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ON THE STRUCTURE OF A REVERSIBLE ENTANGLEMENT GENERATING SET FOR TRIPARTITE STATES

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We show that Einstein–Podolsky–Rosen–Bohm (EPR) and Greenberger–Horne–Zeilinger–Mermin (GHZ) states can not generate, through local manipulation and in the asymptotic limit, all forms of tripartite pure–state entanglement in a reversible way. The techniques that we use indicate that there is a connection between this result and the irreversibility that occurs in the asymptotic preparation and distillation of bipartite mixed states.

Keywords: Multipartite Entanglement, Asymptotic Entanglement Transformations
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1. Introduction

To identify the fundamentally inequivalent ways quantum systems can be entangled [1] is a major goal of quantum information theory. In the case of systems shared by two parties, Alice and Bob, there is only one type of entanglement, namely that contained in the Einstein–Podolsky–Rosen–Bohm (EPR) state

$$|EPR\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \tag{1}$$

in the sense that, in the limit of large N, Alice and Bob can *reversibly* transform N copies of any other entangled state $|\psi\rangle_{AB}$ into EPR states by using only local operations and classical communication (LOCC) [2]. This simple picture becomes much richer in systems shared by more than two parties, since also genuine multipartite entanglement exists [3]. In particular, the Greenberger–Horne–Zeilinger–Mermin (GHZ) state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \tag{2}$$

can not be reversibly generated from EPR states pairwise distributed among Alice, Bob and a third partie Claire (also labelled by A, B and C) [4]. In the terminology of Ref. [3], this implies that EPR states alone do not form a minimal reversible entanglement generating set (MREGS) for tripartite states.

The results of Ref. [4] left open the question whether, instead, the set

$$G_3 = \{|GHZ\rangle, |EPR\rangle_{AB}, |EPR\rangle_{AC}, |EPR\rangle_{BC}\} \tag{3}$$

constitutes a MREGS. Denoting by $\rightleftharpoons$ an asymptotically (i.e. in the large N limit) reversible transformation using LOCC, this question amounts to assessing the feasibility of a transformation of the form

$$\begin{aligned} |\psi\rangle_{ABC}^{\otimes N} \quad \rightleftharpoons \quad & |GHZ\rangle^{\otimes gN} \otimes |EPR\rangle_{AB}^{\otimes xN} \\ & \otimes |EPR\rangle_{AC}^{\otimes yN} \otimes |EPR\rangle_{BC}^{\otimes zN}, \end{aligned} \tag{4}$$

where $g, x, y, z \geq 0$, for any tripartite state $|\psi\rangle_{ABC}$. If this were the case, then entanglement in tripartite systems could be regarded as consisting only of GHZ and EPR correlations.

In the meantime it has been proved that not all four–party states can be reversibly generated from a distribution of EPR and three– and four–party GHZ states [5]. However, no evidence has been found contradicting the following conjecture.

Conjecture: G_3 is a MREGS for tripartite states.

On the contrary, all reversible transformations of tripartite states so far reported, involving Schmidt decomposable states [3], but also a whole class of more elaborated states [6], seem to support it.

In this article we give examples of tripartite states, denoted by $|\Psi_\delta\rangle_{ABC}$, that can not be reversibly generated only with states of the set G_3, thus disproving the above conjecture. We also show that even a reversible transformation of states of G_3 into any of these states *and* states of G_3 is impossible. That is, we show that there are cases where the transformation of Eq. (4) can not be made reversible even if the coefficients g, x, y, z are eventually allowed to be negative [7]. Notice that such a possibility, not previously excluded in four–party systems, would have allowed for a slightly different description of multipartite entanglement, also based exclusively on EPR and GHZ correlations.

These results, therefore, indicate the need to extend the set G_3 in order to eventually obtain a MREGS, either in its original formulation or in the extended sense described above. We would like to note, however, that the notion of a non–trivial MREGS implicitly assumes that the manipulation of multipartite pure states can be made reversible. This is, admittedly, an appealing idea, but has not yet been proved. In this sense, our results can be just interpreted as to indicate that a fundamental irreversibility occurs during the process of combining EPR and GHZ entanglements into any of the tripartite pure states $|\Psi_\delta\rangle$.

It is natural to inquire into the origin of such an irreversibility, which is somewhat analogous to the one that characterizes the cycle of preparing and distilling bipartite mixed states [8]. Actually, the argument that will lead to disprove the above conjecture would fail if mixed–state entanglement could be reversibly distilled. This fact suggests a connection between the two irreversible processes.

2. G_3 and irreversibility in mixed-state entanglement manipulations

In the next sections we provide a way for constructing pure entangled states of three parties, $|\Psi_\delta\rangle$, that cannot be generated reversibly from G_3. Our strategy consists in showing that a conservation law obeyed in reversible asymptotic entanglement transformations [4] would be violated if EPR and GHZ states could generate $|\Psi_\delta\rangle$ reversibly.

2.1. *Relative Entropy of Entanglement and reversible pure-state entanglement transformations*

Let $|\Psi\rangle_{ABC}$ denote an arbitrary tripartite pure state shared by Alice, Bob and Claire, and let ρ_{AB} be the mixed state resulting from tracing out Claire's subsystem. The relative entropy of entanglement of ρ_{AB} [9],

$$E_\Omega(\rho_{AB}) \equiv \min_{\sigma_{AB}\in\Omega} S(\rho_{AB}\,||\,\sigma_{AB}), \tag{5}$$

where Ω is some convex set of states (typically, that of separable states) invariant under LOCC and $S(\rho\,||\,\sigma) \equiv \mathrm{tr}(\rho\log_2\rho - \rho\log_2\sigma)$ is the quantum relative entropy, was originally introduced to quantify the entanglement of bipartite mixed states. Its regularized version,

$$E_\Omega^{reg}(\rho_{AB}) \equiv \lim_{N\to\infty} \frac{E_\Omega(\rho_{AB}^{\otimes N})}{N}, \tag{6}$$

is a lower bound for the entanglement cost E_c [10, 11] of ρ_{AB}, or number of EPR states per copy of ρ_{AB} needed to asymptotically prepare copies of ρ_{AB} [12]. It is also an upper bound for its distillable entanglement E_d [10, 13], or number of EPR states per copy of ρ_{AB} that can be asymptotically distilled from copies of ρ_{AB}. Indeed, E_Ω^{reg} fulfills the postulates required in [14] for an entanglement measure and therefore [14, 15]

$$E_d(\rho_{AB}) \le E_\Omega^{reg}(\rho_{AB}) \le E_c(\rho_{AB}). \tag{7}$$

Particularly relevant in the context of this work will be the fact that, as showed in [4], the relative entropy of entanglement of (say) subsystems AB, $E_\Omega(AB)$ must be conserved during any reversible pure-state transformation of the system ABC [16]. Applied to transformation (4) this law reads

$$E_\Omega(\rho_{AB}^{\otimes N}) = E_\Omega([EPR]_{AB}^{\otimes xN}), \tag{8}$$

$[EPR] \equiv |EPR\rangle\langle EPR|$, where we have used that when tracing out part C, only $|EPR\rangle_{AB}$ gives a non-separable contribution [17]. Thus, in the large N limit we are left with the condition

$$E_\Omega^{reg}(\rho_{AB}) = x, \tag{9}$$

where x is the number of EPR states per copy of ρ_{AB} that should be available on the rhs of Eq. (4), and we have used that $E_\Omega([EPR]_{AB}) = 1$. Similarly, if instead we allow now for states of G_3 to appear simultaneously on both sides of transformation (4), we obtain

$$E_\Omega(\rho_{AB}^{\otimes N} \otimes [EPR]_{AB}^{\otimes x_1 N}) = E_\Omega([EPR]_{AB}^{\otimes x_2 N}), \tag{10}$$

$x_1, x_2 \ge 0$, which implies the condition

$$\lim_{N\to\infty} \frac{E_\Omega(\rho_{AB}^{\otimes N} \otimes [EPR]_{AB}^{\otimes x_1 N})}{N} = x_2. \tag{11}$$

Now, there are several possible elections of the set Ω. Here we will consider only the set Sep of separable states, and the set PPT of states with positive partial transposition. Each of these choices leads to a different constraint. In particular, Eq. (9) becomes two conditions,

$$E^{reg}_{Sep}(\rho_{AB}) = x, \tag{12}$$
$$E^{reg}_{PPT}(\rho_{AB}) = x. \tag{13}$$

Similarly, Eq. (11) also leads to two conditions, that we shall analyze below.

2.2. *Construction of the states*

We construct the tripartite states $|\Psi_\delta\rangle_{ABC} \in \mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ as purifications of any PPT bound–entangled state δ [18] in $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}$ with no products vectors in its range, i.e. $\delta = \mathrm{tr}_C|\Psi_\delta\rangle\langle\Psi_\delta|$. Examples of these states can be found in Refs. [19, 20]. Since there are no PPT entangled states is systems $\mathcal{C}^2 \otimes \mathcal{C}^2$ and $\mathcal{C}^2 \otimes \mathcal{C}^3$, $d_A > 2$ and $d_B > 3$ [21]. The dimension of the third subsystem, d_C, has to be at least equal to the rank of δ.

Since the state δ is PPT, we have $E^{reg}_{PPT}(\delta) = 0$. Next we will prove that $E^{reg}_{Sep}(\delta) > 0$, which leads to the contradiction $0 = x > 0$, indicating that $|\Psi_\delta\rangle_{ABC}$ can not be reversibly generated with states of G_3 [22]. Notice that when applied to the PPT state δ, Eq. (11) for $\Omega = PPT$ implies that $x_1 = x_2$ [23]. We will also prove that

$$\lim_{N\to\infty} \frac{E_{Sep}(\delta^{\otimes N} \otimes [EPR]^{\otimes x_1 N})}{N} > x_1, \tag{14}$$

that by substitution in Eq. (11) for $\Omega = Sep$ implies that $x_2 > x_1$. Therefore, we must have $x_1 = x_2 > x_1$, which is again a contradiction, this time meaning that the states of G_3 can not reversibly generate the state $|\Psi_\delta\rangle$ *and* states of G_3.

In order to proceed, we need the following result.

Theorem 1 [24]: Consider a projector P onto a subspace V of $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}$ that does not contain any product vector. A positive constant $\alpha < 1$ exists such that for all $N \geq 1$,

$$\max_{|a_N \otimes b_N\rangle} \langle a_N \otimes b_N|P^{\otimes N}|a_N \otimes b_N\rangle \leq \alpha^N, \tag{15}$$

where $|a_N \otimes b_N\rangle \in \mathcal{C}^{d_A^N} \otimes \mathcal{C}^{d_B^N}$ denotes a product state.

Proof: P fulfills the following properties: (i) Since there are no product vectors in V, a positive number $\alpha_1 < 1$ exists such that

$$\langle a_1 \otimes b_1|\, P\,|\, a_1 \otimes b_1\rangle \leq \alpha_1 \tag{16}$$

for all product vectors. Indeed, $\langle a_1 \otimes b_1|\, P\,|\, a_1 \otimes b_1\rangle$ defines a continuous function of $|\, a_1 \otimes b_1\rangle$ on a compact set, so it attains its supremum [25]. (ii) A positive number $c > 0$ exists such that $I + cP$ is separable [26]. The proof proceeds using induction over the number of copies in a similar way as in Ref. [8]. Define $\alpha \equiv (1 + \alpha_1\, c)/(1 + c)$, where $\alpha_1 \leq \alpha \leq 1$. Then, Eq. (15) is satisfied for $N = 1$ because of Eq. (16). Now, suppose that it is true for a given N. From the fact that $I + cP$ is separable, i.e. it can be written as $\sum_k \mu_k|\, i_k \otimes j_k\rangle\langle\, i_k \otimes j_k|$ with $\mu_k > 0$, it follows that

$$\langle\, a_{N+1} \otimes b_{N+1}|(I + cP) \otimes \left(\alpha^N I - P^{\otimes N}\right) |\, a_{N+1} \otimes b_{N+1}\rangle \geq 0. \tag{17}$$

Indeed, we can use the separable decomposition of $I + cP$ and define $|a_{k,N}\rangle \equiv \langle i_k|a_{N+1}\rangle$ and $|b_{k,N}\rangle \equiv \langle j_k|b_{N+1}\rangle$, in such a way that Eq. (17) is equal to

$$\sum_k \mu_k \langle\, a_{k,N} \otimes b_{k,N}|\, \left(\alpha^N - P^{\otimes N}\right) |\, a_{k,N} \otimes b_{k,N}\rangle, \tag{18}$$

which, due to the induction hypothesis, is non-negative. Since $P \leq I$, we have

$$(I + cP) \otimes \left(\alpha^N I - P^{\otimes N}\right) \leq \alpha^N (I + cP) \otimes I - (1 + c) P^{\otimes N+1}. \tag{19}$$

Substituting this formula into Eq. (17), we obtain

$$\begin{aligned}
&\langle a_{N+1} \otimes b_{N+1}|P^{\otimes N+1}|a_{N+1} \otimes b_{N+1}\rangle \leq \\
&\frac{\alpha^N}{1+c}\left(1 + c\,\langle a_{N+1} \otimes b_{N+1}|P \otimes I|a_{N+1} \otimes b_{N+1}\rangle\right) \leq \\
&\alpha^N \frac{1 + \alpha_1 c}{1 + c} = \alpha^{N+1},
\end{aligned} \tag{20}$$

and this ends the proof. □

The following theorem provides us with a bound for the relative entropy of entanglement with respect to the set *Sep* and together with theorem 1 is the key to the main result.

Theorem 2: Let P be the projector onto the support of a mixed state ρ_{AB} of a bipartite system $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}$, let $|a \otimes b\rangle \in \mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B}$ denote a product vector and let β be

$$\beta \equiv \max_{|a\otimes b\rangle} \langle a \otimes b|P|a \otimes b\rangle. \tag{21}$$

The relative entropy of entanglement with respect to separable states is bounded below by

$$E_{Sep}(\rho_{AB}) \geq -\log_2 \beta. \tag{22}$$

Proof: Let $\sigma_{AB} \in Sep$ be the separable state such that $E_{Sep}(\rho_{AB}) = S(\rho_{AB}||\sigma_{AB})$. The quantum relative entropy can only decrease under a trace–preserving completely positive map $\mathcal{E}$ [27]. In particular, let us consider

$$\mathcal{E}(\tau) \equiv P\tau P + (I - P)\tau(I - P). \tag{23}$$

We find

$$\begin{aligned}
S(\rho_{AB}||\sigma_{AB}) &\geq S(\mathcal{E}(\rho_{AB})||\mathcal{E}(\sigma_{AB})) = \\
&\mathrm{tr}(\rho_{AB} \log_2 \rho_{AB} - \rho_{AB} \log_2 P\sigma_{AB}P),
\end{aligned} \tag{24}$$

where in the last step we have used that ρ_{AB} is invariant under $\mathcal{E}$ and that we can ignore the contribution $(I - P)\sigma_{AB}(I - P)$ because its support is orthogonal to P. Indeed, notice that for positive operators N, M_1 and M_2, $\log(M_1 \oplus M_2) = \log M_1 \oplus \log M_2$, and therefore $\mathrm{tr}[(N \oplus 0)\log(M_1 \oplus M_2)] = \mathrm{tr}(N \log M_1)$. Define

$$t \equiv \mathrm{tr}(P\sigma_{AB}), \tag{25}$$

$$\sigma'_{AB} \equiv \frac{1}{t} P\sigma_{AB}P. \tag{26}$$

Then, because $\sigma_{AB} = \sum_i p_i|a_i \otimes b_i\rangle\langle a_i \otimes b_i|$ is a separable state, we have that $t \leq \beta$. We finally obtain,

$$S(\rho_{AB}||\sigma_{AB}) \geq \mathrm{tr}(\rho_{AB}\log_2 \frac{\rho_{AB}}{t\sigma'_{AB}}) =$$
$$-\log_2 t + S(\rho_{AB}||\sigma'_{AB}) \geq -\log_2 t \geq -\log_2 \beta, \tag{27}$$

where we have used that for positive operators N, M and a positive constant k, $\mathrm{tr}(N\log kM) = \mathrm{tr}(N\log M) + (\mathrm{tr}N)\log k$, and the positivity of the quantum relative entropy [27]. □

We only need to concatenate theorems 1 and 2 to find that for any state δ

$$E_{Sep}(\delta^{\otimes N}) \geq -\log_2 \alpha^N, \tag{28}$$

and therefore

$$E^{reg}_{Sep}(\delta) \geq -\log_2 \alpha > 0, \tag{29}$$

which disproves the initial conjecture for G_3.

Notice that we can use this result and the inequalities (7) to extend the irreversibility proved in [8] to all the states δ. Indeed, we have $0 = E^{reg}_{PPT}(\delta) < E^{reg}_{Sep}(\delta)$, and both quantities are between the entanglement cost E_c and the distillable entanglement E_d, i.e. $E_d(\delta) < E_c(\delta)$.

Let us move now to prove Eq. (14). We need the following two lemmas.

Lemma 1: Let P be a projector onto a subspace V of $\mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}$, and let $|a\otimes b\rangle \in \mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}$ be a product state. Then

$$\max_{|a\otimes b\rangle}\langle a\otimes b|P|a\otimes b\rangle = \max_{|\psi\rangle\in V} \lambda_1(\psi), \tag{30}$$

where $\lambda_1(\psi)$ denotes the largest coefficient λ_i in the Schmidt decomposition of $|\psi\rangle$, $|\psi\rangle = \sum_i \sqrt{\lambda_i}|u_i\otimes v_i\rangle$, $\lambda_i \geq \lambda_{i+1}$.

Proof: For any product vector $|a\otimes b\rangle$, let us define the normalized vector $|\gamma\rangle \in V$ as $P|a\otimes b\rangle/||P|a\otimes b\rangle||$. Then

$$\langle a\otimes b|P|a\otimes b\rangle = |\langle a\otimes b|\gamma\rangle|^2 \leq \lambda_1(\gamma), \tag{31}$$

where in the last step we have used lemma 1 of [28]. Let $|\psi'\rangle$ be the vector for which the maximum in the rhs of Eq. (30) is attained, and let $\sum_i \sqrt{\lambda'_i}|u'_i\otimes v'_i\rangle$, $\lambda'_i \geq \lambda'_{i+1}$, be its Schmidt decomposition. Then

$$\max_{|\psi\rangle\in V} \lambda_1(\psi) = \lambda'_1 = \langle u'_1\otimes v'_1|P|u'_1\otimes v'_1\rangle, \tag{32}$$

which finishes the proof. □

Lemma 2: Let P be a projector onto a subspace V of $\mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}$ and let P_Φ be a projector onto a bipartite pure state $|\Phi\rangle \in \mathcal{C}^{d'}\otimes\mathcal{C}^{d'}$ with Schmidt decomposition $\sum_{i=1}^{d'}\sqrt{\lambda_i}\ |u_i\rangle\otimes|v_i\rangle$, $\lambda_i \geq \lambda_{i+1}$. Finally, let α_p be

$$\alpha_p \equiv \max_{|a\otimes b\rangle}\ \langle a\otimes b|\ P\ |a\otimes b\rangle, \tag{33}$$

where $|a\otimes b\rangle \in \mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}$ denotes a product state. Then,

$$\max_{|c\otimes d\rangle}\ \ \langle c\otimes d|\ \ P\otimes P_\Phi\ \ |c\otimes d\rangle\ =\ \alpha_p\lambda_1, \tag{34}$$

where the maximization is made over product vectors $|c \otimes d\rangle \in \mathcal{C}^{d_A+d'} \otimes \mathcal{C}^{d_B+d'}$.

Proof: Notice that $P \otimes P_\Phi$ projects onto a subspace spanned by vectors of the form $|\psi\rangle \otimes |\Phi\rangle$, $|\psi\rangle \in V$, and that the largest coefficient λ_1 in a Schmidt decomposition fulfills $\lambda_1(\psi \otimes \Phi) = \lambda_1(\psi)\lambda_1(\Phi)$. Then Eq. (34) follows from lemma 1. □

We would like to bound below the relative entropy of entanglement E_{Sep} of

$$\delta^{\otimes N} \otimes [EPR]^{\otimes M}. \tag{35}$$

The projector onto its support is given by $P_\delta^{\otimes N} \otimes [EPR]^{\otimes M}$, where P_δ is the projector onto the support of δ, and we can use lemma 2 and theorem 1 to obtain

$$\max_{|a\otimes b\rangle} \langle a \otimes b| P_\delta^{\otimes N} \otimes [EPR]^{\otimes M} |a \otimes b\rangle \leq \frac{\alpha^N}{2^M}, \tag{36}$$

where $(1/2)^M$ corresponds to $\lambda_1(EPR^{\otimes M})$. Then we can apply theorem 2 to obtain

$$E_{Sep}(\delta^{\otimes N} \otimes [EPR]^{\otimes M}) \geq -N \log_2 \alpha + M, \tag{37}$$

which implies Eq. (14). This finishes the proof of the fact that it is not possible to reversibly transform states of G_3 into any purification of a state δ and states of G_3.

3. Conclusions

In this work we have showed by means of counter–examples that GHZ and EPR states alone cannot be used to reversibly generate all tripartite pure states. This result leaves several questions open. It would be interesting to understand the mechanisms that lead to this irreversibility. Recall that in the asymptotic limit some non-trivial tripartite states can be reversibly generated from EPR and GHZ states [6]. We ignore which conditions determine whether a tripartite pure–state transformation can be performed in a reversible way. The following two facts suggest, however, that there may be a connection between this question and the irreversibility that takes place during the preparation–distillation cycle of bipartite mixed states:

(i) All known tripartite reversible transformations [3, 6] involve pure states whose bipartite reduced mixed states can be distilled and prepared in a reversible way [29].

(ii) The proof that G_3 is not a MREGS relies on the irreversibility that occurs in bipartite mixed–state manipulation. Indeed, suppose that E_c and E_d would not disagree for the states δ. Then, because of Eq. (7), E^{reg}_{PPT} and E^{reg}_{Sep} would also have been equal, and this would jeopardize our argument.

Finally, a major open question is whether a finite MREGS exists for tripartite states and, if so, which kind of states it must include. These are difficult issues that certainly deserve further investigation. We cautiously conclude the present work by noting that the states of an eventual MREGS must have bipartite reduced density matrices able to reproduce the discrepancies between relative entropies displayed by the states δ, and must therefore carry themselves the signature of bipartite mixed–state irreversibility.

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Note added: after completion of this work, Y. Shi pointed out the relation between the conjecture proved here and his recent work [30]. We have not been able to follow the line of argumentation in such a work.

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1. Consider a composite system of N particles, $\mathcal{C}^{d_1} \otimes \mathcal{C}^{d_2} \otimes \cdots \otimes \mathcal{C}^{d_N}$, where d_i is the dimension of the local space of party i. A pure state, $|\Psi^s\rangle$, belonging to the global space, is said to be separable when it can be written as a tensor product of pure states in each subsystem, i.e. $|\Psi^s\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$. This definition is easily extended to mixed states, and a density matrix ρ is separable when it can be expressed as a convex sum of projectors into separable pure states, i.e. $\rho = \sum_i p_i |\Psi^s_i\rangle\langle\Psi^s_i|$, where $|\Psi^s_i\rangle$ are separable. Physically, separable states are those that the different parties can prepare if they are allowed to use only local operations in each local system and to communicate classically. States that are not separable are called entangled.
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27. See for instance M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).
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29. This can be checked by noticing that the bipartite reduced density matrices of the states discussed in [6] (which contain the Schmidt–decomposable states of [3] as a particular case) consist of a mixture of *locally orthogonal* pure states [6] (either product or entangled). Thus, the entanglement of the mixed state can be distilled without losses by means of a projective local measurement that probabilistically picks up one of the pure states of the mixture.
30. Yu Shi, quant-ph/0201079.