Closed-Form Expressions for Irradiance from Non-Uniform Lambertian Luminaires

Part II : Polynomially-Varying Radiant Exitance

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Abstract

We present new analytical techniques for computing illumination from non-uniform luminaires. The methods are based on new closed-form expressions derived by generalizing the concepts of irradiance tensor and angular moment to rational forms and an arbitrary number of directions, known as *rational irradiance tensors* and *rational angular moments*, respectively. The techniques apply to any emission, reflection or transmission distribution expressed as a polynomial over a polygonal surface, and provide a powerful mathematical tool to handle more complex BRDF's. We derive closed-form expressions for irradiance due to polygonal luminaires with polynomially varying radiant exitance, which satisfy a recurrence relation that subsumes Lambert's formula for uniform luminaires. Our formulas extend the class of available closed-form expressions for computing direct radiative transfer from planar surfaces to points, and can find many potential applications in simulating non-Lambertian illumination and scattering phenomena.

Keywords: Illumination, Rendering, Radiosity, Irradiance Tensor, Angular Moment, Spatially Varying Luminaire

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1 Introduction

The computation of radiant energy transfers is an essential component of physically-based rendering algorithms, both for local and global illumination. In particular, radiative transfers among discrete surface elements arise in finite element methods for global illumination, and in both direct lighting computations and final "gathers" from coarse global solutions [3, 13, 24]. By far the most ubiquitous computations involve transfers from planar areas to points. However, rendering algorithms using deterministic methods are frequently quite limited in the surface reflectance function and luminaires they can accomodate, which stems from the difficulty in evaluating the surface integrals associated with non-uniform illuminations. While a wide assortment of formulas are available for uniform Lambertian environments [14, 6, 20], few tools currently exists for simulating non-Lambertian effects aside from irradiance tensor [3], Monte Carlo [10, 22, 25, 26], Hierarchical subdivision [5] and Numerical quadrature [9, 21].

Currently, few methods exist for computing radiative transfers from area light sources (luminaires) in which the emissive power is non-constant except Monte Carlo. Although the closed-form expressions for irradiance tensor and axial moments derived by Arvo [3] provide an approach for computing direct lighting effects from luminaires with directional variations, they cannot apply for spatially varying luminaires. Such luminaires constitute an important class of light sources in simulating realistic non-Lambertian pheonomena.

In this paper we present the first analytic method for computing direct illumination involving area light sources with polynomially-varying radiant exitance. We derive our result using some of the same mathematical machinery used by Arvo [3]. Specifically, our approach begins with a tensor formulation of irradiance known as *rational irradiance tensor*, with *rational angular moment* as its elements, which is weighted integral of radiance with respect to directions [19]. This representation leads to a general boundary integral using Stokes' theorem, which simplifies to a closed-form solution for rational moments in the case of polyhedral environments. The irradiance due to luminaires with polynomially varying radiant exitance is simply related to a special class of rational moments, which satisfies a recurrence relation subsuming Lambert's formula as its base case.

The idea of using tensor and angular moments to formulate integrals encountered in radiative transfer has been previously used by Arvo [3] to derive algorithms for simulating glossy reflections and illuminations from directional luminaires. Our representation generalizes the concepts of *irradiance tensors* and *axial moments* [3] to accomodate a simple class of rational polynomial functions over the sphere. This new class of functions arises in computing irradiance from non-uniform luminaires, as well as a wide range of surface scattering effects, such as glossy reflection and transmission from non-uniform sources, more complex BRDF models, and so on. Our techniques are quite general and apply to any emission, reflection and transmission distribution expressed as a polynomial over a planar surface, greatly enriches the repertoire of rendering effects that can be computed in closed-form.

Stokes' theorem is a common tool for computing surface integral of this nature. Reduction to a one-dimensional boundary integral is frequently a first step toward closed-form solutions. Previous work in which Stokes' theorem has been directly applied for such transformation includes that of Lambert, Schröder and Hanrahan [20], and Sparrow [23]. Moreover, it is very coincident that just as general patch-to-patch form factors [20] are expressed in terms of the *dilogarithm* [16], our closed-form solution also rests upon a special function called *Clausen integral* [1, 4, 16], which is closely related to the dilogarithm.

Our theoretical contributions are:

1. The derivation of recursive boundary integral formulas for rational irradiance tensors and rational angular moments of any order.
2. The derivation of a *closed-form* expression (including one special function) for rational angular moments for polyhedral environments.
3. Reduction of the single special function to the Clausen integral, a well-known special function that can be evaluated to high accuracy.
4. The derivation of a general tensor recurrence relation for irradiance due to luminaires with polynomially-varying radiant exitance, which extends Lambert's formula.

The remainder of the report is organized as follows. Section 2 introduces the concept of rational irradiance tensor and proves three recursive relations using Stokes' theorem. Using rational tensors, we derive corresponding recursive expressions for rational angular moments in Section 3. In Section 4, we focus on polygonal luminaires and derive closed-form expressions for rational moments by evaluating the boundary integrals exactly or in terms of Clausen integrals. As a concrete example, in Section 5 we formulate the irradiance due to luminaires with polynomially-varying radiant exitance as a special class of rational moments and derive a recursive tensor formula which can be applied to many non-Lambertian simulations. Finally, some additional proofs and derivations are attached in the Appendix.

2 Rational Irradiance Tensor

As a generalization of irradiance tensor $\mathbf{T}^n(A, \mathbf{w})$ [2], we introduce a *rational irradiance tensor* to accomodate spatially varying luminaires. It is defined by

$$\mathbf{T}^{n,q}(A, \mathbf{w}) \;\equiv\; \int_A \frac{\mathbf{u} \otimes \cdots \otimes \mathbf{u}}{\langle \mathbf{w}, \mathbf{u} \rangle^q} \, d\sigma(\mathbf{u}) \tag{1}$$

where $\mathbf{w}$ is a unit vector. In this section, we shall show that the new tensors with rationl elements satisfy a recurrence relation, which involves the original irradiance tensor $\mathbf{T}^n(A, \mathbf{w})$ and $\mathbf{T}^{0,1}(A, \mathbf{w})$ as its base cases. The results can be stated in the following theorems.

Theorem 1 *Let $n \geq 0$ and $q \geq 2$ be integers, then the tensor $\mathbf{T}^{n,q}(A, \mathbf{w})$ satisfies the recurrence relation*

$$\mathbf{T}^{n,q}_{\mathrm{I}} \;=\; \frac{1}{q-1} \left[\sum_{k=1}^{n} \mathbf{w}_{I_k} \mathbf{T}^{n-1,q-1}_{I/k} - (n-q+3)\, \mathbf{T}^{n,q-2}_{I} - \int_{\partial A} \frac{\mathbf{u}^n_I \langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{q-1}} \, ds \right]. \tag{2}$$

where I is an n-index, ds denotes integration with respect to arclength, and $\mathbf{n}$ is the outward normal to the curve ∂A.

Proof: The proof is done by converting the boundary integral in equation (2) to a surface integral using Stokes' theorem. Here, we only show the key steps; the details of the tensor transformations used here are shown in our previous technical report [8].

$$\begin{aligned}
\int_{\partial A} \frac{\mathbf{u}^n_I \langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{q-1}} \, ds \;&=\; \int_{\partial A} \left[\frac{r^{q-1}}{\langle \mathbf{w}, \mathbf{r} \rangle^{q-1}} \right] \left[\frac{\mathbf{r}^n_I}{r^n} \right] \left[\frac{\varepsilon_{kpl} \mathbf{w}_k \mathbf{r}_p \, d\mathbf{r}_l}{r^2} \right] \\
&=\; \int_{\partial A} \left[\frac{r^{q-1}}{\langle \mathbf{w}, \mathbf{r} \rangle^{q-1}} \right] \left[\frac{\varepsilon_{kpl} \mathbf{w}_k \mathbf{r}_p \mathbf{r}^n_I}{r^{n+2}} \right] \, d\mathbf{r}_l \\
&=\; \int_{\partial A} \left[\frac{r^{q-1}}{\langle \mathbf{w}, \mathbf{r} \rangle^{q-1}} \right] A^{n+1}_{Il} \, d\mathbf{r}_l \\
&=\; \int_{\partial A} B^{n+1}_{Il} \, d\mathbf{r}_l,
\end{aligned} \tag{3}$$

where the $(n+1)$-tensors A and B are given by

$$\begin{aligned}
B^{n+1}_{Il} \;&=\; \left[\frac{r^{q-1}}{\langle \mathbf{w}, \mathbf{r} \rangle^{q-1}} \right] A^{n+1}_{Il} \\
A^{n+1}_{Il} \;&=\; \left[\frac{\varepsilon_{kpl} \mathbf{w}_k \mathbf{r}_p \mathbf{r}^n_I}{r^{n+2}} \right].
\end{aligned}$$

Differentiating the tensor B, we get

$$B^{n+1}_{Il,m} \;=\; \frac{\partial}{\partial \mathbf{r}_m}\left(\frac{r}{\langle \mathbf{w},\mathbf{r}\rangle}\right)^{q-1} A^{n+1}_{Il} \;+\; \left(\frac{r}{\langle \mathbf{w},\mathbf{r}\rangle}\right)^{q-1} A^{n+1}_{Il,m}, \tag{4}$$

where

$$\begin{aligned}\frac{\partial}{\partial \mathbf{r}_m}\left(\frac{r}{\langle \mathbf{w},\mathbf{r}\rangle}\right)^{q} &= q\left(\frac{r}{\langle \mathbf{w},\mathbf{r}\rangle}\right)^{q-1}\left[\frac{(\mathbf{r}_m/r)\langle \mathbf{w},\mathbf{r}\rangle - \mathbf{w}_m r}{\langle \mathbf{w},\mathbf{r}\rangle^2}\right]\\ &= \frac{q\, r^{q-2}}{\langle \mathbf{w},\mathbf{r}\rangle^{q+1}}\left[\mathbf{r}_m\langle \mathbf{w},\mathbf{r}\rangle - r^2\mathbf{w}_m\right],\end{aligned}$$

and

$$\begin{aligned}A^{n+1}_{Il,m} &= \varepsilon_{kpl}\mathbf{w}_k\frac{\partial}{\partial \mathbf{r}_m}\left[\frac{\mathbf{r}_p\mathbf{r}^n_I}{r^{n+2}}\right]\\ &= \varepsilon_{kpl}\mathbf{w}_k\frac{(\delta_{pm}\mathbf{r}^n_I + \mathbf{r}_p\mathbf{r}^n_{I,m})r^{n+2} - (n+2)r^n\mathbf{r}_m\mathbf{r}_p\mathbf{r}^n_I}{r^{2n+4}}\\ &= \varepsilon_{kpl}\mathbf{w}_k\left(\frac{\delta_{pm}\mathbf{r}^n_I + \mathbf{r}_p\mathbf{r}^n_{I,m}}{r^{n+2}} - (n+2)\frac{\mathbf{r}_m\mathbf{r}_p\mathbf{r}^n_I}{r^{n+4}}\right).\end{aligned}$$

As usual, we multiply $B^{n+1}_{Il,m}$ by ε_{qml} to complete the 2-form corresponding to $d\omega$. For clarity, we perform this step for each term on the right hand side of equation (4). Beginning with the first term, we have

$$\begin{aligned}&\varepsilon_{qml}\,\frac{\partial}{\partial \mathbf{r}_m}\left(\frac{r}{\langle \mathbf{w},\mathbf{r}\rangle}\right)^{q-1} A^{n+1}_{Il}\\ =\;& \frac{q-1}{\langle \mathbf{w},\mathbf{r}\rangle^q\, r^{n-q+5}}\,\varepsilon_{qml}\varepsilon_{kpl}\left[\mathbf{r}_m\langle \mathbf{w},\mathbf{r}\rangle - r^2\mathbf{w}_m\right]\mathbf{w}_k\mathbf{r}_p\mathbf{r}^n_I\\ =\;& \frac{q-1}{\langle \mathbf{w},\mathbf{r}\rangle^q\, r^{n-q+5}}\left[\delta_{qk}\delta_{pm} - \delta_{pq}\delta_{km}\right]\left[\mathbf{r}_m\langle \mathbf{w},\mathbf{r}\rangle - r^2\mathbf{w}_m\right]\mathbf{w}_k\mathbf{r}_p\mathbf{r}^n_I\\ =\;& \frac{(q-1)\mathbf{r}^n_I}{\langle \mathbf{w},\mathbf{r}\rangle^q\, r^{n-q+2}}\left[r^2 - \langle \mathbf{w},r\rangle^2\right]\left(\frac{\mathbf{r}_q}{r^3}\right)\\ =\;& (q-1)\left[\frac{\mathbf{r}^n_I}{\langle \mathbf{w},\mathbf{r}\rangle^q\, r^{n-q}} - \frac{\mathbf{r}^n_I}{\langle \mathbf{w},\mathbf{r}\rangle^{q-2}\, r^{n-q+2}}\right]\left(\frac{\mathbf{r}_q}{r^3}\right).\end{aligned} \tag{5}$$

For the second term on the right hand side of equation (4), we have

$$
\begin{aligned}
\varepsilon_{qml}\left(\frac{r}{\langle \mathbf{w},\mathbf{r}\rangle}\right)^{q-1} A^{n+1}_{Il,m} &= \frac{\varepsilon_{qml}\varepsilon_{kpl}\,\mathbf{w}_k}{\langle \mathbf{w},\mathbf{r}\rangle^{q-1}\, r^{n-q+3}}\left[\delta_{pm}\mathbf{r}^n_I + \mathbf{r}_p\mathbf{r}^n_{I,m} - (n+2)\frac{\mathbf{r}_m\mathbf{r}_p\mathbf{r}^n_I}{r^2}\right] \\
&= \frac{\mathbf{w}_q\delta_{pm} - \mathbf{w}_m\delta_{pq}}{\langle \mathbf{w},\mathbf{r}\rangle^{q-1}\, r^{n-q+3}}\left[\delta_{pm}\mathbf{r}^n_I + \mathbf{r}_p\mathbf{r}^n_{I,m} - (n+2)\frac{\mathbf{r}_m\mathbf{r}_p\mathbf{r}^n_I}{r^2}\right] \\
&= \frac{1}{\langle \mathbf{w},\mathbf{r}\rangle^{q-1}\, r^{n-q+3}}\left[\mathbf{w}_q\left(-n\,\mathbf{r}^n_I + \mathbf{r}_m\mathbf{r}^n_{I,m}\right) + \mathbf{r}_q\left((n+2)\frac{\langle \mathbf{w},\mathbf{r}\rangle\,\mathbf{r}^n_I}{r^2} - \mathbf{w}_m\mathbf{r}^n_{I,m}\right)\right] \\
&= \left[\frac{(n+2)\,\mathbf{r}^n_I}{\langle \mathbf{w},\mathbf{r}\rangle^{q-2}\, r^{n-q+2}} - \frac{\mathbf{w}_m\mathbf{r}^n_{I,m}}{\langle \mathbf{w},\mathbf{r}\rangle^{q-1}\, r^{n-q}}\right]\left(\frac{\mathbf{r}_q}{r^3}\right). \qquad (6)
\end{aligned}
$$

Here, we have used the fact that

$$
\mathbf{r}_m\mathbf{r}^n_{I,m} \;=\; \mathbf{r}_m\sum_{k=1}^{n}\delta_{mI_k}\mathbf{r}^{n-1}_{I/k} \;=\; \sum_{k=1}^{n}\mathbf{r}_{I_k}\mathbf{r}^{n-1}_{I/k} \;=\; n\,\mathbf{r}^n_I.
$$

Combining equations (5) and (6), we convert the boundary integral (3) into a surface integral in terms of solid angle $d\omega$:

$$
\begin{aligned}
\int_{\partial A} B^{n+1}_{Il}\, d\mathbf{r}_l &= \int_A B^{n+1}_{Il,m}\, d\mathbf{r}_m \wedge\, d\mathbf{r}_l \\
&= (q-1)\left[\mathbf{T}^{n,q-2}_I - \mathbf{T}^{n,q}_I\right] + \sum_{k=1}^{n}\mathbf{w}_{I_k}\mathbf{T}^{n-1,q-1}_{I/k} - (n+2)\mathbf{T}^{n,q-2}_I \\
&= -(q-1)\mathbf{T}^{n,q}_I - (n-q+3)\mathbf{T}^{n,q-2}_I + \sum_{k=1}^{n}\mathbf{w}_{I_k}\mathbf{T}^{n-1,q-1}_{I/k}. \qquad (7)
\end{aligned}
$$

When $q \geq 2$, we can solve for $\mathbf{T}^{n,q}_I$ as

$$
\mathbf{T}^{n,q}_{\mathrm{I}} \;=\; \frac{1}{q-1}\left[\sum_{k=1}^{n}\mathbf{w}_{I_k}\mathbf{T}^{n-1,q-1}_{I/k} - (n-q+3)\mathbf{T}^{n,q-2}_I - \int_{\partial A}\frac{\mathbf{u}^n_I\,\langle \mathbf{w},\mathbf{n}\rangle}{\langle \mathbf{w},\mathbf{u}\rangle^{q-1}}\, ds\right].
$$

which is equation (2). □□□

Equation (2) can be used repeatedly to reduce the order q of the denominator of the rational irradiance tensor to lower-order rational tensors, with $\mathbf{T}^{n,0}$ (i.e, irradiance tensor $\mathbf{T}^n$) and $\mathbf{T}^{n,1}$ as the base cases. Arvo [2, 3] has shown that the irradiance tensor of any order can be represented as a boundary integral by the following recurrence formula

$$
\mathbf{T}^n_{Ij}(A) \;=\; \frac{1}{n+1}\left(\sum_{k=1}^{n-1}\delta_{jI_k}\mathbf{T}^{n-2}_{I/k}(A) - \int_{\partial A}\mathbf{u}^{n-1}_I\mathbf{n}_j\, ds\right). \qquad (8)
$$

Thus, we are left with tensors of the form $\mathbf{T}^{n,1}$, which satisfy a special recurrence relation. It is derived by applying another closely related form of Theorem 1, which we now state.

Theorem 2 *Let $n \geq 1$ and $q \geq 0$ be integers with $q \neq n+1$, then the tensor $\mathbf{T}^{n,q}(A, \mathbf{w})$ satisfies the recurrence relation*

$$\mathbf{T}^{n,q}_{Ij} \;=\; \frac{1}{n-q+1}\left(\sum_{k=1}^{n-1} \delta_{jI_k} \mathbf{T}^{n-2,q}_{I/k} - q\, \mathbf{w}_j \mathbf{T}^{n-1,q+1}_{I} - \int_{\partial A} \frac{\mathbf{u}^{n-1}_I \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u}\rangle^q}\, ds\right), \tag{9}$$

where I is a $(n-1)$-index, ds denotes integration with respect to arclength, and $\mathbf{n}$ is the outward normal to the boundary curve ∂A.

This can be proven directly in much the same way as Theorem 1 [2, pp.193], or by transforming equation (2), as shown in Appendix A. Unfortunately, this recursive formula does not account for all rational irradiance tensors; for example, $\mathbf{T}^{0,q}$ cannot be computed due to the singularity at $n = q - 1$.

Note that two recurrence formulas for rational irradiance tensor $\mathbf{T}^{n,q}$ have different singularities with respect to n and q. Consequently, we can apply the second recurrence (9) to compute $\mathbf{T}^{n,1}$ with $n > 0$:

$$\mathbf{T}^{n,1}_{Ij} \;=\; \frac{1}{n}\left(\sum_{k=1}^{n-1} \delta_{jI_k} \mathbf{T}^{n-2,1}_{I/k} - \mathbf{w}_j \mathbf{T}^{n-1,2}_{I} - \int_{\partial A} \frac{\mathbf{u}^{n-1}_I \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds\right). \tag{10}$$

Then, from the first recurrence formula (2), we have

$$\mathbf{T}^{n-1,2}_{I} \;=\; \sum_{k=1}^{n-1} \mathbf{w}_{I_k} \mathbf{T}^{n-2,1}_{I/k} - n\mathbf{T}^{n-1,0}_{I} - \int_{\partial A} \frac{\mathbf{u}^{n-1}_I \langle \mathbf{w}, \mathbf{n}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds. \tag{11}$$

Substituting equation (11) for $\mathbf{T}^{n-1,2}$ in equation (10), we get a recurrence relation for $\mathbf{T}^{n,1}$. We phrase this as another theorem:

Theorem 3 *The rational irradiance tensor $\mathbf{T}^{n,1}$ satisfies the recurrence relation*

$$\mathbf{T}^{n,1}_{Ij} \;=\; \mathbf{w}_j \mathbf{T}^{n-1}_{I} + \frac{1}{n}\left[\sum_{k=1}^{n-1} \left(\delta_{jI_k} - \mathbf{w}_j \mathbf{w}_{I_k}\right) \mathbf{T}^{n-2,1}_{I/k} + \int_{\partial A} \frac{\mathbf{u}^{n-1}_I \left(\mathbf{w}_j \langle \mathbf{w}, \mathbf{n}\rangle - \mathbf{n}_j\right)}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds\right], \tag{12}$$

or equivalently,

$$\mathbf{T}^{n,1}_{Ij} = \mathbf{w}_j \mathbf{T}^{n-1}_{I} + \frac{1}{n}\left(\delta_{jm} - \mathbf{w}_j\mathbf{w}_m\right)\left[\sum_{k=1}^{n-1} \delta_{mI_k} \mathbf{T}^{n-2,1}_{I/k} - \int_{\partial A} \frac{\mathbf{u}_I^{n-1}\mathbf{n}_m}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds\right], \tag{13}$$

where I is an $(n-1)$-index.

Proof: Equation (12) can be proved directly by converting the boundary integral into a surface integral using Stokes' theorem. For clarity, we split the boundary integral on the right hand side of equation (12) into two terms and perform this step for each one separately.

Step 1: (the first boundary integral)

$$\int_{\partial A} \mathbf{w}_j \frac{\mathbf{u}_I^{n-1}\langle \mathbf{w}, \mathbf{n}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds = \mathbf{w}_j \int_{\partial A} \frac{\mathbf{u}_I^{n-1}\langle \mathbf{w}, \mathbf{n}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds.$$

Notice the boundary integral above is just a special case of equation (3) with $n = n-1$ and $q = 2$, from equation (7), we have

$$\mathbf{w}_j \int_{\partial A} \frac{\mathbf{u}_I^{n-1}\langle \mathbf{w}, \mathbf{n}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds = -\mathbf{w}_j \mathbf{T}^{n-1,2}_{I} - n\mathbf{w}_j \mathbf{T}^{n-1,0}_{I} + \sum_{k=1}^{n-1} \mathbf{w}_j \mathbf{w}_{I_k} \mathbf{T}^{n-2,1}_{I/k}. \tag{14}$$

Step 2: (the second boundary integral)

Letting $q = 1$ in the second recurrence formula (9), we can solve the second boundary integral

$$\int_{\partial A} \frac{\mathbf{u}_I^{n-1}\mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds = \sum_{k=1}^{n-1} \delta_{jI_k} \mathbf{T}^{n-2,1}_{I/k} - \mathbf{w}_j \mathbf{T}^{n-1,2}_{I} - n\mathbf{T}^{n,1}_{Ij}. \tag{15}$$

Using equation (14) and (15), we get

$$\int_{\partial A} \frac{\mathbf{u}_I^{n-1}\left(\mathbf{w}_j\langle \mathbf{w}, \mathbf{n}\rangle - \mathbf{n}_j\right)}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds = -\sum_{k=1}^{n-1}\left(\delta_{jI_k} - \mathbf{w}_j\mathbf{w}_{I_k}\right)\mathbf{T}^{n-2,1}_{I/k} - n\mathbf{w}_j\mathbf{T}^{n-1,0}_{I} + n\mathbf{T}^{n,1}_{Ij}.$$

Solving for $\mathbf{T}^{n,1}_{Ij}$ when $n > 0$, we prove the formula (12). Then, equation (13) can be easily derived by transforming equation (12) using identities

$$\begin{aligned} \delta_{jk} - \mathbf{w}_j \mathbf{w}_k &= (\delta_{ji} - \mathbf{w}_j \mathbf{w}_i)\, \delta_{ik} \\ \mathbf{n}_j - \mathbf{w}_j \langle \mathbf{w}, \mathbf{n} \rangle &= (\delta_{ji} - \mathbf{w}_j \mathbf{w}_i)\, \mathbf{n}_i. \end{aligned}$$

□□□.

Equation (13) holds only when $n > 0$. To compute $\mathbf{T}^{0,1}(A, \mathbf{w})$, we apply the identities [8]

$$\bar{\tau}^k(A, \mathbf{w}) = -\frac{1}{k+1} \int_{\partial A} \frac{1 - \langle \mathbf{w}, \mathbf{u} \rangle^{k+1}}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \langle \mathbf{w}, \mathbf{n} \rangle \; ds,$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - x^{k+1}}{k+1} = -\ln x.$$

to the Taylor expansion of $\mathbf{T}^{0,1}(A, \mathbf{w})$ to obtain

$$\begin{aligned} \mathbf{T}^{0,1}(A, \mathbf{w}) &= \int_A \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle} \, d\sigma(\mathbf{u}) \\ &= \int_A \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \langle \mathbf{w}, \mathbf{u} \rangle^k \; d\sigma(\mathbf{u}) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_A \langle \mathbf{w}, \mathbf{u} \rangle^k \; d\sigma(\mathbf{u}) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bar{\tau}^k(A, \mathbf{w}) \\ &= -\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1} \int_{\partial A} \frac{1 - \langle \mathbf{w}, \mathbf{u} \rangle^{k+1}}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \langle \mathbf{w}, \mathbf{n} \rangle \; ds \\ &= -\int_{\partial A} \frac{\langle \mathbf{w}, \mathbf{n} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 - \langle \mathbf{w}, \mathbf{u} \rangle^{k+1}}{k+1} \right) ds \\ &= \int_{\partial A} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \langle \mathbf{w}, \mathbf{n} \rangle \; ds. \end{aligned} \tag{16}$$

Combining Theorems 1, 2, 3 with equations (8) and (16), we summarize our major results about rational irradiance tensors as follows

$$\mathbf{T}_{\mathrm{I}}^{n,q} \;=\; \frac{1}{q-1}\left[\sum_{k=1}^{n}\mathbf{w}_{I_k}\mathbf{T}_{I/k}^{n-1,q-1}-(n-q+3)\,\mathbf{T}_{I}^{n,q-2}-\int_{\partial A}\frac{\mathbf{u}_I^n\left\langle \mathbf{w},\mathbf{n}\right\rangle}{\left\langle \mathbf{w},\mathbf{u}\right\rangle^{q-1}}\,ds\right] \tag{17}$$

for $n \geq 0$ and $q \geq 2$, and

$$\mathbf{T}_{Ij}^{n,1} \;=\; \mathbf{w}_j\mathbf{T}_I^{n-1}+\frac{1}{n}\left(\delta_{jm}-\mathbf{w}_j\mathbf{w}_m\right)\left[\sum_{k=1}^{n-1}\delta_{mI_k}\mathbf{T}_{I/k}^{n-2,1}-\int_{\partial A}\frac{\mathbf{u}_I^{n-1}\mathbf{n}_m}{\left\langle \mathbf{w},\mathbf{u}\right\rangle}\,ds\right] \tag{18}$$

when $n > 0$, and

$$\mathbf{T}_{Ij}^{n,0} \;=\; \frac{1}{n+1}\left[\sum_{k=1}^{n-1}\delta_{jI_k}\mathbf{T}_{I/k}^{n-2}(A)-\int_{\partial A}\mathbf{u}_I^{n-1}\mathbf{n}_j\,ds\right] \tag{19}$$

when $n > 0$, with $\mathbf{T}_{Ij}^{n,0} = \mathbf{T}_{Ij}^{n}$, and

$$\mathbf{T}_{Ij}^{0,1} = \int_{\partial A}\frac{\ln\left\langle \mathbf{w},\mathbf{u}\right\rangle}{1-\left\langle \mathbf{w},\mathbf{u}\right\rangle^2}\left\langle \mathbf{w},\mathbf{n}\right\rangle\;ds \tag{20}$$

[**Need a better way to organize these formulas...**]

Note that I are n-index and $(n-1)$-index in equation (17) and equation (18), respectively. The base cases for equation (19) are $\mathbf{T}^0(A) = \sigma(A)$ and $\mathbf{T}^{-1}(A) = 0$.

3 Rational Angular Moments

In accordance with the rational irradiance tensor defined in the previous section, we introduce a concept of *rational angular moment* in this section, which can be viewed as an extension of *angular moments* [2] to an arbitrary number of axes and rational functions. The rational angular moments allow us to easily compute the irradiance from polynomially-varying luminaires, which will be discussed in the following sections.

To account for an arbitrary number of axes in *angular moments* [2]

$$\bar{\tau}^n(A,\mathbf{w}) \;\equiv\; \int_A \left\langle \mathbf{w},\mathbf{u}\right\rangle^n\;d\sigma(\mathbf{u}) \tag{21}$$

$$\bar{\bar{\tau}}^{n,m}(A,\mathbf{w},\mathbf{v}) \;\equiv\; \int_A \left\langle \mathbf{w},\mathbf{u}\right\rangle^n\left\langle \mathbf{v},\mathbf{u}\right\rangle^m\;d\sigma(\mathbf{u}), \tag{22}$$

we define a *generalized angular moment* by

$$\tau^n(A, \mathbf{v}_1, \cdots, \mathbf{v}_n) \;\equiv\; \int_A \langle \mathbf{v}_1, \mathbf{u}\rangle \cdots \langle \mathbf{v}_n, \mathbf{u}\rangle \;\, d\sigma(\mathbf{u}). \tag{23}$$

Thus, the *axial moments* (21) and *double-axis moments* (22) are just special cases of τ^n with equal axes. Furthermore, we extend the definition (23) to accomodate rational function and introduce the concept of *rational angular moments.* Given an arbitrary subset $A \subset S^2$ and n vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ and a unit vector $\mathbf{w}$, the *rational angular moment* of A about $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and $\mathbf{w}$ is defined as

$$\tau^{n,q}(A, \mathbf{w}, \mathbf{v}_1, \cdots, \mathbf{v}_n) \;\equiv\; \int_A \frac{\langle \mathbf{v}_1, \mathbf{u}\rangle \cdots \langle \mathbf{v}_n, \mathbf{u}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle^q} \, d\sigma(\mathbf{u}), \tag{24}$$

where integers $n \geq 0$ and $q \geq 0$ denote moment orders for numerator and denominator, respectively. In particular, when $q = 0$, this reduces to our generalized angular moments (23). In Section 5, we show that rational angular moments correspond exactly to the irradiance from a large class of non-uniform luminaires.

To obtain a recurrence formula for $\tau^{n,q}$, we begin by expressing the integrand of equation (24) as a composition of tensors:

$$\frac{\langle \mathbf{v}_1, \mathbf{u}\rangle \cdots \langle \mathbf{v}_n, \mathbf{u}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle^q} \;=\; \frac{\mathbf{u}^n_I}{\langle \mathbf{w}, \mathbf{u}\rangle^q} \, (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n)_I \, ,$$

where the summation convention applies to all repeated pairs of indices. It follows from the definition (1) of rational irradiance tensors that

$$\tau^{n,q} \;=\; \mathbf{T}^{n,q}_I \, (\mathbf{v}_1 \otimes \cdots \mathbf{v}_n)_I \, . \tag{25}$$

Consequently, the recursive formulas for rational angular moments $\tau^{n,q}$ for $n \geq 0$ and $q \geq 0$ can be easily derived using recurrence formulas (17), (18) and (19) to expand the rational irrdiance tensor $\mathbf{T}^{n,q}$ in equation (25). Let $J = (1, 2, \cdots, n)$, $J/k = (1, \cdots, k{-}1, k{+}1, \cdots, n)$, and let $\mathbf{v}_J$ denote the axes set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Then the fundamental recurrence relations for rational angular moments are as follows:

$$\begin{aligned} \tau^{n,q}(A, \mathbf{w}, \mathbf{v}_J) \;&=\; \frac{1}{q-1} \Bigg[\sum_{k=1}^{n} \langle \mathbf{w}, \mathbf{v}_k\rangle \, \tau^{n-1,q-1}(A, \mathbf{w}, \mathbf{v}_{J/k}) - (n-q+3)\tau^{n,q-2}(A, \mathbf{w}, \mathbf{v}_J) \\ &\qquad - \int_{\partial A} \frac{\langle \mathbf{v}_1, \mathbf{u}\rangle \cdots \langle \mathbf{v}_n, \mathbf{u}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle^{q-1}} \, \langle \mathbf{w}, \mathbf{n}\rangle \;\, ds \Bigg] \end{aligned} \tag{26}$$

when $n \geq 0$ and $q \geq 2$, and

$$\tau^{n,1}(A, \mathbf{w}, \mathbf{v}_J) = \langle \mathbf{w}, \mathbf{v}_n \rangle \tau^{n-1}(A, \mathbf{v}_{J/n}) + \frac{\mathbf{v}_n{}^{\mathrm{T}} (\mathbf{I} - \mathbf{w}\mathbf{w}^{\mathrm{T}})}{n} \left[\sum_{k=1}^{n-1} \mathbf{v}_k \tau^{n-2,1}(A, \mathbf{w}, \mathbf{v}_{J/\{k,n\}}) - \int_{\partial A} \frac{\langle \mathbf{v}_1, \mathbf{u} \rangle \cdots \langle \mathbf{v}_{n-1}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{n} \, ds \right] \tag{27}$$

when $n \geq 1$. Here the base cases are $\tau^{0,1}$ and generalized nth order angular moments τ^n. From equation (20), the former moment is given by

$$\tau^{0,1}(A, \mathbf{w}) = \int_{\partial A} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \langle \mathbf{w}, \mathbf{n} \rangle \; ds, \tag{28}$$

and the angular moment τ^n satisfies another recurrence relation from equation (19),

$$\tau^n(A, \mathbf{v}_J) = \frac{1}{n+1} \left[\sum_{k=1}^{n-1} \langle \mathbf{v}_n, \mathbf{v}_k \rangle \tau^{n-2}(A, \mathbf{v}_{J/\{n,k\}}) - \int_{\partial A} \langle \mathbf{v}_1, \mathbf{u} \rangle \cdots \langle \mathbf{v}_{n-1}, \mathbf{u} \rangle \langle \mathbf{v}_n, \mathbf{n} \rangle \; ds \right], \tag{29}$$

where $\tau^0(A) = \sigma(A)$.

4 Closed-Form for Polygons

We have shown that rational angular moments $\tau^{n,q}$ satisfies a recurrence relation that generates a sequence of boundary integrals. In this section, we shall restrict the integration domain A to spherical polygon P, and derive the corresponding closed-form expression for rational moments.

For a spherical polygon P with k edges, the boundary integrals in equations (26), (27) and (29) can be evaluated along each edge ζ of P, which is a great arc connecting two adjacent vertices. Moreover, by using the fact that the outgoing normal $\mathbf{n}$ remains constant along each edge, we can move $\mathbf{n}$ outside the integral. As a result, the closed-form solution of rational moments over a spherical polygon requires evaluating two types of integrals. That is,

$$I_1(\mathbf{w}, \zeta) = \int_{\zeta} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \, ds, \tag{30}$$

$$I_2(\mathbf{w}, \zeta, m, q) = \int_{\zeta} \frac{\langle \mathbf{v}_1, \mathbf{u} \rangle \cdots \langle \mathbf{v}_m, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^q} \, ds \tag{31}$$

for integers $m \geq 0$ and $q \geq 0$. Here, ζ is a great arc corresponding to each spherical polygon edge.

To evaluate these two boundary integrals, we first parametrize the great arc ζ by arclength θ, that is

$$\mathbf{u}(\theta) = \mathbf{s}\cos\theta + \mathbf{t}\sin\theta,$$

where $\mathbf{s}$ and $\mathbf{t}$ are orthonormal vectors in the plane containing the edge and the origin, with $\mathbf{s}$ directed toward the first vertex of the edge [3]. The corresponding parameters are defined as

$$\begin{aligned} a_i &= \langle \mathbf{v}_i, \mathbf{s} \rangle, & b_i &= \langle \mathbf{v}_i, \mathbf{t} \rangle, & c_i &= \sqrt{a_i^2 + b_i^2} \\ a &= \langle \mathbf{w}, \mathbf{s} \rangle, & b &= \langle \mathbf{w}, \mathbf{t} \rangle, & c &= \sqrt{a^2 + b^2}, \end{aligned}$$

and

$$(\cos\phi_i, \sin\phi_i) = \left(\frac{a_i}{c_i}, \frac{b_i}{c_i}\right), \quad (\cos\phi, \sin\phi) = \left(\frac{a}{c}, \frac{b}{c}\right), \tag{32}$$

where $i = 1, 2, \ldots, n$.

4.1 Evaluating the Special integral I_1

We can rewrite the integral (30) as

$$\begin{aligned} \bar{I}_1 &= \int_{-\phi}^{\Theta-\phi} \frac{\ln(c\ \cos\theta)}{1 - c^2\cos^2\theta}\, d\theta \\ &= \Lambda(c, \Theta - \phi) - \Lambda(c, -\phi), \end{aligned} \tag{33}$$

where the two-parameter function $\Lambda(\alpha, \beta)$ is defined by

$$\boxed{\Lambda(\alpha, \beta) \equiv \int_0^\beta \frac{\ln(\alpha\cos\theta)}{1 - (\alpha\cos\theta)^2}\, d\theta} \tag{34}$$

for $0 < \alpha \leq 1, -\pi/2 \leq \beta \leq \pi/2$. This is exactly the special function we encountered in computing the irradiance from linearly-varying luminaires [8]. Figure 1 shows this special

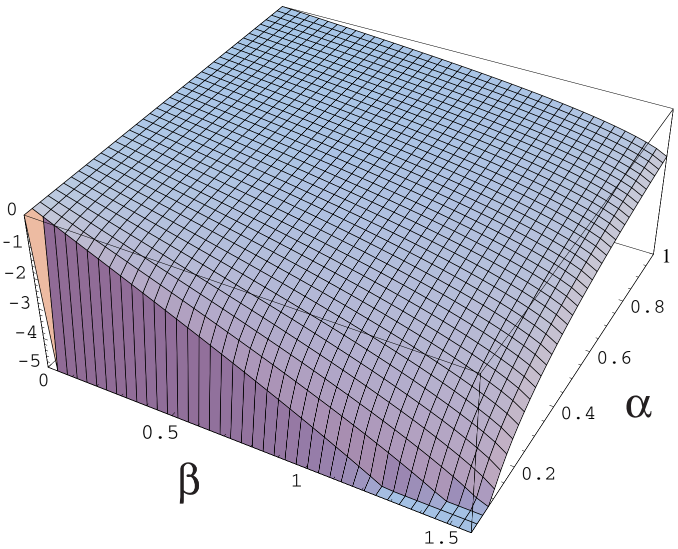

Figure 1: *The special function $\Lambda(\alpha, \beta)$ defined in equation (34), over the range $0 < \alpha \leq 1$, and $0 \leq \beta \leq \pi/2$. When $\alpha \to 0$, $\Lambda(\alpha, \beta)$ tends to $-\infty$.*

function over the domain $(0, 1] \times [0, \pi/2]$. For some special parameters such as $\alpha = 1$, $\Lambda(\alpha, \beta)$ simplifies immediately to known definite integral with exact solutions. For example,

$$\begin{aligned} \Lambda(1, \beta) &= \int_0^\beta \frac{\ln(\cos\theta)}{\sin^2\theta}\, d\theta \\ &= -\int_0^\beta \ln(\cos\theta)\, d(\cot\theta) \\ &= -\beta - \cot\beta \ln(\cos\beta). \end{aligned} \tag{35}$$

However, to our knowledge, $\Lambda(\alpha, \beta)$ has no finite representation in terms of elementary exponential and logarithmic functions in general. In Appendix B, we show that $\Lambda(\alpha, \beta)(\alpha \neq 1)$ can be expressed in terms of a well-known one-parameter special function, called the *Clausen integral* [1, 4, 16]. Specifically,

$$\Lambda(\alpha, \beta) = \frac{1}{4\sqrt{1-\alpha^2}} \left[2\mathrm{Cl}_2(2\mu) - \mathrm{Cl}_2(4\mu - 2\eta) - \mathrm{Cl}_2(2\eta) + 2(\eta - \mu)\ln\gamma \right], \tag{36}$$

where $0 < \alpha < 1, 0 \le \beta \le \pi/2$ and

$$\begin{aligned}
\mu(\alpha,\beta) &= \tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right), \\
\gamma(\alpha) &= \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^2, \\
\eta(\alpha,\beta) &= \tan^{-1}\left(\frac{\sin(2\mu)}{\gamma+\cos(2\mu)}\right).
\end{aligned}$$

The Clausen integral $\mathrm{Cl}_2(x)$ is defined as

$$\mathrm{Cl}_2(x) \equiv -\int_0^x \ln\left|2\sin\frac{\theta}{2}\right| \, d\theta = -\frac{1}{2}\int_0^x \ln\left(4\sin^2\frac{\theta}{2}\right) \, d\theta = \sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$$

for all $x \in \mathbb{R}$ [15, pp. 93]. This reduction is done by relating $\Lambda(\alpha,\beta)$ to the special function $\Upsilon(\mu,\nu)$ introduced by Arvo [2, pp. 112], and the functional relationship is

$$\Lambda(\alpha,\beta) = -\frac{1}{2\sqrt{1-\alpha^2}}\,\Upsilon\left(\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right), \frac{\sqrt{1-\alpha^2}}{\alpha}\right), \tag{37}$$

where $\alpha \in (0,1), \beta \in [0,\pi/2]$ and $\Upsilon(\mu,\nu)$ is defined by

$$\Upsilon(\mu,\nu) \equiv \int_0^\mu \ln\left(1+\nu^2\sec^2\theta\right) \, d\theta.$$

Since $\Upsilon(\mu,\nu)$ can be expressed in terms of three Clausen integrals, as shown in Appendix C, we finally arrive at the formula (36). The relation (37) between Λ and Υ is first derived in Appendix B and then proved directly in Appendix D by a change of variable

$$\tan\theta = \sqrt{1-\alpha^2}\,\tan t.$$

The derivation is performed by cleverly relating $\Lambda(\alpha,\beta)$ to the irradiance from a special class of quadratically-varying luminaires, which is known to have an expression in terms of $\Upsilon(\mu,\nu)$ [2, pp.114].

Although equation (36) only allows the parameter β to vary over the range $[0,\pi/2]$, we can easily obtain the value of Λ when $\beta \in [-\pi/2, 0]$ by using the fact that

$$\Lambda(\alpha,-\beta) = -\Lambda(\alpha,\beta). \tag{38}$$

4.2 Evaluating $\Lambda(\alpha, \beta)$

The integral in equation (30) requires the evaluation of the special function $\Lambda(\alpha, \beta)$, which can be computed directly for each (α, β) by using the reduction formula (36) and equations (35) and (38). The reduction of $\Lambda(\alpha, \beta)$ to the Clausen's integral is advantageous in that $\mathrm{Cl}_2(\theta)$ can be easily evaluated to reasonable accuracy (with relative error less than 0.003%) by applying an approximation formula proposed by Grosjean [12], which consists of elementary functions only:

$$\begin{aligned}\mathrm{Cl}_2(\theta) \;\approx\;& -\theta \ln \sin \frac{1}{2}\theta + \frac{1}{2880}\theta(\pi^2 - \theta^2)(120 - 7\pi^2 + 3\theta^2) + \left(2\ln 2 - \frac{5}{4}\right)\sin\theta \\ & - \left(\frac{89}{128} - \ln 2\right)\sin 2\theta + \left(\frac{2}{3}\ln 2 - \frac{449}{972}\right)\sin 3\theta - \left(\frac{4259}{12288} - \frac{1}{2}\ln 2\right)\sin 4\theta \\ & + \left(\frac{2}{5}\ln 2 - \frac{10397}{37500}\right)\sin 5\theta, \quad 0 \le \theta \le \pi. \end{aligned} \tag{39}$$

Another brute-force but effective method to evaluate Λ is to tabulate the function on a fine regular grid and retrieve values by direct indexing and bilinear interpolation. Unfortunately, it can be seen from the definition (34) that

$$\lim_{\alpha \to 0} \Lambda(\alpha, \beta) \;=\; -\infty,$$

(See Figure 1), which makes it impossible to evaluate Λ over its entire domain using bilinear interpolation. However, notice that $\Lambda(\alpha, \beta)$ varies smoothly in most places except the neighborhood of $\alpha = 0$, we can separate this special region from the domain of bilinear interpolation. For example, by choosing a small $\alpha_\epsilon > 0$, we can tabulate $\Lambda(\alpha, \beta)$ over $[\alpha_\epsilon, 1] \times [0, \pi/2]$ and evaluate $\Lambda(\alpha, \beta)$ using bilinear interpolation when $\alpha \ge \alpha_\epsilon$, while the case of $0 < \alpha < \alpha_\epsilon$ can be directly computed from equation (36). In our experimentation, α_ϵ is chosen as 0.01. After creating a table on a regular 300×300 grid over the domain $[0.01, 1] \times [0, \pi/2]$, we estimate the error by evaluating Λ over 3000×3000 regular grid points. It is shown that such a hybrid approach achieves a maximum relative error 0.2%, which is accurate enough for graphics applications. For simplicity, we only need to tabulate the value of Λ with the parameter β over the range $[0, \pi/2]$, since $\beta \in [-\pi/2, 0]$ can be easily converted to the required domain by a change of variables, that is, $\Lambda(\alpha, -\beta) = -\Lambda(\alpha, \beta)$.

Another alternative approach to avoid the infinity of Λ is to rewrite Λ as

$$\begin{aligned}\Lambda(\alpha, \beta) \;&=\; (\ln \alpha) \int_0^\beta \frac{1}{1 - (\alpha \cos\theta)^2}\, d\theta + \int_0^\beta \frac{\ln(\cos\theta)}{1 - (\alpha\cos\theta)^2}\, d\theta \\ &=\; \lambda(\alpha, \beta) + \Lambda'(\alpha, \beta), \end{aligned} \tag{40}$$

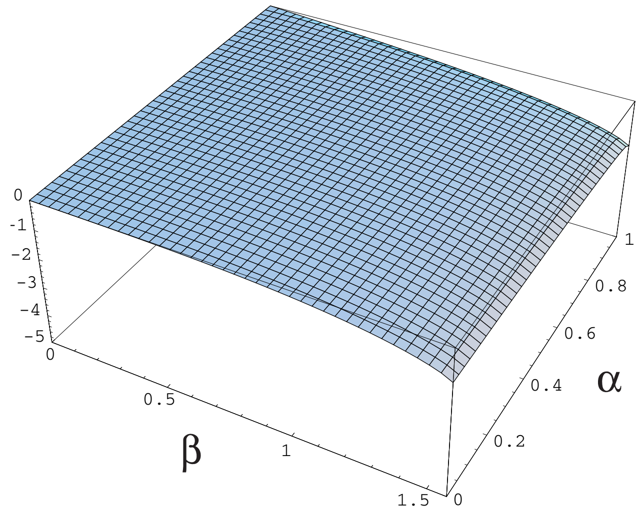

Figure 2: *The special function $\Lambda'(\alpha, \beta)$ defined in equation (42), over the range $0 \le \alpha \le 1$, and $0 \le \beta \le \pi/2$.*

where λ can be evaluated exactly as

$$\lambda(\alpha, \beta) \;=\; \left[\frac{\tan^{-1}(\tan\beta/\sqrt{1-\alpha^2})}{\sqrt{1-\alpha^2}}\right](\ln\alpha) \tag{41}$$

for $0 < \alpha < 1$, and $\lambda(1, \beta) = 0$. The new two-parameter function Λ' is defined as

$$\Lambda'(\alpha, \beta) \;\equiv\; \int_0^\beta \frac{\ln(\cos\theta)}{1-(\alpha\cos\theta)^2}\, d\theta, \tag{42}$$

where $0 \le \alpha \le 1, -\pi/2 \le \beta \le \pi/2$. Thus, by tabulating $\Lambda'(\alpha, \beta)$ instead, we can evaluate $\Lambda(\alpha, \beta)$ in its required domain, since Λ' remains bounded over the required domain, as shown in Figure 1. Moreover, since Λ' is smooth and bounded, it can be easily approximated using numerical quadrature. Alternatively, by incorporating equations (36), (40) and (41), we can easily express $\Lambda'(\alpha, \beta)$ in terms of Clausen integrals. Using either approach, the table for $\Lambda'(\alpha, \beta)$ can be created and stored. Similarly, we only need to tabulate the value of Λ' with the parameter β over the range $[0, \pi/2]$ for the reason of symmetry.

To aid in interpolation, we provide convenient expressions for the four boundary curves of

$\Lambda'(\alpha, \beta)$. Except the boundary $\Lambda'(0, \beta)$, given by

$$\Lambda'(0, \beta) \;=\; \int_0^\beta \ln(\cos\theta)\, d\theta, \tag{43}$$

all the other three boundaries can be expressed in closed-form. That is

$$\begin{aligned}
\Lambda'(1, \beta) &= \Lambda(1, \beta) \;=\; -\beta - \cot\beta \ln(\cos\beta), && (44)\\
\Lambda'(\alpha, 0) &= 0, &&\\
\Lambda'(\alpha, \frac{\pi}{2}) &= -\frac{\pi}{2\sqrt{1-\alpha^2}} \ln\left(1 + \sqrt{1-\alpha^2}\right). && (45)
\end{aligned}$$

To derive the expression for the last curve $\Lambda'(\alpha, \pi/2)$, we apply the formula [2, pp. 113]

$$\Upsilon(\frac{\pi}{2}, \beta) \;=\; \pi \ln\left(\beta + \sqrt{1+\beta^2}\right), \tag{46}$$

which follows from the identity

$$\int_0^{2\pi} \ln\left(a^2\cos^2\theta + b^2\sin^2\theta\right)\, d\theta \;=\; \pi \ln\left(\frac{a+b}{2}\right),$$

given by Carlson [7]. Since

$$\tan^{-1}\left(\frac{\tan(\pi/2)}{\sqrt{1-\alpha^2}}\right) \;=\; \frac{\pi}{2},$$

it follows from equation (37) that

$$\begin{aligned}
\Lambda(\alpha, \frac{\pi}{2}) &= -\frac{1}{2\sqrt{1-\alpha^2}}\, \Upsilon(\frac{\pi}{2}, \frac{\sqrt{1-\alpha^2}}{\alpha})\\
&= -\frac{\pi}{2\sqrt{1-\alpha^2}} \ln\left(\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right)\\
&= \frac{\pi}{2\sqrt{1-\alpha^2}} \ln\alpha - \frac{\pi}{2\sqrt{1-\alpha^2}} \ln\left(1+\sqrt{1-\alpha^2}\right). && (47)
\end{aligned}$$

Also, from equation (40), we have

$$\begin{aligned}
\Lambda(\alpha, \frac{\pi}{2}) &= \lambda(\alpha, \frac{\pi}{2}) + \Lambda'(\alpha, \frac{\pi}{2})\\
&= \frac{\pi}{2\sqrt{1-\alpha^2}} \ln\alpha + \Lambda'(\alpha, \frac{\pi}{2}). && (48)
\end{aligned}$$

Combining equations (47) and (48), we get the exact formula for $\Lambda'(\alpha, \pi/2)$ in equation (45).

As for the first boundary $\Lambda'(0, \beta)$ defined in (43), it is Lobachevsky's integral [11] and can be expressed in terms of Clausen's integral as [18, page 88]

$$\Lambda'(0, \beta) \;=\; \frac{1}{2}\mathrm{Cl}_2(\pi - 2\beta) - \beta \ln 2.$$

Lobachevsky's integral is a well-known special function which cannot be evaluated in finite terms [15]. Since this special function Λ' subsumes Lobachevsky's integral [11] as a special case when $\alpha = 0$, it cannot be evaluated exactly in terms of a finite number of elementary functions. However, since $\alpha = 0$ is only a limit case for $\Lambda(\alpha, \beta)$, we still cannot make a conclusion about whether $\Lambda(\alpha, \beta)$ can be evaluated in finite terms of elementary functions or not.

4.3 Evaluating the Rational Integral I_2

Although we cannot evaluate the special integral exactly, the integral (31) can be computed exactly by means of trigonometric identities. First, we rewrite I_2 in terms of the integral of sine and cosine functions:

$$\begin{aligned}
\bar{I}_2(m, q) &= \frac{c_1 \cdots c_m}{c^q} \int_0^{\Theta} \frac{\cos(\theta - \phi_1) \cdots \cos(\theta - \phi_m)}{\cos^q(\theta - \phi)}\, d\theta \\
&= \frac{c_1 \cdots c_m}{c^q} \int_{-\phi}^{\Theta - \phi} \frac{\cos(\theta + \phi - \phi_1) \cdots \cos(\theta + \phi - \phi_m)}{\cos^q \theta}\, d\theta \\
&= \frac{c_1 \cdots c_m}{c^q} \int_{-\phi}^{\Theta - \phi} \frac{(\alpha_1 \cos\theta + \beta_1 \sin\theta) \cdots (\alpha_m \cos\theta + \beta_m \sin\theta)}{\cos^q \theta}\, d\theta \\
&= \frac{c_1 \cdots c_m}{c^q} \int_{-\phi}^{\Theta - \phi} \frac{\prod_{i=1}^{m} (\alpha_i \cos\theta + \beta_i \sin\theta)}{\cos^q \theta}\, d\theta, \qquad (49)
\end{aligned}$$

where the last step in equation (49) follows from the identity

$$\cos(\theta + \phi_i') \;=\; \cos\theta \cos\phi_i' - \sin\theta \sin\phi_i'$$

and the fact that $\phi_i' = \phi - \phi_i (i = 1, 2, \cdots, m)$ are constant over each edge. Thus,

$$\begin{aligned}
\alpha_i &= \cos\phi_i' \\
\beta_i &= \sin\phi_i'.
\end{aligned}$$

Letting $x = \tan\theta$ and $r_i = \alpha_i/\beta_i = \cot\phi_i'$ $(i = 1, 2, \ldots, m)$, equation (49) can be written as

$$\bar{I}_2(m,q) \;=\; s\int_{-\phi}^{\Theta-\phi} \prod_{i=1}^{m} (x + r_i)\cos^{m-q}\theta \; d\theta, \tag{50}$$

where the scalar coefficient s is given by

$$s \;=\; \frac{c_1 \cdots c_m \; \beta_1 \cdots \beta_m}{c^q}.$$

To compute the indefinite integral

$$I(x) \;=\; \int \prod_{i=1}^{m} (x + r_i)\cos^{m-q}\theta \; d\theta$$

we express $\cos\theta$ in terms of $x = \tan\theta$,

$$\cos\theta \;=\; \frac{1}{\sqrt{1+\tan^2\theta}} \;=\; \frac{1}{\sqrt{1+x^2}},$$

to obtain

$$I(x) \;=\; \int \frac{\prod_{i=1}^{m}(x+r_i)}{\left(\sqrt{1+x^2}\right)^{m-q}} \; d\theta.$$

By the change of variable

$$d\theta \;=\; d(\tan^{-1}x) \;=\; \frac{1}{1+x^2},$$

the integral I reduces to

$$I(x) \;=\; \int P_m(x)\left(1+x^2\right)^{\frac{q-m-2}{2}} \; dx, \tag{51}$$

where $P_m(x)$ is the mth degree monic polynomial with roots $-r_1, -r_2, \cdots, -r_m$. Define

$$G(p,q) \;\equiv\; \int x^p \left(\sqrt{1+x^2}\right)^q \; dx, \tag{52}$$

for integers p, q with $p >= 0$, and denote $P_m(x)$ by

$$P_m(x) \;=\; x^m + \sum_{i=0}^{m-1} p_i x^i. \tag{53}$$

Then equation (51) can be written as

$$I(x) \;=\; \sum_{i=0}^{m} p_i \, G(i, q-m-2). \tag{54}$$

where $p_m = 1$. Consequently, the rational integral (50) simplifies to

$$\bar{I}_2(m,q) \;=\; s \Big[\, I(x) \Big]_{\tan(-\phi)}^{\tan(\Theta-\phi)} . \tag{55}$$

[**Still need some work for the next paragraph, maybe pseudo code...**]
To obtain all the other m polynomial coefficients $p_0, p_1, \cdots, p_{m-1}$, we expand the product

$$\prod_{i=1}^{m} (x + r_i)$$

incrementally. First, we set the current polynomial P_{cur} as the first factor $(x + r_1)$ and the current level $i = 1$; then, at each level i, we multiply the current polynomial P_{cur} (order i) by the next factor $(x + r_i)$. With $2(i+1)$ multiplications and i addition, we get a $(i+1)$th order polynomial as the current. By repeating this process $m-1$ times, finally we obtain $P_m(x)$ and thus its coefficients. Therefore, the total cost for computing all the p_i is

$$\sum_{i=1}^{m-1} \left[2(i+1) + i \right] \;=\; O(m^2).$$

On the other hand, the integral $G(p,q)$ can be evaluated by using the following recursive formula

$$(p+q+1) \; G(p,q) \;=\; x^{p+1} \left(\sqrt{1+x^2} \right)^q + q \; G(p, q-2), \tag{56}$$

which follows from integration by parts. Depending on the sign of q, equation (56) can be applied in either of two ways; that is, reducing the order of q by

$$G(p,q) \;=\; \frac{1}{p+q+1} \left[x^{p+1} \left(\sqrt{1+x^2} \right)^q + q \; G(p, q-2) \right], \tag{57}$$

when $q > 0$, or increasing the order of q by

$$G(p,q) \;=\; \frac{1}{q+2} \left[(p+q+3) G(p, q+2) - x^{p+1} \left(\sqrt{1+x^2} \right)^{q+2} \right], \tag{58}$$

when $q < 0$, which has the singular point $G(p, -2)$. The base cases for the above recurrences are given by

$$\begin{aligned}
G(p,-2) &= \int \frac{x^p}{1+x^2}\, dx \\
&= \int x^{p-2}\left(1 - \frac{1}{1+x^2}\right)\, dx \\
&= \frac{x^{p-1}}{p-1} - G(p-2,-2) \\
G(p,0) &= \frac{x^{p+1}}{p+1} \\
G(p,1) &= \int x^p \sqrt{1+x^2}\, dx \\
&= \int x^{p-2}\sqrt{(1+x^2)^3}\, dx - G(p-2,1) \\
&= \frac{1}{p+2}\left[x^{p-1}\sqrt{(1+x^2)^3} - (p-1)G(p-2,1)\right],
\end{aligned}$$

where the last result follows from integration by parts, and

$$\begin{aligned}
G(1,-2) &= \frac{1}{2}\ln(1+x^2) \\
G(0,-2) &= \tan^{-1}(x) \\
G(1,1) &= \frac{1}{3}\sqrt{(1+x^2)^3} \\
G(0,1) &= \frac{1}{2}\left(x\sqrt{1+x^2} + \sin^{-1}x\right).
\end{aligned}$$

As we can see, the base cases can be computed within $O(p)$ time. Combining with the recurrence formulas (57) and (58), we conclude that evaluating $G(p,q)$ requires at most $O(p + |q|)$ time. It follows from equation (54) and equation (55) that the total cost for evaluating the rational integral $\bar{I}_2(m,q)$ is $O(m^2)$, assuming $q \leq m$, which also accouts for the cost for computing the polynomial coefficients.

5 Spatially-Varying Luminaires

The concept of *rational angular moments* is closely related to computing the irradiance from Lambertian luminaires with *spatially varying radiant exitance*, whose radiance distribution varies with respect to the position. Specifically, for the class of luminaires with polynomially-varying radiant exitance (or *exitance power*), the irradiance reduces to a special order of rational moment about some chosen axes. In this section, we shall formulate the irradiance from *non-uniform* luminaires in terms of rational angular moments, or equivalently, rational irradiance tensors, and then apply the recursive formulas derived in Section 2 and Section 3 to obtain the closed-form solutions for polygonal luminaires. We start with luminaires with linearly-varying radiant exitance, and then extend the result to quadratic and higher order polynomials.

5.1 Irradiance

The irradiance ϕ impinging on a surface at the point $\mathbf{o}$ due to a planar luminaire L is given by

$$\phi(L) = \frac{1}{\pi} \int_L f(\mathbf{x}) \frac{\cos\theta_1 \cos\theta_2}{r^2} \, d\mathbf{x}, \tag{59}$$

where $\mathbf{x}$ is a point on L, r is its length, and $f(\mathbf{x})$ denotes the *radiant exitance* of the luminaire at the point $\mathbf{x}$. Here, we only consider the case that $f(\mathbf{x})$ is a polynomial. If we integrate over the spherical projection of L, denoted by $A = \prod(L)$, we have

$$\begin{aligned} \phi(A) &= \frac{1}{\pi} \int_A f(\mathbf{x}) \cos\theta_1 \, d\sigma(\mathbf{u}) \\ &= \frac{1}{\pi} \int_A f(\mathbf{u}) \langle \mathbf{b}, \mathbf{u} \rangle \, d\sigma(\mathbf{u}), \end{aligned} \tag{60}$$

where $\mathbf{u}$ is the unit vector on the sphere, and $\mathbf{x}$ is the point on L in the direction of $\mathbf{u}$. The measure σ denotes area on the sphere, $f(\mathbf{u})$ is the emissive power distribution function represented in terms of the unit vector $\mathbf{u}$. For simplicity, we still use f to denote it.

5.2 Linearly-Varying Luminaires

It follows from equation (60) that the irradiance at the origin from a luminaire with linearly-varying brightness can be formulated as the integral of a rational function [8]. That is,

$$\phi^1(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) = \frac{1}{\pi} \int_A \frac{\langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \, d\sigma(\mathbf{u}), \tag{61}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{w}$ are three vectors, and $\mathbf{w}$ is normalized. Using the definition of rational moments (24) and its tensor form (25), equation (61) can be simply expressed as

$$\begin{aligned}\phi^1(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) &= \frac{1}{\pi}\tau^{2,1}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) \\ &= \frac{1}{\pi}\mathbf{T}^{2,1}_{ij}\mathbf{a}_i\mathbf{b}_j. \end{aligned} \tag{62}$$

Applying the recursive formula (18), we have

$$\mathbf{T}^{2,1}_{ij} = \mathbf{w}_j\mathbf{T}^1_i + \frac{1}{2}\left(\delta_{jm} - \mathbf{w}_j\mathbf{w}_m\right)\left[\delta_{im}\mathbf{T}^{0,1} - \int_{\partial A}\frac{\mathbf{u}_i\mathbf{n}_m}{\langle \mathbf{w}, \mathbf{u}\rangle}\, ds\right]. \tag{63}$$

Using equation (19) and equation (20), equation (63) becomes

$$\begin{aligned}\mathbf{T}^{2,1}_{ij} &= \frac{1}{2}\int_{\partial A}\left[-\mathbf{w}_j\mathbf{n}_i + \left(\delta_{jm} - \mathbf{w}_j\mathbf{w}_m\right)\left(\delta_{im}\langle \mathbf{w}, \mathbf{n}\rangle\frac{\ln\langle \mathbf{w}, \mathbf{u}\rangle}{1 - \langle \mathbf{w}, \mathbf{u}\rangle^2} - \frac{\mathbf{u}_i\mathbf{n}_m}{\langle \mathbf{w}, \mathbf{u}\rangle}\right)\right] ds \\ &= -\frac{1}{2}\int_{\partial A}\left[\delta_{ik}\mathbf{w}_j - \left(\delta_{jm} - \mathbf{w}_j\mathbf{w}_m\right)\left(\delta_{im}\mathbf{w}_k\,\eta - \frac{\delta_{km}\mathbf{u}_i}{\langle \mathbf{w}, \mathbf{u}\rangle}\right)\right]\mathbf{n}_k\, ds \\ &= -\frac{1}{2}\int_{\partial A}\mathbf{M}^3_{ijk}\mathbf{n}_k\, ds, \end{aligned} \tag{64}$$

where the 3-tensor $\mathbf{M}$, which depends on $\mathbf{w}$ and $\mathbf{u}$, is defined by

$$\boxed{\mathbf{M}^3_{ijk}(\mathbf{w}, \mathbf{u}) = \delta_{ik}\mathbf{w}_j + \left(\delta_{jm} - \mathbf{w}_j\mathbf{w}_m\right)\left(\frac{\delta_{km}\mathbf{u}_i}{\langle \mathbf{w}, \mathbf{u}\rangle} - \delta_{im}\mathbf{w}_k\,\eta\right),} \tag{65}$$

and the scalar-valued function $\eta(\mathbf{w}, \mathbf{u})$ is given by

$$\eta(\mathbf{w}, \mathbf{u}) \equiv \frac{\ln\langle \mathbf{w}, \mathbf{u}\rangle}{1 - \langle \mathbf{w}, \mathbf{u}\rangle^2}.$$

Therefore, the irradiance (62) from luminaires with linearly-varying brightness is given by

$$\boxed{\phi^1(A, \mathbf{w}, \mathbf{a}, \mathbf{b}) = -\frac{1}{2\pi}\int_{\partial A}\mathbf{M}^3_{ijk}\mathbf{a}_i\mathbf{b}_j\mathbf{n}_k\, ds.} \tag{66}$$

In particular, when A is a spherical projection of a polygonal luminaire with s edges, Equation (66) can be evaluated along each edge ζ by considering the outgoing normal $\mathbf{n}$ as constant over the edge. That is,

$$
\begin{aligned}
\phi^1(P, \mathbf{w}, \mathbf{a}, \mathbf{b}) &= -\frac{1}{2\pi} \sum_{e=1}^{s} \left(\mathbf{a}_i \mathbf{b}_j \mathbf{n}_k \int_{\zeta} \mathbf{M}^3_{ijk} \, ds \right) \\
&= -\frac{1}{2\pi} \sum_{e=1}^{s} \left(\mathbf{a}_i \mathbf{b}_j \mathbf{n}_k \left[\delta_{ik} \mathbf{w}_j \Theta^e + (\delta_{jm} - \mathbf{w}_j \mathbf{w}_m) \left(\delta_{km} \int_{\zeta_e} \frac{\mathbf{u}_i}{\langle \mathbf{w}, \mathbf{u} \rangle} \, ds - \delta_{im} \mathbf{w}_k \, I_1^e \right) \right] \right),
\end{aligned}
$$

where I_1^e is the special integral (30) evaluated along the edge ζ_e and Θ^e is its arc length, and they both depend on the edges. By parameterizing each edge ζ by arc length, we get a "closed-form" solution for irradiance from polygonal luminaires with linearly-varying luminaires, given by

$$
\boxed{\phi^1(P) \;=\; -\frac{1}{2\pi} \sum_{e=1}^{s} \Big[\langle \mathbf{a}, \mathbf{n} \rangle \langle \mathbf{b}, \mathbf{w} \rangle \Theta^e + \mathbf{b}^{\mathrm{T}} \left(\mathbf{I} - \mathbf{w}\mathbf{w}^{\mathrm{T}} \right) \left(\bar{I}_2^e(1,1)\, \mathbf{n} - \bar{I}_1^e \langle \mathbf{w}, \mathbf{n} \rangle \, \mathbf{a} \right) \Big],} \tag{67}
$$

where the arc length Θ, the normal $\mathbf{n}$ and two boundary integrals $\bar{I}_1, \bar{I}_2$ all depend on each edge ζ, denoted by the superscript ^{e}. $\bar{I}_1$ is evaluated in terms of our special function in (33), and $\bar{I}_2$ computes the rational boundary integral (31) with $n = q = 1$, that is,

$$
\int_{\zeta} \frac{\langle \mathbf{a}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \, ds,
$$

which can be evaluated exactly [8]

$$
\bar{I}_2^e(1,1) \;=\; \frac{c_1}{c} \left[\cos(\phi')\Theta^e + \sin(\phi') \ln \left(\frac{\cos(\Theta - \phi)}{\cos \phi} \right) \right],
$$

where $\phi' = \phi - \phi_1$ and c_1, c, ϕ_1, ϕ are defined in equation (32).

From equations (65), (66) and (67), we have seen that by fitting the problem of computing irradiance from linearly-varying luminaires into our general framework of rational irradiance tensors and moments, we have obtained the exact same solutions as those derived in our previous report [8] in a much simpler and compact way.

5.3 Quadratically-Varying Luminaires

For luminaires with quadratically-varying brightness, the quadratic terms of $f(\mathbf{x})$ can be expressed as

$$
a_{ij} \mathbf{x}_i \mathbf{x}_j,
$$

where a_{ij} is a scalar coefficient, and $\mathbf{x}_i$, $\mathbf{x}_j$ are the coordinates of $\mathbf{x}$, and $i, j = 1, 2, 3$. Without loss of generality, we assume that

$$f(\mathbf{x}) = \mathbf{x}_i \mathbf{x}_j. \tag{68}$$

Let $\mathbf{e}_i (i = 1, 2, 3)$ denote three coordinate axes, Equation (68) can be expressed as

$$f(\mathbf{x}) = \langle \mathbf{e}_i, \mathbf{x} \rangle \langle \mathbf{e}_j, \mathbf{x} \rangle. \tag{69}$$

By expressing $\mathbf{x}$ in equation (69) in terms of unit vector $\mathbf{u}$ by

$$\mathbf{x} = \frac{h}{\langle \mathbf{w}, \mathbf{u} \rangle} \mathbf{u},$$

we get

$$f(\mathbf{u}) = h^2 \frac{\langle \mathbf{e}_i, \mathbf{u} \rangle \langle \mathbf{e}_j, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2}. \tag{70}$$

Consequently, it follows from equation (60) that the irradiance from luminaires with quadratically-varying brightness can be formalized as the integral of the form

$$\phi^2(A, \mathbf{w}, \mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{\pi} \int_A \frac{\langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{b}, \mathbf{u} \rangle \langle \mathbf{c}, \mathbf{u} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^2} \, d\sigma(\mathbf{u}), \tag{71}$$

where $\mathbf{w}$ is a unit vector and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary vectors. Similarly, we can represent it in terms of rational irradiance tensor or rational moment as

$$\begin{aligned} \phi^2(A, \mathbf{w}, \mathbf{a}, \mathbf{b}, \mathbf{c}) &= \frac{1}{\pi} \, \tau^{3,2}(A, \mathbf{w}, \mathbf{a}, \mathbf{b}, \mathbf{c}) \\ &= \frac{1}{\pi} \, \mathbf{T}^{3,2}_{ijk} \, \mathbf{a}_i \mathbf{b}_j \mathbf{c}_k. \end{aligned} \tag{72}$$

In order to derive the corresponding boundary integral for ϕ^2, we begin with the recurrence formula (17) for $\mathbf{T}^{3,2}$

$$\begin{aligned} \mathbf{T}^{3,2}_{ijk} &= \mathbf{w}_i \mathbf{T}^{2,1}_{jk} + \mathbf{w}_j \mathbf{T}^{2,1}_{ik} + \mathbf{w}_k \mathbf{T}^{2,1}_{ij} - 4\mathbf{T}^3_{ijk} - \int_{\partial A} \frac{\mathbf{u}^3_{ijk} \langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle} \, ds \\ &= -\frac{1}{2} \int_{\partial A} \left(\mathbf{w}_i \mathbf{M}^3_{jkl} + \mathbf{w}_j \mathbf{M}^3_{ikl} + \mathbf{w}_k \mathbf{M}^3_{ijl} + 2 \, \mathbf{w}_l \frac{\mathbf{u}_i \mathbf{u}_j \mathbf{u}_k}{\langle \mathbf{w}, \mathbf{u} \rangle} \right) \mathbf{n}_l \, ds \; - \; 4\mathbf{T}^3_{ijk}, \end{aligned} \tag{73}$$

where we have used the boundary integral representation (64) for $\mathbf{T}^{2,1}$ and the 3-tensor $\mathbf{M}^3$ in the linear case. Then, it follows from equation (19) that

$$\begin{aligned} 4\mathbf{T}^3_{ijk} &= \delta_{ki}\mathbf{T}^1_j + \delta_{kj}\mathbf{T}^1_i - \int_{\partial A} \mathbf{u}^2_{ij}\mathbf{n}_k \, ds \\ &= -\frac{1}{2}\int_{\partial A} \left(\delta_{ki}\delta_{jl} + \delta_{kj}\delta_{il} + 2\,\delta_{kl}\,\mathbf{u}_i\mathbf{u}_j\right)\mathbf{n}_l \, ds. \end{aligned} \tag{74}$$

Substituting equation (74) into equation (73), we get

$$\mathbf{T}^{3,2}_{ijk} = -\frac{1}{2}\int_{\partial A} \mathbf{M}^4_{ijkl}(\mathbf{w},\mathbf{u})\,\mathbf{n}_l \, ds, \tag{75}$$

where the 4-tensor $\mathbf{M}$ depending on $\mathbf{w}$ and $\mathbf{u}$ is defined by

$$\boxed{\mathbf{M}^4_{ijkl} \equiv \mathbf{w}_i\mathbf{M}^3_{jkl} + \mathbf{w}_j\mathbf{M}^3_{ikl} + \mathbf{w}_k\mathbf{M}^3_{ijl} + 2\,\mathbf{w}_l\,\frac{\mathbf{u}_i\mathbf{u}_j\mathbf{u}_k}{\langle\mathbf{w},\mathbf{u}\rangle} - \delta_{ki}\delta_{jl} - \delta_{kj}\delta_{il} - 2\,\delta_{kl}\,\mathbf{u}_i\mathbf{u}_j.} \tag{76}$$

Hence, the irradiance in equation (72) is given by a boundary integral

$$\phi^2(A,\mathbf{w},\mathbf{a},\mathbf{b},\mathbf{c}) = -\frac{1}{2\pi}\int_{\partial A} \mathbf{M}^4_{ijkl}\,\mathbf{a}_i\mathbf{b}_j\mathbf{c}_k\mathbf{n}_l \, ds. \tag{77}$$

5.4 Higher Order Polynomials

We can easily represent the irradiance from luminaires whose spatial variation is a polynomial of order n in a general irradiance tensor formula. That is,

$$\boxed{\phi^n(A,\mathbf{w},\mathbf{a}_1,\cdots,\mathbf{a}_n) = \frac{1}{\pi}\,\mathbf{T}^{n+1,n}_{Ij}(\mathbf{a}_1\cdots\mathbf{a}_n)_{Ij},} \tag{78}$$

where I is a n-index, $\mathbf{w}, \mathbf{a}_i's$ are vectors, and $\mathbf{w}$ is a unit vector.

From equation (64) and equation (75), we have

$$\begin{aligned} \mathbf{T}^{2,1}_{ij} &= -\frac{1}{2}\int_{\partial A} \mathbf{M}^3_{ijk}\,\mathbf{n}_k \, ds \\ \mathbf{T}^{3,2}_{ijk} &= -\frac{1}{2}\int_{\partial A} \mathbf{M}^4_{ijkl}\,\mathbf{n}_l \, ds, \end{aligned}$$

which suggests that the rational irradiance tensor $\mathbf{T}^{n+1,n}$ corresponding to higher order irradiance ϕ^n may also be represented as an integral of a $(n+2)$-tensor $\mathbf{M}$, and the higher-order tensor $\mathbf{M}$ can be computed recursively from lower order $\mathbf{M}$, as in equation (76).

In this section, we show that there actually exists such a general recursive formula for $\mathbf{T}^{n+1,n}$ and the corresponding $(n+2)$-tensor $\mathbf{M}^{n+2}$, where $n \geq 0$.

Theorem 4 *Given an integer $n \geq 0$, the rational irradiance tensor $\mathbf{T}^{n+1,n}$ is given by the boundary integral*

$$\mathbf{T}_I^{n+1,n} = -\frac{1}{2}\int_{\partial A} \mathbf{M}_{Il}^{n+2}(\mathbf{w},\mathbf{u})\ \mathbf{n}_l\ ds, \tag{79}$$

where I is a $(n+1)$-index not including the index l, and the $(n+2)$-tensor $\mathbf{M}$ satisfies a recurrence relation

$$\boxed{\begin{aligned}\mathbf{M}_{Ijl}^{n+2} = \frac{1}{n-1}\Bigg[&\sum_{k=1}^{n}\mathbf{w}_{I_k}\mathbf{M}_{(I/k)jl}^{n+1} + (n-1)\ \mathbf{w}_j\mathbf{M}_{Il}^{n+1} + 2\mathbf{w}_l\frac{\mathbf{u}_{Ij}^{n+1}}{\langle\mathbf{w},\mathbf{u}\rangle^{n-1}} \\ &-\sum_{k=1}^{n}\delta_{jI_k}\mathbf{M}_{(I/k)l}^{n} - 2\delta_{jl}\frac{\mathbf{u}_I^n}{\langle\mathbf{w},\mathbf{u}\rangle^{n-2}}\Bigg]\end{aligned}} \tag{80}$$

when $n \geq 2$, and the base cases are defined by

$$\mathbf{M}_{ijk}^3 = \delta_{ik}\mathbf{w}_j - (\delta_{jl} - \mathbf{w}_j\mathbf{w}_l)\left(\delta_{il}\mathbf{w}_k\ \eta - \frac{\delta_{kl}\mathbf{u}_i}{\langle\mathbf{w},\mathbf{u}\rangle}\right) \tag{81}$$

$$\mathbf{M}_{ij}^2 = \delta_{ij}. \tag{82}$$

Proof: The proof is performed by induction on n, with base cases $n = 0$ and $n = 1$.

Step 1: $(n = 0)$

From Lambert's formula, we have

$$\begin{aligned}\mathbf{T}_i^{1,0} &= -\frac{1}{2}\int_{\partial A}\mathbf{n}_i\ ds \\ &= -\frac{1}{2}\int_{\partial A}\delta_{il}\mathbf{n}_l\ ds \\ &= -\frac{1}{2}\int_{\partial A}\mathbf{M}_{il}^2\mathbf{n}_l\ ds.\end{aligned}$$

Thus, equation (79) holds for $n = 0$.

Step 2: $(n = 1)$

This has been proved in Section 5.2.

Step 3: $(n \geq 2)$

First, comparing equation (80) with equation (76) and accounting for the definition of $\mathbf{M}^2$, it is easy to check equation (79) and equation (80) holds for $\mathbf{T}^{3,2}$.

Now suppose that

$$\mathbf{T}_J^{k+1,k} = -\frac{1}{2}\int_{\partial A} \mathbf{M}_{Jl}^{k+2}\, \mathbf{n}_l \, ds \tag{83}$$

for all $2 \leq k < n$ and $(k+1)$-index J. Applying the recursive formula (17), we have

$$\mathbf{T}_{Ij}^{n+1,n} = \frac{1}{n-1}\left[\sum_{k=1}^{n} \mathbf{w}_{I_k}\mathbf{T}_{(I/k)j}^{n,n-1} + \mathbf{w}_j \mathbf{T}_I^{n,n-1} - 4\mathbf{T}_{Ij}^{n+1,n-2} - \int_{\partial A} \frac{\mathbf{u}_{Ij}^{n+1}\langle \mathbf{w}, \mathbf{n}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle^{n-1}}\, ds\right]. \tag{84}$$

By the induction hypothesis (84), we can replace $\mathbf{T}^{n,n-1}$ by boundary integrals. That is,

$$\mathbf{T}_{Ij}^{n+1,n} = \frac{1}{n-1}\left[-\frac{1}{2}\int_{\partial A}\left(\sum_{k=1}^{n} \mathbf{w}_{I_k}\mathbf{M}_{(I/k)jl}^{n+1} + \mathbf{w}_j\mathbf{M}_{Il}^{n+1} + 2\mathbf{w}_l\frac{\mathbf{u}_{Ij}^{n+1}}{\langle \mathbf{w}, \mathbf{u}\rangle^{n-1}}\right)\mathbf{n}_l\, ds - 4\mathbf{T}_{Ij}^{n+1,n-2}\right]. \tag{85}$$

Then we can apply Theorem 2 for the last term and obtain

$$\begin{aligned} 4\mathbf{T}_{Ij}^{n+1,n-2} &= \sum_{k=1}^{n} \delta_{jI_k}\mathbf{T}_{I/k}^{n-1,n-2} - (n-2)\, \mathbf{w}_j\mathbf{T}_I^{n,n-1} - \int_{\partial A}\frac{\mathbf{u}_I^n \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u}\rangle^{n-2}}\, ds \\ &= -\frac{1}{2}\int_{\partial A}\left(\sum_{k=1}^{n}\delta_{jI_k}\mathbf{M}_{(I/k)l}^{n} - (n-2)\, \mathbf{w}_j\mathbf{M}_{Il}^{n+1} + 2\delta_{jl}\frac{\mathbf{u}_I^n}{\langle \mathbf{w}, \mathbf{u}\rangle^{n-2}}\right)\mathbf{n}_l\, ds. \end{aligned} \tag{86}$$

Substituting equation (86) into equation (85), we get

$$\mathbf{T}_{Ij}^{n+1,n} = -\frac{1}{2}\int_{\partial A}\frac{1}{n-1}\left[\sum_{k=1}^{n}\mathbf{w}_{I_k}\mathbf{M}_{(I/k)jl}^{n+1} + (n-1)\, \mathbf{w}_j\mathbf{M}_{Il}^{n+1} + 2\mathbf{w}_l\frac{\mathbf{u}_{Ij}^{n+1}}{\langle \mathbf{w}, \mathbf{u}\rangle^{n-1}}\right.$$

$$- \sum_{k=1}^{n} \delta_{jI_k} \mathbf{M}^n_{(I/k)l} - 2\delta_{jl} \frac{\mathbf{u}_I^n}{\langle \mathbf{w}, \mathbf{u} \rangle^{n-2}} \Bigg] \mathbf{n}_l \; ds$$

$$= \; -\frac{1}{2} \int_{\partial A} \mathbf{M}^{n+2}_{Ijl} \; \mathbf{n}_l \; ds,$$

where the $(n+2)$-tensor is given by equation (80). Therefore, it follows from induction on n that equation (79) and equation (80) holds for any $n \geq 2$. Consequently, Theorem 4 has been proved for all $n \geq 0$. □□□.

As a corollary of Theorem 4, we have that the irradiance from luminaires with monomially-varying brightness of nth order is given by

$$\boxed{\phi^n(A, \mathbf{w}, \mathbf{a}_1, \cdots, \mathbf{a}_n) \; = \; -\frac{1}{2\pi} \int_{\partial A} \mathbf{M}^{n+2}_{Il} \; (\mathbf{a}_1 \cdots \mathbf{a}_n)_I \mathbf{n}_l \; ds,} \tag{87}$$

where I is a $(n+1)$-index. In particular, when A is a spherical polygon with k edges, we have the irradiance

$$\boxed{\phi^n(P, \mathbf{w}, \mathbf{a}_1, \cdots, \mathbf{a}_n) \; = \; -\frac{1}{2\pi} \left[\sum_{i=1}^{k} \left(\int_{\zeta_i} \mathbf{M}^{n+2}_{Il} \; (\mathbf{a}_1 \cdots \mathbf{a}_n)_I \; ds \right) \mathbf{n}_l \right].} \tag{88}$$

5.5 Time Complexity of Evaluation

In this section, we will discuss the complexity of computing the irradiance from a polygonal luminaire with k edges, with brightness varying as an nth order polynomial. Let T denote the total cost for ϕ^n, and $T(n+2)$ denote the cost for evaluating the line integral

$$\int_{\zeta} \mathbf{M}^{n+2}_{Il} (\mathbf{a}_1 \cdots \mathbf{a}_n)_I \; ds$$

in equation (88). It follows that the time complexity T of evaluating the irradiance ϕ^n in equation (88) is

$$T \; = \; k \, T(n+2). \tag{89}$$

From the recurrence formula (78) for $\mathbf{M}$, we can easily derive the recursive relation satisfied by $T(n+2)$:

$$\begin{aligned} T(n+2) \;&=\; T(n+1) + T(n) + \mathrm{Cost}(\bar{I}_2(n+1, n-1)) + \mathrm{Cost}(\bar{I}_2(n, n-2)) + O(n) \\ &=\; T(n+1) + T(n) + O(n^2), \end{aligned} \tag{90}$$

where we used the complexity result for the rational integral $\bar{I}_2(n, q)$, which was discussed in Section 4. By applying equation (90) repeatedly, we get

$$\begin{aligned} T(n+2) &= (n-1)T(3) + T(2) + O(n^2 + (n-1)^2 + \cdots + 1) \\ &= (n-1)T(3) + T(2) + O(n^3). \end{aligned}$$

Assuming it takes $O(1)$ to evaluate the special integral arising from $\mathbf{M}^3$, we get $T(n+2) = O(n^3)$. Consequently, the time complexity for the irradiance from a polygonal luminaires with polynomially-varying brightness is $O(k\,n^3)$, where k is the polygon side and n is the polynomial order.

6 Conclusion

We have presented a number of new closed-form expressions for computing illumination from luminaires with polynomially-varying radiant exitance, the expressions can be evaluated in $O(kn^3)$ time for arbitrary polygons with k sides, where n is the order of the polynomial. To derive the new expressions, we extend the concept of irradiance tensor to account for rational polynomial functions over the unit sphere. Rational irradiance tensors satisfy several recursive formulas and leads to closed-form expressions for rational angular moments, which are directly related to the irradiance from non-uniform luminaires and other quantities encountered in many non-Lambertian phenomena. In particular, in the case of irradiance due to polynomially-varying luminaires, it reduces to a general recursive tensor formula, which subsumes Lambert's formula.

The new analytical results presented here are of significant theoretical values. There are many potential applications for our closed-form expressions, such as computing direct illumination and glossy reflection or transmission from non-uniform luminaires, handling more complex BRDF models and more realistic illumination models, etc. We believe that the new techniques will give rise to many efficient deterministic algorithms for simulating non-diffuse pheonomena.

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A Transformation Proof for Theorem 2

From the definition of rational irradiance tensors, it follows that

$$\boxed{\mathbf{w}_j \mathbf{T}_{Ij}^{n,q} \;=\; \mathbf{T}_{I}^{n-1,q-1},} \tag{91}$$

where I is a $(n-1)$-index and $n \geq 1$, $q \geq 1$. Then, using equation (91), we can re-express the recurrence formula (2) as

$$\begin{aligned}
(q-1)\,\mathbf{T}_I^{n,q} &= \sum_{k=1}^{n} \mathbf{w}_{I_k} \mathbf{T}_{I/k}^{n-1,q-1} - (n-q+3)\,\mathbf{T}_I^{n,q-2} - \int_{\partial A} \frac{\mathbf{u}_I^n \langle \mathbf{w}, \mathbf{n} \rangle}{\langle \mathbf{w}, \mathbf{u} \rangle^{q-1}}\, ds \\
&= \sum_{k=1}^{n} \delta_{jI_k} \mathbf{w}_j \mathbf{T}_{I/k}^{n-1,q-1} - (n-q+3)\,\mathbf{w}_j \mathbf{T}_{Ij}^{n+1,q-1} - \int_{\partial A} \frac{\mathbf{u}_I^n \mathbf{w}_j \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u} \rangle^{q-1}}\, ds \\
&= \mathbf{w}_j \left[\sum_{k=1}^{n} \delta_{jI_k} \mathbf{T}_{I/k}^{n-1,q-1} - (n-q+3)\,\mathbf{T}_{Ij}^{n+1,q-1} - \int_{\partial A} \frac{\mathbf{u}_I^n \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u} \rangle^{q-1}}\, ds \right]
\end{aligned} \tag{92}$$

Note that $\mathbf{w}_j \mathbf{w}_j = 1$, we can multiply both sides of the above equation by $\mathbf{w}_j$, obtaining

$$(q-1)\,\mathbf{w}_j \mathbf{T}_I^{n,q} = \sum_{k=1}^{n} \delta_{jI_k} \mathbf{T}_{I/k}^{n-1,q-1} - (n-q+3)\,\mathbf{T}_{Ij}^{n+1,q-1} - \int_{\partial A} \frac{\mathbf{u}_I^n \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u} \rangle^{q-1}}\, ds. \tag{93}$$

By changing variables $n = n+1$ and $q = q-1$, Equation (93) becomes

$$q\,\mathbf{w}_j \mathbf{T}_I^{n-1,q+1} = \sum_{k=1}^{n-1} \delta_{jI_k} \mathbf{T}_{I/k}^{n-2,q} - (n-q+1)\,\mathbf{T}_{Ij}^{n,q} - \int_{\partial A} \frac{\mathbf{u}_I^{n-1} \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u} \rangle^{q}}\, ds. \tag{94}$$

When $q \neq n+1$, we can solve for $\mathbf{T}_{Ij}^{n,q}$ from equation (94):

$$\mathbf{T}_{Ij}^{n,q} = \frac{1}{n-q+1} \left(\sum_{k=1}^{n-1} \delta_{jI_k} \mathbf{T}_{I/k}^{n-2,q} - q\,\mathbf{w}_j \mathbf{T}_I^{n-1,q+1} - \int_{\partial A} \frac{\mathbf{u}_I^{n-1} \mathbf{n}_j}{\langle \mathbf{w}, \mathbf{u} \rangle^{q}}\, ds \right),$$

which is Theorem 2. □□□

B Reduction of $\Lambda(\alpha, \beta)$ to Clausen Integrals

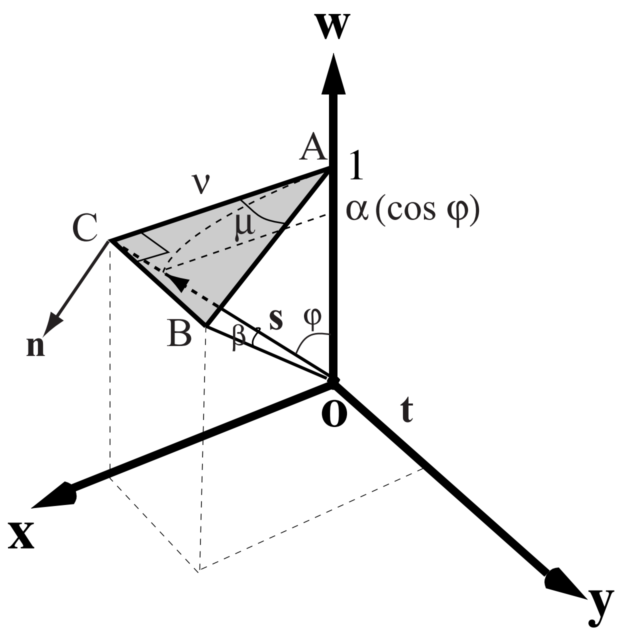

Figure 3: *The irradiance at the origin due to a triangular luminaire with quadratically varying radiant exitance can be represented in terms of either* $\Upsilon(\mu, \nu)$ *or* $\Lambda(\alpha, \beta)$.

In this appendix, we will show how to reduce the special function Λ, defined in equation (34), to a well-known special function called *Clausen integral*. To show this, we reduce the problem of computing our special function $\Lambda(\alpha, \beta)$ to that of computing the irradiance due to a two-parameter family of luminaires with spatially varying radiant exitance, which in turn may be expressed in terms of the Clausen integral.

Figure 3 shows a triangular luminaire in the $z = 1$ plane with two parameters: the angle μ and the edge length ν. Let P denote the spherical projection of this triangle, and $\phi(\mu, \nu)$ denote the irradiance at the origin due to this luminaire whose radiance distribution is given by

$$f(\mathbf{r}) \;\equiv\; \langle \mathbf{r}, \mathbf{r} \rangle \,, \tag{95}$$

where $\mathbf{r}$ is the position vector of the point on the luminaire plane. Expressing $\mathbf{r}$ in terms of the unit vector $\mathbf{u}$ by using

$$\mathbf{r} = \frac{h}{\langle \mathbf{w}, \mathbf{u} \rangle}\mathbf{u} = \frac{\mathbf{u}}{\langle \mathbf{w}, \mathbf{u} \rangle},$$

where $h = 1$ is the distance from the origin to the planar luminaire, Equation (95) becomes

$$f(\mathbf{u}) = \left\langle \frac{\mathbf{u}}{\langle \mathbf{w}, \mathbf{u} \rangle}, \frac{\mathbf{u}}{\langle \mathbf{w}, \mathbf{u} \rangle} \right\rangle = \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle^2}.$$

Here, $\mathbf{w}$ is the unit vector from the origin orthogonal to the luminaire plane. For this special configuration, $\mathbf{w}$ happens to be coincident with the $\mathbf{z}$ axis. It follows that the irradiance at the origin due to this family of luminaires is given by

$$\begin{aligned} \phi(\mu, \nu) &= \int_P f(\mathbf{u}) \cos\theta \; d\sigma(\mathbf{u}) \\ &= \int_P \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle^2} \langle \mathbf{w}, \mathbf{u} \rangle \; d\sigma(\mathbf{u}) \\ &= \int_P \frac{1}{\langle \mathbf{w}, \mathbf{u} \rangle} \; d\sigma(\mathbf{u}), \end{aligned} \tag{96}$$

where we used the fact that the surface normal at the origin is $\mathbf{w}$, and thus $\cos\theta = \langle \mathbf{w}, \mathbf{u} \rangle$. The right hand side in equation (96) is exactly the base case of our rational moments, $\tau^{0,1}(P, \mathbf{w})$, given by

$$\tau^{0,1}(P, \mathbf{w}) = \int_{\partial P} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \langle \mathbf{w}, \mathbf{n} \rangle \; ds,$$

which reduces to a summation of three line integrals along three spherical edges of the luminaire projection. That is,

$$\phi(\mu, \nu) = \sum_{i=1}^{3} \langle \mathbf{w}, \mathbf{n}_i \rangle \int_{\zeta_i} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2} \; ds, \tag{97}$$

where $\zeta_1, \zeta_2, \zeta_3$ are the great arcs corresponding to the three edges AB, BC and CA, respectively, and $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are their outgoing normals. Let A, B, C denote the position vectors for the three triangle vertices, then

$$\begin{aligned} \mathbf{n}_1 &= \text{Normalize}(A \times B) \\ \mathbf{n}_2 &= \text{Normalize}(B \times C) \\ \mathbf{n}_3 &= \text{Normalize}(C \times A). \end{aligned}$$

Notice that $\mathbf{w}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_3$, we have $\langle \mathbf{w}, \mathbf{n}_1 \rangle = \langle \mathbf{w}, \mathbf{n}_3 \rangle = 0$. Then, two terms on the right hand side of equation (97) vanish, and it simplifies to

$$\phi(\mu, \nu) \;=\; \langle \mathbf{w}, \mathbf{n} \rangle \int_{\zeta} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2}\, ds, \tag{98}$$

where we have omitted the subscripts for $\mathbf{n}$ and ζ. To evaluate the integral on the right hand side of equation (98), we parameterize the great arc ζ by

$$\mathbf{u} \;=\; \mathbf{s} \cos\theta + \mathbf{t} \sin\theta,$$

where $\mathbf{s}$ is the unit vector from the origin to the vertex C, and $\mathbf{t}$ is $\mathbf{y}$ axis. Then, $\langle \mathbf{w}, \mathbf{t} \rangle = 0$, let

$$\alpha \;=\; \langle \mathbf{w}, \mathbf{s} \rangle \;=\; \cos\varphi,$$

where φ is the angle between vector $\mathbf{w}$ and $\mathbf{s}$ (see Figure 3), we get

$$\begin{aligned} \int_{\zeta_2} \frac{\ln \langle \mathbf{w}, \mathbf{u} \rangle}{1 - \langle \mathbf{w}, \mathbf{u} \rangle^2}\, ds \;&=\; \int_0^{\beta} \frac{\ln(\alpha\ \cos\theta)}{1 - \alpha^2 \cos^2\theta}\, d\theta \\ &=\; \Lambda(\alpha, \beta), \end{aligned}$$

where β is the arc length of ζ. It follows that the irradiance $\phi(\mu, \nu)$ in equation (98) can be represented in terms of Λ as

$$\phi(\mu, \nu) \;=\; \langle \mathbf{w}, \mathbf{n} \rangle\, \Lambda(\alpha, \beta). \tag{99}$$

In this special configuration, $\mathbf{n}$ is perpendicular to the triangle OBC, and thus orthogonal to the $\mathbf{y}$ axis. Consequently, $\mathbf{n}$ lies on the XZ plane and points in the outwards direction shown in Figure 3, then

$$\langle \mathbf{w}, \mathbf{n} \rangle \;=\; \cos\left(\frac{\pi}{2} + \varphi\right) \;=\; -\sin\varphi \;=\; -\sqrt{1 - \alpha^2}.$$

Therefore, equation (99) can be written as

$$\boxed{\Lambda(\alpha, \beta) \;=\; -\frac{\phi(\,\mu(\alpha, \beta),\ \nu(\alpha, \beta)\,)}{\sqrt{1 - \alpha^2}}} \tag{100}$$

for $\alpha \neq 1$. Equation (100) shows an important result that any method for computing irradiance from quadratically varying polygonal luminaires must also be capable of evaluating Λ over its entire domain. Thus, Λ is an inescapable component of such irradiance computations.

On the other hand, using the indentity between 2-form $d\omega$ and the differential area dA

$$d\omega = \frac{\cos\theta}{r^2} dA,$$

where r is the distance from the origin to the point on the luminaire, and the fact that $\langle \mathbf{w}, \mathbf{u} \rangle = \cos\theta$ in this configuration, we can convert the integral (96) over solid angle to an integral over the area of the luminaire. That is,

$$\begin{aligned} \phi(\mu,\nu) &= \int\!\!\int_{\triangle} \frac{1}{r^2}\, dA \\ &= \int\!\!\int_{\triangle} \frac{1}{x^2+y^2+1}\, dxdy, \end{aligned} \tag{101}$$

where $(x, y, 1)$ represents a point in the luminaire. Finally, by changing variables to polar coordinates, the integral (101) reduces to $\Upsilon(\mu,\nu)$ [2, pp. 112], defined by

$$\boxed{\Upsilon(\mu,\nu) \equiv \int_0^{\mu} \ln\left(1+\nu^2\sec^2\theta\right)\, d\theta,} \tag{102}$$

where $0 \leq \mu \leq \pi/2$ and $\nu \geq 0$. The reduction is done as follows:

$$\begin{aligned} \phi(\mu,\nu) &= \frac{1}{2}\int_0^{\mu}\int_0^{\nu\sec\theta} \frac{2}{1+\rho^2}\, \rho\, d\rho d\theta \\ &= \frac{1}{2}\int_0^{\mu} \ln(1+\rho^2)\,\Big|_0^{\nu\sec\theta}\, d\theta \\ &= \frac{1}{2}\int_0^{\mu} \ln(1+\nu^2\sec^2\theta)\, d\theta \\ &= \frac{1}{2}\Upsilon(\mu(\alpha,\beta),\ \nu(\alpha,\beta)), \end{aligned} \tag{103}$$

where variables μ and ν are dependent on α and β. It is obvious from the geometry in Figure 3 that these variables are related by

$$\begin{aligned} \alpha &= \cos\varphi = \frac{1}{\sqrt{1+\nu^2}}, \\ \tan\beta &= \frac{\nu\tan\mu}{\sqrt{1+\nu^2}}. \end{aligned} \tag{104}$$

Solving for μ and ν, we get

$$\boxed{\begin{aligned} \mu(\alpha,\beta) &= \tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right), \\ \nu(\alpha) &= \frac{\sqrt{1-\alpha^2}}{\alpha}. \end{aligned}} \tag{105}$$

It is easily checked that when (μ,ν) ranges over $[0,\pi/2]\times[0,\infty)$, the corresponding parameter (α,β) of Λ varies over the range of $0\le\alpha\le 1$ and $0\le\beta\le\pi/2$. It now follows from equations (100),(103) and (105) that $\Lambda(\alpha,\beta)$ can be evaluated in terms of $\Upsilon(\mu,\nu)$ by

$$\boxed{\Lambda(\alpha,\beta) = -\frac{1}{2\sqrt{1-\alpha^2}}\,\Upsilon\left(\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right),\ \frac{\sqrt{1-\alpha^2}}{\alpha}\right),} \tag{106}$$

where $\alpha\neq 1$. When $\alpha=1$, $\Lambda(\alpha,\beta)$ can be evaluated exactly, as shown in equation (44). That is,

$$\Lambda(1,\beta) = -\beta-\cot\beta\ln(\cos\beta).$$

Equation (106) can also be proved directly by change of variables (see Appendix D). However, such a change of variable would be extremely difficult to discover without the aid of the construction of Figure 3.

In Appendix C, we have demonstrated how to reduce the function to the *Clausen integral* $\mathrm{Cl}_2(x)$, and the formula is given by

$$\Upsilon(\mu,\nu) = -\frac{1}{2}\left[2\mathrm{Cl}_2(2\mu)-\mathrm{Cl}_2(4\mu-2\eta)-\mathrm{Cl}_2(2\eta)+2(\eta-\mu)\,\ln\gamma\right], \tag{107}$$

where

$$\begin{aligned} \gamma &= (\nu-\sqrt{\nu^2+1})^2 = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^2, \\ \eta &= \tan^{-1}\left(\frac{\sin(2\mu)}{\gamma+\cos(2\mu)}\right), \end{aligned} \tag{108}$$

and the Clausen integral $\mathrm{Cl}_2(x)$ is defined by

$$\mathrm{Cl}_2(x) \equiv -\int_0^x \ln\left|2\sin\frac{\theta}{2}\right|\,d\theta = -\frac{1}{2}\int_0^x \ln\left(4\sin^2\frac{\theta}{2}\right)\,d\theta = \sum_{n=1}^{\infty}\frac{\sin nx}{n^2} \tag{109}$$

for all $x \in \mathbb{R}$ [15, pp.93]. Our derivation follows the procedure proposed by Arvo [2] and further simplifies the resulting formula.

Incorporating equations (106) and (107), we can represent the special function Λ in terms of Clausen integrals:

$$\boxed{\Lambda(\alpha,\beta) \quad = \quad \frac{1}{4\sqrt{1-\alpha^2}}\left[2\mathrm{Cl}_2(2\mu) - \mathrm{Cl}_2(4\mu - 2\eta) - \mathrm{Cl}_2(2\eta) + 2(\eta - \mu)\ln\gamma\right],} \tag{110}$$

where $\alpha \neq 1$ and μ, ν, γ, η are defined in equations (105) and (108).

C Reduction of $\Upsilon(\mu,\nu)$ to Clausen Integrals

To represent $\Upsilon(\mu,\nu)$ in terms of the Clausen integral, where

$$\Upsilon(\mu,\nu) \;\equiv\; \int_0^{\mu} \ln(1+\nu^2\sec^2\theta)\,d\theta, \tag{111}$$

we shall make use of two intermediate functions. One is Lobachevsky's function [11, pp. 941], defined by

$$L(x) \;\equiv\; \int_0^{x} \ln(\cos\theta)\,d\theta, \tag{112}$$

and the other is a two-parameter function, $F(\alpha,x)$, defined by

$$F(\alpha,x) \;\equiv\; \int_0^{x} \ln(1+\sin\alpha\cos\theta)\,d\theta. \tag{113}$$

In the mid 19th century, F.W.Newman [17, 18] showed that many closely related logarithmic integrals can be reduced to Clausen's integral. In particular, for the Lobachevsky's function (112), we have

$$\boxed{L(x) \;=\; \frac{1}{2}\,\mathrm{Cl}_2(\pi-2x) - x\ln 2.} \tag{114}$$

Proof: From the series representation of Clausen integral in equation (109), it is obvious that $\mathrm{Cl}_2(\pi)=0$. Thus,

$$\begin{aligned}
\mathrm{Cl}_2(\pi-2x) &= -\int_0^{\pi-2x} \ln\left(2\sin\frac{\theta}{2}\right)\,d\theta \\
&= -\mathrm{Cl}_2(\pi) - \int_{\pi}^{\pi-2x} \ln\left(2\sin\frac{\theta}{2}\right)\,d\theta \\
&= 2x\ln 2 - \int_{\pi}^{\pi-2x} \ln\left(\sin\frac{\theta}{2}\right)\,d\theta \\
&= 2x\ln 2 + 2\int_0^{x} \ln(\cos t)\,dt \quad (\theta=\pi-2t) \\
&= 2x\ln 2 + 2\,L(x).
\end{aligned}$$

Solve for $L(x)$, we get equation (114). □□□

The other identity for $F(\alpha, x)$ was derived by Newman [18, pp. 89], given by

$$\boxed{F(\alpha, x) \;=\; x \ln\left(\sin^2 \frac{\alpha}{2}\right) - \eta \ln\left(\tan^2 \frac{\alpha}{2}\right) - \mathrm{Cl}_2(2x) + \mathrm{Cl}_2(2x - 2\eta) + \mathrm{Cl}_2(2\eta),} \tag{115}$$

where

$$\tan \eta \;\equiv\; \frac{\sin x}{\tan \frac{\alpha}{2} + \cos x}. \tag{116}$$

Our derivation closely follows Newman's, which utilized several clever trigonometric substitutions. For example,

$$\sin \alpha \;=\; 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \;=\; \frac{2 \tan \frac{\alpha}{2}}{\sec^2 \frac{\alpha}{2}} \;=\; \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}. \tag{117}$$

Proof: Let

$$\tan \gamma \;=\; \frac{\sin \theta}{m + \cos \theta} \tag{118}$$

where $m = \tan \frac{\alpha}{2}$. Then, we can write

$$m \;=\; \frac{\sin(\theta - \gamma)}{\sin \gamma}.$$

From identity (117) and other elementary trigonometric identities, we have

$$\begin{aligned}
\sin \alpha \;&=\; \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \;=\; \frac{2m}{1 + m^2}, \\
m + \cos \theta \;&=\; \frac{\sin(\theta - \gamma)}{\sin \gamma} + \cos \theta \;=\; \frac{\sin \theta \cos \gamma}{\sin \gamma}, \\
1 + m^2 \;&=\; \sec^2 \frac{\alpha}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
1 + \sin \alpha \cos \theta \;&=\; \frac{1 + 2m \cos \theta + m^2}{1 + m^2} \\
&=\; \frac{\sin^2 \theta + (\cos \theta + m)^2}{1 + m^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sin^2\theta}{\sin^2\gamma\sec^2\frac{\alpha}{2}} \\
&= \frac{\sin^2\theta}{\sin^2\gamma}\cos^2\frac{\alpha}{2}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
F(\alpha,x) &= \int_0^x \left(\ln(\sin^2\theta) - \ln(\sin^2\gamma) + \ln\left(\cos^2\frac{\alpha}{2}\right)\right)\, d\theta \\
&= \int_0^x \left(\ln\left(4\sin^2\theta\right) - \ln\left(4\sin^2\gamma\right) + \ln\left(\cos^2\frac{\alpha}{2}\right)\right)\, d\theta \\
&= x\ln\left(\cos^2\frac{\alpha}{2}\right) - \mathrm{Cl}_2(2x) - \int_0^x \ln\left(4\sin^2\gamma\right)\, d\theta. \qquad (119)
\end{aligned}
$$

Consider $\gamma(\theta)$ as a C^1 function with respect to θ. By letting

$$
d\theta = d\gamma(\theta) + d(\theta - \gamma(\theta)), \qquad (120)
$$

the last integral above becomes

$$
\begin{aligned}
\int_0^x \ln(4\sin^2\gamma)\, d\theta &= \int_0^x \ln[\,4\sin^2\gamma(\theta)\,]\, d\gamma(\theta) + \int_0^x \ln[\,4\sin^2\gamma(\theta)\,]\, d(\theta-\gamma(\theta)) \\
&= \int_0^{\gamma(x)} \ln[\,4\sin^2 t\,]\, dt + \int_0^x \ln[\,4\sin^2\gamma(\theta)\,]\, d(\theta-\gamma(\theta) \\
&= -\mathrm{Cl}_2(2\gamma(x)) + 2\int_0^x \ln\left[\frac{4\sin^2(\theta-\gamma(\theta))}{m^2}\right]\, d(\theta-\gamma(\theta)) \\
&= -\mathrm{Cl}_2(2\gamma(x)) + \int_0^{x-\gamma(x)} \ln\left[4\sin^2 t\right]\, dt - [x-\gamma(x)]\ln m^2 \\
&= -\mathrm{Cl}_2(2\gamma(x)) - \mathrm{Cl}_2(2x-2\gamma(x)) - (x-\gamma(x))\ln\left(\tan^2\frac{\alpha}{2}\right),
\end{aligned}
$$

where we require that γ vanishes when $\theta = 0$ in the change of variables; that is, $\gamma(0) = 0$. Let $\eta = \gamma(x)$ and thus

$$
\tan\eta = \frac{\sin x}{\tan\frac{\alpha}{2} + \cos x}.
$$

From equation (119), we have

$$
\begin{aligned}
F(\alpha,x) &= x\ln\left(\cos^2\frac{\alpha}{2}\right) + (x-\eta)\ln\left(\tan^2\frac{\alpha}{2}\right) - \mathrm{Cl}_2(2x) + \mathrm{Cl}_2(2\eta) + \mathrm{Cl}_2(2x-2\eta) \\
&= x\ln\left(\sin^2\frac{\alpha}{2}\right) - \eta\ln\left(\tan^2\frac{\alpha}{2}\right) - \mathrm{Cl}_2(2x) + \mathrm{Cl}_2(2\eta) + \mathrm{Cl}_2(2x-2\eta),
\end{aligned}
$$

which establishes equation (115). □□□

The function η in the identity (115), which depends on α and x, is only implicitly defined by equation (116). To determine η explicitly and uniquely from $\tan\eta$, we must use two requirements satisfied by $\gamma(\theta)$ ($\eta = \gamma(x)$) enforced by our proof above. That is,

- $\gamma(x)$ must be a C^1 function in θ to make equation (120) hold.
- $\gamma(0) = 0$.

Specifically, the ranges of α and x we are interested in are $0 \le \alpha \le \pi/2$ and $0 \le x \le \pi$, where

$$0 \le m \; = \; \tan\frac{\alpha}{2} \le 1.$$

Then,

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{m+\cos\theta}\right) \; = \; \frac{m\cos\theta+1}{(m+\cos\theta)^2} \; \ge \; 0$$

for all $0 \le \theta \le \pi$ and $\theta \ne \theta_0 = \cos^{-1}(-m)$, that is, the denominator $m + \cos\theta \ne 0$. Therefore, $\tan\gamma$ is non-negative when $\theta \in [0, \theta_0)$ and is non-positive when $\theta \in (\theta_0, \pi]$. It follows from the two conditions of $\gamma(\theta)$ that γ should be in $[0, \pi]$ instead of $[-\pi/2, \pi/2]$, the range of $\tan^{-1}$ routine. Therefore,

$$\eta \; = \; \begin{cases} \tan^{-1}\left(\dfrac{\sin x}{\tan(\alpha/2)+\cos x}\right) & \tan\eta \ge 0 \\[2ex] \pi + \tan^{-1}\left(\dfrac{\sin x}{\tan(\alpha/2)+\cos x}\right) & \tan\eta < 0 \end{cases} \tag{121}$$

Using the identities (114) and (115), we can easily represent $\Upsilon(\mu,\nu)$ in terms of Clausen integrals as follows:

$$\begin{aligned} \Upsilon(\mu,\nu) \; &= \; \int_0^\mu \ln\left(1+\frac{\nu^2}{\cos^2\theta}\right)\, d\theta \\ &= \; \int_0^\mu \ln\left(\nu^2+\cos^2\theta\right)\, d\theta - 2\int_0^\mu \ln\left(\cos\theta\right)\, d\theta \\ &= \; \frac{1}{2}\int_0^{2\mu} \ln\left[\frac{\cos\theta+2\nu^2+1}{2}\right]\, d\theta - 2\,L(\mu) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{2\mu} \ln\left(1+\frac{\cos\theta}{2\nu^2+1}\right)\, d\theta + \mu\, \ln\left(\frac{2\nu^2+1}{2}\right) - 2\,L(\mu) \\
&= \frac{1}{2}F\left(\sin^{-1}\frac{1}{2\nu^2+1}, 2\mu\right) + \mu\, \ln\left(\frac{2\nu^2+1}{2}\right) - 2\,L(\mu).
\end{aligned}
$$

Replacing F, L above by (115) and (114), we get

$$
\begin{aligned}
\Upsilon(\mu,\nu) &= \mu\, \ln\left(\sin^2\frac{a}{2}\right) - \frac{1}{2}\eta\, \ln\left(\tan^2\frac{a}{2}\right) + \mu\, \ln\left(4\nu^2+2\right) \\
&\quad -\frac{1}{2}\left[\mathrm{Cl}_2(4\mu) - \mathrm{Cl}_2(4\mu-2\eta) - \mathrm{Cl}_2(2\eta) + 2\mathrm{Cl}_2(\pi-2\mu)\right] \\
&= -\frac{1}{2}\left[\mathrm{Cl}_2(4\mu) - \mathrm{Cl}_2(4\mu-2\eta) - \mathrm{Cl}_2(2\eta) + 2\mathrm{Cl}_2(\pi-2\mu)\right. \\
&\quad \left. -2\mu\, \ln\left(\left(4\nu^2+2\right)\sin^2\frac{a}{2}\right) + \eta\, \ln\left(\tan^2\frac{a}{2}\right)\right], \qquad (122)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \sin^{-1}\frac{1}{2\nu^2+1} \\
\eta &= \tan^{-1}\left(\frac{\sin(2\mu)}{\tan\frac{a}{2}+\cos(2\mu)}\right).
\end{aligned}
$$

Here, $0 \le a \le \pi/2$ and $0 \le \mu \le \pi/2$ from equation (105), and the inverse function $\tan^{-1}$ are interpreted as equation (121).

Using the following identity for Clausen integrals [15, pp. 94],

$$
\frac{1}{2}\mathrm{Cl}_2(2\theta) = \mathrm{Cl}_2(\theta) - \mathrm{Cl}_2(\pi-\theta), \qquad (123)
$$

equation (122) simplifies to

$$
\begin{aligned}
\Upsilon(\mu,\nu) &= -\frac{1}{2}\left[2\mathrm{Cl}_2(2\mu) - \mathrm{Cl}_2(4\mu-2\eta) - \mathrm{Cl}_2(2\eta)\right. \\
&\quad \left. -2\mu\, \ln\left(\left(4\nu^2+2\right)\sin^2\frac{a}{2}\right) + \eta\, \ln\left(\tan^2\frac{a}{2}\right)\right] \qquad (124)
\end{aligned}
$$

Furthermore, since

$$\sin a = \frac{1}{2\nu^2+1}$$

$$\cos a = \frac{2\nu\sqrt{\nu^2+1}}{2\nu^2+1},$$

we can simplify $\tan(a/2)$ as

$$\begin{aligned}\tan\frac{a}{2} &= \frac{1-\cos a}{\sin a} \\ &= (\nu - \sqrt{\nu^2+1})^2.\end{aligned}$$

Let $\gamma = (\nu - \sqrt{\nu^2+1})^2$, then

$$\eta = \tan^{-1}\left(\frac{\sin(2\mu)}{\gamma + \cos(2\mu)}\right)$$

$$\sin^2\frac{a}{2} = \frac{\tan^2(a/2)}{\sec^2(a/2)} = \frac{\gamma^2}{1+\gamma^2}.$$

Notice that

$$\gamma = \frac{\sqrt{\nu^2+1}-\nu}{\sqrt{\nu^2+1}+\nu},$$

the above formula for $\sin^2\frac{a}{2}$ can be further simplified as

$$\sin^2\frac{a}{2} = \frac{\gamma^2}{1+\gamma^2} = \frac{\gamma}{4\nu^2+2}. \tag{125}$$

It follows from equation (124) that

$$\Upsilon(\mu,\nu) = -\frac{1}{2}[2\mathrm{Cl}_2(2\mu) - \mathrm{Cl}_2(4\mu - 2\eta) - \mathrm{Cl}_2(2\eta) + 2(\eta-\mu)\ln\gamma]. \tag{126}$$

Here is another two simpler ways to derive equation (125) by using various trigonometric identities.

1. $\sin^2\frac{a}{2} = \frac{1}{2}\tan\frac{a}{2}\sin a = \frac{\gamma}{4\nu^2+2}.$
2. It follows from the identity $\cos a = 1 - 2\sin^2\frac{a}{2}$ that

$$\sin^2\frac{a}{2} = \frac{1-\cos a}{2} = \frac{\gamma}{4\nu^2+2}.$$

D Direct Proof of Equation (106)

In this appendix, we show how to directly prove the following formula

$$\Lambda(\alpha,\beta) \;=\; -\frac{1}{2\sqrt{1-\alpha^2}}\,\Upsilon\left(\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right),\; \frac{\sqrt{1-\alpha^2}}{\alpha}\right),$$

which was arrived at using the construction shown on page 33.

Proof: The proof is done by change of variables. Let

$$\tan\theta \;=\; \sqrt{1-\alpha^2}\,\tan t,$$

then, we can easily derive that

$$\begin{aligned}
t &= \tan^{-1}\left(\frac{\tan\theta}{\sqrt{1-\alpha^2}}\right)\\
\cos^2\theta &= \frac{\cos^2 t}{1-\alpha^2\sin^2 t}\\
d\theta &= \sqrt{1-\alpha^2}\,\frac{\cos^2\theta}{\cos^2 t}\,dt\\
&= \frac{\sqrt{1-\alpha^2}}{1-\alpha^2\sin^2 t}\,dt.
\end{aligned}$$

It follows that

$$\alpha^2\cos^2\theta \;=\; \frac{\alpha^2\cos^2 t}{1-\alpha^2\sin^2 t},$$

and

$$\begin{aligned}
\Lambda(\alpha,\beta) &= \int_0^\beta \frac{\ln(\alpha\cos\theta)}{1-\alpha^2\cos^2\theta}\,d\theta\\
&= \frac{1}{2}\int_0^\beta \frac{\ln(\alpha^2\cos^2\theta)}{1-\alpha^2\cos^2\theta}\,d\theta\\
&= \frac{1}{2}\int_0^{\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right)} \ln\left(\frac{\alpha^2\cos^2 t}{1-\alpha^2\sin^2 t}\right)\left[\frac{1-\alpha^2\sin^2 t}{1-\alpha^2}\right]\left[\frac{\sqrt{1-\alpha^2}}{1-\alpha^2\sin^2 t}\right]\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{1-\alpha^2}} \int_0^{\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right)} \ln\left(\frac{\alpha^2\cos^2 t}{1-\alpha^2\sin^2 t}\right)\, dt \\
&= -\frac{1}{2\sqrt{1-\alpha^2}} \int_0^{\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right)} \ln\left(\frac{1-\alpha^2\sin^2 t}{\alpha^2\cos^2 t}\right)\, dt \\
&= -\frac{1}{2\sqrt{1-\alpha^2}} \int_0^{\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right)} \ln\left(1+\frac{1-\alpha^2}{\alpha^2}\sec^2 t\right)\, dt \\
&= -\frac{1}{2\sqrt{1-\alpha^2}}\, \Upsilon\left(\tan^{-1}\left(\frac{\tan\beta}{\sqrt{1-\alpha^2}}\right),\ \frac{\sqrt{1-\alpha^2}}{\alpha}\right).
\end{aligned}
$$

□□□

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