-t') + h_2(t-t')\right] \cdot \frac{d^2}{dt'^2}\left\{\sin\left[\Theta(t') + \sigma_2\right]\right\} dt' .$$

It will suffice to treat the typical element x_1. Its integral representation may be recast in the form

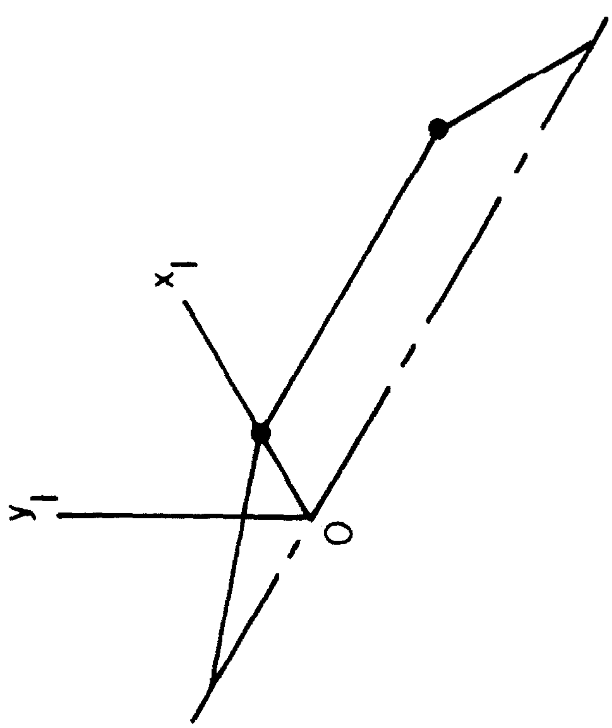


FIG. 6.3a

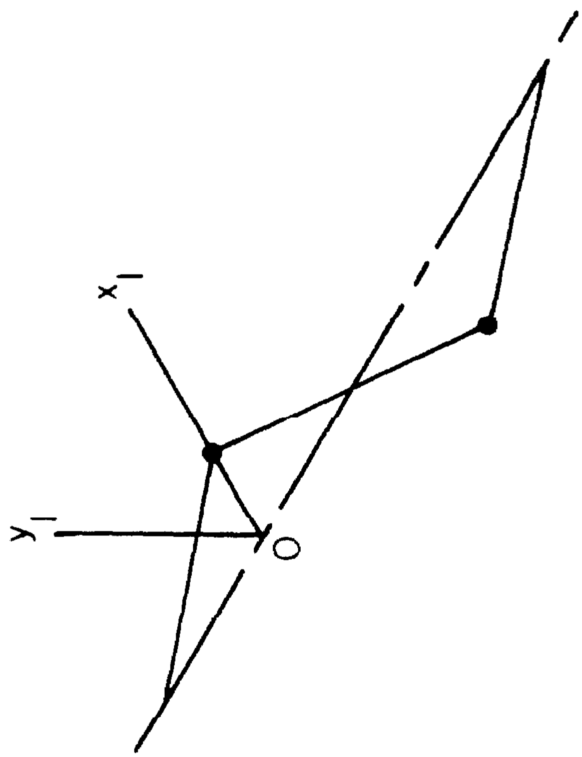


FIG. 6.3b

$$x_1 = \frac{A_1}{2}\int_{-\infty}^{t} h_1(t-t')\,\frac{d^2}{dt'^2}\left\{\cos[\Theta(t')+\psi_1]\right\}dt'$$
$$+\frac{A_2}{2}\int_{-\infty}^{t} h_2(t-t')\,\frac{d^2}{dt'^2}\left\{\cos[\Theta(t')+\psi_2]\right\}dt' \quad ; \tag{6.7}$$

$$A_1 = \left[e_1^2 + e_2^2 + 2e_1e_2\cos(\sigma_1-\sigma_2)\right]^{1/2} \quad ,$$

$$A_2 = \left[e_1^2 + e_2^2 - 2e_1e_2\cos(\sigma_1-\sigma_2)\right]^{1/2} \quad ,$$

$$\psi_1 = \tan^{-1}\left[(e_1\sin\sigma_1 + e_2\sin\sigma_2)/(e_1\cos\sigma_1 + e_2\cos\sigma_2)\right] \quad ,$$

$$\psi_2 = \tan^{-1}\left[(e_1\sin\sigma_2 - e_2\sin\sigma_2)/(e_1\cos\sigma_1 - e_2\cos\sigma_2)\right] \quad .$$

If the mass center of M_1 relative to M_2 is disposed such that $A_1 \sim A_2$, then x_1 will manifest resonant behavior of the same order for excitation frequencies equal to ω_1 and ω_2.

The first mode, associated with ω_1, and with the integral multiplying A_1, is shown in Fig. 6.3a; the second mode, associated with ω_2, and with the integral multiplying A_2, is shown in Fig. 6.3b.

Two integrations by parts in (6.7) lead to the result

$$x_1 = \frac{A_1}{2}\left\{\cos[\Theta(t)+\psi_1] - \frac{\omega_1}{m_1}\int_{-\infty}^{t} e^{-\zeta_1\omega_1(t-t')}\sin[m_1\omega_1(t-t')+\delta_1]\cdot \cos[\Theta(t')+\psi_1]\,dt'\right\}$$

$$+ \frac{A_2}{2}\left\{\cos[\Theta(t)+\psi_2] - \frac{\omega_2}{m_2}\int_{-\infty}^{t} e^{-\zeta_2\omega_2(t-t')}\sin[m_2\omega_2(t-t')+\delta_2]\,.\right.$$

$$\left.\cos[\Theta(t')+\psi_2]\,dt'\right\} \quad ,$$

$$(6.8)$$

$$\delta_1 = \tan^{-1}\left(\frac{2\zeta_1 m_1}{1-2\zeta_1^2}\right) \quad , \qquad \delta_2 = \tan^{-1}\left(\frac{2\zeta_2 m_2}{1-2\zeta_2^2}\right) \quad .$$

After a trigonometric substitution, each of the integrals in (6.8) may be written

$$I_i = \frac{1}{2}\int_{-\infty}^{t} e^{-\zeta_i\omega_i(t-t')}\sin\left[m_i\omega_i(t-t') + \Theta(t') + \delta_i + \psi_i\right]dt'$$

$$+ \frac{1}{2}\int_{-\infty}^{t} e^{-\zeta_i\omega_i(t-t')}\sin\left[m_i\omega_i(t-t') - \Theta(t') + \delta_i - \psi_i\right]dt' \quad , \tag{6.9}$$

$$i = 1, 2 \quad .$$

Trigonometric time dependence may be factored from the integrands of (6.9), and I_i may be written per (2.11) as the product of a time varying modulus and a trigonometric term:

$$I_i = \frac{1}{2}\,C_i(t)\,.\,\sin[\omega_i m_i t + B_i(t)] \quad .$$

The modulus, C_i, will consist of the four integrals

$$\int_{-\infty}^{t} e^{-\zeta_i\omega_i(t-t')}\,\begin{matrix}\sin\\ \cos\end{matrix}\left\{m_i\omega_i t' - \delta_i \pm [\Theta(t') + \psi_i]\right\}dt' \quad . \tag{6.10}$$

If it is required that the excitation frequency, $\Theta'(t)$, sweeps monotonically from $+\infty$ at $t = -\infty$, to 0 at some time t_o in the future, then the instantaneous frequency of the integrands of (6.10):

$$\varphi_{1,3} = m_i\omega_i + \Theta'(t) \quad ,$$

$$\varphi_{2,4} = m_i\omega_i - \Theta'(t)$$

will appear as represented in Fig. 2.10.

The integrals

$$\int_{-\infty}^{t} e^{-\zeta_i\omega_i(t-t')} \begin{matrix}\sin\\ \cos\end{matrix}\left\{ m_i\omega_i t' - \delta_i + [\Theta(t') + \psi_i] \right\} dt' \quad , \qquad (6.11)$$

having integrand frequency

$$\varphi_{1,3} = m_i\omega_i + \Theta'(t) \quad ,$$

will attain their maximum value at the end of the interval of interest, $t = t_o$. These maxima will be bounded by expression (2.20), modified slightly to suit the present situation, and will be $0\left[1/\omega_i\right]$. These integrals will be still smaller for $t < t_o$ as indicated below.

The integrals

$$\int_{-\infty}^{t} e^{-\zeta_i\omega_i(t-t')} \begin{matrix}\sin\\ \cos\end{matrix}\left\{ \omega_i m_i t' - \delta_i - [\Theta(t') + \psi_i] \right\} dt' \quad , \qquad (6.12)$$

on the other hand, have a stationary phase point when

$$\varphi_{2,4} = 0 \quad , \quad \Theta'(t_i) = m_i\omega_i \quad ,$$

and will be $0\left[1/\sqrt{\Theta''(t_i)}\right]$ shortly after $t = t_i$. For cases of interest, the integrals (6.12) will be much larger than the integrals (6.11) near $t = t_i$.

The modulus, C_i, may then be approximated near the i^{th} mode resonance by the square root of the sum of the squares of the integrals (6.12) which, in turn, may be approximated per Chapter II by R_i, equal to the square root of the sum of the squares of expressions similar to (2.21a) and (2.21b). The modulus of I_i near t_i may then be estimated by

$$I_i(t_i)_{mod} \cong \frac{1}{2} R_i \quad . \tag{6.13}$$

The frequency of the prescribed sweeping excitation will first pass through ω_2 at $t = t_2$ and then through ω_1 at $t = t_1 > t_2$. The estimated second mode contribution to the response amplitude, $x_{1_{mod}}$, for excitation frequency near ω_2 may be written using (6.8) and (6.13),

$$\frac{A_2 \omega_2 R_2}{4m_2} \quad .$$

The integrals (6.11) associated with the first mode will be $0\left[\frac{1}{\omega_1+\omega_2}\right]$ for $t \cong t_2$ and the remaining integrals, (6.12), corresponding to ω_1 will be bounded by expression (2.20), again appropriately modified; their bound, r_1, will be $0\left[\frac{1}{\omega_1 - \omega_2}\right] < R_1$.

The response amplitude for $t \cong t_2$ in terms of these Chapter II expressions then will be approximated by

$$x_{1_{mod}} \cong \frac{A_2 \omega_2 R_2}{4m_2} + \frac{A_1 \omega_1 r_1}{4m_1} \quad .$$

Estimating the response amplitude for times shortly after the excitation has a frequency equal to ω_1, $t > t_1$ presents some difficulty. To be sure, the resonant mode contribution, coming now from the first mode, will be estimated as before by (6.13), and the integrals

(6.11) associated with the second mode will be $0\left[\frac{1}{\omega_1+\omega_2}\right]$. The integrals (6.12) corresponding to ω_2 will, however, no longer be bounded by an expression similar to the r_1 used above, since the significant portion of the integrals comes not only from the end-point of the interval, t_1, but also from the neighborhood of the included stationary phase point t_2--solving $\Theta'(t_2) = m_2\omega_2$. The latter contribution will, like any other transient behavior, be diminished exponentially in importance as a function of the time difference $(t_1 - t_2)$, the frequency ω_2, and the coefficient of damping ζ_2, whereas the former contribution, like r_1, will be $0\left[\frac{1}{\omega_1 - \omega_2}\right] < R_2$.

It will be assumed in view of the above that the modulus of I_2 near t_1 will be approximated by

$$I_2(t_1)_{mod} \cong \frac{1}{2}\left[e^{-\zeta_2\omega_2(t_1-t_2)} R_2 + r_2\right] \quad ,$$

in which r_2, computed as before, estimates or bounds the integral I_2 over (t_2, t_1) excluding stationary phase behavior.

The response amplitude for $t \cong t_1$ may then be estimated by

$$x_{1_{mod}} \cong \frac{A_1\omega_1 R_1}{4m_1} + \frac{A_2\omega_2}{4m_2}\left[e^{-\zeta_2\omega_2(t_1-t_2)} R_2 + r_2\right] \ .$$

The peak corresponding to the first mode will be influenced by the pass through the second mode resonant frequency to an extent determined by the factor

$$\frac{A_2\omega_2}{4m_2} e^{-\zeta_2\omega_2(t_1-t_2)} \cdot R_2 \quad .$$

This factor has no counterpart in the expression for steady-state response amplitude as a function of excitation frequency. Its size will depend on the level of the response peak corresponding to ω_2, the separation of resonant frequencies, sweep rate, second mode damping ratio, and second mode natural frequency.

It should be added parenthetically that, when the excitation frequency sweeps linearly in time:

$$\Theta(t) = \pi h t^2 + Pt + Q \ , \tag{6.14}$$

the integrals (6.7) will be quite similar to the solution integral obtained for the unbalanced mass exciter problem of Chapter IV. The solution to the present problem, given (6.14), may then be written in terms of tabulated functions.

The value of such an exercise may be questioned, however, since rotating systems influenced by constant angular acceleration are rarely found in practice. Macchia[18] points out, though, that for small values of the radius of imbalance, e, the solution to the constant angular acceleration case approaches that of the more important constant applied torque case.

The outlined estimating (and bounding) method may be extended to those systems most appropriately approximated by certain simple continuous models.

Consider, for instance, a clamped, uniformly stretched membrane S bounded by the closed curve Γ, and forced by an excitation $F(x_1, x_2) \sin G(t)$. The equation of motion for the system will be

$$T\nabla_2^2 u + F(x_1, x_2) \sin G(t) = \rho(x_1, x_2) \frac{\partial^2 u}{\partial t^2} \quad , \tag{6.15}$$

$$u_\Gamma = 0 \quad ,$$

where

T	the tension
∇_2^2	the Laplacian operator in the planar coordinants x_1, x_2
u	the membrane deflection in a direction perpendicular to the plane of x_1, x_2
$\rho(x_1, x_2)$	the mass per unit area

The solution to the homogeneous equation:

$$T\nabla_2^2 u = \rho(x_1, x_2) \frac{\partial^2 u}{dt^2} \quad ,$$

may be obtained by the <u>Ansatz</u>: $u = X(x_1, x_2) e^{j\omega t}$.

The separation constant ω^2 will have an infinity of values, taken here as distinct, ω_{mn}^2, $m, n = 1, 2, 3, \ldots$, corresponding to an infinity of solutions X_{mn} to

$$T\nabla_2^2 X_{mn} + \omega_{mn}^2 \rho(x_1, x_2) X_{mn} = 0 \quad , \tag{6.16}$$

satisfying the boundary condition

$$(X_{mn})_\Gamma = 0 \quad .$$

It may be shown that, when properly normalized, the X_{mn} will have the orthonormality property

$$\int_S \rho(x_1, x_2) X_{mn} X_{k\ell} \, ds = \delta_{mk} \delta_{n\ell} \quad .$$

If, in the inhomogeneous case, the solution

$$u = \sum_{m,n} \xi_{mn}(t)\, X_{mn}(x_1, x_2) \quad ,$$

is assumed, (6.15) becomes

$$T \sum_{m,n} \xi_{mn} \nabla_2^2 X_{mn} + F(x_1, x_2) \sin G(t) = \rho(x_1, x_2) \sum_{m,n} \ddot{\xi}_{mn} X_{mn} \tag{6.17}$$

In (6.17) $T\nabla_2^2 X_{mn}$ may be replaced by its equivalent expression, $-\omega_{mn}^2 \rho(x_1, x_2) X_{mn}$, from (6.16). Premultiplication of (6.17) by $X_{k\ell}$, and integration over S give the equation for $\xi_{k\ell}$:

$$\ddot{\xi}_{k\ell} + \omega_{k\ell}^2 \xi_{k\ell} = a_{k\ell} \sin G(t) \quad ;$$

$$a_{k\ell} = \int_S X_{k\ell}\, F(x_1, x_2)\, ds \quad .$$

The steady-state solution for $\xi_{k\ell}$:

$$\xi_{k\ell} = a_{k\ell} \int_{-\infty}^{t} \frac{\sin \omega_{k\ell}(t-t')}{\omega_{k\ell}} \cdot \sin G(t')\, dt' \quad ,$$

provides an expression for the steady-state response:

$$u = \sum_{k,\ell}^{\infty} a_{k\ell} \left[\int_{-\infty}^{t} \frac{\sin \omega_{k\ell}(t-t')}{\omega_{k\ell}} \sin G(t')\, dt' \right] X_{k\ell} \quad ,$$

which may be treated by a mode-by-mode application of the method of Chapter II.

VII. SUMMARY AND CONCLUSIONS

SUMMARY

It was shown that, although formal solutions for the response of viscous damped, linear systems to general sweeping excitations can often be written, closed-form solutions are possible only in certain cases. These and other cases, for which potentially useful series expressions may be developed, were discussed briefly in Chapter IV, and in Chapter VI.

The failure of exact analysis for the general sweeping excitation suggested analysis by approximate techniques, in addition to direct solution for several sufficiently different excitations as to permit intelligent inferences concerning the response of known systems subject to any sweeping excitation.

Approximate analysis in the present study included the following:

1. A method developed in Chapter II for deriving a hard, time dependent bound on the response of an undamped single degree of freedom system to sweeping excitations was used to estimate the response of a damped single degree of freedom system. It was shown in Chapter VI that the method could also be applied to estimate the response of classically damped multi-degree of freedom systems, and simple continuous systems.

2. Two perturbation schemes were derived in Chapter III and were used to formulate expressions describing response envelope behavior for a single degree of freedom system influenced by the general slowly sweeping excitation. The experimental aspect of the problem was considered, and expressions were formulated for estimating system damping and natural frequency using quantities measured from a record of system response to slowly sweeping excitation.

3. In Appendix B, the methods of stationary phase and saddle-point integration were applied to estimate the maximum response of damped and undamped systems. Failure of the methods was discussed and ascribed to the fact that neither took into account the important end-point contributions of the integrals forming the expression for system response.

To complement the approximate analysis, a study was performed to investigate--by analog means--the response of viscous damped single degree of freedom systems to excitations sweeping linearly and exponentially in time. Experimental curves were presented providing information concerning response behavior near system resonance. Curves were given to provide a means whereby sweeping response records may be employed to determine system damping.

CONCLUSIONS

The discussion of Chapter II indicates that the response of an undamped system will be bounded provided that the excitation frequency equals the system natural frequency (frequencies) for a finite total time.

Accordingly, sweeping excitations producing an unbounded response would include those in which the frequency of excitation approaches a system natural frequency asymptotically, or cycles an infinite number of times through a system natural frequency. Actual response in the latter case, as has been shown in Chapter IV, will be bounded except for specific values of the sweep parameters.

The criterion applied to undamped continuous systems suggests also that response will be unbounded when the system is influenced by an excitation sweeping from zero frequency to infinite frequency in an infinite time.

A common contention has long been that the maximum response of a damped system to sweeping excitation will be bounded by the system's steady-state response maximum. It was shown in Chapter III that response maxima exceeding steady-state values may be obtained by permitting the excitation frequency to sweep slowly down through resonance. These excesses will be very small, however, for lightly damped systems and, in addition, will occur over a very limited range of sweep rate.

The perturbation analysis and the experimental study of Chapter V indicate that, for useful damping ratios ($\zeta < .10$), and for sweep rates ranging to the fastest generally attained, the measured response behavior--resonance peak amplitude, center frequency of the peak, and width of the peak--for identically damped systems, excited by linearly and exponentially sweeping excitations, will be identical, for all practical purposes, provided that the relevant sweep factors are equal.

The dissimilarity in the time dependence of exponentially, and linearly sweeping excitations constrasts with the similarity of the above results, suggesting the insensitivity, again for practical purposes, of the system to the manner in which the excitation frequency sweeps through resonance. One is led to conclude that two identically damped systems, excited by smoothly sweeping but otherwise different excitations, will manifest identical resonance peak amplitude, center frequency of peak, and width of peak if

$$\frac{1}{N_1^2} \cdot \frac{1}{2\pi} \frac{d\Omega_1}{dt}\bigg|_{t_{r_1}} = \frac{1}{N_2^2} \cdot \frac{1}{2\pi} \frac{d\Omega_2}{dt}\bigg|_{t_{r_2}}, \qquad (7.1)$$

in which N_i is the natural frequency of the system in cps, $\frac{1}{2\pi}\frac{d\Omega_i}{dt}$ is the real time derivative of the excitation frequency in cps/s, and t_{r_i} is the time of resonance. It will be required here and in the following that $\frac{d\Omega}{dt} \neq 0$.

It may by concluded further that, since system response near resonance appears to depend primarily upon the first two terms of the Taylor series expansion of the excitation frequency about system resonance for a wide range of sweep rate and damping, analytical results for the linear sweep may be applied to obtain response characteristics of systems subject to any smoothly varying excitation.

Analytical and experimental work suggest, in addition, the importance of the factor

$$\frac{1}{N^2\zeta^2} \cdot \frac{1}{2\pi} \frac{d\Omega}{dt}\bigg|_{t_r} , \tag{7.2}$$

in relation to the resonance peak amplitude and peak width. Experimental results predict that, within the ability to measure, resonance peak amplitude and peak width are functions only of the parameter (7.2) for both linearly and exponentially sweeping excitations.

It is then concluded that the above resonant response behavior of a viscous damped single degree of freedom system influenced by a smoothly sweeping excitation will be the same as that of any other viscous damped single degree of freedom system influenced by any other smoothly sweeping excitation provided that the parameters (7.2) are equal. Functional dependence of the resonance peak amplitude on (7.2) will be given in Fig. 5.3; dependence of the peak width on (7.2) will be given in Figs. 5.5a, b.

The conclusion may be extended, per Chapter VI, to the resonant behavior of classically damped multi-degree of freedom systems provided that system resonances are widely separated.

It should be added here that for damping ratios less than ten percent of critical, the resonance peak amplitude will depend on the absolute value of the time rate of change of excitation frequency. For all practical purposes, sweeping up through resonance will give the same peak amplitude as will sweeping down through resonance.

The similarity of the estimates developed per the analysis of Chapter II for the peak center frequency of the undamped system influenced by linearly and exponentially sweeping excitations leads to an expression for the peak center frequency, N_{max}, for the undamped system influenced by the general sweeping excitation:

$$N_{max} = N + .34 \operatorname{sgn}\left(\left. \frac{d\Omega}{dt} \right|_{t_r} \right) \cdot \sqrt{\left| \left. \frac{d\Omega}{dt} \right|_{t_r} \right|} \qquad (7.3)$$

Experimental results indicate that (7.3) may be used to overestimate the peak shift, $|N - N_{max}|$, for damped systems.

The phenomenon of beating was treated only briefly in the preceding chapters. Such experimental evidence as was gathered leads to the conclusion, however, that beating will appear in response records when the parameter (7.2) exceeds a value between three and four. The level of the first peak in the beat pattern at first appearance will be the highest attained for any practical sweep rate--about thirty-five percent of the steady-state response maximum. Its value relative to that of the resonance peak will generally increase as the sweep rate increases but will not exceed $y_{max}/\sqrt{2}$ for $\zeta > .01$.

The question of what excitation sweep rates qualify as slow may be answered on the basis of the foregoing work. The effect of sweep rate upon the response of a system depends not only upon the time derivative of excitation frequency, but also upon system natural frequency and damping. It should be required for slow sweeping that

$$\frac{1}{N^2\zeta^2} \cdot \frac{1}{2\pi} \left.\frac{d\Omega}{dt}\right|_{t_r} < 1 \qquad (7.4)$$

Sweeping excitations satisfying (7.4) will produce amplitude of response records approximating steady-state response amplitude-excitation frequency curves to good order. Corrections to these records may be made by application of the perturbation methods of Chapter III.

For multi-degree of freedom and continuous systems it should be required, additionally, that

$$t_{r_{i+1}} - t_{r_i} >> \frac{1}{2\pi\zeta_i N_i} \quad ,$$

where t_{r_i} is the time at which the excitation frequency equals the i^{th} resonant frequency, N_i, to be excited during a particular test, and where ζ_i is the modal damping associated with the i^{th} mode to be excited. The time difference, $t_{r_{i+1}} - t_{r_i}$, will depend upon modal separation and sweep rate.

REMARKS

Several interesting possibilities for sweep testing physical structures are suggested by the included work:

1. Modal damping determination in the field has always presented difficulties. Suppose, however, that an experimenter had the means to sweep exponentially at various speeds through the resonance for which the modal damping were desired. He could then vary the rate of sweeping until beats appeared and, if he accepted the linear viscous damped model as representative of his system, solve the

equation based on Fig. 5.8:

$$\frac{1}{N^2\zeta^2} \cdot \frac{1}{2\pi}\frac{d\Omega}{dt}\bigg|_{t_r} = 3.46 \quad ,$$

for ζ, using the quantities N and $\left.\frac{d\Omega}{dt}\right|_{t_r}$ which may, in general, be determined to good order.

2. Sweeping excitations other than those described in the foregoing might be of potential use in the laboratory. Suppose, for instance, that the experimenter desired to determine modal damping and steady-state peak response of a system directly by application of constant corrections multiplying peak amplitudes and peak widths measured from the record of system response to a rapid--in the sense of (7.4)--sweep test. Previous work indicates that this would require, for the i^{th} mode, that

$$\frac{1}{N_i^2\zeta_i^2} \cdot \frac{1}{2\pi}\frac{d\Omega}{dt}\bigg|_{t_{r_i}} = \text{constant} \qquad i = 1, 2, 3, \ldots \quad . \tag{7.5}$$

It will be possible to satisfy (7.5) with an excitation having a frequency defined by

$$\int \frac{d\Omega}{\zeta^2(\Omega)\cdot\Omega^2} = Ct + D$$

The proper sweeping excitation thus requires knowledge of the law governing the modal damping of the system to be tested.

The above suggests an iterative process in which the experimenter assumes a damping law, carries out the test, and tries the validity of his assumption by applying

$$\frac{N_m \zeta_m}{N_n \zeta_n} \cong \frac{W_m}{W_n} \quad ,$$

where W_m is the width of the m^{th} peak measured between the half-power points and expressed in frequency, and where ζ_m is the m^{th} modal damping computed from the assumed damping law. If the approximate equality holds, the experimenter has guessed the applicable damping law and may apply a constant correction for the chosen value of (7.5) from Fig. 5.4 to all the measured peaks for the purpose of determining steady-state peak response.

APPENDIX A

AN ELECTRONIC DEVICE FOR PRODUCING LOW FREQUENCY SWEEPING SINUSOIDAL EXCITATIONS IN THE LABORATORY

The device to be described was designed and built by the author and Mr. A. N. Schmitt, then a technician in the Vibration Laboratory at the California Institute of Technology.

A prototype model was completed on October 9th, 1963, and a patent disclosure form was filed with the Institute on November 6th, 1963. At the time, no such device was commercially available.

The sweeping function generator consists of a modified commercial function generator, the Hewlett-Packard 202A, and associated circuitry. The input to the device may be any low frequency voltage varying between zero and two volts peak. The output will be an oscillating signal with a frequency proportional to the instantaneous level of the input voltage. The relation between input voltage and output frequency is linear to within ± one percent over the range $10 \leq f \leq 110$ cps. The output, for a fixed input voltage, will be sinusoidal with less than one percent total harmonic distortion, and its amplitude may be varied from zero to thirty volts peak-to-peak into a load of 3000 ohms or greater.

The block diagram, Fig. A. 1, indicates the essential components of the Hewlett-Packard function generator. The generator synthesizes its sine-wave output from a square-wave produced by a bi-stable circuit of the Eccles-Jordan type. In its first stable mode of operation, the bi-stable circuit produces a positive D. C. signal of fixed level. This signal is attenuated by the frequency controlling potentiometer and passes to the linear integrator through R_a. The linear integrator produces a ramp signal with a slope proportional to the level of the input. The instantaneous level of the linear integrator output is compared to a reference signal by a "Multiar" voltage comparitor. When these two signals are equal, the "Multiar" issues a pulse switching the bi-stable circuit to its alternate stable mode in which a negative D. C. voltage of fixed level is produced. During the course of the process described above, the input to the sine-synthesizing circuit will be a positive-going ramp. The duration of the foregoing process will be determined by the absolute value of the D. C. input to the linear integrator.

The process will be similar for the alternate stable mode of the bi-stable circuit except that the input to the sine-synthesizing circuit will be a negative-going ramp. Switching back to the first stable mode will reinitiate the cycle and continuous repetition will produce a triangular-wave input to the sine-synthesizer. The latter is a nonlinear circuit which presents the input signal with an apparent resistance that is related to input voltage level.

A more thorough description of circuit operation may be found in the operating and service manual for this instrument[19].

The modification to the circuit described above, permitting frequency modulation of the output signal, provides an additional current path at point "A" in Fig. A. 1. An increased current flow in R_a due to this additional path causes an increased voltage drop across R_a and the signal presented to the integrator is attenuated accordingly. In the circuit illustrated in Fig. A. 2, connected at point "A" in Fig. A. 1, the tube and transistor are arranged to provide a current flow that is proportional to the applied control voltage, E_c. Since the frequency of the output of the function generator is linearly related to the voltage level of the input to the integrator, it will thus be linearly related to the control voltage, E_c. The associated circuitry illustrated in Fig. A. 3 simply provides a means for producing the required floating voltage, E_c, proportional to the input voltage, E_s. The voltage E_s is used to modulate the amplitude of a 5000 cps square-wave produced by the T-104 free-running multivibrator. This modulated A. C. signal is applied to the primary of an isolation transformer. The output of the transformer is full-wave rectified, and filtered to produce E_c. Capacitances in this circuit have been kept small to produce a small overall time constant.

The value of the speed-up capacitor C_ρ in Fig. A. 3 is chosen to provide the cleanest square-wave output from the D. C. amplifier. The values of R_s and R_s' depend upon the impedances of the D. C. amplifier and the volt-meter. For an Alinco 516-Al D. C. amplifier with 51-1 attenuator, R_s is 300K. For a 1000 Ω/volt voltmeter, R_s' is 1.5K.

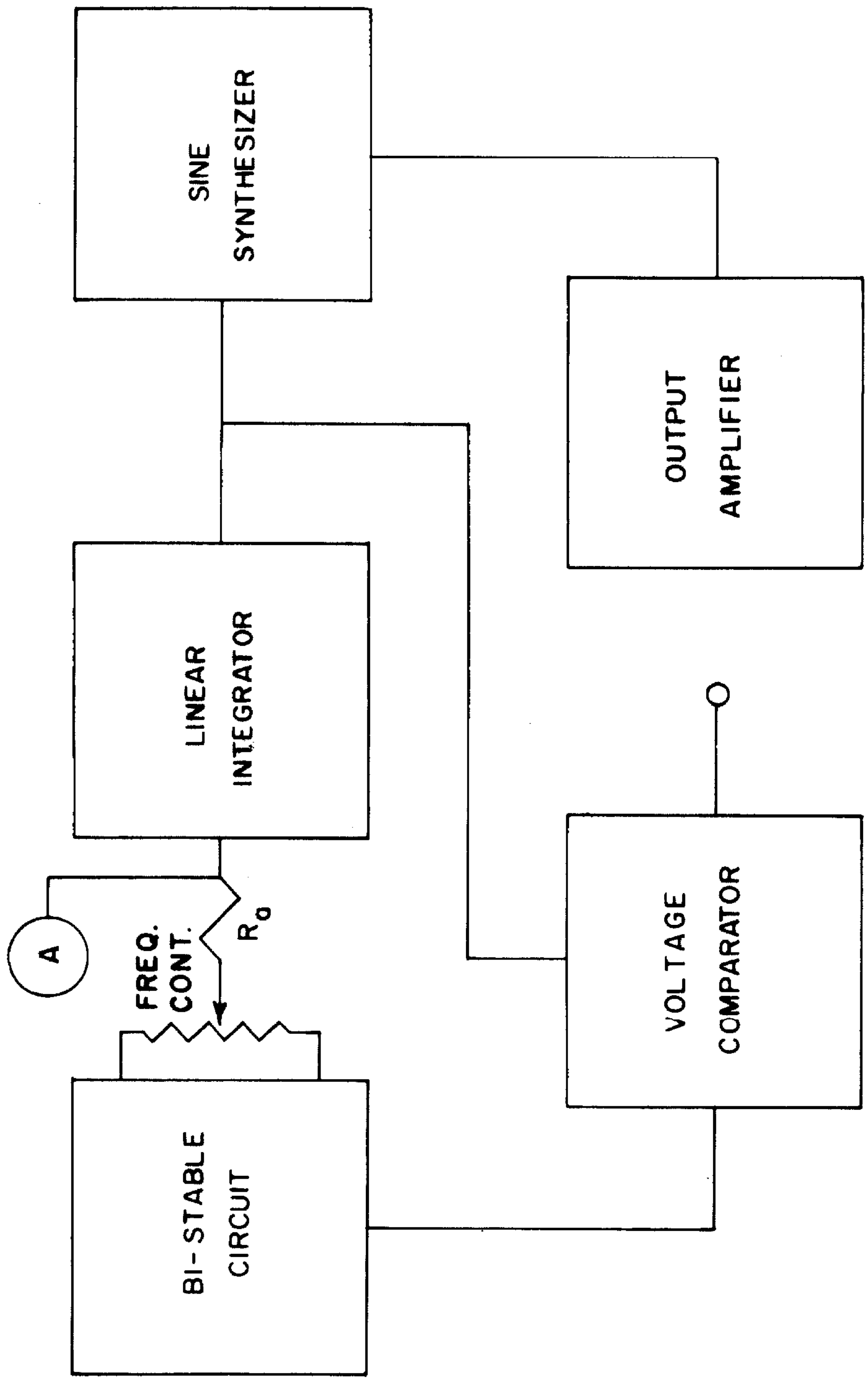

SINE SYNTHESIZER
OUTPUT AMPLIFIER
LINEAR INTEGRATOR
VOLTAGE COMPARATOR
BI-STABLE CIRCUIT
A
FREQ. CONT.
R_a

FIG. A.1

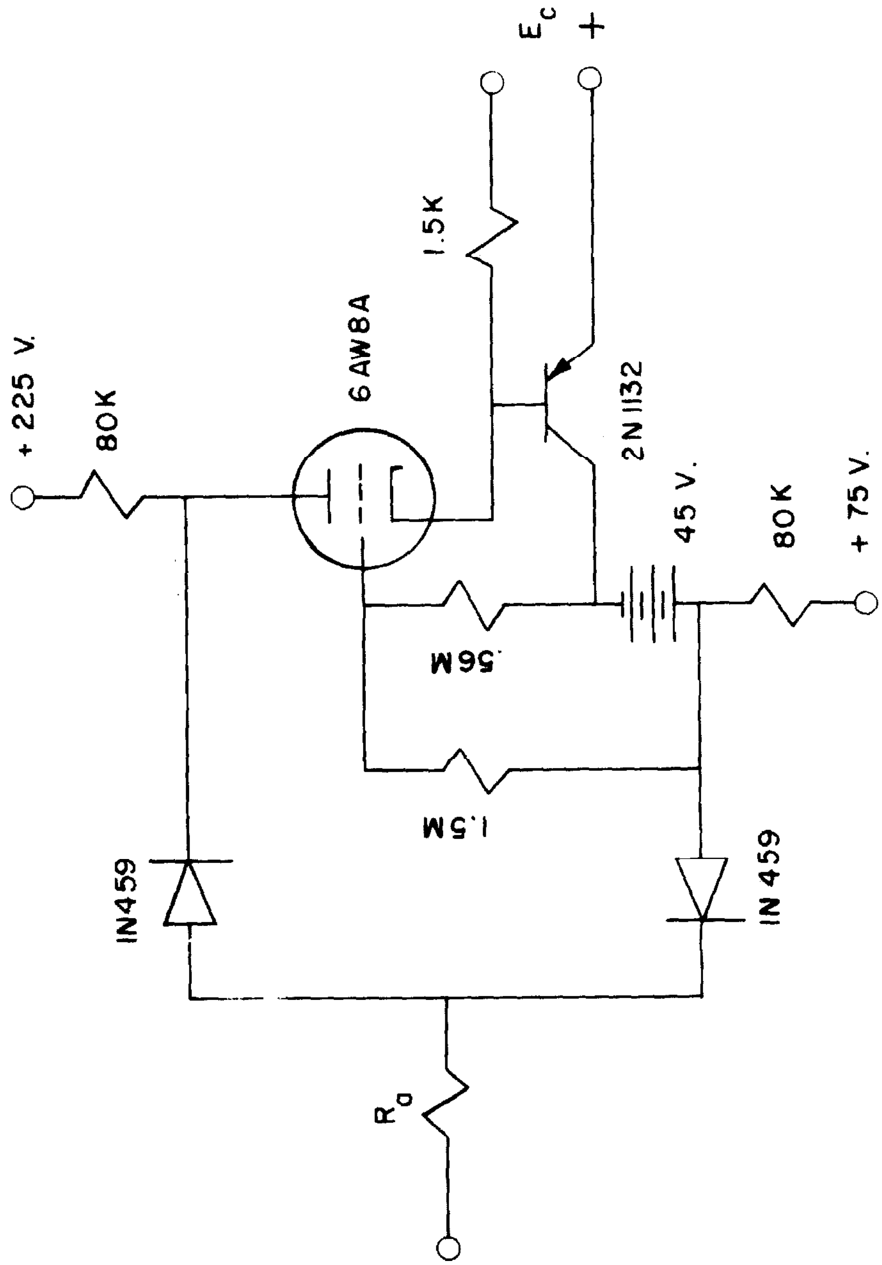
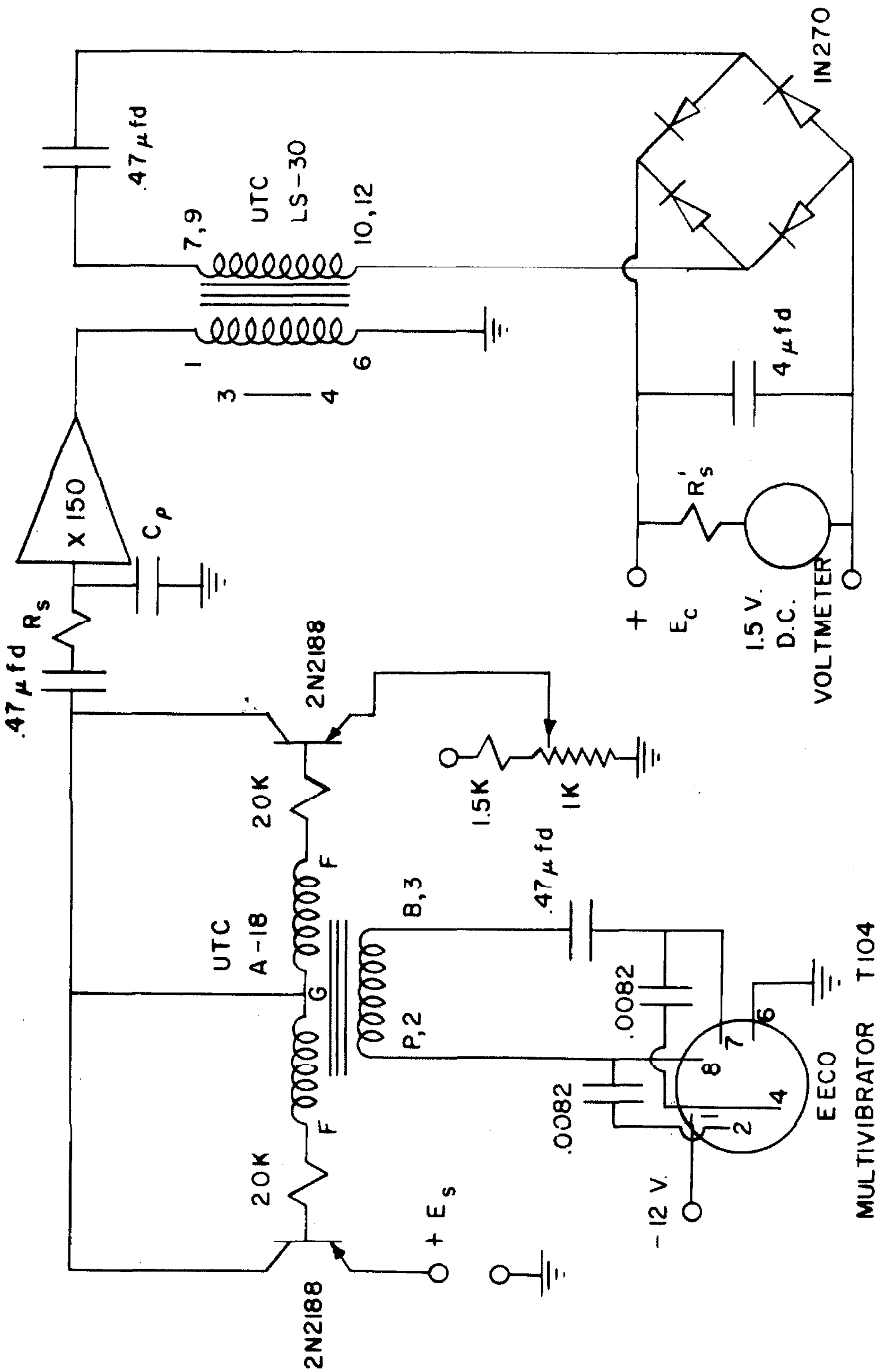

+ 225 V.
80K
6 AW8A
1.5K
E_c
+
2N1132
45 V.
80K
+ 75 V.
.56M
1.5M
IN459
IN 459
R_a

FIG. A.2



.47 μfd
UTC
LS-30
7,9
10,12
1
6
3
4
IN270
X 150
C_p
R_s
.47 μfd
4 μfd
R_s'
+
E_c
1.5 V.
D.C.
VOLTMETER
2N2188
1.5K
1K
20K
UTC
A-18
G
F
B,3
P,2
.47 μfd
.0082
.0082
EECO
MULTIVIBRATOR T104
− 12 V.
+ E_s
2N2188

FIG. A.3

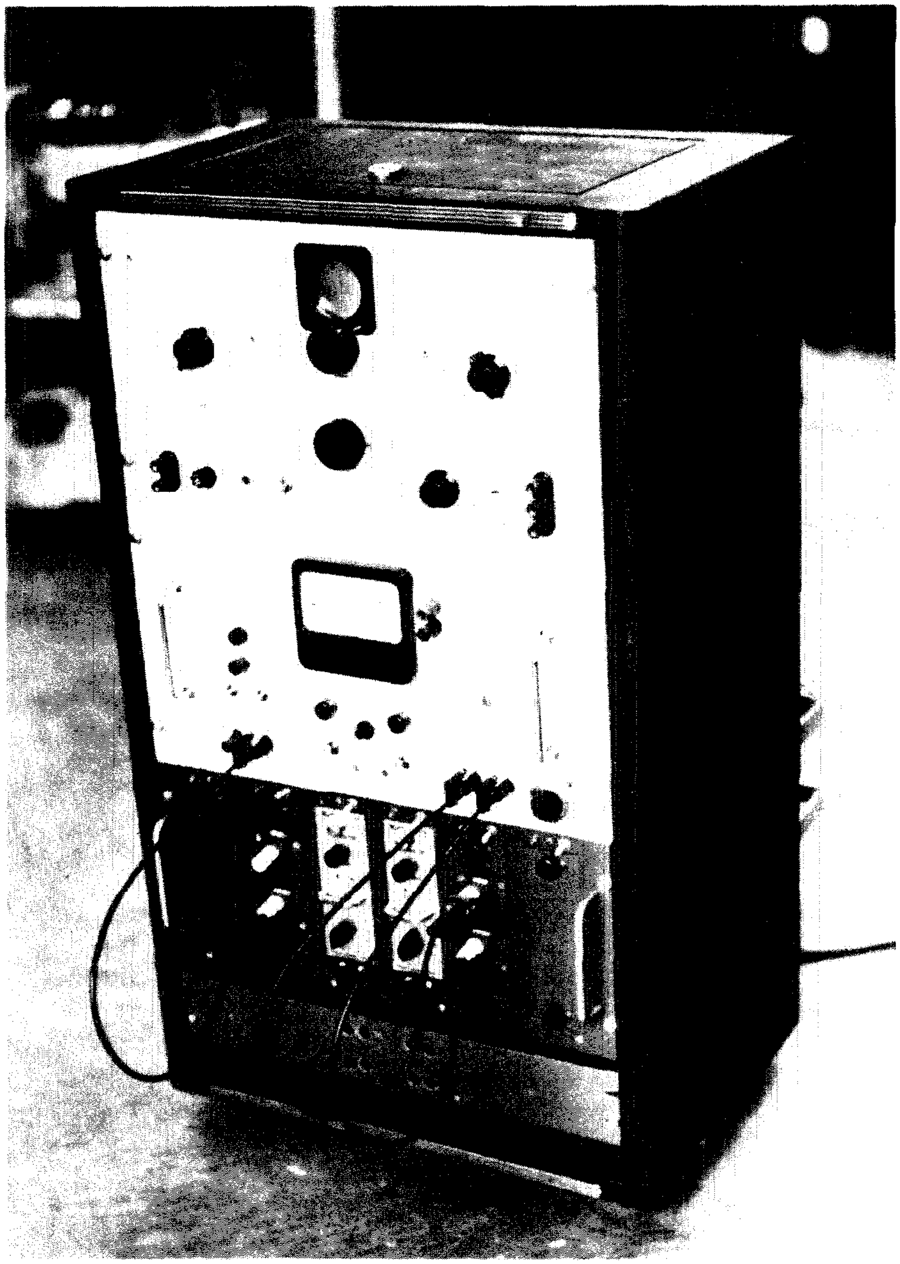


FIG. A.4

The sweeping function generator is illustrated in Fig. A.4.

A device for producing the input frequency control voltage, E_s, was also designed and constructed. Its operating principle is similar to that of the spectrum analyzer described by Caughey, Hudson and Powell[20] insofar as the required frequency behavior is plotted radially as a function of θ on a disc. The disc is rotated about a vertical axis, and the radial displacement, proportional to E_s, is read by an optical displacement follower. The filtered, amplified output voltage of the displacement follower is supplied to the sweeping function generator described above.

The block diagram, Fig. A.5, illustrates the components of this device.

The circuit diagram of the single revolution passing device is given in Fig. A.6. The cam-operated micro-switches, "B", in Fig. A.6 may be seen in the photograph, Fig. A.7. This circuit is designed to pass, on command, the output of the optical displacement follower for one revolution of the disc. At other times the circuit passes an adjustable base voltage to the sweep function generator.

Rotational speed of the disc may be controlled over the range 2 rpm $< N <$ 200 rpm to provide a one parameter family of E_s for a given record.

Records are drawn twice size and photographically reduced. The resulting positive transparencies are backed up by a polished, chrome-plated disc to present the optical displacement follower with a high

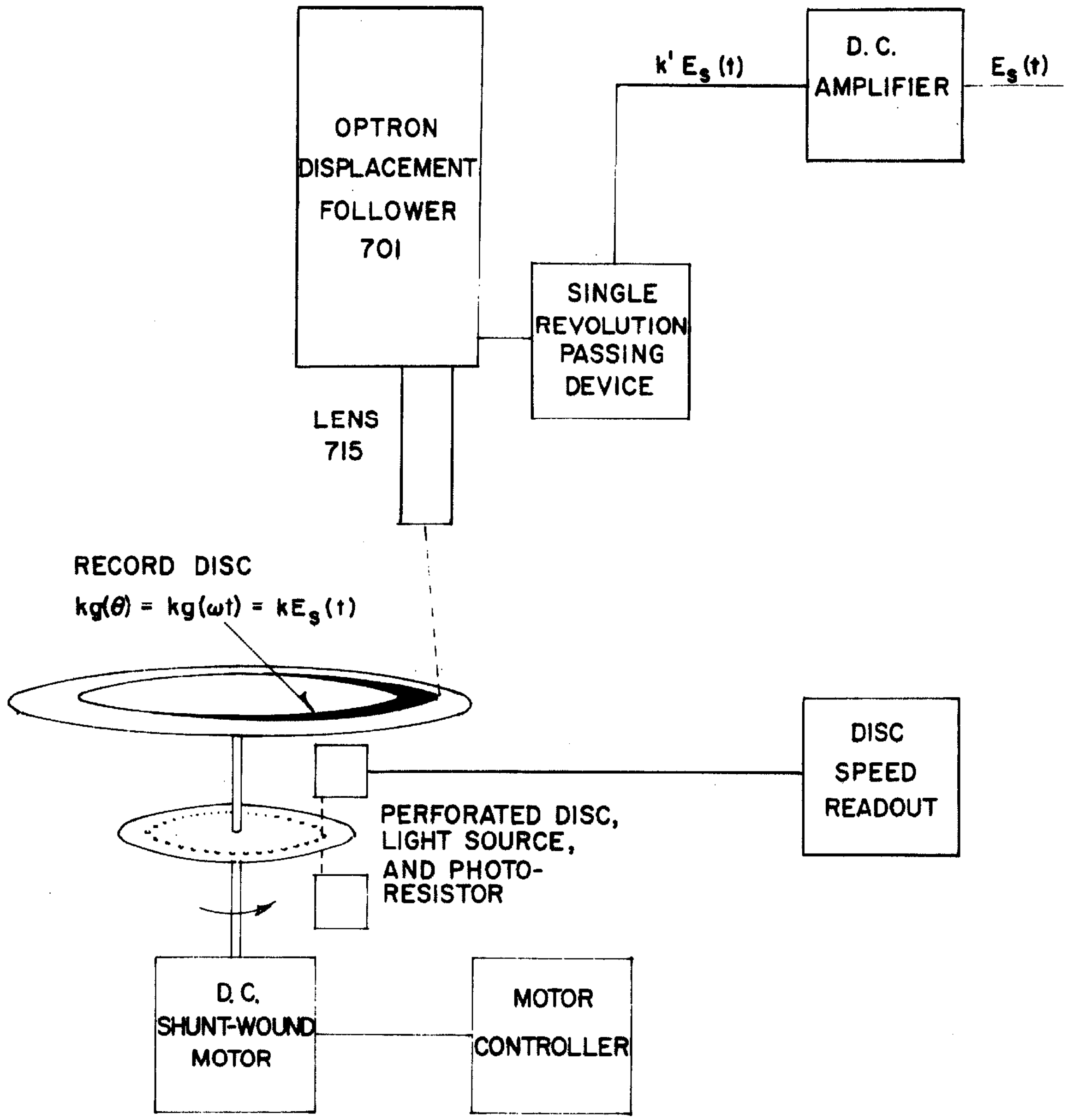

OPTRON
DISPLACEMENT
FOLLOWER
701
k' E_s (t)
D. C.
AMPLIFIER
E_s (t)
SINGLE
REVOLUTION
PASSING
DEVICE
LENS
715
RECORD DISC
$kg(\theta) = kg(\omega t) = kE_s(t)$
DISC
SPEED
READOUT
PERFORATED DISC,
LIGHT SOURCE,
AND PHOTO-
RESISTOR
D. C.
SHUNT-WOUND
MOTOR
MOTOR
CONTROLLER

FIG. A:5

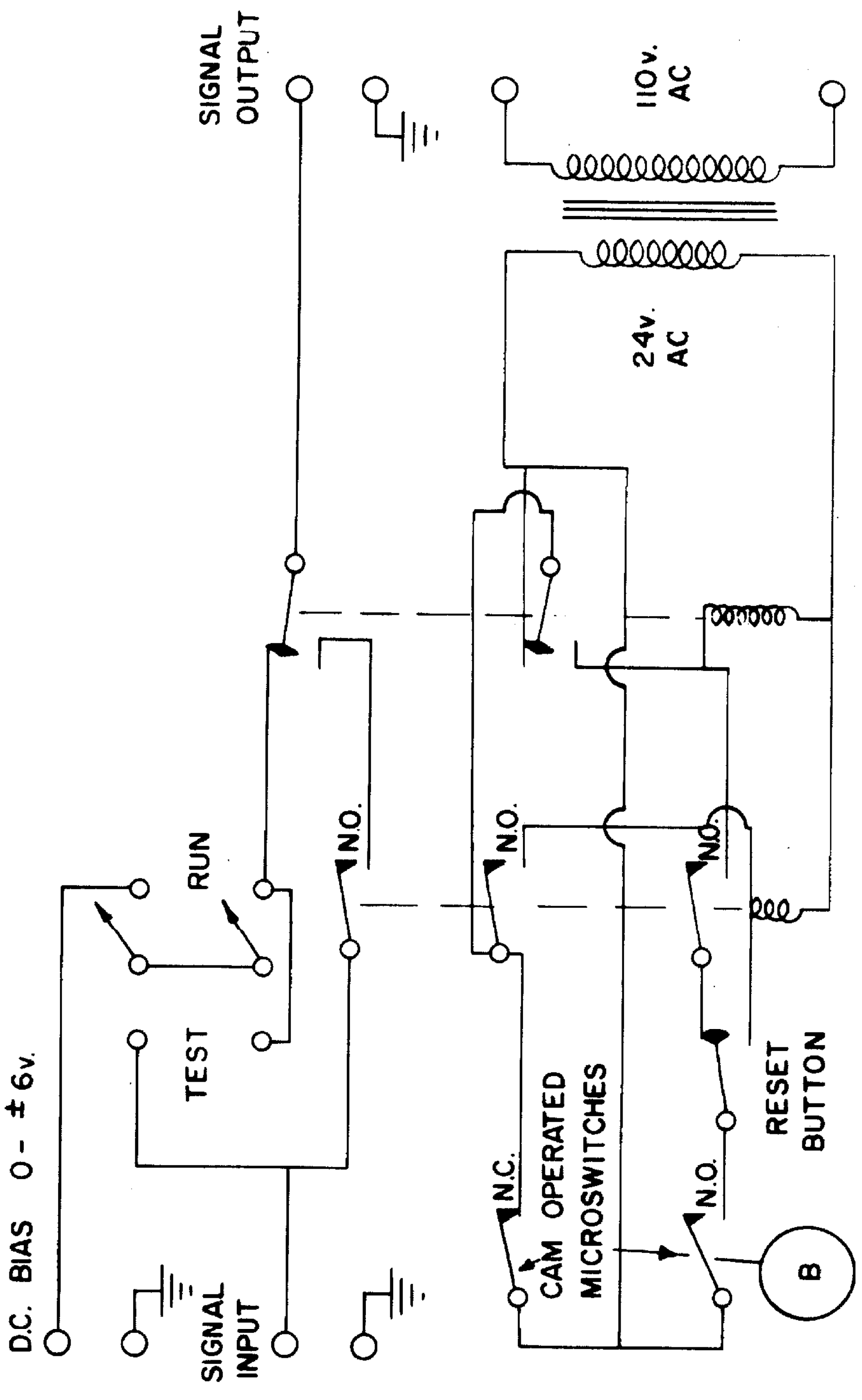

SIGNAL OUTPUT
110v. AC
24v. AC
RUN
TEST
N.O.
N.O.
N.O.
N.C.
N.O.
D.C. BIAS 0- ± 6v.
SIGNAL INPUT
CAM OPERATED MICROSWITCHES
RESET BUTTON
B

FIG. A.6

quality target. The back-up disc in Fig. A.7 is vacuum metalized Lucite. It proved to be unsuitable due to surface dust inclusions, and susceptivity to scratches.

This optical device for producing input frequency control voltages was not used in the experiments discussed in Chapter V.

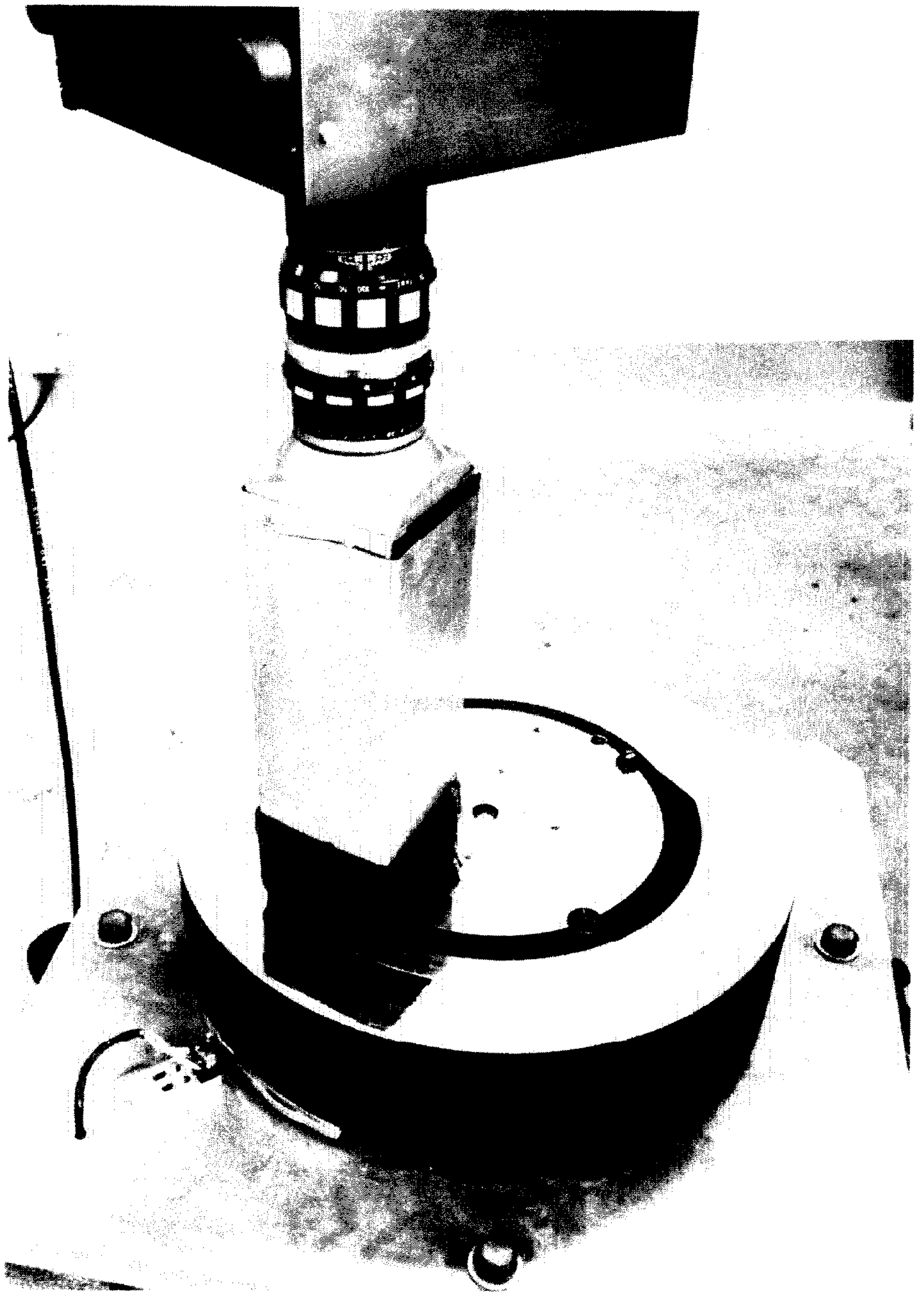


FIG. A.7

APPENDIX B

APPROXIMATE ANALYSIS OF SYSTEM RESPONSE TO SWEEPING EXCITATIONS BY THE METHODS OF STATIONARY PHASE AND SADDLE-POINT INTEGRATION

It was implied in Chapter II that the integrals (2. 11) or (2. 19) constituting the response, y, are likely candidates for approximation by the method of stationary phase. It is indeed true that the method may be used to predict behavior of undamped systems in the late post-resonance era of the excitation, and also to provide an estimate for the response maximum in undamped and lightly damped systems under sweeping conditions. The former knowledge is of little practical value, since all physical systems possess a finite amount of damping. The stationary phase estimate of the peak response will be generally inferior to one derived by careful use of the method outlined in Chapter II.

The method does, however, warrant discussion not only because of its historical significance in the approximate evaluation of the type of integrals at hand, but because its shortcomings may serve to clarify the behavior of these integrals.

The result of Kelvin[21] as generalized by Watson[7] may be summarized per Erdelyi[22]:

$$\int_a^b k(t)\, e^{j\chi h(t)}\, dt \sim k(\rho)\sqrt{\frac{2\pi}{\chi h''(\rho)}} \cdot \exp\left[j\chi h(\rho) + j\,\frac{\pi}{4}\right] , \qquad \text{(B. 1)}$$

$$\chi \to \infty \ .$$

The integration is taken along the real axis and $h(t)$ is a real function. Erdelyi requires that: χ is a large positive variable; $k(t)$ is continuous; $h(t)$ is of the class C^2; there exists a single point, ρ, within the interval (a, b) where $h'(t) = 0$; and $h''(\rho) > 0$. This is not the most general statement of the Watson result but it will suffice for the present.

The method assumes that the prime contribution to the integral over the interval (a, b) comes from the neighborhood of ρ, the point at which the frequency of the trigonometric portion of the integrand is zero. Positive and negative contributions for t displaced from ρ in (a, b) will tend to cancel one another.

The function $k(t)$ and the trigonometric argument, $h(t)$, are replaced by their Taylor series expansions about ρ; the first term of the series for $k(t)$ is preserved, and the first three terms of the series for $h(t)$ are preserved (recall that $h'(\rho) = 0$). Since it is argued that contributions for t displaced from ρ are small, the integral will be relatively insensitive to the upper and lower limits of integration, and the range of integration is taken $(-\infty, \infty)$. The result (B. 1) follows immediately.

The expression for the response:

$$y = \frac{1}{m} \int_0^{\infty} e^{-\zeta u} \sin mu \cdot \sin g(\tau - u)\, du \tag{2. 9}$$

is expanded by trigonometric substitution:

$$y = \frac{1}{2m}\left\{\int_0^\infty e^{-\zeta u}\cos[mu-g(\tau-u)]du - \int_0^\infty e^{-\zeta u}\cos[mu+g(\tau-u)]du\right\}. \quad \text{(B. 2)}$$

As in the discussion of Chapter II, the excitation will be taken such that its instantaneous frequency will be everywhere positive, and will equal the system natural frequency only once for $-\infty < \tau < \infty$. It may be shown, then, that over the interval $(0, \infty)$ only the latter of the integrals in (B. 2) has a stationary phase point, $\rho = \tau - \mu$. By previous arguments, the contribution of the other integral will be small, and the method will yield the approximate expression

$$y \cong -\frac{e^{-\zeta(\tau-\mu)}}{m}\sqrt{\frac{\pi}{2|g''(\mu)|}} \cdot \cos[m\tau-m\mu+g(\mu) \pm \frac{\pi}{4}] \quad , \quad \text{(B. 3)}$$

for: $\tau >> \mu$. A positive phase factor $\pi/4$ in (B. 3) corresponds to $g''(\mu) > 0$, and a negative phase factor $\pi/4$ to $g''(\mu) < 0$.

The response of the undamped system for times long after the excitation frequency equals system natural frequency will be approximated by

$$y \cong -\sqrt{\frac{\pi}{2|g''(\mu)|}} \cos[\tau-\mu+g(\mu) \pm \frac{\pi}{4}] \quad .$$

The response maximum obtains for $\tau = \mu$. It will be equal to

$$y_{max} \cong \frac{1}{m}\sqrt{\frac{\pi}{2|g''(\mu)|}} \quad . \quad \text{(B. 4)}$$

The expression (B. 4) suggests that for infinitely slow sweep rates $(g''(\mu) \to 0)$, the response of even the undamped system will become unbounded.

If an excitation with linearly decreasing frequency is applied to the system, the peak response approximation will be

$$y_{max} \cong \frac{\pi}{m}\sqrt{q} \quad . \tag{B.5}$$

The estimate (B. 5) may be compared to the estimate for the undamped system,(2. 16). The present expression under-predicts (2. 16) by ~ 15%.

As may be seen in (B. 3), the stationary phase approximation applied to the present problem suggests that prior to the time the excitation frequency equals the system natural frequency, the system is at the state of rest. When the excitation frequency is equal to the system natural frequency, the system is displaced from rest with the initial conditions

$$y(\mu) = -\frac{1}{m}\sqrt{\frac{\pi}{2\left|g''(\mu)\right|}} \cos\left[g(\mu) \pm \frac{\pi}{4}\right] ,$$

$$y'(\mu) = \sqrt{\frac{\pi}{2\left|g''(\mu)\right|}} \cdot \left\{\frac{\zeta}{m} \cos\left[g(\mu) \pm \frac{\pi}{4}\right] + \sin\left[g(\mu) \pm \frac{\pi}{4}\right]\right\} .$$

After resonance, the excitation makes no further contribution to the system, and the system oscillates freely; the amplitude of oscillation decays at a rate prescribed by the damping.

The method may be generalized, of course, for a finite number of passes through system resonance. Expression (B. 3) then becomes

$$y \cong -\frac{1}{m}\sqrt{\frac{\pi}{2}} \cdot \sum_{i=1}^{N} \frac{e^{-\zeta(\tau-\mu_i)}}{\sqrt{\left|g''(\mu_i)\right|}} \cdot \cos\left[m(\tau-\mu_i) + g(\mu_i) + \operatorname{sgn} g''(\mu_i) \cdot \frac{\pi}{4}\right] , \tag{B.6}$$

for

$$\tau >> \mu_N > \mu_{N-1} > \cdot \cdot \cdot > \mu_1$$

and related predictions are altered accordingly.

The fact that the addition of damping to the system has little effect on the estimate for the response maximum is not surprising, since the decaying exponential behavior conferred on the integrand of (2.9) by non-zero damping is factored out of the integral. The integral from which the approximate expression (B.1) is derived depends not on the behavior of k(t) for convergence, but on h(t) the argument of the trigonometric portion of the integrand.

The discussion concerning expression (B.3) indicates why the expressions (B.3) and (B.6) represent unrealistic pictures of the post-resonant system response. The sum contribution of the excitation is expressed in terms of starting conditions for damped free oscillation. The actual form of the response for times after the excitation frequency equals the system natural frequency will, of course, be determined by the persisting excitation, <u>and</u> the decaying free vibration associated with the high amplitude response near system resonance.

Since a quadratic function is identical to the first three terms of its Taylor series expansion about any value of the independent variable, the stationary phase approximation for the response of an undamped system influenced by a linearly sweeping excitation will differ from the actual expression for the response only in the limits of integration. The disparity between the estimate for the response maximum (B.5) and the estimate (2. 16) must then be a consequence of this change in the end points of integration.

It may be seen in (B. 1) that the contribution to the integral from the neighborhood of the stationary phase point is $0\left[\frac{1}{\sqrt{\chi}}\right]$. The contribution over the remainder of the interval may be written in terms of the large variable χ by an integration by parts:

$$\int_{a}^{\rho-\epsilon} k(t)e^{j\chi h(t)}dt + \int_{\rho+\epsilon}^{b} k(t)e^{j\chi h(t)}dt \sim \frac{1}{j\chi}\left[\frac{k(b)e^{j\chi h(b)}}{h'(b)} - \frac{k(a)e^{j\chi h(a)}}{h'(a)}\right] + 0\left[\frac{1}{\chi^2}\right]. \tag{B.7}$$

The stipulation succeeding (B. 1) requires that $h'(t) \neq 0$ over the interval of (B. 7). The error in extending the limits of integration is then $0\left[\frac{1}{\chi}\right]$, providing that $k(a)$ and $h'(a)$ are $0[1]$. The response maximum, however, obtains shortly after the excitation frequency equals the system natural frequency, and the lower limit of integration in the exact expression for the response maximum is close to the stationary phase point. For this reason $h'(a)$ will be quite small. The error in neglecting this end point contribution will then be rather large.

One is led, by the failure of the stationary phase approximation in the present study, to suggest that the more sophisticated saddle-point method be used. This method certainly appears to take damping more fully into account, since it does preserve the decaying exponential behavior within the integral. It would be expected then that a better representation for the damped response and the damped response maximum would result.

Saddle-point integration consists of deforming the original path of integration of

$$I = \int_C e^{f(z)}\,dz \quad ,$$

in which $z = x + jy$ and $f(z) = u(x,y) + jv(x,y)$, into an equivalent path passing through the point ρ, where

$$\frac{df(z)}{dz} = 0 \quad .$$

The new path of integration is chosen such that $u(x,y)$ has a maximum at ρ. The prime contribution to the integral is assumed to come from this point, and the same simplifications as made in the stationary phase method are introduced.

The details of the method are rather complicated and are discussed by many authors including: Erdelyi[22], Copson[23], and Cerrillo[24].

It is unfortunate that saddle-point integration fails to significantly improve on the previously discussed method for predicting system response.

A comparison of the methods may be carried out by rewriting the latter integral of (B. 2) in complex form:

$$y \simeq -\frac{1}{2m}\,\mathrm{Re}\int_0^\infty \exp\left\{-\,\zeta u + j[mu + g(\tau - u)]\right\} du \quad ,$$

and noting that the stationary phase point, $\rho_1 = \tau - \mu_1$, will satisfy

$$g'(\rho_1) = m \quad . \tag{B.8}$$

The point at which $\frac{df(z)}{dz} = 0$, $\rho_2 = \tau - \mu_2$, will satisfy

$$g'(\rho_2) = m + j\zeta \quad . \tag{B.9}$$

The envelope for the stationary phase approximation is given in (B. 3) as

$$y_{mod_1} \cong \frac{1}{m} \cdot \sqrt{\frac{\pi}{2\,|g''(\mu_1)|}} \cdot e^{-\zeta(\tau-\mu_1)} \quad .$$

The corresponding envelope for the saddle-point approximation may be derived after a small amount of work:

$$y_{mod_2} \cong \frac{1}{m}\sqrt{\frac{\pi}{2\,|g''(\mu_2)|}} \cdot \exp\left\{-\zeta\,\mathrm{Re}(\rho_2) - m\,\mathrm{Im}(\rho_2) - \mathrm{Im}[g(\mu_2)]\right\} . \tag{B.10}$$

An alteration to the path of integration merely serves to introduce an additional phase factor which has no effect on the envelope expression.

The identity

$$g'(\rho_2) \equiv g'(\rho_1) + (\rho_2 - \rho_1) \cdot g''(\widetilde{\rho}) \quad ,$$

may be applied to (B. 8) and (B. 9) to produce the result

$$\rho_2 = \rho_1 + \frac{\zeta\,\mathrm{Im}[g''(\widetilde{\rho})]}{|g''(\widetilde{\rho})|^2} + \frac{j\zeta\,\mathrm{Re}[g''(\widetilde{\rho})]}{|g''(\widetilde{\rho})|^2} \quad . \tag{B.11}$$

An expansion of $g(\mu_2)$ and $g''(\mu_2)$ about μ_1 in (B. 10) and substitution of (B. 11) lead to the following expression

$$y_{mod_2} \cong \frac{1}{m}\sqrt{\frac{\pi}{2\left|g''(\mu)\right|}} \cdot \left\{1 + 0\left[\zeta^2\right]\right\} \cdot e^{-\zeta\{\tau - \mu_1 + 0[\zeta]\}} ,$$

and

$$y_{mod_2} \cong y_{mod_1} + 0[\zeta^2] \quad .$$

REFERENCES

1. Lewis, F. M., "Vibration During Acceleration Through a Critical Speed," Transactions of the American Society of Mechanical Engineers, vol. 54 (1932), pp. 253-261.

2. Barber, N. F. and Ursell, F., "The Response of a Resonant System to a Gliding Tone," Philosophical Magazine, series 7, vol. 39 (1948), pp. 354-361.

3. Hok, G., "Response of Linear Resonant Systems to Excitation of a Frequency Varying Linearly with Time," Journal of Applied Physics, vol. 19 (1948), pp. 242-250.

4. Baker, J. G., "Mathematical-Machine Determination of the Vibration of Accelerated Rotor," Journal of Applied Mechanics, Trans., A. S. M. E., vol. 61 (1939), pp. A-145 to A-150.

5. Crede, C. E., "Research on Failure of Equipment When Subject to Vibration," California Institute of Technology Dynamics Laboratory Report, First Annual Report under Contract No. NAS 8-2451 with Marshall Space Flight Center, N. A. S. A. (1962).

6. Hartenstein, R., "TR-10 Computer Solution to Q, Freq, Sweep Rate Problem," United States Government Memorandum, Nov. 6th, 1962.

7. Watson, G. N., "The Limit of Applicability of the Principle of Stationary Phase," Cambridge Philosophical Society, Proceedings, vol. 19 (1916-1919), pp. 49-55.

8. Fried, B. D. and Conte, S. D., "The Plasma Dispersion Function," Physical Research Laboratory, Space Technology Laboratories, Inc., Los Angeles, California (1960).

9. Karpov, K. A., "Tables of the Function $F(z) = \int_0^z e^{x^2} dx$ in the Complex Domain," (in Russian), Akademia Nauk, S. S. S. R. (1958).

10. Martz, C. W., "Tables of the Complex Fresnel Integral," N. A. S. A. Publication SP-3010 (1964).

11. Nielsen, N. N., "Dynamic Response of Multistory Buildings," Ph. D. Thesis, California Institute of Technology (1964), pp. 168-189.

12. Parker, A. V., "The Response of a Vibrating System to Several Time Dependent Frequency Excitations," M. S. Thesis, Iowa State University of Science and Technology, Ames, Iowa (1962).

13. Hawkes, P., "Response of a Single Degree of Freedom System to Exponential Sweep Rates (U)," Lockheed Missiles and Space Company, Report No. LSMC/A362881 (1963).

14. Erdelyi, A. et al. (eds.), Higher Transcendental Functions, vol. 2, Bateman Manuscript Project, McGraw-Hill Book Company, Inc., New York (1953), p. 42.

15. Watson, G. N., A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge University Press, London (1944), pp. 320-324.

16. McCann, G. D. and Bennett, R. R., "Vibration of Multifrequency Systems During Acceleration Through Critical Speeds," Journal of Applied Mechanics, vol. 16 (1949), pp. 375-382.

17. Caughey, T.K. and O'Kelly, M.E.J., "General Theory of Damped Linear Dynamic Systems," California Institute of Technology Dynamics Laboratory Report (1963).

18. Macchia, D., "Acceleration of an Unbalanced Rotor Through the Critical Speed," A.S.M.E. Paper No. 63-WA-9, Winter Annual Meeting, Nov. 17-22, 1963.

19. "Operating and Service Manual, Model 202A, Low Frequency Function Generator," Hewlett-Packard Company, Palo Alto, California (1959).

20. Caughey, T.K., Hudson, D.E., and Powell, R.V., "The C.I.T. Mark II Electric Analog Type Response Spectrum Analyzer for Earthquake Excitation Studies," California Institute of Technology Earthquake Engineering Research Laboratory Report (1960).

21. Thomson, W., Lord Kelvin, "On the Waves Produced by a Single Impulse in Water of Any Depth, or in a Dispersive Medium," Philosophical Magazine, series V, vol. 23 (1887), pp. 252-255.

22. Erdelyi, A., Asymptotic Expansions, Dover Publications, Inc., New York, N.Y. (1956).

23. Copson, E.T., "The Asymptotic Expansion of a Function Defined by a Definite Integral or Contour Integral," Department of Scientific Research and Experiment, Ref. No. S.R.E./ACS 106, The Admiralty, London (1946).

24. Cerrillo, M. V., "On the Evaluation of Integrals of the Type $f(\tau_1, \tau_2 \ldots \tau_n) = \frac{1}{2\pi i} \int F(s) \exp \omega(s, \tau_1, \tau_2 \ldots \tau_n) ds$ and the Mechanism of Transient Phenomena," Massachusetts Institute of Technology Research Laboratory of Electronics, Technical Report N. 55:2A (1950).