FLUX NORM APPROACH TO HOMOGENIZATION PROBLEMS WITH NON-SEPARATED SCALES

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Flux norm approach to homogenization problems with non-separated scales.

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Abstract

We consider linear divergence-form scalar elliptic equations and vectorial equations for elasticity with rough ($L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$) coefficients $a(x)$ that, in particular, model media with non-separated scales and high contrast in material properties. While the homogenization of PDEs with periodic or ergodic coefficients and well separated scales is now well understood, we consider here the most general case of arbitrary bounded coefficients. For such problems we introduce explicit finite dimensional approximations of solutions with controlled error estimates, which we refer to as homogenization approximations. In particular, this approach allows one to analyze a given medium directly without introducing the mathematical concept of an ϵ family of media as in classical periodic homogenization. We define the flux norm as the L^2 norm of the potential part of the fluxes of solutions, which is equivalent to the usual H^1-norm. We show that in the flux norm, the error associated with approximating, in a properly defined finite-dimensional space, the set of solutions of the aforementioned PDEs with rough coefficients is equal to the error associated with approximating the set of solutions of the same type of PDEs with smooth coefficients in a standard space (e.g., piecewise polynomial). We refer to this property as the *transfer property*. A simple application of this property is the construction of finite dimensional approximation spaces with errors independent of the regularity and contrast of the coefficients and with optimal and explicit convergence rates. This transfer property also provides an alternative to the global harmonic change of coordinates for the homogenization of elliptic operators that can be extended to elasticity equations. The proofs of these homogenization results are based on a new class of elliptic inequalities which play the same role in our approach as the div-curl lemma in classical homogenization.

Contents

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1 Introduction

In this paper we are interested in finite dimensional approximations of solutions of scalar and vectorial divergence form equations with rough coefficients in $\Omega \subset \mathbb{R}^d$, $d \geq 2$. More precisely, in the *scalar case*, we consider the partial differential equation

$$\begin{cases} -\operatorname{div}\Big(a(x)\nabla u(x)\Big) = f(x) \quad x \in \Omega; f \in L^2(\Omega),\ a(x) = \{a_{ij} \in L^\infty(\Omega)\} \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^d$ with a smooth boundary (e.g.,C^2) and a is symmetric and uniformly elliptic on Ω. It follows that the eigenvalues of a are uniformly bounded from below and above by two strictly positive constants, denoted by $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$. Precisely, for all $\xi \in \mathbb{R}^d$ and $x \in \Omega$,

$$\lambda_{\min}(a)|\xi|^2 \le \xi^T a(x)\xi \le \lambda_{\max}(a)|\xi|^2. \tag{1.2}$$

In the *vectorial case*, we consider the equilibrium deformation of an inhomogeneous elastic body under a given load $b \in (L^2(\Omega))^d$, described by

$$\begin{cases} -\operatorname{div}(C(x) : \varepsilon(u)) = b(x) & x \in \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $C(x) = \{C_{ijkl}(x)\}$ is a 4th order tensor of elastic modulus (with the associated symmetries), $u(x) \in \mathbb{R}^d$ is the displacement field, and for $\psi \in (H_0^1(\Omega))^d$, $\varepsilon(\psi)$ is the symmetric part of $\nabla\psi$, namely,

$$\varepsilon_{ij}(\psi) = \frac{1}{2}\Big(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\Big). \tag{1.4}$$

We assume that C is uniformly elliptic and $C_{ijkl} \in L^\infty(\Omega)$. It follows that the eigenvalues of C are uniformly bounded from below and above by two strictly positive constants, denoted by $\lambda_{\min}(C)$ and $\lambda_{\max}(C)$.

The analysis of finite dimensional approximations of scalar divergence form elliptic, parabolic and hyberbolic equations with rough coefficients that in addition satisfy a Cordes-type condition in arbitrary dimensions has been performed in [48, 49, 45]. In these works, global harmonic coordinates are used as a coordinate transformation. We also refer to the work of Babuška, Caloz, and Osborn [7, 8] in which a harmonic change of coordinates is introduced in one dimensional and quasi-one dimensional divergence form elliptic problems.

In essence this harmonic change of coordinates allows for the mapping of the operator $L_a := \operatorname{div}(a\nabla)$ onto the operator $L_Q := \operatorname{div}(Q\nabla)$ where Q is symmetric positive and divergence-free. This latter property of Q implies that L_Q can be written in both a divergence form and a non-divergence form operator. Using the $W^{2,2}$ regularity of solutions of $L_Q v = f$ (for $f \in L^2$) one is able to obtain homogenization results for the operator L_a in the sense of finite dimensional approximations of its solution space (this relation with homogenization theory will be discussed in detail in section 6).

This harmonic change of coordinates provides the desired approximation in two-dimensional scalar problems but there is no analog of such a change of coordinates for vectorial elasticity equations. One goal of this paper is to obtain an analogous homogenization approximation without relying on any coordinate change and therefore allowing for treatment of both scalar and vectorial problems in a unified framework.

In section 2, we introduce a new norm, called the flux norm, defined as the L^2-norm of the potential component of the fluxes of solutions of (1.1) and (1.3). We show that this norm is equivalent to the usual H^1-norm. Furthermore this new norm allows for the transfer of error estimates associated with a given elliptic operator $\operatorname{div}(a\nabla)$ and a given approximation space V onto error estimates for another given elliptic operator $\operatorname{div}(a'\nabla)$ with another approximation space V' provided that the potential part of the fluxes of elements of V and V' span the same linear space. In this work this transfer/mapping property will replace the transfer/mapping property associated with a global harmonic change of coordinates.

In section 3, we show that a simple and straightforward application of the flux-norm transfer property, is to obtain finite dimensional approximation spaces for solutions of (1.1) and (1.3) with "optimal" approximation errors independent of the regularity and contrast of the coefficients and the regularity of $\partial\Omega$.

Another application of the transfer property of the flux norm is given in section 5 for controlling the approximation error associated with discontinuous Galerkin solutions of (1.1) and (1.3). In this context, for elasticity equations, harmonic coordinates are replaced by harmonic displacements. The estimates introduced in section 5 are based on mapping onto divergence-free coefficients via the flux-norm and a new class of inequalities introduced in section 4. We believe that these inequalities are of independent interest for PDE theory and could be helpful in other problems.

Connections between this work, homogenization theory and other related works will be discussed in section 6.

2 The flux norm and its properties

In this section we will introduce the flux-norm for solutions of (1.1) (and (1.3)). This flux-norm is equivalent to the usual $H^1_0(\Omega)$-norm (or $(H^1_0(\Omega))^d$-norm for solutions to the vectorial problem) but leads to error estimates that are independent of the material contrast. Furthermore it allows for the transfer of error estimates associated with a given elliptic operator $\operatorname{div}(a\nabla)$ and a given approximation space V onto error estimates for another given elliptic operator $\operatorname{div}(a'\nabla)$ with another approximation space V' provided that the potential part of the fluxes of elements of V and V' span the same linear space. In [48] approximation errors have been obtained for finite element solutions of (1.1) with arbitrarily rough coefficients a. These approximation errors are based on the mapping of the operator $-\operatorname{div}(a\nabla)$ onto an non-divergence form operator $-Q_{i,j}\partial_i\partial_j$ using global harmonic coordinates as a change of coordinates. It is not clear how to extend this change of coordinates to elasticity equations whereas the flux-norm approach has a natural extension to systems of equations and can be used to link error estimates on two separate operators.

2.1 Scalar case.

For $k \in (L^2(\Omega))^d$, denote by k_{pot} the potential portion of the Weyl-Helmholtz decomposition of k (the orthogonal projection of k onto the closure of the space $\{\nabla f \ : \ f \in C_0^\infty(\Omega)\}$ in $(L^2(\Omega))^d$). For $\psi \in H_0^1(\Omega)$, define

$$\|\psi\|_{a\text{-flux}} := \|(a\nabla\psi)_{pot}\|_{(L^2(\Omega))^d}. \tag{2.1}$$

The potential part appears in this definition, since flux across $\partial\Omega$ is determined solely by the potential part of the vector field due to the Divergence theorem and it is also used for the transfer property given in theorem 2.1.

Proposition 2.1. *$\|.\|_{a\text{-flux}}$ is a norm on $H_0^1(\Omega)$. Furthermore, for all $\psi \in H_0^1(\Omega)$*

$$\lambda_{\min}(a)\|\nabla\psi\|_{(L^2(\Omega))^d} \le \|\psi\|_{a\text{-flux}} \le \lambda_{\max}(a)\|\nabla\psi\|_{(L^2(\Omega))^d} \tag{2.2}$$

Proof. The proof of the left hand side of inequality (2.2) follows by observing that

$$\int_\Omega (\nabla\psi)^T a\nabla\psi = \int_\Omega (\nabla\psi)^T (a\nabla\psi)_{pot} \tag{2.3}$$

from which we deduce by Cauchy Schwartz inequality that

$$\int_\Omega (\nabla\psi)^T a\nabla\psi \le \|\nabla\psi\|_{L^2(\Omega)}\|\psi\|_{a\text{-flux}}. \tag{2.4}$$

□

The proof of the main theorem of this section will require

Lemma 2.1. *For $f \in L^2(\Omega)$, let u be the solution of (1.1). Then,*

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \sup_{w\in H^2(\Omega)\cap H_0^1(\Omega)} \inf_{v\in V} \frac{\|(\nabla w - a\nabla v)_{pot}\|_{(L^2(\Omega))^d}}{\|\Delta w\|_{L^2(\Omega)}} \tag{2.5}$$

Proof. Since $f \in L^2(\Omega)$, it is known that there exists $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta w = f & x \in \Omega \\ w = 0 \quad \text{on} & \partial\Omega. \end{cases} \tag{2.6}$$

We conclude by observing that for $v \in V$,

$$\|(\nabla w - a\nabla v)_{pot}\|_{(L^2(\Omega))^d} = \|(a\nabla u - a\nabla v)_{pot}\|_{(L^2(\Omega))^d}. \tag{2.7}$$

□

For V, a finite dimensional linear subspace of $H_0^1(\Omega)$, we define

$$(\operatorname{div} a\nabla V) := \operatorname{span}\{\operatorname{div}(a\nabla v) \ : \ v \in V\}. \tag{2.8}$$

Note that $(\operatorname{div} a\nabla V)$ is a finite dimensional subspace of $H^{-1}(\Omega)$.

The following theorem establishes the transfer property of the flux norm which is pivotal for our analysis.

Theorem 2.1. (Transfer property of the flux norm) *Let V' and V be finite-dimensional subspaces of $H^1_0(\Omega)$. For $f \in L^2(\Omega)$ let u be the solution of (1.1) with conductivity a and u' be the solution of (1.1) with conductivity a'. If* $(\operatorname{div} a\nabla V) = (\operatorname{div} a'\nabla V')$, *then*

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \sup_{f\in L^2(\Omega)} \inf_{v\in V'} \frac{\|u'-v\|_{a'\text{-flux}}}{\|f\|_{L^2(\Omega)}}. \tag{2.9}$$

Remark 2.1. The usefulness of (2.9) can be illustrated by considering $a' = I$ so that $\operatorname{div} a'\nabla = \Delta$. Then $u' \in H^2$ and therefore V' can be chosen as, e.g., the standard piecewise linear FEM space with nodal basis $\{\varphi_i\}$. The space V is then defined by its basis $\{\psi_i\}$ determined by

$$\operatorname{div}(a\nabla\psi_i) = \Delta\varphi_i \tag{2.10}$$

with Dirichlet boundary conditions (see details in section 3.1.1). Furthermore, equation (2.9) shows that the error estimate for a problem with arbitrarily rough coefficients is equal to the well-known error estimate for the Laplace equation.

Corollary 2.1. *Let X and V be finite-dimensional subspaces of $H^1_0(\Omega)$. For $f \in L^2(\Omega)$ let u be the solution of (1.1) with conductivity a. If* $(\operatorname{div} a\nabla V) = (\operatorname{div} \nabla X)$ *then*

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \sup_{w\in H^1_0(\Omega)\cap H^2(\Omega)} \inf_{v\in X} \frac{\|\nabla w - \nabla v\|_{(L^2(\Omega))^d}}{\|\Delta w\|_{L^2(\Omega)}} \tag{2.11}$$

Equation (2.11) can be obtained by setting $a' = I$ in theorem 2.1 and applying lemma 2.1.

Theorem 2.1 is obtained from the following proposition by noting that the right hand side of equation (2.12) is the same for pairs (a, V) and (a', V') whenever $\operatorname{div}(a\nabla V) = \operatorname{div}(a'\nabla V')$.

Proposition 2.2. *For $f \in L^2(\Omega)$ let u be the solution of (1.1). Then,*

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \sup_{z\in(\operatorname{div} a\nabla V)^\perp} \frac{\|z\|_{L^2(\Omega)}}{\|\nabla z\|_{(L^2(\Omega))^d}}, \tag{2.12}$$

where

$$(\operatorname{div} a\nabla V)^\perp := \{z \in H^1_0(\Omega) \,:\, \forall v \in V, (\nabla z, a\nabla v) = 0\}. \tag{2.13}$$

Proof. For $w \in H^2(\Omega)$, define

$$I := \inf_{v\in V} \|(\nabla w - a\nabla v)_{pot}\|_{(L^2(\Omega))^d}. \tag{2.14}$$

Observe that

$$I = \inf_{v\in V, \xi\in(L^2(\mathbb{R}^d))^d \,:\, \operatorname{div}(\xi)=0} \|\nabla w - a\nabla v - \xi\|_{(L^2(\Omega))^d}. \tag{2.15}$$

Additionally, observing that the space spanned by ∇z for $z \in (\operatorname{div} a\nabla V)^{\perp}$ is the orthogonal complement (in $(L^2(\Omega))^d$) of the space spanned by $a\nabla v + \xi$, we obtain that

$$I = \sup_{z \in (\operatorname{div} a\nabla V)^{\perp}} \frac{(\nabla w, \nabla z)}{\|\nabla z\|_{(L^2(\Omega))^d}}. \tag{2.16}$$

Integrating by parts and applying the Cauchy-Schwartz inequality yields

$$I \leq \|\Delta w\|_{L^2(\Omega)} \sup_{z \in (\operatorname{div} a\nabla V)^{\perp}} \frac{\|z\|_{L^2(\Omega)}}{\|\nabla z\|_{(L^2(\Omega))^d}}. \tag{2.17}$$

which proves

$$\sup_{f \in L^2(\Omega)} \inf_{v \in V} \frac{\|u - v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} \leq \sup_{z \in (\operatorname{div} a\nabla V)^{\perp}} \frac{\|z\|_{L^2(\Omega)}}{\|\nabla z\|_{(L^2(\Omega))^d}}, \tag{2.18}$$

Dividing by $\|\Delta w\|_{L^2(\Omega)}$, integrating by parts, and taking the supremum over $w \in H^2(\Omega) \cap H^1_0(\Omega)$, we get

$$\sup_{w \in H^2(\Omega) \cap H^1_0(\Omega)} \frac{I}{\|\Delta w\|_{L^2(\Omega)}} = \sup_{z \in (\operatorname{div} a\nabla V)^{\perp}} \sup_{w \in H^2(\Omega) \cap H^1_0(\Omega)} -\frac{(\Delta w, z)}{\|\nabla z\|_{(L^2(\Omega))^d} \|\Delta w\|_{L^2(\Omega)}}. \tag{2.19}$$

we conclude the theorem by choosing $-\Delta w = z$. □

The transfer property (2.9) for solutions can be complemented by an analogous property for fluxes. To this end, for a finite dimensional linear subspace $\mathcal{V} \subset (L^2(\Omega))^d$ define

$$(\operatorname{div} a\mathcal{V}) := \{\operatorname{div}(a\zeta) \,:\, \zeta \in \mathcal{V}\}. \tag{2.20}$$

Observe that $(\operatorname{div} a\mathcal{V})$ is a finite dimensional subspace of $H^{-1}(\Omega)$. The proof of the following theorem is similar to the proof of theorem 2.1.

Theorem 2.2. (Transfer property for fluxes) *Let $\mathcal{V}'$ and $\mathcal{V}$ be finite-dimensional subspaces of $(L^2(\Omega))^d$. For $f \in L^2(\Omega)$ let u be the solution of (1.1) with conductivity a and u' be the solution of (1.1) with conductivity a'. If $(\operatorname{div} a\mathcal{V}) = (\operatorname{div} a'\mathcal{V}')$ then*

$$\sup_{f \in L^2(\Omega)} \inf_{\zeta \in \mathcal{V}} \frac{\|(a(\nabla u - \zeta))_{pot}\|_{(L^2(\Omega))^d}}{\|f\|_{L^2(\Omega)}} = \sup_{f \in L^2(\Omega)} \inf_{\zeta \in \mathcal{V}'} \frac{\|(a'(\nabla u' - \zeta))_{pot}\|_{(L^2(\Omega))^d}}{\|f\|_{L^2(\Omega)}} \tag{2.21}$$

Corollary 2.2. *Let $\mathcal{V}$ be a finite-dimensional subspace of $(L^2(\Omega))^d$ and X a finite-dimensional subspace of $H^1_0(\Omega)$. For $f \in L^2(\Omega)$ let u be the solution of (1.1) with conductivity a. If $(\operatorname{div} a\mathcal{V}) = (\operatorname{div} \nabla X)$ then*

$$\sup_{f \in L^2(\Omega)} \inf_{\zeta \in \mathcal{V}} \frac{\|(a(\nabla u - \zeta))_{pot}\|_{(L^2(\Omega))^d}}{\|f\|_{L^2(\Omega)}} = \sup_{w \in H^1_0(\Omega) \cap H^2(\Omega)} \inf_{v \in X} \frac{\|\nabla w - \nabla v\|_{(L^2(\Omega))^d}}{\|\Delta w\|_{L^2(\Omega)}} \tag{2.22}$$

2.2 Vectorial case.

For $k \in (L^2(\Omega))^{d\times d}$, denote by k_{pot} the potential portion of the Weyl-Helmholtz decomposition of k (the orthogonal projection of k onto the closure of the space $\{\nabla f \,:\, f \in (C_0^\infty(\Omega))^d\}$ in $(L^2(\Omega))^{d\times d}$). Define

$$\|\psi\|_{C\text{-flux}} := \|(C : \varepsilon(\psi))_{pot}\|_{(L^2(\Omega))^{d\times d}}. \tag{2.23}$$

Proposition 2.3. *$\|.\|_{C\text{-flux}}$ is a norm on $(H_0^1(\Omega))^d$. Furthermore, for all $\psi \in (H_0^1(\Omega))^d$*

$$\lambda_{\min}(C)\|\varepsilon(\psi)\|_{(L^2(\Omega))^{d\times d}} \le \|\psi\|_{C\text{-flux}} \le \lambda_{\max}(C)\|\varepsilon(\psi)\|_{(L^2(\Omega))^{d\times d}}. \tag{2.24}$$

Proof. The proof of the left hand side of inequality (2.24) follows by observing that

$$\int_\Omega (\varepsilon(\psi))^T : C : \varepsilon(\psi) \le \|\varepsilon(\psi)\|_{(L^2(\Omega))^{d\times d}}\|\psi\|_{C\text{-flux}}. \tag{2.25}$$

The fact that $\|\psi\|_{C\text{-flux}}$ is a norm follows from the left hand side of inequality (2.24) and Korn's inequality [34]: i.e., for all $\psi \in (H_0^1(\Omega))^d$,

$$\|\nabla\psi\|_{(L^2(\Omega))^{d\times d}} \le \sqrt{2}\|\varepsilon(\psi)\|_{(L^2(\Omega))^{d\times d}}. \tag{2.26}$$

□

For V, a finite dimensional linear subspace of $(H_0^1(\Omega))^d$, we define

$$(\operatorname{div} C : \varepsilon(V)) := \operatorname{span}\{\operatorname{div}(C : \varepsilon(v)) \,:\, v \in V\}. \tag{2.27}$$

Observe that $(\operatorname{div} C : \varepsilon(V))$ is a finite dimensional subspace of $(H^{-1}(\Omega))^d$. Similarly for X, a finite dimensional linear subspace of $(H_0^1(\Omega))^d$, we define

$$\Delta X := \operatorname{span}\{\Delta v \,:\, v \in X\}. \tag{2.28}$$

Theorem 2.3. *Let V' and V be finite-dimensional subspaces of $(H_0^1(\Omega))^d$. For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3) with elasticity C and u' be the solution of (1.3) with elasticity C'. If $(\operatorname{div} C : \varepsilon(V)) = (\operatorname{div} C' : \varepsilon(V'))$ then*

$$\sup_{b\in(L^2(\Omega))^d} \inf_{v\in V} \frac{\|u-v\|_{C\text{-flux}}}{\|b\|_{(L^2(\Omega))^d}} = \sup_{b\in(L^2(\Omega))^d} \inf_{v\in V'} \frac{\|u'-v\|_{C'\text{-flux}}}{\|b\|_{(L^2(\Omega))^d}} \tag{2.29}$$

Corollary 2.3. *Let X and V be finite-dimensional subspaces of $(H_0^1(\Omega))^d$. For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3) with elasticity tensor C. If $(\operatorname{div} C : \varepsilon(V)) = \Delta X$ then*

$$\sup_{b\in(L^2(\Omega))^d} \inf_{v\in V} \frac{\|u-v\|_{C\text{-flux}}}{\|b\|_{(L^2(\Omega))^d}} = \sup_{w\in(H_0^1(\Omega)\cap H^2(\Omega))^d} \inf_{v\in X} \frac{\|\nabla w-\nabla v\|_{(L^2(\Omega))^{d\times d}}}{\|\Delta w\|_{(L^2(\Omega))^d}} \tag{2.30}$$

The proof of theorem 2.3 is analogous to the proof of theorem 2.1.

For $\mathcal{V}$ a finite dimensional linear subspace of $(L^2(\Omega))^{d\times d}$ we define

$$(\operatorname{div} C : \mathcal{V}) := \operatorname{span}\{\operatorname{div}(C : \zeta) \,:\, \zeta \in \mathcal{V}\}. \tag{2.31}$$

Observe that $(\operatorname{div} C : \mathcal{V})$ is a finite dimensional subspace of $(H^{-1}(\Omega))^d$. The proof of the following theorem is analogous to the proof of theorem 2.1.

Theorem 2.4. *Let $\mathcal{V}'$ and $\mathcal{V}$ be finite-dimensional subspaces of $(L^2(\Omega))^{d\times d}$. For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3) with conductivity C and u' be the solution of (1.3) with conductivity C'. If $(\operatorname{div} C : \mathcal{V}) = (\operatorname{div} C' : \mathcal{V}')$ then*

$$\sup_{b\in(L^2(\Omega))^d}\inf_{\zeta\in\mathcal{V}}\frac{\|(C:(\varepsilon(u)-\zeta))_{pot}\|_{(L^2(\Omega))^{d\times d}}}{\|b\|_{(L^2(\Omega))^d}} = \sup_{b\in(L^2(\Omega))^d}\inf_{\zeta\in\mathcal{V}'}\frac{\|(C':(\varepsilon(u)-\zeta))_{pot}\|_{(L^2(\Omega))^{d\times d}}}{\|b\|_{(L^2(\Omega))^d}} \tag{2.32}$$

Corollary 2.4. *Let $\mathcal{V}$ be a finite-dimensional subspace of $(L^2(\Omega))^{d\times d}$ and X a finite-dimensional subspace of $(H^1_0(\Omega))^d$. For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3) with elasticity C. If $(\operatorname{div} C : \mathcal{V}) = (\Delta X)$ then*

$$\sup_{b\in L^2(\Omega)}\inf_{\zeta\in\mathcal{V}}\frac{\|(C:(\varepsilon(u)-\zeta))_{pot}\|_{(L^2(\Omega))^{d\times d}}}{\|b\|_{(L^2(\Omega))^d}} = \sup_{w\in H^1_0(\Omega)\cap H^2(\Omega)}\inf_{v\in X}\frac{\|\nabla w-\nabla v\|_{(L^2(\Omega))^{d\times d}}}{\|\Delta w\|_{(L^2(\Omega))^d}} \tag{2.33}$$

3 Application to finite element methods with accuracy independent of material contrast.

In this section we will show how, as a very simple and straightforward application, the flux norm can be used to obtain finite dimensional approximation spaces for solutions of (1.1) and (1.3) with errors independent of the regularity and contrast of the coefficients and the regularity of $\partial\Omega$ (for the basis defined in subsection 3.1.2). A similar approximation problem can be found in the work of Melenk [40] where subsets of L^2 such as piecewise discontinuous polynomials have been used as an approximation basis (for the right hand side of (1.1)). The main difference between [40] and this section lies in the introduction of the flux-norm ($\|.\|_{a\text{-flux}}$), which plays a key role in our analysis, since it puts the approximation error of the space V_h on solutions of the operator $\operatorname{div}(a\nabla)$ in relation with the approximation error of the space V'_h on solutions of the operator $\operatorname{div}(a'\nabla)$ provided that $\operatorname{div}(a\nabla V_h) = \operatorname{div}(a'\nabla V'_h)$. Moreover this allows us to obtain an *explicit and optimal* constant in the rate of convergence (theorem 3.3 and 3.4). To our knowledge no explicit optimal error constant has been obtained. This question of optimal approximation with respect to a linear finite dimensional space is related to the Kolmogorov n-width [51] which measures how accurately a given set of functions can be approximated by linear spaces of dimension n in a given norm. A surprising result of the theory of n-widths is the non-uniqueness of the space realizing the optimal approximation [51]. A related work is also [9] in which errors in approximations to solutions of $\operatorname{div}(a\nabla u) = 0$ from linear spaces generated by a finite set of boundary conditions are analyzed as functions of the distance to the boundary (the penetration function).

3.1 Scalar divergence form equation

3.1.1 Approximation with piecewise linear nodal basis functions of a regular tessellation of Ω

In this subsection we will assume $\partial\Omega$ to be of class C^2. Let Ω_h be a regular tessellation of Ω of resolution h. Let $\mathcal{L}_0^h$ be the set of piecewise linear functions on Ω_h with Dirichlet boundary conditions. Denote by φ_k the piecewise linear nodal basis elements of $\mathcal{L}_0^h$, which are *localized*. Here we will express the error estimate in terms of h to emphasize the analogy with classical FEM (it could be expressed in terms of $N(h)$, see below if needed).

Let Φ_k be the functions associated with the piecewise linear nodal basis elements φ_k through the equation

$$\begin{cases} -\operatorname{div}\left(a(x)\nabla\Phi_k(x)\right) = \Delta\varphi_k & \text{in } \Omega \\ \Phi_k = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.1}$$

Define

$$V_h := \operatorname{span}\{\Phi_k\}, \tag{3.2}$$

Theorem 3.1. *For any $f \in L^2(\Omega)$, let u be the solution of (1.1). Then,*

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V_h} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} \leq Ch \tag{3.3}$$

where C depends only on Ω and the aspect ratios of the tetrahedra of Ω_h.

Proof. Theorem 3.1 is a straightforward application of the equation (2.11) and the fact that one can approximate H^2 functions by functions from $\mathcal{L}_0^h$ in the H^1 norm with $\mathcal{O}(h)$ accuracy (since $\partial\Omega$ is of class C^2 solutions of the Laplace-Dirichlet operator with L^2 right hand sides are in H^2). □

Corollary 3.1. *For $f \in L^2(\Omega)$, let u be the solution of (1.1) in $H_0^1(\Omega)$ and u_h the finite element solution of (1.1) in V_h. Then,*

$$\sup_{f\in L^2(\Omega)} \frac{\|u-u_h\|_{H_0^1(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq \frac{C}{\lambda_{\min}(a)} h \tag{3.4}$$

where C depends only on Ω and the aspect ratios of the tetrahedra of Ω_h.

Proof. Corollary 3.1 is a straightforward application of theorem 3.1 and inequality (2.2). □

Let Q be a symmetric, uniformly elliptic, *divergence-free* (as defined in section 4) matrix with entries in $L^\infty(\Omega)$. We note that this matrix will be chosen below so that the solutions of $\operatorname{div} Q\nabla u = f$ are in $H^2(\Omega)$ if $f \in L^2(\Omega)$ and therefore can be approximated by functions from $\mathcal{L}_0^h$ in H^1 norm with $O(h)$ accuracy. It follows from [10] this is not possible for the solutions of (1.1). In particular, in some cases Q can be chosen to be the identity.

Let Φ_k^Q be the functions associated with the piecewise linear nodal basis elements φ_k through the equation

$$\begin{cases} -\operatorname{div}\Big(a(x)\nabla\Phi_k^Q(x)\Big) = \operatorname{div}(Q\nabla\varphi_k) & \text{in } \Omega \\ \Phi_k^Q = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.5}$$

Define

$$V_h^Q := \operatorname{span}\{\Phi_k^Q\}, \tag{3.6}$$

Theorem 3.2. *For $f \in L^2(\Omega)$, let u be the solution of (1.1) in $H_0^1(\Omega)$ and u_h the finite element solution of (1.1) in V_h^Q. If Q satisfies one of the inequalities of theorem 4.1 or theorem 4.2 then*

$$\sup_{f\in L^2(\Omega)} \frac{\|u-u_h\|_{H_0^1(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq \frac{C}{\lambda_{\min}(a)} h \tag{3.7}$$

where C depends only on Ω and the aspect ratios of the tetrahedra of Ω_h.

Proof. The proof follows from the fact that if Q satisfies one of the inequalities of theorem 4.1 or theorem 4.2 then solutions of $-\operatorname{div}(Q\nabla u) = f$ with Dirichlet boundary conditions are in H^2. The rest of the proof is similar to that of the previous corollary. □

Remark 3.1. The basis elements Φ_k^Q are solutions of (3.5), which, in general, are globally supported in the entire domain Ω. In numerical implementations it is advantageous to have a locally supported basis. To this end we note that the support of $\operatorname{div}(Q\nabla\varphi_k)$ is localized to the support of φ_k, denoted by Ω_k. Then, if there exists an enlargement of Ω_k, which we denote by Ω_k', such that $\Phi_k^Q = 0$ and $n \cdot Q\nabla\Phi_k^Q = 0$ on $\partial\Omega_k'$ then $\Phi_k^Q = 0$ on $\Omega - \Omega_k'$, i.e. the support of Φ_k^Q is localized to Ω_k'. Given the sub-domains Ω_k', one can look at equations (3.5) as linear equations for two unknown functions Φ_k^Q, and Q under the constraints that $\Phi_k^Q = 0$ and $n \cdot Q\nabla\Phi_k^Q = 0$ on $\partial\Omega_k'$ and that Q is divergence-free. If these equations admit a solution under the constraint that Q is uniformly elliptic (a quadratic condition), then the support of elements Φ_k^Q are localized to sub-domains Ω_k'.

3.1.2 Approximation with eigenfunctions of the Laplace-Dirichlet operator.

In this sub-section we assume the minimal regularity condition on the boundary $\partial\Omega$ such that the Weyl formula holds (we refer to [43] and references therein).

Denote by Ψ_k the eigenfunctions associated with the Laplace-Dirichlet operator in Ω and λ_k the associated eigenvalues–i.e., for $k \in \mathbb{N}^* = \{1, 2, \cdots\}$

$$\begin{cases} -\Delta\Psi_k = \lambda_k\Psi_k \quad x \in \Omega \\ \Psi_k = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.8}$$

We assume that the eigenvalues are ordered–i.e., $\lambda_k \leq \lambda_{k+1}$.

Let θ_k be the functions associated with the Laplace-Dirichlet eigenfunctions Ψ_k ((3.8)) through the equation

$$\begin{cases} -\operatorname{div}\left(a(x)\nabla\theta_k(x)\right) = \lambda_k \Psi_k & \text{in } \Omega \\ \theta_k = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.9}$$

Here, λ_k is introduced on the right hand side of (3.9) in order to normalize θ_k ($\theta_k = \Psi_k$, if $a(x) = I$) and can be otherwise ignored since only the span of $\{\theta_k\}$ matters. Define

$$\Theta_h := \operatorname{span}\{\theta_1, \ldots, \theta_{N(h)}\}, \tag{3.10}$$

where $N(h)$ is the integer part of $|\Omega|/h^d$. The motivation behind our definition of Θ_h is that its dimension corresponds to the number of degrees of freedom of piecewise linear functions on a regular triangulation (tessellation) of Ω of resolution h.

Theorem 3.3. *For $f \in L^2(\Omega)$, let u be the solution of (1.1). Then,*

$$\lim_{h\to 0} \sup_{f\in L^2(\Omega)} \inf_{v\in\Theta_h} \frac{\|u-v\|_{a\text{-flux}}}{h\|f\|_{L^2(\Omega)}} = \frac{1}{2\sqrt{\pi}}\Big(\frac{1}{\Gamma(1+\frac{d}{2})}\Big)^{\frac{1}{d}}. \tag{3.11}$$

Furthermore the space Θ_h leads (asymptotically as $h \to 0$) to the smallest possible constant in the right hand side of (3.11) among all subspaces of $H^1_0(\Omega)$ with $N(h)$, the integer part of $|\Omega|/h^d$, elements.

Remark 3.2. The constant in the right hand side of (3.11) is the classical Kolmogorov n-width $d_n(A, X)$ understood in the asymptotic sense as $h \to 0$ (because the Weyl formula is asymtotic). Recall that the n-width measures how accurately a given set of functions $A \subset X$ can be approximated by linear spaces E_n of dimension n

$$d_n(A, X) = \inf_{E_n} \sup_{w\in A} \inf_{g\in E_n} \|w-g\|_X$$

for a normed linear space X. In our case $X = H^1_0(\Omega)$, A being the set of all solutions of (1.1) as f spans $L^2(\Omega)$ for a given $a(x)$ and Ω. It should be observed there is a slight difference with classical Kolmogorov n-width, indeed the flux norm $\|.\|_{a\text{-flux}}$ used in (3.11) depends on a (as opposed to the $H^1_0(\Omega)$-norm). A surprising result of the theory of n-widths [51] is that the space realizing the optimal approximation is not unique, therefore there may be subspaces, other than Θ_h, providing the same asymptotic constant.

Remark 3.3. Whereas the constant in (3.11) depends only on the dimension d, the estimate for finite h given by (3.3) depends explicitly on the aspect ratios of the tetrahedra of Ω_h (the uniform bound on the ratio between the outer and inner radii of those tetrahedra). A non-asymptotic version of error estimate (3.11) is given by the equation (3.21).

Proof. Let V_h be a subspace of $H^1_0(\Omega)$ with $[|\Omega|/h^d]$ elements. Let v_k be a basis of V_h.

Let v'_k be the functions associated with the basis elements v_k through the equation

$$\begin{cases} \Delta v'_k = -\operatorname{div}(a(x)\nabla v_k(x)) & \text{in } \Omega \\ v'_k = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.12}$$

It follows from equation (2.11) of theorem 2.1 that the following *transfer equation* holds

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V_h} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \sup_{w\in H^2\cap H^1_0(\Omega)} \inf_{v'\in V'_h} \frac{\|\nabla w - \nabla v'\|_{(L^2(\Omega))^d}}{\|\Delta w\|_{L^2(\Omega)}} \tag{3.13}$$

where for $f \in L^2(\Omega)$, u is the solution of (1.1). Using the eigenfunctions Ψ_k of the Laplace-Dirichlet operator, we arrive at

$$\frac{\|\nabla w - \nabla v'\|^2_{(L^2(\Omega))^d}}{\|\Delta w\|_{L^2(\Omega)}2} = \frac{\sum_{k=1}^{\infty} \frac{1}{\lambda_k}(\Delta w - \Delta v', \Psi_k)^2}{\sum_{k=1}^{\infty}(\Delta w, \Psi_k)^2} \tag{3.14}$$

When the supremum is taken with respect to $w \in H^2 \cap H^1_0(\Omega)$, the right hand side of (3.14) can be minimized by taking V'_h to be the linear span of the first $[|\Omega|/h^d]$ eigenfunctions of the Laplace-Dirichlet operator on Ω, because with such a basis the first $N(h)$ coefficients of Δw are canceled, i.e.

$$\inf_{v'\in V'_h} \frac{\|\nabla w - \nabla v'\|^2_{(L^2(\Omega))^d}}{\|\Delta w\|_{L^2(\Omega)}2} = \frac{\sum_{k=N(h)+1}^{\infty} \frac{1}{\lambda_k}(\Delta w, \Psi_k)^2}{\sum_{k=1}^{\infty}(\Delta w, \Psi_k)^2} \tag{3.15}$$

with $N(h) = [|\Omega|/h^d]$. Then

$$\inf_{V'_h,\, dim(V'_h)=N(h)} \sup_{w\in H^2\cap H^1_0(\Omega)} \inf_{v'\in V'_h} \frac{\|\nabla w - \nabla v'\|^2_{(L^2(\Omega))^d}}{\|\Delta w\|^2_{L^2(\Omega)}} = \frac{1}{\lambda_{N(h)+1}}. \tag{3.16}$$

This follows by noting that the right hand side of equation (3.15) is less than or equal to $\frac{1}{\lambda_{N(h)+1}}$ and that equality is obtained for $w = \Psi_{N(h)+1}$.

The optimality of the constant in V'_h translates into the optimality of the constant associated with V_h using the transfer equation (3.13), i.e.

$$\inf_{V_h,\, dim(V_h)=N(h)} \sup_{f\in L^2(\Omega)} \inf_{v\in V_h} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \frac{1}{\lambda_{N(h)+1}} \tag{3.17}$$

We obtain the constant in (3.11) by using Weyl's asymptotic formula for the eigenvalues of the Laplace-Dirichlet operator on Ω [56].

$$\lambda_k \sim 4\pi\Big(\frac{\Gamma(1+\frac{d}{2})k}{|\Omega|}\Big)^{\frac{2}{d}}, \tag{3.18}$$

In equation (3.18) $|\Omega|$ is the volume of Ω, d is the dimension of the physical space and Γ is the Gamma function defined by $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt$. It follows from equation (3.12) that by defining $V_h = \Theta_h$ one obtain the smallest asymptotic constant in the right hand side of (3.11). This being said, it should be recalled that the space Θ_h is not the unique space achieving this optimal constant [51].

For the sake of clarity, an alternate (but similar) proof is provided below. By proposition 2.2

$$\sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \sup_{z\in(\operatorname{div} a\nabla V)^\perp} \frac{\|z\|_{L^2(\Omega)}}{\|\nabla z\|_{(L^2(\Omega))^d}}, \tag{3.19}$$

Taking *inf* of both sides, we have

$$\inf_{V,\dim(V)=N(h)} \sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \inf_{V_h,\dim(V_h)=N(h)} \sup_{z\in(\operatorname{div} a\nabla V)^\perp} \frac{\|z\|_{L^2(\Omega)}}{\|\nabla z\|_{(L^2(\Omega))^d}}, \tag{3.20}$$

Notice that the right hand side is the inverse of Rayleigh quotient, and $(\operatorname{div} a\nabla V)^\perp$ is a co-dimension $N(h)$ space, then by Courant-Fischer min-max principle for the eigenvalues, we have

$$\inf_{V,\dim(V)=N(h)} \sup_{f\in L^2(\Omega)} \inf_{v\in V} \frac{\|u-v\|_{a\text{-flux}}}{\|f\|_{L^2(\Omega)}} = \frac{1}{\lambda_{N(h)+1}} \tag{3.21}$$

Taking V to be Θ_h, then the optimal constant can be achieved asymptotically as $h \to 0$. □

Remark 3.4. Theorem 3.3 is related to Melenk's n-widths analysis for elliptic problems [40] where subsets of L^2 such as piecewise discontinuous polynomials have been used as an approximation basis. The main difference between [40] and this section lies in the introduction of and the emphasis on the flux-norm ($\|.\|_{a\text{-flux}}$) with respect to which errors become independent of the contrast of the coefficients and the regularity of a. Moreover this allows us to obtain an *explicit* and optimal constant in the rate of convergence.

Remark 3.5. The space Θ_h also satisfies, for $\nu \in [0,1)$.

$$\lim_{h\to 0} \sup_{g\in H^{-\nu}(\Omega)} \inf_{v\in\Theta_h} \frac{\|u-v\|_{a\text{-flux}}}{h^{1-\nu}\|g\|_{H^{-\nu}(\Omega)}} = \Big(\frac{1}{2\sqrt{\pi}}\Big(\frac{1}{\Gamma(1+\frac{d}{2})}\Big)^{\frac{1}{d}}\Big)^{1-\nu}. \tag{3.22}$$

3.2 Vectorial elasticity equations.

Let $(e_1,\ldots,e_d)$ be an orthonormal basis of $\mathbb{R}^d$. For $j \in \{1,\ldots,d\}$ and $k \in \mathbb{N}^* = \{1,2,\cdots\}$, let τ_k^j be the solution of

$$\begin{cases} -\operatorname{div}\Big(C:\varepsilon(\tau_k^j)\Big) = e_j\lambda_k\Psi_k, & \text{in } \Omega, \\ \tau_k^j = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.23}$$

where Ψ_k are the eigenfunctions (3.8) of the scalar Laplace-Dirichlet operator in Ω. Let $M := \big[|\Omega|/h^d\big]$ be the integer part of $|\Omega|/h^d$ and T_h be the linear space spanned by τ_k^j for $k \in \{1,\ldots,M\}$ and $j \in \{1,\ldots,d\}$.

Theorem 3.4. *For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3). Then,*

$$\lim_{h\to 0} \sup_{b\in(L^2(\Omega))^d} \inf_{v\in T_h} \frac{\|u-v\|_{C\text{-flux}}}{h\|b\|_{(L^2(\Omega))^d}} = \frac{1}{2\sqrt{\pi}}\Big(\frac{1}{\Gamma(1+\frac{d}{2})}\Big)^{\frac{1}{d}}. \tag{3.24}$$

Furthermore the space T_h leads (asymptotically) to the smallest possible constant in the right hand side of (3.24) among all subspaces of $H^1_0(\Omega)$ with $O(|\Omega/h^d|)$ elements.

Proof. Theorem 3.4 is a straightforward application of equation (2.30) of theorem 2.3 and Weyl's estimate (3.18) (the proof is similar to the scalar case). □

Defining φ_k as in subsection 3.1.1, for $j \in \{1,\dots,d\}$ let Φ^j_k be the solution of

$$\begin{cases} -\operatorname{div}\Big(C(x):\varepsilon(\Phi^j_k)\Big) = e_j\Delta\varphi_k, & \text{in } \Omega,\\ \Phi^j_k = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.25}$$

Define

$$W_h := \operatorname{span}\{\Phi^j_k\}, \tag{3.26}$$

Theorem 3.5. *For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3). Then,*

$$\sup_{b\in(L^2(\Omega))^d} \inf_{v\in W_h} \frac{\|u-v\|_{C\text{-flux}}}{\|b\|_{(L^2(\Omega))^d}} \leq Kh. \tag{3.27}$$

where K depends only on Ω and the aspect ratios of the tetrahedra of Ω_h.

Proof. Theorem 3.5 is a straightforward application of equation (2.30) and the fact that one can approximate H^2 functions in the H^1 norm by functions from $\mathcal{L}^h_0$ with $\mathcal{O}(h)$ accuracy. □

Corollary 3.2. *For $b \in (L^2(\Omega))^d$ let u be the solution of (1.3) and u_h the finite element solution of (1.3) in W_h. Then,*

$$\sup_{b\in(L^2(\Omega))^d} \inf_{v\in W_h} \frac{\|u-u_h\|_{(H^1_0(\Omega))^d}}{\|b\|_{(L^2(\Omega))^d}} \leq \frac{K}{\lambda_{\min}(C)}h. \tag{3.28}$$

where K depends only on Ω and the aspect ratios of the tetrahedra of Ω_h.

Proof. Corollary 3.2 is a straightforward application of theorem 3.5, inequality (2.24) and Korn's inequality (2.26). □

4 A new class of inequalities

The flux-norm (and harmonic coordinates in the scalar case [48]) can be used to map a given operator $\operatorname{div}(a\nabla)$ ($\operatorname{div}(C:\varepsilon(u)$ for elasticity)) onto another operator $\operatorname{div}(a'\nabla)$ ($\operatorname{div}(C':\varepsilon(u))$). Among all elliptic operators those with divergence-free coefficients (as defined below) play a very special role in the sense that they can be written in both a divergence-form and a non-divergence form operator. We introduce a new class of inequalities for these operators. We show that these inequalities hold under Cordes type conditions on coefficients and conjecture that they hold without these conditions.

These inequalities will be required to hold only for divergence-free conductivities because, by using the flux-norm through the transfer property defined in section 2 or harmonic coordinates as in [48] (for the scalar case), we can map non-divergence free conductivities onto divergence-free conductivities and hence deduce homogenization results on the former from inequalities on the latter.

4.1 Scalar case.

Let a be the conductivity matrix associated with equation (1.1). In this subsection, we will assume that a is uniformly elliptic, with bounded entries and divergence free–i.e., for all $l\in\mathbb{R}^d$, $\operatorname{div}(a.l)=0$ (that is each column of a is div free); alternatively, for all $\varphi\in C_0^\infty(\Omega)$

$$\int_\Omega \nabla\varphi.a.l=0. \tag{4.1}$$

Assume that Ω is a bounded domain in $\mathbb{R}^d$. For a $d\times d$ matrix M, define

$$\text{Hess}:M:=\sum_{i,j=1}^{d}\partial_i\partial_j M_{i,j}. \tag{4.2}$$

We will also denote by $\Delta^{-1}M$ the $d\times d$ matrix defined by

$$(\Delta^{-1}M)_{i,j}=\Delta^{-1}M_{i,j}. \tag{4.3}$$

Theorem 4.1. *Let a be a divergence free conductivity matrix. Then the following statements are equivalent for the same constant C:*

- *There exists $C>0$ such that for all $u\in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)}\leq C\big\|\Delta^{-1}\operatorname{div}(a\nabla u)\big\|_{L^2(\Omega)}. \tag{4.4}$$

- *There exists $C>0$ such that for all $u\in H_0^1(\Omega)$,*

$$\big\|(\operatorname{div}(a\nabla))^{-1}\Delta u\big\|_{L^2(\Omega)}\leq C\big\|u\big\|_{L^2(\Omega)}. \tag{4.5}$$

- *Writing θ_i the solutions of (3.9). For all $(U_1, U_2, \dots) \in \mathbb{R}^{\mathbb{N}^*}$,*

$$\| \sum_{i=1}^{\infty} U_i \theta_i \|_{L^2(\Omega)}^2 \leq C^2 \sum_{i=1}^{\infty} U_i^2. \tag{4.6}$$

- *The inverse of the operator $-\operatorname{div}(a\nabla)$ (with Dirichlet boundary conditions) is a continuous and bounded operators from H^{-2} onto L^2. Moreover, for $u \in H^{-2}(\Omega)$,*

$$\big\|(\operatorname{div} a\nabla)^{-1} u\big\|_{L^2(\Omega)} \leq C \|\Delta^{-1} u\|_{L^2(\Omega)}. \tag{4.7}$$

- *There exists $C > 0$ such that for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)}^2 \leq C^2 \sum_{i=1}^{\infty} \Big\langle \operatorname{div}(a\nabla \frac{\Psi_i}{\lambda_i}), u \Big\rangle_{H^{-1}, H^1}^2. \tag{4.8}$$

- *There exists $C > 0$ such that*

$$\frac{1}{C} \leq \inf_{u \in H_0^1(\Omega)} \sup_{z \in H^2(\Omega) \cap H_0^1(\Omega)} \frac{(\nabla z, a \nabla u)_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)} \|\Delta z\|_{L^2(\Omega)}}. \tag{4.9}$$

- *There exists $C > 0$ such that for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C \big\|\Delta^{-1} \operatorname{Hess} : (au)\big\|_{L^2(\Omega)}. \tag{4.10}$$

- *There exists $C > 0$ such that for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C \big\| \operatorname{Hess} : (\Delta^{-1}(au))\big\|_{L^2(\Omega)}. \tag{4.11}$$

Remark 4.1. Theorem 4.1 can be related to the work of Conca and Vanninathan [22], on uniform H^2-estimates in periodic homogenization, which established a similar result in the periodic homogenization setting.

Proof. Let $U_j \in \mathbb{R}$. Observe that

$$-\operatorname{div}\big(a\nabla \sum_{j=1}^{\infty} \theta_j U_j\big) = \sum_{j=1}^{\infty} \Psi_j \lambda_j U_j, \tag{4.12}$$

hence

$$-\Delta^{-1} \operatorname{div}\big(a\nabla \sum_{j=1}^{\infty} \theta_j U_j\big) = \sum_{j=1}^{\infty} \Psi_j U_j. \tag{4.13}$$

Identifying u with $\sum_{j=1}^{\infty} \theta_j U_j$, it follows that

$$\inf_{u \in L^2(\Omega)} \frac{\big\|\Delta^{-1} \operatorname{div}(a\nabla u)\big\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \geq \frac{1}{C} \tag{4.14}$$

is equivalent to

$$\|\sum_{j=1}^{\infty}\theta_j U_j\|_{L^2(\Omega)}^2 \leq C^2 U^T U. \tag{4.15}$$

Observe that equation (4.4) is also equivalent to

$$\big\|(\operatorname{div} a\nabla)^{-1}u\big\|_{L^2(\Omega)} \leq C\|\Delta^{-1}u\|_{L^2(\Omega)}, \tag{4.16}$$

which is equivalent to the fact that the inverse of the operator $-\operatorname{div}(a\nabla)$ (with Dirichlet boundary conditions) is a continuous and bounded operator from H^{-2} onto L^2. Finally, the equivalence with (4.8) is a consequence of equation (4.6). Let us now prove the equivalence with equations (4.10) and (4.11). Observe that if a is a divergence free $d\times d$ symmetric matrix and $u\in H^2(\Omega)\cap H^1_0(\Omega)$, then

$$\operatorname{div}(a\nabla u) = \text{Hess} : (au), \tag{4.17}$$

since

$$\text{Hess} : (au) = \sum_{i,j=1}^{d} a_{i,j}\partial_i\partial_j u + \sum_{j=1}^{d}\sum_{i=1}^{d}\partial_i a_{i,j}\partial_j u + \sum_{i=1}^{d}\sum_{j=1}^{d}\partial_j a_{i,j}\partial_i u, \tag{4.18}$$

$\sum_{i=1}^{d}\partial_i a_{i,j}=0$ and $\sum_{j=1}^{d}\partial_j a_{i,j}=0$. It follows that

$$\Delta^{-1}\operatorname{div}(a\nabla u) = \Delta^{-1}\,\text{Hess} : (au) = \text{Hess} : \Delta^{-1}(au), \tag{4.19}$$

which concludes the proof of the equivalence between the statements. □

Theorem 4.2. *If a is divergence-free, then the statements of theorem 4.1 are implied by the following equivalent statements with the same constant C.*

- *For all $u\in H^1_0(\Omega)\cap H^2(\Omega)$,*

$$\|\Delta u\|_{L^2(\Omega)} \leq C\|a : \text{Hess}(u)\|_{L^2(\Omega)}. \tag{4.20}$$

- *There exists $C>0$ such that for $u\in C_0^{\infty}(\Omega)$,*

$$\|k^2\mathcal{F}(u)\|_{L^2(\Omega)} \leq C\|k^T.\mathcal{F}(au).k\|_{L^2}, \tag{4.21}$$

  where $\mathcal{F}(u)$ is the Fourier transform of u.

Proof. Equation (4.10) is equivalent to

$$\frac{1}{C} \leq \inf_{u\in H^1_0(\Omega)}\sup_{\varphi\in L^2(\Omega)}\frac{\big(\varphi,\Delta^{-1}\,\text{Hess} : (au)\big)_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}}. \tag{4.22}$$

Denoting by ψ the solution of $\Delta\psi=\varphi$ in $H^1_0(\Omega)\cap H^2(\Omega)$, we obtain that (4.22) is equivalent to

$$\frac{1}{C} \leq \inf_{u\in H_0^1(\Omega)} \sup_{\psi\in H_0^1(\Omega)\cap H^2(\Omega)} \frac{\big(\psi, \text{Hess} : (au)\big)_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}\|\Delta\psi\|_{L^2(\Omega)}}. \tag{4.23}$$

Integrating by parts, we obtain that (4.23) is equivalent to

$$\frac{1}{C} \leq \inf_{u\in H_0^1(\Omega)} \sup_{\psi\in H_0^1(\Omega)\cap H^2(\Omega)} \frac{\big(a : \text{Hess}(\psi), u\big)_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}\|\Delta\psi\|_{L^2(\Omega)}}. \tag{4.24}$$

Since a is divergence free, $a : \text{Hess} = \text{div}(a\nabla .)$ and so there exists ψ such that $a : \text{Hess}(\psi) = u$ with Dirichlet boundary conditions. For such a ψ, we have

$$\frac{\big(a : \text{Hess}(\psi), u\big)_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}\|\Delta\psi\|_{L^2(\Omega)}} = \frac{\|a : \text{Hess}(\psi)\|_{L^2(\Omega)}}{\|\Delta\psi\|_{L^2(\Omega)}}. \tag{4.25}$$

It follows that inequality (4.24) is implied by the inequality

$$\frac{1}{C} \leq \inf_{\psi\in H_0^1(\Omega)\cap H^2(\Omega)} \frac{\|a : \text{Hess}(\psi)\|_{L^2(\Omega)}}{\|\Delta\psi\|_{L^2(\Omega)}}. \tag{4.26}$$

The equivalence with (4.21) follows from $a : \text{Hess}(u) = \text{Hess} : (au)$ and the conservation of the L^2-norm by the Fourier transform. □

Theorem 4.3. *Let a be a divergence free conductivity matrix.*

- *If $d = 1$, then the statements of theorem 4.2 are true.*
- *If $d = 2$ and Ω is convex then the statements of theorem 4.2 are true.*
- *If $d \geq 3$, Ω is convex and the following Cordes condition is satisfied*

  $$\text{esssup}_{x\in\Omega}\Big(d - \frac{\big(\text{Trace}[a(x)]\big)^2}{\text{Trace}[a^T(x)a(x)]}\Big) < 1 \tag{4.27}$$

  then the the statements of theorem 4.2 are true.
- *If $d \geq 2$, Ω is non convex then there exists $C_\Omega > 0$ such that if the following Cordes condition is satisfied*

  $$\text{esssup}_{x\in\Omega}\Big(d - \frac{\big(\text{Trace}[a(x)]\big)^2}{\text{Trace}[a^T(x)a(x)]}\Big) < C_\Omega \tag{4.28}$$

  then the the statements of theorem 4.2 are true.

Proof. In dimension one, if a is divergence free then it is a constant and the statements of theorem 4.2 are trivially true. Define

$$\beta_a := \mathrm{esssup}_{x\in\Omega}\Big(d - \frac{\big(\mathrm{Trace}[a(x)]\big)^2}{\mathrm{Trace}[a^T(x)a(x)]}\Big) \tag{4.29}$$

Theorem 1.2.1 of [39] implies that if Ω is convex and $\beta_a < 1$ then inequality (4.20) is true. In dimension 2, if a is uniformly elliptic and bounded then $\beta_a < 1$. It follows that if $d = 2$ and Ω is convex or if $d \geq 3$, Ω is convex, and $\beta_a < 1$ then the statements of theorem 4.2 are true. The last statement of theorem 4.3 is a direct consequence of corollary 4.1 of [36].

For the sake of completeness we will include the proof of three bullet points here (Ω convex). Write $\mathcal{L}$ the differential operator from $H^2(\Omega)$ onto $L^2(\Omega)$ defined by:

$$\mathcal{L}u := \sum_{i,j} a_{ij}\partial_i\partial_j u \tag{4.30}$$

Let us consider the equation

$$\begin{cases} \mathcal{L}u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.31}$$

The following lemma corresponds to theorem 1.2.1 of [39] (and a does not need to be divergence free for the validity of the following theorem). For the convenience of the reader, we will recall its proof in subsection 7.1 of the appendix.

Lemma 4.1. *Assume Ω to be convex with C^2-boundary. If $\beta_a < 1$ then (4.31) has a unique solution and*

$$\|u\|_{H^2\cap H^1_0(\Omega)} \leq \frac{\mathrm{esssup}_\Omega \alpha(x)}{1-\sqrt{\beta_a}}\|f\|_{L^2(\Omega)} \tag{4.32}$$

where $\alpha(x) := (\Sigma_{i=1}^d a_{ii}(x))/\sum_{i,j=1}^d (a_{ij}(x))^2$

β_a is a measure of the anisotropy of a. In particular, for the identity matrix one has $\beta_{I_d} = 0$. Furthermore in dimension 2

$$\beta_a = 1 - \mathrm{essinf}_{x\in\Omega}\frac{2\lambda_{\min}(a(x))\lambda_{\max}(a(x))}{(\lambda_{\min}(a(x)))^2 + (\lambda_{\max}(a(x)))^2} \tag{4.33}$$

and one always have $\beta_a < 1$ provided that a is uniformly elliptic and bounded. The first three bullet points of theorem 4.3 follow by observing that if $\beta_a < 1$ then

$$\|u\|_{H^2\cap H^1_0(\Omega)} \leq C\|\sum_{i,j} a_{ij}\partial_i\partial_j u\|_{L^2(\Omega)} \tag{4.34}$$

which implies inequality (4.20).

□

4.1.1 A brief reminder on the mapping using harmonic coordinates.

Consider the divergence-form elliptic scalar problem (1.1). Let F denote the harmonic coordinates associated with (1.1)–i.e., $F(x) = \big(F_1(x), \ldots, F_d(x)\big)$ is a d-dimensional vector field whose entries satisfy

$$\begin{cases} \operatorname{div} a\nabla F_i = 0 \quad \text{in} \quad \Omega \\ F_i(x) = x_i \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.35}$$

It is easy to show that F is a mapping from Ω onto Ω. In dimension one, F is trivially a homeomorphism. In dimension two this property still holds for convex domains [1, 4]. In dimensions three and higher, F may be non-injective (even if a is smooth, we refer to [4], [19]).

Define Q to be the positive symmetric $d \times d$ matrix defined by

$$Q := \frac{(\nabla F)^T a \nabla F}{\det \nabla F} \circ F^{-1}. \tag{4.36}$$

It is shown in [48] that Q is divergence free. Moreover, writing $\|u\|_a := \int_\Omega \nabla u \cdot a \nabla u$ one has for $v \in H^1_0(\Omega)$

$$\|u - v\|_a = \|\hat{u} - \hat{v}\|_Q, \tag{4.37}$$

where $\hat{v} := v \circ F^{-1}$ and $\hat{u} := u \circ F^{-1}$ solves

$$-\sum_{i,j} Q_{i,j} \partial_i \partial_j \hat{u} = \frac{g}{\det(\nabla F)} \circ F^{-1} \tag{4.38}$$

Note that (4.37) allows one to transfer the error for a general conductivity matrix a to a special divergent-free conductivity matrix Q. The approximation results obtained in [48] are based on (4.37) and can also be derived by using the new class of inequalities described above for Q.

4.2 Tensorial case.

Let C be the elastic stiffness matrix associated with equation (1.3). In this subsection, we will assume that C is uniformly elliptic, has bounded entries and is divergence free–i.e., C is such that for all $l \in \mathbb{R}^{d\times d}$, $\operatorname{div}(C : l) = 0$; alternatively, for all $\varphi \in (C^\infty_0(\Omega))^d$,

$$\int_\Omega (\nabla \varphi)^T : C : l = 0. \tag{4.39}$$

The inequalities given below will allow us to deduce homogenization results for arbitrary elasticity tensors (not necessarily divergence-free) by using harmonic displacements and the flux-norm to map non-divergence free tensors onto divergence-free tensors.

For a $d \times d \times d$ tensor M, denote by Hess : M the vector

$$(\text{Hess} : M)_k := \sum_{i,j=1}^{d} \partial_i \partial_j M_{i,j,k}. \tag{4.40}$$

Let $\Delta^{-1}M$ denote the $d \times d \times d$ tensor defined by

$$(\Delta^{-1}M)_{i,j,k} = \Delta^{-1}M_{i,j,k}. \tag{4.41}$$

The proof of the following theorem is almost identical to the proof of theorem 4.1.

Theorem 4.4. *Let C be a divergence free elasticity tensor. The following statements are equivalent for the same constant γ:*

- *There exists $\gamma > 0$ such that for all $u \in (H_0^1(\Omega))^d$,*

$$\|u\|_{(L^2(\Omega))^d} \leq \gamma \big\|\Delta^{-1} \operatorname{div}(C : \varepsilon(u))\big\|_{(L^2(\Omega))^d}. \tag{4.42}$$

- *There exists $\gamma > 0$ such that for all $u \in (H_0^1(\Omega))^d$,*

$$\big\|(\operatorname{div}(C : \varepsilon(.)))^{-1}\Delta u\big\|_{(L^2(\Omega))^d} \leq \gamma \big\|u\big\|_{(L^2(\Omega))^d}. \tag{4.43}$$

- *For all $(U_1, U_2, \ldots) \in (\mathbb{R}^d)^{\mathbb{N}^*}$,*

$$\|\sum_{k=1}^{\infty}\sum_{j=1}^{d} U_k^j \tau_k^j\|_{L^2(\Omega)}^2 \leq \gamma^2 \sum_{k=1}^{\infty} U_k^2, \tag{4.44}$$

  where τ_k^j is the superior basis defined in 3.23.

- *The inverse of the operator $-\operatorname{div}(C : \varepsilon(.))$ (with Dirichlet boundary conditions) is a continuous and bounded operator from $(H^{-2})^d$ onto $(L^2)^d$. Moreover, for $u \in (H^{-2}(\Omega))^d$,*

$$\big\|(\operatorname{div} C : \varepsilon(.))^{-1}u\big\|_{(L^2(\Omega))^d} \leq \gamma \|\Delta^{-1}u\|_{(L^2(\Omega))^d}. \tag{4.45}$$

- *There exists $\gamma > 0$ such that for all $u \in (H_0^1(\Omega))^d$,*

$$\|u\|_{(L^2(\Omega))^d}^2 \leq \gamma^2 \sum_{i=1}^{\infty}\sum_{j=1}^{d} \left\langle (\operatorname{div}\big(C : (\frac{\nabla \Psi_i}{\lambda_i} \otimes e_j)\big), u \right\rangle_{(H^{-1},H^1)}^2. \tag{4.46}$$

- *There exists $\gamma > 0$ such that*

$$\frac{1}{\gamma} \leq \inf_{u \in (H_0^1(\Omega))^d} \sup_{z \in (H^2(\Omega) \cap H_0^1(\Omega))^d} \frac{((\nabla z)^T : C : \varepsilon(u))_{L^2(\Omega)}}{\|u\|_{(L^2(\Omega))^d}\|\Delta z\|_{(L^2(\Omega))^d}}. \tag{4.47}$$

- *There exists $\gamma > 0$ such that for all $u \in (H_0^1(\Omega))^d$,*

$$\|u\|_{(L^2(\Omega))^d} \leq \gamma \big\|\Delta^{-1} \operatorname{Hess} : (u.C)\big\|_{L^2(\Omega)}. \tag{4.48}$$

- *There exists $\gamma > 0$ such that for all $u \in (H_0^1(\Omega))^d$,*

$$\|u\|_{(L^2(\Omega))^d} \leq \gamma \big\| \operatorname{Hess} : (\Delta^{-1}(u.C)) \big\|_{(L^2(\Omega))^d}. \tag{4.49}$$

Theorem 4.5. *If C is divergence-free, the statements of theorem 4.4 are implied by the following statement with the same constant γ.*

- *For all $u \in (H_0^1(\Omega) \cap H^2(\Omega))^d$,*

$$\|\Delta u\|_{(L^2(\Omega))^d} \leq \gamma \| \operatorname{Hess} : (u.C) \|_{(L^2(\Omega))^d}. \tag{4.50}$$

Proof. The proof is similar to that of theorem 4.2. □

4.2.1 A Cordes Condition for tensorial non-divergence form elliptic equations

Let us now show that the inequality in theorem 4.5, and hence the inequalities of theorem 4.4, are satisfied if C satisfies a Cordes type condition. The proof of the following theorem is an adaptation of the proof of theorem 1.2.1 of [39] (note that C does not need to be divergence free in order for the the following theorem to be valid).

Let $\mathcal{L}$ denote the differential operator from $(H^2(\Omega)^d$ onto $(L^2(\Omega))^d$ defined by

$$(\mathcal{L}u)_j := \sum_{i,k,l} C_{ijkl} \partial_i \partial_k u_l. \tag{4.51}$$

Let us consider the equation

$$\begin{cases} \mathcal{L}u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.52}$$

Let B be the $d \times d$ matrix defined by $B_{jm} = \sum_{k=1}^d C_{kmkj}$. Let A be the $d \times d$ matrix defined by $A_{j'm} = \sum_{i,k,l=1}^d C_{imkl} C_{ij'kl}$. Define

$$\beta_C := d^2 - \operatorname{Trace}[BA^{-1}B^T]. \tag{4.53}$$

Theorem 4.6. *Assume Ω is convex with a C^2-boundary. If $\beta_C < 1$, then (4.52) has a unique solution and*

$$\|u\|_{(H^2 \cap H_0^1(\Omega))^d} \leq K \|f\|_{(L^2(\Omega))^d}, \tag{4.54}$$

where K is a function of β_C and $\|BA^{-1}\|_{(L^\infty(\Omega))^{d\times d}}$.

Remark 4.2. β_C is a measure of the anisotropy of C. In particular, for the identity tensor, one has $\beta_{I_d} = 0$.

Proof. Let u be the solution of $\mathcal{L}u = f$ with Dirichlet boundary conditions (assuming that it exists). Let α be a field of $d \times d$ invertible matrices. Observe that (4.52) is equivalent to

$$\Delta u = \alpha f + \Delta u - \alpha \mathcal{L} u. \tag{4.55}$$

Consider the mapping $T : (H^2 \cap H^1_0(\Omega))^d \to (H^2 \cap H^1_0(\Omega))^d$ defined by $v = Tw$, where v be the unique solution of the Dirichlet problem for Poisson equation

$$\Delta v = \alpha f + \Delta w - \alpha \mathcal{L} w. \tag{4.56}$$

Let us now choose α so that T is a contraction.

Note that

$$\big\|Tw_1 - Tw_2\big\|_{(H^2\cap H^1_0(\Omega))^d} = \|v_1 - v_2\|_{(H^2\cap H^1_0(\Omega))^d}. \tag{4.57}$$

Using the convexity of Ω, one obtains the following classical inequality satisfied by the Laplace operator (see lemma 1.2.2 of [39]):

$$\|v_1 - v_2\|_{(H^2\cap H^1_0(\Omega))^d} \leq \|\Delta(v_1 - v_2)\|_{(L^2(\Omega))^d}. \tag{4.58}$$

Hence,

$$\begin{aligned}\big\|Tw_1 - Tw_2\big\|^2_{(H^2\cap H^1_0(\Omega))^d} \leq & \|\Delta(w_1 - w_2) - \alpha\mathcal{L}(w_1 - w_2)\|^2_{(L^2(\Omega))^d} \\ = & \Big\| \sum_{i,j,k,l=1}^{d} e_j \big(\delta_{jl}\delta_{ki} - \sum_{j'=1}^{d} \alpha_{jj'} C_{ij'kl}\big)\partial_i\partial_k(w_1^l - w_2^l)\Big\|^2_{(L^2(\Omega))^d}.\end{aligned} \tag{4.59}$$

Using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}\big\|Tw_1 - Tw_2\big\|^2_{(H^2\cap H^1_0(\Omega))^d} \leq \int_\Omega & \Big(\sum_{i,j,k,l=1}^{d} (\delta_{jl}\delta_{ki} - \sum_{j'=1}^{d} \alpha_{jj'} C_{ij'kl})^2\Big) \\ & \Big(\sum_{i,k,l=1}^{d} (\partial_i\partial_k(w_1^l - w_2^l))^2\Big).\end{aligned} \tag{4.60}$$

Hence, writing

$$\beta_{\alpha,C} := \sum_{i,j,k,l=1}^{d} (\delta_{jl}\delta_{ki} - \sum_{j'=1}^{d} \alpha_{jj'} C_{ij'kl})^2, \tag{4.61}$$

we obtain that

$$\big\|Tw_1 - Tw_2\big\|^2_{(H^2\cap H^1_0(\Omega))^d} \leq \operatorname{esssup}_{x\in\Omega}\beta_{\alpha,C}(x)\big\|w_1 - w_2\big\|^2_{(H^2\cap H^1_0(\Omega))^d}. \tag{4.62}$$

Observe that

$$\beta_{\alpha,C} := d^2 - 2\sum_{j',j,k=1}^{d} \alpha_{jj'}C_{kj'kj} + \sum_{i,j,k,l=1}^{d} (\sum_{j'=1}^{d} \alpha_{jj'} C_{ij'kl})^2. \tag{4.63}$$

Taking variations with respect to α, one must have, at the minimum, that for all j, m,

$$\sum_{i,k,l=1}^{d} C_{imkl}(\sum_{j'=1}^{d} \alpha_{jj'} C_{ij'kl}) = \sum_{k=1}^{d} C_{kmkj}. \tag{4.64}$$

Hence,

$$\sum_{j'=1}^{d} \alpha_{jj'} \sum_{i,k,l=1}^{d} C_{imkl} C_{ij'kl} = \sum_{k=1}^{d} C_{kmkj}. \tag{4.65}$$

Let B be the matrix defined by $B_{jm} = \sum_{k=1}^{d} C_{kmkj}$. Let A be the matrix defined by $A_{j'm} = \sum_{i,k,l=1}^{d} C_{imkl} C_{ij'kl}$. Then (4.65) can be written as

$$\alpha A = B, \tag{4.66}$$

which leads to

$$\alpha^* = BA^{-1}. \tag{4.67}$$

For such a choice, on has

$$\sum_{i,j,k,l=1}^{d} (\sum_{j'=1}^{d} \alpha^*_{jj'} C_{ij'kl})^2 = \sum_{j,m,k=1}^{d} \alpha^*_{jm} C_{kmkj}. \tag{4.68}$$

Hence, at the minimum, $\beta_{\alpha,C} = \beta_C$ with

$$\beta_C := d^2 - \mathrm{Trace}[BA^{-1}B^T]. \tag{4.69}$$

For that specific choice of α, if $\beta_C < 1$, then T is a contraction and we obtain the existence and solution of (4.52) through the fixed point theorem. Moreover,

$$\|\Delta u\|_{(L^2(\Omega))^d} \leq \|\alpha^* f\|_{L^2(\Omega)} + \beta_C^{\frac{1}{2}} \|\Delta u\|_{(L^2(\Omega))^d}, \tag{4.70}$$

which concludes the proof.

□

As a direct consequence of theorem 4.5 and theorem 4.6, we obtain the following theorem.

Theorem 4.7. *Let C be a divergence free bounded, uniformly elliptic, fourth order tensor. Assume Ω is convex with a C^2-boundary. If β_C, defined by (4.53), is strictly bounded from above by one, then the inequalities of theorem 4.5 and theorem 4.4 are satisfied.*

5 Application of the flux-norm to non-conforming Galerkin.

The change of coordinates used in [48] (see also subsection 4.1.1) to obtain error estimates for finite element solutions of scalar equation (1.1) in two-dimenisons admits no straightforward generalization for vectorial elasticity equations. In this section we show how the flux-norm can be used to obtain error estimates for discontinuous Galerkin solutions of (1.1) and (1.3). These estimates are based on the inequalities introduced in section 4 and the control of the non-conforming error associated with the discontinuous Galerkin method. The control of the non-conforming error could be implemented by methods such as the penalization method. Its analysis is, however, difficult in general and will not be done here. In the scalar case we refer to [47] for the control of the non-conforming error.

5.1 Scalar equations

Let $w \in H^2 \cap H^1_0(\Omega)$ such that $-\Delta w = f$. Let u be the solution in $H^1_0(\Omega)$ of

$$\int_\Omega \nabla\varphi a \nabla u = \int_\Omega \nabla\varphi\nabla w \quad \varphi \in H^1_0(\Omega) \tag{5.1}$$

Let $\mathcal{V}$ be a finite dimensional linear subspace of $(L^2(\Omega))^d$. Let $w \in H^2 \cap H^1_0(\Omega)$. We write $\zeta_\mathcal{V}$ the Galerkin solution of (1.1) in $\mathcal{V}$ –i.e., $\zeta_\mathcal{V}$ is defined such that for all $\eta \in \mathcal{V}$,

$$\int_\Omega \eta a \zeta_\mathcal{V} = \int_\Omega \eta\nabla w. \tag{5.2}$$

For $\xi \in (L^2(\Omega))^d$, denote by $\xi = \xi_{curl} + \xi_{pot}$ the Weyl-Helmholtz decomposition of ξ (ξ_{curl} is divergence-free).

Definition 5.1. Write

$$\mathcal{K}_\mathcal{V} := \sup_{\zeta\in\mathcal{V}} \frac{\|\zeta_{curl}\|_{(L^2(\Omega))^d}}{\|\zeta\|_{(L^2(\Omega))^d}}. \tag{5.3}$$

$\mathcal{K}_\mathcal{V}$ is related to the "non-conforming error" associated with $\mathcal{V}$ (see for instance [18] chapter 10). If $\mathcal{K}_\mathcal{V} > 0$ then the space $\mathcal{V}$ must contain functions that are not exact gradients. Moreover, it determines the "distance" between $\mathcal{V}$ and $(L^2_{pot})^d$.

Definition 5.2. Write

$$\mathcal{D}_\mathcal{V} := \inf_{a',\mathcal{V}' : \operatorname{div}(a'\mathcal{V}')=\operatorname{div}(a\mathcal{V})} \sup_{w'\in H^2(\Omega)\cap H^1_0(\Omega)} \inf_{\zeta'\in\mathcal{V}'} \frac{\big\|(a'(\nabla u' - \zeta'))_{pot}\big\|_{(L^2(\Omega))^d}}{\|\Delta w'\|_{L^2(\Omega)}} \tag{5.4}$$

The first minimum in (5.4) is taken with respect to all finite dimensional linear subspaces *$\mathcal{V}'$ of $(L^2(\Omega))^d$, and all bounded uniformly elliptic matrices a' ($a'_{ij} \in L^\infty(\Omega)$) such that $\operatorname{div}(a'\mathcal{V}') = \operatorname{div}(a\mathcal{V})$. Furthermore u' in (5.4) is defined as the (weak) solution of $\operatorname{div}(a'\nabla u') = \Delta w'$ with Dirichlet boundary condition on $\partial\Omega$. Due to Theorem 2.1 the $\inf_{a',\mathcal{V}' : \operatorname{div}(a'\mathcal{V}')=\operatorname{div}(a\mathcal{V})}$ can be dropped. However, we keep it to emphasize the independence of the choice of $\mathcal{V}'$ and a' as long as they satisfy $\operatorname{div}(a'\mathcal{V}') = \operatorname{div}(a\mathcal{V})$.

Theorem 5.1. *There exists a constant $C^* > 0$ depending only on $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$ such that for $\mathcal{K}_{\mathcal{V}} \leq C^*$,*

$$\|\nabla u - \zeta_{\mathcal{V}}\|_{(L^2(\Omega))^d} \leq C\|f\|_{L^2(\Omega)}\big(\mathcal{D}_{\mathcal{V}} + \mathcal{K}_{\mathcal{V}}\big) \tag{5.5}$$

where u is the solution of (5.1), $\zeta_{\mathcal{V}}$ the solution of (5.2) and C is a constant depending only on $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

Remark 5.1. Theorem 5.1 is in essence stating that the approximation error associated with $\mathcal{V}$ and the operator $\operatorname{div}(a\nabla)$ is composed of two terms: $\mathcal{D}_{\mathcal{V}}$ and $\mathcal{K}_{\mathcal{V}}$. $\mathcal{K}_{\mathcal{V}}$ is related to the non-conforming error associated to $\mathcal{V}$. $\mathcal{D}_{\mathcal{V}}$ is the minimum (over a', $\mathcal{V}'$ such that $\operatorname{div}(a'\mathcal{V}') = \operatorname{div}(a\mathcal{V})$) approximation error associated to $\mathcal{V}'$ and the operator $\operatorname{div}(a'\nabla)$. Hence $\mathcal{D}_{\mathcal{V}}$ and the transfer property allow us to equate the accuracy of a scheme associated with $\mathcal{V}'$ and a conductivity a' to the accuracy of the scheme associated with $\mathcal{V}$ and the conductivity a provided that $\operatorname{div}(a'\mathcal{V}') = \operatorname{div}(a\mathcal{V})$.

Remark 5.2. In fact, it is possible to deduce from theorem 5.1 that the maximum approximation error associated to $\mathcal{V}$ and the operator $\operatorname{div}(a\nabla)$ can be bounded from below by a multiple of $\big(\mathcal{D}_{\mathcal{V}} + \mathcal{K}_{\mathcal{V}}\big)$ (see also equation (10.1.6) of [18]).

Remark 5.3. When $\mathcal{V}$ is the set local gradients discontinuous Galerkin elements we can replace the right hand side of (5.2) by $-\int_\Omega v\Delta w$ if $\eta = \sum_{\tau\in\Omega_h} 1_{(x\in\tau)}\nabla v$ (see subsection 1.3 of [48]). This modification doesn't affect the validity of (5.5) since the difference between the two terms remains controlled by $\mathcal{D}_{\mathcal{V}}$. For the clarity of the presentation we have used the formulation (5.2).

In order to prove theorem 5.1, we will need the following lemma

Lemma 5.1. *There exists C depending only on $\lambda_{\min}(a), \lambda_{\max}(a)$ such that for $u \in H^1_0(\Omega)$ and $\zeta \in (L^2(\Omega))^d$*

$$\|\nabla u - \zeta\|_{(L^2(\Omega))^d} \leq C\Big(\big\|(a(\nabla u - \zeta))_{pot}\big\|_{(L^2(\Omega))^d} + \|\zeta_{curl}\|_{(L^2(\Omega))^d}\Big) \tag{5.6}$$

$$\|\nabla u - \zeta\|_{(L^2(\Omega))^d} \geq \frac{1}{C}\Big(\big\|(a(\nabla u - \zeta))_{pot}\big\|_{(L^2(\Omega))^d} + \|\zeta_{curl}\|_{(L^2(\Omega))^d}\Big) \tag{5.7}$$

Proof. The proof of (5.7) is straightforward. For (5.6) observe that

$$\int_\Omega (\nabla u - \zeta)^T a(\nabla u - \zeta) = \int_\Omega (\nabla u - \zeta_{pot})^T \big(a(\nabla u - \zeta)\big)_{pot} + \int_\Omega \zeta_{curl}^T a(\nabla u - \zeta) \tag{5.8}$$

It follows from Cauchy-Schwartz inequality that

$$\begin{aligned}\lambda_{\min}(a)\|\nabla u - \zeta\|_{(L^2(\Omega))^d} \leq & \frac{\|\nabla u - \zeta_{pot}\|_{(L^2(\Omega))^d}}{\|\nabla u - \zeta\|_{(L^2(\Omega))^d}}\big\|(a(\nabla u - \zeta))_{pot}\big\|_{(L^2(\Omega))^d} \\ & + \lambda_{max}(a)\|\zeta_{curl}\|_{(L^2(\Omega))^d}\end{aligned} \tag{5.9}$$

□

We also need the following lemma which corresponds to lemma (10.1.1) of [18]

Lemma 5.2. *Let H be a Hilbert space, V and V_h be subspaces of H (V_h may not be a subset of V). Assume that $a(.,.)$ is continuous bilinear form on H which is coercive on V_h, with respective continuity and coercivity constants C and γ. Let $u \in V$ solve*

$$a(u,v) = F(v) \quad \forall v \in V \tag{5.10}$$

where $F \in H'$. Let $u_h \in V_h$ solve

$$a(u_h,v) = F(v) \quad \forall v \in V_h \tag{5.11}$$

Then

$$\|u-u_h\|_H \leq \big(1+\frac{C}{\gamma}\big) \inf_{w\in V_h} \|u-w\|_H + \frac{1}{\gamma} \sup_{w\in V_h\setminus\{0\}} \frac{a(u-u_h,w)}{\|w\|_H} \tag{5.12}$$

We now proceed by proving theorem 5.1.

Proof. Using lemma 5.1 we obtain that

$$\|\nabla u - \zeta\|_{(L^2(\Omega))^d} \leq C\Big(\big\|(a(\nabla u-\zeta))_{pot}\big\|_{(L^2(\Omega))^d} + \|\zeta\|_{(L^2(\Omega))^d}\mathcal{K}_{\mathcal{V}}\Big) \tag{5.13}$$

Using the triangle inequality $\|\zeta\|_{(L^2(\Omega))^d} \leq \|\nabla u - \zeta\|_{(L^2(\Omega))^d} + \|\nabla u\|_{(L^2(\Omega))^d}$ we obtain that

$$\|\nabla u - \zeta\|_{(L^2(\Omega))^d} \leq \frac{C}{1-C\mathcal{K}_{\mathcal{V}}}\Big(\big\|(a(\nabla u-\zeta))_{pot}\big\|_{(L^2(\Omega))^d} + \|\nabla u\|_{(L^2(\Omega))^d}\mathcal{K}_{\mathcal{V}}\Big) \tag{5.14}$$

from which we deduce that

$$\inf_{\zeta\in\mathcal{V}} \|\nabla u - \zeta\|_{(L^2(\Omega))^d} \leq \frac{C}{1-C\mathcal{K}_{\mathcal{V}}} \inf_{\zeta\in\mathcal{V}} \Big(\big\|(a(\nabla u-\zeta))_{pot}\big\|_{(L^2(\Omega))^d} + \|\nabla u\|_{(L^2(\Omega))^d}\mathcal{K}_{\mathcal{V}}\Big) \tag{5.15}$$

(C^* in the statement of the theorem is chosen so that $\mathcal{K}_{\mathcal{V}} < C^*$ implies $C\mathcal{K}_{\mathcal{V}} < 0.5$). We obtain from lemma 5.2 that (observe that the last term in equation (5.12) is the non-conforming error and that it is bounded by $C\|\nabla u\|_{(L^2(\Omega))^d} \sup_{\zeta\in\mathcal{V}} \frac{\|\zeta_{curl}\|_{(L^2(\Omega))^d}}{\|\zeta\|_{(L^2(\Omega))^d}}$ for an appropriate constant C).

$$\|\nabla u - \zeta\|_{(L^2(\Omega))^d} \leq C\Big(\inf_{\zeta\in\mathcal{V}} \|\nabla u - \zeta\|_{(L^2(\Omega))^d} + \|\nabla u\|_{(L^2(\Omega))^d} \sup_{\zeta\in\mathcal{V}} \frac{\|\zeta_{curl}\|_{(L^2(\Omega))^d}}{\|\zeta\|_{(L^2(\Omega))^d}}\Big) \tag{5.16}$$

Combining (5.15) with (5.16), we conclude using theorem 2.2 and the Poincaré inequality. □

Let us now show how theorem 5.1 can be combined with the new class of inequalities obtained in sub-section 4.1 to obtain homogenization results for arbitrary rough coefficients a. Let M be a uniformly elliptic $d \times d$ matrix (observe that uniform ellipticity of

M implies its invertibility) and $\mathcal{V}'$ be a finite dimensional linear subspace of $(L^2(\Omega))^d$. Define

$$\mathcal{V} := \{M\zeta' \,:\, \zeta' \in \mathcal{V}'\} \tag{5.17}$$

Assume furthermore that for all $w \in H_0^1 \cap H^2(\Omega)$,

$$\inf_{\zeta' \in \mathcal{V}'} \|\nabla w - \zeta'\|_{(L^2(\Omega))} \leq Ch\|\Delta w\|_{L^2(\Omega)} \tag{5.18}$$

where h is a small parameter (the resolution of the tessellation associated to $\mathcal{V}'$ for instance). We remark here that $\mathcal{V}'$ can be viewed as the coarse scale h approximation space (see example below). The fine scale information from coefficients $a(x)$ is contained in the elements of the matrix M. This is illustrated in the example below where $M = \nabla F$ for harmonic coordinates F. Therefore matrix M is determined by d harmonic coordinates that are analogues of d cell problems in periodic homogenization and we call space $\mathcal{V}$ the "minimal pre-computation space" since it requires minimal (namely d) pre-computation of fine scales.

Then we have the following theorem

Theorem 5.2. Approximation by "minimal pre-computation space" *If*

- *$a \cdot M$ is divergence free (as defined in sub-section 4.1).*
- *The symmetric part of $a \cdot M$ satisfies the Cordes condition (4.27) or the symmetric part of $a \cdot M$ satisfies one of the inequalities of theorem 4.2.*
- *The non-conforming error satisfies $\mathcal{K}_{\mathcal{V}} \leq Ch^\alpha$ for some constant $C > 0$ and $\alpha \in (0, 1]$,*

then

$$\|\nabla u - \zeta_{\mathcal{V}}\|_{(L^2(\Omega))^d} \leq C\|f\|_{L^2(\Omega)} h^\alpha \tag{5.19}$$

where u is the solution of (1.1) and $\zeta_{\mathcal{V}}$ the solution of (5.2),

Remark 5.4. The error estimate is given in the L^2 norm of $\nabla u - \zeta_{\mathcal{V}}$ because we wish to give a strong error estimate and if $\zeta_{\mathcal{V}}$ is not a gradient then the L^2 norm of $(a(\nabla u - \zeta_{\mathcal{V}}))_{pot}$ is not equivalent to the L^2 norm $\nabla u - \zeta_{\mathcal{V}}$.

Remark 5.5. It is in fact sufficient that the symmetric part of $a \cdot M$ satisfies one of the inequalities of theorem 4.1 instead of 4.2. For the sake of clarity we have used inequalities of theorem 4.2.

Proof. The proof is a direct consequence of theorem 5.1, we simply need to bound $\mathcal{D}_{\mathcal{V}}$. Since $\operatorname{div}(a\mathcal{V}) = \operatorname{div}(a \cdot M\mathcal{V}')$ It follows from equation 5.4 that

$$\mathcal{D}_{\mathcal{V}} \leq \sup_{w' \in H^2(\Omega) \cap H_0^1(\Omega)} \inf_{\zeta' \in \mathcal{V}'} \frac{\left\|(a \cdot M(\nabla u' - \zeta'))_{pot}\right\|_{(L^2(\Omega))^d}}{\|\Delta w'\|_{L^2(\Omega)}} \tag{5.20}$$

where u' in (5.4) is defined as the (weak) solution of $\operatorname{div}(a \cdot M\nabla u') = \Delta w'$ with Dirichlet boundary condition on $\partial\Omega$. Now, if symmetric part of $a \cdot M$ satisfies the Cordes condition (4.27) or the symmetric part of $a \cdot M$ satisfies one of the inequalities of theorem 4.2 then $\|u'\|_{H^2} \leq C\|\Delta w'\|_{L^2}$ and we conclude using the approximation property (5.18).

□

An example of $\mathcal{V}$ can be found in the discontinuous Galerkin method introduced in subsection 1.3 of [48]. This method is also a generalization of the method II of [8] to non-laminar media. In that method, we pre-compute F denote the harmonic coordinates associated with (1.1)–i.e., $F(x) = \big(F_1(x), \dots, F_d(x)\big)$ is a d-dimensional vector field whose entries satisfy

$$\begin{cases} \operatorname{div} a\nabla F_i = 0 & \text{in} \quad \Omega \\ F_i(x) = x_i & \text{on} \quad \partial\Omega. \end{cases} \tag{5.21}$$

Introducing Ω_h, a regular tessellation of Ω of resolution h, the elements of $\mathcal{V}$ are defined as $\nabla F(\nabla_c F)^{-1}\nabla\varphi$, where φ is a piecewise linear function on Ω_h with Dirichlet boundary condition on Ω_h and $\nabla_c F$ is the gradient of the linear interpolation of F over Ω_h. In that example $a \cdot \nabla F$ is divergence-free and ∇F plays the role of M. The non-conforming error is controlled by the aspect ratios of the images of the triangles of Ω_h by F. In [48] the estimate 5.19 is obtained using F as a global change of coordinates that has no clear equivalent for tensorial equations whereas the proof based on the flux-norm can be extended to tensorial equations.

Remark 5.6. One may ask whether it is possible to solve (5.21) with L^∞ coefficients. In real applications a is restricted to piecewise constant values on a very fine mesh of Ω and results given here guarantee error bounds independent of the size of the fine mesh. Hence in practice solutions of (3.1) and (5.21) are (pre)computed using classical methods such as multigrid [38, 17] or $H - matrices$ [12, 30].

5.2 Tensorial equations.

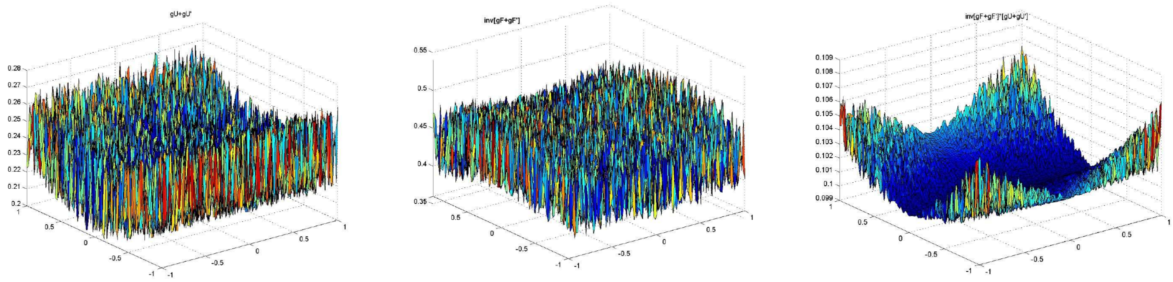
The generalization of the results of this section to elasticity equations doesn't pose any difficulty. This generalization is simply based on theorem 2.4 and the new class of inequalities introduced in subsection 4.2. An example of numerical scheme can be found in [37] for (non-linear) elasto-dynamics with rough elasticity coefficients. With elasticity equations harmonic coordinates are replaced by harmonic displacements, i.e. solutions of

$$\begin{cases} -\operatorname{div}(C(x)\nabla F^{kl}) = 0 & x \in \Omega \\ F^{kl} = \frac{x_k e_l + x_l e_k}{2} & \text{on } \partial\Omega. \end{cases} \tag{5.22}$$

and strains $\varepsilon(u)$ are approximated by a finite dimension linear space $\mathcal{V}$ with elements of the form $\varepsilon(F) : (\varepsilon_c F)^{-1}(\varepsilon(\varphi))$ where the φ are piecewise linear displacements on Ω_h, $\varepsilon_c F$ is the strain of the linear interpolation of F over Ω_h and $\varepsilon(F)$ denotes the $d \times d \times d \times d$ tensor with entries

$$\varepsilon(F)_{i,j,k,l} := \frac{\partial_i F^{kl}_j + \partial_j F^{kl}_i}{2}. \tag{5.23}$$

Here $C : \varepsilon(F)$ is divergence-free and plays the role of M, furthermore, the regularization property observed in the scalar case [48] is also observed in the tensorial case by taking the product $(\varepsilon(F))^{=1}\varepsilon(u)$ (figure 1).



(a) (b) (c)

Figure 1: *Computation by Lei Zhang. The elasticity stiffness is obtained by choosing its coefficients to be random and oscillating over many overlapping scales. Figure (a) and (b) show wild oscillations of one of the components of the strain tensor $\nabla u + \nabla u^T$ (u solves* (1.3)*) and one of the components of $(\nabla F + \nabla F^T)^{-1}$ ($F = \{F_{ij}\}$ is defined by* (5.22)*). Figure (c) illustrates one of the components of the product $(\nabla F + \nabla F^T)^{-1}(\nabla u + \nabla u^T)$, which is smooth if compared to (a) and (b). There is no smoothing near the boundary due to sharp corners.*

6 Relation with homogenization theory and other works.

Recall that classical homogenization theory (e.g.,[13], [33], [11], [21], [2]) has two main objectives: (i) finding the effective constitutive relation or equivalently the effective (homogenized) PDE, and (ii) finding a coarse scale approximation to the solution of the original problem that has both fine and coarse scales *and evaluate the error. In the simplest case of second order divergence form PDE, this can be schematically illustrated as follows.

$$\partial_i \left(a_{ij} \left(\frac{x}{\epsilon} \right) \partial_j u_\epsilon \right) = f \to \hat{a}_{ij} \partial_i \partial_j \hat{u} = f \to \hat{u} \approx \hat{u}_h$$

Here, the original elliptic problem with periodic coefficients (fine scale ϵ, coarse scale $\mathcal{O}(1)$) is approximated by a coarse scale problem with constant coefficients $\hat{a}$, so that u_ϵ is approximated by $\hat{u}$, which is much easier to compute. However, actual computation of $\hat{u}$ requires further discretization on a coarse computational scale h. Namely, one has to introduce u_h, which we call the *finite-dimensional homogenization approximation* to u_ε. This approximation is what is actually used in practical applications. In this work we do not address the first objective (i) but rather focus solely on the second objective (ii). In short, we start from the PDE $\partial_i (a_{ij} \partial_j u) = f$ with, in the most general case, measurable coefficients $a_{ij}(x)$, and construct a finite dimensional homogenization approximation u_h

directly, bypassing the derivation of the homogenized PDE. Moreover, this is done with a controlled and explicit error estimate in the H^1 norm.

Homogenization with scale separation and in low contrast materials/media is now well understood. One of our purposes in this paper was to introduce a geometric description (see, e.g., thin subspace concept below) of homogenization theory that can be used to extend its results to high contrast media/materials with non separated scales. This geometric description is introduced in such a way that it can be applied to both scalar and vectorial based equations such as heat conduction, reservoir modeling, and elasticity equations.

We will describe the essence of the connections between the approach presented here and classical homogenization theory using the classical scalar parabolic divergence form equation

$$\begin{cases} \partial_t u(x,t) - \operatorname{div}\Big(a(x)\nabla u(x,t)\Big) = g(x,t) \quad x \in \Omega; g \in L^2(\Omega \times [0,T]), \\ u = 0 \quad \text{on} \quad \Omega \times \{t=0\} \cup \partial\Omega \times [0,T], \end{cases} \tag{6.1}$$

where Ω is a bounded subset of $\mathbb{R}^d$ with a smooth boundary and a is symmetric and uniformly elliptic on Ω with coefficients that are only bounded $a(x) = \{a_{ij} \in L^\infty(\Omega)\}$. It follows that the eigenvalues of a are uniformly bounded from below and above by two strictly positive constants, denoted by $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$, i.e. for all $\xi \in \mathbb{R}^d$ and $x \in \Omega$,

$$\lambda_{\min}(a)|\xi|^2 \le \xi^T a(x)\xi \le \lambda_{\max}(a)|\xi|^2 \tag{6.2}$$

We are interested in obtaining a numerical solution for this problem. Since the coefficients $a(x)$ have no regularity, the computational complexity can be enormous. We are interested in *in constructing a finite dimensional approximation to the solution of this problem that allows for a reduction of the computational complexity while controlling its accuracy.*

Assume initially that $a(x) = B(\frac{x}{\epsilon})$ where $B(y)$ is a symmetric uniformly elliptic matrix with bounded periodic entries (i.e. $B_{i,j} \in L^\infty(\mathbb{T}^d)$ where $\mathbb{T}^d$ is the unit torus of dimension d). Then $u = v_\epsilon$ and from classical homogenization theory [33, 13] it is known that v_ϵ can be approximated by v_0 where v_0 is the solution of the problem:

$$\begin{cases} \partial_t v_0(x,t) - \operatorname{div}\Big(\bar{B}\nabla v_0(x,t)\Big) = g(x,t) \quad x \in \Omega; \\ v_0 = 0 \quad \text{on} \quad \Omega \times \{t=0\} \cup \partial\Omega \times [0,T], \end{cases} \tag{6.3}$$

and $\bar{B}$ called the homogenized matrix, is elliptic and has constant entries. In this way we reduce computational complexity drastically. Indeed, numerically solving problem (6.1) requires the resolution of both fine scales of order ϵ and coarse scales of order 1. In contrast, numerically solving problem (6.3) involves only resolution of coarse scales of order 1. It is well known that the irregularity of the right hand side of the equations contributes nothing to the computational complexity of the problem (i.e., g is not an issue and the reader may assume that $g \in L^2(\Omega \times [0,T])$ for simplicity). The essence of homogenization theory can thus be summarized as reducing the complexity of the problem due to the roughness in material properties of the medium, while external fields are "reasonably regular". The price to pay for this reduction in complexity lies in the fact that in order to find $\bar{B}$ one has to solve, for $i \in \{1, \ldots, d\}$ the following so called cell problems:

$$\begin{cases} \operatorname{div}\Big(B(y)\nabla(\chi_i(y) + y_i)\Big) = 0 \quad y \in \Omega; \\ \chi_i \in H^1(\mathbb{T}^d) \end{cases} \tag{6.4}$$

*Here $y = (y_1, ..y_d)$, $\bar{B}$ is defined via solution of the cell problem χ_i according classical formulas of homogenization theory [13], [33],[11], [21], [?]. Observe that the cell problem involves only the coefficients of B and not the right hand side $g(x,t)$, nor the boundary conditions on $\partial\Omega$. Indeed, cell problem has zero right hand side and standard (e.g. periodic) boundary conditions. In other words, the reduction of complexity requires resolution of the microstructure *d-times*. Here we are using standard terminology from homogenization literature by referring to the coefficients of $B(y)$ as the microstructure since they describe the material properties of the medium.

The next level of difficulty is to consider problem (6.1) with $a(x) = B(\frac{x}{\epsilon}, \omega)$ where B is no longer periodic but it is a stationary ergodic random field of uniformly elliptic matrices (ω stands for the particular realization of the random field). Note that the stationarity condition can be viewed as a generalized periodicity, since it implies "statistical translational invariance". The solution of (6.1) will depend on ϵ and ω i.e., $u = v_\epsilon$–and classical homogenization theory states that v_ϵ can be approximated by v_0, where v_0 is the solution of (6.3), as $\epsilon \downarrow 0$.

In order to obtain the homogenized matrix $\bar{B}$, one has to solve d elliptic problems in the whole space $\mathbb{R}^d$ with coefficients $B(\frac{x}{\epsilon}, \omega)$ for a "typical" realization ω that occurs with probability one ([35, 50]). In practical computations one approximates $\bar{B}$ by solving d elliptic equations (still called cell problems, since they are a generalization of the periodic cell problems) on a "large enough" hypercube of size R ($R \to \infty$ gives $\bar{B}$) subject to standard boundary conditions (e.g., linear/periodic analogous to the periodic case) [46]. So here, again, one has to resolve d elliptic problems with full computational complexity due to the coefficients, but these problems do not depend on the domain Ω and the right hand side g. In short, again one has to resolve the random microstructure d times. Under additional assumptions on the mixing properties of the ergodic field a one can obtain the rate at which the approximate effective conductivities converge to the homogenized matrix and solve numerically those d elliptic on a sub-domain of Ω (which could be much smaller than Ω, [15, 24, 25]).

In many practical situations, one has to deal with a medium (rather than a sequence of media) that has no periodicity or ergodicity property. Moreover, it may not be possible to distinguish finitely many well separated scales (e.g., different lengths of oscillations). In this paper, we have considered this next level of difficulty where *no assumptions are made on a* except the generic requirements of boundedness and uniform ellipticity.

The theory of homogenization in its most general formulation is based on abstract operator convergence, –i.e., G-convergence for symmetric operators, H-convergence for non-symmetric operators and Γ-convergence for variational problems. We refer to the work of De Giorgi, Spagnolo, Murat, Tartar, Pankov and many others [41, 29, 23, 53, 52, 42, 16]). H, G and Γ-convergence allows one to obtain the convergence a family of operators parameterized by ϵ under very weak assumptions on the coefficients.

The main difference with this work is that instead of characterizing the limit of an ϵ-family of operators we are approximating a given operator with a finite-dimensional operator with explicit error estimates. Indeed, given a medium that is not periodic or stationary ergodic, it is not clear how to define a family of operators A_ϵ. Moreover, the definition of oscillating test functions involves the limiting (homogenized) operator $\hat{A}$. While this works well for the proof of the abstract convergence results, in practice only the coefficients A are known (computing $\hat{A}$ may not be possible), and our approach allows one to construct the approximate (upscaled) solution from the given coefficients without constructing $\hat{A}$.

Furthermore, in most engineering problems, one has to deal with a given medium and not with an family of media, and this is the situation addressed by this paper. In particular, for our problem, it is not possible to find a small parameter ϵ intrinsic to the medium with respect to which one could perform an asymptotic analysis. We call such coefficients a, *"arbitrarily complex"*, which strictly speaking, means that no assumptions are made beyond the boundedness and uniform ellipticity.

A key ingredient of our approach is an understanding of homogenization as seeking an approximate solution in a thin subspace of H^1, which is isomorphic to H^2 (the true solution is in H^1). More precisely, consider equation (1.1). It is known that for $g \in H^{-1}(\Omega)$, the solution u of (1.1) belongs to $H^1_0(\Omega)$. When g spans $L^2(\Omega)$, u spans a subspace V of Ω. How "thin" is that space compared to $H^1_0(\Omega)$? For $a = I_d$ we know that $V = H^1_0(\Omega) \cap H^2(\Omega)$, whose elements can be approximated in H^1-norm with accuracy h by piecewise linear functions on a regular triangulation of Ω with resolution h (involving $\frac{|\Omega|}{h^d}$ degrees of freedom). Section 3 shows that when the entries of a are only assumed to be bounded, V is isomorphic to $H^1_0(\Omega) \cap H^2(\Omega)$. Moreover, its elements can be approximated in H^1-norm with accuracy h by elements of the linear span of a basis composed of $\frac{|\Omega|}{h^d}$ functions (this analysis is also related to the n-widths for elliptic problems [40]). We show how to compute a "superior basis" (involving $\frac{|\Omega|}{h^d}$ standard solutions that are analogous to solutions of cell problems) such that the accuracy of the approximation expressed in terms of the L^2-norm of the flux is independent of a and Ω (allowing for high material contrast).

[48] formulates the conditions under which these functions can be constructed from any set of d "linearly independent" solutions of (1.1) (harmonic coordinates, for instance). For elasticity problems, $d \times (d+1)/2$ "linearly independent" solutions are required (section 5).

Related work. By now, the field of asymptotic homogenization with non periodic coefficients has become large enough that it is not possible to cite all contributors. Therefore, we will restrict our attention to works directly related to our work.

- In the work [7, 8], a change of coordinates is introduced in one dimensional and quasi-one dimensional divergence form elliptic problems, allowing for efficient finite dimensional approximations.

- In the work of [32, 57], oscillating test functions are introduced in the numerical homogenization of divergence form elliptic equations. The idea of oscillating test functions in the context of homogenization theory appeared in [41] (see also related work on G-convergence [52, 29]). In [3] the solution space is constructed by composing splines with local harmonic coordinates (leading to higher accuracy). More recently, in [27, 26], the idea of a global change of coordinates was implemented numerically in order to up-scale porous media flows.

- In the work of [24, 28], the structure of the medium is numerically decomposed into a micro-scale and a macro-scale (meso-scale) and solutions of cell problems are computed on the micro-scale, providing local homogenized matrices that are transferred (up-scaled) to the macro-scale grid. This procedure allows one to obtain rigorous homogenization results with controlled error estimates for non periodic media of the form $a(x, \frac{x}{\epsilon})$ (where $a(x, y)$ is assumed to be smooth in x and periodic or ergodic with specific mixing properties in y). Moreover, it is shown that the numerical algorithms associated with HMM and MsFEM can be implemented for a broader class of coefficients $a(x, \frac{x}{\epsilon})$.

- More recent work includes an adaptive projection based method [44], which is consistent with homogenization when there is scale separation, leading to adaptive algorithms for solving problems with no clear scale separation; fast and sparse chaos approximations of elliptic problems with stochastic coefficients [55, 31]; finite difference approximations of fully nonlinear, uniformly elliptic PDEs with Lipschitz continuous viscosity solutions [20] and operator splitting methods [6, 5].

- For a series of computational papers on the cost versus accuracy capabilities for the generalized FEM we refer to [54] and the references therein.

- We also refer to [14] for adaptive FEM for elliptic equations with non-smooth coefficients.

7 Appendix

7.1 Proof of lemma 4.1

Let u be the solution of $\mathcal{L}u = f$ with Dirichlet boundary condition (assume that it exists). Since $\alpha > 0$ the solvability of (4.31) is equivalent to finding $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\Delta u = \alpha f + \Delta u - \alpha \mathcal{L} u \tag{7.1}$$

Consider the mapping $T : H^2 \cap H^1_0(\Omega) \to H^2 \cap H^1_0(\Omega)$ defined by $v = Tw$ where v be the unique solution of the Dirichlet problem for Poisson equation

$$\Delta v = \alpha f + \Delta w - \alpha \mathcal{L} w \tag{7.2}$$

Let us now show that for $\beta_a < 1$, T is a contraction.

$$\|Tw_1 - Tw_2\|_{H^2\cap H_0^1(\Omega)} = \|v_1 - v_2\|_{H^2\cap H_0^1(\Omega)} \tag{7.3}$$

Using the convexity of Ω one obtains the following classical inequality satisfied by the Laplace operator (see lemma 1.2.2 of [39])

$$\|v_1 - v_2\|_{H^2\cap H_0^1(\Omega)} \leq \|\Delta(v_1 - v_2)\|_{L^2(\Omega)} \tag{7.4}$$

Hence

$$\begin{aligned}\|Tw_1 - Tw_2\|^2_{H^2\cap H_0^1(\Omega)} &\leq \|\Delta(w_1 - w_2) - \alpha\mathcal{L}(w_1 - w_2)\|^2_{L^2(\Omega)} \\ &\leq \Big\| \sum_{i,j=1}^{d} \big(\delta_{ij} - \alpha a_{ij}\big)\partial_i\partial_j(w_1 - w_2)\Big\|^2_{L^2(\Omega)}\end{aligned} \tag{7.5}$$

Using Cauchy-Schwartz inequality we obtain that

$$\begin{aligned}\|Tw_1 - Tw_2\|^2_{H^2\cap H_0^1(\Omega)} \leq \int_\Omega \Big(\sum_{i,j=1}^{d}(\delta_{ij} - \alpha a_{ij})^2\Big) \\ \Big(\sum_{i,j=1}^{d}(\partial_i\partial_j(w_1^l - w_2^l))^2\Big)\end{aligned} \tag{7.6}$$

Hence observing that

$$\text{esssup}_\Omega\Big(\sum_{i,j=1}^{d}(\delta_{ij} - \alpha a_{ij})^2\Big) = \beta_a \tag{7.7}$$

we obtain that

$$\|Tw_1 - Tw_2\|^2_{H^2\cap H_0^1(\Omega)} \leq \text{esssup}_{x\in\Omega}\beta_a(x)\|w_1 - w_2\|^2_{H^2\cap H_0^1(\Omega)} \tag{7.8}$$

It follows that if $\beta_C < 1$, then T is a contraction and we obtain the existence and solution of (4.31) through the fixed point theorem. Moreover

$$\|\Delta u\|_{L^2(\Omega)} \leq \|\alpha f\|_{L^2(\Omega)} + \beta_a^{\frac{1}{2}}\|\Delta u\|_{L^2(\Omega)} \tag{7.9}$$

which concludes the proof.

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