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Part I : Linearly-Varying Radiant Exitance

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Abstract

We present a closed-form expression for the irradiance at a point on a surface due to an arbitrary polygonal Lambertian luminaire with linearly-varying radiant exitance. The solution consists of elementary functions and a single well-behaved special function that can be either approximated directly or computed exactly in terms of classical special functions such as Clausen's integral or the closely related dilogarithm. We first provide a general boundary integral that applies to all planar luminaires and then derive the closed-form expression that applies to arbitrary polygons, which is the result most relevant for global illumination. Our approach is to express the problem as an integral of a simple class of rational functions over regions of the sphere, and to convert the surface integral to a boundary integral using a generalization of irradiance tensors. The result extends the class of available closed-form expressions for computing direct radiative transfer from finite areas to differential areas. We provide an outline of the derivation, a detailed proof of the resulting formula, and complete pseudo-code of the resulting algorithm. Finally, we demonstrate the validity of our algorithm by comparison with Monte Carlo. While there are direct applications of this work, it is primarily of theoretical interest as it introduces much of the machinery needed to derive closed-form solutions for the general case of luminaires with radiance distributions that vary polynomially in both position and direction.

Keywords: Illumination, Rendering, Radiosity, Solid Angle, Irradiance, Monte Carlo

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1 Introduction

The computation of radiant energy transfers is an essential component of physically-based rendering algorithms, both for local and global illumination. In particular, radiative transfers among discrete surface elements arise in finite element methods for global illumination, and in both direct lighting computations and final gathers from coarse global solutions [2, 8, 14]. By far the most ubiquitous computations involve transfers from finite planar areas to differential areas. In diffuse piecewise-constant environments, these transfers correspond to point-to-area form-factors [3], for which a wide assortment of formulas are available.

In this paper we present a direct means of computing radiative transfers from Lambertian area light sources (luminaires) in which the radiant exitance is non-uniform, varying spatially. Few tools currently exist for handling this type of luminaire aside from Monte Carlo or numerical quadrature [6]. We derive our result with an approach similar to that used by Arvo [2]; specifically, we extend the concept of *axial moments* to accommodate a simple class of rational functions over the sphere. This new class of functions allows us to compute irradiance from non-uniform luminaires. One direct application of our result is the exact computation of transfers from linear polygonal elements to differential areas [10].

Common to all methods for computing surface integrals of this nature is Stokes' theorem [13], by which surface integrals can be converted into boundary integrals. Such reductions are generally advantageous, as boundary integrals are better suited for computation and frequently lead to closed-form solutions.

Our approach is to use Stokes' theorem to derive a recurrence relation for a tensor form of irradiance; a similar approach was previously used to derive closed-form expressions for simulating glossy reflection and illumination from directional luminaries [2]. Our representation leads to a general boundary integral for expressing irradiance from linearly-varying luminaires, which departs from that derived by DiLaura [6] in that our boundary integral is around the spherical projection of a luminaire with respect to the receiver point, and results in a closed-form solution in the case of polygonal luminaires. One complication that arises in our solution is the appearance of a *special function* that has no finite representation in terms of elementary functions. This also occurs in computing patch-to-patch form factors [12], which requires a special function known as the dilogarithm [9]. Interestingly, the special function encountered in our case can also be evaluated in terms of the dilogarithm, or a somewhat simpler function known as the Clausen integral.

The specific contributions of this paper are

1. The derivation of a boundary integral for computing the irradiance due to planar luminaires of arbitrary shape and linearly-varying radiant exitance,

2. The derivation of a *closed-form* expression (including one special function) for the restricted case of polygonal luminaires with linearly-varying radiant exitance,

3. A simple evaluation method that is suitable for irradiance computations in image synthesis.

The theoretical contribution of this work is somewhat broader, however, in that the same approach can be applied to luminaires with non-linear spatial variation and directional variation as well. All the cases of polynomial variation admit similar closed-form solutions, which rest on the same special function that emerges in the linear case [5].

The report is organized as follows. First, we introduce triple-axis moments and derive its recurrence relations by applying and extending angular moments. Then, in section 3 and 4, we reformulate the irradiance due to a Lambertian luminaire with linearly-varying radiant exitance as the integral of a simple class of rational functions over spherical regions, and represent it as an infinite summation of triple-axis moments. By means of assorted summation identities shown in section 5, we simplifies the infinite summation to obtain a general boundary integral for all luminaire shapes and specialize the result to polygonal luminaires, and describe various techniques to evaluate one special function arising in our formula. In section 8, we demonstrate the applications of our new formula in computer graphics and validate it with Monte Carlo. Some future work are discussed in the end.

2 Triple-axis Moments

We define the *triple-axis moment* of a region $A \subset S^2$ with respect to three axes $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$ by

$$\overset{\equiv}{T}^{i,j,k}(A, \mathbf{r}, \mathbf{s}, \mathbf{t}) \equiv \int_A \langle \mathbf{r}, \mathbf{u} \rangle^i \langle \mathbf{s}, \mathbf{u} \rangle^j \langle \mathbf{t}, \mathbf{u} \rangle^k \, d\omega,$$

which is a natural generalization of the *angular moments* used by Arvo [1] for simulating non-Lambertian phenomena. We shall consider a special case where $j = k = 1$; that is

$$\overset{\equiv}{T}^{n,1,1}(A, \mathbf{r}, \mathbf{s}, \mathbf{t}) \equiv \int_A \langle \mathbf{r}, \mathbf{u} \rangle^n \langle \mathbf{s}, \mathbf{u} \rangle \langle \mathbf{t}, \mathbf{u} \rangle \, d\omega.$$

By expressing the integrand as a tensor composition:

$$\langle \mathbf{r},\mathbf{u}\rangle^n \langle \mathbf{s},\mathbf{u}\rangle \langle \mathbf{t},\mathbf{u}\rangle = (\mathbf{u}\otimes\cdots\otimes\mathbf{u})^{n+2}_{Iij}(\mathbf{r}\otimes\cdots\otimes\mathbf{r})^n_I \mathbf{s}_i\mathbf{t}_j,$$

this triple-axis moment $\bar{\bar{\bar{\tau}}}^{n,1,1}$ can be represented in terms of irradiance tensor $\mathbf{T}$ [2, 1]:

$$\bar{\bar{\bar{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) = \mathbf{T}^{n+2}(A)(\mathbf{r}\otimes\cdots\mathbf{r}\otimes\mathbf{s}\otimes\mathbf{t}). \tag{1}$$

Here the irradiance tensor $\mathbf{T}^n(A)$ satisfies a recurrence relation [2, 1]

$$\mathbf{T}^{n+2}_{Iij}(A) = \frac{1}{n+3}\left[\sum_{k=1}^{n}\delta_{jI_k}\mathbf{T}^n_{(I/k)i}(A) + \delta_{ji}\mathbf{T}^n_I(A) - \int_{\partial A}\mathbf{u}^{n+1}_{Ii}\mathbf{n}_j ds\right], \tag{2}$$

where I is an n-index, I_k is its kth index, and I/k is the $(n-1)$-index obtained by deleting the kth index of I. In the above equations, and throughout the paper, we employ the summation convention, where repeated subscripts imply summation from 1 to 3.

Substituting equation (2) into (1), we have

$$\begin{aligned}
(n+3)\bar{\bar{\bar{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= (n+3)\mathbf{T}^{n+2}_{Iij}(A)(\mathbf{r}\otimes\cdots\otimes\mathbf{r})_I\mathbf{s}_i\mathbf{t}_j \\
&= \left[\sum_{k=1}^{n}\delta_{jI_k}\mathbf{T}^n_{(I/k)i}(A) + \delta_{ji}\mathbf{T}^n_I(A) - \int_{\partial A}\mathbf{u}^{n+1}_{Ii}\mathbf{n}_j ds\right](\mathbf{r}\otimes\cdots\otimes\mathbf{r})_I\mathbf{s}_i\mathbf{t}_j \\
&= \sum_{k=1}^{n}\delta_{jI_k}\mathbf{t}_j\mathbf{r}_{I_k}\mathbf{T}^n_{(I/k)i}(A)(\mathbf{r}\otimes\cdots\otimes\mathbf{r})_{I/k}\mathbf{s}_i + \\
&\quad \delta_{ji}\mathbf{s}_i\mathbf{t}_j\mathbf{T}^n_I(A)(\mathbf{r}\otimes\cdots\otimes\mathbf{r})_I - \\
&\quad \int_{\partial A}\mathbf{u}^{n+1}_{Ii}(\mathbf{r}\otimes\cdots\otimes\mathbf{r})_I\mathbf{s}_i\mathbf{t}_j\mathbf{n}_j ds,
\end{aligned}$$

which yields a recurrence formula for the triple-axis moments:

$$\boxed{\begin{aligned}
(n+3)\bar{\bar{\bar{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= n\langle\mathbf{r},\mathbf{t}\rangle\,\bar{\bar{\tau}}^{n-1,1}(A,\mathbf{r},\mathbf{s}) + \langle\mathbf{s},\mathbf{t}\rangle\,\bar{\tau}^n(A,\mathbf{r}) - \\
&\quad \int_{\partial A}\langle\mathbf{r},\mathbf{u}\rangle^n\langle\mathbf{s},\mathbf{u}\rangle\langle\mathbf{t},\mathbf{n}\rangle\,ds.
\end{aligned}} \tag{3}$$

The *axial moments*, $\bar{\tau}^n$ and $\bar{\bar{\tau}}^{n-1,1}$ appearing in the above expression each satisfy simpler recurrence relations when $\mathbf{r}$ is a unit vector [1, 2]; in particular, we have

$$(n+2)\bar{\bar{\tau}}^{n,1}(A,\mathbf{r},\mathbf{s}) = n\langle\mathbf{r},\mathbf{s}\rangle\,\bar{\tau}^{n-1}(A,\mathbf{r}) - \int_{\partial A}\langle\mathbf{r},\mathbf{u}\rangle^n\langle\mathbf{s},\mathbf{n}\rangle\,ds \tag{4}$$

$$
\begin{aligned}
(n+1)\bar{\tau}^n(A,\mathbf{r}) &= (n-1)\bar{\tau}^{n-2}(A,\mathbf{r}) - \int_{\partial A} \langle \mathbf{r},\mathbf{u}\rangle^{n-1} \langle \mathbf{r},\mathbf{n}\rangle \, ds \\
&= \bar{\tau}^q - \int_{\partial A} \left[\langle \mathbf{r},\mathbf{u}\rangle^{n-1} + \langle \mathbf{r},\mathbf{u}\rangle^{n-3} + \cdots + \langle \mathbf{r},\mathbf{u}\rangle^{q+1} \right] \times \\
& \quad \langle \mathbf{r},\mathbf{n}\rangle \, ds, \qquad (5)
\end{aligned}
$$

In equation (5), $q = 0$ if n is even, and $q = -1$ if n is odd. Moreover, $\bar{\tau}^{-1} = 0$ and $\bar{\tau}^0(A) = \sigma(A)$. It follows from the partial sum formula for a geometric series that

$$
\langle \mathbf{r},\mathbf{u}\rangle^{n-1} + \langle \mathbf{r},\mathbf{u}\rangle^{n-3} + \cdots + \langle \mathbf{r},\mathbf{u}\rangle^{q+1} = \begin{cases} \dfrac{1-\langle \mathbf{r},\mathbf{u}\rangle^{n+1}}{1-\langle \mathbf{r},\mathbf{u}\rangle^2} - \dfrac{1}{1+\langle \mathbf{r},\mathbf{u}\rangle} & \text{even } n \\[2ex] \dfrac{1-\langle \mathbf{r},\mathbf{u}\rangle^{n+1}}{1-\langle \mathbf{r},\mathbf{u}\rangle^2} & \text{odd } n \end{cases}
$$

The axial moment in equation (5) can be expressed in terms of solid angle and a boundary integral:

$$
(n+1)\bar{\tau}^n(A,\mathbf{r}) = \begin{cases} \left(\sigma(A) + \displaystyle\int_{\partial A} \frac{\langle \mathbf{r},\mathbf{n}\rangle}{1+\langle \mathbf{r},\mathbf{u}\rangle} \, ds \right) - \displaystyle\int_{\partial A} \frac{1-\langle \mathbf{r},\mathbf{u}\rangle^{n+1}}{1-\langle \mathbf{r},\mathbf{u}\rangle^2} \langle \mathbf{r},\mathbf{n}\rangle \; ds & \text{even } n \\[2ex] -\displaystyle\int_{\partial A} \frac{1-\langle \mathbf{r},\mathbf{u}\rangle^{n+1}}{1-\langle \mathbf{r},\mathbf{u}\rangle^2} \langle \mathbf{r},\mathbf{n}\rangle \; ds & \text{odd } n \end{cases} .
$$

However, this expression can be simplified using the identity

$$
-\int_{\partial A} \frac{\langle \mathbf{w},\mathbf{n}\rangle}{1+\langle \mathbf{w},\mathbf{u}\rangle} \, ds \; = \; \sigma(A), \qquad (6)
$$

for any unit vector $\mathbf{w}$ such that $-\mathbf{w} \notin A$, which is proven in Appendix B. It follows that

$$
(n+1)\bar{\tau}^n(A,\mathbf{r}) \; = \; -\int_{\partial A} \frac{1-\langle \mathbf{r},\mathbf{u}\rangle^{n+1}}{1-\langle \mathbf{r},\mathbf{u}\rangle^2} \langle \mathbf{r},\mathbf{n}\rangle \; ds \qquad (7)
$$

for any unit vector $\mathbf{r}$ such that $-\mathbf{r} \notin A$.

The formula (7) for axial moments is more convenient than the previous one given eariler in (5) in the sense that it subsumes solid angle, and there are no even/odd differences. More importantly, it provides an alternate approach to evaluate axial moments by expressing the integral in terms of hypergeometric functions, which may be more efficient than the existing algorithm of cost $O(n)$ [1].

Remark:

- The boundary integral in (5) is zero when $q+1>n-1$.
- These angular moments incorporate solid angle $\sigma(A)$ in their base cases:

$$\begin{aligned}\bar{\tau}^{-1}(A,\mathbf{r}) &= 0\\ \bar{\tau}^{0}(A,\mathbf{r}) &= \sigma(A)\\ \bar{\bar{\tau}}^{-1,1}(A,\mathbf{r},\mathbf{s}) &= 0\\ \bar{\bar{\tau}}^{0,1}(A,\mathbf{r},\mathbf{s}) &= \bar{\tau}^{1}(A,\mathbf{s}),\end{aligned}$$

- The base case of the triple-axis moment $\bar{\bar{\bar{\tau}}}^{n,1,1}$ reduces to a double-axis moment from the definition:

$$\bar{\bar{\bar{\tau}}}^{0,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) = \bar{\bar{\tau}}^{1,1}(A,\mathbf{s},\mathbf{t}).$$

Based on these recurrence relations, we can easily expand $\bar{\bar{\bar{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t})$ in equation (3) in terms of solid angle and boundary integrals. By rewriting equation (3) as

$$\begin{aligned}\bar{\bar{\bar{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= \frac{1}{n+3}\Big(n\left\langle\mathbf{r},\mathbf{t}\right\rangle \bar{\bar{\tau}}^{n-1,1}(A,\mathbf{r},\mathbf{s}) + \left\langle\mathbf{s},\mathbf{t}\right\rangle \bar{\tau}^{n}(A,\mathbf{r}) - \\ &\int_{\partial A}\left\langle\mathbf{r},\mathbf{u}\right\rangle^{n}\left\langle\mathbf{s},\mathbf{u}\right\rangle\left\langle\mathbf{t},\mathbf{n}\right\rangle ds\Big),\end{aligned}$$

and replacing $\bar{\bar{\tau}}^{n-1,1}$ by its recurrence formula from equation (4):

$$\bar{\bar{\tau}}^{n-1,1}(A,\mathbf{r},\mathbf{s}) = \frac{1}{n+1}\left((n-1)\left\langle\mathbf{r},\mathbf{s}\right\rangle \bar{\tau}^{n-2}(A,\mathbf{r}) - \int_{\partial A}\left\langle\mathbf{r},\mathbf{u}\right\rangle^{n-1}\left\langle\mathbf{s},\mathbf{n}\right\rangle ds\right),$$

we have

$$\begin{aligned}\bar{\bar{\bar{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= \frac{n(n-1)}{(n+3)(n+1)}\left\langle\mathbf{r},\mathbf{t}\right\rangle\left\langle\mathbf{r},\mathbf{s}\right\rangle \bar{\tau}^{n-2}(A,\mathbf{r}) + \frac{1}{n+3}\left\langle\mathbf{s},\mathbf{t}\right\rangle \bar{\tau}^{n}(A,\mathbf{r}) - \\ &\int_{\partial A}\Big[\frac{1}{n+3}\left\langle\mathbf{r},\mathbf{u}\right\rangle^{n}\left\langle\mathbf{s},\mathbf{u}\right\rangle\left\langle\mathbf{t},\mathbf{n}\right\rangle + \\ &\frac{n}{(n+3)(n+1)}\left\langle\mathbf{r},\mathbf{u}\right\rangle^{n-1}\left\langle\mathbf{r},\mathbf{t}\right\rangle\left\langle\mathbf{s},\mathbf{n}\right\rangle\Big]\, ds. \qquad (8)\end{aligned}$$

Applying the recurrence formula (5) for $\bar{\tau}^n$, we obtain

$$\begin{aligned}
\overline{\overline{\overline{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= \left(\frac{n-1}{(n+3)(n+1)}\left(n\left\langle \mathbf{r},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{s}\right\rangle + \left\langle \mathbf{s},\mathbf{t}\right\rangle\right)\right)\bar{\tau}^{n-2}(A,\mathbf{r}) - \\
&\int_{\partial A}\left[\frac{1}{n+3}\left\langle \mathbf{r},\mathbf{u}\right\rangle^n \left\langle \mathbf{s},\mathbf{u}\right\rangle \left\langle \mathbf{t},\mathbf{n}\right\rangle + \right.\\
&\left.\left\langle \mathbf{r},\mathbf{u}\right\rangle^{n-1}\left(\frac{n\left\langle \mathbf{r},\mathbf{t}\right\rangle \left\langle \mathbf{s},\mathbf{n}\right\rangle + \left\langle \mathbf{s},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{n}\right\rangle}{(n+3)(n+1)}\right)\right] ds. \qquad (9)
\end{aligned}$$

Using equation (7), we obtain a boundary integral representation for $\overline{\overline{\overline{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t})$:

$$\begin{aligned}
\overline{\overline{\overline{\tau}}}^{n,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= -\frac{1}{n+3}\left(\int_{\partial A}\left[\left\langle \mathbf{r},\mathbf{u}\right\rangle^n \left\langle \mathbf{s},\mathbf{u}\right\rangle \left\langle \mathbf{t},\mathbf{n}\right\rangle + \left\langle \mathbf{r},\mathbf{u}\right\rangle^{n-1}\frac{n\left\langle \mathbf{r},\mathbf{t}\right\rangle \left\langle \mathbf{s},\mathbf{n}\right\rangle + \left\langle \mathbf{s},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{n}\right\rangle}{n+1} + \right.\right.\\
&\left.\left.\frac{1-\left\langle \mathbf{r},\mathbf{u}\right\rangle^{n-1}}{1-\left\langle \mathbf{r},\mathbf{u}\right\rangle^2}\left(\frac{n\left\langle \mathbf{r},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{s}\right\rangle + \left\langle \mathbf{s},\mathbf{t}\right\rangle}{n+1}\left\langle \mathbf{r},\mathbf{n}\right\rangle\right)\right] ds\right) - \\
&= -\frac{1}{n+3}\left(\int_{\partial A}\left[\ \left\langle \mathbf{r},\mathbf{u}\right\rangle^n \left\langle \mathbf{s},\mathbf{u}\right\rangle \left\langle \mathbf{t},\mathbf{n}\right\rangle\ + \right.\right.\\
&\frac{n}{n+1}\left\langle \mathbf{r},\mathbf{u}\right\rangle^{n-1}\left\langle \mathbf{r},\mathbf{t}\right\rangle\left(\left\langle \mathbf{s},\mathbf{n}\right\rangle - \left\langle \mathbf{r},\mathbf{s}\right\rangle \left\langle \mathbf{r},\mathbf{n}\right\rangle\right) + \\
&\left.\left.\frac{1-\left\langle \mathbf{r},\mathbf{u}\right\rangle^{n+1}}{1-\left\langle \mathbf{r},\mathbf{u}\right\rangle^2}\left(\frac{n\left\langle \mathbf{r},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{s}\right\rangle + \left\langle \mathbf{s},\mathbf{t}\right\rangle}{n+1}\left\langle \mathbf{r},\mathbf{n}\right\rangle\right)\right] ds\right), \qquad (10)
\end{aligned}$$

where $-\mathbf{r} \notin A$. Although we assumed $n \geq 2$ in the equation (9), the equation (10) holds for all $n \geq 0$. This can be easily verified by examining several base cases:

<u>$n = 0:$</u>

$$\begin{aligned}
\overline{\overline{\overline{\tau}}}^{0,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) &= \overline{\overline{\tau}}^{1,1}(A,\mathbf{s},\mathbf{t}) \\
&= \frac{1}{3}\left\langle \mathbf{s},\mathbf{t}\right\rangle \sigma(A) - \int_{\partial A}\frac{\left\langle \mathbf{s},\mathbf{u}\right\rangle \left\langle \mathbf{t},\mathbf{n}\right\rangle}{3} ds \\
&= -\frac{1}{3}\int_{\partial A}\left(\left\langle \mathbf{s},\mathbf{u}\right\rangle \left\langle \mathbf{t},\mathbf{n}\right\rangle + \frac{\left\langle \mathbf{s},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{n}\right\rangle}{1+\left\langle \mathbf{r},\mathbf{u}\right\rangle}\right) ds.
\end{aligned}$$

<u>$n = 1:$</u>

$$\overline{\overline{\overline{\tau}}}^{1,1,1}(A,\mathbf{r},\mathbf{s},\mathbf{t}) = -\frac{1}{4}\int_{\partial A}\left[\left\langle \mathbf{r},\mathbf{u}\right\rangle \left\langle \mathbf{s},\mathbf{u}\right\rangle \left\langle \mathbf{t},\mathbf{n}\right\rangle + \frac{\left\langle \mathbf{r},\mathbf{t}\right\rangle \left\langle \mathbf{s},\mathbf{n}\right\rangle + \left\langle \mathbf{s},\mathbf{t}\right\rangle \left\langle \mathbf{r},\mathbf{n}\right\rangle}{2}\right] ds.$$

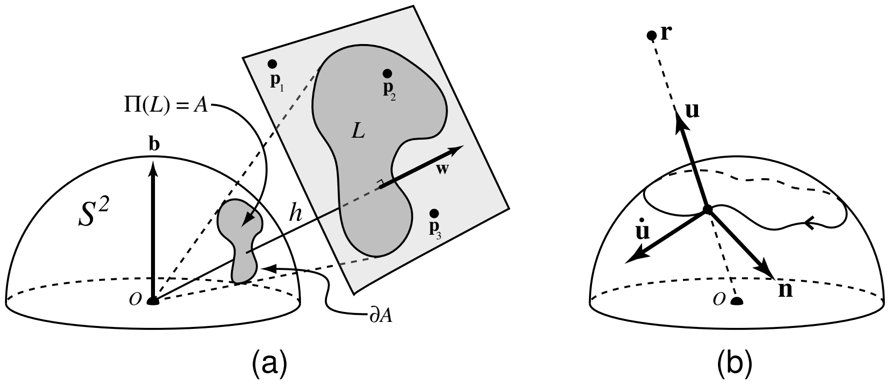


Figure 1: *(a) The irradiance at* $\mathbf{o}$ *due to a planar figure* L *with linearly varying radiant exitance is computed by a boundary integral. The radiant exitance variation is uniquely determined by any three non-collinear points* $\mathbf{p}_1$*,* $\mathbf{p}_2$*, and* $\mathbf{p}_3$ *on the luminaire plane. (b) The outward normal* $\mathbf{N}$ *is tangent to the sphere and orthogonal to the projection curve.*

3 Luminaires with Varying Radiant Exitance

By *Spatially varying luminaires* we mean the class of *non-uniform luminaires* whose radiance distributions are a function of position. Handling this type of spatially varying luminarie is an important generalization with immediate applications to high-order finite element methods for simulating global illumination. To our knowledge, no closed-form solutions for such luminaires have been previously reported. In this section, we consider a class of luminaries with linearly-varying radiant exitance and reformulate the problem as the integral of a rational function over regions of the sphere.

The irradiance ϕ impinging on a surface at the point $\mathbf{o}$ due to a planar luminaire L is given by

$$\phi(L) \;=\; \frac{1}{\pi} \int_L f(\mathbf{x}) \frac{\cos\theta_1 \cos\theta_2}{r^2}\, d\mathbf{x}, \tag{11}$$

where $\mathbf{x}$ is a point on L, r is its distance from $\mathbf{o}$, and $f(\mathbf{x})$ denotes the *radiant exitance* of the luminaire at the point $\mathbf{x}$. Here θ_1 denotes the angle between $\mathbf{x} - \mathbf{o}$ and the receiver normal $\mathbf{b}$, and θ_2 denotes the angle between $\mathbf{x} - \mathbf{o}$ and the luminaire normal. Without loss of generality, we shall assume that $\mathbf{o}$ is the origin throughout the remainder of the report.

If we integrate over the spherical projection of L, denoted by $A = \prod(L)$, we have

$$\begin{aligned}\phi(A) &= \frac{1}{\pi}\int_A f(\mathbf{x})\cos\theta_1\, d\sigma(\mathbf{u}) \\ &= \frac{1}{\pi}\int_A f(\mathbf{x})\,\langle \mathbf{b},\mathbf{u}\rangle\ d\sigma(\mathbf{u}),\end{aligned} \tag{12}$$

where $\mathbf{u}$ is the unit vector on the sphere, and $\mathbf{x}$ is the point on L in the direction of $\mathbf{u}$. The measure σ denotes area on the sphere. The class of luminaires that we consider are those that are planar and with linearly-varying radiant exitance, or equivalently, for which the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is linear. Thus, if $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are three non-collinear points on L with radiant exitance w_1, w_2 and w_3, respectively, then $f(\mathbf{x})$ can be expressed as a linear combination of w_1, w_2 and w_3:

$$f(\mathbf{x}) = [w_1\ w_2\ w_3]\,\hat{\mathbf{x}}, \tag{13}$$

where $\hat{\mathbf{x}}$ is barycentric coordinate vector of $\mathbf{x}$ with respect to $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$; that is,

$$\hat{\mathbf{x}} = [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}\,\mathbf{x},$$

where $[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]$ is the matrix with $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ as its columns. Thus, Equation (13) becomes

$$f(\mathbf{x}) = [w_1\ w_2\ w_3]\,[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}\,\mathbf{x}. \tag{14}$$

Let $\mathbf{w}$ denote the unit vector orthogonal to the planar luminaire, and let h denote the distance from the origin to the plane containing the luminaire. See Figure 1a. By expressing $\mathbf{x}$ in equation (14) in terms of the unit vector $\mathbf{u}$,

$$\mathbf{x} = \frac{h}{\langle \mathbf{w},\mathbf{u}\rangle}\mathbf{u}, \tag{15}$$

we have

$$f(\mathbf{u}) = \frac{\langle \mathbf{a},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle},$$

where the vector $\mathbf{a}$ is given by

$$\mathbf{a} = h\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}^{-1}. \tag{16}$$

Consequently, the irradiance formula (12) becomes the integral of a simple rational function over A. Specifically,

$$\phi(A) = \frac{1}{\pi}\int_A \frac{\langle \mathbf{a},\mathbf{u}\rangle\,\langle \mathbf{b},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle}\ d\sigma(\mathbf{u}). \tag{17}$$

As for a receiver point $\mathbf{q}$ other than the origin $\mathbf{o}$, the vector $\mathbf{a}$ can be computed by translating the luminaire by $-\mathbf{q}$, that is

$$\mathbf{a} = h \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 - \mathbf{q} & \mathbf{p}_2 - \mathbf{q} & \mathbf{p}_3 - \mathbf{q} \end{bmatrix}^{-1}. \tag{18}$$

However, this translation method is inefficient since it requires inverting a different matrix at each receiver point. In fact, the vector argument $\mathbf{a}$ corresponding to an arbitrary receiver point $\mathbf{q}$ can be updated immediately from that in the origin. Let

$$\mathbf{a}_0 = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}^{-1},$$

we have shown in Appendix G that the vector $\mathbf{a}$ in equation (17) which is associated with an arbitrary receiver point $\mathbf{q}$ can be expressed as

$$\mathbf{a}(\mathbf{q}) = \langle \mathbf{a}_0, \mathbf{q} \rangle \ \mathbf{w} + h(\mathbf{q})\, \mathbf{a}_0, \tag{19}$$

where $h(\mathbf{q})$ is the distance from $\mathbf{q}$ to the luminaire plane. Obviously, equation (19) is more efficient than equation (18) in that only one matrix inverse is involved for a linearly-varying luminaire.

4 Irradiance from Linearly-Varying Luminaires

In Section 2, we have introduced triple-axis moments and a basic recurrence relation that it satisfies. Here, we employ this extension of axial moments to accommodate the integral of a simple class of rational functions arising in the irradiance formula (17), which provides a crucial step to pursuing the closed-form formula on computing the irradiance ϕ from linearly-varying luminaires. The underlying mathematical tool we used here is Taylor expansion.

To begin, we expand the denominator of the rational integrand in equation (17) into an infinite series using Taylor's theorem and the binomial theorem. Let $y = \langle \mathbf{w}, \mathbf{u} \rangle$, then the series at $y = y_0$ (y_0 is a scalar constant to be chosen) can be computed as:

$$\begin{aligned} \frac{1}{y} &= \frac{1}{y_0 + (y - y_0)} \\ &= \frac{1}{y_0} \left[\sum_{n=0}^{\infty} \left(-\frac{y - y_0}{y_0} \right)^n \right] \\ &= \frac{1}{y_0} \left[\sum_{n=0}^{\infty} \left(1 - \frac{y}{y_0} \right)^n \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{y_0}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k}\left(\frac{y}{y_0}\right)^k\right] \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{y_0^{k+1}}\binom{n}{k} y^k. \qquad (20)
\end{aligned}
$$

for $|y - y_0| \leq y_0$. As a result, the integrand $\langle \mathbf{a}, \mathbf{u}\rangle \langle \mathbf{b}, \mathbf{u}\rangle \,/\, \langle \mathbf{w}, \mathbf{u}\rangle$ can also be expanded into an infinite series, obtaining

$$
\begin{aligned}
\phi(A) &= \frac{1}{\pi}\int_A \frac{1}{y}\langle \mathbf{a}, \mathbf{u}\rangle \langle \mathbf{b}, \mathbf{u}\rangle \, d\omega \\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{y_0^{k+1}}\binom{n}{k}\int_A \langle \mathbf{w}, \mathbf{u}\rangle^k \langle \mathbf{a}, \mathbf{u}\rangle \langle \mathbf{b}, \mathbf{u}\rangle \, d\omega.
\end{aligned}
$$

Notice that

$$
\bar{\bar{\bar{\tau}}}^{k,1,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) = \int_A \langle \mathbf{w}, \mathbf{u}\rangle^k \langle \mathbf{a}, \mathbf{u}\rangle \langle \mathbf{b}, \mathbf{u}\rangle \, d\omega,
$$

the irradiance turns into an infinite summation of triple-axis moments:

$$
\phi(A) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{y_0^{k+1}}\binom{n}{k}\bar{\bar{\bar{\tau}}}^{k,1,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}). \qquad (21)
$$

To be complete, we have to show that there exists a scalar y_0 at which the Taylor series (20) converges. Or equivalently, show that there exists a unit vector $\mathbf{u}_0$ over the sphere such that

$$
y_0 = \langle \mathbf{w}, \mathbf{u}_0\rangle \qquad (22)
$$

and

$$
|\langle \mathbf{w}, \mathbf{u}\rangle - y_0| \quad \leq \quad y_0 \qquad (23)
$$

for all $\mathbf{u}$ over the spherical projection A. In deriving the irradiance formula (17) from a linearly-varying luminaire, we define $\mathbf{w}$ as the unit vector from the origin orthogonal to the planar luminaire. By choosing the fixed vector $\mathbf{u}_0 = \mathbf{w}$, we get $y_0 = 1$. Let φ $(0 \leq \varphi \leq \pi)$ denote the angle between any unit vector $\mathbf{u}$ over the sphere and the perpendicular vector $\mathbf{w}$, the convergence condition (23) is actually equivalent to

$$
|\cos\varphi - 1| \leq 1, \qquad (24)
$$

which is obvious true for the illuminated hemisphere where $0 \leq \varphi \leq \pi/2$. Otherwise, $\mathbf{u}$ cannot hit the planar luminaire. Thus, our Taylor expansion of $\frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle}$ around $\mathbf{u}_0 = \mathbf{w}$ converges for planar luminaires. With $y_0 = 1$, Equation (21) simplifies to

$$\boxed{\phi(A) = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{\overline{\overline{\mathcal{T}}}}^{k,1,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}).} \tag{25}$$

In order to obtain a closed-form expression for $\phi(A)$, we have to get rid of the infinite summation, which can be done by incorporating the summation into the boundary integral representation (10) of the triple-axis moment. This step rests upon assorted binomial summation identities and series identities, which will be described in the next section.

5 Assorted Summation Identities

First, we provide some miscellaneous transformations and series arising frequently in our simplification, some are Taylor series.

$$\frac{1}{(n+3)(n+2)(n+1)} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}\right) \tag{26}$$

$$\sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = -\ln x \tag{27}$$

$$\sum_{n=0}^{\infty} \frac{(1-x)^{n+2}}{n+2} = x - 1 - \ln x \tag{28}$$

$$\sum_{n=0}^{\infty} \frac{(1-x)^{n+3}}{n+3} = -\frac{(x-3)(x-1)}{2} - \ln x \tag{29}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}\right) = \frac{1}{2}. \tag{30}$$

Next, we list some important binomial identities. Their proofs are shown in Appendix A.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k+1} = \frac{1 - (1-x)^{n+1}}{(n+1)x} \tag{31}$$

$$
\begin{aligned}
\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{x^k}{k+3} &= \frac{1}{x^3}\left(\frac{2}{(n+1)(n+2)(n+3)} - \frac{(1-x)^{n+3}}{n+3} + \right. \\
&\left. \frac{2(1-x)^{n+2}}{n+2} - \frac{(1-x)^{n+1}}{n+1}\right) && (32) \\
\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{k+1} &= \frac{1}{n+1} && (33) \\
\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{k+3} &= \frac{2}{(n+3)(n+2)(n+1)} && (34)
\end{aligned}
$$

Finally, by applying the inifinite summation over the index n to the the binomial identities above, we easily derive the following double summation identities (See Appendix A for details):

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1-x^{k+1}}{k+1} = -\ln x \tag{35}
$$

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{x^k}{k+3} = \frac{1}{2x} \tag{36}
$$

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{k+3} = \frac{1}{2} \tag{37}
$$

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1-x^{k+1}}{k+3} = 0 \tag{38}
$$

In all the identities, we assumes that $0 < x \leq 2$, which guarantees convergence of the Taylor series. With all these identities described, we are prepared to derive our new closed-form expression for the irradiance ϕ from linearly-varying luminaires.

6 The General Boundary Integral

The simplification of the infinite summation

$$
\phi(A) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \overset{\equiv}{\mathcal{T}}^{k,1,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b})
$$

is performed by applying assorted binomial identities discussed in the previous section. In this section we show each step in detail.

Notice that for $\mathbf{w}$ is the unit vector orthogonal to the planar luminaires, it obviously satisfies $-\mathbf{w} \notin A$ and $0 < \langle \mathbf{w}, \mathbf{u} \rangle \leq 1$. In Section 2, we have shown that the triple-axis moment $\overline{\overline{\overline{\tau}}}^{k,1,1}$ with respect to the spherical projection A and such a unit vector $\mathbf{w}$ can be computed as boundary integrals. That is,

$$
\begin{aligned}
\overline{\overline{\overline{\tau}}}^{k,1,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) \;=\; & -\frac{1}{k+3}\left(\int_{\partial A}\Bigg[\; \langle \mathbf{w}, \mathbf{u} \rangle^k \langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{n} \rangle \;+\right. \\
& \frac{k}{k+1} \langle \mathbf{w}, \mathbf{u} \rangle^{k-1} \langle \mathbf{w}, \mathbf{b} \rangle \left(\langle \mathbf{a}, \mathbf{n} \rangle - \langle \mathbf{w}, \mathbf{a} \rangle \langle \mathbf{w}, \mathbf{n} \rangle\right) + \\
& \left.\frac{1 - \langle \mathbf{w}, \mathbf{u} \rangle^{k+1}}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2}\left(\frac{k \langle \mathbf{w}, \mathbf{b} \rangle \langle \mathbf{w}, \mathbf{a} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle}{k+1} \langle \mathbf{w}, \mathbf{n} \rangle\right)\Bigg] \; ds\right), \qquad (39)
\end{aligned}
$$

Let us define

$$
\begin{aligned}
T_1 &= \int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{u} \rangle^k}{k+3} \langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{n} \rangle \; ds \\
T_2 &= \int_{\partial A} \frac{k}{(k+3)(k+1)} \langle \mathbf{w}, \mathbf{u} \rangle^{k-1} \langle \mathbf{w}, \mathbf{b} \rangle \left(\langle \mathbf{a}, \mathbf{n} \rangle - \langle \mathbf{w}, \mathbf{a} \rangle \langle \mathbf{w}, \mathbf{n} \rangle\right) \; ds \\
T_3 &= \int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{n} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \frac{1 - \langle \mathbf{w}, \mathbf{u} \rangle^{k+1}}{(k+3)(k+1)} \left(k \langle \mathbf{w}, \mathbf{b} \rangle \langle \mathbf{w}, \mathbf{a} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle\right) \; ds
\end{aligned}
$$

Then, Equation (39) can be expressed as

$$
\overline{\overline{\overline{\tau}}}^{k,1,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) \;=\; -\left(T_1 + T_2 + T_3\right),
$$

And the irradiance ϕ becomes

$$
\phi(A) \;=\; -\frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(T_1 + T_2 + T_3\right). \qquad (40)
$$

The three resulting infinite summations are reduced to finite terms by using the double summation identities described in the previous section. That is,

(1) Term One

$$
\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \; T_1 \;&=\; \int_{\partial A} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\langle \mathbf{w}, \mathbf{u} \rangle^k}{k+3} \langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{n} \rangle \; ds \\
&=\; \frac{1}{2} \int_{\partial A} \frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \langle \mathbf{b}, \mathbf{n} \rangle \; ds. \qquad (41)
\end{aligned}
$$

(2) Term Two

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} T_2 \\
= & \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \int_{\partial A} \langle \mathbf{w},\mathbf{b}\rangle \left(\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle\right) \frac{k}{(k+3)(k+1)} \langle \mathbf{w},\mathbf{u}\rangle^{k-1}\, ds \\
= & \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \int_{\partial A} \frac{\langle \mathbf{w},\mathbf{b}\rangle \left(\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle\right)}{\langle \mathbf{w},\mathbf{u}\rangle^2} \left(\frac{3}{k+3} - \frac{1}{k+1}\right) \langle \mathbf{w},\mathbf{u}\rangle^{k+1}\, ds \\
= & \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \int_{\partial A} \frac{\langle \mathbf{w},\mathbf{b}\rangle \left(\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle\right)}{\langle \mathbf{w},\mathbf{u}\rangle^2} \times \\
& \qquad \left(-\frac{1}{k+1} + \frac{3\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+3} + \frac{1-\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+1}\right) ds.
\end{aligned}
$$

In addition, it follows from Stokes' theorem that

$$
\int_{\partial A} \frac{\langle \mathbf{v},\mathbf{n}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2}\, ds = \langle \mathbf{w},\mathbf{v}\rangle \int_{\partial A} \frac{\langle \mathbf{w},\mathbf{n}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2}\, ds \tag{42}
$$

for any arbitrary vector $\mathbf{v}$ and unit vector $\mathbf{w}$ in S^2. See Appendix C for the proof of this identity. Thus,

$$
\int_{\partial A} \frac{\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2} \;=\; 0.
$$

Consequently, we have

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} T_2 \\
= & \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \int_{\partial A} \frac{\langle \mathbf{w},\mathbf{b}\rangle \left(\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle\right)}{\langle \mathbf{w},\mathbf{u}\rangle^2} \times \\
& \qquad \left(\frac{3\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+3} + \frac{1-\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+1}\right) ds \\
= & \frac{1}{2}\int_{\partial A} \frac{\langle \mathbf{w},\mathbf{b}\rangle \left(\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle\right)}{\langle \mathbf{w},\mathbf{u}\rangle^2} \times \\
& \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(\frac{3\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+3} + \frac{1-\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+1}\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial A} \frac{\langle \mathbf{w},\mathbf{b}\rangle \left(\langle \mathbf{a},\mathbf{n}\rangle - \langle \mathbf{w},\mathbf{a}\rangle \langle \mathbf{w},\mathbf{n}\rangle\right)}{\langle \mathbf{w},\mathbf{u}\rangle^2} \left(\frac{3}{2} - \ln\langle \mathbf{w},\mathbf{u}\rangle\right)\, ds \\
&= \frac{\langle \mathbf{w},\mathbf{b}\rangle}{2}\int_{\partial A} \left(\langle \mathbf{w},\mathbf{a}\rangle \langle \mathbf{w},\mathbf{n}\rangle - \langle \mathbf{a},\mathbf{n}\rangle\right) \frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2}\, ds. \qquad (43)
\end{aligned}
$$

(3) Term Three

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k}\ T_3 \\
= &\int_{\partial A} \frac{\langle \mathbf{w},\mathbf{n}\rangle}{1-\langle \mathbf{w},\mathbf{u}\rangle^2} \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left[\left(1-\langle \mathbf{w},\mathbf{u}\rangle^{k+1}\right)\left(\frac{k\langle \mathbf{w},\mathbf{b}\rangle\langle \mathbf{w},\mathbf{a}\rangle}{(k+3)(k+1)} + \frac{\langle \mathbf{a},\mathbf{b}\rangle}{(k+3)(k+1)}\right)\right] ds \\
= &\frac{1}{2}\int_{\partial A} \frac{\langle \mathbf{w},\mathbf{n}\rangle}{1-\langle \mathbf{w},\mathbf{u}\rangle^2} \Bigg[\left(3\langle \mathbf{w},\mathbf{b}\rangle\langle \mathbf{w},\mathbf{a}\rangle - \langle \mathbf{a},\mathbf{b}\rangle\right) \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1-\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+3} + \\
&\left(\langle \mathbf{a},\mathbf{b}\rangle - \langle \mathbf{w},\mathbf{b}\rangle\langle \mathbf{w},\mathbf{a}\rangle\right) \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1-\langle \mathbf{w},\mathbf{u}\rangle^{k+1}}{k+1}\Bigg] ds \\
= &\frac{\langle \mathbf{w},\mathbf{b}\rangle\langle \mathbf{w},\mathbf{a}\rangle - \langle \mathbf{a},\mathbf{b}\rangle}{2}\int_{\partial A} \langle \mathbf{w},\mathbf{n}\rangle \frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{1-\langle \mathbf{w},\mathbf{u}\rangle^2}\, ds. \qquad (44)
\end{aligned}
$$

Incorporating equations (41), (43) and (44), Equation (40) simplifies to

$$
\begin{aligned}
\phi(A) \;=\; &-\frac{1}{2\pi}\int_{\partial A}\Bigg(\frac{\langle \mathbf{a},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle}\langle \mathbf{b},\mathbf{n}\rangle + \left[\langle \mathbf{w},\mathbf{b}\rangle\langle \mathbf{w},\mathbf{a}\rangle - \langle \mathbf{a},\mathbf{b}\rangle\right]\langle \mathbf{w},\mathbf{n}\rangle \frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{1-\langle \mathbf{w},\mathbf{u}\rangle^2} \\
&+ \left[\langle \mathbf{w},\mathbf{a}\rangle\langle \mathbf{w},\mathbf{n}\rangle - \langle \mathbf{a},\mathbf{n}\rangle\right]\langle \mathbf{w},\mathbf{b}\rangle \frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2}\Bigg)\, ds \qquad (45)
\end{aligned}
$$

In Appendix D, we have shown that another identity

$$
\int_{\partial A}\left[\langle \mathbf{a},\mathbf{n}\rangle - \frac{\langle \mathbf{a},\mathbf{u}\rangle\langle \mathbf{w},\mathbf{n}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle}\right] ds \;=\; \int_{\partial A}\left[\langle \mathbf{a},\mathbf{w}\rangle\langle \mathbf{w},\mathbf{n}\rangle - \langle \mathbf{a},\mathbf{n}\rangle\right]\frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2}\, ds \qquad (46)
$$

holds for an arbitrary vector $\mathbf{a}$ and a unit vector $\mathbf{w}$. Consequently, Equation (45) can be further simplified to

$$
\phi(A) \;=\; -\frac{1}{2\pi}\int_{\partial A}\Bigg(\frac{\langle \mathbf{a},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle}\langle \mathbf{b},\mathbf{n}\rangle + \left[\langle \mathbf{w},\mathbf{b}\rangle\langle \mathbf{w},\mathbf{a}\rangle - \langle \mathbf{a},\mathbf{b}\rangle\right]\langle \mathbf{w},\mathbf{n}\rangle \frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{1-\langle \mathbf{w},\mathbf{u}\rangle^2}
$$

$$
\begin{aligned}
&+ \langle \mathbf{b}, \mathbf{w} \rangle \left[\langle \mathbf{a}, \mathbf{n} \rangle - \frac{\langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \right] \Bigg) \; ds \\
= \;& -\frac{1}{2\pi} \int_{\partial A} \Bigg(\langle \mathbf{b}, \mathbf{w} \rangle \langle \mathbf{a}, \mathbf{n} \rangle + [\langle \mathbf{b}, \mathbf{n} \rangle - \langle \mathbf{b}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{n} \rangle] \frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \\
&+ [\langle \mathbf{a}, \mathbf{w} \rangle \langle \mathbf{b}, \mathbf{w} \rangle - \langle \mathbf{a}, \mathbf{b} \rangle] \langle \mathbf{w}, \mathbf{n} \rangle \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \Bigg) \; ds.
\end{aligned} \tag{47}
$$

Notice that

$$
\begin{aligned}
\langle \mathbf{b}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{n} \rangle - \langle \mathbf{b}, \mathbf{n} \rangle &= \mathbf{b}^{\mathrm{T}} \mathbf{w} \mathbf{w}^{\mathrm{T}} \mathbf{n} - \mathbf{b}^{\mathrm{T}} \mathbf{n} \\
&= \mathbf{b}^{\mathrm{T}} \left(\mathbf{w} \mathbf{w}^{\mathrm{T}} - \mathbf{I} \right) \mathbf{n},
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \mathbf{a}, \mathbf{w} \rangle \langle \mathbf{b}, \mathbf{w} \rangle - \langle \mathbf{a}, \mathbf{b} \rangle &= \mathbf{b}^{\mathrm{T}} \mathbf{w} \mathbf{w}^{\mathrm{T}} \mathbf{a} - \mathbf{b}^{\mathrm{T}} \mathbf{a} \\
&= \mathbf{b}^{\mathrm{T}} \left(\mathbf{w} \mathbf{w}^{\mathrm{T}} - \mathbf{I} \right) \mathbf{a},
\end{aligned}
$$

Equation (47) becomes

$$
\phi(A) = -\frac{1}{2\pi} \int_{\partial A} \left(\langle \mathbf{b}, \mathbf{w} \rangle \langle \mathbf{a}, \mathbf{n} \rangle + \mathbf{b}^{\mathrm{T}} \left(\mathbf{I} - \mathbf{w} \mathbf{w}^{\mathrm{T}} \right) \left[\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{n} - \eta(\mathbf{w}, \mathbf{u}) \; \langle \mathbf{w}, \mathbf{n} \rangle \, \mathbf{a} \right] \right) \, ds, \tag{48}
$$

where the scalar-valued function η is dependent on $\mathbf{w}$ and $\mathbf{u}$ and given by

$$
\eta(\mathbf{w}, \mathbf{u}) \equiv \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \tag{49}
$$

with $\langle \mathbf{w}, \mathbf{u} \rangle > 0$ for the illuminated hemisphere and $\eta = 0$ at the singular point $\langle \mathbf{w}, \mathbf{u} \rangle = 1$. Rewriting the integrand in equation (48) in a tensor notation, we have

$$
\begin{aligned}
&\langle \mathbf{b}, \mathbf{w} \rangle \langle \mathbf{a}, \mathbf{n} \rangle + \mathbf{b}^{\mathrm{T}} \left(\mathbf{I} - \mathbf{w} \mathbf{w}^{\mathrm{T}} \right) \left[\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{n} - \eta \; \langle \mathbf{w}, \mathbf{n} \rangle \, \mathbf{a} \right] \\
= \;& \delta_{ik} \mathbf{a}_{\mathrm{i}} \mathbf{n}_{\mathrm{k}} \mathbf{b}_{\mathrm{j}} \mathbf{w}_{\mathrm{j}} + \mathbf{b}_{\mathrm{j}} \left(\delta_{jm} - \mathbf{w}_{\mathrm{j}} \mathbf{w}_{\mathrm{m}} \right) \left[\frac{\mathbf{a}_{\mathrm{i}} \mathbf{u}_{\mathrm{i}}}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{n}_{\mathrm{m}} - \eta \; \mathbf{w}_{\mathrm{k}} \mathbf{n}_{\mathrm{k}} \mathbf{a}_{\mathrm{m}} \right] \\
= \;& \left[\delta_{ik} \mathbf{w}_{\mathrm{j}} + \left(\delta_{jm} - \mathbf{w}_{\mathrm{j}} \mathbf{w}_{\mathrm{m}} \right) \left(\frac{\delta_{km} \mathbf{u}_{\mathrm{i}}}{\langle \mathbf{w}, \mathbf{u} \rangle} - \delta_{im} \mathbf{w}_{\mathrm{k}} \; \eta \right) \right] \mathbf{a}_{\mathrm{i}} \mathbf{b}_{\mathrm{j}} \mathbf{n}_{\mathrm{k}}.
\end{aligned} \tag{50}
$$

It follows from equations (48) and (50) that the irradiance at the origin due to a Lambertian luminaire L with linearly-varying radiant exitance is given by the boundary integral

$$
\boxed{\phi(A) = -\frac{1}{2\pi} \int_{\partial A} \mathbf{M}_{\mathrm{ijk}}(\mathbf{w}, \mathbf{u}) \, \mathbf{a}_{\mathrm{i}} \, \mathbf{b}_{\mathrm{j}} \, \mathbf{n}_{\mathrm{k}} \; ds(\mathbf{u})} \tag{51}
$$

where $A \subset S^2$ is the spherical projection of L, it denote a region on the sphere with a *rectifiable* boundary; that is, a boundary for which an outward normal is defined almost everywhere. ds denotes integration with respect to arclength, $\mathbf{n}$ is the outward-pointing normal of the curve ∂A; thus $\mathbf{n}$ is always tangent to the sphere and orthogonal to the curve on the sphere. See Figure 1b. $\mathbf{M}$ is a 3-tensor that depends on unit vectors $\mathbf{w}$ and $\mathbf{u}$, given by

$$\boxed{\mathbf{M}_{\mathrm{ijk}}(\mathbf{w},\mathbf{u}) \;\equiv\; \delta_{ik}\mathbf{w}_{\mathrm{j}} + \left(\frac{\delta_{km}\mathbf{u}_{\mathrm{i}}}{\langle\mathbf{w},\mathbf{u}\rangle} - \delta_{im}\mathbf{w}_{\mathrm{k}}\,\eta\right)(\delta_{jm} - \mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{m}})} \tag{52}$$

with η defined in equation (49).

It is advantageous to express the irradiance $\phi(A)$ in a tensor form as in equation (52), various irradiance vectors can be easily derived from here. Among these irradiance vectors, most frequently used is defined as

$$\vec{\phi}(A) \;=\; \frac{1}{\pi}\int_A \frac{\mathbf{u}\,\langle\mathbf{b},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\,d\sigma(\mathbf{u}),$$

which is very convenient in some graphics application. For example, suppose we want to compute the irradiance at a receiver point due to a luminaire with several linear variations superimposed together. Since different variations only differ in the vector argument $\mathbf{a}$, by computing $\vec{\phi}(A)$, the computation cost is almost the same as computing the irradiance from one variation. The scalar irradiance value from a particular linear variation can be obtained by the inner product of $\vec{\phi}(A)$ and the corresponding $\mathbf{a}$. The formula for $\vec{\phi}(A)$ follows easily from equation (52), that is,

$$\vec{\phi}(A) \;=\; -\frac{1}{2\pi}\int_{\partial A}\mathbf{M}_{\mathrm{ijk}}(\mathbf{w},\mathbf{u})\,\mathbf{b}_{\mathrm{j}}\,\mathbf{n}_{\mathrm{k}}\,ds(\mathbf{u}). \tag{53}$$

Remark: In general, it is very difficult to find the boundary integral formula for a surface integral by derivation. However, once we find the formula, we can easily prove it by directly applying Stokes' theorem. Combined with equation (17), equation (51) corresponds to an identity relating a surface integral on the left hand side to a boudary integral on the right hand side, thus, we can verify its correctness by direct proof. See Appendix F.

Observe that for luminaires with constant radiant exitance c, we have $w_1 = w_2 = w_3 = c$. Let $\mathbf{p}_1(x_1, y_1, z_1)$, $\mathbf{p}_2(x_2, y_2, z_2)$ and $\mathbf{p}_3(x_3, y_3, z_3)$ be the three non-collinear points on a luminaire L with constant radiant exitance c, then equation (16) gives us

$$\mathbf{a} \;=\; c\,h\,[1\;1\;1]\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}^{-1}.$$

$$= \frac{c\,h}{\triangle}\,(\triangle_1, \triangle_2, \triangle_3), \tag{54}$$

where $\triangle_1, \triangle_2, \triangle_3$ and $\triangle$ are the determinants

$$\triangle = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

and

$$\triangle_1 = \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \quad \triangle_2 = -\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \quad \triangle_3 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Let V denote the volume of the tetrahedron defined by $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ from the origin $\mathbf{o}$, then

$$V = \frac{1}{6}\triangle = \frac{1}{3}\,S\,h,$$

which is derived from the fact that the volume of a tetrahedron specified by the three edge vectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ from a given vector is given by

$$V = \frac{1}{6}\left|\mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)\right|.$$

Thus, we have

$$\mathbf{a} = \frac{c}{2S}(\triangle_1, \triangle_2, \triangle_3), \tag{55}$$

where S is the area of $\triangle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ [4], and thus

$$S = \frac{1}{2}\left|(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)\right|.$$

Notice that $(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)$ is just the vector $(\triangle_1, \triangle_2, \triangle_3)$, we have

$$S = \frac{1}{2}\sqrt{\triangle_1^2 + \triangle_2^2 + \triangle_3^2}.$$

In addition, it is easy to check that the vector $(\triangle_1, \triangle_2, \triangle_3)$ is perpendicular to the triangle plane $\mathbf{p}_1\mathbf{p}_1\mathbf{p}_3$. Thus, we get[1]

$$(\triangle_1, \triangle_2, \triangle_3) = 2S\,\mathbf{w},$$

[1]Depending on the order of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, it may have a negative sign.

where $\mathbf{w}$ is the unit vector orthogonal to the planar luminaire. As a result, it follows from equation (55) that

$$\mathbf{a} = c\,\mathbf{w}.$$

It then follows easily that equation (52) simplifies to

$$\phi(A) = -\frac{c}{2\pi} \int_{\partial A} \langle \mathbf{b}, \mathbf{n} \rangle \; ds, \tag{56}$$

which is the continuous version of Lambert's well-known formula. That is, when A is a spherical polygon, Equation (56) reduces to

$$\phi(A) = \frac{c}{2\pi} \sum_{i=1}^{n} \Theta_i \, \Gamma_i, \tag{57}$$

which is the form that is applied most frequently in computer graphics [3].

7 Restriction to Polygonal Luminaires

We now specialize the result given in the previous section to spherical polygons, which results from projecting simple planar polygons onto the sphere. The resulting formula will allow us to compute the exact irradiance due to a polygonal luminaire with linearly-varying radiant exitance.

For a spherical polygons P with s edges, the boundary integrals in equation (51) can be evaluated along each edge ζ of P, which is the great arc connecting two adjacent vertices. That is,

$$\phi(P) = -\frac{1}{2\pi} \sum_{m=1}^{s} \left[\int_{\zeta_m} \mathbf{M}_{\mathrm{ijk}} \, \mathbf{a}_{\mathrm{i}} \, \mathbf{b}_{\mathrm{j}} \; ds \right] \, \mathbf{n}_{\mathrm{k}}^{m}, \tag{58}$$

where the normal $\mathbf{n}$ can be moved outside the integral since it is constant along each ζ_m; this is precisely the property that allows boundary integrals of this form to simplify for polygons. Consequently, the integral in equation (58) simplifies to

$$\begin{aligned} & \int_{\zeta} \mathbf{M}_{\mathrm{ijk}} \, \mathbf{a}_{\mathrm{i}} \, \mathbf{b}_{\mathrm{j}} \, \mathbf{n}_{\mathrm{k}} \; ds \\ = \; & \int_{\zeta} \left[\delta_{ik} \mathbf{a}_{\mathrm{i}} \mathbf{n}_{\mathrm{k}} \mathbf{b}_{\mathrm{j}} \mathbf{w}_{\mathrm{j}} + \left(\frac{\delta_{km} \mathbf{u}_{\mathrm{i}}}{\langle \mathbf{w}, \mathbf{u} \rangle} - \delta_{im} \mathbf{w}_{\mathrm{k}} \, \eta \right) \mathbf{a}_{\mathrm{i}} \mathbf{n}_{\mathrm{k}} \mathbf{b}_{\mathrm{j}} \left(\delta_{jm} - \mathbf{w}_{\mathrm{j}} \mathbf{w}_{\mathrm{m}} \right) \right] \; ds \end{aligned}$$

$$
\begin{aligned}
&= \int_\zeta \left[\langle \mathbf{a}, \mathbf{n} \rangle \langle \mathbf{b}, \mathbf{w} \rangle + \left(\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{n}_{\mathrm{m}} - \eta \ \langle \mathbf{w}, \mathbf{n} \rangle \, \mathbf{a}_{\mathrm{m}} \right) \mathbf{b}_{\mathrm{j}} \left(\delta_{jm} - \mathbf{w}_{\mathrm{j}} \mathbf{w}_{\mathrm{m}} \right) \right] \, ds \\
&= \int_\zeta \left[\langle \mathbf{a}, \mathbf{n} \rangle \langle \mathbf{b}, \mathbf{w} \rangle + \mathbf{b}^{\mathrm{T}} (\mathbf{I} - \mathbf{w}\mathbf{w}^{\mathrm{T}}) \left(\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{n} - \eta \ \langle \mathbf{w}, \mathbf{n} \rangle \, \mathbf{a} \right) \right] \, ds \\
&= \langle \mathbf{a}, \mathbf{n} \rangle \langle \mathbf{b}, \mathbf{w} \rangle \, \Theta + \mathbf{b}^{\mathrm{T}} (\mathbf{I} - \mathbf{w}\mathbf{w}^{\mathrm{T}}) \left[B_1 \mathbf{n} - B_2 \langle \mathbf{w}, \mathbf{n} \rangle \, \mathbf{a} \right] \qquad (59)
\end{aligned}
$$

for each m, where the boundary integrals are defined by

$$
\begin{aligned}
B_1 &\equiv \int_\zeta \frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \, ds, \\
B_2 &\equiv \int_\zeta \eta(\mathbf{w}, \mathbf{u}) \, ds.
\end{aligned}
$$

Parametrizing the great arc ζ by arclength θ, we have

$$
\mathbf{u}(\theta) = \mathbf{s} \cos\theta + \mathbf{t} \sin\theta,
$$

where $\mathbf{s}$ and $\mathbf{t}$ are orthonormal vectors in the plane containing the edge and the origin, with $\mathbf{s}$ directed toward the first vertex of the edge [2]. Using this parameterization, we can evaluate B_1 and B_2 as follows:

$$
\begin{aligned}
\overline{B}_1 &= \int_0^\Theta \frac{a_2 \cos\theta + b_2 \sin\theta}{a_1 \cos\theta + b_1 \sin\theta} \, d\theta \\
&= \frac{c_2}{c_1} \int_0^\Theta \frac{\cos(\theta - \phi_2)}{\cos(\theta - \phi_1)} \, d\theta \\
&= \frac{c_2}{c_1} \int_{-\phi_1}^{\Theta - \phi_1} \frac{\cos(\theta + \phi_1 - \phi_2)}{\cos\theta} \, d\theta \\
&= \frac{c_2}{c_1} \int_{-\phi_1}^{\Theta - \phi_1} \frac{\cos\theta \cos(\phi_1 - \phi_2) - \sin\theta \sin(\phi_1 - \phi_2)}{\cos\theta} \, d\theta \\
&= \frac{c_2}{c_1} \int_{-\phi_1}^{\Theta - \phi_1} \left[\cos(\phi_1 - \phi_2) - \sin(\phi_1 - \phi_2) \frac{\sin\theta}{\cos\theta} \right] \, d\theta \\
&= \frac{c_2}{c_1} \left[\cos(\phi_1 - \phi_2)\Theta + \sin(\phi_1 - \phi_2) \left[\ln\cos\theta \right]_{-\phi_1}^{\Theta - \phi_1} \right] \\
&= \frac{c_2}{c_1} \left[\cos(\phi_1 - \phi_2)\Theta + \sin(\phi_1 - \phi_2) \ln \frac{\cos(\Theta - \phi_1)}{\cos\phi_1} \right] \\
&= \frac{c_2}{c_1} \left[\cos(\phi')\Theta + \sin(\phi') \ln \frac{\cos(\Theta - \phi_1)}{\cos\phi_1} \right], \qquad (60)
\end{aligned}
$$

and

$$
\overline{B}_2 = \int_0^\Theta \frac{\ln\left(a_1 \cos\theta + b_1 \sin\theta \right)}{1 - \left(a_1 \cos\theta + b_1 \sin\theta \right)^2} \, d\theta
$$

$$
\begin{aligned}
&= \int_0^{\Theta} \frac{\ln\left(c_1 \cos(\theta - \phi_1)\right)}{1 - \left(c_1 \cos(\theta - \phi_1)\right)^2}\, d\theta \\
&= \int_{-\phi_1}^{\Theta - \phi_1} \frac{\ln\left(c_1 \cos\theta\right)}{1 - \left(c_1 \cos\theta\right)^2}\, d\theta \\
&= \Lambda(c_1, \Theta - \phi_1) - \Lambda(c_1, -\phi_1), \qquad (61)
\end{aligned}
$$

where Θ is the angle subtended by the edge ζ, $\phi' = \phi_1 - \phi_2$, and ϕ_1 and ϕ_2 satisfy

$$
(\cos\phi_i, \sin\phi_i) = \left(\frac{a_i}{c_i}, \frac{b_i}{c_i}\right)
$$

for $i = 1, 2$, and

$$
\begin{aligned}
a_1 = \langle \mathbf{w}, \mathbf{s} \rangle\,,\ \ b_1 = \langle \mathbf{w}, \mathbf{t} \rangle\,,\ \ c_1 = \sqrt{a_1^2 + b_1^2}, \\
a_2 = \langle \mathbf{a}, \mathbf{s} \rangle\,,\ \ b_2 = \langle \mathbf{a}, \mathbf{t} \rangle\,,\ \ c_2 = \sqrt{a_2^2 + b_2^2}.
\end{aligned}
$$

For a unit vector $\mathbf{w}$, we have $0 \le c_1 \le 1$. The two-parameter function Λ appearing in the expression for $\overline{B}_2$ is defined by

$$
\boxed{\Lambda(\alpha, \beta) \ \equiv\ \int_0^{\beta} \frac{\ln(\alpha\cos\theta)}{1 - (\alpha\cos\theta)^2}\, d\theta,} \qquad (62)
$$

where $0 < \alpha \le 1, -\pi/2 \le \beta \le \pi/2$.

To our knowledge, $\Lambda(\alpha, \beta)$ has no finite representation in terms of elementary exponential and logarithmic functions, and therefore qualifies as a *higher transcendendal function*, also known as a *special function*. Figure 2 shows $\Lambda(\alpha, \beta)$ over its required domain, $(0, 1] \times [0, \pi/2]$. As shown in a subsequent report [5], this integral is closely related to the function studied by Powell [11] in 1943, which he showed to be equivalent to the *Airey radiation integrals* that arise in both astrophysics and quantum mechanics, and is subsequently named as the *dilogarithm* [9]. We have shown [5] that they satisfy the following relationship:

$$
\boxed{\Lambda(\alpha, \beta) \ =\ \frac{2(\eta - \mu)\ln\gamma + 2\mathrm{Cl}_2(2\mu) - \mathrm{Cl}_2(4\mu - 2\eta) - \mathrm{Cl}_2(2\eta)}{4\sqrt{1 - \alpha^2}},} \qquad (63)
$$

where the Clausen's integral $\mathrm{Cl}_2(x)$ is a simpler form of dilogarithm. Therefore, it is not surprising that such a function appears in the problem we have addressed, as dilogarithms

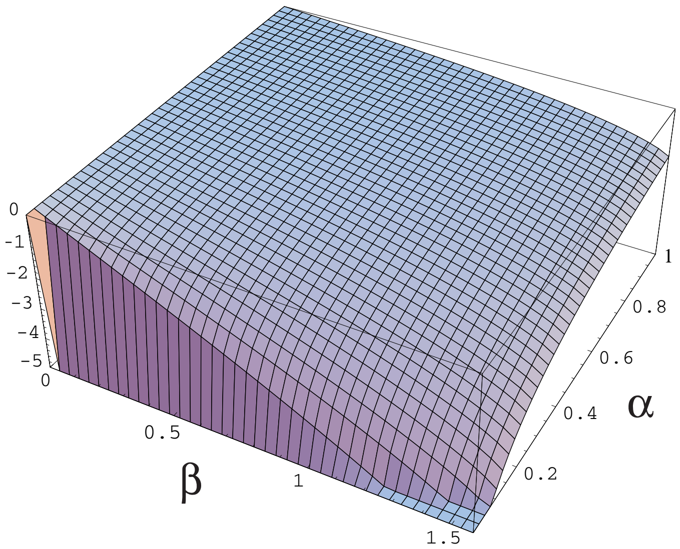


Figure 2: *The special function $\Lambda(\alpha, \beta)$ defined in equation (62), over the range $0 < \alpha \leq 1$, and $0 \leq \beta \leq \pi/2$. When $\alpha \to 0$, $\Lambda(\alpha, \beta)$ tends to $-\infty$.*

and related functions arise in many problems of radiative transfer [12, 1]. We will discuss the evaluation of this integral in detail in a subsequent technical report [5].

Combining Equations (58), (59), (60) and (61), the "closed-form" solution for a polygonal luminaire P with s edges is given by

$$\boxed{\phi(P) = -\frac{1}{2\pi} \sum_{m=1}^{s} \left[\langle \mathbf{a}, \mathbf{n} \rangle \langle \mathbf{b}, \mathbf{w} \rangle \Theta + \mathbf{b}^{\mathrm{T}} (\mathbf{I} - \mathbf{w}\mathbf{w}^{\mathrm{T}}) \left(\overline{B}_1 \mathbf{n} - \overline{B}_2 \langle \mathbf{w}, \mathbf{n} \rangle \mathbf{a} \right) \right],} \tag{64}$$

where $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$ and $\mathbf{n}$ vary over each edge ζ_m.

To compute the irradiance vector $\vec{\phi}$ due to a polygonal luminaire, it is required to evaluate the vector form of the boundary integral B_1, denoted by $\mathbf{B}_1$, defined as

$$\mathbf{B}_1 \quad \equiv \quad \int_\zeta \frac{\mathbf{u}}{\langle \mathbf{w}, \mathbf{u} \rangle}\, ds.$$

Using the same parameterization discussed above, we can evaluate $\mathbf{B}_1$ as

$$
\begin{aligned}
\overline{\mathbf{B}}_1 &= \int_0^{\Theta} \frac{\mathbf{s}\cos\theta + \mathbf{t}\sin\theta}{a_1\cos\theta + b_1\sin\theta}\, d\theta \\
&= \frac{1}{c_1}\int_0^{\Theta} \frac{\mathbf{s}\cos\theta + \mathbf{t}\sin\theta}{\cos(\theta-\phi_1)}\, d\theta \\
&= \frac{1}{c_1}\int_{-\phi_1}^{\Theta-\phi_1} \frac{\mathbf{s}\cos(\theta+\phi_1) + \mathbf{t}\sin(\theta+\phi_1)}{\cos\theta}\, d\theta \\
&= \frac{1}{c_1}\int_{-\phi_1}^{\Theta-\phi_1} \frac{(\mathbf{s}\cos\phi_1 + \mathbf{t}\sin\phi_1)\cos\theta - (\mathbf{s}\sin\phi_1 - \mathbf{t}\cos\phi_1)\sin\theta}{\cos\theta}\, d\theta \\
&= \frac{1}{c_1}\int_{-\phi_1}^{\Theta-\phi_1} \left[(\mathbf{s}\cos\phi_1 + \mathbf{t}\sin\phi_1) - (\mathbf{s}\sin\phi_1 - \mathbf{t}\cos\phi_1)\frac{\sin\theta}{\cos\theta}\right] d\theta \\
&= \frac{1}{c_1}\left[(\mathbf{s}\cos\phi_1 + \mathbf{t}\sin\phi_1)\Theta + (\mathbf{s}\sin\phi_1 - \mathbf{t}\cos\phi_1)\left[\ln\cos\theta\right]_{-\phi_1}^{\Theta-\phi_1}\right] \\
&= \frac{1}{c_1}\left[(\mathbf{s}\cos\phi_1 + \mathbf{t}\sin\phi_1)\Theta + (\mathbf{s}\sin\phi_1 - \mathbf{t}\cos\phi_1)\ln\frac{\cos(\Theta-\phi_1)}{\cos\phi_1}\right].
\end{aligned}
$$

Thus, corresponding to equation (64), we have

$$
\vec{\phi}(P) = -\frac{1}{2\pi}\sum_{m=1}^{s}\left[\mathbf{n}\langle\mathbf{b},\mathbf{w}\rangle\,\Theta + \mathbf{b}^{\mathrm{T}}(\mathbf{I}-\mathbf{w}\mathbf{w}^{\mathrm{T}})\left(\mathbf{n}\,\overline{\mathbf{B}}_1 - \overline{B}_2\langle\mathbf{w},\mathbf{n}\rangle\right)\right]. \tag{65}
$$

The pseudo-code for computing the irradiance at the origin $\mathbf{o}$ due to a linearly-varying luminaire P is shown below. Here, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are any three non-colinear points on P with radiant exitance w_1, w_2 and w_3, respectively.

$\underline{\text{ComputeIrradiance}(\ P,\ \mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3,\ w_1,\ w_2,\ w_3)}$

$$
\begin{array}{l}
\mathbf{w} \leftarrow \mathbf{unit}[\,(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_2 - \mathbf{p}_3)\,] \\
\textbf{if } \langle \mathbf{w}, \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \rangle < 0 \\
\quad \mathbf{w} \leftarrow\ -\mathbf{w} \\
\textbf{endif} \\
\mathbf{b} \leftarrow \textit{surface normal at the origin } \mathbf{o} \\
\mathbf{a} \leftarrow\ \langle \mathbf{p}_1, \mathbf{w} \rangle\, [w_1\ w_2\ w_3]\ [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1} \\
s \leftarrow 0 \\
\textbf{for each } \text{edge } \mathbf{AB} \text{ in } P \textbf{ do} \\
\quad \mathbf{s} \leftarrow \mathbf{unit}[\,\mathbf{A}\,] \\
\quad \mathbf{t} \leftarrow \mathbf{unit}[\,(\mathbf{I} - \mathbf{s}\mathbf{s}^{\mathrm{T}})\mathbf{B}\,] \\
\quad a_1 \leftarrow\ \langle \mathbf{w}, \mathbf{s} \rangle\ ;\ a_2 \leftarrow\ \langle \mathbf{a}, \mathbf{s} \rangle \\
\quad b_1 \leftarrow\ \langle \mathbf{w}, \mathbf{t} \rangle\ ;\ b_2 \leftarrow\ \langle \mathbf{a}, \mathbf{t} \rangle \\
\quad c_1 \leftarrow\ \sqrt{a_1^2 + b_1^2};\ \ c_2 \leftarrow\ \sqrt{a_2^2 + b_2^2} \\
\quad \phi_1 \leftarrow \mathbf{sign}[\,b_1\,] * \cos^{-1}(a_1 / c_1) \\
\quad \phi_2 \leftarrow \mathbf{sign}[\,b_2\,] * \cos^{-1}(a_2 / c_2) \\
\quad \Theta \leftarrow \textit{angle between } \mathbf{A} \textit{ and } \mathbf{B} \\
\quad \mathbf{n} \leftarrow \mathbf{unit}[\,\mathbf{A} \times \mathbf{B}\,] \\
\quad \mathbf{v} \leftarrow\ (\mathbf{I} - \mathbf{w}\mathbf{w}^{\mathrm{T}})\,\mathbf{b} \\
\quad s_0 \leftarrow\ \langle \mathbf{a}, \mathbf{n} \rangle * \langle \mathbf{b}, \mathbf{w} \rangle * \Theta \\
\quad s_1 \leftarrow \overline{B}_1(c_1, c_2, \Theta, \phi_1, \phi_2) * \langle \mathbf{v}, \mathbf{n} \rangle \\
\quad s_2 \leftarrow \overline{B}_2(c_1, \Theta, \phi_1) * \langle \mathbf{w}, \mathbf{n} \rangle * \langle \mathbf{v}, \mathbf{a} \rangle \\
\quad s \leftarrow s + s_0 + s_1 - s_2 \\
\textbf{endfor} \\
\textbf{return } -s \,/\, 2\pi \\
\textbf{end}
\end{array}
$$

Here, the operator **unit** normalizes a given vector. We assume that the polygon vertices of P are ordered *counterclockwise* viewed from the receiver point $\mathbf{o}$, which guarantees that $\mathbf{n}$ computed above is outward pointing and the irradiance computed is positive. For vertices that are oriented clockwise, the resulting irradiance will be negative. Preserving the sign is useful for integrating over polygons with holes. Note that the ordering of the points $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are immaterial, as the $\mathbf{w}$ vector is always adjusted to point toward the polygon.

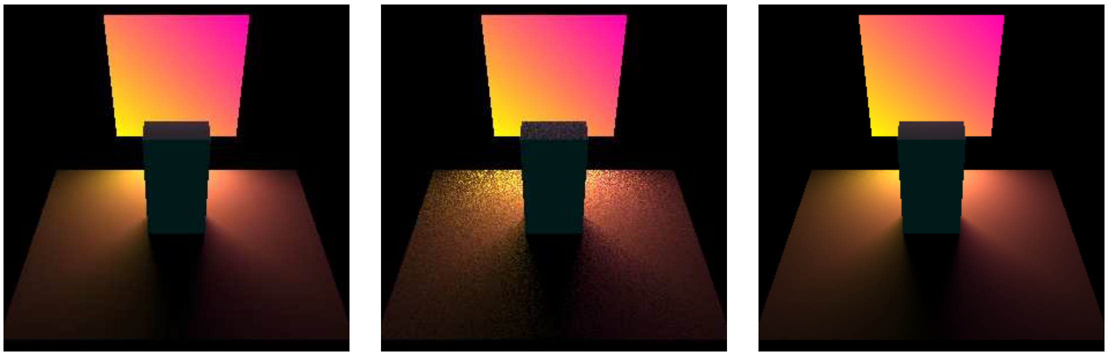
$\overline{B}_1$ and $\overline{B}_2$ in ComputeIrradiance are evaluated using equations (60) and (61), respectively. The special function $\Lambda(\alpha, \beta)$ involved in $\overline{B}_2$ can be computed directly from equation (63), where the Clausen integral can be approximated using the pseudo-code below, which uses a formula derived by Grosjean [7].

ApproxClausen(**double** x)

$c_1 \leftarrow \quad 3.472222222E-04$
$c_2 \leftarrow \quad 9.869604401E+00$
$c_3 \leftarrow \quad 5.091276919E+01$
$c_4 \leftarrow \quad 1.362943611E-01$
$c_5 \leftarrow -2.165319440E-03$
$c_6 \leftarrow \quad 1.639639947E-04$
$c_7 \leftarrow -2.471701169E-05$
$c_8 \leftarrow \quad 5.538890645E-06$
if $x == 0$
    return 0.0
endif
if $x > \pi$
    return -ApproxClausen($2\pi - x$)
endif
return
    $c_1 * x * (c_2 - x^2) * (c_3 + 3x^2)$
    $+c_4 * \sin(x) + c_5 * \sin(2x)$
    $+c_6 * \sin(3x) + c_7 * \sin(4x)$
    $+c_8 * \sin(5x)$
    $-x * \ln(\sin(x/2))$
end

8 Results

Figures 3 and 4 shows two simple scenes, each illuminated by an area light source with two linearly-varying superimposed colors. This effect is simulated by integrating two linearly-varying scalar values over the luminaire and weighting the corresponding colors by them. Both scenes were rendered using the new technique and by Monte Carlo for comparison. In implementing our analytical method, several optimizations mentioned in previous sections were employed: 1) computing the vector **a** at each receiver point incrementally using equation (19) rather than inverting a matrix at each point, 2) computing an irradiance



Analytic (1.7m) Monte Carlo (2.6m) Monte Carlo (42m)

Figure 3: *Images of a polygonal environment generated using the analytic solution and Monte Carlo. The numbers beneath each image indicate the computation time. (Left) The analytic solution is applied by clipping the luminaire against all blockers with respect to each point on the receiving surface and computing the contributions from the remaining polygons. (Middle) Monte Carlo solution using four stratified samples per pixel, which requires a comparable amount of time. (Right) Monte Carlo solution using 100 stratified samples per pixel, which produces an image comparable to the analytic solution.*

vector of the form in equation (65) to save the computation cost for two color variations and 3) bilinear interpolation for Λ evaluation. Stratified sampling were used for the Monte Carlo solutions. All the images have a resolution of 300×300, and were generated on a SGI Indigo2 working at 175 Mhz $R10000$. The numbers beneath each picture indicate the computation time in minutes.

Figure 3 depicts a box blocker in front of a luminaire. Polygonal occlusions are handled by clipping the luminaire against all blockers and computing the contribution from each visible portion using our closed-form solution. Our noise-free result clearly matches the Monte Carlo image generated by using 100 stratified samples per pixel, yet is over 20 times faster. Moreover, our method results in much higher quality than the simple Monte Carlo solution, using nearly the same amount of time.

For the leaf-like polygonal luminaire in Figure 4, our proposed algorithm shows a significant advantage over Monte Carlo in that it can handle non-convex polygons directly, and produce a noise-free image in far less time, as shown in the top row. The Monte Carlo images are rendered by stratified sampling the bounding rectangle of the leaf-shape polygon. In the bottom, we rendered the same scene using a semi-analytical method for further comparison.

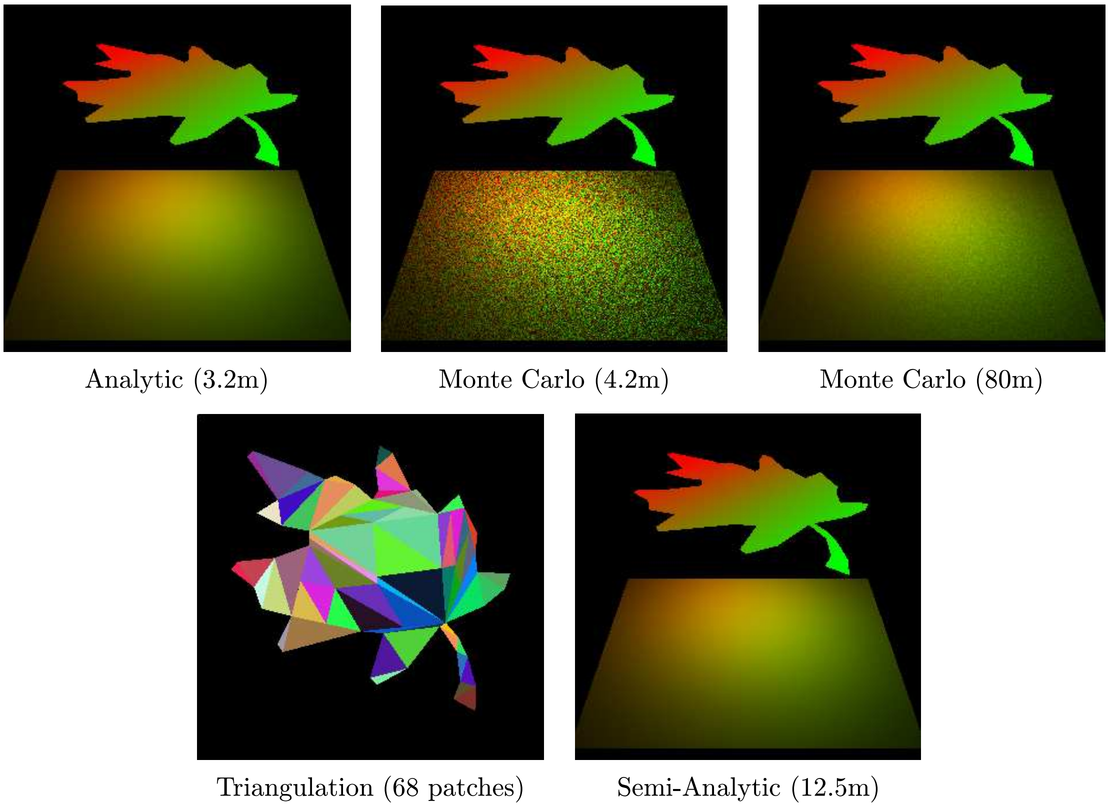


Figure 4: *Non-convex polygonal luminaires with linearly-varying colors can be efficiently handled with our proposed algorithm. (Top Row) comparison similar to that shown in Figure 3. (Bottom Row) Image generated by subdividing the luminaire into small regions that are treated as constant. Although each pixel of the resulting image is within 2% of the exact solution, this approach is slower than the analytic solution. Moreover, the accuracy depends on an appropriate level of subdivision.*

This approach subdivides the luminaire into small pieces, and computes the contribution from each piece using Lambert's formula by assuming they are uniform. Although each pixel of the resulting image by breaking the leaf into 68 triangles shown in the left is within 2% of the exact solution from our new method, this finite element approach is much slower than the analytic solution. Moreover, the accuracy depends on an appropriate level of subdivision, which is hard to choose.

These experiments provide numerical evidence of the correctness of our formula, and also demonstrate its application to computing direct illumination. The approach is also applicable to indirect illumination, however, as it provides a means of computing form-factors involving non-constant surface elements [10].

9 Conclusions

We have introduced a new deterministic technique for computing irradiance from polygonal luminaires with linearly-varying radiant exitance. Our solution is closed-form, but for a single special function which is a close relative of other well-known special functions that arise in mathematical physics. Indeed, our function $\Lambda(\alpha, \beta)$ can be evaluated in terms of the dilogarithm or Clausen's lintegral [9], as we shall demonstrate in a subsequent report [5]. We believe that our results can extend to include directional variation as well as spatial variation, which would accommodate non-Lambertian phenomena such as glossy reflection and transmission of *non-uniform* luminaires. The most immediate extension of this work is to luminaires with polynomially-varying radiant exitance, which will be the topic of the following report [5].

Acknowledgements

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A Proof of Binomial Identities (31) to (38)

In this appendix, we are going to prove the four binomial identities shown in section 5, that is, equations (31), (32), (33) and (34). Each identity involves a denominator of either $(k+1)$ or $(k+3)$. Our approach is to use *generating function*.

$$\boxed{\begin{aligned} \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{x^k}{k+1} &= \frac{1-(1-x)^{n+1}}{(n+1)x} \\ \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{k+1} &= \frac{1}{n+1} \end{aligned}} \tag{66}$$

Proof: Integrating both sides of

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} x^k = (1-x)^n$$

with respect to x, we have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{x^{k+1}}{k+1} = c(n) - \frac{(1-x)^{n+1}}{n+1}.$$

where $c(n)$ is a function only dependent on n. Letting $x=0$, we get

$$c(n) = \frac{1}{n+1}.$$

Thus,

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{x^k}{k+1} = \frac{1-(1-x)^{n+1}}{(n+1)x}.$$

By letting $x=1$ above, we get another identity:

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{k+1} = \frac{1}{n+1}.$$

□□□

$$\boxed{\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k+3} &= \frac{1}{x^3}\left(\frac{2}{(n+1)(n+2)(n+3)}-\right.\\ &\qquad\left.\frac{(1-x)^{n+3}}{n+3}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+1}}{n+1}\right)\\ \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+3} &= \frac{2}{(n+3)(n+2)(n+1)}.\end{aligned}} \quad (67)$$

Proof: Similarly, integrating both sides of

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{k+2} = (1-x)^n x^2$$

with respect to the variable x, we get

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^{k+3}}{k+3} &= \int(1-x)^n x^2 dx + c(n)\\ &= \int(1-x)^n(1-x-1)^2 dx + c(n)\\ &= \int\Big[(1-x)^{n+2}-2(1-x)^{n+1}+(1-x)^n\Big]dx + c(n)\\ &= -\frac{(1-x)^{n+3}}{n+3}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+1}}{n+1}+c(n).\end{aligned}$$

where $c(n)$ is a function only dependent on n. Letting $x=0$, we get

$$c(n) = \frac{1}{n+3}-\frac{2}{n+2}+\frac{1}{n+1} = \frac{2}{(n+1)(n+2)(n+3)}.$$

Thus, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k+3} = \frac{1}{x^3}\left(\frac{2}{(n+1)(n+2)(n+3)}-\frac{(1-x)^{n+3}}{n+3}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+1}}{n+1}\right).$$

By letting $x=1$, we can get another identity. □□□

Using these binomial identities, we can easily derive the four double summation identities (35), (36), (37) and (38) as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1-x^{k+1}}{k+1} &= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1-(1-x)^{n+1}}{n+1}\right)\\
&= \sum_{n=0}^{\infty}\frac{(1-x)^{n+1}}{n+1}\\
&= -\ln x\\
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k+3} &= \frac{1}{x^3}\left(\sum_{n=0}^{\infty}\frac{2}{(n+3)(n+2)(n+1)}-\sum_{n=0}^{\infty}\frac{(1-x)^{n+3}}{n+3}+\right.\\
&\quad\left.\sum_{n=0}^{\infty}\frac{2(1-x)^{n+2}}{n+2}-\sum_{n=0}^{\infty}\frac{(1-x)^{n+1}}{n+1}\right)\\
&= \frac{1}{x^3}\left(\frac{1}{2}+\frac{(x-3)(x-1)}{2}+\ln x-2(1-x)-2\ln x+\ln x\right)\\
&= \frac{1}{2x}.\\
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+3} &= \sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)(n+3)}\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}\right)\\
&= \frac{1}{2}.\\
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1-x^{k+1}}{k+3} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+3}-x\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k+3}\\
&= 0.
\end{aligned}$$

B Proof of Identity (6)

$$-\int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{n} \rangle}{1 + \langle \mathbf{w}, \mathbf{u} \rangle}\, ds \;=\; \sigma(A),$$

where $\sigma(A)$ is the area of $A \subset S^2$, or equivalently, the solid angle subtended by A, and $\mathbf{w}$ is any unit vector such that $-\mathbf{w} \notin A$.

Proof: The proof is done by using Stokes' Theorem, which can be stated in three steps:

Step 1: (Rewrite the boundary integral)

Let $r = ||\,\mathbf{r}\,||$, we can use the identities

$$\begin{aligned} \mathbf{n}\, ds &= \frac{\mathbf{r} \times\, d\mathbf{r}}{r^2} \\ \mathbf{u} &= \frac{\mathbf{r}}{r} \end{aligned}$$

to rewrite the boundary integral on the left in terms of the position vector $\mathbf{r}$ and its derivatives, obtaining

$$\begin{aligned} \int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{n} \rangle}{1 + \langle \mathbf{w}, \mathbf{u} \rangle}\, ds &= \int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{r} \times\, d\mathbf{r} \rangle}{(r + \langle \mathbf{w}, \mathbf{r} \rangle)\, r} \\ &= \int_{\partial A} \frac{\varepsilon_{jpl} \mathbf{w}_{\mathrm{j}} \mathbf{r}_{\mathrm{p}}\, d\mathbf{r}_{\mathrm{l}}}{(r + \langle \mathbf{w}, \mathbf{r} \rangle)\, r} \\ &= \int_{\partial A} B_{\mathrm{l}}\, d\mathbf{r}_{\mathrm{l}}, \end{aligned}$$

where

$$B_{\mathrm{l}} = \frac{\varepsilon_{jpl} \mathbf{w}_{\mathrm{j}} \mathbf{r}_{\mathrm{p}}}{(r + \langle \mathbf{w}, \mathbf{r} \rangle)\, r}.$$

Step 2: (Convert boundary integral into surface integral)

To do this, we need to compute the partial derivatives of B_{l}. Since

$$\frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}} \left(\frac{1}{(r + \langle \mathbf{w}, \mathbf{r} \rangle)\, r} \right) = -\frac{2r\mathbf{r}_{\mathrm{m}} + r^2 \mathbf{w}_{\mathrm{m}} + \langle \mathbf{w}, \mathbf{r} \rangle\, \mathbf{r}_{\mathrm{m}}}{(r + \langle \mathbf{w}, \mathbf{r} \rangle)^2\, r^3},$$

the partial derivative $B_{l,m}$ can be computed as

$$\begin{aligned} B_{l,m} &= \varepsilon_{jpl}\mathbf{w}_{\mathrm{j}}\left(\frac{\delta_{pm}}{(r+\langle\mathbf{w},\mathbf{r}\rangle)\,r}-\frac{2r\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}+r^2\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{m}}+\langle\mathbf{w},\mathbf{r}\rangle\,\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{(r+\langle\mathbf{w},\mathbf{r}\rangle)^2\,r^3}\right) \\ &= \varepsilon_{jpl}\mathbf{w}_{\mathrm{j}}\left(\frac{\delta_{pm}\,(r+\langle\mathbf{w},\mathbf{r}\rangle)\,r^2-2r\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}-r^2\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{m}}-\langle\mathbf{w},\mathbf{r}\rangle\,\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{(r+\langle\mathbf{w},\mathbf{r}\rangle)^2\,r^3}\right) \end{aligned}$$

Then, we can use Stokes' Theorem to convert the boundary integral into a surface integral over A:

$$\begin{aligned} \int_{\partial A}\frac{\langle\mathbf{w},\mathbf{n}\rangle}{1+\langle\mathbf{w},\mathbf{u}\rangle}\,ds &= \int_A B_{l,m}\,d\mathbf{r}_{\mathrm{m}}\wedge\,d\mathbf{r}_{\mathrm{l}} \\ &= \int_A \varepsilon_{kml}B_{l,m}\left[\frac{\varepsilon_{kst}\,d\mathbf{r}_{\mathrm{s}}\wedge\,d\mathbf{r}_{\mathrm{t}}}{2}\right], \end{aligned}$$

where the last step follows from properties of wedge product and tensor identities:

$$\begin{aligned} d\mathbf{r}_{\mathrm{m}}\wedge\,d\mathbf{r}_{\mathrm{l}} &= \frac{d\mathbf{r}_{\mathrm{m}}\wedge\,d\mathbf{r}_{\mathrm{l}}-\,d\mathbf{r}_{\mathrm{l}}\wedge\,d\mathbf{r}_{\mathrm{m}}}{2} \\ &= \left[\frac{\delta_{sm}\delta_{tl}-\delta_{sl}\delta_{tm}}{2}\right]\,d\mathbf{r}_{\mathrm{s}}\wedge\,d\mathbf{r}_{\mathrm{t}} \\ &= \varepsilon_{kml}\left[\frac{\varepsilon_{kst}\,d\mathbf{r}_{\mathrm{s}}\wedge\,d\mathbf{r}_{\mathrm{t}}}{2}\right]. \end{aligned} \tag{68}$$

Step 3: (Express the surface integral in terms of solid angle)

Notice that the 2-form $d\omega$ has a tensor form [13, page 131]

$$d\omega = -\frac{\varepsilon_{qlm}\mathbf{r}_{\mathrm{q}}\,d\mathbf{r}_{\mathrm{l}}\wedge\,d\mathbf{r}_{\mathrm{m}}}{2r^3},$$

We can express

$$\begin{aligned} \varepsilon_{kml}B_{l,m} &= \varepsilon_{kml}\varepsilon_{jpl}\mathbf{w}_{\mathrm{j}}\left(\frac{\delta_{pm}\,(r+\langle\mathbf{w},\mathbf{r}\rangle)\,r^2-2r\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}-r^2\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{m}}-\langle\mathbf{w},\mathbf{r}\rangle\,\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{(r+\langle\mathbf{w},\mathbf{r}\rangle)^2\,r^3}\right) \\ &= \frac{\delta_{pm}\delta_{jk}-\delta_{pk}\delta_{jm}}{(r+\langle\mathbf{w},\mathbf{r}\rangle)^2\,r^3}\mathbf{w}_{\mathrm{j}}\left(\delta_{pm}\,(r+\langle\mathbf{w},\mathbf{r}\rangle)\,r^2-2r\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}-r^2\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{m}}-\langle\mathbf{w},\mathbf{r}\rangle\,\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}\right) \\ &= \frac{1}{(r+\langle\mathbf{w},\mathbf{r}\rangle)^2\,r^3}\Big[3\mathbf{w}_{\mathrm{k}}\,(r+\langle\mathbf{w},\mathbf{r}\rangle)\,r^2-2r^3\mathbf{w}_{\mathrm{k}}-r^2\,\langle\mathbf{w},\mathbf{r}\rangle\,\mathbf{w}_{\mathrm{k}}-r^2\,\langle\mathbf{w},\mathbf{r}\rangle\,\mathbf{w}_{\mathrm{k}} \end{aligned}$$

$$
\begin{aligned}
& -\mathbf{w}_{\mathrm{k}}\left(r+\langle\mathbf{w},\mathbf{r}\rangle\right)r^{2}+2r\langle\mathbf{w},\mathbf{r}\rangle\mathbf{r}_{\mathrm{k}}+r^{2}\langle\mathbf{w},\mathbf{w}\rangle\mathbf{r}_{\mathrm{k}}+\langle\mathbf{w},\mathbf{r}\rangle^{2}\mathbf{r}_{\mathrm{k}}\Big] \\
= & \frac{\mathbf{r}_{\mathrm{k}}}{\left(r+\langle\mathbf{w},\mathbf{r}\rangle\right)^{2}r^{3}}\left[2r\langle\mathbf{w},\mathbf{r}\rangle+r^{2}+\langle\mathbf{w},\mathbf{r}\rangle^{2}\right] \\
= & \frac{\mathbf{r}_{\mathrm{k}}}{r^{3}}.
\end{aligned}
$$

Here, we used the following tensor properties:

$$
\begin{aligned}
\delta_{ij}\delta_{ij} &= 3 \\
\varepsilon_{pjl}\varepsilon_{kml} &= \delta_{pk}\delta_{jm}-\delta_{pm}\delta_{jk} \\
\delta_{ij}\mathbf{A}_{i}\mathbf{B}_{j} &= \langle\mathbf{A},\mathbf{B}\rangle\,.
\end{aligned}
$$

Incorporating all the above three steps, we get

$$
\begin{aligned}
-\int_{\partial A}\frac{\langle\mathbf{w},\mathbf{n}\rangle}{1+\langle\mathbf{w},\mathbf{u}\rangle}\,ds &= -\int_{A}\frac{\varepsilon_{kst}\mathbf{r}_{\mathrm{k}}\;d\mathbf{r}_{\mathrm{s}}\wedge\;d\mathbf{r}_{\mathrm{t}}}{2r^{3}} \\
&= \int_{A}d\omega \\
&= \sigma(A).
\end{aligned}
$$

□□□

Equation (6) can be viewed as a generalization of Girard's formula for the area of spherical triangles to arbitrary regions on the sphere. More importantly, it tells us that although the 2-form $d\omega$ is not *exact* over the whole sphere S^2 [13], it is *locally exact* over $S^2 - \mathbf{p}$, that is, the sphere minus one point, and thus can be represented as a boundary integral over any spherical region, except S^2.

C Proof of Identity (42)

$$\int_{\partial A} \frac{\langle \mathbf{v}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2}\, ds = \langle \mathbf{w}, \mathbf{v} \rangle \int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2}\, ds,$$

where $\mathbf{w}$ and $\mathbf{v}$ are arbitrary unit vectors in S^2.

Proof: Similarly, using Stokes' theorem, we can convert the boundary integral on both sides into surface integrals, the procedure is almost the same as we did in proving equation (6). Then, the left hand side yields

$$\begin{aligned}
\int_{\partial A} \frac{\langle \mathbf{v}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2}\, ds &= \int_{\partial A} \frac{\langle \mathbf{v}, \mathbf{r} \times\, d\mathbf{r} \rangle}{\langle \mathbf{w}, \mathbf{r} \rangle^2} \\
&= \int_{\partial A} \frac{\varepsilon_{jpl} \mathbf{v}_{\mathrm{j}} \mathbf{r}_{\mathrm{p}}\, d\mathbf{r}_{\mathrm{l}}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} \\
&= \int_A \varepsilon_{jpl} \mathbf{v}_{\mathrm{j}} \left(\frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}} \left[\frac{\mathbf{r}_{\mathrm{p}}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} \right] \right) d\mathbf{r}_{\mathrm{m}} \wedge\, d\mathbf{r}_{\mathrm{l}} \\
&= \int_A \varepsilon_{jpl} \mathbf{v}_{\mathrm{j}} \left[\frac{\delta_{pm}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} - \frac{2\mathbf{w}_{\mathrm{m}} \mathbf{r}_{\mathrm{p}}}{\langle \mathbf{w}, \mathbf{r} \rangle^3} \right] d\mathbf{r}_{\mathrm{m}} \wedge\, d\mathbf{r}_{\mathrm{l}} \\
&= \int_A \varepsilon_{kml} \varepsilon_{jpl} \mathbf{v}_{\mathrm{j}} \left[\frac{\delta_{pm}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} - \frac{2\mathbf{w}_{\mathrm{m}} \mathbf{r}_{\mathrm{p}}}{\langle \mathbf{w}, \mathbf{r} \rangle^3} \right] \left[\frac{\varepsilon_{kst}\, d\mathbf{r}_{\mathrm{s}} \wedge\, d\mathbf{r}_{\mathrm{t}}}{2} \right] \quad \text{(Equation (68))} \\
&= \int_A (\delta_{pm}\delta_{jk} - \delta_{pk}\delta_{jm})\, \mathbf{v}_{\mathrm{j}} \left[\frac{\delta_{pm}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} - \frac{2\mathbf{w}_{\mathrm{m}} \mathbf{r}_{\mathrm{p}}}{\langle \mathbf{w}, \mathbf{r} \rangle^3} \right] \left[\frac{\varepsilon_{kst}\, d\mathbf{r}_{\mathrm{s}} \wedge\, d\mathbf{r}_{\mathrm{t}}}{2} \right] \\
&= \int_A \left[\frac{3\mathbf{v}_{\mathrm{k}}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} - \frac{2\langle \mathbf{w}, \mathbf{r} \rangle \mathbf{v}_{\mathrm{k}}}{\langle \mathbf{w}, \mathbf{r} \rangle^3} - \frac{\mathbf{v}_{\mathrm{k}}}{\langle \mathbf{w}, \mathbf{r} \rangle^2} + \frac{2\mathbf{r}_{\mathrm{k}} \langle \mathbf{w}, \mathbf{v} \rangle}{\langle \mathbf{w}, \mathbf{r} \rangle^3} \right] \left[\frac{\varepsilon_{kst}\, d\mathbf{r}_{\mathrm{s}} \wedge\, d\mathbf{r}_{\mathrm{t}}}{2} \right] \\
&= \langle \mathbf{w}, \mathbf{v} \rangle \int_A \frac{2\mathbf{r}_{\mathrm{k}}}{\langle \mathbf{w}, \mathbf{r} \rangle^3} \left[\frac{\varepsilon_{kst}\, d\mathbf{r}_{\mathrm{s}} \wedge\, d\mathbf{r}_{\mathrm{t}}}{2} \right].
\end{aligned}$$

This formula holds for all vectors $\mathbf{w}$ and $\mathbf{v}$. Let $\mathbf{v} = \mathbf{w}$ above and notice that $\langle \mathbf{w}, \mathbf{w} \rangle = 1$, we get

$$\int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2}\, ds = \int_A \frac{2\mathbf{r}_{\mathrm{k}}}{\langle \mathbf{w}, \mathbf{r} \rangle^3} \left[\frac{\varepsilon_{kst}\, d\mathbf{r}_{\mathrm{s}} \wedge\, d\mathbf{r}_{\mathrm{t}}}{2} \right].$$

Therefore, Equation (42) holds. □□□

D Proof of Identity (46)

In this appendix, we shall show the detailed proof of the following identity:

$$\boxed{\int_{\partial A}\left[\langle\mathbf{a},\mathbf{n}\rangle-\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{w},\mathbf{n}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\right]\,ds \;=\; \int_{\partial A}\left[\langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{w},\mathbf{n}\rangle-\langle\mathbf{a},\mathbf{n}\rangle\right]\frac{\ln\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle^2}\,ds.} \tag{69}$$

Proof: This is done by converting the boundary integrals on both sides into their corresponding surface integrals and then comparing them.

Step 1: (The left hand side)

$$\begin{aligned}\int_{\partial A}\left[\langle\mathbf{a},\mathbf{n}\rangle-\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{w},\mathbf{n}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\right]\,ds &= \int_{\partial A}\langle\mathbf{a},\mathbf{n}\rangle\,ds-\int_{\partial A}\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{w},\mathbf{n}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\,ds\\ &= A_1-A_2.\end{aligned} \tag{70}$$

Next, we perform the boundary-to-surface conversion for the two terms on the right hand side of equation (70).

$$\begin{aligned}A_1 &= \int_{\partial A}\langle\mathbf{a},\mathbf{n}\rangle\,ds\\ &= \int_{A}\varepsilon_{kpl}\mathbf{a}_k\left[\frac{\mathbf{r}_p}{r^2}\right]\,d\mathbf{r}_l\\ &= \int_{A}\varepsilon_{qml}\varepsilon_{kpl}\mathbf{a}_k\frac{\partial}{\partial\mathbf{r}_m}\left[\frac{\mathbf{r}_p}{r^2}\right]\left[\frac{\varepsilon_{qst}\,d\mathbf{r}_s\wedge\,d\mathbf{r}_t}{2}\right]\\ &= \int_{A}\left[\mathbf{a}_q\delta_{pm}-\mathbf{a}_m\delta_{pq}\right]\left[\frac{\delta_{pm}}{r^2}-\frac{2\mathbf{r}_p\mathbf{r}_m}{r^4}\right]\left[\frac{\varepsilon_{qst}\,d\mathbf{r}_s\wedge\,d\mathbf{r}_t}{2}\right]\\ &= \int_{A}\left[-2\langle\mathbf{a},\mathbf{u}\rangle\right]\left[-\frac{\mathbf{r}_q}{r^3}\right]\left[\frac{\varepsilon_{qst}\,d\mathbf{r}_s\wedge\,d\mathbf{r}_t}{2}\right]\\ &= -\int_{A}2\langle\mathbf{a},\mathbf{u}\rangle\,d\omega,\\ A_2 &= \int_{\partial A}\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{w},\mathbf{n}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\,ds\\ &= \int_{\partial A}\varepsilon_{kpl}\mathbf{w}_k\left[\frac{\langle\mathbf{a},\mathbf{r}\rangle}{\langle\mathbf{w},\mathbf{r}\rangle}\right]\left[\frac{\mathbf{r}_p}{r^2}\right]\,d\mathbf{r}_l\end{aligned}$$

$$
\begin{aligned}
&= \int_A \varepsilon_{qml}\varepsilon_{kpl}\mathbf{w}_k \frac{\partial}{\partial \mathbf{r}_m}\left(\frac{\langle \mathbf{a},\mathbf{r}\rangle}{\langle \mathbf{w},\mathbf{r}\rangle}\left[\frac{\mathbf{r}_p}{r^2}\right]\right)\left[\frac{\varepsilon_{qst}\, d\mathbf{r}_s \wedge\, d\mathbf{r}_t}{2}\right] \\
&= \int_A \left[\mathbf{w}_q\delta_{pm} - \mathbf{w}_m\delta_{pq}\right]\left[\left(\frac{\mathbf{a}_m}{\langle \mathbf{w},\mathbf{r}\rangle} - \frac{\mathbf{w}_m \langle \mathbf{a},\mathbf{r}\rangle}{\langle \mathbf{w},\mathbf{r}\rangle^2}\right)\frac{\mathbf{r}_p}{r^2} + \frac{\langle \mathbf{a},\mathbf{r}\rangle}{\langle \mathbf{w},\mathbf{r}\rangle}\left(\frac{\delta_{pm}}{r^2} - \frac{2\mathbf{r}_p\mathbf{r}_m}{r^4}\right)\right]\left[\frac{\varepsilon_{qst}\, d\mathbf{r}_s \wedge\, d\mathbf{r}_t}{2}\right] \\
&= \int_A \left[\frac{\langle \mathbf{a},\mathbf{w}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle} - \frac{\langle \mathbf{a},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2} - 2\langle \mathbf{a},\mathbf{u}\rangle\right]\left[-\frac{\mathbf{r}_q}{r^3}\right]\left[\frac{\varepsilon_{qst}\, d\mathbf{r}_s \wedge\, d\mathbf{r}_t}{2}\right] \\
&= \int_A \left[\frac{\langle \mathbf{a},\mathbf{w}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle} - \frac{\langle \mathbf{a},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2} - 2\langle \mathbf{a},\mathbf{u}\rangle\right] d\omega.
\end{aligned}
$$

Thus, we have

$$
\int_{\partial A}\left[\langle \mathbf{a},\mathbf{n}\rangle - \frac{\langle \mathbf{a},\mathbf{u}\rangle\langle \mathbf{w},\mathbf{n}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle}\right] ds \;=\; \int_A \left[\frac{\langle \mathbf{a},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^2} - \frac{\langle \mathbf{a},\mathbf{w}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle}\right] d\omega, \tag{71}
$$

which expresses the left hand side of integral (70) as a surface integral.

Step 2: (The right hand side)

Define

$$
\xi_n \;\equiv\; \frac{\ln\langle \mathbf{w},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^n},
$$

we can write the right hand side integral as

$$
\begin{aligned}
\int_{\partial A} \xi_2\left[\langle \mathbf{a},\mathbf{w}\rangle\langle \mathbf{w},\mathbf{n}\rangle - \langle \mathbf{a},\mathbf{n}\rangle\right] ds \;&=\; \langle \mathbf{a},\mathbf{w}\rangle \int_{\partial A}\xi_2 \langle \mathbf{w},\mathbf{n}\rangle\, ds - \int_{\partial A}\xi_2\langle \mathbf{a},\mathbf{n}\rangle\, ds \\
&=\; \langle \mathbf{a},\mathbf{w}\rangle B_1 - B_2.
\end{aligned} \tag{72}
$$

Similarly, we shall apply Stokes' theorem to each term on the right hand side and then combine them. Notice that B_1 is just a special case of B_2 with $\mathbf{a} = \mathbf{w}$, we only need to convert the boundary integral B_2. In computing the partial derivative of ξ_2 in the following, we have used the following formula

$$
\xi_{n,m} \;=\; \frac{\partial \xi_n}{\partial \mathbf{r}_m} \;=\; \frac{1 - n\ln\langle \mathbf{w},\mathbf{u}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^{n+1} r^2}\left[r\mathbf{w}_m - \langle \mathbf{w},\mathbf{u}\rangle\,\mathbf{r}_m\right]. \tag{73}
$$

The proof of this identity is done by induction in Appendix E. Then, from equation (73), we have

$$
\begin{aligned}
B_2 &= \int_{\partial A} \xi_2 \langle \mathbf{a}, \mathbf{n} \rangle \; ds \\
&= \int_{\partial A} \varepsilon_{kpl} \mathbf{a}_k \left[\xi_2 \, \frac{\mathbf{r}_p}{r^2} \right] \; d\mathbf{r}_l \\
&= \int_A \varepsilon_{qml} \varepsilon_{kpl} \mathbf{a}_k \left[\frac{1 - 2 \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^3 r^2} \left(r \mathbf{w}_m - \mathbf{r}_m \langle \mathbf{w}, \mathbf{u} \rangle \right) \frac{\mathbf{r}_p}{r^2} \right. \\
&\quad \left. + \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2} \left(\frac{\delta_{pm}}{r^2} - \frac{2 \mathbf{r}_p \mathbf{r}_m}{r^4} \right) \right] \left[\frac{\varepsilon_{qst} \; d\mathbf{r}_s \wedge \; d\mathbf{r}_t}{2} \right] \\
&= \int_A \left[\mathbf{a}_q \delta_{pm} - \mathbf{a}_m \delta_{pq} \right] \left[\frac{1 - 2 \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^3 r^3} \mathbf{w}_m \mathbf{r}_p - \frac{\mathbf{r}_p \mathbf{r}_m}{\langle \mathbf{w}, \mathbf{u} \rangle^2 r^4} + \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2 r^2} \delta_{pm} \right] \left[\frac{\varepsilon_{qst} \; d\mathbf{r}_s \wedge \; d\mathbf{r}_t}{2} \right] \\
&= \int_A \left[\frac{1 - 2 \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^3} \langle \mathbf{a}, \mathbf{w} \rangle - \frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2} \right] \left[- \frac{\mathbf{r}_q}{r^3} \right] \left[\frac{\varepsilon_{qst} \; d\mathbf{r}_s \wedge \; d\mathbf{r}_t}{2} \right] \\
&= \int_A \left[\frac{1 - 2 \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^3} \langle \mathbf{a}, \mathbf{w} \rangle - \frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2} \right] \; d\omega.
\end{aligned}
$$

Letting $\mathbf{a} = \mathbf{w}$ above, we have

$$
B_1 = \int_A \left[\frac{1 - 2 \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^3} - \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle} \right] \; d\omega.
$$

Then, it follows from equation (72) that

$$
\int_{\partial A} \xi_2 \left[\langle \mathbf{a}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{n} \rangle - \langle \mathbf{a}, \mathbf{n} \rangle \right] \; ds \; = \; \int_A \left[\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2} - \frac{\langle \mathbf{a}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \right] \; d\omega. \tag{74}
$$

Comparing equation (71) and equation (74), we get the identity (69).

E Proof of Equation (73)

Let

$$\xi_n \equiv \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^n},$$

then, the partial derivative of ξ_n with respect to $\mathbf{r}_m$ is given by

$$\boxed{\xi_{n,m} = \frac{1 - n \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{n+1} r^2} \left[r\mathbf{w}_m - \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{r}_m \right]} \tag{75}$$

for any integer $n \geq 0$.

Proof: The proof is done by induction.

$n = 0:$

$$\begin{aligned} \frac{\partial}{\partial \mathbf{r}_m} \ln \langle \mathbf{w}, \mathbf{u} \rangle &= \frac{\mathbf{w}_m}{\langle \mathbf{w}, \mathbf{r} \rangle} - \frac{\mathbf{r}_m}{r^2} \\ &= \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle r^2} \left[r\mathbf{w}_m - \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{r}_m \right]. \end{aligned}$$

Thus, equation (75) holds for $n = 0$.

$n = k$**:** Suppose equation (75) holds for $n = k - 1$, that is,

$$\xi_{k-1,m} = \frac{1 - (k-1) \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^k r^2} \left[r\mathbf{w}_m - \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{r}_m \right]. \tag{76}$$

Then, the partial derivative of ξ_k can be computed as:

$$\begin{aligned} \xi_{k,m} &= \frac{\partial}{\partial \mathbf{r}_m} \left[\xi_{k-1} \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle} \right] \\ &= \xi_{k-1,m} \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle} + \xi_{k-1} \frac{\partial}{\partial \mathbf{r}_m} \left[\frac{r}{\langle \mathbf{w}, \mathbf{r} \rangle} \right] \\ &= \frac{1 - (k-1) \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{k+1} r^2} \left[r\mathbf{w}_m - \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{r}_m \right] + \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{k-1}} \cdot \frac{\mathbf{r}_m \langle \mathbf{w}, \mathbf{u} \rangle - r\mathbf{w}_m}{\langle \mathbf{w}, \mathbf{u} \rangle^2 r^2} \\ &= \frac{1 - k \ln \langle \mathbf{w}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{k+1} r^2} \left[r\mathbf{w}_m - \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{r}_m \right]. \end{aligned}$$

Therefore, equation (75) also holds for $n = k$ if it holds for $n = k - 1$.

From induction, equation (75) is true for all natural numbers $n \geq 0$. □□□

F Direct Verification of the Boundary Integral (52)

While deriving the formula for the irradiance $\phi(A)$ from a luminaire with linearly-varying radiant exitance, we first formulated it as a surface integral (17) of a rational function, then by means of various binomial identities and triple-axis moments, we arrive at a boundary integral formula (52) for $\phi(A)$. To verify the correctness of our new boundary integral formula, here we may prove it directly by using Stokes' theorem. By equating equation (17) to equation (52), the identity to be proved becomes:

$$\boxed{\int_A \frac{\langle \mathbf{a}, \mathbf{u}\rangle \langle \mathbf{b}, \mathbf{u}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\, d\omega \;=\; -\frac{1}{2}\int_{\partial A} \mathbf{M}_{\mathrm{ijk}}\, \mathbf{a}_{\mathrm{i}}\, \mathbf{b}_{\mathrm{j}}\, \mathbf{n}_{\mathrm{k}}\, ds,} \tag{77}$$

where the 3-tensor $\mathbf{M}$ is defined as

$$\mathbf{M}_{\mathrm{ijk}} \;\equiv\; \delta_{ik}\mathbf{w}_{\mathrm{j}} + \left(\frac{\delta_{km}\mathbf{u}_{\mathrm{i}}}{\langle \mathbf{w}, \mathbf{u}\rangle} - \delta_{im}\mathbf{w}_{\mathrm{k}}\,\eta\right)\left(\delta_{jm} - \mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{m}}\right). \tag{78}$$

Proof: The proof is done by applying Stokes' Theorem on the right hand side and changing it into a surface integral, which can be stated in the following steps:

Step 1: (Rewrite the boundary integral)

As shown in Figure 1b,

$$\begin{aligned} \mathbf{n}\, ds &= \frac{\mathbf{r} \times\, d\mathbf{r}}{r^2} \\ \mathbf{u} &= \frac{\mathbf{r}}{r}, \end{aligned}$$

we express the integrand on the right hand in terms of the position vector $\mathbf{r}$ and its derivatives:

$$\begin{aligned} \int_{\partial A} \mathbf{M}_{\mathrm{ijk}}\, \mathbf{a}_{\mathrm{i}}\, \mathbf{b}_{\mathrm{j}}\, \mathbf{n}_{\mathrm{k}}\, ds &= \int_{\partial A} \mathbf{M}_{\mathrm{ijk}}\, \mathbf{a}_{\mathrm{i}}\, \mathbf{b}_{\mathrm{j}} \frac{\varepsilon_{kpl}\mathbf{r}_{\mathrm{p}}\, d\mathbf{r}_{\mathrm{l}}}{r^2} \\ &= \int_{\partial A} B_{\mathrm{l}}\, d\mathbf{r}_{\mathrm{l}}. \end{aligned} \tag{79}$$

Here, we have introduced a vector B_{l}, given by

$$B_{\mathrm{l}} \;=\; \frac{\varepsilon_{kpl}\mathbf{M}_{\mathrm{ijk}}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\mathbf{r}_{\mathrm{p}}}{r^2},$$

and it follows from equation (78) that

$$\mathbf{M}_{\mathrm{ijk}} \;=\; \delta_{\mathrm{ik}}\mathbf{w}_{\mathrm{j}} + \frac{\delta_{\mathrm{jk}} - \mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}}}{\langle \mathbf{w}, \mathbf{u}\rangle}\mathbf{u}_{\mathrm{i}} - (\delta_{\mathrm{ij}}\mathbf{w}_{\mathrm{k}} - \mathbf{w}_{\mathrm{i}}\mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}})\eta. \tag{80}$$

Step 2: (Convert the boundary integral to a surface integral)

This step follows from Stokes' Theorem and the derivative of a differential form

$$d[f_{\mathrm{i}}(\mathbf{r})\ d\mathbf{r}_{\mathrm{i}}] \;=\; f_{\mathrm{i,m}}(\mathbf{r})\ d\mathbf{r}_{\mathrm{m}} \wedge\ d\mathbf{r}_{\mathrm{i}}.$$

To convert the boundary integral in equation (79) into a surface integral, we need to compute the partial derivative of B_{l} with respect to $\mathbf{r}_{\mathrm{m}}$, yields

$$B_{\mathrm{l,m}} \;=\; \varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}}\left[\frac{\mathbf{M}_{\mathrm{ijk}}\mathbf{r}_{\mathrm{p}}}{r^2}\right]. \tag{81}$$

However,

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}}\left[\frac{\mathbf{r}_{\mathrm{p}}}{r^2}\right] &= \frac{\delta_{pm}r^2 - 2\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{r^4} \\
\frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}}[\mathbf{M}_{\mathrm{ijk}}] &= (\delta_{jk} - \mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}})\frac{\delta_{im}\langle \mathbf{w}, \mathbf{r}\rangle - \mathbf{r}_{\mathrm{i}}\mathbf{w}_{\mathrm{m}}}{\langle \mathbf{w}, \mathbf{r}\rangle^2} - \eta_{\mathrm{m}}(\delta_{ij}\mathbf{w}_{\mathrm{k}} - \mathbf{w}_{\mathrm{i}}\mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}}),
\end{aligned}$$

where η_{m} denotes the partial derivatives of η with respect to $\mathbf{r}_{\mathrm{m}}$, which can be computed as follows:

$$\begin{aligned}
\eta_{\mathrm{m}} &= \frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}}\left[\frac{\ln\langle \mathbf{w}, \mathbf{u}\rangle}{1 - \langle \mathbf{w}, \mathbf{u}\rangle^2}\right] \\
&= \frac{\partial}{\partial \mathbf{r}_{\mathrm{m}}}\left[\frac{(\ln\langle \mathbf{w}, \mathbf{r}\rangle - \ln r)r^2}{r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2}\right] \\
&= \frac{\left(r^2\mathbf{w}_{\mathrm{m}}/\langle \mathbf{w}, \mathbf{r}\rangle - \mathbf{r}_{\mathrm{m}} + 2\mathbf{r}_{\mathrm{m}}\ln\langle \mathbf{w}, \mathbf{u}\rangle\right)\left(r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2\right) - \left(2\mathbf{r}_{\mathrm{m}} - 2\langle \mathbf{w}, \mathbf{r}\rangle\mathbf{w}_{\mathrm{m}}\right)r^2\ln\langle \mathbf{w}, \mathbf{u}\rangle}{\left(r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2\right)^2} \\
&= \frac{r^2\mathbf{w}_{\mathrm{m}}\left(r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2 + 2\langle \mathbf{w}, \mathbf{r}\rangle^2\ln\langle \mathbf{w}, \mathbf{u}\rangle\right)/\langle \mathbf{w}, \mathbf{r}\rangle - \mathbf{r}_{\mathrm{m}}\left(r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2 + 2\langle \mathbf{w}, \mathbf{r}\rangle^2\ln\langle \mathbf{w}, \mathbf{u}\rangle\right)}{\left(r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2\right)^2} \\
&= \frac{r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2 + 2\langle \mathbf{w}, \mathbf{r}\rangle^2\ln\langle \mathbf{w}, \mathbf{u}\rangle}{\left(r^2 - \langle \mathbf{w}, \mathbf{r}\rangle^2\right)^2}\left(\frac{r^2\mathbf{w}_{\mathrm{m}}}{\langle \mathbf{w}, \mathbf{r}\rangle} - \mathbf{r}_{\mathrm{m}}\right).
\end{aligned} \tag{82}$$

Thus, Equation (81) becomes

$$
\begin{aligned}
B_{\mathrm{l,m}} &= \varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\left[\mathbf{M}_{\mathrm{ijk}}\frac{\delta_{pm}r^2 - 2\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{r^4} + (\delta_{jk} - \mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}})\frac{\delta_{im}\mathbf{r}_{\mathrm{p}}\langle\mathbf{w},\mathbf{r}\rangle - \mathbf{r}_{\mathrm{i}}\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{m}}}{r^2\langle\mathbf{w},\mathbf{r}\rangle^2}\right. \\
&\qquad \left. - \frac{\mathbf{r}_{\mathrm{p}}}{r^2}\,\eta_{\mathrm{m}}(\delta_{ij}\mathbf{w}_{\mathrm{k}} - \mathbf{w}_{\mathrm{i}}\mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}})\right] \\
&= \varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\left[A_1 + A_2 + A_3\right], \qquad (83)
\end{aligned}
$$

where A_1, A_2, A_3 are given by

$$
\begin{aligned}
A_1 &= \mathbf{M}_{\mathrm{ijk}}\frac{\delta_{pm}r^2 - 2\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{r^4} \\
A_2 &= (\delta_{jk} - \mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}})\frac{\delta_{im}\mathbf{r}_{\mathrm{p}}\langle\mathbf{w},\mathbf{r}\rangle - \mathbf{r}_{\mathrm{i}}\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{m}}}{r^2\langle\mathbf{w},\mathbf{r}\rangle^2} \\
A_3 &= -\frac{\mathbf{r}_{\mathrm{p}}}{r^2}\,\eta_{\mathrm{m}}(\delta_{ij}\mathbf{w}_{\mathrm{k}} - \mathbf{w}_{\mathrm{i}}\mathbf{w}_{\mathrm{j}}\mathbf{w}_{\mathrm{k}}).
\end{aligned}
$$

and $\mathbf{M}$, η_{m} are given by equations (80), (82), respectively. Consequently, we have

$$
\begin{aligned}
\int_{\partial A} B_{\mathrm{l}}\, d\mathbf{r}_{\mathrm{l}} &= \int_A B_{\mathrm{l,m}}\, d\mathbf{r}_{\mathrm{m}} \wedge\, d\mathbf{r}_{\mathrm{l}} \\
&= \int_A \varepsilon_{qml}\varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\left[A_1 + A_2 + A_3\right]\left[\frac{\varepsilon_{qst}\, d\mathbf{r}_{\mathrm{s}} \wedge\, d\mathbf{r}_{\mathrm{t}}}{2}\right]. \qquad (84)
\end{aligned}
$$

where equation (68) is applied.

Step 3: (Simplification of the surface integral)

Using the tensor identity

$$
\varepsilon_{qml}\varepsilon_{kpl} = \delta_{qk}\delta_{pm} - \delta_{pq}\delta_{km},
$$

the three terms in equation (84) simplify to

$$
\begin{aligned}
\varepsilon_{qml}\varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}A_1 &= \left(\delta_{qk}\delta_{pm} - \delta_{pq}\delta_{km}\right)\frac{\delta_{pm}r^2 - 2\mathbf{r}_{\mathrm{p}}\mathbf{r}_{\mathrm{m}}}{r^4}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\mathbf{M}_{\mathrm{ijk}} \\
&= \frac{\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\mathbf{M}_{\mathrm{ijk}}}{r^4}\left(3\delta_{qk}r^2 - 2\delta_{qk}r^2 - \delta_{qk}r^2 + 2\mathbf{r}_{\mathrm{q}}\mathbf{r}_{\mathrm{k}}\right) \\
&= \frac{2\mathbf{r}_{\mathrm{q}}}{r^4}\left(\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}\mathbf{r}_{\mathrm{k}}\mathbf{M}_{\mathrm{ijk}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\mathbf{r}_{\mathrm{q}}}{r^4}\left(\langle\mathbf{a},\mathbf{r}\rangle\langle\mathbf{b},\mathbf{w}\rangle + \frac{\langle\mathbf{b},\mathbf{r}\rangle - \langle\mathbf{w},\mathbf{b}\rangle\langle\mathbf{w},\mathbf{r}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\langle\mathbf{a},\mathbf{u}\rangle\right.\\
&\qquad \left.-\eta\left(\langle\mathbf{a},\mathbf{b}\rangle\langle\mathbf{w},\mathbf{r}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\langle\mathbf{r},\mathbf{w}\rangle\right)\right)\\
&= \frac{2\mathbf{r}_{\mathrm{q}}}{r^3}\left[\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{b},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle} - \eta\left(\langle\mathbf{a},\mathbf{b}\rangle\langle\mathbf{w},\mathbf{u}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\langle\mathbf{w},\mathbf{u}\rangle\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{qml}\varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}A_2 &= \frac{\delta_{qk}\delta_{pm} - \delta_{pq}\delta_{km}}{r^2\langle\mathbf{w},\mathbf{r}\rangle^2}\left[(\mathbf{b}_{\mathrm{k}} - \mathbf{w}_{\mathrm{k}}\langle\mathbf{b},\mathbf{w}\rangle)(\mathbf{a}_{\mathrm{m}}\mathbf{r}_{\mathrm{p}}\langle\mathbf{w},\mathbf{r}\rangle - \mathbf{w}_{\mathrm{m}}\mathbf{r}_{\mathrm{p}}\langle\mathbf{a},\mathbf{r}\rangle)\right]\\
&= \frac{\mathbf{r}_{\mathrm{q}}}{r^3}\left[-\frac{\langle\mathbf{a},\mathbf{b}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle} + \frac{\langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\right],
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{qml}\varepsilon_{kpl}\mathbf{a}_{\mathrm{i}}\mathbf{b}_{\mathrm{j}}A_3 &= -\frac{\varepsilon_{qml}\varepsilon_{kpl}}{r^2}\,\eta_{\mathrm{m}}\left(\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{k}}\langle\mathbf{a},\mathbf{b}\rangle - \mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{k}}\langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\right)\\
&= -\frac{\left(r^2 - \langle\mathbf{w},\mathbf{r}\rangle^2 + 2\langle\mathbf{w},\mathbf{r}\rangle^2\ln\langle\mathbf{w},\mathbf{u}\rangle\right)\left(\langle\mathbf{a},\mathbf{b}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\right)}{r^2\left(r^2 - \langle\mathbf{w},\mathbf{r}\rangle^2\right)^2}\times\\
&\qquad\left[\left(\delta_{qk}\delta_{pm} - \delta_{pq}\delta_{km}\right)\mathbf{r}_{\mathrm{p}}\mathbf{w}_{\mathrm{k}}\left(\frac{r^2\mathbf{w}_{\mathrm{m}}}{\langle\mathbf{w},\mathbf{r}\rangle} - \mathbf{r}_{\mathrm{m}}\right)\right]\\
&= -\frac{\left(r^2 - \langle\mathbf{w},\mathbf{r}\rangle^2 + 2\langle\mathbf{w},\mathbf{r}\rangle^2\ln\langle\mathbf{w},\mathbf{u}\rangle\right)\left(\langle\mathbf{a},\mathbf{b}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\right)}{r^2\left(r^2 - \langle\mathbf{w},\mathbf{r}\rangle^2\right)^2}\times\\
&\qquad\left[\mathbf{r}_{\mathrm{q}}\langle\mathbf{w},\mathbf{r}\rangle - \mathbf{r}_{\mathrm{q}}\frac{r^2}{\langle\mathbf{w},\mathbf{r}\rangle}\right]\\
&= \frac{\left(1 - \langle\mathbf{w},\mathbf{u}\rangle^2 + 2\langle\mathbf{w},\mathbf{u}\rangle^2\ln\langle\mathbf{w},\mathbf{u}\rangle\right)\left(\langle\mathbf{a},\mathbf{b}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\right)}{r^2 - \langle\mathbf{w},\mathbf{r}\rangle^2}\cdot\frac{\mathbf{r}_{\mathrm{q}}}{\langle\mathbf{w},\mathbf{r}\rangle}\\
&= \frac{1 - \langle\mathbf{w},\mathbf{u}\rangle^2 + 2\langle\mathbf{w},\mathbf{u}\rangle^2\ln\langle\mathbf{w},\mathbf{u}\rangle}{r^3\left(1 - \langle\mathbf{w},\mathbf{u}\rangle^2\right)\langle\mathbf{w},\mathbf{u}\rangle}\mathbf{r}_{\mathrm{q}}\left[\langle\mathbf{a},\mathbf{b}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\right]\\
&= \frac{\mathbf{r}_{\mathrm{q}}}{r^3}\left[\left(\frac{1}{\langle\mathbf{w},\mathbf{u}\rangle} + 2\eta\langle\mathbf{w},\mathbf{u}\rangle\right)\left(\langle\mathbf{a},\mathbf{b}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\right)\right]\\
&= \frac{\mathbf{r}_{\mathrm{q}}}{r^3}\left[\frac{\langle\mathbf{a},\mathbf{b}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle} - \frac{\langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle} + \right.\\
&\qquad\left. 2\eta\left(\langle\mathbf{a},\mathbf{b}\rangle\langle\mathbf{w},\mathbf{u}\rangle - \langle\mathbf{a},\mathbf{w}\rangle\langle\mathbf{b},\mathbf{w}\rangle\langle\mathbf{w},\mathbf{u}\rangle\right)\right]
\end{aligned}
$$

It follows that

$$\varepsilon_{qml}\varepsilon_{kql}\mathbf{a}_{\rm i}\mathbf{b}_{\rm j}\left[A_1 + A_2 + A_3\right] \;=\; \frac{2\mathbf{r}_{\rm q}}{r^3}\left[\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{b},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\right].$$

Equation (84) thus reduces to

$$\int_{\partial A} B_{\rm l}\; d\mathbf{r}_{\rm l} \;=\; 2\int_A \left[\frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{b},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle}\right]\frac{\mathbf{r}_{\rm q}}{r^3}\left[\frac{\varepsilon_{qst}\; d\mathbf{r}_{\rm s}\wedge\; d\mathbf{r}_{\rm t}}{2}\right]. \tag{85}$$

<u>**Step** 4: (Express the surface integral in terms of solid angle)</u>

By representing the differential 2-form $d\omega$ relating to the solid angle as [13, page 131]

$$d\omega = -\frac{\varepsilon_{qst}\mathbf{r}_{\rm q}\; d\mathbf{r}_{\rm s}\wedge\; d\mathbf{r}_{\rm t}}{2r^3},$$

The right hand side of equation (85) can be written as

$$-2\int_A \frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{b},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle} d\omega.$$

Thus, we get

$$\int_A \frac{\langle\mathbf{a},\mathbf{u}\rangle\langle\mathbf{b},\mathbf{u}\rangle}{\langle\mathbf{w},\mathbf{u}\rangle} d\omega \;=\; -\frac{1}{2}\int_{\partial A} B_{\rm l}\; d\mathbf{r}_{\rm l} \;=\; -\frac{1}{2}\int_{\partial A} \mathbf{M}_{\rm ijk}\mathbf{a}_{\rm i}\mathbf{b}_{\rm j}\mathbf{n}_{\rm k}\; ds,$$

which proves the formula (77). □□□

G Derivation of Formula (19)

In this Appendix, we shall show that the irradiance at an arbitrary receiver point $\mathbf{q}$ due to a linearly-varying luminaire is also given by an integral of a rational polynomial, that is

$$\phi(A) = \frac{1}{\pi} \int_A \frac{\langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \, d\sigma(\mathbf{u}), \tag{86}$$

where $\mathbf{b}$ is the normal at the receiver point, $\mathbf{w}$ is the unit vector from $\mathbf{q}$ orthogonal to the planar luminaire, and the vector argument $\mathbf{a}$ varies with $\mathbf{q}$, satisfying

$$\mathbf{a} = \langle \mathbf{a}_0, \mathbf{q} \rangle \; \mathbf{w} + h(\mathbf{q}) \, \mathbf{a}_0. \tag{87}$$

For a linearly-varying luminaire specified by three non-collinear points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ with corresponding radiosity values w_1, w_2 and w_3, $\mathbf{a}_0$ is defined as

$$\mathbf{a}_0 = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}^{-1}.$$

As shown in section 3, the radiant exitant distribution function of the given linearly-varying luminaire can be expressed as

$$f(\mathbf{x}) = \langle \mathbf{a}_0, \mathbf{x} \rangle ,$$

where $\mathbf{x}$ is the point on the luminaire, denoted by its coordinates with respect to the coordinate system at the origin $\mathbf{o}$. By rewriting $\mathbf{x} = \mathbf{q} + (\mathbf{x} - \mathbf{q})$, we have

$$f(\mathbf{x}) = \langle \mathbf{a}_0, \mathbf{q} \rangle + \langle \mathbf{a}_0, \mathbf{x} - \mathbf{q} \rangle .$$

Consequently, the irridance at $\mathbf{q}$ due to the linearly-varying luminaire is given by

$$\phi(L, \mathbf{q}) = \frac{1}{\pi} \int_L f(\mathbf{x}) \frac{\cos\theta_1 \cos\theta_2}{r^2} \, d\mathbf{x} \tag{88}$$

$$= \frac{1}{\pi} \left[\int_L \langle \mathbf{a}_0, \mathbf{q} \rangle \frac{\cos\theta_1 \cos\theta_2}{r^2} \, d\mathbf{x} + \int_L \langle \mathbf{a}_0, \mathbf{x} - \mathbf{q} \rangle \frac{\cos\theta_1 \cos\theta_2}{r^2} \, d\mathbf{x} \right]. \tag{89}$$

If we integrate over the spherical projection $A = \prod(L)$, equation (88) becomes

$$\phi(A, \mathbf{q}) = \frac{1}{\pi} \left[\langle \mathbf{a}_0, \mathbf{q} \rangle \int_A \langle \mathbf{b}, \mathbf{u} \rangle \; d\sigma(\mathbf{u}) + \int_A \left\langle \mathbf{a}_0, \frac{h(\mathbf{q})}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{u} \right\rangle \langle \mathbf{b}, \mathbf{u} \rangle \; d\sigma(\mathbf{u}) \right]$$

$$= \frac{1}{\pi} \left[\langle \mathbf{a}_0, \mathbf{q} \rangle \int_A \langle \mathbf{b}, \mathbf{u} \rangle \; d\sigma(\mathbf{u}) + h(\mathbf{q}) \int_A \frac{\langle \mathbf{a}_0, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \, d\sigma(\mathbf{u}) \right], \tag{90}$$

where $h(\mathbf{q})$ is the distance from $\mathbf{q}$ to the planar luminaire, and we used the relation (15) between a position vector on the luminaire and the unit vector in the same direction. By letting

$$\mathbf{a} \;=\; \langle \mathbf{a}_0, \mathbf{q} \rangle \; \mathbf{w} + h(\mathbf{q})\, \mathbf{a}_0,$$

it can be easily verified that equation (90) reduces to the irradiance formula (86).

References

[1] James Arvo. *Analytic Methods for Simulated Light Transport.* PhD thesis, Yale University, December 1995.

[2] James Arvo. Applications of irradiance tensors to the simulation of non-Lambertian phenomena. In *Computer Graphics* Proceedings, Annual Conference Series, ACM SIGGRAPH, pages 335–342, August 1995.

[3] Daniel R. Baum, Holly E. Rushmeier, and James M. Winget. Improving radiosity solutions through the use of analytically determined form-factors. *Computer Graphics*, 23(3):325–334, July 1989.

[4] Marcel Berger. *Geometry*, volume 2. Springer-Verlag, New York, 1987. Translated by M. Cole and S. Levy.

[5] Min Chen and James Arvo. Closed-form expressions for irradiance from non-uniform lambertian luminaires, Part 2: Polynomially-varying radiant exitance. Technical Report CS-TR-00-04, Caltech, CA, March 2000.

[6] David L. DiLaura. Non-diffuse radiative transfer 3: Inhomogeneous planar area sources and point receivers. *Journal of the Illuminating Engineering Society*, 26(1), 1997.

[7] C. C. Grosjean. Formulae concerning the computation of the clausen integral $cl_2(\theta)$. *Journal of Computational and Applied Mathematics*, 11:331–342, 1984.

[8] David Hart, Philip Dutré, and Donald P. Greenberg. Direct illumination with lazy visibility evaluation. In *Computer Graphics* Proceedings, Annual Conference Series, ACM SIGGRAPH, pages 147–154, August 1999.

[9] Leonard Lewin. *Polylogarithms and associated functions.* North Holland, New York, 1981.

[10] Nelson Max and Michael Allison. Linear radiosity approximation using vertex-to-vertex form factors. In David Kirk, editor, *Graphics Gems III.* Academic Press, New York, 1992.

[11] E. O. Powell. An integral related to the radiation integrals. *Philosophical Magazine*, 34(236):600–607, September 1943.

[12] Peter Schröder and Pat Hanrahan. On the form factor between two polygons. In *Computer Graphics* Proceedings, Annual Conference Series, ACM SIGGRAPH, pages 163–164, August 1993.

[13] Michael Spivak. *Calculus on Manifolds.* Benjamin/Cummings, Reading, Massachusetts, 1965.

[14] Michael M. Stark, Elaine Cohen, Tom Lynche, and Richard F. Riesenfeld. Computing exact shadow irradiance using splines. In *Computer Graphics* Proceedings, Annual Conference Series, ACM SIGGRAPH, pages 155–164, August 1999.