Transistor Sizing of Energy-Delay–Efficient Circuits

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Abstract

This paper studies the problem of transistor sizing of CMOS circuits optimized for energy-delay efficiency, i.e., for optimal Et^n where E is the energy consumption and t is the delay of the circuit, while n is a fixed positive optimization index that reflects the chosen trade-off between energy and delay.

We propose a set of analytical formulas that closely approximate the optimal transistor sizes. We then study an efficient iteration procedure that can further improve the original analytical solution. Based on these results, we introduce a novel transistor sizing algorithm for energy-delay efficiency.

1. Introduction

The rapidly increasing complexity of VLSI systems has made it necessary to pay ever more attention to design issues that affect energy consumption. One of the original motivations for CMOS technology was its low energy consumption, and today, there are still no alternatives that approach it in energy efficiency. Nevertheless, energy consumption is more and more often the factor that limits the performance of contemporary CMOS systems.

In order to compare designs that run at different speeds *and* consume different amounts of energy, we have to combine the energy, E, and the delay, t, into a single metric δ. The authors have previously proposed $\delta = Et^2$ as an energy-delay–efficiency metric for VLSI computation [1, 2, 17]. The main reason for choosing this metric over others is that δ is to first order constant when we vary the supply voltage of a CMOS system: the delay falls roughly linearly with supply voltage, and the energy consumption increases roughly quadratically; therefore, Et^2 stays roughly constant. Hence, the δ metric allows the designer to factor "runtime" voltage scaling out of consideration. The authors have argued that, owing to its voltage independence, the δ metric is superior to other efficiency metrics found in the literature, such as E or Et [3].

In practice, we can achieve any desired target speed or target energy consumption by adjusting the supply voltage. If we desire to change to a particular delay target t, we adjust the voltage to meet it, and a circuit optimized for δ would have the best E for that t. Likewise, we may choose an energy target E and get the best t instead.

The Et^2 metric is a special case of a wider class of energy metrics, which includes E and Et, among others. The authors have shown that a metric of the more general form Et^n for $n \geq 0$ characterizes *any* feasible trade-off, not only the trade-off through voltage scaling, between the energy and the delay of a computation [4]. For example, any problem of minimizing the energy of a circuit for a given target delay can be restated as minimizing Et^n for a certain n. We call n the *energy-delay efficiency index*.

In this paper, we study the problem of transistor sizing for energy-delay efficient circuits. Given a transistor netlist where each transistor i has width w_i and length l_i, transistor sizing finds the values of w_i and l_i that optimize the target function—in our case Et^n. While it is true that most layout systems demand that transistor sizes be quantized to some grid, we ignore this constraint.

Also, we can remove the l_is from consideration since there is usually no reason to set the lengths of transistors in a digital circuit to anything other than the minimum allowed by the fabrication technology: increasing the length increases both the resistance and the capacitance and hence worsens both the energy and delay.

The sized transistors of a circuit are connected to each other through wires. The capacitance of these wires leads to additional energy and delay. (We ignore wire resistance in this paper.) For delay-only optimization, which can be phrased as the minimization of the metric Et^n for very large n, the wire capacitance can be overcome by increasing transistor sizes where appropriate. Conversely, for energy-only optimization, when $n = 0$, the transistor widths can be chosen to be minimum size, independently of the wire capacitance. In contrast to these special cases, for n small but nonzero, wire capacitance cannot be ignored or overcome in a straightforward way, and the optimal transistor sizes depend strongly on this capacitance.

In this paper, we propose an analytic formula for transistor sizing. If the approximate solution is acceptable for the given application, the formula can be used as is (no numerical optimization is then needed); however, if more accuracy is required, the formula can be used to provide a good starting point for numerical optimization. Later in the paper, we propose an efficient iteration procedure that

can further improve the accuracy of the original analytical solution. Based on these results, we introduce a novel transistor-sizing algorithm for energy-delay efficient circuits.

The proofs of properties and theorems have been omitted owing to space limitations. They can be found in the first author's Ph.D. dissertation [16].

2. Previous Work

Classical numerical methods, such as the conjugate gradient descent method, have been applied to the transistor-sizing problem: there exist several transistor sizing programs that minimize power consumption while maintaining performance specifications [5, 6, 7]. More recently, several specialized numerical techniques have been proposed [8, 9, 10]. On the analytical side, Cong and Koh have studied the related problem of simultaneous gate and wire optimization for optimal delay and power [13]. Cong and Koh's solution space and optimization metric are different from what we shall see in the present paper. A different analytic approach to the transistor sizing problem, for the performance metric Et, is given by Hu [11] and another by Horowitz, Indermaur, and Gonzalez [12]. Both Hu and Horowitz *et al.* present qualitative results; they only analyze basic inverter gates. To the best of the authors' knowledge, the present paper is the first one that goes beyond such a qualitative approach, both in terms of the generality of the optimization metric and in terms of the generality of the considered circuits.

3. Et^n-optimal circuits

Let t be the cycle time of the critical cycle of the circuit whose transistor sizes we wish to optimize. We assume that the circuit is designed so that all cycles are critical; this is true in many well designed circuits, and it is true for any optimally sized circuit in the absence of additional constraints on transistor sizes (such as minimum-size constraints or slew-rate constraints). Let E be the energy consumption of the critical cycle. Let us further assume that E is a constant proportion of the total energy consumption; in this case, optimizing the energy E of the critical cycle optimizes the total energy of the circuit, and vice-versa.

Using the τ-model [14, 15], we can write the energy as

$$E = \sum_{i=0}^{m-1} (w_{ni} + w_{pi} + p_i), \tag{1}$$

and the delay as

$$t = \sum_{i=0}^{m-1} \frac{k_{ni} f_{i+1} (w_{n(i+1)} + w_{p(i+1)} + p_{i+1})}{w_{ni}} + \sum_{i=0}^{m-1} \frac{\mu k_{pi} f_{i+1} (w_{n(i+1)} + w_{p(i+1)} + p_{i+1})}{w_{pi}}, \tag{2}$$

where w_{ni} and w_{pi} are the nFET and pFET (nMOS and pMOS transistor) widths of logic gate i; $k_{ni}, k_{pi} > 0$ are the numbers of nFETs and pFETs in series in logic gate i; $p_{i+1} > 0$ represents the wire parasitics at the output of logic gate i; $f_{i+1} > 0$ is the fanout of logic gate i; μ is the ratio of electron mobility to hole mobility; m is the length of the cycle, and $i \in 0..m-1$ with all indices modulo m. In writing Equations 1 and 2 we have made several simplifying assumptions. We ignored the energy consumption due to short-circuit and leakage currents. Furthermore, we have constrained all devices in a series transistor network to have the same width. Finally, we have ignored the wire RC and time-of-flight delays.

4. Properties of transistor sizes in Et^n-optimal circuits

Property 1 *If w_i are the values that minimize Et^n for a given set of wire parasitics p_i and gate topologies k_i, then αw_i, $\alpha > 0$ are the values that minimize Et^n for the set of wire parasitics αp_i and gate topologies k_i.*

Property 2 *If w_i are the values that minimize Et^n for a given set of wire parasitics p_i and gate topologies k_i, then w_i also minimize Et^n for the set of wire parasitics p_i and gate topologies αk_i.*

If we ignore special constraints on transistor sizes, such as minimum-size and minimum–slew-rate constraints, and if we further assume that every transition on every circuit node matters to the circuit's overall speed (this last assumption is especially relevant in asynchronous circuits), then we can show that, when a system is optimized for Et^n, the widths of the nFETs and pFETs of each gate i are related as follows [16]:

$$w_{pi} = w_{ni} \sqrt{\mu \frac{k_{pi}}{k_{ni}}}. \tag{3}$$

Equation 3 is a local relationship; it does not depend on either E, t or n. Equation 3 allows us to eliminate either the nFETs or the pFETs from the transistor-sizing problem. In particular, with the notation

$$w_i = w_{ni} + w_{pi} = w_{ni} \left(1 + \sqrt{\frac{\mu k_{pi}}{k_{ni}}} \right) \tag{4}$$

and

$$k_i = f_{i+1} k_{ni} \left(1 + \sqrt{\frac{\mu k_{pi}}{k_{ni}}} \right)^2, \tag{5}$$

by eliminating the pFET sizes from Equations 1 and 2, we get

$$E = \sum_{i=0}^{m-1} (w_i + p_i) \tag{6}$$

and

$$t = \sum_{i=0}^{m-1} k_i \frac{w_{i+1} + p_{i+1}}{w_i}. \tag{7}$$

We shall use these simpler formulas in the expressions for Et^n.

5. Preliminaries for Et^n-optimal transistor sizing

We formalize the sizing problem of a transistor netlist for minimal Et^n as the minimization, over the w_is, of Et^n where E and t are given by Equations 6 and 7.

$$Et^n = \left(\sum_{i=0}^{m-1}(w_i + p_i)\right)\left(\sum_{i=0}^{m-1} k_{i-1}\frac{w_i + p_i}{w_{i-1}}\right)^n \quad (8)$$

Note that Equation 8 holds not only for a ring, but also for a chain of gates, as long as the widths and parasitics for the input of the chain are equal to the widths and parasitics for the output of the chain (since in this case the E and t for a chain have the same form as the ones for a ring). This is an important observation, as it makes our results for transistor sizing applicable both to latency and cycle-time minimization.

Et^n is a *posynomial* function of the transistor widths. A posynomial in variables w_i is a function of the form $\sum_{0<i\leq q}\alpha_i w_0^{\beta_i^0} w_1^{\beta_i^1}...w_{m-1}^{\beta_i^{m-1}}$ where $\alpha_i \geq 0$. A *posynomial problem* is the minimization of one posynomial while simultaneously satisfying a set of upper-bound constraints on other posynomials. With the substitution $w_i = e^{x_i}$, any posynomial can be transformed into a convex function; therefore the unique optimum of Et^n is achieved when $\forall i : \frac{\partial Et^n}{\partial w_i} = 0$.

This implies that the optimum is achieved when

$$\begin{aligned}\forall i: \quad & \frac{k_{i-1}}{w_{i-1}} - \frac{k_i(w_{i+1}+p_{i+1})}{w_i^2} \\ = \quad & -\frac{1}{n}\frac{\sum_{i=0}^{m-1} k_{i-1}\frac{w_i+p_i}{w_{i-1}}}{\sum_{i=0}^{m-1}(w_i+p_i)} = -\frac{1}{n}\frac{1}{P}, \end{aligned} \quad (9)$$

where $P = E/t$ is the power consumption of the chosen cycle. If $\forall i : p_i = 0$ (no wire parasitics) and n is very large (delay-only optimization), Equation 9 reduces to

$$k_i\frac{w_{i+1}}{w_i} = k_{i-1}\frac{w_i}{w_{i-1}},$$

which is the known condition of equal stage delays for delay-only transistor sizing [14]. If we were able to solve Equation 9 analytically for any p_is and k_is, we could compute the optimal w_is directly and our transistor-sizing problem would be solved. Unfortunately, this is not the case. We can compute an exact analytical solution of Equation 9 only for a restricted class of p_is and k_is [16]. In particular, we can show that if $\forall i : k_i = k$, i.e., the case of homogeneous circuits, and $\forall i : p_i = p$ then

$$\forall i : w_i = np \;\; \forall p, k > 0. \quad (10)$$

Equation 10 states that the transistor widths w_i of a homogeneous circuit with equal wire parasitics p, optimized for Et^n, are all equal to np, independently of k [17, 18].

6. Et^n-optimal transistor sizes

So far we have explored some general properties of transistor sizes for circuits optimized for Et^n. Based on these properties, we now develop a simple analytical formula that approximates the transistor sizes of an Et^n-optimal circuit.

We start by finding an approximate formula for the transistor sizes w_i that optimize Et^n, in Equation 8, when $\forall i : k_i = k$. We then extend this formula to the case when the k_is are no longer equal to each other.

6.1. Homogeneous Circuits

For the case when $\forall i : k_i = k$, we propose an approximate solution of the w_is, of the following form:

$$w_i = \alpha_1 p_{i+1} + \alpha_2 p_{Avg} \quad (11)$$

where

$$p_{Avg} = \frac{1}{m}\sum p_i \quad (12)$$

and α_1 and α_2 are constants to be determined later. First, let us motivate Equation 11. Based on Property 2, we know that finding the w_is when $\forall i : k_i = k$ is equivalent to finding the w_is when $\forall i : k_i = 1$. In other words, the value of the w_is is independent of the k_is, when all k_is are equal. Conversely, based on Property 1, we know that the w_is scale linearly with the p_is. This suggests that the w_is should not have terms that are independent of the p_is. Based on our experience of sizing, we know that—while the transistor sizes of gate i depend mostly on p_{i+1}—the effect of a particular p_i gets distributed to some degree to all other gates. As a consequence, we would like Equation 11 to depend linearly on both p_{i+1} and some average of all other p_is and one such choice is $\alpha_1 p_{i+1} + \alpha_2 p_{Avg}$. We use the arithmetic mean for p_{Avg} since the p_is correspond physically to wire capacitances that are manipulated additively both in terms of delay and energy. With these clarifications in mind, we state the following:

Theorem 1 *For a neighborhood $\mathcal{V}_p = [p-\eta, p+\eta]$ of $p > 0$, $\eta > 0$, the values of α_1 and α_2 that minimize Et^n given the w_is of the form defined by Equation 11, where $\forall i : p_i \in \mathcal{V}_p$, $k_i = k > 0$ and $\eta \to 0$, are*

$$\alpha_1 = \frac{\frac{1}{2}}{\frac{1}{n}+\frac{m}{m-1}} \text{ and } \alpha_2 = n - \frac{\frac{1}{2}}{\frac{1}{n}+\frac{m}{m-1}}.$$

If the problem is large, i.e., $m \to \infty$, $\frac{m}{m-1} \approx 1 \Rightarrow \alpha_1 = \frac{n}{2(1+n)}$ and $\alpha_2 = \frac{n(1+2n)}{2(1+n)}$, thus $\frac{\alpha_2}{\alpha_1} = 1+2n$. What is particularly surprising about Equation 11 is that the strength of a given gate depends far more strongly (5× for Et^2 optimization) on the *average* parasitic load ($\alpha_2 = 5/3$) than it does on the load on that *particular* gate ($\alpha_1 = 1/3$). Furthermore, $\lim_{n\to 0}\alpha_1 = \lim_{n\to 0}\alpha_2 = 0 \Rightarrow \forall i : w_i = 0$ for $n = 0$ regardless of the p_is. In other words, for energy-only optimization, Equation 11 yields minimum-size transistors, as one might expect.

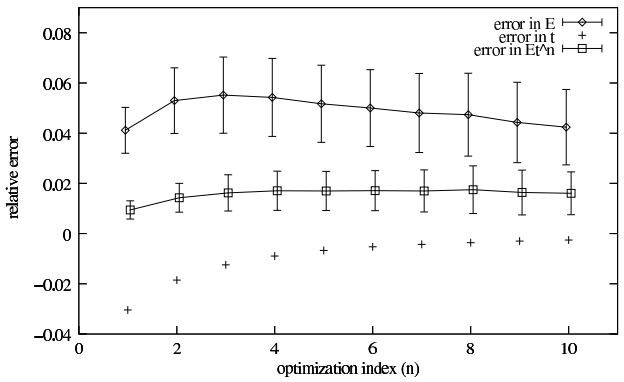

Figure 1. Accuracy in E, t and Et^n of Equation 11 with α_1 and α_2 given by Theorem 1.

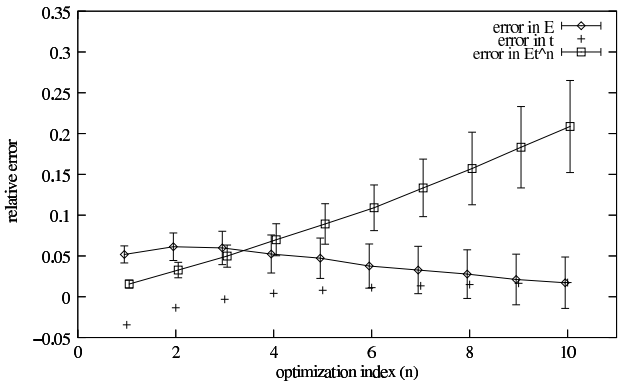

Figure 2. Accuracy in E, t and Et^n of Equation 13 with α_1, α_2, and r_is given by Theorem 2.

Theorem 1 yields the optimal values of α_1 and α_2 in a close neighborhood of p, or equivalently when the p_is are close to each other. We want to check now if the form of the w_is given by Equation 11 and Theorem 1 yields a practical approximation of the w_is when $\forall i : k_i = k$ but the p_is are no longer close to each other. We use a numerical optimizer to compute the error between the optimal and the predicted Et^n for a given n, m and a set of p_is. We varied $m \in [2,1000]$, $n \in [1,10]$ and used three different distributions (uniform, uniform-squared, and uniform-cubed) for $p_i \in [1,100]$. The observed errors are practically independent of the problem size m and the distribution chosen for the p_is; the errors only depend on n. Figure 1 shows the relative error in E, t and Et^n for $m = 31$, $n \in [1,10]$, $k_i = 1$ and $p_i \in [1,100]$ chosen randomly through a uniform-squared distribution. The average error in E is between 4.1% and 5.5%, the average error in t is between -3.0% and -0.3%, and the average error in Et^n is between 1.0% and 1.7%.

6.2. Non-homogeneous Circuits (first form)

The formula resulting from Theorem 1 yields excellent results when all k_is are equal. We would like to extend it to incorporate the case when the k_is are no longer all equal. To do this, we assume that the cumulative effect of the p_is and the k_is on the w_is can be viewed as the product between the individual effect of the p_is (wire capacitances) on the w_is and the individual effect of the k_is (gate topologies) on the w_is. Hence, we propose an approximate solution of the w_is of the following form:

$$w_i = (\alpha_1 p_{i+1} + \alpha_2 p_{Avg})\ r_i(k_0, k_1, \ldots, k_{m-1}) \quad (13)$$

where α_1 and α_2 are given by Theorem 1, while functions r_i will be determined later. When all gates are identical, i.e., $\forall i : k_i = k$, we know from Equation 10 that the w_is are independent of the k_is. For this reason, we choose $r_i(k_0, k_1, \ldots, k_{m-1})$ such that $\forall k :\ r_i(k, k, \ldots, k) = 1$.

Based on our experience on delay-only transistor sizing, we know that—while the transistor sizes of gate i depend strongly on k_i—the effect of a particular k_i gets distributed to some degree to all other gates. As a consequence, we would like r_i to depend on both k_i and some average of all other k_is. We use the geometric mean $k_{Avg} = \sqrt[m]{\prod k_i}$ as an average of the k_is, since it has physical meaning—it is proportional to the theoretical minimal delay of the cycle. In this context, we introduce the following

Theorem 2 *For a neighborhood $\mathcal{V}_p = [p-\eta, p+\eta]$ of $p > 0$, $\eta > 0$, and a neighborhood $\mathcal{V}_k = [k-\eta, k+\eta]$ of $k > 0$, the values of α_1, α_2, β_1 and β_2 that minimize Et^n given the w_is of the form defined by Equation 13 with $r_i(k_0, k_1, \ldots, k_{m-1}) = \beta_1 \frac{k_i}{k_{Avg}} + \beta_2$, where $\forall i : p_i \in \mathcal{V}_p$, $k_i \in \mathcal{V}_k$, and $\eta \to 0$, are*

$$\alpha_1 = \frac{\frac{1}{2}}{\frac{1}{n} + \frac{m}{m-1}}, \alpha_2 = n - \frac{\frac{1}{2}}{\frac{1}{n} + \frac{m}{m-1}}, \text{ and } \beta_1 = \beta_2 = \frac{1}{2}.$$

Theorem 2 yields the optimal values of α_1, α_2, β_1 and β_2 when the p_is are in a close neighborhood of p, and the k_is are in a close neighborhood of k. We would like to verify now how good these values are in minimizing Et^n when the p_is and the k_is are no longer close to each other. We use again a numerical optimizer to compute the error between the optimal and the estimated Et^n for a given n, m and a set of p_is and k_is. We vary $m \in [2,1000]$, $n \in [1,10]$ and use three different distributions (uniform, uniform-squared, and uniform-cubed) for $p_i \in [1,100]$ and $k_i \in [1,3.3]$ (if we assume $k_{ni} \in [1,6]$ and $k_{pi} \in [1,2]$, then with $\mu = 2.5$ we get $k_i \in [6.66, 21.95]$ or equivalently, using Property 2, $k_i \in [1,3.3]$). As for Equation 11, the observed errors are practically independent of the problem size m and the distribution chosen for the p_is and the k_is; the errors only depend on n. Figure 2 shows the relative error in E, t and Et^n for $m = 31$, $n \in [1,10]$ and $p_i \in [1,100]$, $k_i \in [1,3.3]$ chosen randomly through a uniform-squared distribution. The average error in E is between 1.7% and 6.1%, the average error in t is between -3.4% and 1.7%, while the average error in Et^n is about 3.3% for $n = 2$, but increasing about linearly with n, ow-

ing to the error amplifying artifact of Et^n (if $t = t_0(1+\Delta)$ $\Rightarrow t^n \approx t_0^n(1+n\Delta)$ for small Δ).

6.3. Non-homogeneous Circuits (second form)

The main intended use of Equation 13 in energy-delay efficient design is to find approximate transistor sizes when $n \approx 2$, i.e., when voltage scaling is a design parameter. As Figure 2 shows, the equation stated by Theorem 2, i.e., a particular case of Equation 13, does this reasonably well—i.e., within a few percent of the optimum. On the other hand, one might want to use such a sizing formula for large n as well—i.e., predominantly delay-only optimization. Getting a close approximation of Et^n when n is large requires a very good delay estimate, since even a small error Δ in t gets linearly amplified to $n\Delta$ in Et^n. For this reason, we study the behavior of Equation 13 and the delay estimate resulting from it, when $n \to \infty$.

For now, consider a simpler problem, namely finding the transistor widths $w_{\infty i}$ that minimize t given by Equation 7. This is a special case of the Et^n optimization problem for $n \to \infty$. In [16] we have shown that the optimal delay $t_\infty = mk_{Avg}$ is reached for transistor widths that have the property

$$\forall i : \frac{w_{\infty(i+1)}}{w_{\infty i}} = \frac{k_{Avg}}{k_i}. \tag{14}$$

We would like the w_is given by Equation 13 to have property (14) for large n. More precisely,

$$\lim_{n\to\infty} \frac{w_{i+1}}{w_i} = \frac{w_{\infty(i+1)}}{w_{\infty i}} \tag{15}$$

or equivalently, using α_1 and α_2 given by Theorem 2,

$$\lim_{n\to\infty} \frac{w_{i+1}}{w_i} = \lim_{n\to\infty} \frac{r_{i+1}(k_0,k_1,\ldots,k_{m-1})}{r_i(k_0,k_1,\ldots,k_{m-1})} = \frac{w_{\infty(i+1)}}{w_{\infty i}}. \tag{16}$$

Condition 16 guarantees that the delay estimate resulting from Equation 13 is optimal for large n. An obvious choice of the r_is that fulfills (16) is $\forall i$: $r_i(k_0,k_1,\ldots,k_{m-1}) = \beta w_{\infty i}$, where $\beta > 0$ is a constant scaling factor. The role of β is to normalize the w_is to the right energy level; its optimal value is stated by the following

Theorem 3 *For a neighborhood $\mathcal{V}_p = [p-\eta, p+\eta]$ of $p > 0$, $\eta > 0$, and a neighborhood $\mathcal{V}_k = [k-\eta, k+\eta]$ of $k > 0$, the values of α_1, α_2, β that minimize Et^n given the w_is of the form defined by Equation 13 with $r_i(k_0,k_1,\ldots,k_{m-1}) = \beta w_{\infty i}$, where $\forall i : p_i \in \mathcal{V}_p$, $k_i \in \mathcal{V}_k$, and $\eta \to 0$, are*

$$\alpha_1 = \frac{\frac{1}{2}}{\frac{1}{n} + \frac{m}{m-1}}, \alpha_2 = n - \frac{\frac{1}{2}}{\frac{1}{n} + \frac{m}{m-1}}, \text{ and}$$

$$\beta = \frac{S_2}{2}\left(\left(1-\frac{1}{n}\right) + \sqrt{\left(1-\frac{1}{n}\right)^2 + \frac{4}{nS_1S_2}}\right),$$

where

$$S_1 = \frac{1}{m}\sum_{i=0}^{m-1} w_{\infty i} \quad \text{and} \quad S_2 = \frac{1}{m}\sum_{i=0}^{m-1} \frac{1}{w_{\infty i}}.$$

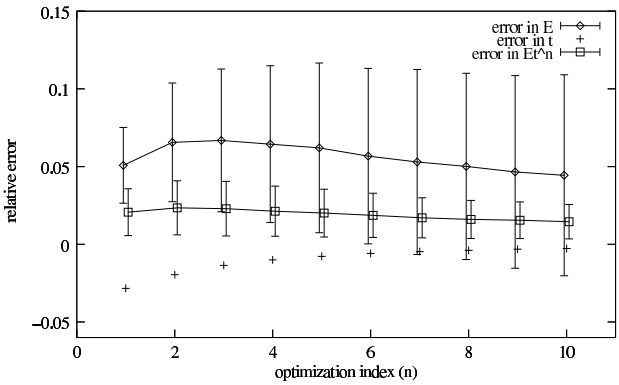

Figure 3. Accuracy in E, t and Et^n of the approximation given by Theorem 3.

Assuming $\forall i : p_i = p$, and the w_is given by Theorem 3, Equations 6 and 7 yield

$$E = (1+nS_1\beta)E_0 \quad \text{and} \quad t = \left(1+\frac{S_2}{n\beta}\right)t_\infty,$$

where E_0 is the theoretical minimal energy (i.e. total switched wire parasitic) and t_∞ is the theoretical minimal delay. In [16, 17] we have shown that for a wide class of circuits

$$E \approx (1+n)E_0 \quad \text{and} \quad t \approx \left(1+\frac{1}{n}\right)t_\infty.$$

Given the value of β from Theorem 3, we have that $\forall n \geq 0 : \frac{1}{S_1} \leq \beta \leq S_2$ with $\beta = \frac{1}{S_1}$ if $n \to 0$ and $\beta = S_2$ if $n \to \infty$. If we choose $\beta = \frac{1}{S_1}$, the error in E is reduced by bringing E close to $(1+n)E_0$, while if we choose $\beta = S_2$, the error in t is reduced by bringing t close to $(1+\frac{1}{n})t_\infty$.

The formula resulting from Theorem 3 works extremely well in practice for small m, i.e., it keeps the error in Et^n very low for the entire range of n, including large n. However, for m large the accuracy of the formula deteriorates somewhat due to the fact that E becomes consistently overestimated, while the estimate in t stays very accurate. This is a consequence of the choice of the r_is, where we have intentionally favored the accuracy of the delay estimation. For large ms, the difference between $\frac{1}{S_1}$ and S_2 becomes large enough so that the resulting β pulls E noticeably away from the optimum $(1+n)E_0$.

Figure 3 shows the relative error in E, t and Et^n for the approximation given by Theorem 3 for $m = 9$ (an 18 transitions per cycle circuit), $n \in [1,10]$ and $p_i \in [1,100]$, $k_i \in [1,3.3]$ chosen randomly through a uniform-squared distribution. The average error in E is between 4.4% and 6.7%, the average error in t is between -0.2% and -2.8%, and the average error in Et^n is between 1.4% and 2.3%. It is interesting to point out that for $n = 100$, the average error in E is about 1.2%, the average error in t is about -0.003%, and the average error in Et^n is about 0.5%.

For clarity, Theorems 1, 2, and 3 were formulated to refer to the transistor sizing problem of a single-cycle system. However, these theorems can be easily extended to multi-cycle systems. We extend formula 11, and as a consequence Theorem 1, to multi-cycle systems by redefining p_{Avg} for each gate i to be the average parasitic of all simple cycles gate i is part of. Theorem 2 extends to multi-cycle systems by substituting mk_{Avg} with t_∞ (the minimum achievable delay of the circuit). Given the definition of $w_{\infty i}$s and p_{Avg}, Theorem 3 generalizes straightforwardly to multi-cycle systems, with the only remark that m—in the expression of S_1 and S_2—represents the total number of transistors in the considered circuit, not just the ones on a given cycle.

Remembering the derivation of Section 4, the values of the w_is are per gate i; but they can be transformed into the effective nFET and pFET sizes directly, using Equations 3, 4 and 5.

7. An iterative approach to Et^n-optimal transistor sizing

With the help of Theorems 2 and 3, we can compute approximate transistor sizes of an Et^n-optimal circuit. As we have seen, the approximate solution yields energy and delay values within a few percent of the optimum. However, if the accuracy of such a solution is not acceptable for the given application, one might wish to employ an iterative procedure to "fine tune" the initial transistor sizes.

Using Equation 9, we can compute w_i—for a fixed i—as a function of the other ws. More precisely, if we call

$$a_2 = \frac{b_1 + nb_0b_2}{(n+1)b_2},\ a_1 = \frac{(-n+1)b_3}{(n+1)b_2},\ \text{and}\ a_0 = \frac{-nb_0b_3}{(n+1)b_2},$$

where

$$b_0 = \sum_{i=1, i\neq j}^{m} w_j + \sum_{i=1}^{m} p_j,$$

$$b_1 = \sum_{i=1, i\neq j, i\neq j+1}^{m} k_{j-1}\frac{w_j + p_j}{w_{j-1}} + k_{i-1}\frac{p_i}{w_{i-1}},$$

$$b_2 = \frac{k_{i-1}}{w_{i-1}},$$

and

$$b_3 = k_i(w_{i+1} + p_{i+1});$$

we can compute w_i as the positive solution to the cubic equation

$$w_i^3 + a_2w_i^2 + a_1w_i + a_0 = 0. \qquad (17)$$

(Equation 17 has a single positive root for $n \geq 1$; this can be found using Cardan's method.)

The iterative procedure starts with an initial solution and then repetitively computes each w_i as the positive solution of Equation 17 with coefficients computed from the current value of all other ws. It is easy to see that such a procedure converges to the Et^n-optimal solution. First, the recomputed value of w_i yields a better Et^n than the pre-iteration value. This is because $\frac{\partial Et^n}{\partial w_i} = 0$, i.e., the new w_i is Et^n-optimal when all other ws are fixed at their current value. Secondly, the Et^n optimization problem is convex in the ws, hence a local minimum reached by the iteration procedure is indeed the global minimum.

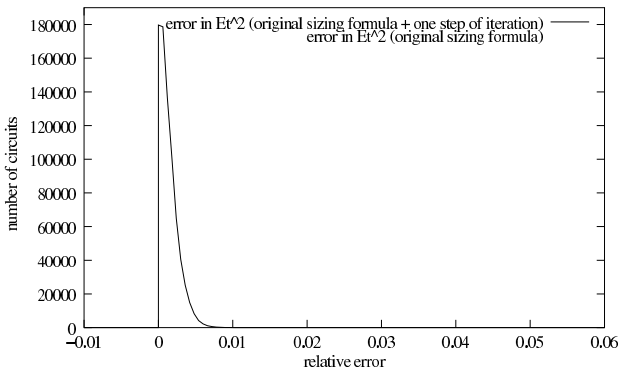

Figure 4. Error in Et^2 when exhaustively simulating an entire class of circuits.

To fully appreciate the benefit of the proposed iteration procedure when applied to the initial solution given by Theorems 2 or 3, we exhaustively analyze a particular case of Equation 8 with $n = 2, m = 5, p_i \in \{1,2,3,4,5\}$ and $k_i \in \{1,2,3\}$. Figure 4 shows a histogram of the relative error in Et^2 between the optimal values (computed with an optimization algorithm) and the estimated values based on Theorem 3, and also between the optimal values and the values computed by one step of the iteration procedure starting with the approximate solution given by Theorem 3. One step of iteration assigns one new value to each w_i. We observe that the already small maximal error of the original sizing formula is reduced about ten-fold by a single step of the iteration procedure. Of course, one can repeat the same procedure and get an even smaller error. However, this second step does not have the same impact on reducing the error as the first step had. Given that the transistor sizes of a real circuit are integer multiples of a technology dependent constant, there is not much point in trying to find the zero-error solution. That solution is unlikely to be implementable in practice, since it will likely have non-integer components.

We have done several experiments in which we tested the dependence of the iteration procedure on the initial starting point. We have found that the applicability of the method strongly depends on the initial solution's proximity to the optimal solution. Without a good initial solution like the one given by Theorem 2 or 3, the method still converges eventually to the optimum. However, the first step of iteration yields a solution that has an error spread two orders of magnitude greater than the solution resulting from the first step of the iteration executed on the good initial solution.

8. An algorithm for Et^n-optimal sizing

As we have seen, the transistor sizes w_i of a system optimized for Et^n depend strongly on the wire parasitics p_i. Unfortunately, these parasitics are not known *a priori*, since they are attributes of wires that connect transistors whose dimensions have not yet been found.

A two-phase algorithm solves the problem of the unknown parasitics. In the first phase, given the transistor netlist, each wire is assigned an initial wiring cost. The more is known about the structure of the transistor netlist and about a future floorplan, the more accurate such an assignment will be. Based on these initial wire parasitics, we can then compute an initial estimate for the w_is with the formulas established by Theorems 2 and 3.

In the second phase, we wire up the pre-sized transistors and extract the actual wire capacitances from the layout. With these new parasitics, we recompute the transistor widths w_i. Finally, we may fine-tune the solution by iterating once as described in Section 7.

If the accuracy of the final solution should not be deemed acceptable, we can add a pass through a classical numerical optimizer. Given the proximity of the current solution to the optimum, such an optimization will converge quickly. In this last phase, a more accurate transistor model (e.g., a BSIM model) can be employed, so as to bridge the gap between the simplified transistor model used in this paper and the actual transistor behavior.

9. Conclusions

We have proposed a set of analytical formulas that closely approximate the optimal transistor sizes for circuits optimized for Et^n. We have justified the validity of these formulas both mathematically and experimentally. We have proposed an iterative procedure that can further improve the accuracy of the original analytical solution. Experiments show that, when the procedure is applied on the analytical solution, it converges much more quickly then with an arbitrary starting point. Based on these results, we have introduced a novel transistor sizing algorithm for energy-delay efficiency.

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