



A Primer for Program Composition Notation

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Abstract

This primer describes a notation for program composition. Program composition is putting programs together to get larger ones. PCN (Program Composition Notation) is a programming language that allows programmers to compose programs so that composed programs execute efficiently on uniprocessors, distributed-memory multicomputers or shared-memory multiprocessors.

The programs that are put together using PCN can be in PCN itself or in C or in Fortran. Later implementations of PCN will allow composition of programs in notations in addition to C and Fortran.

PCN is implemented on a variety of sequential and concurrent architectures including networks of UNIX-based workstations (Sun and NeXT), Symult 2010, Intel iPSC, BBN Butterfly, and Sequent Symmetry.

Several programming examples are presented in the primer. The examples are presented with methods for reasoning about the correctness of PCN programs.

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1 Overview

PCN is based on UNITY [3], a theory and a notation for concurrent programming, and on *Strand*[6]. Composition in PCN is motivated by, but is different from, composition in CSP [8]. The motivation for PCN and a comparison of PCN with other notations is found in [4]. A programming environment and the run-time support system for PCN are described in [2] and [7], respectively.

1.1 New Concepts

PCN has a three concepts that are not in languages such as C or Fortran. Next, these concepts are discussed very briefly and informally. Readers may want to skim through sections describing familiar material so as to spend more time on the new concepts.

1. **Mutables and Definition Variables:** PCN has two kinds of variables: mutables and definition variables. Mutables are variables as in C. Definition variables are different from variables in C; values of definition variables can be algebraic formulae (such as $y + z$), the initial value of a definition variable is a special symbol indicating that it is undefined, and a definition variable is defined at most once. For most programmers, the concept of mutable variables is familiar and the concept of definition variables is new.
2. **Composition Operators:** A program in PCN is a program heading (program name and arguments), a declaration of types of mutables and a block. A block is an elementary block or a composed block. An elementary block is an assignment statement (similar to assignments in C or Fortran), or a definition statement that defines definition variables, or a call to a program written in PCN, C, or Fortran. Later implementations will allow composition of programs in other languages. A composed block is a composition operator followed by a list of blocks or guarded-blocks (a block preceded by a boolean expression); the composition operator specifies how the blocks are to be put together.

The only things programmers can do in PCN are:

- *put blocks together using composition operators, or*
- *define elementary blocks.*

PCN has three composition operators: sequential composition, choice composition and parallel composition.

Sequential composition of a list of blocks executes the blocks in sequence, just as in C or Fortran. Choice composition is an extension of **if-then-else** and guarded commands. In a parallel composition of a list of blocks, all blocks in the list are executed in parallel, and the parallel composition completes execution when all its constituent blocks complete execution. For most programmers, sequential and choice composition are familiar concepts, while parallel composition is new.

The central concept of PCN is that of composition — putting blocks together — and once that is mastered all forms of composition are equally easy.

3. **Tuples:** PCN has a data type called a tuple which is a sequence of items between braces ‘{’ and ‘}’. Linear lists, circular lists, trees and other such linked structures are constructed using tuples.

The concepts of mutables, definition variables and composition are different from concepts found in conventional notations, and therefore, readers should focus attention on these ideas.

1.2 Highlighting Examples, Syntax and Operation

Most of this primer consists of examples; a large number are found in the *Simple Programming Examples* section. There are times when readers will want to study examples carefully and there are other times when readers will want to skip examples. To help identify examples, an example is placed between lines as in:

Examples of tuples

The empty tuple: {}.

Most examples are on odd-numbered pages, and are therefore on right-hand side pages with most text on left-hand side pages.

For ease in identification, syntax is placed between lines as in:

tuple :: { < *term* > } | ...

The operation of statements in PCN are described in terms of operations in familiar languages such as C. Operational descriptions of PCN statements are placed between lines as in:

repeat skip until *rhs* is reducible;
assign the reduced value of *rhs* to *m*.

2 Syntax

In this document we use a stylized syntax to convey the central concepts; a complete formal BNF is given in Appendix A. All nonterminal symbols are in italics, and all terminal symbols are in plain type. The notation $\langle su \rangle$, where su is a syntactic unit, represents a list of zero or more instances of the syntactic unit, with multiple instances separated by commas. The notation $\langle su \rangle^{(1)}$ is a list of one or more instances of su separated by commas. The notation $\langle su \rangle$ denotes an optional syntactic unit su .

Variable names, comparison operators, macros, and file inclusion are as in C. A comment begins with `/*` and ends with `*/` as in C. Expressions in PCN have the same syntax as arithmetic expressions in C, except that the only operators in PCN are `%`, `+`, `-`, `*` and `/`.

A variable name is a sequence of characters where the first character is a letter, and a character is a letter or a digit. A letter is an upper case or lower case letter of the alphabet, or it is the symbol “_”. An upper case letter is different from a lower case letter.

3 Data Types

3.1 Conventional Data Types

Conventional data types, such as in C, are also data types in PCN:

1. **char** for character,
2. **int** for integer,
3. **double** for double-precision floating point number.

PCN has **one-dimensional arrays** of these data types. Arrays in PCN and C are treated in the same way and are indexed from zero.

Strings in PCN are treated in exactly the same way as in C. A string S is an array A of char, where the characters of S are $A[i]$, in increasing order of i starting with $i = 0$ and ending with $i = k$, where k is the smallest index such that $A[k + 1]$ is the *null character* $\backslash 0$. If $A[0]$ is the null character, S is the empty string. A constant string can be denoted by placing the characters of the string between quotes; for example "PCN" is a string consisting of the three characters: P, C and N. The empty string is " ".

In this document, a **number** is an integer or a double-precision floating point value. We define a **simple-value** as a number or a character or an array of numbers or a string.

Qualifiers for int in C (such as short, long, and unsigned) are not available in PCN, nor are single-precision floating point numbers and structures.

3.2 Tuples and Lists

A tuple is a pointer to a possibly empty sequence of **terms**, where a term is a simple-value, an expression, or a tuple. A tuple is represented in a program as a sequence of terms between braces — '{' and '}' — where terms are separated by commas.

A list is a special case of tuple. A list is:

1. The empty tuple, {}, or

Examples of tuples

The empty tuple: $\{\}$.

A 1-tuple: $\{\{x\}\}$, where the single element of the tuple is itself a 1-tuple, $\{x\}$.

A 2-tuple: $\{"msg", 3\}$.

Examples of lists

The empty list: $\{\}$

A single-element list: $\{d, \{\}\}$ is a list containing a single element, d .

A four-element list: $\{a, \{b, \{c, \{d, \{\}\}\}\}\}$ is a list containing the sequence of 4 elements, a , b , c and d , in that order.

2. A 2-tuple, $\{a, b\}$, where the second element of the tuple, b , is a list.

PCN has a more succinct notation for lists: a list consisting of a sequence of zero or more elements can be represented by the sequence of elements between the enclosing brackets '[' and ']'. Also, for brevity, we can employ the notation, $[L_1, L_2, \dots, L_k \mid z]$, to represent the tuple, $\{L_1, \{L_2, \{\dots \{L_k, z\} \dots\}\}\}$.

For convenience, the notation $h(x_0, \dots, x_k)$, where h is an identifier and x_0, \dots, x_k are tuple-elements, denotes the tuple $\{h, x_0, \dots, x_k\}$; this allows programs to be represented as data.

Syntax of Tuples A tuple has the following syntax:

<i>tuple</i>	:: $\{ \prec term \succ \} \mid$ $[\prec term \succ] \mid$ $[\prec term \succ^{(1)} \mid \prec term \succ] \mid$ $identifier(\prec term \succ)$
<i>term</i>	:: <i>simple-value</i> \mid <i>expression</i> \mid <i>tuple</i>

Examples of list notation

The empty list: The empty list is $[]$; hence $[] = \{\}$.

A single-element list: $[d]$ is a list containing a single element, d ; hence $[d] = \{d, \{\}$.

A four-element list: $[a, b, c, d]$ is a list containing the sequence of 4 elements, a , b , c and d , in that order; hence $[a, b, c, d] = \{a, \{b, \{c, \{d, \{\}\}\}\}$.

Catenation of lists: A list, y , consisting of a sequence of values, u , v , w , followed by another list, z , is represented by: $[u, v, w \mid z]$; thus, if $z = [b, c, d]$, and $y = [u, v, w \mid z]$, then $y = [u, v, w, b, c, d]$.

More examples of tuples

$g(x)$ and $\{ "g", x \}$ denote the same tuple.

$f(x, y)$ and $\{ "f", x, y \}$ denote the same tuple.

The length Function PCN includes a function *length* which has a single argument and returns:

- the number of elements in the argument if its argument is a tuple or an array, and
- 1 (one) if its argument is a single number or character.

The argument of *length* must be a variable. Note the difference between the *length* function in PCN and the *sizeof* function in C: in C the function returns the size of its argument in bytes, whereas in PCN the function returns the number of elements.

Elements of a tuple are referenced in the same way as elements of an array in C: $t[i]$ is element i of a tuple t , for $0 \leq i < \text{length}(t)$.

Examples of length function

Arrays of numbers:

```
/* declare  $u$  to be an array of 10 doubles */  
double  $u[10]$ ;  
 $length(u)$  is 10.
```

A single number:

```
/* declare  $i$  to be an integer */  
int  $i$ ;  
 $length(i)$  is 1.
```

A tuple: Let z be the tuple $\{x, y\}$; then $length(z)$ is 2.

A string:

```
/* declare  $D$  to be an array of 10 chars */  
char  $D[10]$ ;  
 $length(D)$  is 10.
```

4 Variables

4.1 Values of Variables

At each point in a computation, a variable has precisely one **value**. The value of a variable is a term. The value of a variable in PCN can be an expression such as $y + z$. In C, execution of the assignment $x = y + z$ causes the value of x to become the value of y plus the value of z , and thus the values of variables x , y and z in C are always numbers; the execution of the assignment in C does not make the value of x become the formula $y + z$. Indeed, in most notations, values of variables are numbers or characters, but not expressions. Variables in PCN can have expressions as values which allows for a degree of symbolic computation in addition to the usual numeric computation of C and Fortran.

Notation for Value of a Variable In C, $x = 2$ denotes that x has value 2. We need additional notation to denote the value of a variable in PCN as illustrated by the following example. In PCN, $u = x + y + z$ does not necessarily imply that the value of u is the expression $x + y + z$, because it is possible that the value of u is the expression $v + z$, and the value of v is the expression $x + y$. To avoid this ambiguity, we denote the value of a variable x by $value(x)$. Instead of writing, $x = 2$, to denote that x has value 2, we shall say that $value(x)$ is 2. Thus, $value$ is a function that maps from variables to terms.

At any point in a computation, we can substitute the value of x for x . Therefore, if u has value $v + z$ and v has value $x + y$ at some point in a computation, then $u = v + z$ and $v = x + y$ and hence, $u = (x + y) + z$ at that point. If u and v are undefined, we are not permitted to conclude anything about the relationship between u and v ; in particular we cannot conclude that $u = v$.

Classes of Variables A variable in PCN is either a **mutable** or a **definition variable**. Informally, a mutable is similar to variables in C and a definition variable is a variable that is assigned at most once.

Examples of Values of Variables

Possible values of a variable x are presented next.

Simple-Value: $value(x)$ is 2.

Expression as Value: $value(x)$ is $y + z$.

Expression as Value: $value(x)$ is $u + v * (y + z)$.

Tuple as Value: $value(x)$ is $\{y\}$.

Tuple as Value: $value(x)$ is $\{2, "A", y, z\}$.

4.2 Mutables

The type of a mutable is declared in programs in which it is used, its initial value is an arbitrary value of its declared type, and its value can be changed arbitrarily often during a computation by execution of assignment statements that assign values to it. Type declarations are as in C. A mutable type is a C type (i.e., char, int, or double).

4.3 Definition Variables

A definition variable is different from variables used in C. A definition variable is either **undefined** or **defined**; it can be defined *at most once* in a computation. A definition variable is defined to be a *term* by executing a **definition statement** in which the definition variable appears on the left-hand side; definition statements are described later.

The value of an undefined definition variable is undefined. The value of a defined definition variable is its definition. The value of a definition variable does not reference mutables.

Definition variables are not declared.

4.4 Review of Differences between Definition Variables and Mutables

Declaration Mutables are declared. Definition variables are not declared.

Initial Value The initial value of a mutable is an arbitrary value of its declared type. The initial value of a definition variable is a special symbol indicating that it is undefined.

Expressions as Values The value of a mutable cannot be an expression. The value of a definition variable can be an expression.

Tuples as Values The elements of a tuple are initially definitions variables and can be defined using definition statements. The value of a definition variable can be a tuple; however, no element of a tuple can be a mutable.

Changes in Values The value of a mutable can be changed arbitrarily many times. The value of a definition variable can be changed at most once, from undefined to a defined value. Once a definition variable is defined, its value remains unchanged forever thereafter.

5 Programs

A program consists of a heading followed by a declaration section followed by a **block**. The heading is the program name and a list of formal parameters, as in C. In PCN all parameters are passed by reference, unlike in C where parameters can be passed by value. The syntax of a declaration section is identical to that in C. The scope of a variable is the program in which it appears: all variables that appear in a program are either formal parameters or local variables of the program. All mutables referenced in a program are declared in the declaration section of the program; definition variables are not declared.

Local variables in PCN are local to the program in which they are declared, whereas local variables in C can be declared to be local to *blocks* within programs. Also, C allows programs to access variables that are not formal parameters or local variables of the program, whereas PCN does not.

The dimensions of a local array of a program can change from one call of the program to the next, unlike in C where some of the dimensions of a multidimensional local array must remain unchanged in all calls.

The syntax of a block is:

<i>block</i>	::	<i>elementary-block</i> <i>composed-block</i>
<i>elementary-block</i>	::	<i>definition-statement</i> <i>assignment-statement</i> <i>program-call</i>

Composed blocks are discussed later; the next few sections discuss each of the forms of elementary blocks.

An Example of Formal Parameters and Local Variables

```
p(sum, x)  
int sum, v;  
{? x ?= [m | xs] ->  
      {; v := m, sum := sum + v, p(sum, xs)}  
}
```

The operators in this example are not important here; only the heading and the declarations are relevant. The first line is the heading for a program with name, p , and two formal parameters, sum and x . The second line declares sum and v to be integer. Therefore, sum and v are mutable. Since the types of x , xs and m are not declared, they are definition variables. Since xs , m and v are not formal parameters, they are local variables of program p .

6 Definition statements

A definition statement has the following syntax:

definition-statement :: *definition-variable* = *term*

The execution of the definition statement $x = rhs$ completes, and at completion, $value(x)$ is rhs' where rhs' is obtained by substituting $value(v)$ for each mutable v in rhs . Mutables do not appear in rhs' , and hence the value of a definition variable does not name mutables.

The value of an expression in PCN can be an array. For example if mutable m is declared to be an array of integers, then the value of the expression ' m ' is an array. Hence, upon completion of the execution of the statement $x = m$, the value of x is an array.

Arithmetic operators in PCN are identical to those in C, and hence their operands are numbers, not arrays of numbers. So the expression ' $m + 1$ ', where m is an array, is incorrect.

Examples of Definition Statements Without Mutables

In these examples, z is a definition variable.

number: Execution of the definition statement

$z = 3.0$

terminates with

$value(z) = 3.0$.

string: Execution of the definition statement,

$z = \text{"abc "}$

terminates with

$value(z) = \text{"abc"}$.

Examples of Definition Statements With Mutables

In these examples, z and y are definition variables, and m is a mutable with value 2 at the point in the computation at which the definition statements are executed.

expression: Execution of the definition statement

$$z = y + m + 5$$

terminates with

$$\text{value}(z) = y + 2 + 5.$$

tuple: Execution of the definition statement

$$z = \{y, \{m\}, 5\}$$

terminates with

$$\text{value}(z) = \{y, \{2\}, 5\}.$$

tuple: Execution of the definition statement

$$z = [m, "b"]$$

terminates with

$$\text{value}(z) = [2, "b"].$$

Examples of Definition Statements With Arrays

If A is a 2-element integer array with $A[0] = 1$, and $A[1] = 2$, then the definition statement, $z = A$ terminates with z defined as a 2-element integer array with $value(z[0]) = 1$, and $value(z[1]) = 2$.

7 Reducibility

The meaning of an assignment is based on the concept of reducibility. For a term e , the *reduced value of e* is a simple-value or tuple x , where $x = e$. A term e is *reducible at a point t in a computation* if and only if a reduced value of e can be computed at t , i.e.,

1. e is a simple-value or a tuple, in which case the reduced value of e is e ,
or
2. $value(e)$ is f and f is reducible, in which case the reduced value of e is the reduced value of f , or
3. e is an expression and all variables in e are reducible, in which case the reduced value of e is computed by substituting the reduced value of v for each variable v in e and evaluating.

7.1 Properties of Reducibility

Value Equals Reduced Value For all variables x , the value of x is equal to its reduced value. This is because the reduced value of x is obtained from the value of x by substitution and expression evaluation.

Unique Reduced Values The reduced value of an element e at a point in a computation is unique. For example, if the reduced value of e is 2 at a point in a computation then the reduced value of e cannot also be 3 at that point.

Mutables Are Reducible A mutable is reducible at all points in computation and the reduced value of a mutable is its value.

Undefined Definition Variables Are Not Reducible An undefined definition variable is not reducible because the value of an undefined definition variable is a special symbol indicating that it is undefined, and hence none of the rules of reducibility can be employed to compute a simple value or tuple for it.

Examples of Reducibility

Example 1

Let z be a definition variable, and let m be a mutable that is declared to be an integer. If at a point t in a computation, $value(z)$ is 1 and $value(m)$ is 2, then $z + m$ is reducible and its reduced value is 3, at t .

If $m = 4$ at a later point t' , then the reduced value of $z + m$ is 5 at t' .

Example 2

If at a point t in a computation, y is undefined, then y is not reducible at t . If at t , the value of z is $y + 1$, then z is not reducible at t because y is not reducible at t .

Example 3

If at a point t in a computation, $value(z)$ is $y + 1$, and $value(y)$ is 0, then z is reducible and its reduced value is 1 at t . Furthermore, if z is a definition variable, the reduced value of z remains 1 at all points after t .

Defined Definition Variables A defined definition variable may or may not be reducible. For example, if $value(x) = y + z$, where x , y and z are definition variables, then x is not reducible if y or z is not reducible. If, however, y and z are reducible with reduced values (say) 1 and 2 respectively, then x is reducible and has reduced value 3.

Once Reducible, Remains Reducible Once a term is reducible it remains reducible forever thereafter. The reasons for this are as follows. Mutables, simple-values and tuples are always reducible. Once a definition variable is reducible it remains reducible because its value remains unchanged. Once an expression (that can name mutables and definition variables or both) is reducible it remains reducible.

Reduced Values of Definition Variables The reduced value of a definition variable remains unchanged. For example, if the reduced value of z is 2 at some point in a computation, then the reduced value of z remains 2 forever thereafter. Likewise, the reduced value of an expression, that does not name mutables, remains unchanged. For example, if the reduced value of $y + z$ is 3 at some point in a computation (where y and z are definition variables), then the reduced value of $y + z$ remains 3 thereafter.

Reduced Values of Mutables The reduced value of a mutable can change; for instance mutable m can have value 2 at some point in a computation and value 3 at a later point. Likewise, the reduced value of an expression that names mutables can change. For example, the value of expression $y + z + m$ can change, where m is a mutable, because m can change value.

More Examples of Reducibility

Example 4

If at a point t in a computation, $value(y) = \{1, z\}$, then y is reducible and its reduced value is $\{1, z\}$ at t . Note that y is reducible even if z is not reducible.

Example 5

If at a point t in a computation, $value(y) = A$, where A is an integer array, then y is reducible, and its reduced value is A at t .

Example 6

If at a point t in a computation, $value(y) = x+1$, and $value(x) = 2*y-2$, then both x and y are nonreducible at t . (Note that from the mathematics of simultaneous equations we can conclude $x = 0$ and $y = 1$, but according to our definitions x and y are not reducible.)

8 Assignment Statements

The syntax of an assignment statement is:

assignment-statement :: *mutable* := *expression*

The execution of the assignment statement $m := rhs$ where rhs is an expression and m is a mutable variable, declared to be one of the types in C , is as follows:

repeat skip until rhs is reducible;
assign the reduced value of rhs to m .

A skip is an operation that does nothing, and it is sometimes referred to as a ‘no-op.’ Therefore, while rhs is not reducible, the program executes operations that do nothing — in other words, the program waits. When rhs becomes reducible, the reduced value of rhs (coerced to be the same type as m) is assigned to m , and the assignment completes. If rhs never becomes reducible the assignment does not complete, and execution of the assignment statement is an infinite number of skips. Note that if rhs does not reference definition variables, the assignment statement is executed without skips, because rhs is reducible; in this case the assignment is executed in the same way as assignments in C .

Simple Examples of Assignment

Right-Hand Side Does Not Reference Definition Variables

```
int m, i, j, u[10], v[10];  
..., m := i + j, ..., u := v, ...
```

The assignment $m := i + j$ is executed in the same way as the assignment $m = i + j$ in C: the sum of the values of the integers i and j is assigned to m .

The assignment $u := v$ makes u become the array v .

Right-Hand Side References Definition Variables

```
int m;  
..., m := z, ...
```

The assignment, $m := z$, where m is an integer mutable, and z is a definition variable, is executed as follows. While z is not reducible, skip. When z becomes reducible, assign its value (coerced to integer) to m . If z never becomes reducible, execution of the assignment does not complete.

9 Program Calls

The syntax of a program call is:

program-call :: *program-name*(\prec *term* \succ) |
 '*definition-variable*(\prec *term* \succ)

A program call *program-name*(\prec *term* \succ) is the same as a function call in C, except that all parameter passing is by reference, and the program does not return a value (in the way that a function does). Later, we will describe two modifications to this syntax that allow programmers to specify processors on which called programs are to be spawned, and modules (files) that contain the source texts of the called programs; these do not change the essential meaning of program calls.

9.1 Calling C programs from PCN

PCN programs can call C programs. The actual parameters in a call to a C program can be definition variables or mutables. All parameter passing in PCN is by reference; hence, the arguments of the C program must be *pointers* to simple-values (i.e., char, int, or double). The execution of a C program call is as follows:

repeat skip until all actual parameters are reducible;
execute the C program.

We do not specify the behavior of PCN programs that call C programs which continue execution for ever; therefore, programmers must ensure that C programs terminate.

Examples of Calls to Programs

Consider the C program:

```
q(v, w, x)
int *v, *w, *x;
{ *v = *v - *x; *w = *w + *x }
```

The execution of the call $q(a, b, z)$, where a and b are mutable integer variables and z is a definition variable, is as follows: repeat skip until z is reducible; when z becomes reducible, execute $q(v, w, z')$ where z' is the reduced value of z . Note that even though z is a definition variable, and its value cannot be changed by the C program, z is passed by reference and not by value. That is why the type declaration of formal parameter, x , is 'int *x', and not 'int x'. Definition variable z must reduce to an integer value because the corresponding formal parameter x is a pointer to an integer.

Consider the PCN program:

```
p(w, x, y, z)
{|| y = w + x, z = w - x}
```

A call $p(a, b, c, d)$ of program p causes program p to be executed, *even if actual parameters are not reducible*. We will see later that the program completes, and at completion $value(c)$ is $a + b$ and $value(d)$ is $a - b$, regardless of whether a or b are reducible.

9.2 Calling Fortran Programs from PCN

Fortran programs are called in the same way as C programs. Parameter passing in Fortran is by reference, as it is in PCN.

9.3 Calling PCN Programs from PCN

PCN programs can call PCN programs; the actual parameters of the call can be mutables or definition variables. The called program is initiated even if some or all of the actual parameters are not reducible.

There is a difference between the execution of calls to C programs and PCN programs. A called C program is initiated only when all its actual parameters are reducible. A called PCN program is initiated without waiting for all its actual parameters to be reducible. (A strict semantic is used for C calls and a nonstrict semantic for PCN calls.)

Types of actual parameters should be the same as types of corresponding formal parameters in calls to PCN programs. In particular, an actual parameter should be a definition variable if and only if the corresponding formal parameter is a definition variable.

9.4 Program Names as Actual Parameters of Programs

A program-name can be passed as an argument of a program. To distinguish a *variable* whose value is a program name from a *program name*, the symbol ' is employed. For example $y(x)$ is a call of program y , whereas $'y(x)$ is a call to a program g , where the reduced value of y is "g".

Recall that $f(x_0, \dots, x_k)$ represents the tuple $\{f, x_0, \dots, x_k\}$. Similarly, $'f(x_0, \dots, x_k)$ represents the tuple $\{f, x_0, \dots, x_k\}$.

The program call $'y(x)$ where x is a list of actual parameters and y is a definition variable, is executed as follows:

repeat skip until y is reducible;
execute $g(x)$ where "g" is the reduced value of y .

Example of a Program Name as A Parameter

Next, consider a program *map*, which has a formal parameter *operator* defined as a string that is a name of a program.

```
map(operator, lst, result)
int result;
{? lst ?= [head | tail] ->
    {; 'operator(head, result),
      map(operator, tail, result)
    }
}
```

Passing a Program Name as a Parameter

Calling the preceding program with *map("add", L, R)*, causes the following block to be executed:

```
{? L ?= [head | tail] ->
    {; add(head, R), map("add", tail, R)}
}
```

10 Composed Blocks

A composed block has the following syntax.

$$\begin{aligned} \text{composed-block} &:: \{ ; \prec \text{block} \succ^{(1)} \} \mid \\ &\{ \parallel \prec \text{block} \succ^{(1)} \} \mid \\ &\{ ? \prec \text{guard} \rightarrow \text{block} \succ^{(1)} \} \end{aligned}$$

where ‘;’ denotes sequential composition, ‘||’ denotes parallel composition, and ‘?’ denotes choice composition.

11 Sequential Composition

Let d be the block $\{ ; b_1, \dots, b_k \}$, where $k > 0$. The execution of d is a sequential execution of b_i , in order, from $i = 1$ to $i = k$.

12 Parallel Composition

Let d be the block $\{ \parallel b_1, \dots, b_k \}$, where $k > 0$, and for all i where $0 < i \leq k$, b_i is a block. In an execution of d , all blocks b_i are executed in parallel. Block d terminates when the computations of b_i terminate for all i . (A computation of d is a fair interleaving of computations of b_i , for all i , $0 < i \leq k$. Execution is fair in the following sense: For all i , it is always the case that eventually computation of b_i will progress if b_i has not terminated.)

Programmers must ensure that the following condition about shared variables is satisfied.

Restriction on Shared Variables in Parallel Composition *Shared mutables must not change value during parallel composition.*

Example of Sequential Composition

```
p(j, k, x, y)
int j, k, m;
/* Let the value of j be J. */
/* m is a local integer mutable of p */

{
  m := 2,
  /* value(m) is 2 */

  x = m + 1,
  /* The reduced value of x is 3. */

  k := y + j
  /* value(k) is sum of the */
  /* reduced values of y and j. */
}
```

First m becomes 2, then x is defined as $2 + 1$ (and hence its reduced value is 3), and then, after executing skips until y becomes reducible, x becomes the sum of the values of y and j .

This restriction is equivalent to: In a parallel composition block d defined as $\{\parallel b_1, \dots, b_k\}$,

for each variable v accessed in distinct blocks b_i and b_j :

1. v remains unchanged during the execution of d , or
2. v is a definition variable.

For this purpose, each element of a shared array is treated as separate variable; therefore, blocks composed in parallel can modify a shared array, but each *element* of the array must remain unchanged or be accessed by at most one block. Similarly, each element of a tuple is treated as a separate element.

An important consequence of this restriction is that blocks, composed in parallel, interact with each other in a disciplined manner. Consider a predicate z that references only variables that appear in one of the constituent blocks, say b_i of parallel composition block d . For example, z could be $u = v + 1$, where u and v are variables referenced in block b_i . Suppose we can reason from the text of block b_i (i.e., by considering block b_i in isolation, independent of the blocks that b_i is composed with) that predicate z holds at some point p in b_i . In our reasoning we are not permitted to conclude that definition variables are undefined if they are undefined in b_i (because definition variables can be defined in blocks that are composed with b_i). Then, no matter what blocks are composed in parallel with b_i , we are assured that our reasoning is valid, and z holds at p .

An equally important consequence of this restriction is that we do not have to be concerned about atomicity in parallel composition.

Example of Parallel Composition

$\{\parallel p(j, k, x, y), y = x + j\}$

(Program p is defined in the previous example.) This parallel composition block obeys our convention about shared variables. The only shared mutable is j which is not modified in the block. Let the value of j be 1 when this block is executed. The computation of this block terminates, and at termination, the reduced value of x is 3, and y is 4, and k is 5.

A possible computation of the parallel composition block is: y is defined as $x + 1$ (assuming $j = 1$), then, in program p : m becomes 2, then x becomes defined as $2+1$, and finally k becomes 5. Note that y can become defined before x becomes defined.

13 Choice Composition

A choice composition block is similar to a guarded command [5]. A guard in a choice composition block is a boolean expression or the keyword **default**. At a point in a computation, a boolean expression is:

1. not reducible, or
2. reducible and has value *true*, or
3. reducible and has value *false*.

We shall see later that once a guard is reducible it remains reducible for ever thereafter.

There can be at most one default guard in a choice composition block. A choice block without a default guard is equivalent to a choice block with the addition of the guarded block: `default -> skip`. therefore, we can restrict attention to choice blocks that contain precisely one default guard.

The basic idea about execution of the choice block:

$$\{? \text{ default} \rightarrow b_0, G_1 \rightarrow b_1, \dots, G_k \rightarrow b_k \}$$

is as follows:

1. If all guards are *false* then execute b_0 ; execution of the choice block terminates when execution of b_0 terminates.
2. If at least one guard is *true* then execute any block b_i where G_i is *true*; execution of the choice block terminates when execution of b_i terminates.
3. Because guards can be nonreducible, there is a third possibility: at least one guard is nonreducible and no guard is *true*. In this case the program repeatedly executes skips until one of the first two conditions holds.

Examples of Choice Blocks

Example 1

Consider the choice-block:

$$\left\{ \begin{array}{l} x \geq 0 \rightarrow y = x + 1, \\ x \leq 0 \rightarrow y = x - 1 \end{array} \right\}$$

The execution of this block is as follows. While x is irreducible, skip. When x becomes reducible, if $x > 0$, then only the first guard holds, and hence y is defined as $x + 1$; if $x < 0$, then only the second guard holds, and hence y is defined as $x - 1$; if $x = 0$, then both guards hold, and a nondeterministic choice is made to define y either as $x + 1$, or as $x - 1$. Execution of the block terminates after y is defined.

Example 2

Consider the choice-block:

$$\{? \ x \geq 0 \rightarrow y = x + 1 \}$$

The execution of this block is as follows. While x is irreducible, skip. When x becomes reducible: if $x \geq 0$ then y is defined as $x + 1$ else y is left unchanged; execution of the block terminates.

13.1 Guards

A guard is either a sequence of one or more guard elements or *default*. The syntax of a guard is:

$$\textit{guard} \quad :: \prec \textit{guard-element} \succ^{(1)} \mid \textit{default}$$

Case 1: If All Guard-Elements are Reducible If all the guard-elements of a guard G are reducible, then G is reducible, and the value of G is a ‘conditional-and’ of its guard-elements: Each of the guard-elements in the sequence is evaluated in order until all guard-elements in the sequence are evaluated or a guard-element evaluates to *false*; if all guard-elements evaluate to *true* the value of the guard is *true*, otherwise the value of the guard is *false*. The evaluation of a guard is similar to the evaluation, in C, of an expression consisting of the sequence of guard-elements with the logical connective `&&` between guard-elements.

Case 2: At Least One of the Guard-Elements is Irreducible Next, consider the case where at least one of G ’s guard-elements is irreducible: If all guard-elements before the first nonreducible element of G evaluate to *true*, then G is not reducible; otherwise, $G = \textit{false}$.

13.2 Guard-Elements

The syntax of a guard-element is:

$$\textit{guard-element} \quad :: \textit{type-check} \mid \textit{comparison} \mid \\ \textit{data-check} \mid \textit{pattern-match}$$

Another Example of a Choice Block

Example 3

Consider the choice-block:

$$\left\{ \begin{array}{l} x \geq 0 \rightarrow y = x + 1, \\ z \geq 0 \rightarrow y = z + 1 \end{array} \right\}$$

Repeat skips until both guards are reducible and both have reduced value *false*, or at least one of the guards is reducible and has reduced value *true*. In the former case the choice block terminates without changing the value of any variable. In the latter case, if both guards have reduced value *true* then execute either $y = x + 1$ or $y = z + 1$, if only $x \geq 0$ has reduced value *true* then execute $y = x + 1$, and if only $z \geq 0$ has reduced value *true* then execute $y = z + 1$.

Type Checks The syntax for *type-check* is:

type-check :: *type-name*(*definition-variable*)
type-name :: int | double | char | tuple

If x is not reducible, *type-check* $h(x)$ is not reducible. If x is reducible, *type-check* $h(x)$ evaluates to:

1. *true* if the reduced value of x is of type h or is an array of type h ,
2. *false* otherwise.

For example, if x is reducible, then $\text{int}(x)$ holds if and only if the reduced value of x is an integer or an array of integers.

Comparison The syntax of a comparison is:

comparison :: *term equality-test term* |
 expression ordering expression
equality-test :: == | !=
ordering :: < | <= | > | >=

A comparison $x \alpha y$, where α is an ordering, is reducible if and only if both x and y are reducible; the reduced values of x and y must be numbers or arrays. Numbers and strings contained in character arrays are compared as in C; integer and double arrays are compared consecutively by element.

A comparison $x \alpha y$, where α is an equality-test, is reducible only if both x and y are reducible. An equality-test $x == y$, where the reduced values of

Example of Type Check and Comparison

$\{? \text{ int}(x), x \geq y \rightarrow z = x + 1 \}$

Consider the case where y is irreducible and x is reducible. If the reduced value of x is an integer, then the guard, is not reducible because the first guard-element evaluates to *true* and the second guard-element is not reducible. If the reduced value of x is a character, then the guard evaluates to *false*, because the first guard-element evaluates to *false*.

Next consider the same program except that the order of guard-elements is reversed.

$\{? x \geq y, \text{int}(x) \rightarrow z = x + 1 \}$

As before, consider the case where y is irreducible and x is reducible. As in the last example, if the reduced value of x is an integer, then the guard, is not reducible because the first guard-element is not reducible. If the reduced value of x is a character then the guard is not reducible for the same reason. Note that in the previous example, if x is a character the guard *is* reducible. This example shows that the ordering of guard-elements can make a difference to the computation.

x and y are tuples, is equivalent to the following sequence of equality-tests:

$length(x) == length(y)$, and for all i where $0 \leq i < length(x)$: $x[i] == y[i]$.

Thus, equality tests of tuples are equivalent to sequences of equality-tests without tuples. A comparison $x == y$ is reducible if x and y reduce to simple values. Equality of characters and numbers is as in C.

Inequality is defined as the negation of equality.

Data Check The syntax of a data-check is:

data-check :: data(*definition-variable*)

If x is reducible then $data(x) = true$. If x is not reducible, then $data(x)$ is also not reducible. The value of $data(x)$ is never *false*.

Testing Equality of Tuples

$\{? x == y \rightarrow p(x) \}$

In the equality test, x or y can be tuples (and therefore can be lists). If $x = [0, 1, 2]$, then the equality test succeeds only if y is equal to the same list. If y is $[0, 1|z]$, where z is a nonreducible definition variable, then $x == y$ is not reducible.

As another example, consider the case where the reduced value of x is z and the reduced value of y is also z . The equality test is reducible if and only if z is reducible. Of course, if z is reducible, the equality test succeeds.

13.3 Pattern Matches

A pattern-match is merely a syntactic convenience for operating on tuples. It has the following syntax:

```
pattern-match  :: variable ?= pattern
pattern       :: {< pattern-element >}
pattern-element :: simple-value | definition-variable | pattern
```

A pattern-match $x \text{ ?= } pat$ succeeds (i.e., has reduced value *true*) if the reduced value of x is a pattern of the same ‘form’ as the pattern, pat , on the right. For example, if pat is $\{u, v\}$, then the match succeeds if the reduced value of x is a tuple of size 2. If a match succeeds, a variable in the pattern become an alias for the corresponding element of the tuple for the remainder of the guard and its associated block. Thus the pattern-match $x \text{ ?= } \{u, v\}$ succeeds if x is the 2-tuple $\{x[0], x[1]\}$, and if the match succeeds then u becomes an alias for $x[0]$, and v becomes an alias for $x[1]$ for the remainder of the guard and its associated block. Next, matches are discussed in more detail.

Evaluation of a Pattern-Match A definition variable that appears in a pattern must be undefined when the pattern-match is evaluated. *Mutables cannot appear in patterns.*

A pattern match, $z \text{ ?= } pat$ can be transformed into a sequence of guard-elements without the pattern match by the following syntactic transformation: Replace the pattern match by,

$$tuple(z), (length(z) == length(pat))$$

and for each i , where $0 \leq i < length(z)$, if $pat[i]$ is a

pattern add the pattern-match, $z[i] \text{ ?= } pat[i]$,

Simple Patterns

In the following example, z , hd and tl are definition variables that are undefined at the point at which the pattern matches are executed.

$$z \text{ ?= } \{hd, tl\} \quad -> \{\mid hd = u, v = tl\}$$

is equivalent to:

$$\text{tuple}(z), \text{length}(z) == 2 \quad -> \{\mid z[0] = u, v = z[1]\}$$

The match $z \text{ ?= } \{hd, tl\}$ succeeds if z is a tuple of the same form as $\{hd, tl\}$, i.e., if z is a 2-tuple. If the match succeeds, then hd is an alias for $z[0]$, and tl is an alias for $z[1]$ in the remainder of the guard and its associated block.

simple-value add the equality-test, $z[i] == pat[i]$,

definition variable replace all instances of $pat[i]$ by $z[i]$ in the remainder of the guard and its associated block.

Patterns with Strings and Numbers

$$z \text{ ?= } \{v, \{y\}, 2, \text{"msg"}\} \rightarrow \{\| v = y\}$$

is equivalent to

$$\begin{aligned} & \text{tuple}(z), (\text{length}(z) == 4), \\ & (\text{tuple}(z[1]), (\text{length}(z[1]) == 1)), \\ & (z[2] == 2), (z[3] == \text{"msg"}) \rightarrow \\ & \{\| z[0] = z[1][0]\} \end{aligned}$$

The match succeeds if the reduced value of z is the same form as the pattern, i.e., if z 's reduced value is a 4-tuple where $z[3]$ and $z[4]$ are the integer 2 and the string "msg", respectively, and where the form of $z[1]$ is the pattern $\{y\}$. If the match succeeds, v and $z[0]$ are aliases of each other, and likewise, y and $z[1][0]$ are aliases of each other.

Patterns with Lists

$$z \text{ ?= } [w \mid x] \rightarrow \{; \text{sum} := \text{sum} + w, p(\text{sum}, x)\}$$

is equivalent to

$$\begin{aligned} & \text{tuple}(z), \text{length}(z) == 2, \rightarrow \\ & \{; \text{sum} := \text{sum} + z[0], p(\text{sum}, z[1])\} \end{aligned}$$

14 Syntactic Sugar: Composition of Guarded Statements

The syntax that we gave for composed blocks was:

$$\begin{aligned} \textit{composed-block} &:: \{ ; \prec \textit{block} \succ^{(1)} \} \mid \\ &\quad \{ || \prec \textit{block} \succ^{(1)} \} \mid \\ &\quad \{ ? \prec \textit{guard} \rightarrow \textit{block} \succ^{(1)} \} \end{aligned}$$

Thus parallel and sequential composition compose *blocks* but not *guard* \rightarrow *blocks*. For notational convenience we relax this syntax to allow parallel and sequential composition of *guard* \rightarrow *blocks*. Each *guard* \rightarrow *block* is transformed into the choice block $\{ ? \textit{guard} \rightarrow \textit{block} \}$. The sugared syntax is:

$$\begin{aligned} \textit{composed-block} &:: \{ ; \prec \textit{unit} \succ^{(1)} \} \mid \\ &\quad \{ || \prec \textit{unit} \succ^{(1)} \} \mid \\ &\quad \{ ? \prec \textit{guard} \rightarrow \textit{block} \succ^{(1)} \} \end{aligned}$$
$$\textit{unit} :: \textit{block} \mid \textit{guard} \rightarrow \textit{block}$$

We may find it convenient to think of *all* composition blocks as consisting of a composition operator operating on a list of *guard* \rightarrow *block*, where some *guards* can be *true*.

The Variable Name ‘_’ There are a few places where we would like to refer to a variable that we do not wish to use later. For instance, we may want a pattern with 3 elements, but we wish to use only the first of the 3 elements. Instead of coming up with names for the remaining two elements we can use ‘_’ as a name for both elements. Each instance of ‘_’ is treated as the name of a *new* unique variable.

15 Built-In Programs and I/O

make_tuple One of the built-in programs provided in PCN is *make_tuple* which has two arguments, n and x where x is a definition variable that is defined by *make_tuple* and n is a variable unchanged by the program. The program defines x to be a tuple of n elements, where n is a nonnegative integer, where each element of x is a definition variable. (If $n \leq 0$ then x is defined to be the empty tuple.) The elements of tuple x are left undefined by *make_tuple*. For example, *make_tuple*(2, x) defines x to be a 2-tuple, leaving definition variables $x[0]$ and $x[1]$ undefined.

The program call *make_tuple*(n, x) is executed as follows:

```
repeat skip until  $n$  is reducible;  
coerce  $n$  to integer (if necessary);  
if( $n \geq 0$ ) define  $x$  to be a tuple with  $n$  elements  
else define  $x$  to be a tuple with 0 elements.
```

I/O A standard I/O library is available in the system library *stdio.pcn*. This library deals with formatted I/O as in C but also allows for manipulation of definition variables.

An Example Using Variable Name ‘_’

```
p(x, z)
int n;
{? x ?= [{"msg", m, _} | xs] ->
    {; f(m, n), z = [n | zs], p(xs, zs)},
  x ?= [{"val", _, m} | xs] ->
    {; g(m, n), z = [n | zs], p(xs, zs)}
}
```

This program has an input argument x and an output argument z . The input is a list of 3-element tuples, where the first element of each tuple is the string "msg" or the string "val". If the first element is the string "msg" then the second element of the tuple is used by the program to compute n by executing $f(m, n)$, and then n is placed in list z . If the second element is the string "val" then the third element of the tuple is used by the program to compute n by executing $g(m, n)$, and then n is placed in list z .

16 Architectures, Implementation and Efficiency

The speed of execution of a PCN program depends on its implementation. To understand how to develop programs that execute quickly, programmers need to understand something about the implementation of PCN. The implementation may change in future releases, but the central ideas about efficiency are not likely to change significantly.

Uniprocessors, Multiprocessors, and Multicomputers PCN programs run on uniprocessors, multiprocessors or multicomputers. A multiprocessor is a collection of two or more (uni)processors where all processors access a common address space. A multicomputer is a collection of computers where different computers in the collection have disjoint address spaces[10]. A node of a multicomputer (i.e., one of the computers in the collection that forms the multicomputer) can be a uniprocessor, a multiprocessor or a multicomputer. One of the processors in a multiprocessor or a multicomputer is designated the *host* processor; this is the processor with which programmers interact. (The host processor in a uniprocessor is the uniprocessor itself.) A computer with n processors (in addition to the host) has its processors indexed 0 through $n - 1$, the host is indexed n . In systems where the host is itself one of the nodes then the host is numbered $n - 1$.

Mutables, Definition Variables and Address Spaces A mutable resides in precisely one address space. The implementation does not make copies of mutables. By contrast, a definition variable can have several copies that reside in arbitrarily many address spaces. Since a definition variable is either undefined, or defined and unchanging, all copies of a definition variable are consistent in the following sense: if two copies of a definition variable differ in value then in one of the copies the definition variable is undefined. A program does not make use of undefined definition variables; it merely waits for the definition variable to become defined. Therefore, no problems are created if the value of a definition variable is defined in one copy and undefined in another.

Consider a block b in a program p . If b references mutables (declared in p) then b is executed in the address space in which p is initiated. This is because there is only one copy of a mutable, and all blocks that access a mutable are executed in the address space in which the mutable resides. If b does not reference mutables then b can be executed in any address space; copies of definition variables are made in the address space in which b is executed, as needed by b . The greatest degree of concurrent execution is achieved by employing parallel composition in which the blocks composed in parallel do not share mutables; this allows blocks to be spawned on arbitrary address spaces and thus employs concurrency in multicomputers and multiprocessors.

Granularity If the execution time of a block is small, the time required to spawn the block in a remote address space may exceed the time gained from concurrent execution of the block. Therefore, programmers should ensure that block granularity in parallel composition is appropriate for the target architecture.

17 Simple Programming Examples

17.1 Membership in a List

Develop a program *member* with arguments x , m and r , where x is a list, m and r are mutables, and at termination of execution of the program, $r = true$ if and only if m appears in list x . Mutable m is to be left unchanged by *member*. Of course, definition variable x must be left unchanged by *member*.

```
member( $x, m, r$ )
int  $m, r$ ;
{?  $x$  ?= [] ->  $r := false$ ,
    $x$  ?= [ $v|xs$ ],  $v == m$  ->  $r := true$ ,
    $x$  ?= [ $v|xs$ ],  $v != m$  -> member( $xs, m, r$ )
}
```

Assume that *false* = 0 and *true* = 1, to be consistent with C.

Operation of the Program

1. If x is the empty list, then r becomes *false* and the program terminates.
2. If x is nonempty, let the head of x be v , and let the tail of x be xs ; if $v = m$ then r becomes *true* and the program terminates.
3. If x is nonempty and the head of x is not m , then the value of r is set by *member*(xs, m, r), and *member*(x, m, r) terminates execution when *member*(xs, m, r) does.

Reasoning About the Program In many examples we reason about the correctness (and the efficiency) of programs by induction. In this example, we carry out induction on the length of x . (The length of a list is the number of elements in it.)

Base Case: If x is the empty list then r is *false* upon termination because m does not appear in an empty list.

Induction Step: Assume that $member(x, m, r)$ is correct (i.e., it terminates with the correct values of its arguments) for all lists x that have at most k elements, for some $k \geq 0$, and prove that it is correct for lists with $k+1$ elements. If x has $k+1$ elements it has a head element. Let the head of x be v and let the tail of x be xs ; then xs has k elements. If $v = m$ then r must be *true* at termination of the program because m is in list x . If $v \neq m$, then at termination r is *true* if and only if m is in xs ; by the induction assumption, $member(xs, m, r)$ sets r to *true* if and only if m is in xs .

A Program with Definition Variables Now consider a program with the same specification, except that m and r are definition variables, where m is left unchanged by the program, and r is defined by the program. All we need to do is to remove the declaration of m and r , and replace assignment statements by definition statements.

```
member1(x, m, r)
{?  x  ?=  []          ->  r = false,
    x  ?=  [v|xs], v == m ->  r = true,
    x  ?=  [v|xs], v != m ->  member1(xs, m, r)
}
```

17.2 Sum all Elements in a List

Develop a program sum with arguments x and r , where x is a list of integers and r is a mutable integer. At termination of execution of the program, r is required to be the initial value of r plus the sum of the elements of x . List x is to be left unchanged. For example, if $x = [1, 2, 3]$ and $r = 4$ initially, then $r = 10$ at termination of the program.

```
sum(x, r)
int r;
{?  x  ?=  [v|xs]    ->  {; r := r + v, sum(xs, r)}
}
```

Operation of the Program In this program:

1. If x is the empty list the program terminates leaving r unchanged.
2. If x is nonempty, let v be the head of x and let xs be the tail of x ; mutable r becomes $r + v$, and then $sum(xs, r)$ is executed.

Reasoning About the Program Let r^i be the initial value of r , and let r^f be the value of r when program sum terminates execution. We reason about the program by inducting on the length of x .

Base Case: If x is the empty list, the program terminates and r is left unmodified, and hence $r^f = r^i$, as required.

Induction Step: Assume that program $sum(x, r)$ is correct for all lists x with length at most k , for some $k \geq 0$, and prove that it is correct for lists x with length $k + 1$. If the length of x is $k + 1$ then x is nonempty; let v be the head of x and let xs be the tail of x . List xs has length k . From the induction assumption, $sum(xs, r)$ is correct. Hence $r^f = r^i + v + \text{sum of elements in } xs$, and hence r^f is the sum of r^i and the sum of all elements of x , as required by the specification.

Another Summation Example Next we write a program *total* to define a definition variable z as the sum of all the elements of a list x . The difference between this program and the previous one is that z is definition variable whereas r is mutable, and furthermore z is to be the sum of the elements of x , whereas the sum of the elements of x was added to r in the previous program.

```
total(x, z)
int r;
{; r := 0, sum(x, r), z = r}
```

An alternate version *total1*, using only definition variables and parallel composition, is given next.

$$\begin{array}{l}
total1(x, z) \\
\{? \quad x ?= [] \quad \rightarrow z = 0, \\
\quad \quad x ?= [v|xs] \quad \rightarrow \{\parallel z = zs + v, total1(xs, zs)\} \\
\}
\end{array}$$

Operation of the Program The operation of this program is as follows.

1. If x is the empty list then definition variable z is defined to be 0.
2. If x is nonempty, let v be the head of x and let xs be the tail of x . Define definition variable z to be the expression $zs + v$, where zs is defined by $total1(xs, zs)$.

Reasoning About the Program We are obliged to ensure that blocks in parallel composition do not modify shared mutables. This program has no mutables, and so the restriction about shared variables holds vacuously.

We reason by induction on the length of x , as in the previous example. The reasoning is not given here because it is largely a repetition of the arguments given earlier.

Difference Between Programs Let x be the list $[1, 2]$. At the termination of $total(x, z)$, definition variable z is defined to be the number 3. At the termination of $total1(x, z)$, definition variable z is defined by the following equations:

$$\begin{array}{l}
z = 1 + a, \text{ where } a \text{ is defined by} \\
a = 2 + b, \text{ where } b \text{ is defined by} \\
b = 0.
\end{array}$$

If we now execute $m := z$, where m is a mutable, and then print m , we will get the same answer whether we use $total$ or $total1$ because the reduced value of z is the same in both cases.

At first glance, sequential programs such as $total$ may appear more efficient in memory and time than parallel programs such as $total1$. Consider the following variation in which a parallel implementation can require less time than a sequential implementation.

17.3 Sum Function of Elements in a List

Let x be a list. Define z as follows: z is the sum over all elements of the list of a function g applied to each element. For example, if x is $[1, 2]$ and g is the square operation, then $z = 5$.

A sequential program sigma , analogous to sum in the previous subsection, is given next.

```

sigma( $x, r$ )
int  $r, w$ ;
{?  $x \neq []$    -> {;  $f(v, w), r := r + w, \mathit{sigma}(xs, r)$ }
}

```

Program $f(v, w)$ sets the value of w to be $g(v)$.

Program Operation The operation of this program is as follows. If x is empty the program terminates with r unchanged. If x is not empty, let v be the head of x , and let xs be the tail of x ; first compute w , then increment r and then call $\mathit{sigma}(xs, r)$ to sum the remainder of the list.

Program tote , given next, is analogous to total of the previous subsection.

```

tote( $x, z$ )
int  $r$ ;
{;  $r := 0, \mathit{sigma}(x, r), z = r$ }

```

A version using parallel composition is given next.

```

totpar( $x, z$ )
int  $w$ ;
{?  $x \neq []$    ->  $z = 0,$ 
   $x \neq [v|xs]$  -> {|| {;  $f(v, w), y = w,$ 
                        $z = z + y,$ 
                       totpar( $xs, z$ )
                       }
}
}

```

This program spawns programs $f(v, w)$, for each element v of list x , in parallel. Thus if f takes a long time to execute, and x is a large list, and there are a large number of processors, the parallel version will execute faster than the sequential version, because execution of f for different elements of the list will proceed in parallel.

In program *totpar* we are obliged to ensure that no mutable shared variable in a parallel composition block is modified. There are no mutable shared variables in the parallel composition block, and hence the restriction is satisfied.

Consider an erroneous program in which the parallel composition block of *totpar* is replaced:

```

error_totpar(x, z)
int w;
{?  x ?= []      -> z = 0,
   x ?= [v|xs]   -> {|| f(v, w),
                    z = zs + w,
                    error_totpar(xs, zs)
                    }
}

```

In this program, w is a mutable shared by blocks $f(v, w)$ and $z = zs + w$ in a parallel composition block, and w is modified by $f(v, w)$; this violates the restriction on shared mutables in parallel composition. The reason for the restriction is that program *error_totpar*(x, z) does not specify what value of w is to be used in the definition $z = zs + w$: is it the value of w before, during or after the execution of $f(v, w)$? This problem does not arise in *totpar* because the shared definition variable y is used in place of shared mutable w , and y is defined as the correct value of w — the value of w after $f(v, w)$ is executed — by means of the sequential composition block $\{; f(v, w), y = w\}$.

17.4 Reverse a List

Develop a program *rev* with arguments x , e and b , where x and e are lists that are to be left unchanged by *rev*, and b is to be defined by *rev* to be the list of elements in x , in reverse order, concatenated with e . For example, if $x = ["A", "B"]$, and $e = ["C", "D"]$, then b is to be defined as $["B", "A", "C", "D"]$.

}

}

Operation of the Program

1. If t is the empty tuple, in which case t is the empty tree, its height z is defined to be 0.
2. If t is not the empty tree, then t is a 3-tuple. Let t be the tuple $\{left, val, right\}$. Define definition variables l and r by $ht(left, l)$ and $ht(right, r)$, respectively. If l exceeds r then define z as the expression $1 + l$. If r exceeds l then define z as the expression $1 + r$. If $r = l$, then define z either as $1 + l$ or $1 + r$.

Reasoning About the Program We check that mutable shared variables are not modified in parallel composition. We reason about the program by induction on the height of tree t .

Base Case: If t is the empty tree, then its height z is defined (correctly) as 0.

Induction Step: Assume that the program is correct for all trees t with height at most k for some $k \geq 0$, and prove that the program is correct for all trees with height $k + 1$. If t is a tree with height $k + 1$, then t is a tuple of the form $\{left, val, right\}$ where the heights of trees $left$ and $right$ are at most k . By the induction assumption, $ht(left, l)$ and $ht(right, r)$ correctly define l and r to be the heights of the left and right subtrees of t . Hence z is 1 more than the height of the higher subtree.

17.6 Preorder Traversal of a Binary Tree

Develop a program *preorder* with arguments t , b and e , where t is a binary tree, b and e are lists. Binary trees are represented using tuples as in the last example. Parameters t and e are to be left unchanged by the program. List b is to be the list consisting of the *val* of all nodes of the tree in preorder,

concatenated with list e . (A traversal of a tree in preorder visits the root, then the left subtree, and finally the right subtree.)

```

preorder( $t, b, e$ )
{?  $t$  ?= { }           ->  $b = e$ ,
   $t$  ?= { $left, val, right$ } -> {||  $b = [val | l]$ ,
                                preorder( $left, l, m$ ),
                                preorder( $right, m, e$ )
                                }
}

```

Operation of the Program

1. If t is the empty tree then b is defined as e .
2. If t is not empty, then it is a tuple of the form $\{left, val, right\}$. In parallel, define m by $preorder(right, m, e)$, and l by $preorder(left, l, m)$, and b as val followed by l .

Reasoning About the Program First check that shared mutables are not modified in parallel composition.

We prove correctness of the program by induction on the height of tree t .

Base Case: If t is the empty tree, then $b = e$, because there are no nodes to traverse.

Induction Step: Assume that $preorder(t, b, e)$ is correct for trees of height at most k , $k \geq 0$, and show that it is also correct for trees of height $k + 1$. Let t be a tree of height $k + 1$. Then $t = \{left, val, right\}$, where $left$ and $right$ are trees of height at most k . Define a definition variable m by $preorder(right, m, e)$. By the induction assumption, m is the preorder traversal of the right subtree concatenated with e . Define a definition variable l by $preorder(left, l, m)$. By the induction assumption, l is the preorder traversal of the left subtree concatenated with m . Define b as $[val | l]$; hence, b is val

followed by the preorder traversal of the left subtree of t followed by the preorder traversal of the right subtree of t , and hence b is the preorder traversal of t .

17.7 Quicksort with Copying

In this section we present C.A.R.Hoare's quicksort [9] program, $q0$, that uses lists (of definition variables); later, we discuss an *in place* quicksort, $q1$, that uses arrays. (The quicksort algorithm is discussed in most texts on algorithms such as [1].)

In this section, when we refer to a list of numbers we mean a list of definition variables that (eventually) reduce to numbers. Program $q0$ has two input variables, x and end , and one output variable, z : variables x and end are definition variables that are not defined by the program, and z is a definition variable that is defined by the program. All three variables are lists of numbers. The output z is specified to be the list x sorted in increasing order concatenated with list end . For example if $end = [5, 4]$ and $x = [2, 1]$, then $z = [1, 2, 5, 4]$. If end is the empty list, then z is x sorted in increasing order.

```

q0(x, end, z)
{?  x  ?= []          -> z = end,
   x  ?= [mid | xs]  -> {|| part(mid, xs, left, right),
                        q0(left, [mid | r], z),
                        q0(right, end, r)
                        }
}

```

Operation of the Program The operation of program $q0$ is as follows. If x is the empty list then z is defined to be end . If x is nonempty, let mid be the first element of x , and let xs be the remaining elements of x . The call $part(mid, xs, left, right)$ defines $left$ to be the list of values of xs that are at most mid , and $right$ to be the list of values of xs that exceed mid . Call $q0(right, end, r)$, thus defining r to be the sorted list of $right$ appended to end . Call $q0(left, [mid | r], z)$, thus defining z to be the sorted list of $left$ followed by mid followed by r .

Reasoning About the Program There are no shared mutables in the parallel composition, and hence the restrictions on parallel composition are satisfied. We reason about this program by induction on the length of x . The reasoning is straightforward and is left to the reader.

Next, we discuss program *part*. Program *part* inspects each element of xs in turn, placing elements that are at most mid in *left* and all other elements in *right*.

```

part(mid, xs, left, right)
{?  xs  ?=  []                                ->  {|| left = [], right = []},
   xs  ?=  [hd | tl], hd <= mid ->  {|| left = [hd | ls],
                                     part(mid, tl, ls, right)
                                     },
   xs  ?=  [hd | tl], hd > mid  ->  {|| right = [hd | rs],
                                     part(mid, tl, left, rs)
                                     }
}

```

Operation of the Program If xs is the empty list, define *left* and *right* to be empty lists. If xs is not empty, then let hd and tl be the head and tail (respectively) of xs . If hd is at most mid , define ls and $right$ by $part(mid, tl, ls, right)$, and define $left$ as hd followed by ls . If hd exceeds mid , define $left$ and rs by $part(mid, tl, left, rs)$, and define $right$ as hd followed by rs .

Reasoning About the Program We check that shared mutables are not modified in parallel composition, and reason about the program by induction on the length of xs .

Base Case: If xs is the empty list, then *left* and *right* are correctly defined as empty lists.

Induction Step: Assume that the program is correct for all mid , xs , where the length of xs is at most k , for some $k \geq 0$, and prove that the program is correct for all xs of length $k + 1$. If the length of xs is $k + 1$, then

xs is nonempty; let hd and tl be the head and tail (respectively) of xs . The length of tl is k . Next, consider the case where hd is at most mid . From the induction assumption, $part(mid, tl, ls, right)$ defines ls and $right$ to be the elements of tl that are at most mid , and that exceed mid , respectively. Since, xs is hd followed by tl , and $hd \leq mid$, it follows that the sequence of elements of xs that are at most mid is hd followed by the sequence of elements of tl that are at most mid , and the sequence of elements of xs that exceed mid is the sequence of elements of tl that exceed mid . Hence, in this case, the definitions of $left$ and $right$ are correct. A similar argument applies for the case where $hd > mid$.

17.8 In Place Quicksort

Program $q1$ Program $q1$ has two input parameters, l , and r , both of which are definition variables, and it has one input-output parameter C which is a one-dimensional array of numbers. Let C^{init} be the initial value of C , and let C^{final} be the value of C on termination of the program. Then C^{final} is to be a permutation of C^{init} , where $C^{final}[l, \dots, r]$ is $C^{init}[l, \dots, r]$ in increasing order, and the other elements of C are to remain unchanged. (If $l \geq r$ then C^{final} is C^{init} .)

```

q1(l, r, C)
int C[];
{? l < r -> {; split(l, r, C, mid),
              {|| q1(l, mid - 1, C), q1(mid + 1, r, C)}
              }
}

```

Execution of $split(l, r, C, mid)$ permutes C and assigns a value to mid such that $l \leq mid \leq r$, and such that all elements in $C[l, \dots, mid - 1]$ are at most $C[mid]$, and all elements in $C[mid + 1, \dots, r]$ exceed $C[mid]$.

Operation of the Program If $l \geq r$, then $q1$ takes no action, leaving C unchanged. If $l < r$, then $split$ is called, and after $split$ terminates execution, $C[l, \dots, mid - 1]$ and $C[mid + 1, \dots, r]$ are sorted in parallel.

Reasoning About the Program First check that shared mutables in parallel composition are not modified. Array C is shared by $q1(l, mid - 1, C)$ and $q1(mid + 1, r, C)$, but no element of C is shared by both blocks. Hence the restriction on parallel composition is satisfied.

We reason about the program by induction on $r - l$.

Base Case: If $r - l \leq 0$, then C is left unchanged, and this is correct according to our specifications.

Induction Step: Assume that $q1(l, r, C)$ is correct for all l, r , and C for which $r - l \leq k$, for some $k \geq 0$, and prove that the program is correct for $r - l = k + 1$. If $r - l = k + 1$ then $l < r$. In this case *split* is called, and execution of *split*(l, r, C, mid) permutes C and assigns a value to mid such that $l \leq mid \leq r$, and such that all elements in $C[l, \dots, mid - 1]$ are at most $C[mid]$, and all elements in $C[mid + 1, \dots, r]$ exceed $C[mid]$. Hence, $mid - 1 - l \leq k$, and $r - (mid + 1) \leq k$. From the induction assumption, $q1(l, mid - 1, C)$ and $q1(mid + 1, r, C)$ are correct, and sort $C[l, \dots, mid - 1]$ and $C[mid + 1, \dots, r]$. Therefore, $C[l, \dots, r]$ is sorted correctly.

Program *split*

```

split(l, r, C, mid)
int C[ ], left, right, temp;
{? l <= r ->
    {; left := l + 1, right := r, s = C[l],
      part1(l, r, C, s, left, right), temp := l,
      swap(temp, right, C), mid = right
    }
}

```

Operation of the Program If $l > r$ then *split* terminates execution without taking any action. If $l \leq r$, then program *split*(l, r, C, mid) calls *part1*($l, r, C, s, left, right$) after setting $left = l + 1$, $right = r$ and $s = C[l]$; program *part* leaves s unchanged, modifies $left$ and $right$, and permutes elements of $C[l + 1, \dots, r]$ so that, at termination of *part1*, $left = right + 1$,

and all elements in $C[l + 1, \dots, right]$ are at most s , and all elements in $C[right + 1, \dots, r]$ exceed s .

After termination of *part1*, program *swap* is called to exchange $C[l]$ (which is s) with $C[right]$. After the swap, all elements in $C[l, \dots, right - 1]$ are at most s , and $C[right] = s$, and all elements in $C[right + 1, \dots, r]$ exceed s . The program terminates after *mid* is defined as *right*.

Program *part1* Program *part1* moves *left* rightwards and moves *right* leftwards until they cross (i.e., $left = right + 1$), so that the following invariant is maintained:

invariant:

$l + 1 \leq left \leq r + 1$ and $l \leq right \leq r$, and
 all elements of $C[l + 1, \dots, left - 1]$ are at most s ,
 and all elements of $C[right + 1, \dots, r]$ exceed s .

From the invariant it follows that $left \leq right + 1$.

```

part1(l, r, C, s, left, right)
int C[], left, right;
{? left <= right -> {;
    {|| left_rightwards(r, C, s, left),
      right_leftwards(l + 1, C, s, right)
    },
    left <= right -> {; swap(left, right, C),
                      left := left + 1,
                      right := right - 1
                    },
    part1(l, r, C, s, left, right)
  }
}

```

```

left_rightwards(r, C, s, left)
int C[], left;
{? left <= r, C[left] <= s ->

```

```

    {; left := left + 1, left_rightwards(r, C, s, left)}
}

right_leftwards(l, C, s, right)
int C[ ], right;
{? right >= l, C[right] > s ->
    {; right := right - 1, right_leftwards(l, C, s, right)}
}

swap(i, j, C)
int i, j, C[ ], temp;
{; temp := C[i], C[i] := C[j], C[j] := temp}

```

Operation of the Program The invariant holds initially because $left = l + 1$ and $right = r$. Programs $left_rightwards(r, C, s, left)$ and $right_leftwards(l + 1, C, s, right)$ are executed in parallel. The only mutables that change value in these programs are $left$ and $right$, and these mutables are not shared. Therefore, the restriction on parallel composition is satisfied.

Program $left_rightwards(r, C, s, left)$ moves $left$ rightwards from $l + 1$, i.e., it increases $left$, until $left = r + 1$ or $C[left] > s$, and it maintains the invariant. Likewise, $right_leftwards(l + 1, C, s, right)$ moves $right$ leftwards from r , i.e., it decreases $right$, until $right = l$ or $C[right] \leq s$, and it maintains the invariant. If $left = r + 1$ or $right = l$ then $left > right$, and the program terminates execution. Consider the case where $C[right] \leq s < C[left]$, and $left \leq right$ at termination of the parallel composition block. In this case, $C[left]$ and $C[right]$ are exchanged, and then $left$ is incremented and $right$ is decremented, maintaining the invariant. Then $part1$ calls itself.

Reasoning About the Program We reason about the program by induction on $right + 1 - left$.

Base Case: Consider the case where $right + 1 - left = 0$. In this case the program terminates. If the invariant holds, the program terminates correctly. (The program may not terminate correctly if the invariant does not hold.)

Induction Step: Assume that the program is correct, for all parameters provided $right+1-left \leq k$, for some $k \geq 0$, provided the invariant holds when the program is initiated, and prove the program correct for $right+1-left = k+1$ provided the invariant holds when the program is initiated.

If $right+1-left = k+1$ then $left \leq right$. In this case the parallel composition of *right_leftwards* and *left_rightwards* maintains the invariant and decreases $right+1-left$, or it leaves *left* and *right* unchanged. In the latter case, $C[right] \leq s < C[left]$, and therefore, $C[left]$ and $C[right]$ are exchanged, and then *left* is incremented and *right* is decremented, maintaining the invariant, and reducing $right+1-left$. Therefore, when *part1* is called recursively, the invariant is maintained, and $right+1-left \leq k$. From the induction assumption, the recursive call to *part1* terminates execution with the correct values for C , *left* and *right*.

18 Instructions About Where to Execute Programs

For purposes of efficiency, programmers may want to specify the processors on which programs are to be executed. For this purpose the annotation '@' is added as a suffix to the program call [6]. Specifying processors does not change the semantics of programs.

The syntax of an annotated program call is:

```
program-call :: program-call@location
location     :: variable | integer | relative-location
```

A PCN program runs on a computer in which processors are numbered $0 \dots n$, where 0 is the host machine. A program p can be executed on a processor numbered i , for any integer i where $0 < i \leq n$ by executing $p@i$ or $p@v$, where v is a variable with reduced value i .

PCN can execute on a network of processors where the topology of the network can take any form: It can be a local area network connecting workstations, or a hypercube, or a mesh, to name a few examples. Programmers find it convenient to develop their programs for a given topology, and to have their topology mapped to that of the machine on which their programs execute [6].

The processor in the virtual network at which a program is to be executed can be specified by appending $@relative-location$ to the program call. For example, on a ring:

relative-location can be **fwd** for forward, **bwd** for backward, or **random**;

The relative-location specifies a processor in the virtual network by specifying its relationship (forward, backward, or random) with respect to the

processor executing the parallel composition block. For instance, execution of a parallel composition statement $\{\parallel p(left)@bwd, p(right)@fwd\}$ in a processor q in a virtual ring would cause $p(left)$ and $p(right)$ to be executed on the processor following q and preceding q , respectively, on the virtual ring.

The *location* at which a program $p(\dots)$ is to be executed can be specified at run-time by executing the statement $p(\dots)@'x$, where x is a definition variable. Execution of $p(\dots)@'x$, is as follows:

repeat skip until x is reducible;
execute $p@loc$ where " loc " is the reduced value of x .

For instance, if the reduced value of x is " fwd ", then $p(\dots)@'x$ is executed as $p(\dots)@fwd$.

If a program call is executed in an address space then the called program will not be executed in another address space if any argument of the called program is a mutable.

19 Compilation and Modules

Collections of related PCN programs are written in files, for convenience. All PCN programs can be put in a single file, but it is usually more convenient to partition programs in some logical manner, and put each set of related programs in a separate file [6]. A file containing PCN programs is called a **module**, and the name of the file is also the module name. The syntax of a module is:

```
module:: -exports(< program-name >(1))
         -foreign(< library-name >(1))
         < program >
```

where *program-name* is the name of one of the programs in the module, and *library-name* is the name of a library containing C object code, and all names are quoted.

An example of a module is:

```
-exports("member", "sum")
-foreign("algebra.a", "diffeqns.o")

.... definition of programs including member and sum ...
```

If there is no export statement in a module then all programs are exported. A program *p* within a module *m* can be called by programs in other modules if and only if *p* is exported by *m*. A program in a module references a PCN program in another module by prefixing the name of the referenced program

with the name of the module in which the referenced program is located, followed by the symbol ':'. For instance, a program in a module *B* references a program *p* in another module *D* as *D:p*. A program references another program in the same module without the '*module-name:*' prefix.

PCN programs can also call C and Fortran programs. Consider a PCN program *p* that calls a C program *f* that appears in a library *L*. The module in which *p* appears must have library *L* declared as foreign, by including *L* in the `-foreign(< library-name >)` statement that appears in the module.

A PCN source file can begin with macro definitions and file inclusion statements as in C.

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A Complete PCN Syntax

Non-terminal symbols begin with an uppercase letter. Token classes such as an integer, identifier (ID), or string are signified in uppercase and are lexically identical to C. All other symbols are terminal.

```
ModulePart :: Form | ModulePart Form
Form       :: Program | Directive
Directive  :: -exports Args | -capabilities Args | -foreign Args

Program    :: Heading Declarations Implication
Heading    :: ID Names
Names      :: ( ) | ( NameList )
NameList   :: ID | NameList , ID

Declarations :: empty | Declarations Declaration
Declaration :: Type Mutables ;
Mutables    :: Mutable | Mutables , Mutable
Mutable     :: ID | ID Dimension
Dimension   :: [ ] | [ INTEGER ] | [ Var ]

Implication :: Block | Guard -> Block
Block       :: Var := Exp | Var = Def | ID | Call | { Op Blocks }
Blocks      :: Implication | Blocks , Implication
Def         :: Exp | Tuple | List | STRING | Call
Op          :: | | ? | ;
Guard       :: Tests | default
Tests       :: Test | Tests , Test
Test        :: ID ?= Rhs | Con Eq Con | Exp Ar Exp | Type ( Def ) | data ( Def )
Con         :: Exp | STRING | [ ] | { }
Rhs         :: List | Tuple | Call
Eq          :: == | !=
Ar          :: < | > | <= | >=
Type        :: int | double | char | tuple
```

Call :: LocalCall | RemoteCall
LocalCall :: QID : QID Args | QID Args
QID :: ID | ' ID
RemoteCall :: LocalCall @ INTEGER | LocalCall @ QID
Args :: () | (ArgList)
ArgList :: Def | ArgList , Def

Exp :: Term | Exp + Term | Exp - Term
Term :: Factor | Term * Factor | Term Factor | Term % Factor
Factor :: Num | (Exp) | length (ID)
Num :: Numeric | - Numeric | Var
Numeric :: INTEGER | REAL
Var :: ID | ID Subscript
Subscript :: [INTEGER] | [Var]

List :: [] | [Elements] | [Elements | Element]
Tuple :: { Elements } | { }
Elements :: Element | Elements , Element
Element :: Num | STRING | List | Tuple | Call