LECTURE 16

Squeezed Light and its Potential Use in LIGO

Lecture by H. Jeff Kimble

Assigned Reading:

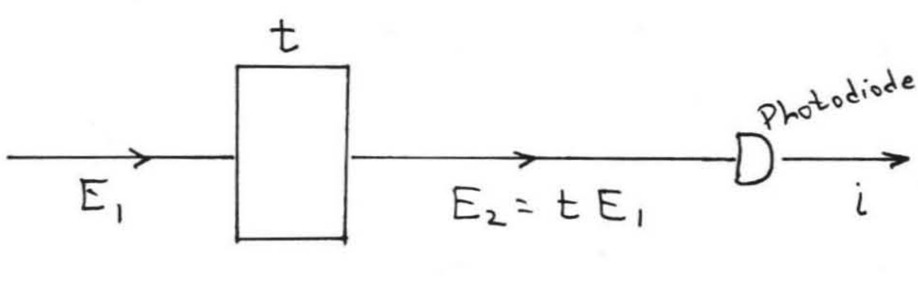
TT. C. M. Caves, "Quantum mechanical noise in an interferometer," *Phys. Rev. D*, **23**, 1693–1708 (1981).
UU. D. F. Walls, "Squeezed states of light," *Nature*, **306**, 141–146 (1983).
VV. M. Xiao, L. A. Wu, and H. J. Kimble, "Precision measurement beyond the shot-noise limit," *Phys. Rev. Lett.*, **59**, 278–281 (1987).

Suggested Supplementary Reading:

l. H. J. Kimble, "Quantum fluctuations in quantum optics—Squeezing and related phenomena," in *Fundamental Systems in Quantum Optics*, eds. J. Dalibard, J. M. Raimond, and J. Zinn-Justin, (Elsevier, Amsterdam, 1992), pp. 545–674.
m. "Squeezed States of the Electromagnetic Field," Feature Issue, *J. Opt. Soc. Amer.*, **B4**, 1450–1741 (1987).
n. "Squeezed Light," Special Issue, *J. Modern Optics*, **34**, 709–1020 (1987).
o. "Quantum Noise Reduction," Special Issue, *Appl. Phys. B*, **55**, 189ff. (1992).
p. S. Reynaud, A. Heidman, E. Giacobino, and C. Fabre, "Quantum fluctuations in optical systems," in *Progress in Optics*, XXX, ed. E. Wolf (Elsevier, 1992), pp. 1–85.

A Few Suggested Problems:

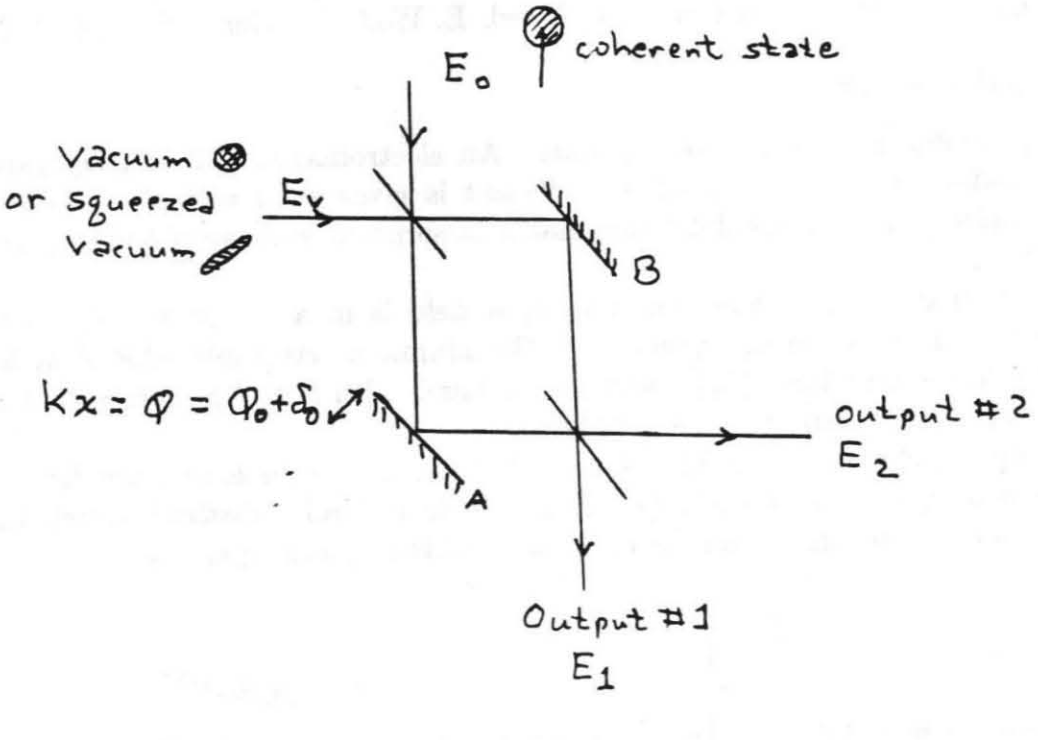
1. *Detection of Modulation in a Squeezed State.* An electromagnetic field propagates through a medium whose transmission coefficient is given by $t = t_0 e^{-\gamma(t)}$, where $\gamma(t) \equiv \gamma_0 \cos(\Omega_0 t)$ (i.e., sinusoidally modulated absorption with amplitude γ_0 and frequency Ω_0).
   a. Assuming that $\gamma_0 \ll 1$ and that the input field is in a coherent state (with frequency $\gg \Omega_0$), derive an expression for the minimum detectable value of γ_0, for a fixed input energy flux $\langle |E_1|^2 \rangle$ and a fixed bandwidth $B \equiv \Delta f$ (corresponding to a photodiode integration time $\hat{\tau} = 1/B$).
   b. If the input field instead is in a squeezed state, derive an expression for the minimum detectable amplitude γ_0. Illustrate in a "ball-and-stick" sketch the dependence of your answer on the orientation of the squeezing ellipse.



2. *Squeezed Vacuum in an Interferometer.* In Part IV of Kimble's lecture transparencies, he sketches a calculation of the minimum detectable phase deviation δ_0 when a coherent state is put into one port of the Mach-Zehnder interferometer shown below, and either the vacuum state or the squeezed vacuum state is put into the other port. His answer was $\delta_0 = 1/\sqrt{N}$ for the vacuum state, and $\delta_0 = (1+\xi S)^{1/2}/\sqrt{N}$ for the squeezed vacuum, where N is the total number of available photons, S is the squeeze factor ($-1 < S \leq 0$), and $\xi < 1$ is the efficiency of the squeezing. Show, in a phasor diagram, the relative phase relationships for the fields that emerge from the outputs, and from your diagrams infer that to achieve the above optimal sensitivities with readout at output #1, the unperturbed position of mirror A should be adjusted so that the phase difference between the two paths along the two arms is $\phi_0 = \pi/2$. More specifically:

   a. Show the orientation of the squeezing ellipses relative to the coherent amplitudes for each of the two fields E_a, E_b that contribute to the total field E_1 at the output #1.

   b. Show how these two fields with their fluctuations sum to give a resultant E_1 that (for $\phi_0 = \pi/2$) produces noise in the photodetector below the standard shot-noise level $1/\sqrt{N}$ and a signal proportional to the phase deviation δ_0.



c. Note that for an efficiency $\xi \to 1$ and for perfect squeezing $S \to -1$, the above analysis and diagrams predict that the minimum detectable phase deviation becomes arbitrarily small, $\delta_0 \to 0$. Show that, in fact, if the interferometer system is perfectly lossless, and δ_0 is modulated so $\delta_0 = \Delta_0 \cos(\Omega_0 t)$, the minimum detectable modulation amplitude Δ_0 is actually $\Delta_0 \sim 1/N$. Calculate the corresponding length sensitivity Δx for the displacement of mirror A. Estimate the laser power required to achieve the sensitivity of the advanced LIGO, *if* this limit could be achieved.

d. In the above discussion it was tacitly assumed that the interferometer mirrors are so massive that light pressure fluctuations do not disturb them significantly. Suppose now that mirror A has a finite, small mass and is free to move in response to light pressure, and that we apply a feedback force to the back of the mirror, to counteract the time-averaged light-pressure force on its front. Show, using the phasor diagrams of parts a. and b., that when we improve our measurement of δ_0 (and hence of the mirror position x) by increasing the amount of squeezing, we increase the random light-pressure perturbations of the mirror, thereby enforcing the uncertainty principle. Relate this result to the standard quantum limit for sensing the position of the small mass, and thence to the curve labeled "Quantum Limit" in the plots of LIGO noise sources that were shown in earlier lectures. [For a quantitative analysis, in the context of a Michelson interferometer, see C. M. Caves, *Phys. Rev. D*, **23**, 1693 (1981). In this problem you are supposed to be ignoring the possibility of going beyond the standard quantum limit as discussed by Jackel and Reynaud, *Europhys. Lett.*, **13**, 301 (1990).]

Lecture 16
Squeezed Light and its Potential Use in LIGO

by Jeff Kimble, 18 & 20 May 1994

Kimble lectured from the following transparencies, which Kip has annotated a bit. Kimble's lecture came in two parts, one covering the second half of the class on 18 May; the other covering the full class on 20 May.

LECTURE 16

"Squeezed Light and its Potential Use in LIGO"

- PART I

by

JEFF KIMBLE

5-18, 5-20-94

Quantum $\left\{\begin{matrix}\text{Measurement} \\ \text{Fluctuations} \\ \text{Noise}\end{matrix}\right\}$ in Quantum Optics

I. A vocabulary for quantum noise

- Wigner distributions

II. Generation, propagation, & detection of squeezed light

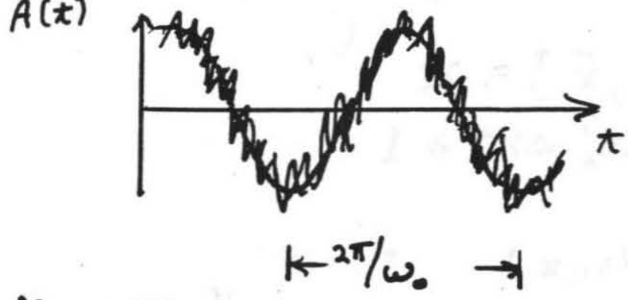
III. Quantum measurement with Squeezed Light

- Interferometry

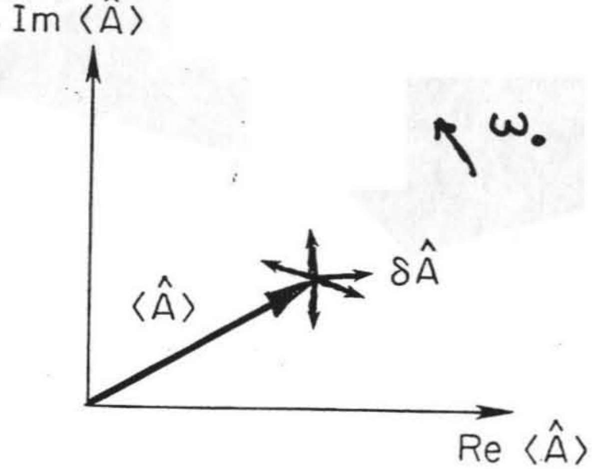
IV. Beyond the SQL for a free mass

H. J. Kimble
Caltech

I. Field Fluctuations



Phasor diagram-



$\langle \hat{A} \rangle$ - mean amplitude of field

$\delta\hat{A}$ - fluctuations of field

"Distribution" of fluctuations?

- Begin with Wigner phase space function

$$W(X_+, X_-)$$

where

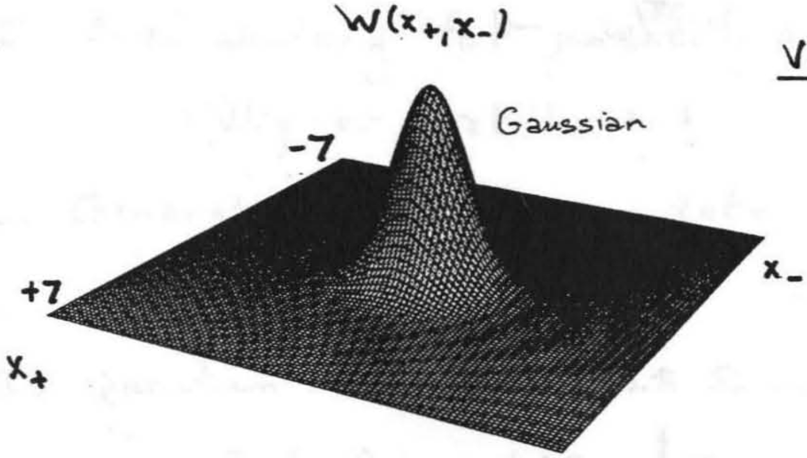
$$\left.\begin{aligned} \hat{X}_+ &\equiv \delta\hat{A} + \delta\hat{A}^\dagger \\ \hat{X}_- &\equiv \tfrac{1}{i}(\delta\hat{A} - \delta\hat{A}^\dagger) \end{aligned}\right\}$$

Note - $\hat{X}_+, \hat{X}_-$ canonical variables with $[\hat{X}_+, \hat{X}_-] = 2i$

Wigner Distributions - $W(x_+, x_-)$

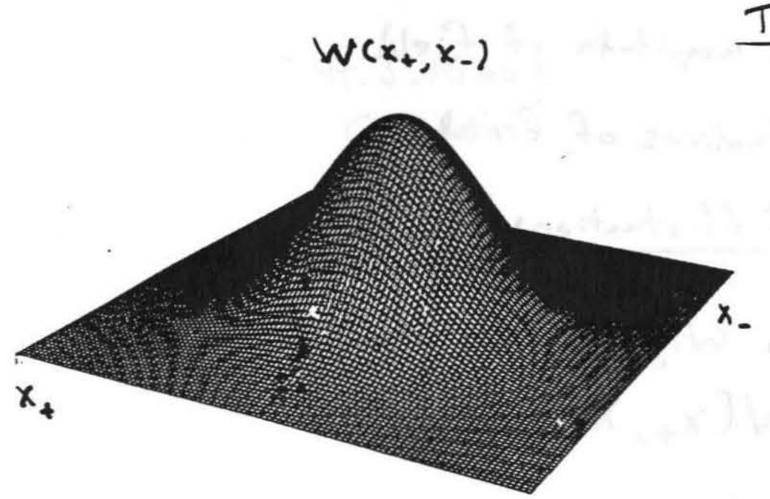
Recall that $[\hat{x}_+, \hat{x}_-] = 2i$

$$\Rightarrow \quad \Delta x_+^2 \, \Delta x_-^2 \geq 1$$



Vacuum State $|0\rangle$

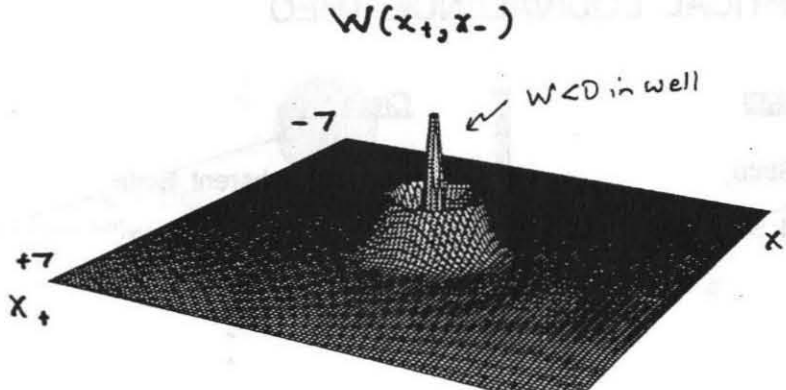
- $\langle \hat{A} \rangle = 0$
- Zero-point fluctuations $\sigma_\pm^2 = 1$
- $\langle n \rangle = 0$ photons



Thermal Field $\hat{\rho}_{th}$

- $\langle \hat{A} \rangle = 0$
- increased, symmetric fluctuations $\sigma_\pm^2 = 5$
- $\langle n \rangle = 2$ photons

$W(x_+, x_-)$, cont.



Fock (number) State

$|n\rangle = |2\rangle$

- $\langle \hat{A} \rangle = 0$
- $n = 2$ exactly

Squeezed State

$|r\rangle$

- $\langle \hat{A} \rangle = 0$
  "Squeezed vacuum"
- phase dependent redistribution of quantum fluctuation

  $\sigma_+^2 = 1/10$

  $\sigma_-^2 = 10$

  $\{\sigma_+ \sigma_- = 1\}$
- $\langle n \rangle = 2$ photons

Manifestly Quantum or Nonclassical States of the Electromagnetic Field

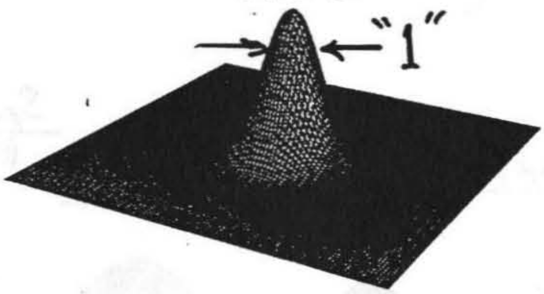
OPTICAL EQUIVALENCE THEOREM

Manifestly Quantum	Classical
Fock State	Coherent State
Squeezed State	Thermal State
⋮	⋮
Los Angeles	Boston

Vacuum State

"1"

Scale for variation $\ll 1$



Scale for variation $\gg 1$

Fields from classical (stochastic) current sources

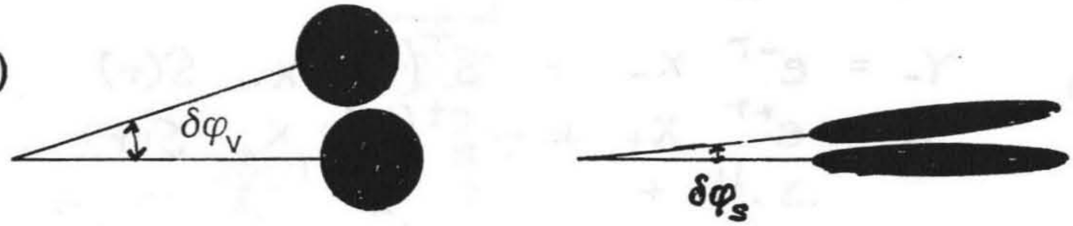
Fields which require quantum probability amplitudes

Squeezed (Nonclassical) Light for Sensitivity Beyond Standard Quantum Limit

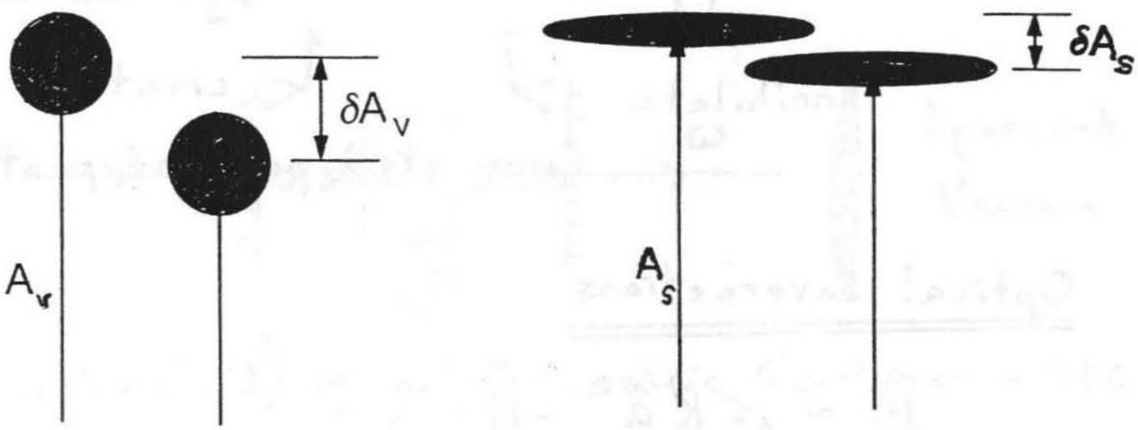
Phase Measurements

$$\delta\varphi_v \simeq \frac{1}{\sqrt{N}} \qquad\qquad \delta\varphi_s \simeq \frac{\Delta X_-}{\sqrt{N}}$$

(a)



(b)

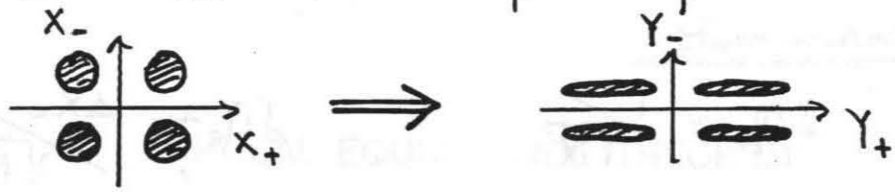


Amplitude Measurements

$$\frac{\delta A_v}{A_v} \simeq \frac{1}{\sqrt{N}} \qquad\qquad \frac{\delta A_s}{A_s} \simeq \frac{\Delta X_+}{\sqrt{N}}$$

Generation of Squeezed States

→ "Elastic" deformation of phase space



$$\text{with} \quad \begin{aligned} Y_- &= e^{-r}\, X_- = S^\dagger(r)\, X_-\, S(r) \\ Y_+ &= e^{+r}\, X_+ = S^\dagger(r)\, X_+\, S(r) \end{aligned}$$

Squeezing generated by transformation

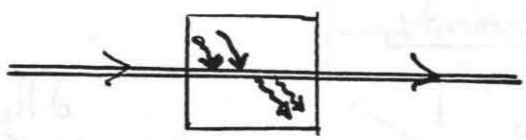
$$S(r) = \exp\left[\tfrac{1}{2}\left(r\,\hat{a}^2 - r\,\hat{a}^{\dagger 2}\right)\right]$$

Annihilate ↗   ↖ create

Correlated pairs of photons

Optical Interactions

$$\hat{H}_I \sim i\hbar K \hat{a}^2 - i\hbar K \hat{a}^{\dagger 2}$$

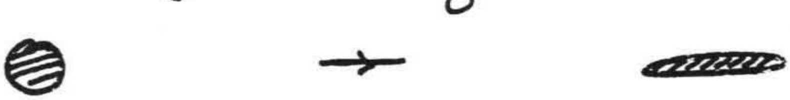
In  Out

K

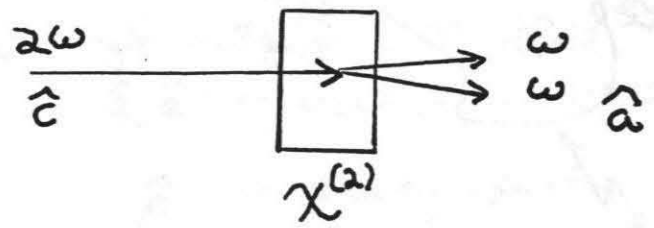
Time evolution operator

$$U(t) \sim \exp\left\{ (Kt)\hat{a}^2 - (Kt)\,\hat{a}^{\dagger 2} \right\} \longleftrightarrow S(Kt)$$

$U(t)$ generates squeezed state!

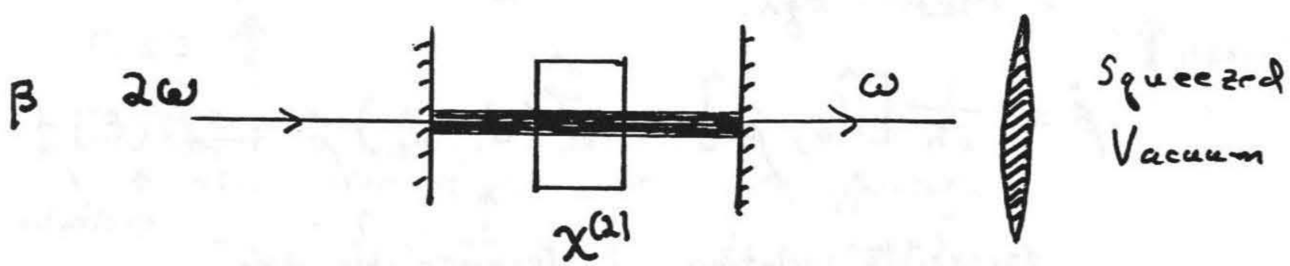
In  out

Squeezed State Generation by Parametric Down Conversion

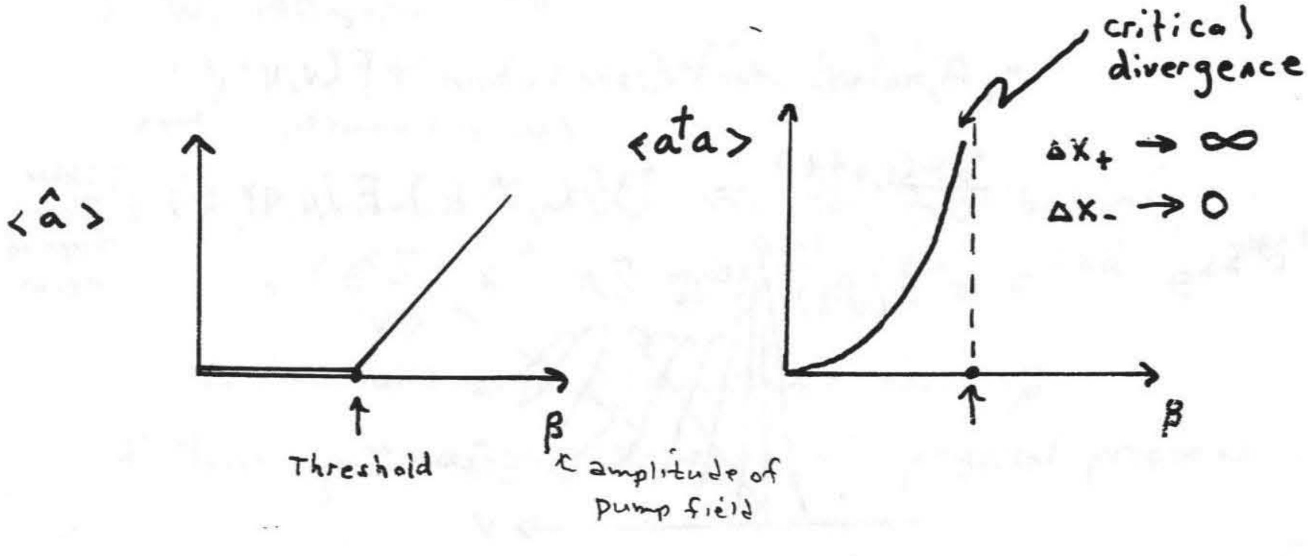


$$\hat{H}_0 \sim \chi^{(2)} \, \hat{c} \, \hat{a}^{\dagger 2} + \text{H.c.}$$

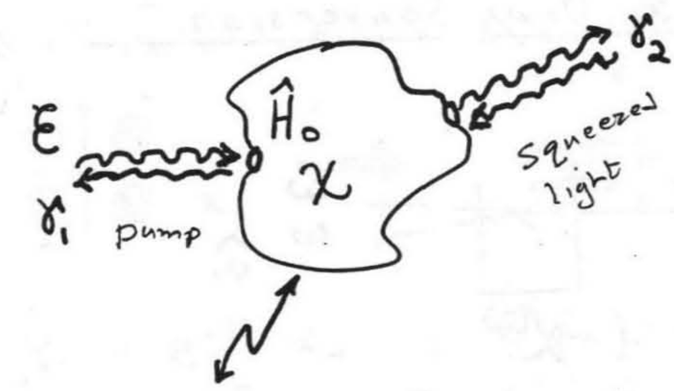
Experiment



(Subthreshold) Optical Parametric Oscillator - OPO



Dynamics of Open Quantum Systems



$\hat{\rho}$ - Reduced density operator

<u>A well-worn path -</u>

- Schroedinger Eqn.

$$\Big\downarrow \quad \omega_0 \gg (\chi, \gamma)$$

- Master Eqn.

$$\dot{\hat{\rho}} = \underset{\substack{\uparrow \\ \text{reversible evolution}}}{\frac{1}{i\hbar}\left[\hat{H}_0, \hat{\rho}\right]} + \underset{\substack{\uparrow \\ \text{irreversible decay}}}{\hat{\mathcal{L}}(\gamma_1, \gamma_2)\hat{\rho}} + \hat{\mathcal{G}}\underset{\substack{\uparrow \\ \text{excitation}}}{(\varepsilon)}\hat{\rho}$$

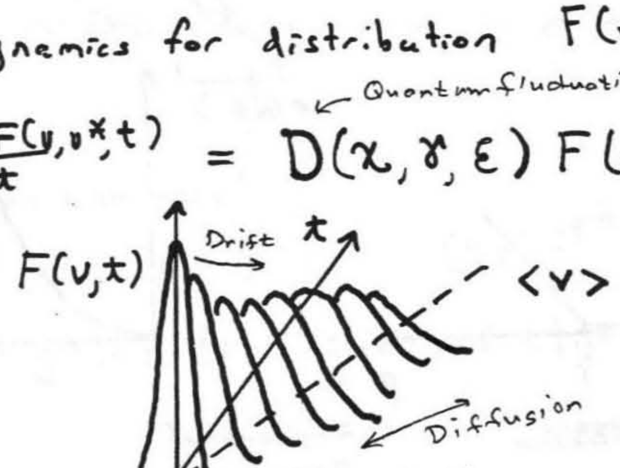
$$\Big\downarrow \quad \chi \ll \gamma$$

- Dynamics for distribution $F(v, v^*)$ ← Wigner Distribution

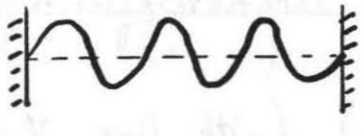
$$\frac{\partial F(v, v^*, t)}{\partial t} = D(\chi, \gamma, \varepsilon)\, F(v, v^*, t)$$

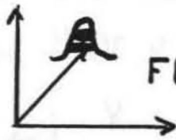
← Quantum fluctuations are here

Fokker-Planck type of equation



Quasiprobability Distributions





- Consider single mode of EM field

$\hat{a}, \hat{a}^{+} \leftrightarrow$ creation, annihilation operators

$\hat{\rho} \leftrightarrow$ density operator

$= |\psi\rangle\langle\psi|$ for pure state

- "Distribution" $F(v, v^*)$ from $\hat{\rho}$?

$$F(v, v^*) = \frac{1}{\pi^2} \int d^2 z \, e^{-i(z^* v^* + z v)} \, \mathrm{Tr}\left[\hat{\rho} \, e^{i(z^* \hat{a}^{+} + z \hat{a})}\right]$$

C-#'s ↑ ↑ operators

- Note

1. Association of c-numbers with operators

e.g. $\left\{ \begin{array}{l} v \leftrightarrow \hat{a} \\ v^* \leftrightarrow \hat{a}^{+} \end{array} \right\}, \left\{ \begin{array}{l} x_+ \leftrightarrow \hat{X}_+ \\ x_- \leftrightarrow \hat{X}_- \end{array} \right\}$

2. Wavefunction $|\psi\rangle$?

e.g. $|\psi(x_{\pm})|^2 = \int W(x_+, x_-) \, dx_{\mp}$

3. Field commutation relations for "distributions"

$$e^{i(z^* \hat{a}^{+} + z \hat{a})} \neq e^{i z^* \hat{a}^{+}} e^{i z \hat{a}} \neq e^{i z \hat{a}} \, e^{i z^* \hat{a}^{+}}$$

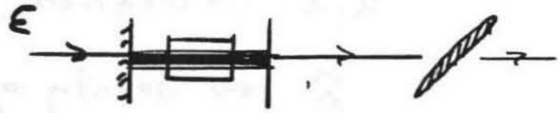
∴ Can build many distributions from $\hat{\rho}$

* Nonuniqueness in discussion of physical processes

Positive P Representation for Optical Parametric Oscillator (OPO)*

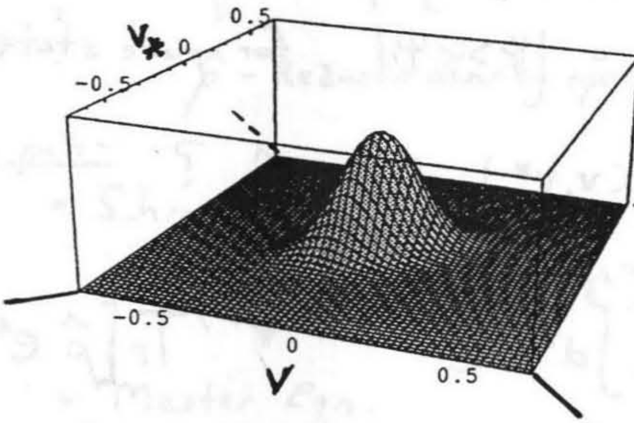
$$\left.\begin{matrix} v \\ v_* \end{matrix}\right\} \rightarrow P(v, v_*) \text{ , with line } v_* = v \text{ as "classical" dimension}$$
$$\text{and } v_* = -v \text{ as "nonclassical" dimension}$$

$\mathcal{E}$



Below threshold –

$\mathcal{E}/\mathcal{E}_{th} = 0.5$



classical

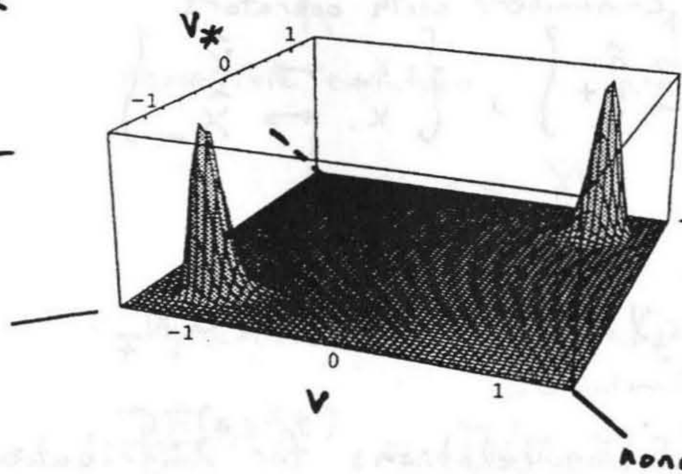
nonclassical

$$g \equiv \frac{\chi}{\sqrt{\kappa \gamma_2^2 \gamma_1}} \sim \frac{1}{\sqrt{n_0}}$$

$$g = 0.2$$

Above threshold –

$\mathcal{E}/\mathcal{E}_{th} = 2$



classical

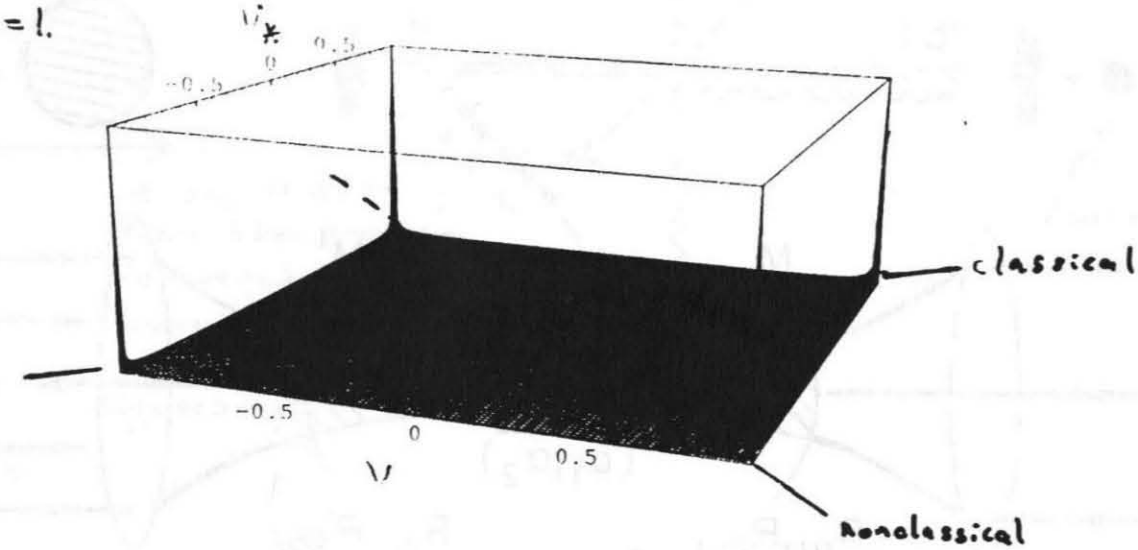
nonclassical

* Wolinsky and Carmichael, PRL 60, 1836 (1988).

From Squeezed States to Coherent Superpositions

Strong Coupling $\quad g \sim \frac{\chi}{\gamma} \sim \frac{1}{\sqrt{n_0}}$ $\quad g = 5$ $\Big\}$ Pathological Distribution! Positive $P(v, v_*)$

$\mathcal{E}/\mathcal{E}_{th} = 1.$



$P(v, v_*) \rightarrow$ delta functions at 4 corners

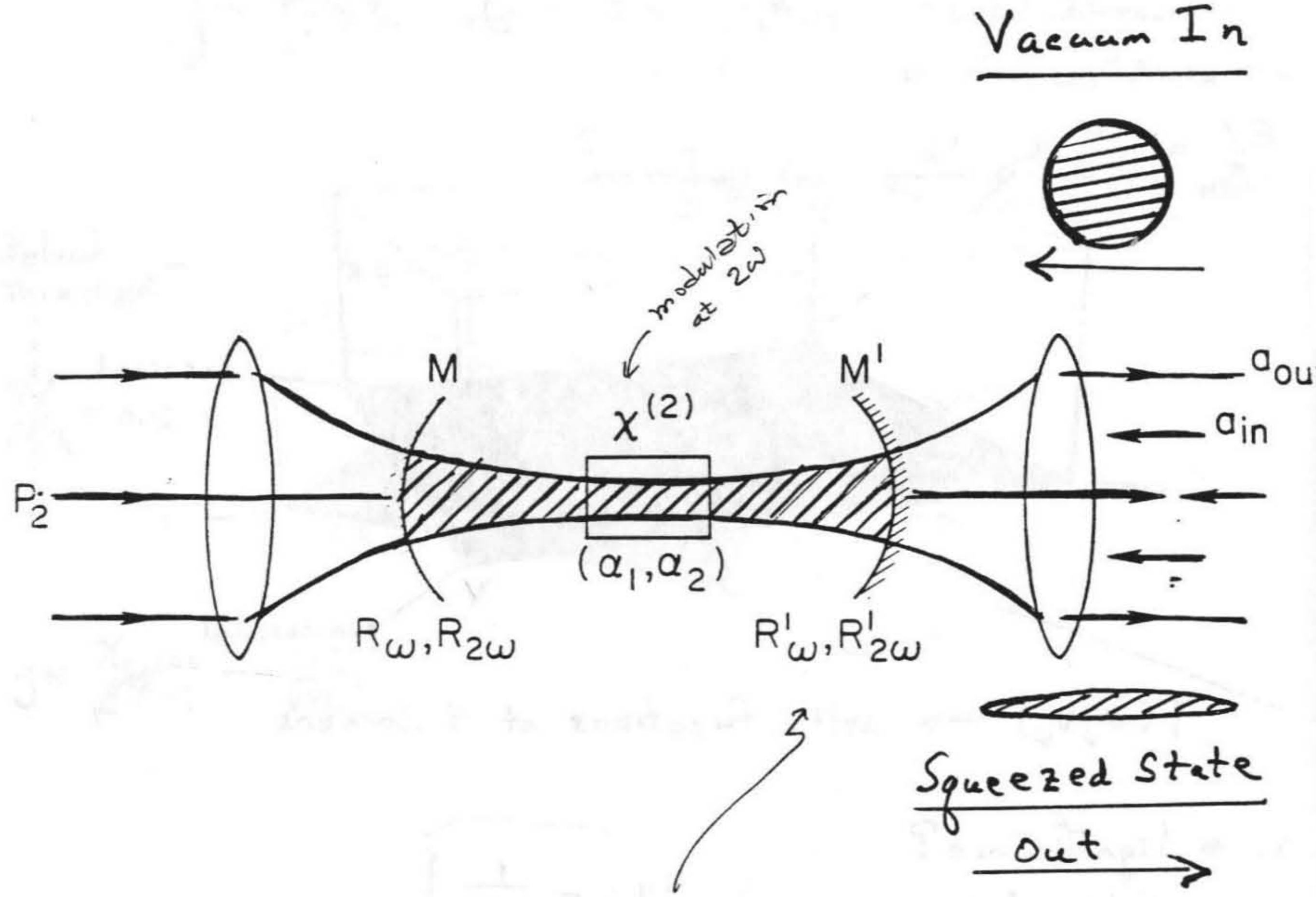
- Significance?
  Coherent superposition $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\left[|\alpha_0\rangle + |-\alpha_0\rangle\right]$

  $\alpha_0 = 1/g$

From a dissipative dynamical system!
{See also Carmichael, Haroche, Meystre, ... within context of cavity Q.E.D. for large g }

Wolinsky ; Carmichael

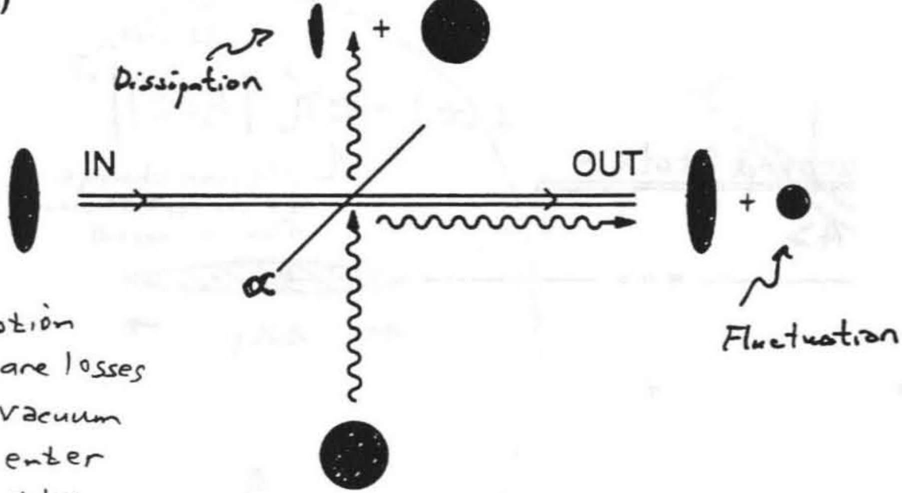
Generation of Squeezed Light



Would like to just wiggle this mirror, but cannot do so fast enough

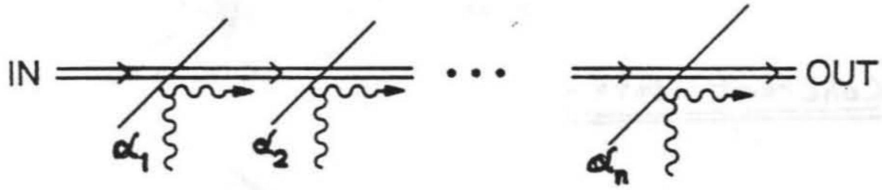
Fluctuation - Dissipation Theorem - The Grim Reaper for Nonclassical Fields

(a)



At any location where there are losses (dissipation), vacuum fluctuations enter and degrade the squeezing

(b)



$$\Delta X^2_{out} = (1-\bar{\alpha}) + \bar{\alpha}\,\Delta X^2_{in}$$

$\bar{\alpha}$ - overall efficiency $\quad 0 \le \bar{\alpha} \le 1$

$$\bar{\alpha} = \prod_{i=1}^{n} \alpha_i$$

Many tiny losses kill you! Every kind of loss is bad news!

$\underline{\underline{\bar{\alpha} \to 0}}$

$$\Rightarrow \quad \Delta X^2_{out} \longrightarrow 1$$

↑ vacuum-state limit

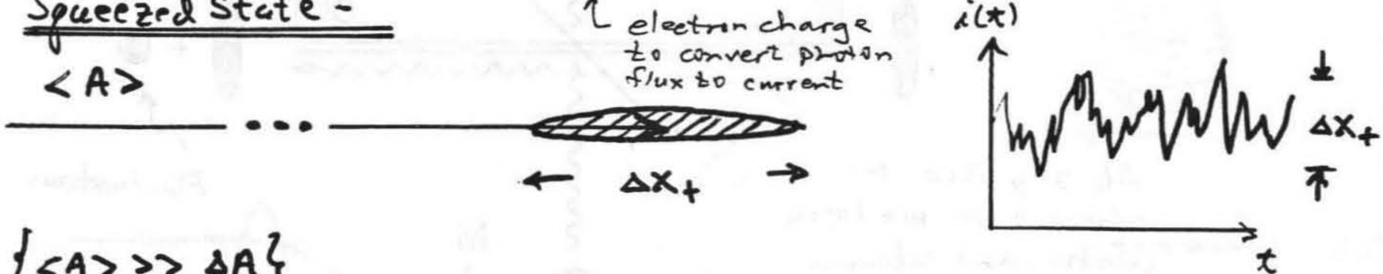
Photoelectric Detection of Squeezed Light

$$i(t) \sim e\,\eta\,|A(t)|^2$$

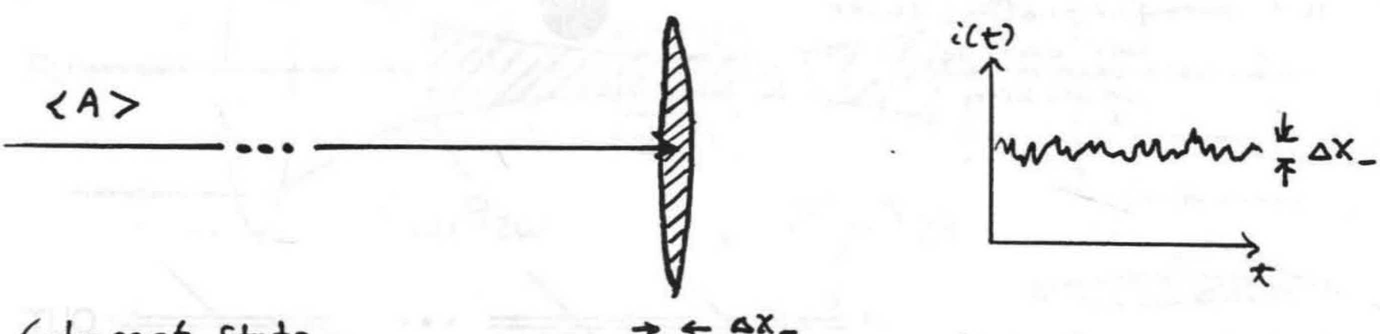
efficiency

electron charge to convert photon flux to current

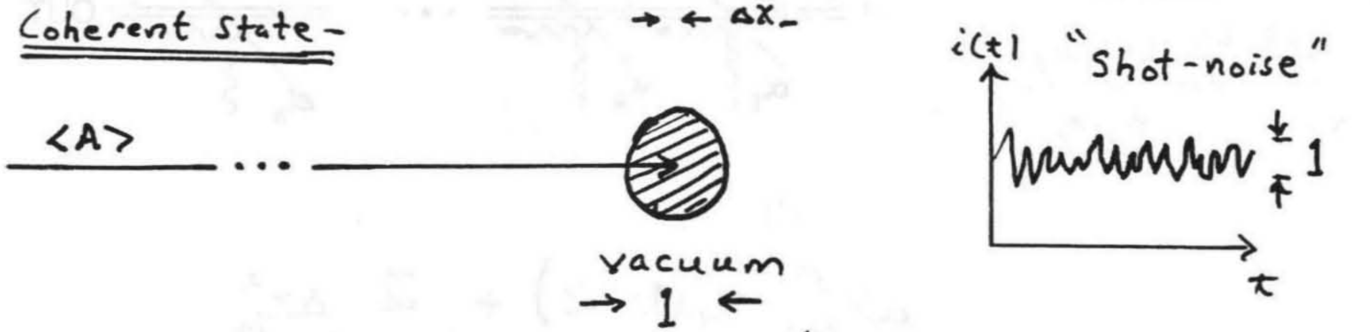
Squeezed State -



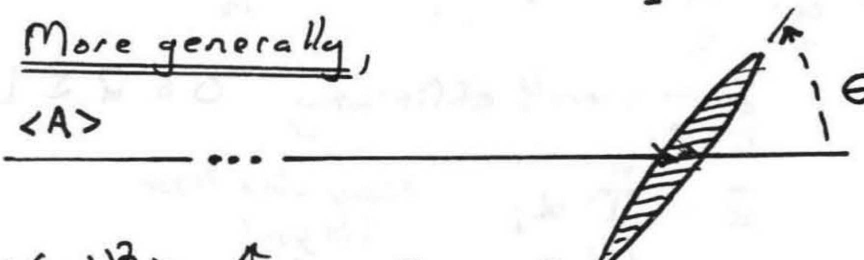
$\{\langle A \rangle \gg \Delta A\}$



Coherent State -

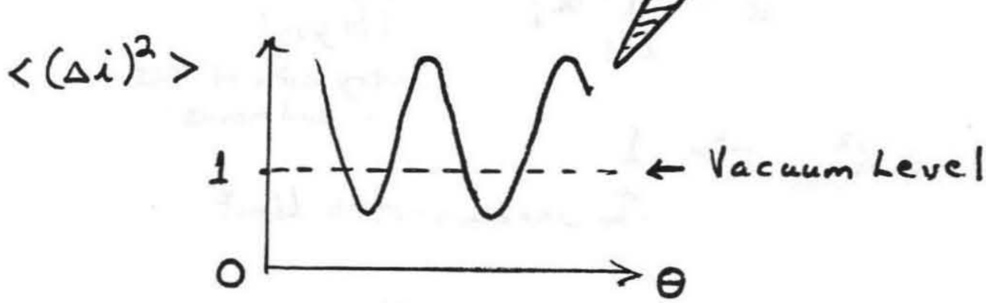


More generally,

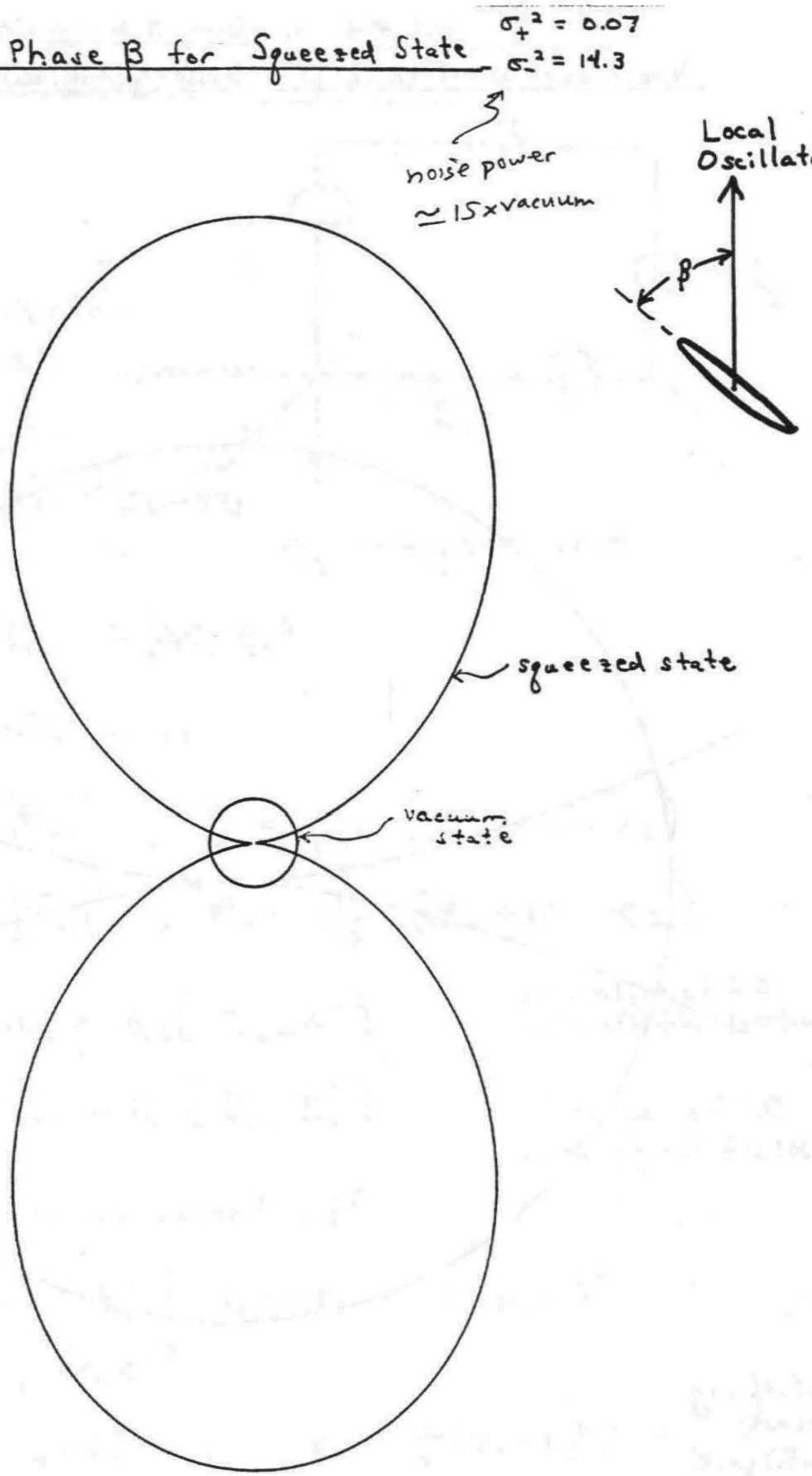


"Quantum Shish Kebab"

gradually change θ to see the squeezing

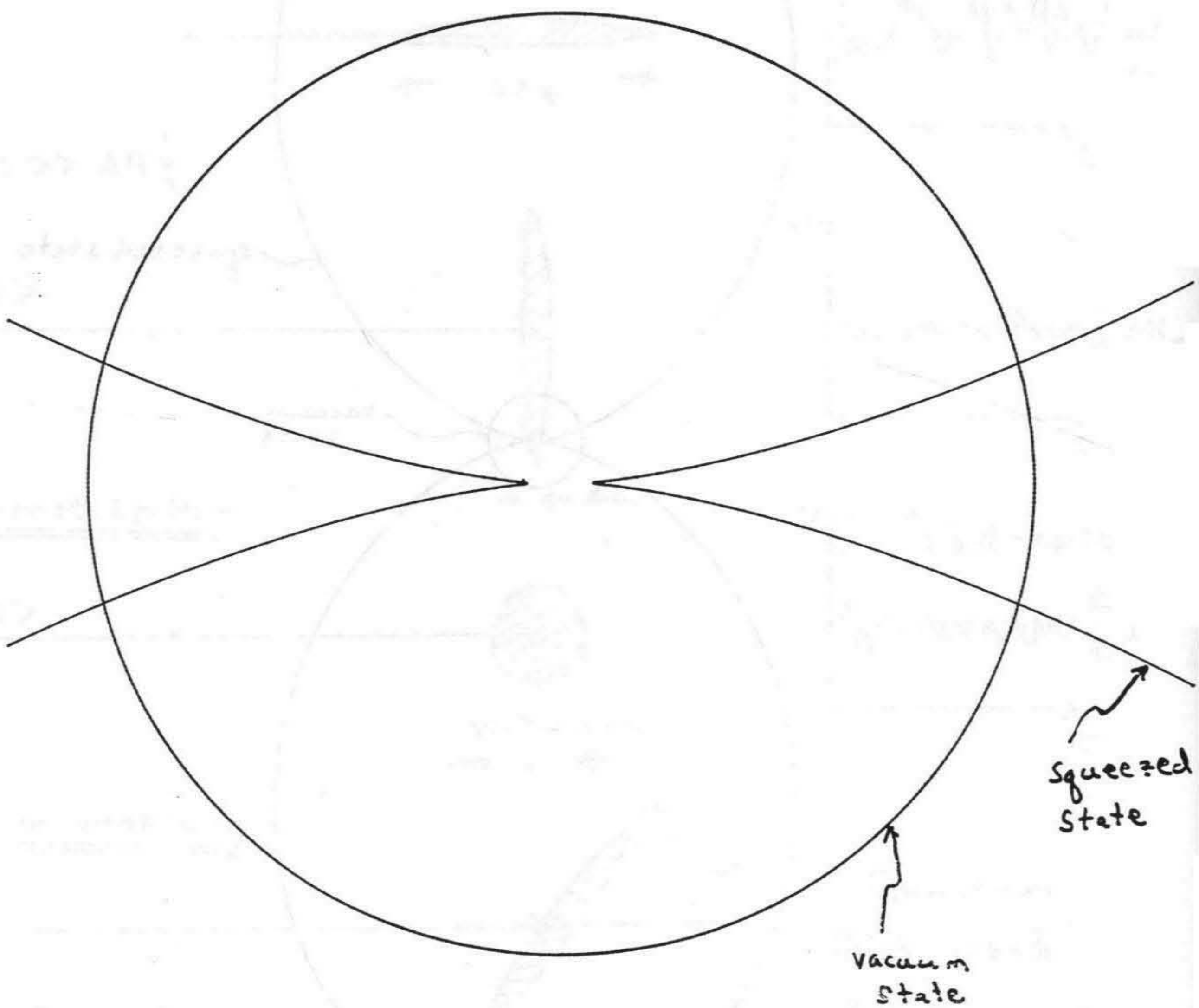


Variance vs. Phase β for Squeezed State



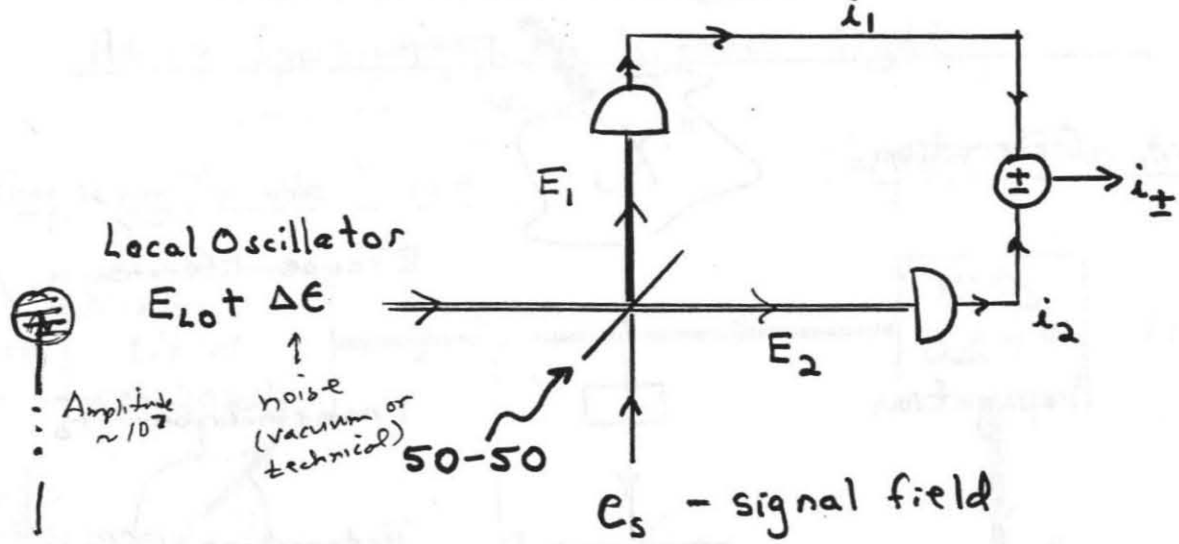
Variance vs. Phase β – Enlarged View

$\sigma_+^2 = 0.07$
$\sigma_-^2 = 14.3$



Balanced Homodyne Detector -

The Basic Idea



Assume $E_{LO} >> (\Delta\epsilon, e_s)$

Photocurrents i_1, i_2

$$i_1 \propto |E_1|^2 \quad , \quad E_1 = \frac{1}{\sqrt{2}}\left(E_{LO} + \Delta\epsilon + e_s\right)$$

$$i_2 \propto |E_2|^2 \quad , \quad E_2 = \frac{1}{\sqrt{2}}\left(E_{LO} + \Delta\epsilon - e_s\right)$$

Hence $\Delta i_+ \sim \mathrm{Re}\left[E_{LO}\Delta\epsilon^*\right]$ Homodyne of LO with its fluctuations

$\Delta i_- \sim \mathrm{Re}\left[E_{LO} e_s^*\right]$ Homodyne of LO with signal field

Noise power in photocurrent $\sim i^2$

Hence $\Delta i_-^2 \sim \mathcal{E}_0^2 \left[X_+ \cos\theta + X_- \sin\theta \right]^2$

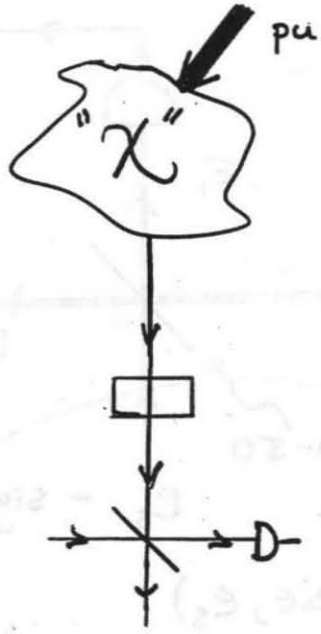
where $E_{LO} = \mathcal{E}_0 e^{i\theta}$

$$X_+ = e_s + e_s^* \quad , \quad X_- = \frac{1}{i}(e_s - e_s^*) \longleftrightarrow \text{quadrature phase amplitudes}$$

* Interrogate signal quadratures by varying LO phase β

* Quality of balancing

Generic Experiment



Generation

Escape Efficiency ρ

Propagation — Transmission T_0

Heterodyne Efficiency η

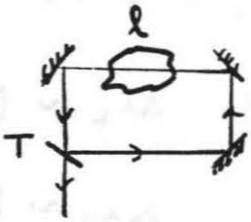
Detection — Detector q.e. α

Recall $\quad R = \underbrace{\xi}_{\text{efficiency}} \Delta X^2 + (1-\xi)$

with $\quad \xi = \rho T_0 \eta^2 \alpha$

$\rho \leftrightarrow$ losses in medium

* $\rho \simeq 0.98$ [-17 dB]



$l \sim 2\times10^{-3}$
$T \sim 0.1$

$\rho \sim \frac{T}{T+l}$

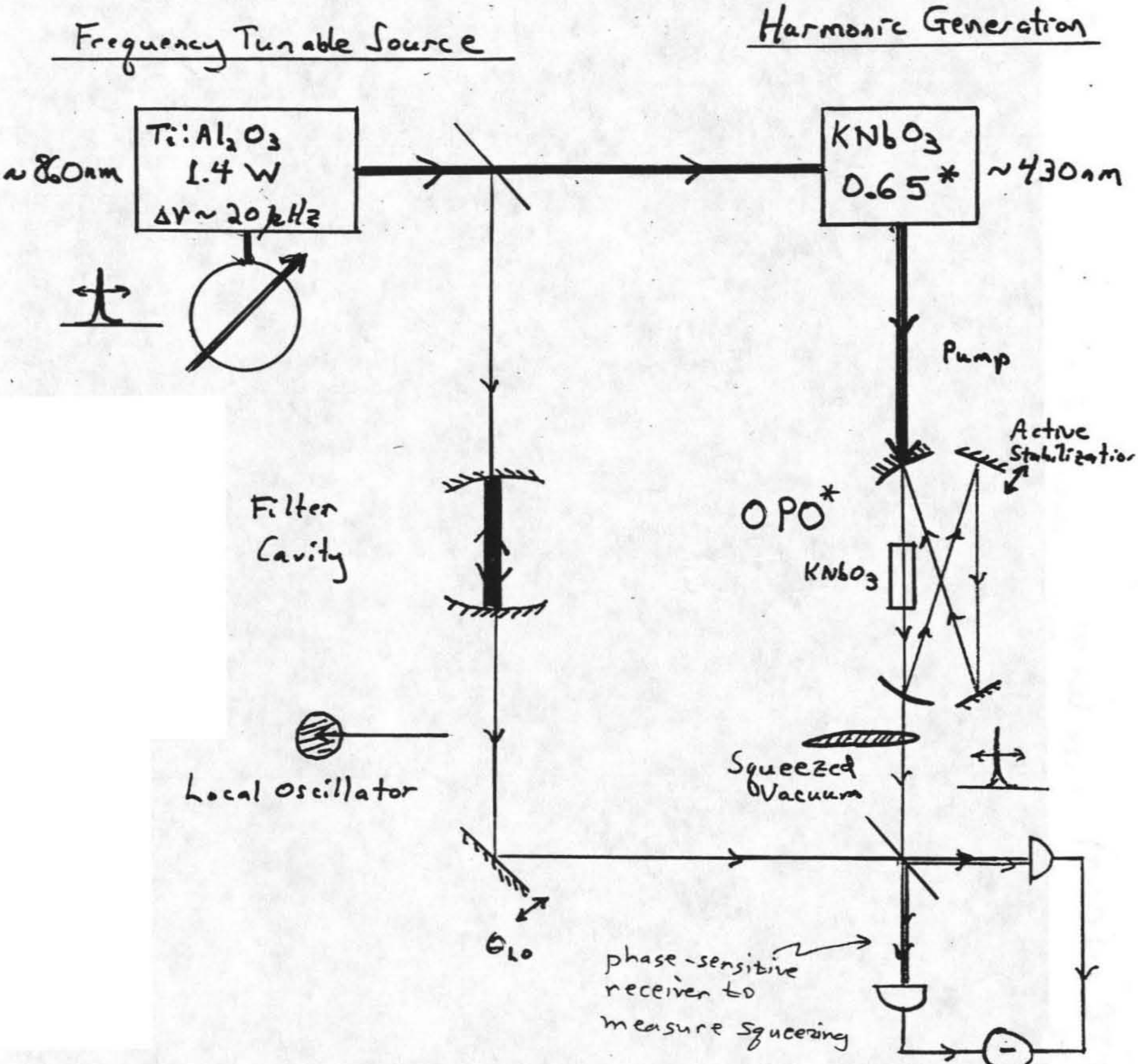
$T_0 \lesssim 0.99$ Transmission

$\eta \lesssim 0.99$ Heterodyne efficiency

$\alpha \lesssim 0.99$ Photodetector efficiency

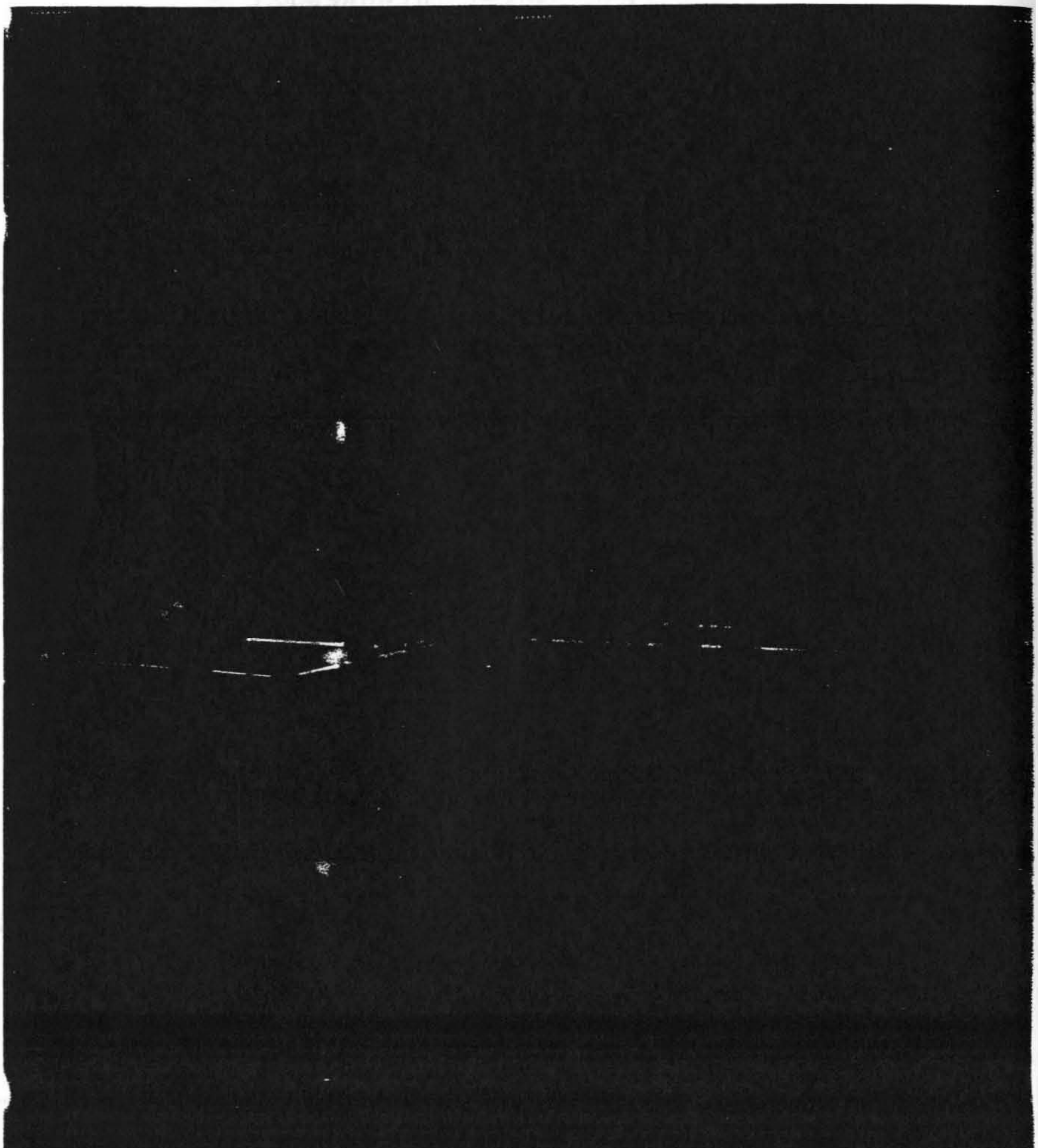
Total $\xi \lesssim 0.94$ $\{17\times, -12\,dB\}$
observed

Experimental Arrangement -
Atomic Spectroscopy with Squeezed Light
[in Kimble's lab]

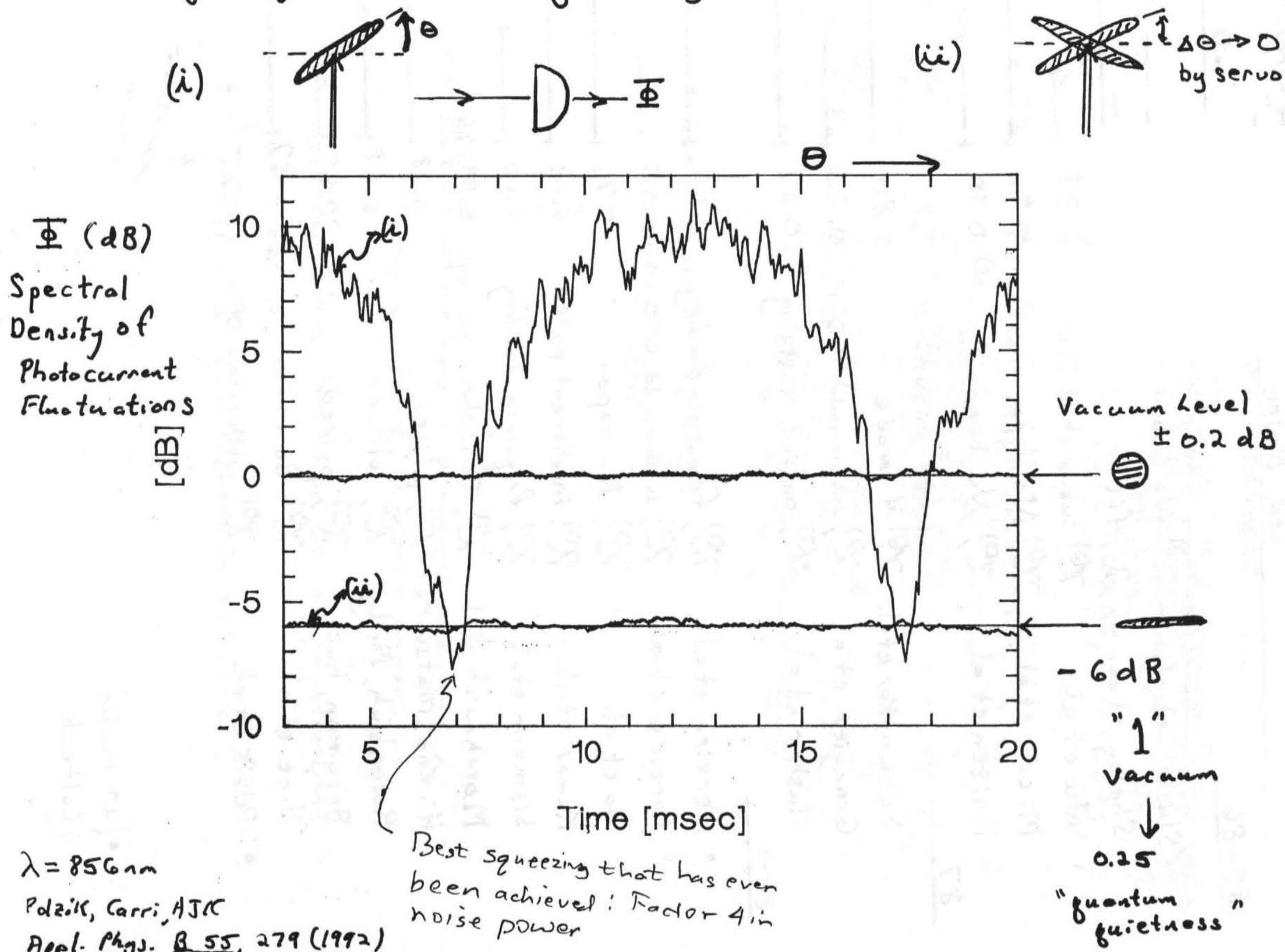


* E.S. Polzik & HJK, Opt. Lett. 16, 1400 (1991)
E.S. Polzik, J. Carri, & HJK, PRL 68, 3020 (1992)
and * Appl. Phys. B (Oct., 1992)

a-1=17



Frequency Tunable Squeezing



$\lambda = 856\,nm$

Polzik, Carri, HJK

Appl. Phys. B 55, 279 (1992)

Status of Observed Noise Levels R

[circa 1992]

85-86		Vacuum-State Limit	1.0
Slusher et al.	$\chi^{(3)}$, Na beam	0.75	
Shelby et al.	$\chi^{(3)}$, fiber	0.87	
• Wu et al.	$\chi^{(2)}$, subthreshold OPO	0.37	
Maeda et al.	$\chi^{(3)}$, Na Vapor	0.96	
• Raizen et al.	$\chi^{(3)}$, Na beam	0.70	
87			
Schumaker et al.	$\chi^{(3)}$, 4-mode	0.80	
Grangier et al.	$\chi^{(2)}$, subthreshold OPO	0.63	
Slusher et al.	$\chi^{(2)}$, pulsed squeezing	0.87	
88→			
• Pereira et al.	$\chi^{(2)}$, frequency doubling	0.87	
• Pereira et al.	$\chi^{(2)}$, subthreshold OPO	0.45	
Ho et al.	$\chi^{(3)}$, Na vapor	0.75	
Kumar et al.	$\chi^{(2)}$, incoherent pulse	0.83	
Sizmann et al.	$\chi^{(2)}$, frequency doubling	0.60	
Movshovich et al.	$\chi^{(2)}$, Josephson paramp	0.99896	
Hirano, Matsuoka	$\chi^{(2)}$, pulsed	0.78	
Rosenbluh, Shelby	$\chi^{(3)}$, solitons	0.68	
Bergman, Haus	$\chi^{(3)}$, pulsed	0.32	
Hope et al.	$\chi^{(3)}$, Ba beam	0.87	
• Polzik et al.	$\chi^{(2)}$, subthreshold OPO	0.25	

⋮

↑ power below vacuum

• { UT Austin, Caltech }

Omnivorous Creatures

Not Included in Theoretical Model Community

[Practical factors that impede better squeezing]

- Thermally excited noise {TEXAS, GAWBS, ...}

- Transverse effects in propagation
  - $\chi^{(2)}$ - Slusher et al.
  - $\chi^{(3)}$ - Shapiro et al.

- Raman Scattering
  - $\chi^{(3)}$ - Solitons (Shelby, Drummond, ...)

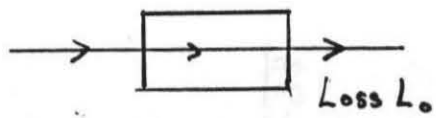
- Light induced absorption
  - $\chi^{(2)}$ - Polzik

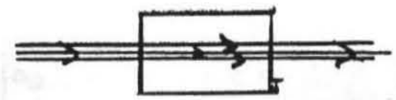
⋮

→ Nature of microscopic processes underlying χ
Thus far only for $\chi^{(3)}$ in atomic vapors

→ Self consistent treatment of propagation and quantization
Solitons; quantization in dielectric

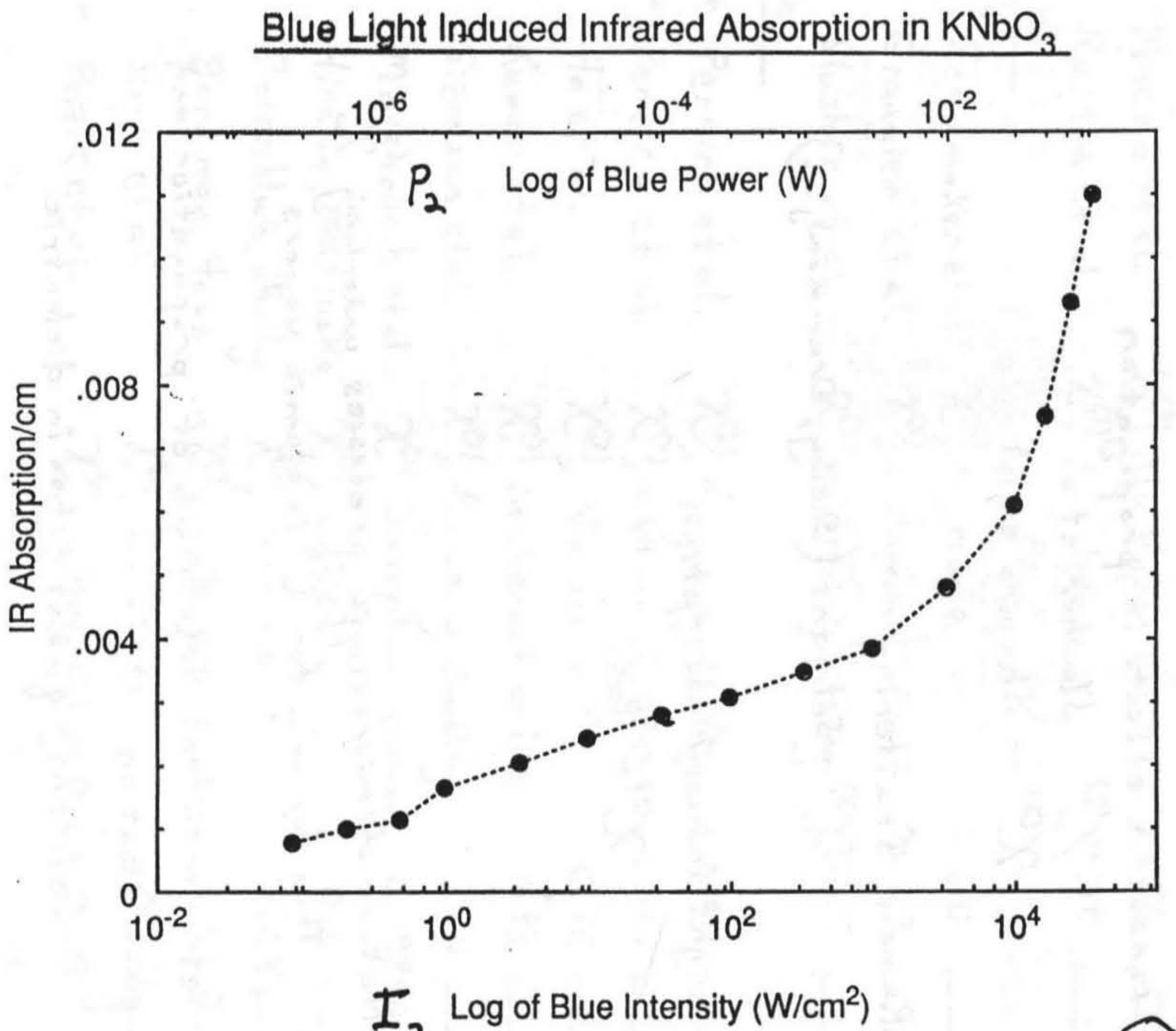
Trouble in River City!

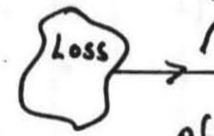




$L_2(P_2)$

(1/cm)





30x squeezing until turn up the power

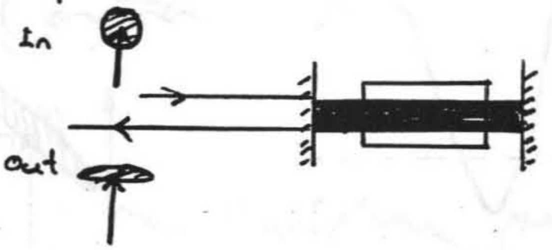
$\rho = 0.972$

$\rho(P_2) \simeq 0.875$

result of turning up power

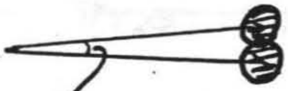
Moral?

→ Squeezing with "large" coherent amplitude invites other problems



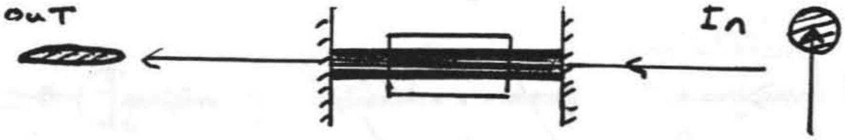
Technical noise makes squeezing much harder when the amplitude is large than when squeezing the vacuum.

Allowed $\Delta\varphi$?



$\Delta\varphi < 1/\sqrt{N}$

→ By contrast, squeezed vacuum by way of parametric down conversion

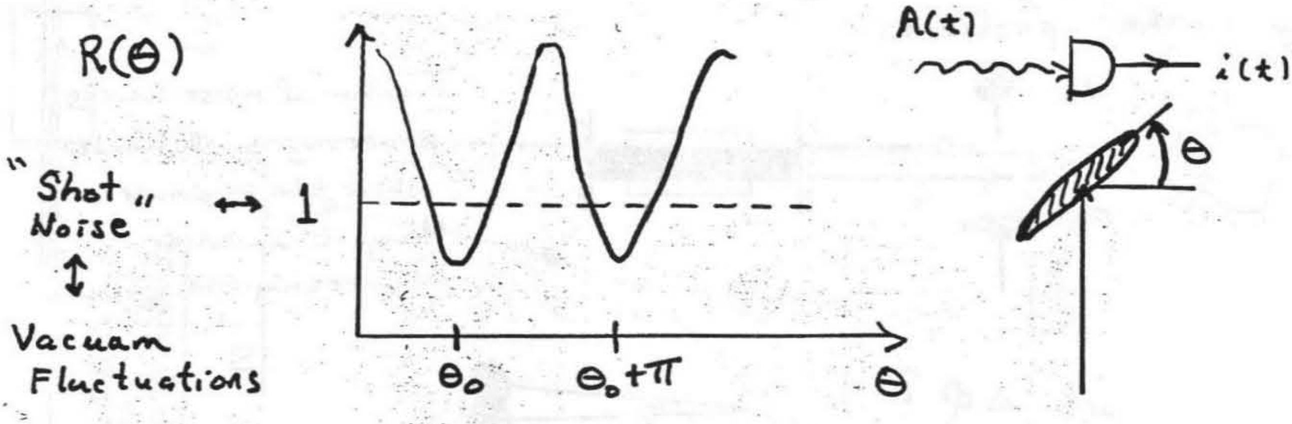


Allowed $\Delta\varphi$?

$$\Delta\varphi \sim \left[R_- / R_+ \right]^{1/2}$$

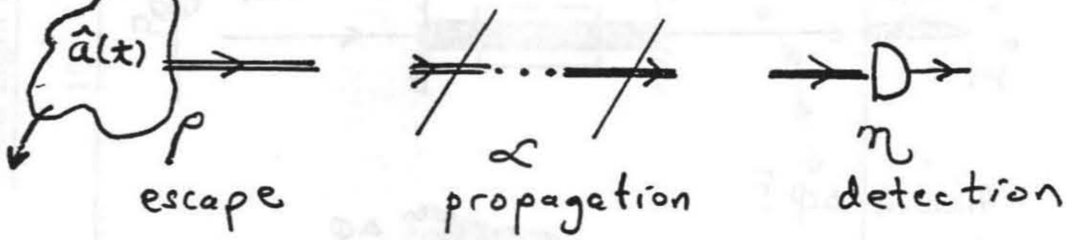
{Recall pump quantization, Caves et al.}
laser linewidth effects, Walls et al.

Inference from Photocurrent Statistics to Field Statistics



$$R(\theta) = \left[(1-\xi) + \xi \langle (\Delta X(\theta))^2 \rangle \right]$$

ξ - overall system efficiency



$$\xi = \rho \alpha \eta$$

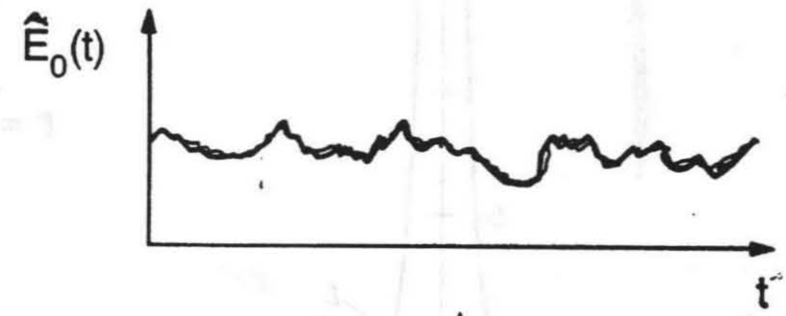
- From $R(\theta)$ and ξ extract $\langle (\Delta X(\theta))^2 \rangle$

$\theta = 0$, ΔX_+

$\theta = \pi/2$, ΔX_-

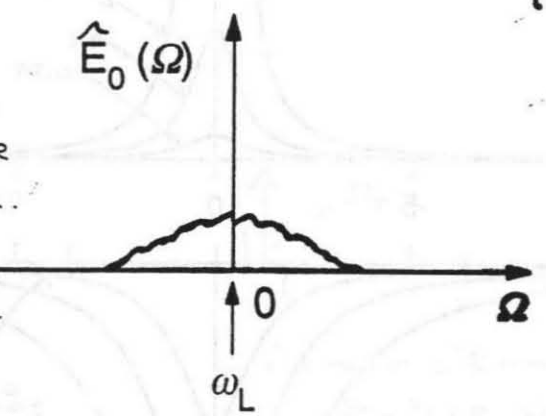
Minor Complication –
Distribution in frequency Ω of field fluctuations

At plane Γ, take $\hat{E}_0 e^{-i\omega_L t}$



Fourier transform

In practice, one does not use a degenerate pump; instead $\omega \pm \Omega$. Squeezing correlates the fields at $\omega+\Omega$ and $\omega-\Omega$



- Spectrum of Squeezing $S(\Omega)$*

annihilation at Ω — creation at $-\Omega$

Quadrature amplitudes $\hat{X}_\theta(\Omega) = \hat{E}_0(\Omega)e^{-i\theta} + \hat{E}_0^+(-\Omega)e^{i\theta}$

$$\Rightarrow \quad \langle \hat{X}_\theta(\Omega), \hat{X}_\theta(\Omega') \rangle = \left[1 + S_\theta(\Omega)\right] \delta(\Omega+\Omega')$$

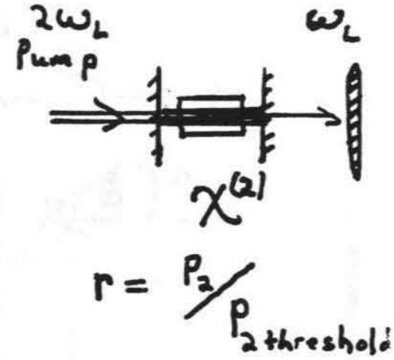
Spectral density for quantum noise at $\pm\Omega$ relative to ω_L

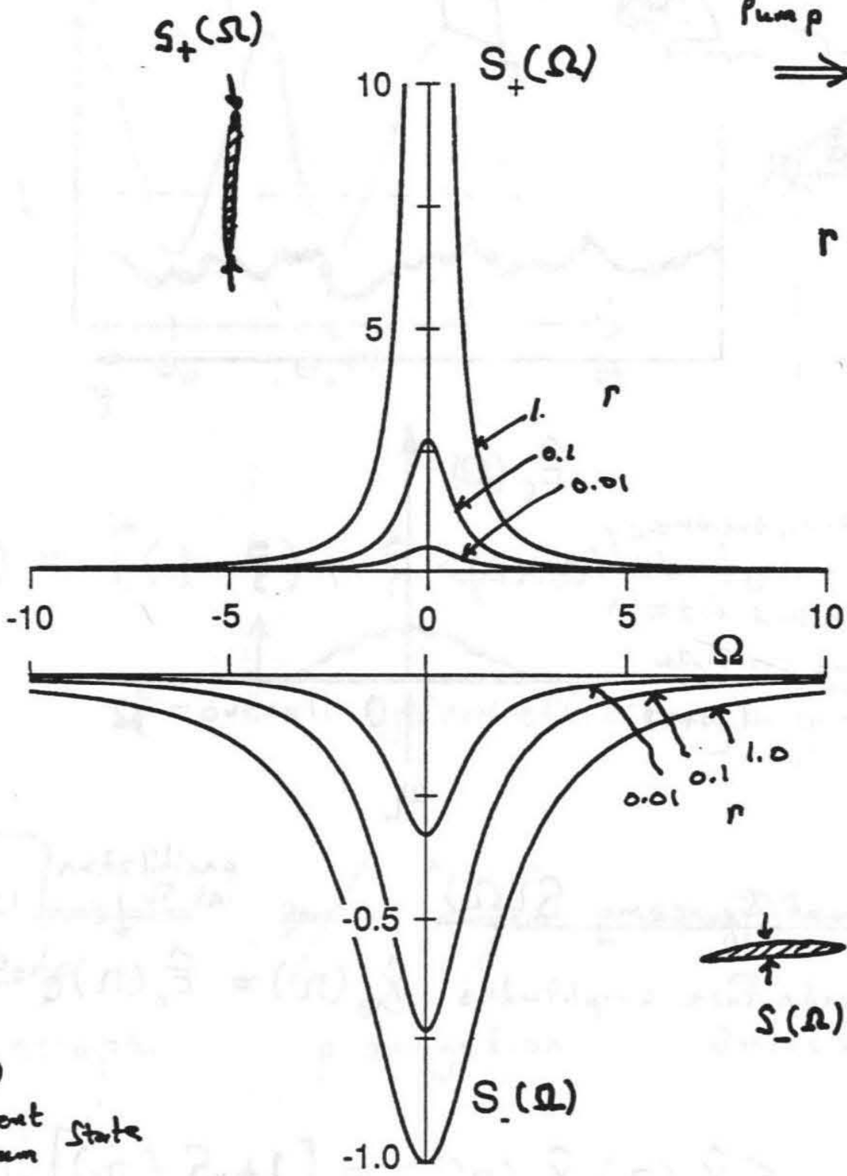
Units ~ (photons/sec)/bandwidth

– dimensionless

* Gardiner et al.

Spectrum of Squeezing $S_{\pm}(\Omega)$
for Subthreshold OPO





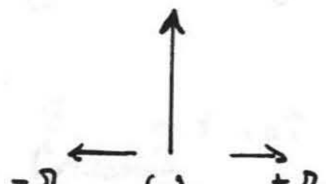
Note -

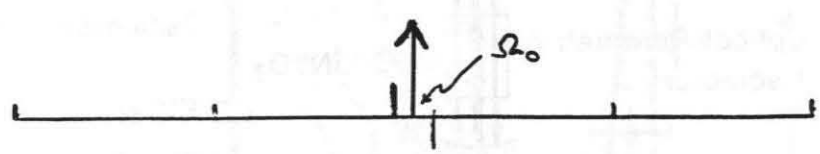
$S(\Omega) = 0$
coherent state
vacuum

$S(\Omega, \theta) \to -1$

$S(\Omega, \theta \pm \frac{\pi}{2}) \to +\infty$

limit of perfect squeezing





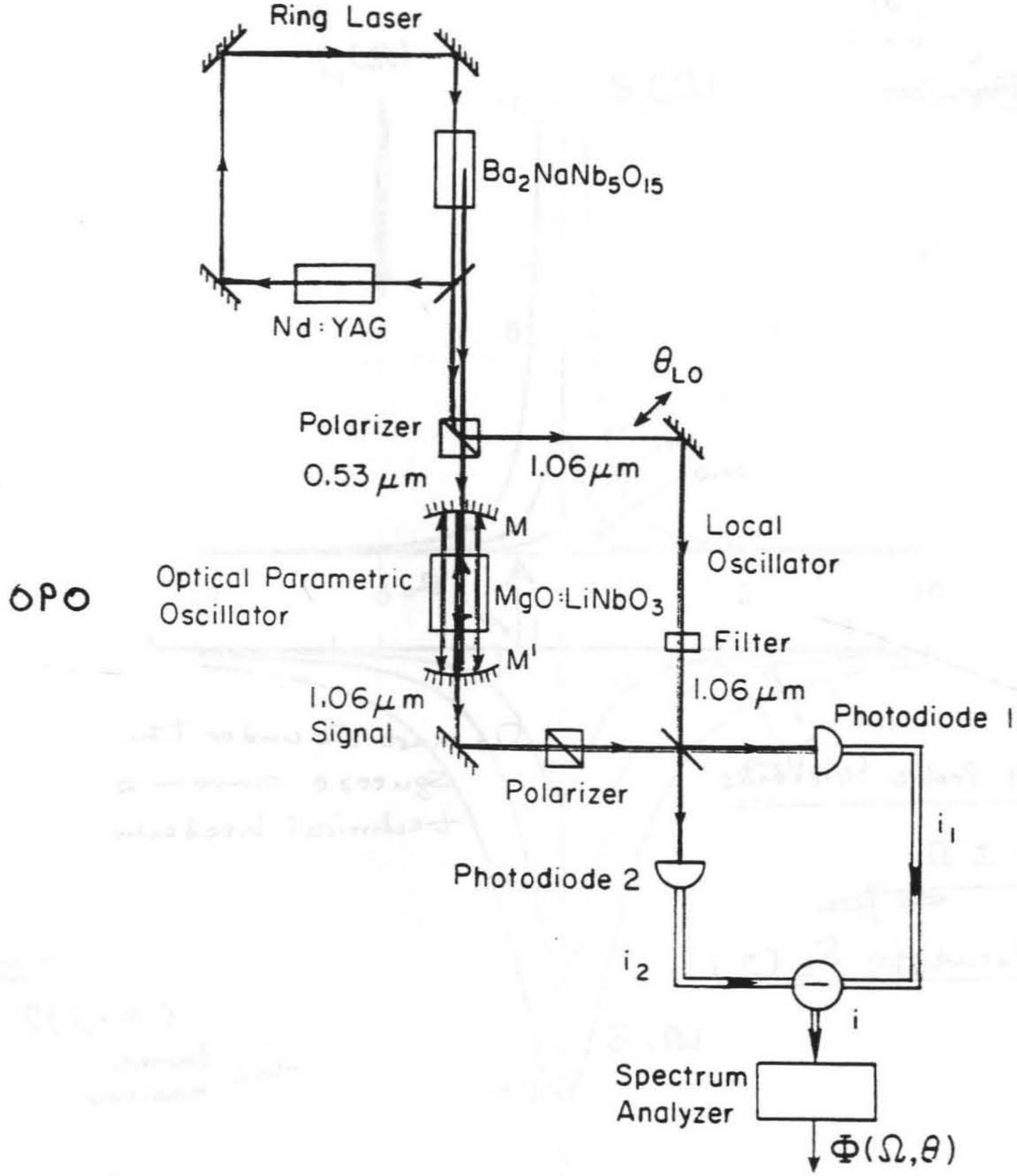
Ω_0 must sit under the squeeze curve — a technical headache

FM Probe Sidebands

at $\pm \Omega_0$

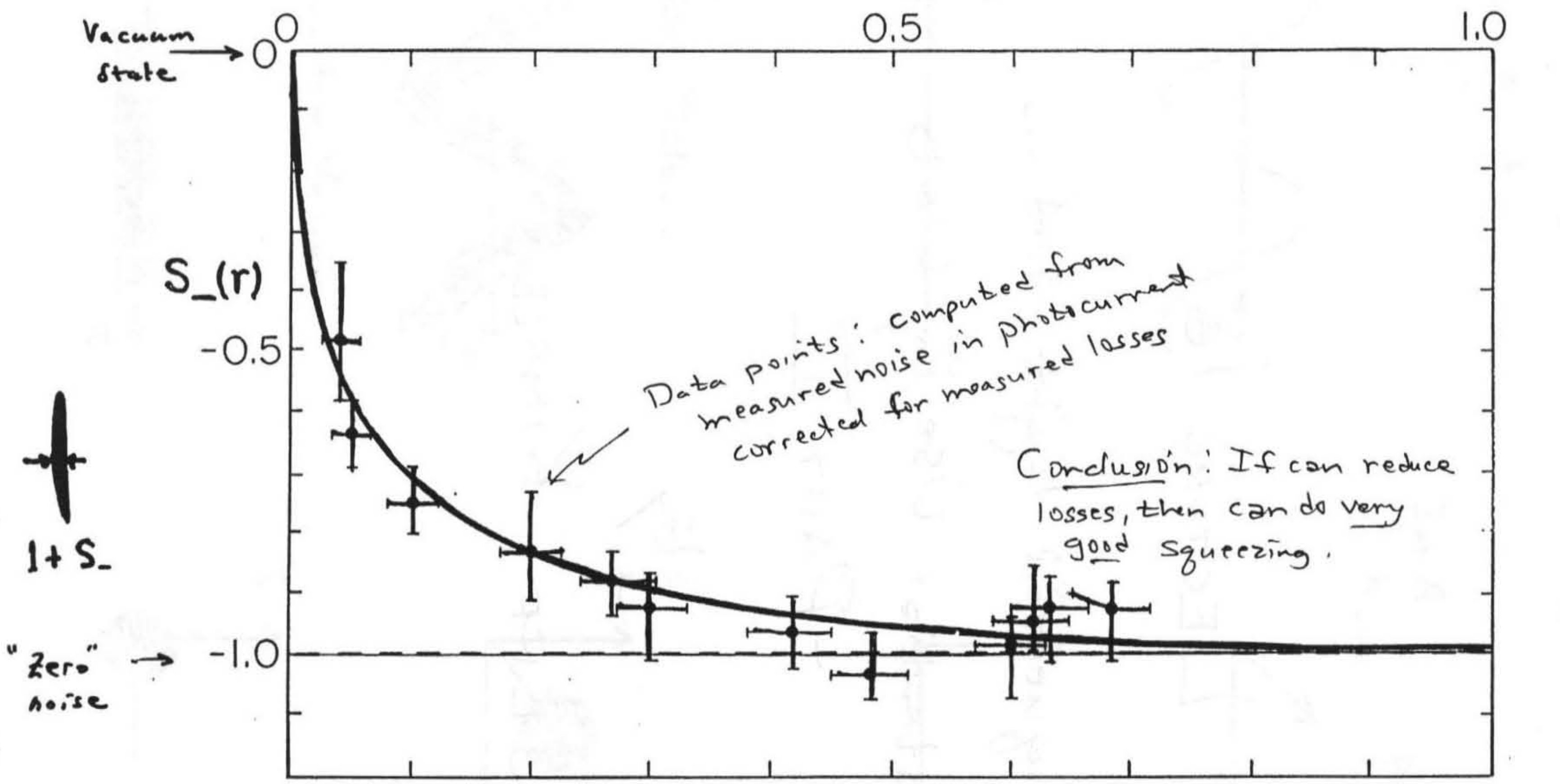
Relative to $S_\pm(\Omega)$

Experiment – Squeezing with an OPO



<u>Inferred</u> Degree of <u>Squeezing</u> vs. <u>Pump Power</u> – <u>Absolute Measurement</u>

How good would the amount of squeezing be if we could get rid of all the losses?



$R_- = 1 + \xi S_-$

$S_- = (R_- - 1)/\xi$

Wu et al., PRL 57, 2520 (1986).

LECTURE 16

"Squeezed Light and its Potential Use in LIGO

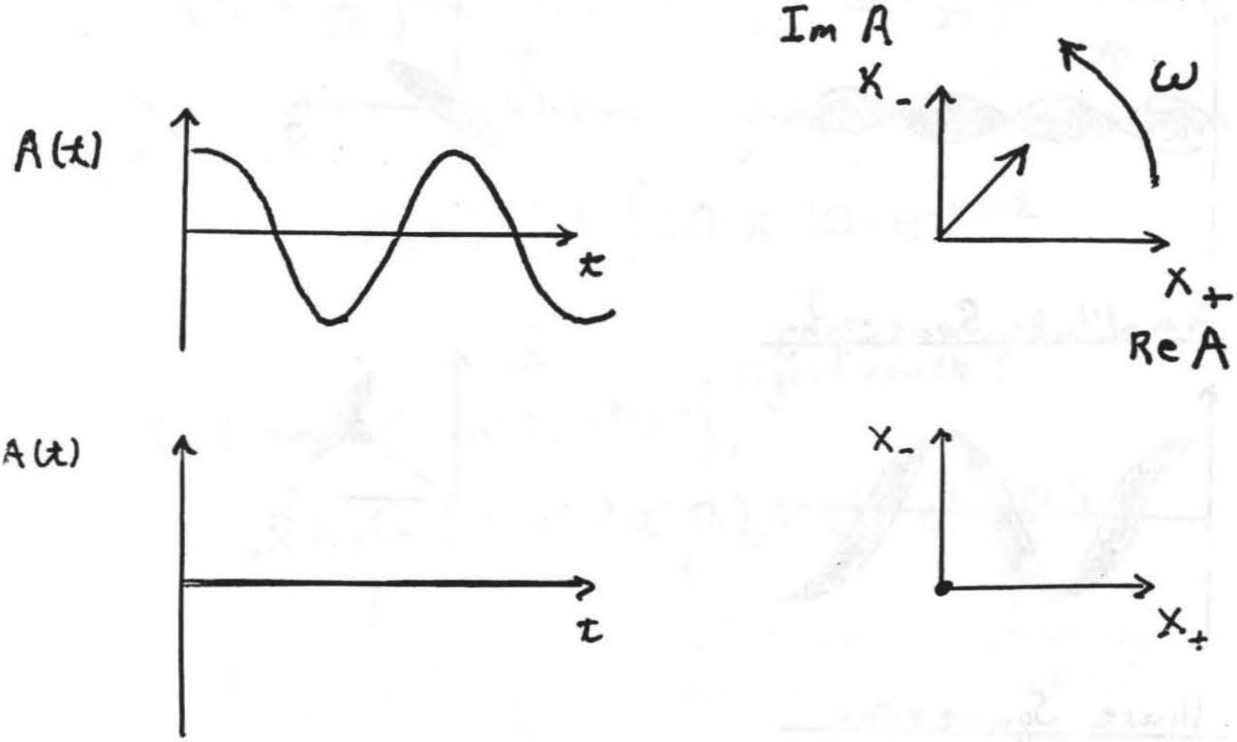
- PART II

by

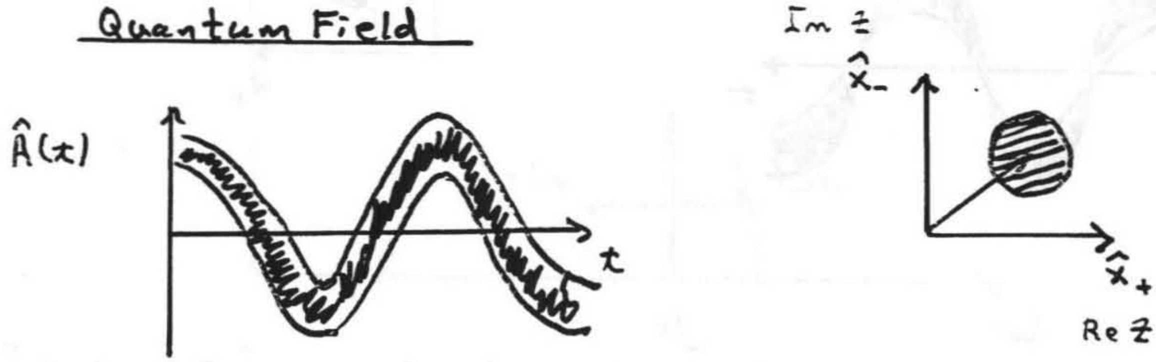
JEFF KIMBLE

Pictographs for Squeezing

Classical Field



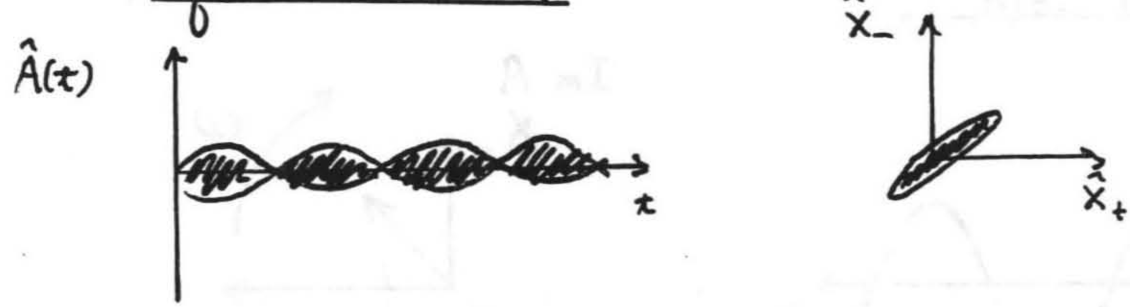
Quantum Field



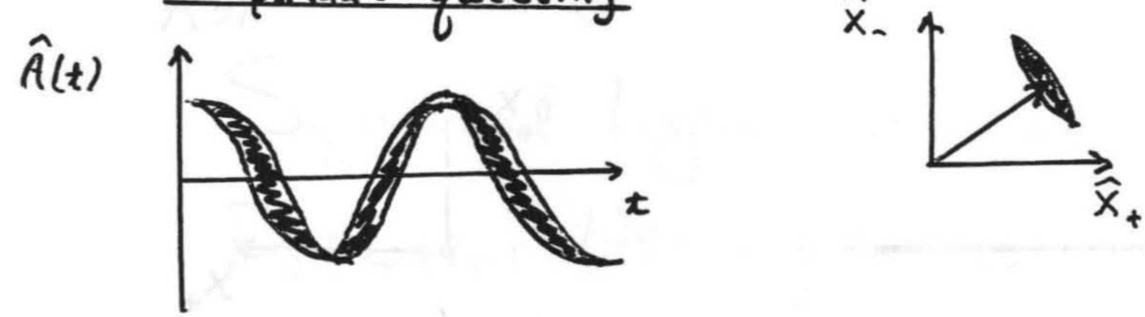
Quantum Vacuum

Other Possibilities

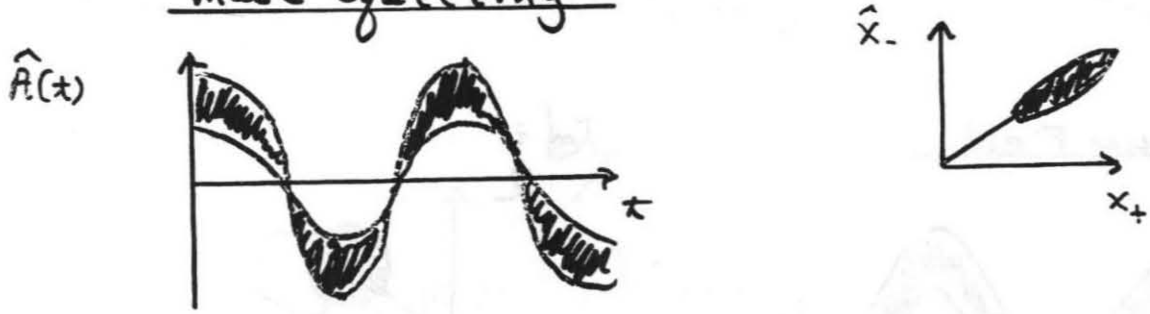
Squeezed Vacuum



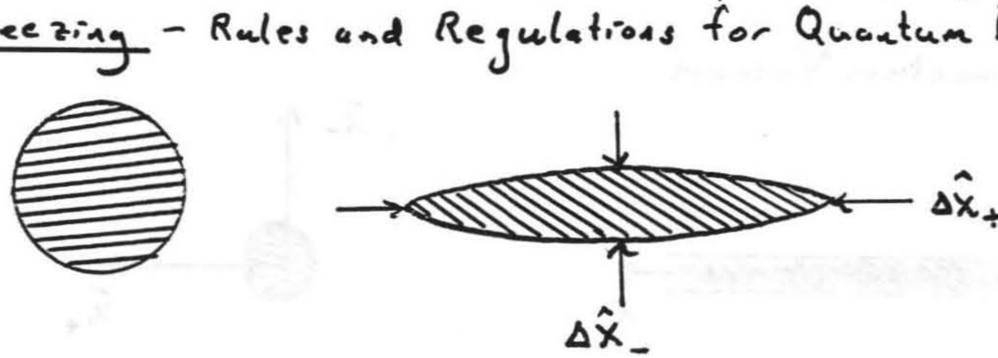
Amplitude Squeezing



Phase Squeezing



Squeezing - Rules and Regulations for Quantum "Fuzz Balls"



$$\langle(\Delta\hat{X}_+)^2\rangle\langle(\Delta\hat{X}_-)^2\rangle \geq 1$$

Frequency Correlations for Squeezed Light

Introduce spectral decomposition of field

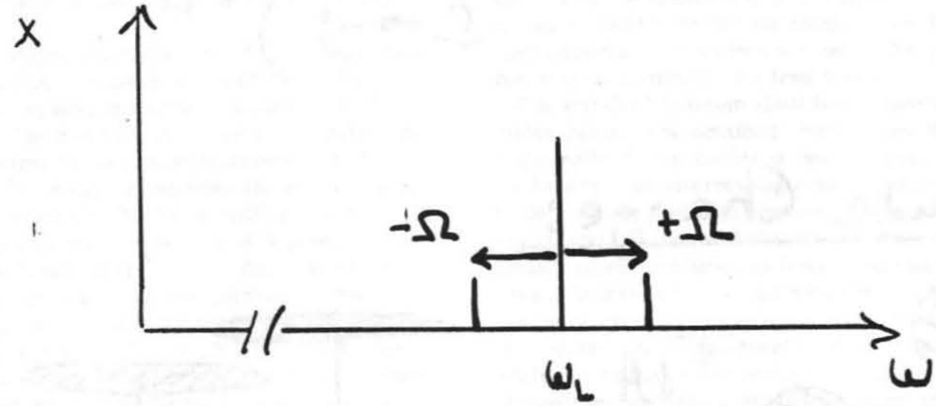
$$a(t) = \frac{1}{2\pi}\int d\Omega\, \underset{\substack{\uparrow \\ \text{annihilation}}}{a(\Omega)}\, e^{-i\Omega t} \quad , \quad a^{\dagger}(t) = \frac{1}{2\pi}\int d\Omega\, \underset{\substack{\uparrow \\ \text{creation}}}{a^{\dagger}(\Omega)}\, e^{i\Omega t}$$

and

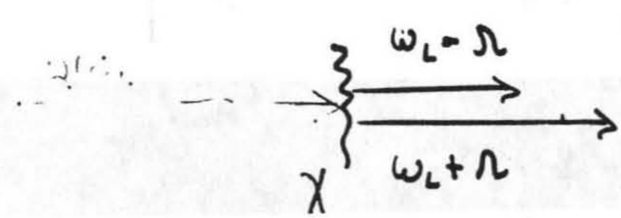
$$X_{\theta}(t) = \frac{1}{2\pi}\int d\Omega\, \underset{\substack{\uparrow \\ \text{significance?}}}{X_{\theta}(\Omega)}\, e^{-i\Omega t} .$$

In terms of $\{a(\Omega), a^{\dagger}(\Omega)\}$,

$$\underset{\substack{\uparrow \\ \text{[illegible]}}}{X_{\theta}(\Omega)} = e^{-i\theta}\, \underset{\substack{\uparrow \\ \text{annihilation} \\ \text{at } \Omega}}{a(\Omega)} + e^{i\theta}\, \underset{\substack{\uparrow \\ \text{creation} \\ \text{at } -\Omega}}{a^{\dagger}(-\Omega)}$$



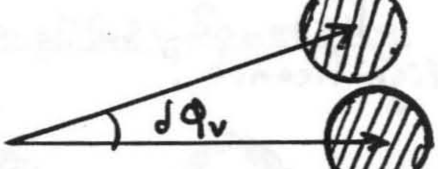
* Parametric process introduces correlations between fields at $\omega_L + \Omega$ and $\omega_L - \Omega$

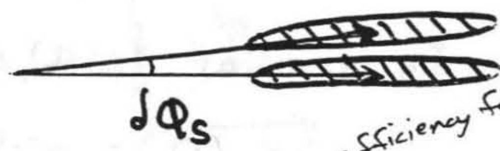


Squeezed Light for

Sensitivity Beyond the Vacuum-State Limit

Phase Changes





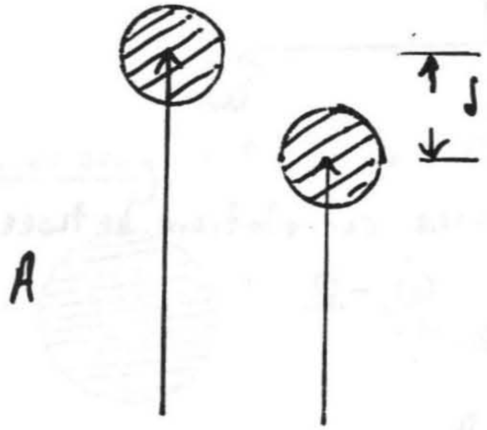
$$\delta\varphi_v \sim 1/\sqrt{N}$$

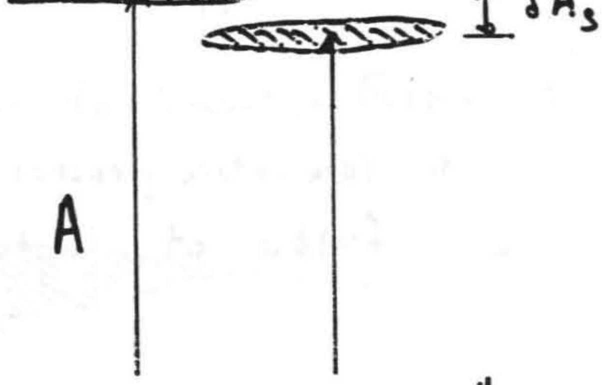
efficiency factor

$$\delta\varphi_s \sim [1+\alpha S_-]^{1/2}/\sqrt{N}$$

$$\left.\begin{array}{l}\alpha \simeq 1 \\ S_- \to -1\end{array}\right\} \delta\varphi_s \to 0$$

Amplitude Changes





$$\delta A_v/A \sim 1/\sqrt{N}$$

$$\delta A_s/A \sim [1+\alpha S_+]^{1/2}/\sqrt{N}$$

J Kible

PHYSICAL REVIEW D VOLUME 23, NUMBER 8 15 APRIL 1981

Quantum-mechanical noise in an interferometer

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(Received 15 August 1980)

The interferometers now being developed to detect gravitational waves work by measuring the relative positions of widely separated masses. Two fundamental sources of quantum-mechanical noise determine the sensitivity of such an interferometer: (i) fluctuations in number of output photons (photon-counting error) and (ii) fluctuations in radiation pressure on the masses (radiation-pressure error). Because of the low power of available continuous-wave lasers, the sensitivity of currently planned interferometers will be limited by photon-counting error. This paper presents an analysis of the two types of quantum-mechanical noise, and it proposes a new technique—the "squeezed-state" technique—that allows one to decrease the photon-counting error while increasing the radiation-pressure error, or vice versa. The key requirement of the squeezed-state technique is that the state of the light entering the interferometer's normally unused input port must be not the vacuum, as in a standard interferometer, but rather a "squeezed state"—a state whose uncertainties in the two quadrature phases are unequal. Squeezed states can be generated by a variety of nonlinear optical processes, including degenerate parametric amplification.

I. INTRODUCTION

The task of detecting gravitational radiation is driving dramatic improvements in a variety of technologies for detecting very weak forces.[1] These improvements are forcing a careful examination of quantum-mechanical limits on the accuracy with which one can monitor the state of a macroscopic body on which a weak force acts.[2] One promising technology uses an interferometer to monitor the relative positions of widely separated masses. This paper analyzes the quantum-mechanical limits on the performance of interferometers, and it introduces a new technique that might lead to improvements in their sensitivity.

The prototypal interferometer for gravitational-wave detection is a two-arm, multireflection Michelson system, powered by a laser (see Fig. 3 below). The intensity in either of the interferometer's output ports provides information about the difference $z \equiv z_2 - z_1$ between the end mirrors' positions relative to the beam splitter, and changes in z reveal the passing of a gravitational wave. The first interferometer for gravitational-wave detection was built and operated at the Hughes Research Laboratories in Malibu, California, in the early 1970's (Ref. 3); this first effort was small-scale and had modest sensitivity. Now several groups around the world are developing interferometers of greatly improved sensitivity.[4-6] A long-range goal is to construct large-scale interferometers, with baselines $l \sim 1$ km, in order to achieve a strain sensitivity $\Delta z/l \sim 10^{-21}$ for frequencies from about 30 Hz to 10 kHz. This sensitivity goal is based on estimates for the strength of gravitational waves that pass the Earth reasonably often.[1]

It has been known for some time that quantum mechanics limits the accuracy with which an interferometer can measure z—or, indeed, the accuracy with which any position-sensing device can determine the position of a free mass.[2,5,7] In a measurement of duration τ, the probable error in the interferometer's determination of z can be no smaller than the "standard quantum limit":

$$(\Delta z)_{\rm SQL} = (2\hbar\tau/m)^{1/2}\,, \tag{1.1}$$

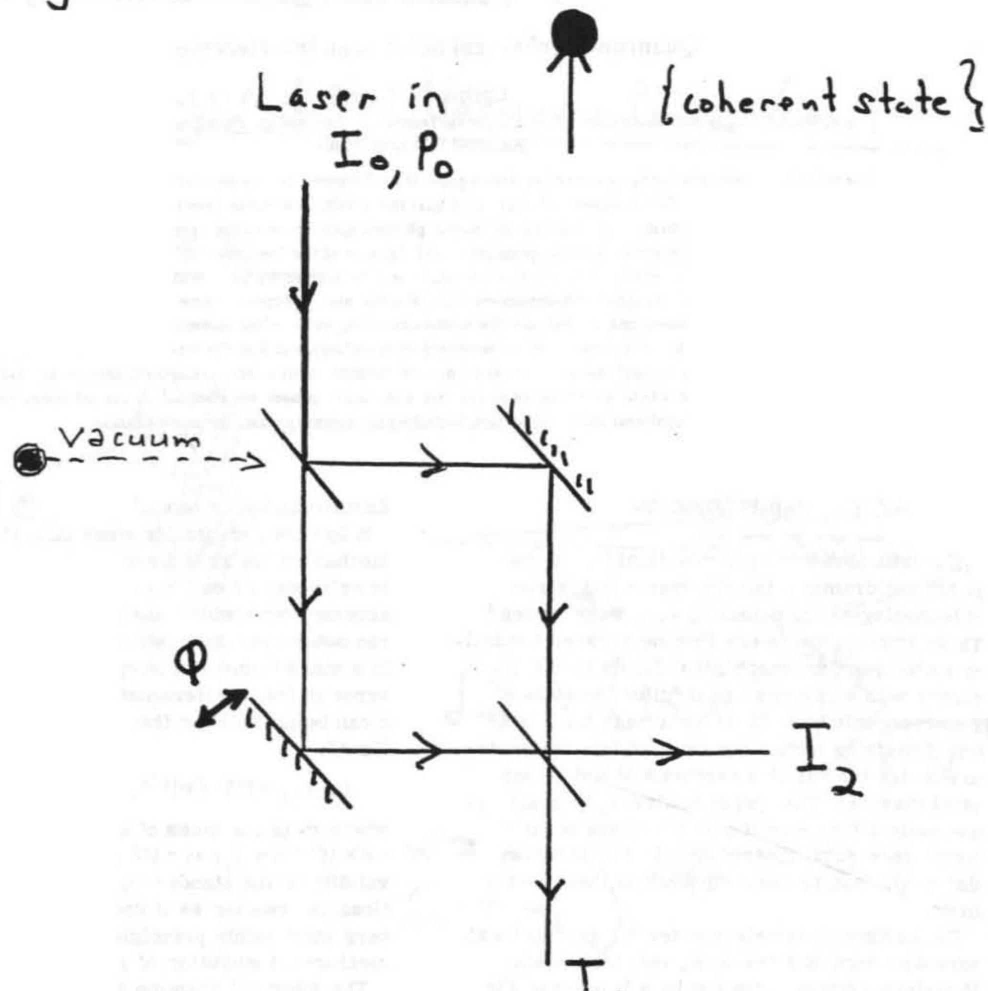
where m is the mass of each end mirror [$(\Delta z)_{\rm SQL} \sim 6\times10^{-18}$ cm for $m \sim 10^5$ g, $\tau \sim 2\times10^{-3}$ sec]. The validity of the standard quantum limit is unquestionable, resting as it does solely on the Heisenberg uncertainty principle applied to the quantum-mechanical evolution of a free mass.

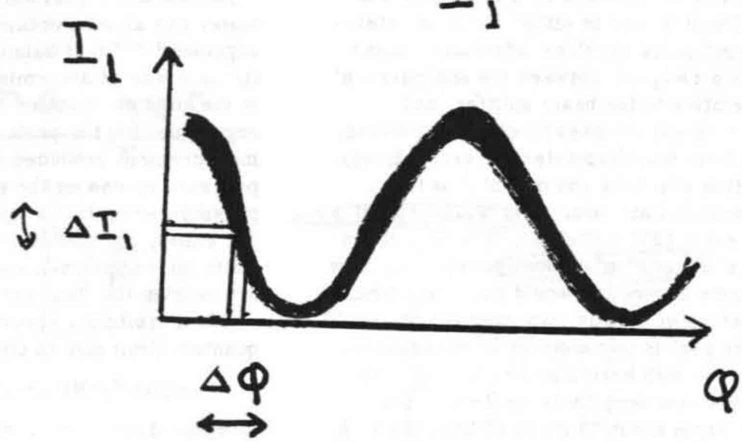
The standard quantum limit for an interferometer can also be obtained from a more detailed argument[5,8-10] that balances two sources of error: (i) the error in determining z due to fluctuations in the number of output photons (photon-counting error) and (ii) the perturbation of z during a measurement produced by fluctuating radiation-pressure forces on the end mirrors (radiation-pressure error). As the input laser power P increases, the photon-counting error decreases, while the radiation-pressure error increases. Minimizing the total error with respect to P yields a minimum error of order the standard quantum limit and an optimum input power[9,11]

$$P_0 \simeq \tfrac{1}{2}(mc^2/\tau)(1/\omega\tau)(1/b^2) \tag{1.2}$$

at which the minimum error can be achieved. Here ω is the angular frequency of the light, and b is the number of bounces at each end mirror.

IV. Interferometry Beyond Vacuum-State Limit ↔ Caves

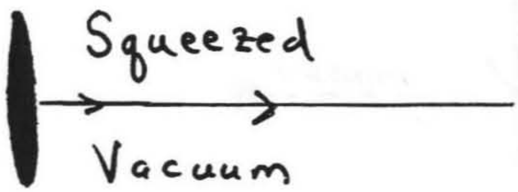


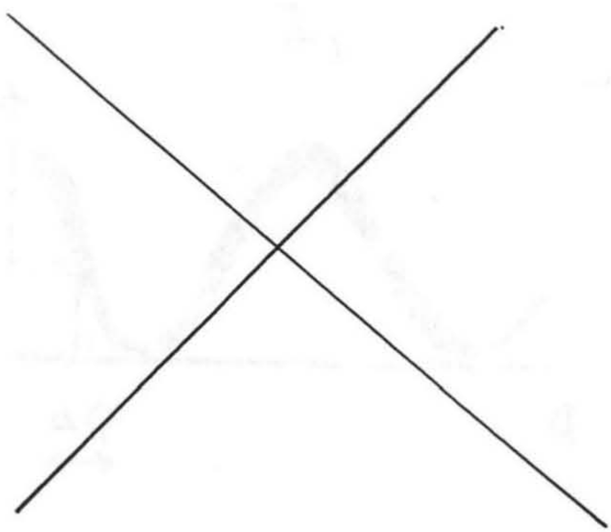


[Overlay for previous transparency]

Vacuum

[overlay again]





Xiao, Wu, Kimble
PRL 41, 278 (1987).

Vacuum-State / Shot-Noise } Limit

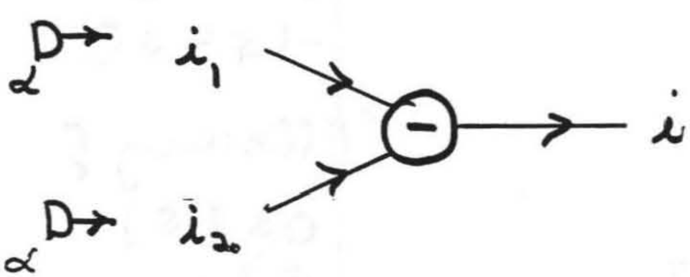
(How one does the calculations corresponding to previous diagrams.)

Power $P_{1,2} = T\frac{P_0}{2}\left[1 \pm \cos\varphi\right]$

$$\varphi = \frac{\pi}{2} + 2\delta_0 \cos\Omega t$$

↑ (sit on side of fringe)

Current $i_{1,2} = \alpha e P_{1,2}$



Signal Coherent modulation at Ω – $i_S = \sqrt{2}\, e T \alpha P_0 \delta_0$

Noise "Shot-Noise" $i_n^2 = 2 e i B$

Signal to Noise $\psi = i_S^2 / i_n^2$

$$= \delta_0^2 N$$

where $N = T \alpha P_0 B^{-1}$ = (number of photons that get turned into photoelectrons)

$$\Rightarrow \psi = 1 \text{ for } \delta_0 = \frac{1}{\sqrt{N}}$$

(Overlay)

Improvement with Squeezed Light

$\rightarrow | \leftarrow 1+\xi S$

Spectrum of Squeezing S

$-1 \leq S \leq 0$

Efficiency ξ

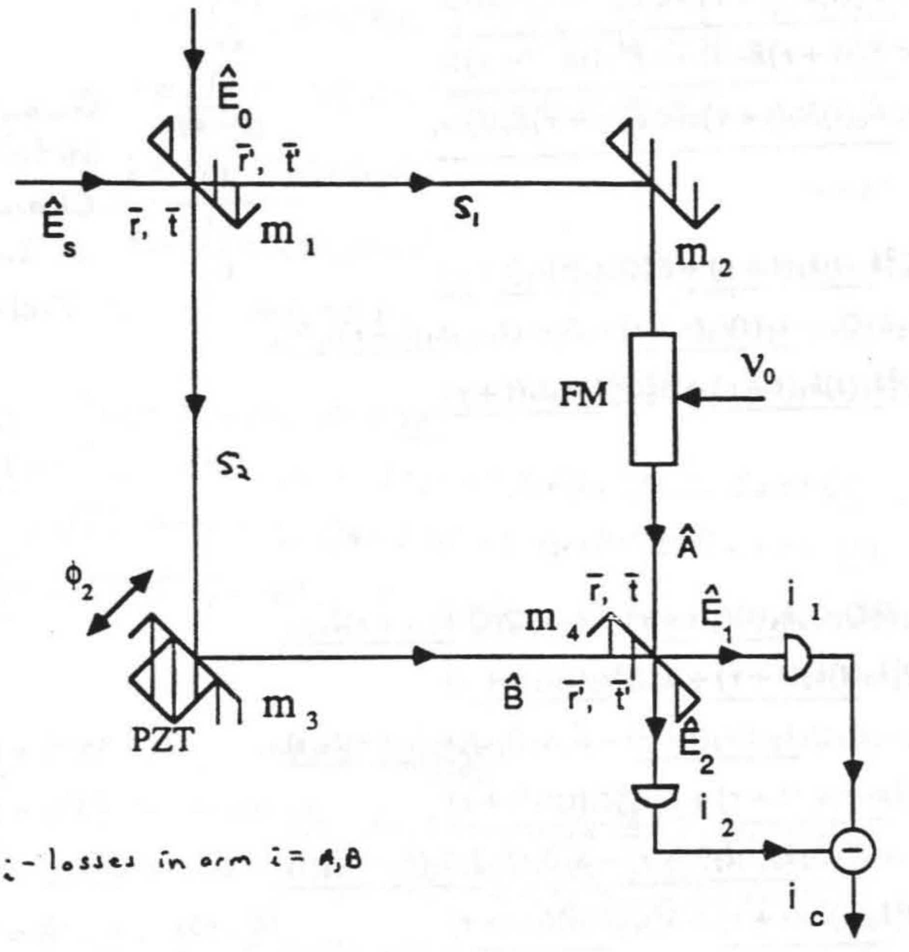
$0 \leq \xi \leq 1$

$[1+\xi S]$

$/[1+\xi S]$

$[1+\xi S]^{1/2}$

The Real Story



S_i - losses in arm $i = A, B$

Photocurrent Fluctuations $<\Delta i(t)\Delta i(t+\tau)>$

$$<\Delta i_c(t)\Delta i_c(t+\tau)> = R_1(t)Q_1^2\delta(\tau) + R_2(t)Q_2^2\delta(\tau)$$
$$+ H_1[<T:\hat{E}_s^\dagger(t)\hat{E}_s^\dagger(t+\tau)\hat{E}_s(t+\tau)\hat{E}_s(t)> - <\hat{E}_s^\dagger(t)\hat{E}_s(t)>^2]$$
$$+ H_2[<T:\hat{E}_0^\dagger(t)\hat{E}_0^\dagger(t+\tau)\hat{E}_0(t+\tau)\hat{E}_0(t)> - <\hat{E}_0^\dagger(t)\hat{E}_0(t)>^2]$$
$$+ H_3 <\hat{E}_0(t+\tau)\hat{E}_0(t)><\hat{E}_s^\dagger(t)\hat{E}_s^\dagger(t+\tau)>$$
$$+ H_4 <\hat{E}_0^\dagger(t)\hat{E}_0^\dagger(t+\tau)><\hat{E}_s(t+\tau)\hat{E}_s(t)>$$
$$+ H_5 <\hat{E}_0^\dagger(t+\tau)\hat{E}_0(t)><\hat{E}_s^\dagger(t)\hat{E}_s(t+\tau)>$$
$$+ H_6 <\hat{E}_0^\dagger(t)\hat{E}_0(t+\tau)><\hat{E}_s^\dagger(t+\tau)\hat{E}_s(t)>, \qquad (4-44)$$

Quantum Statistical Characteristics of Incident Fields

where

$$H_1 \equiv \bar{\alpha}_1^2Q_1^2k_1(t)k_1(t+\tau) + \bar{\alpha}_2^2Q_2^2\ell_1(t)\ell_1(t+\tau)$$
$$- \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_1(t)\ell_1(t+\tau) - \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_1(t+\tau)\ell_1(t),$$
$$H_2 \equiv \bar{\alpha}_1^2Q_1^2k_4(t)k_4(t+\tau) + \bar{\alpha}_2^2Q_2^2\ell_4(t)\ell_4(t+\tau)$$
$$- \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_4(t)\ell_4(t+\tau) - \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_4(t+\tau)\ell_4(t),$$
$$H_3 \equiv \bar{\alpha}_1^2Q_1^2k_2(t)k_2(t+\tau) + \bar{\alpha}_2^2Q_2^2\ell_2(t)\ell_2(t+\tau)$$
$$- \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_2(t)\ell_2(t+\tau) - \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_2(t+\tau)\ell_2(t),$$
$$H_4 \equiv \bar{\alpha}_1^2Q_1^2k_3(t)k_3(t+\tau) + \bar{\alpha}_2^2Q_2^2\ell_3(t)\ell_3(t+\tau)$$
$$- \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_3(t)\ell_3(t+\tau) - \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_3(t+\tau)\ell_3(t),$$
$$H_5 \equiv \bar{\alpha}_1^2Q_1^2k_2(t)k_3(t+\tau) + \bar{\alpha}_2^2Q_2^2\ell_2(t)\ell_3(t+\tau) \qquad (4-45)$$
$$- \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_2(t)\ell_3(t+\tau) - \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_3(t+\tau)\ell_2(t),$$
$$H_6 \equiv \bar{\alpha}_1^2Q_1^2k_2(t+\tau)k_3(t) + \bar{\alpha}_2^2Q_2^2\ell_2(t+\tau)\ell_3(t)$$
$$- \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_3(t)\ell_2(t+\tau) - \bar{\alpha}_1\bar{\alpha}_2Q_1Q_2k_2(t+\tau)\ell_3(t),$$

Photocurrent Fluctuations, cont.

$\hat{E}_s(r,t) = \hat{E}_s(t)\, V(\vec{r})$
$\hat{E}_o(r,t) = \hat{E}_o(t)\, U(\vec{r})$

$$
\begin{aligned}
\bar{\eta} &\equiv \left|\iint_S V_i^*(\vec{r})U_i(\vec{r})dS\right| = \left|\iint_S U_i^*(\vec{r})V_i(\vec{r})dS\right|,\\
\bar{\nu} &\equiv \left|\iint_S V_i^*(\vec{r})U_j(\vec{r})dS\right| = \left|\iint_S U_i^*(\vec{r})V_j(\vec{r})dS\right|,\\
\bar{\mu} &\equiv \left|\iint_S V_i^*(\vec{r})V_j(\vec{r})dS\right|, \qquad\qquad (4-34)\\
\bar{\epsilon} &\equiv \left|\iint_S U_i^*(\vec{r})U_j(\vec{r})dS\right|,\\
i,j &= 1,2 \quad \text{and} \quad i \neq j.
\end{aligned}
$$

$$
\begin{aligned}
\underline{k_1(t)} &= \bar{R}\bar{T}\left[S_1^2 + S_2^2 + 2S_1S_2\bar{\mu}\,\underline{\cos\Delta\phi(t)}\right]\\
\underline{k_2(t)} &= \sqrt{\bar{R}\bar{T}}e^{i(-\varphi_1+\varphi_2)}\left[\bar{R}S_1^2\bar{\eta} - \bar{T}S_2^2\bar{\eta} - S_1S_2\bar{\nu}\left(\bar{T}e^{-\underline{i\Delta\phi(t)}} - \bar{R}e^{\underline{i\Delta\phi(t)}}\right)\right]\\
\underline{k_3(t)} &= \sqrt{\bar{R}\bar{T}}e^{-i(-\varphi_1+\varphi_2)}\left[\bar{R}S_1^2\bar{\eta} - \bar{T}S_2^2\bar{\eta} + S_1S_2\bar{\nu}\left(\bar{R}e^{-\underline{i\Delta\phi(t)}} - \bar{T}e^{\underline{i\Delta\phi(t)}}\right)\right]\\
\underline{k_4(t)} &= \bar{R}^2S_1^2 + \bar{T}^2S_2^2 - 2\bar{R}\bar{T}S_1S_2\bar{\epsilon}\,\underline{\cos\Delta\phi(t)}, \qquad\qquad (4-38)
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\ell_1(t)} &= \bar{T}^2S_1^2 + \bar{R}^2S_2^2 - 2\bar{R}\bar{T}S_1S_2\bar{\mu}\,\underline{\cos\Delta\phi(t)}\\
\underline{\ell_2(t)} &= \sqrt{\bar{R}\bar{T}}e^{i(-\varphi_1+\varphi_2)}\left[\bar{T}S_1^2\bar{\eta} - \bar{R}S_2^2\bar{\eta} + S_1S_2\bar{\nu}\left(\bar{T}e^{-\underline{i\Delta\phi(t)}} - \bar{R}e^{\underline{i\Delta\phi(t)}}\right)\right]\\
\underline{\ell_3(t)} &= \sqrt{\bar{R}\bar{T}}e^{-i(-\varphi_1+\varphi_2)}\left[\bar{T}S_1^2\bar{\eta} - \bar{R}S_2^2\bar{\eta} - S_1S_2\bar{\nu}\left(\bar{R}e^{-\underline{i\Delta\phi(t)}} - \bar{T}e^{\underline{i\Delta\phi(t)}}\right)\right]\\
\underline{\ell_4(t)} &= \bar{R}\bar{T}\left[S_1^2 + S_2^2 + 2S_1S_2\bar{\epsilon}\,\underline{\cos\Delta\phi(t)}\right]. \qquad\qquad (4-39)
\end{aligned}
$$

______ signal

Min Xiao, PhD Thesis

Interferometry Beyond The Vacuum State Limit → Caves

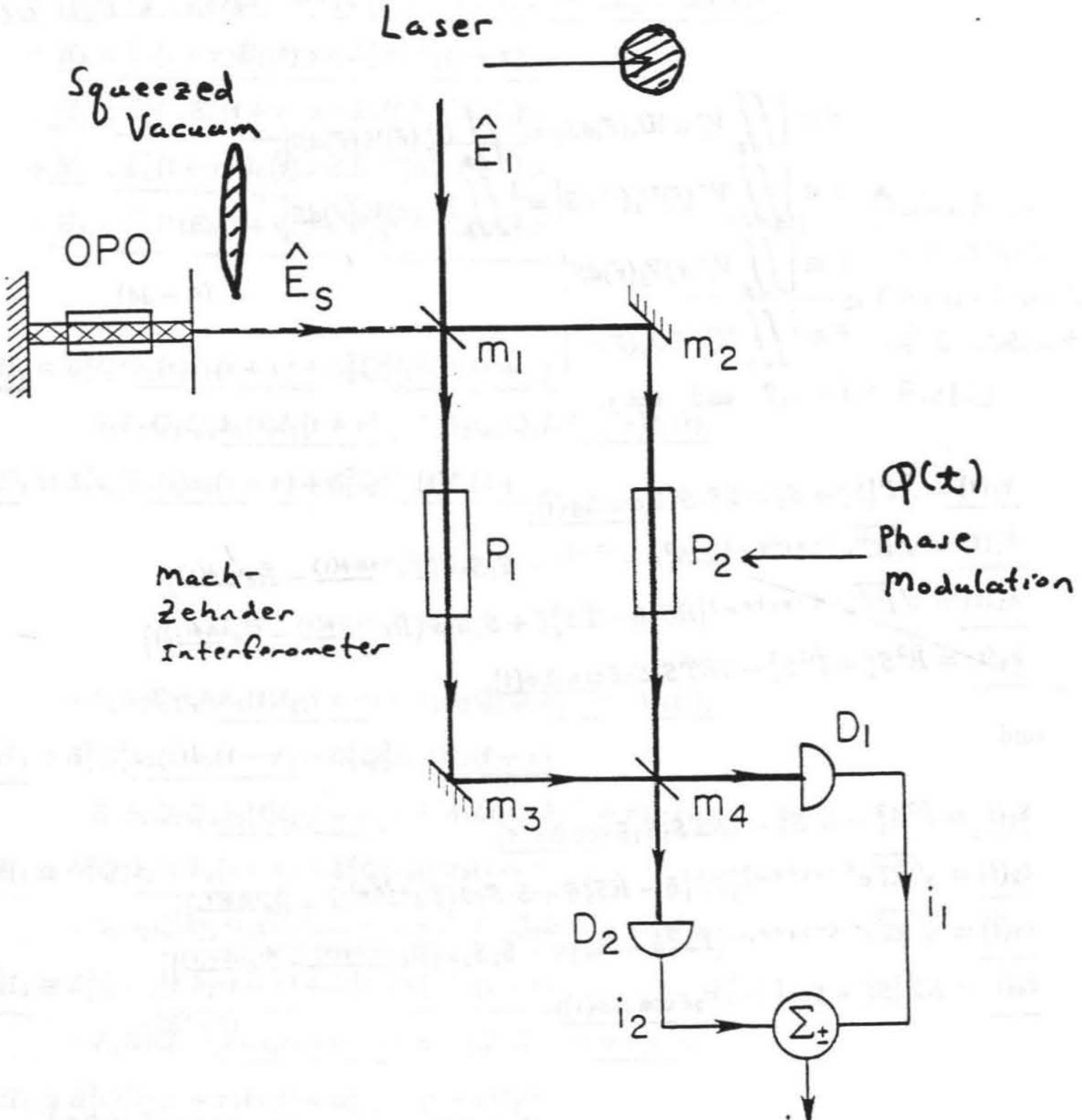


FIGURE 1

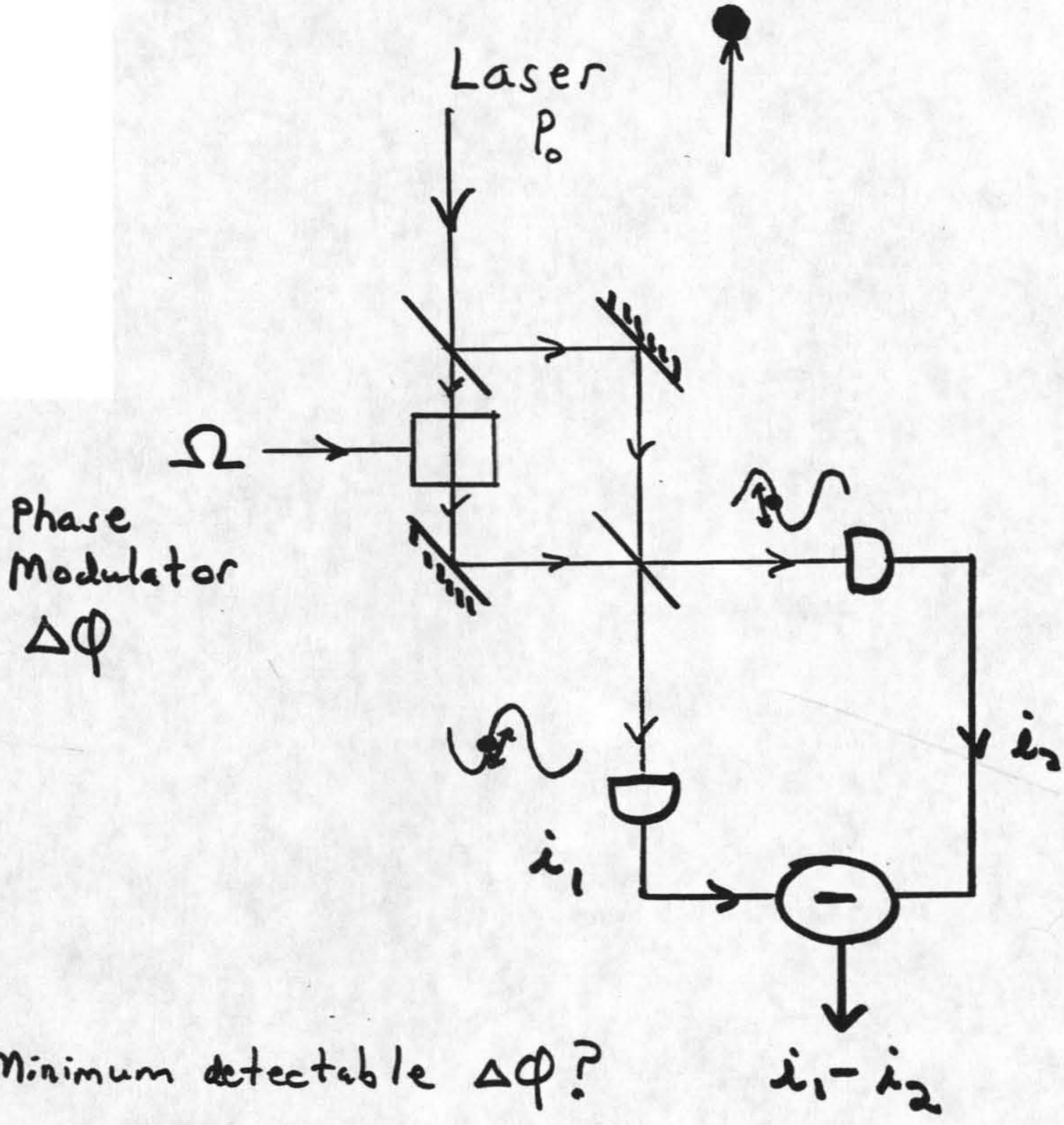
Photocurrent

Signal $\varphi(t)$

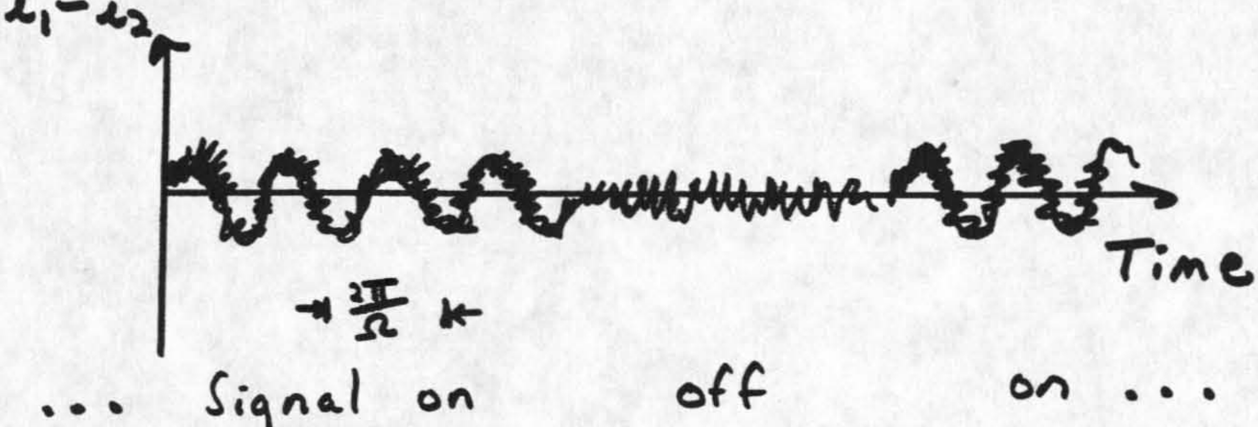
Noise

Xiao, Wu, HJK, Phys. Rev. Lett. 59, 278 (1987)

Overview of Experiment {Caves}

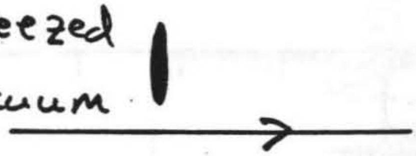


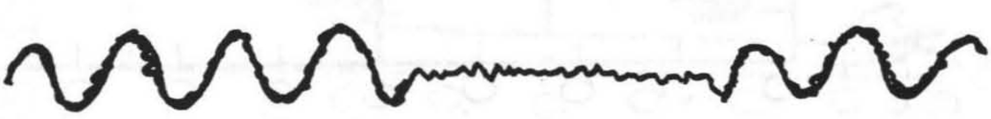
Minimum detectable $\Delta\phi$?



(Overlay)

Squeezed
Vacuum





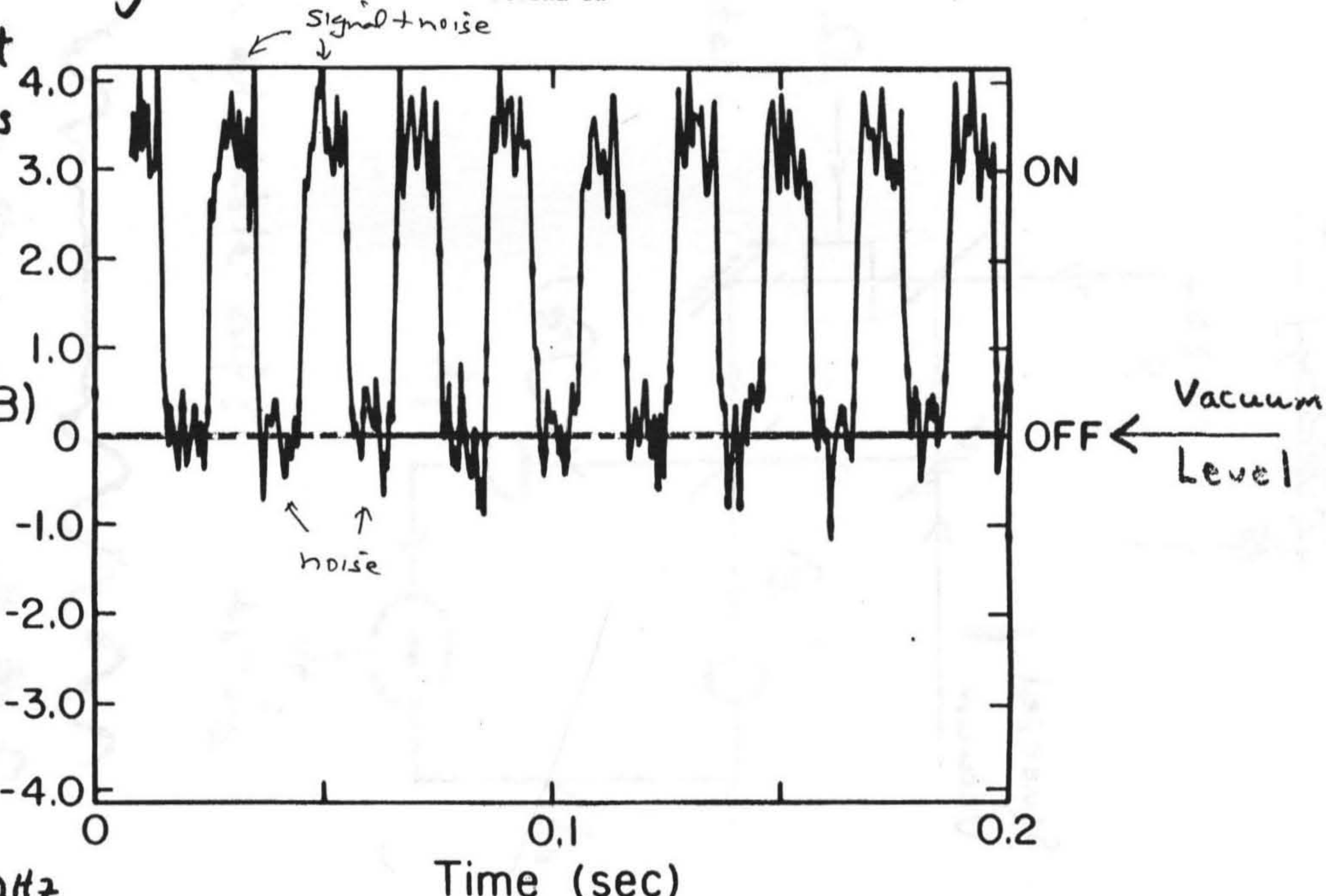
Vacuum State Input ●

$\Rightarrow$ Shot-Noise Limit ~ 5 μrad rms

Spectral Density of Photocurrent Fluctuations

$i_1 - i_2$

FIGURE 2a



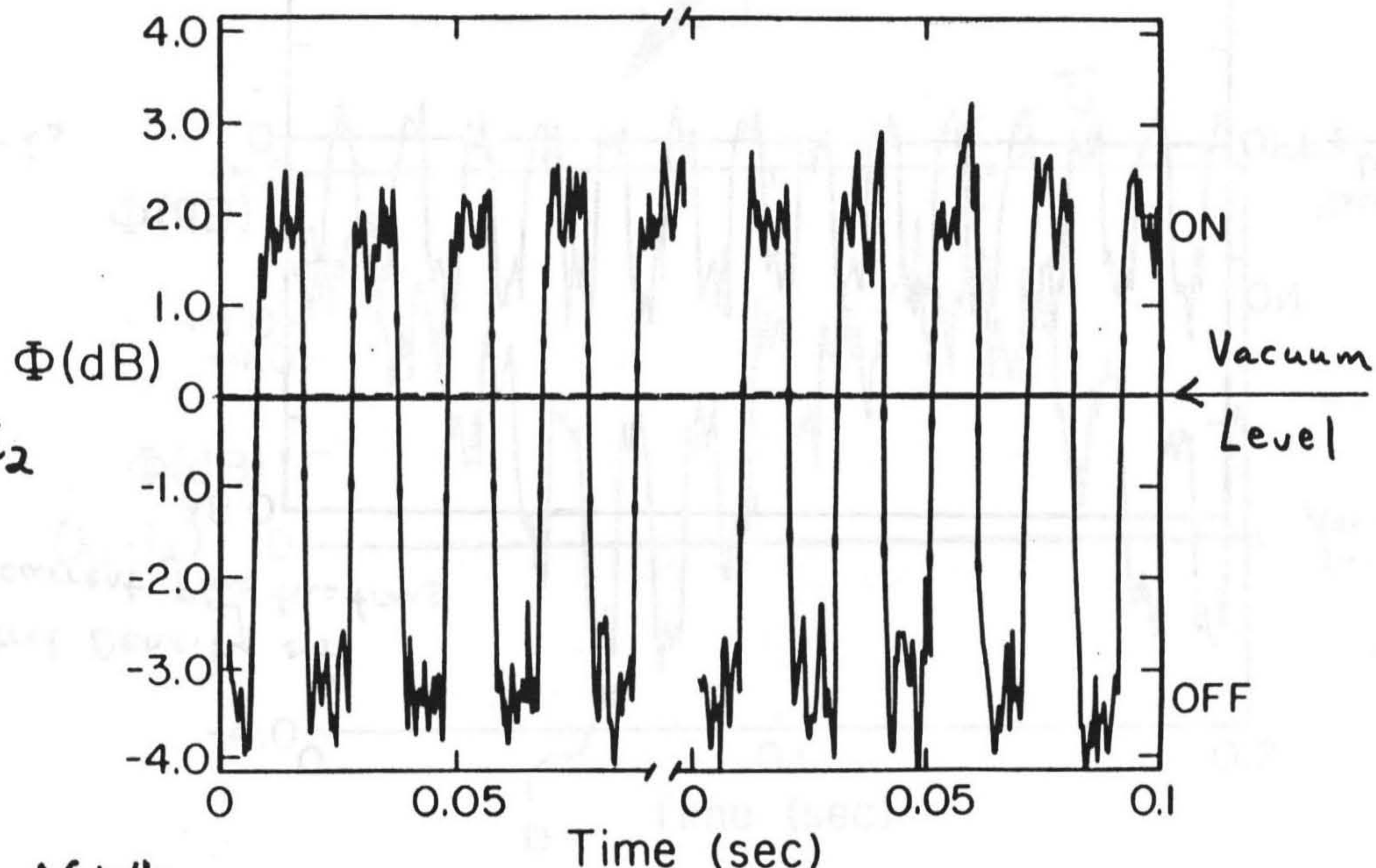
$\Omega/2\pi = 1.6$ MHz

$\Delta f = 100$ kHz $\quad P = 800\ \mu W$

Squeezed Input {~10^{-12} WATT}

⇒ Improvement in S/N of 3.0 dB beyond Shot-Noise Limit

FIGURE 2b



$i_1 - i_2$

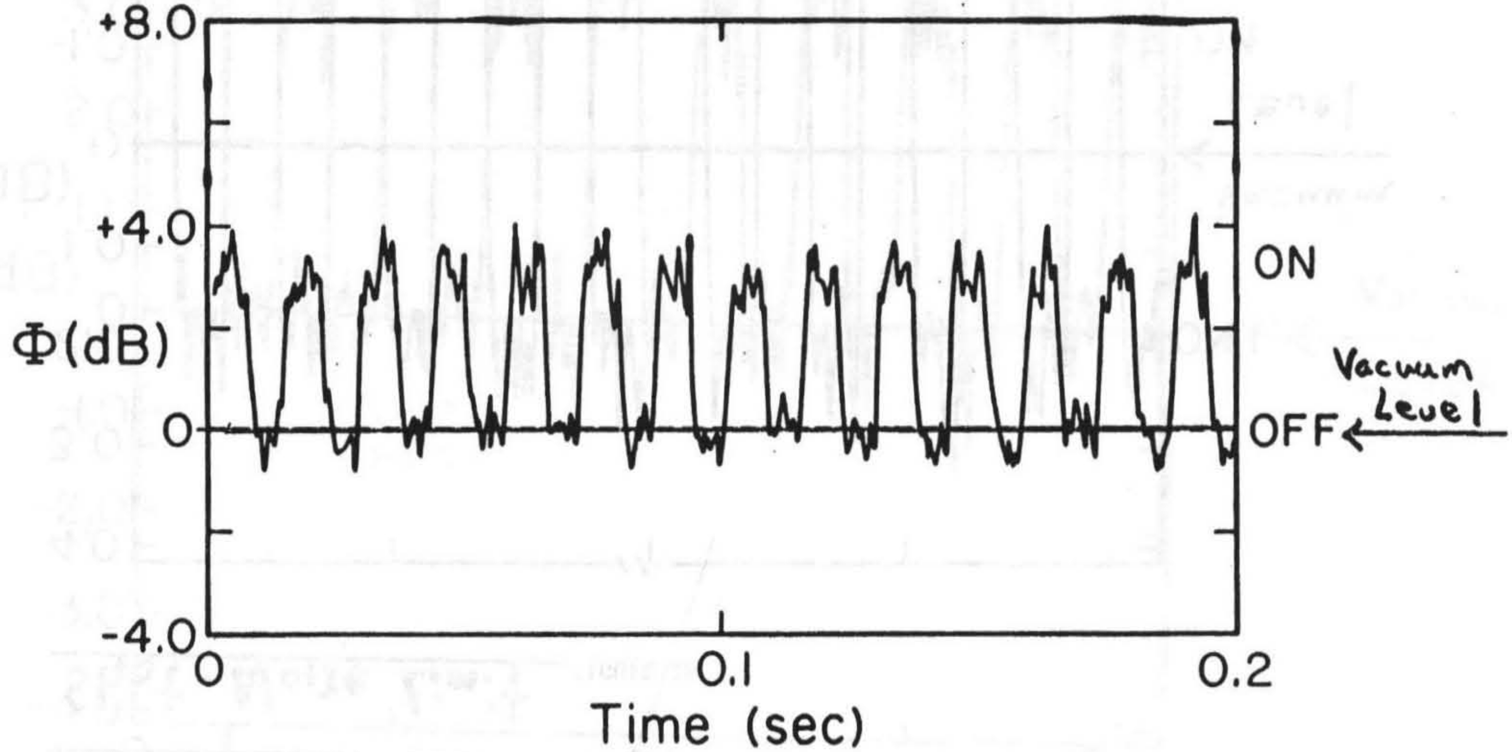
$\Omega/2\pi = 1.6$ mHz

$\Delta f = 100$ kHz

Vacuum State Input

Spectral Density of
Photocurrent Fluctuations

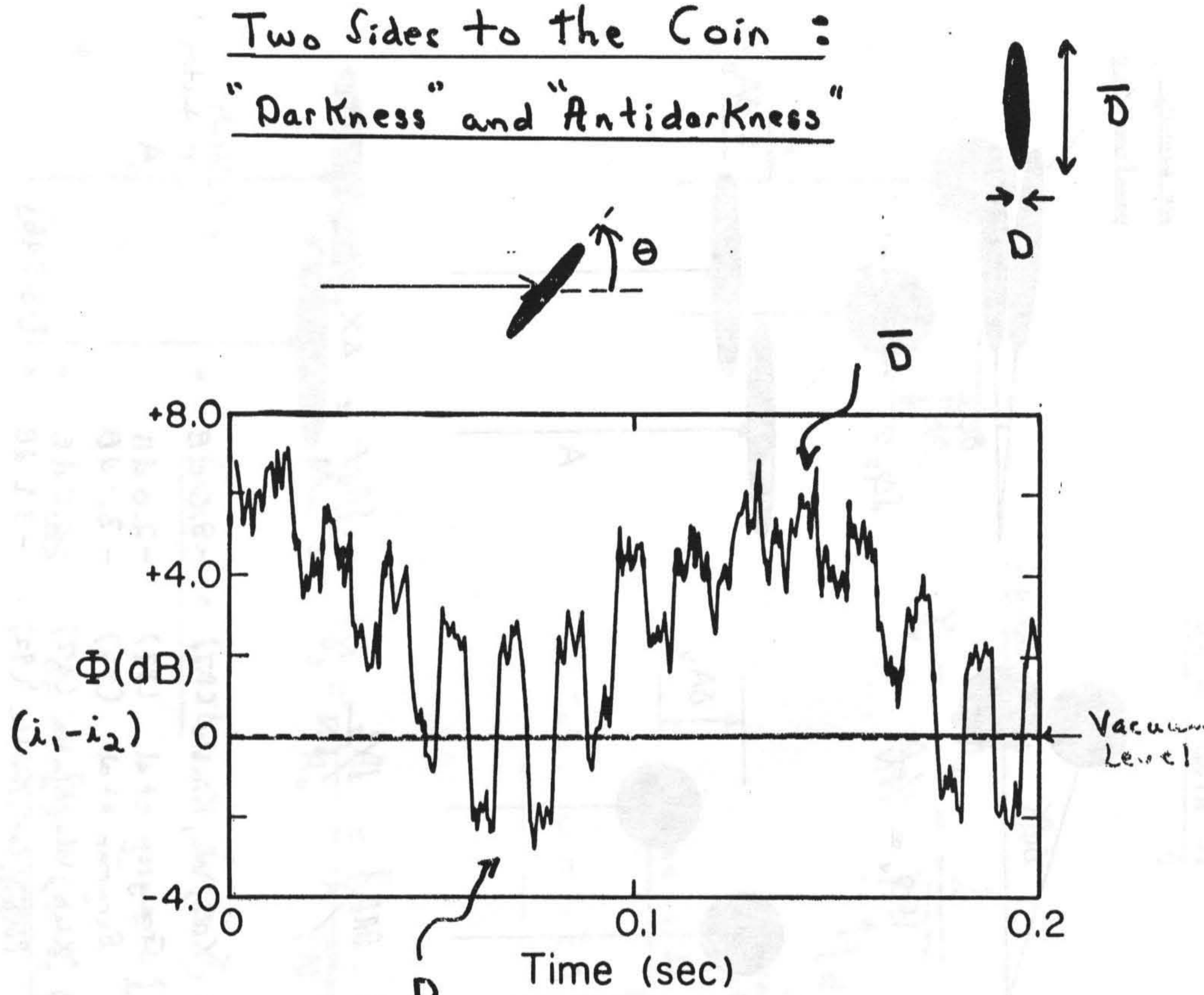
$i_1 - i_2$



$\Omega/2\pi \simeq 1.6$ MHz
$\Delta f = 100$ kHz

Two Sides to the Coin :

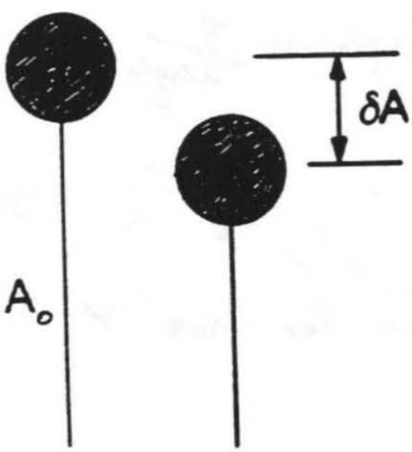
"Darkness" and "Antidarkness"



$\Omega/2\pi = 1.6$ MHz
$\Delta f = 100$ kHz

$\longrightarrow$ Phase Θ

"Ultimate" Limit for Sensitivity Enhancement with Squeezing



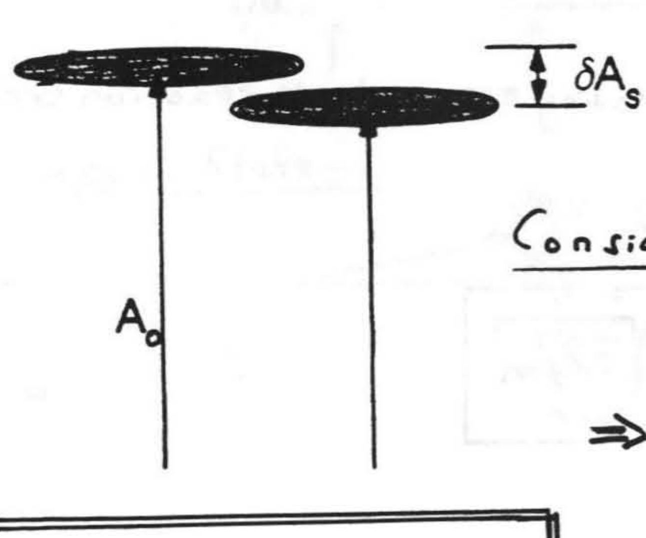
Sample loss γ

$$\gamma_v = \frac{\delta A_v}{A_0} \sim \frac{1}{\sqrt{N}}$$

photons in measurement interval

Degree of Squeezing

Total loss $= 1 - \xi$



$$\gamma_s = \frac{\delta A_s}{A_0} \sim \frac{\left[1 + \xi S_-\right]^{1/2}}{\sqrt{N}}$$

Consider limit $1 - \xi \to \gamma_s$

$S_- \to -1$

$$\Rightarrow \quad \gamma_s \sim \frac{\left[1 - (1 - \gamma_s)\right]^{1/2}}{\sqrt{N}}$$

$$\gamma_s \sim \frac{1}{N}$$

Enhancement

$$\gamma_v / \gamma_s \sim \sqrt{N}\,!$$

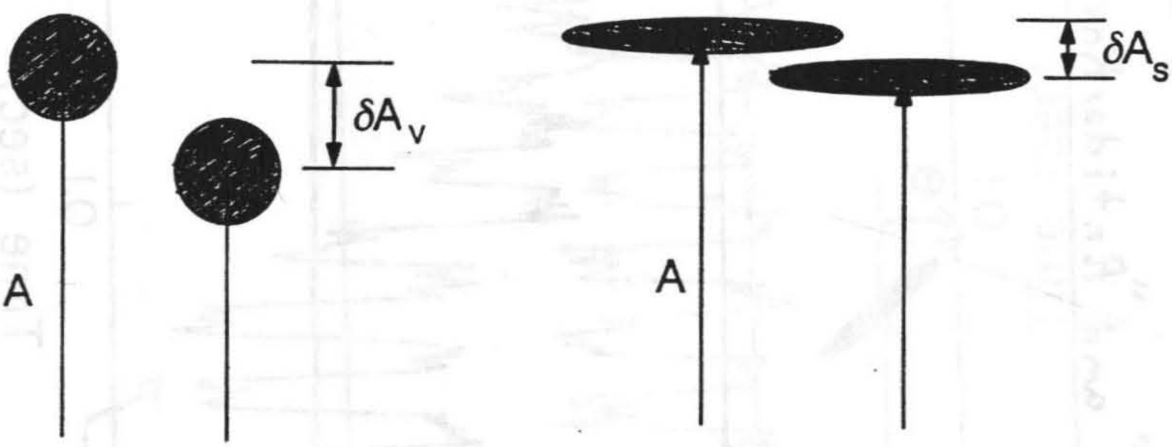
Squeezed Light for Sensitivity Beyond Vacuum-State Limit

aka: Shot-Noise Limit,
Coherent-state Limit,
Standard Quantum Limit

(a)

$\delta\varphi_v \simeq 1/\sqrt{N}$ $\qquad$ $\delta\varphi_s \simeq \Delta X_- / \sqrt{N}$

(b)



$\delta A_v / A \simeq 1/\sqrt{N}$ $\qquad$ $\delta A_s / A \simeq \Delta X_+ / \sqrt{N}$

(a) Xiao, Wu, Kimble (87) $\quad$ −3.0 dB •

• Caltech
UT Austin

Grangier et al. (87) $\quad$ −2.0 dB

Bergman et al. (92) $\quad$ −3. dB

(b) Xiao, Wu, Kimble (87) $\quad$ −2.5 dB •

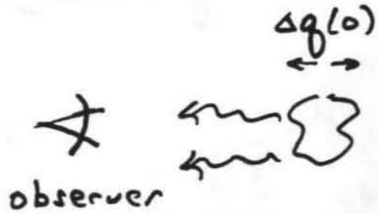
Polzik, Cari, Kimble (92) $\quad$ −3.1 dB • [−3.8 dB]

Also, Nabors & Shelby (90) – Twin beams

Rarity & Tapster (90) – Photon pairs

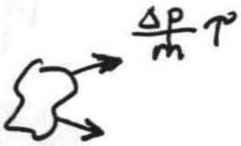
Hong, Friberg, Mandel (85) – Photon pairs

The Standard Quantum Limit (SQL) for the Position of a Free Mass

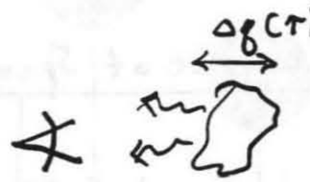
1. 

Measurement with accuracy $\Delta q(0)$

$$\Rightarrow \quad \Delta p(0) \simeq \frac{\hbar}{2\Delta q(0)}$$

2. 

Free evolution for time τ

3. 

2^{nd} measurement with accuracy

$$\Delta q^2(\tau) = \Delta q^2(0) + \frac{\Delta p^2(0)}{m^2}\tau^2$$

$$\geq 2\,\underbrace{\Delta q(0)}_{\text{sensing error}}\;\underbrace{\Delta p(0)}_{\text{back reaction error}}\frac{\tau}{m}$$

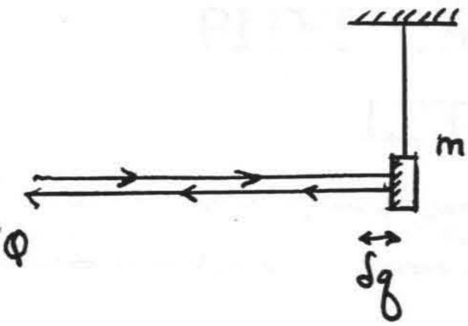
$$\geq \hbar\tau/m$$

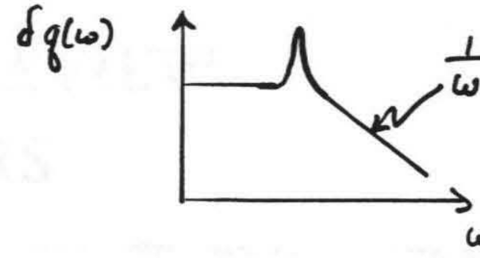
SQL -

$$\boxed{\Delta q_{SQL} \simeq \sqrt{\hbar\tau/m}}$$

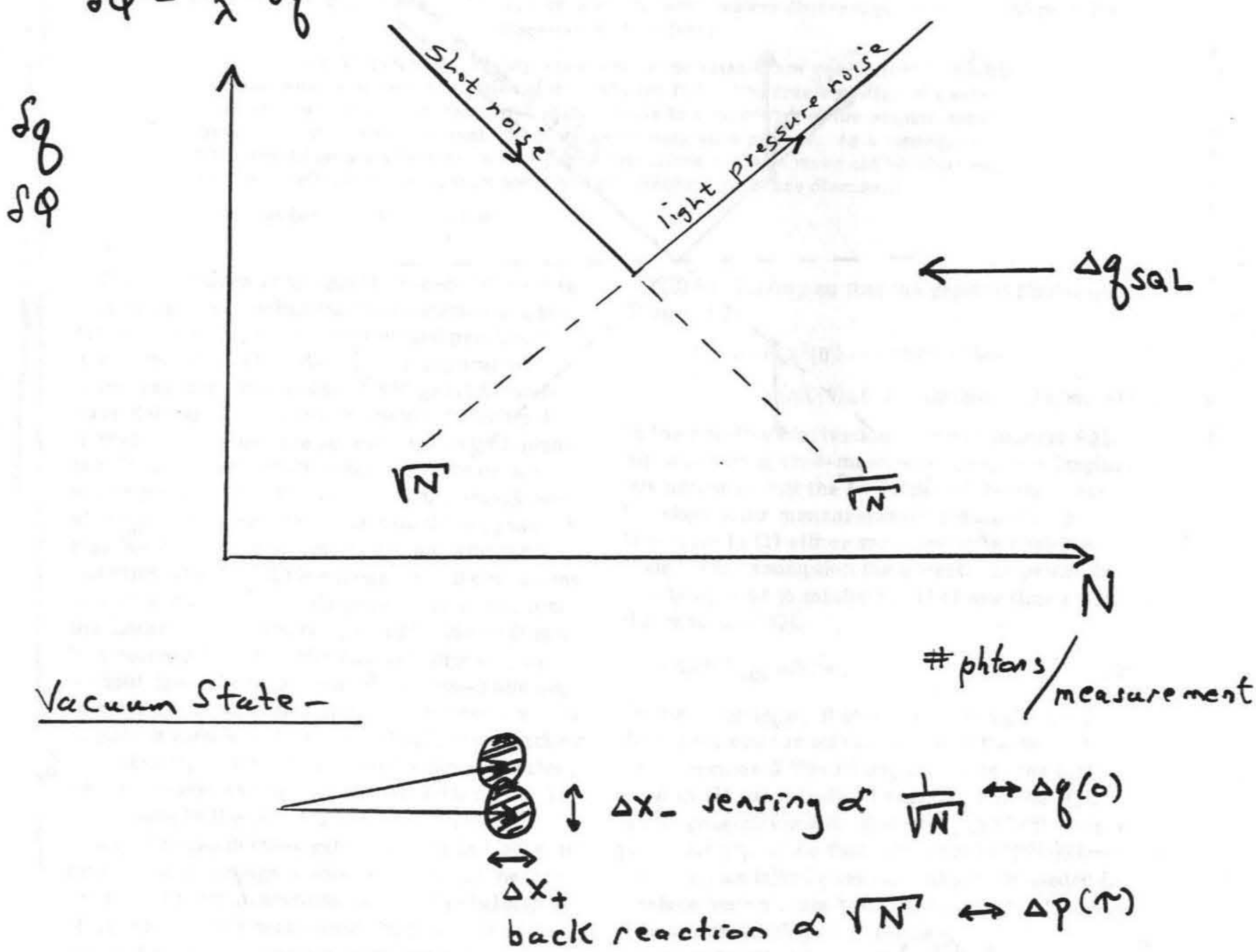
Expressed Explicitly for Interferometry

"Free" Mass



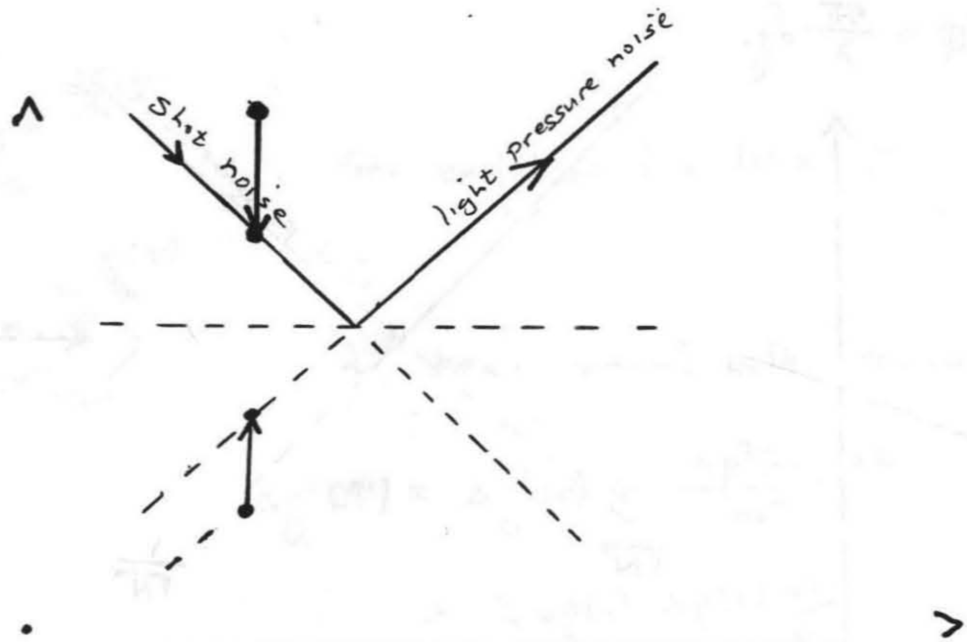


$$\delta\phi = \frac{4\pi}{\lambda}\,\delta q$$

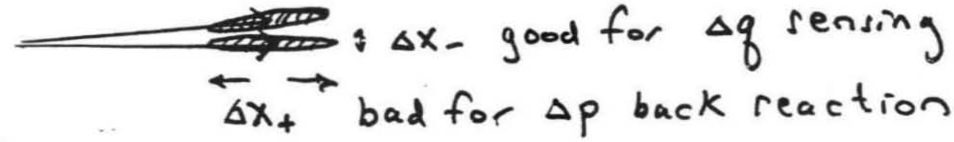


Vacuum State -

(Overlay)



Squeezed State –



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Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions

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(Received 27 June 1983)

The familiar minimum-uncertainty wave packets for masses are generalized in analogy with the two-photon coherent states of the radiation field. The free evolution of a subclass of these states, the contractive states, leads to a narrowing of the position uncertainty in contrast with the usual minimum-uncertainty wave packets. As a consequence the standard quantum limit for monitoring the positions of a free mass can be breached. Further implications on quantum nondemolition measurements are discussed.

PACS numbers: 03.65.Bz, 04.80.+z

There has been considerable recent interest in ascertaining and achieving the fundamental quantum limits on signal processing and precision measurements, in particular for applications to optical communications[1-3] and gravitational-wave detection.[4-8] A major result of this work is that one can beat the so-called standard quantum limit for amplitude measurements on harmonic oscillators. However, for the gravitational-wave interferometer[9] it is usually supposed[7,8] that the resolution is limited by the "standard quantum limit" (SQL) for measuring the positions of a free mass.[4-5] In this paper it is shown that the latter SQL is also *not* generally valid; it can be breached by a specific quantum measurement without special preparation of the free-mass quantum state. Toward this end I will describe a class of generalized minimum-uncertainty wave packets for masses, to be called twisted coherent states, which are also of interest in their own right. The breakdown of the SQL for free-mass position measurements demonstrates the fact that back actions from a conjugate observable do *not* necessarily, at least in accordance with the principle of quantum mechanics, limit the accuracy of subsequent measurements on an observable.

The evolution of a free mass is given by $X(t) = X(0) + P(0)t/m$, so that the position fluctuation at time t is

$$\langle \Delta X^2(t)\rangle = \langle \Delta X^2(0)\rangle + \langle \Delta P^2(0)\rangle t^2/m^2 + \langle \Delta X(0)\Delta P(0) + \Delta P(0)\Delta X(0)\rangle t/m. \quad (1)$$

In the previous derivation[4-5] of the general SQL for monitoring free-mass positions, it is implicitly assumed that the $t=0$ state of the mass (or the state after measurement) is such that the last term in (1) either vanishes or is positive. Under this assumption the uncertainty principle can be applied to minimize (1) at any time t with the resulting SQL

$$\langle \Delta X^2(t)\rangle_{\mathrm{SQL}} = \hbar t/m. \quad (2)$$

On the other hand, it is clear that $\langle \Delta X^2(t_0)\rangle = 0$ the initial state is an eigenstate of the self-adjoint operator $X(0) + P(0)t_0/m$. Thus, the last term in (1) can surely be negative and the SQL is not generally valid. However, $\langle \Delta X^2(t)\rangle = 0$ implies $\langle \Delta P^2(t)\rangle = \infty$ so that $\langle P^2(0)/2m\rangle = \langle P^2(t)/2m\rangle = \infty$, i.e., an infinite average energy is needed to produce such a state.[3] A more realistic description can be developed as follows.

For an oscillator of mass m and frequency ω, the twisted or two-photon coherent states (TCS)[1,3]

$|\mu\nu\alpha\rangle$ are the eigenstates of $\mu a+\nu a^\dagger$:

$$(\mu a+\nu a^\dagger)|\mu\nu\alpha\rangle=(\mu\alpha+\nu\alpha^*)|\mu\nu\alpha\rangle,\quad |\mu|^2-|\nu|^2=1, \tag{3}$$

where a is the annihilation operator of the oscillator mode. Here we adopt them to yield a class of states for a mass m with position X and momentum P. Define the following operator a on the Hilbert space of states for the mass:

$$a\equiv X(m\omega/2\hbar)^{1/2}+iP/(2\hbar m\omega)^{1/2},\quad [a,a^\dagger]=I, \tag{4}$$

where ω is now an *arbitrary* parameter with unit $\sec^{-1}$. The *twisted coherent states* (TCS) $|\mu\nu\alpha\omega\rangle$ of a mass are defined to be the eigenstates of $\mu a+\nu a^\dagger$, $|\mu|^2-|\nu|^2=1$, in analogy with (3) but with a given by (4). The free-mass Hamiltonian can be expressed

$$H=P^2/2m=\tfrac{1}{2}\hbar\omega(a^\dagger a-\tfrac{1}{2}a^2-\tfrac{1}{2}a^{\dagger 2}+\tfrac{1}{2}). \tag{5}$$

The wave function $\langle x|\mu\nu\alpha\omega\rangle$, $X|x\rangle=x|x\rangle$, can be found through Eq. (3.24) of Ref. 1. Within the choice of a constant phase it is given by

$$\langle x|\mu\nu\alpha\omega\rangle=\left[\frac{m\omega}{\pi\hbar|\mu-\nu|^2}\right]^{1/4}\exp\left\{-\frac{m\omega}{2\hbar}\frac{1+i\xi}{|\mu-\nu|^2}\left[x-\left(\frac{2\hbar}{m\omega}\right)^{1/2}\alpha_1\right]^2+i\left(\frac{2m\omega}{\hbar}\right)^{1/2}\alpha_2\left[x-\left(\frac{2\hbar}{m\omega}\right)^{1/2}\alpha_1\right]\right\}, \tag{6}$$

where

$$\xi\equiv\operatorname{Im}(\mu^*\nu);\quad \alpha\equiv\alpha_1+i\alpha_2,\quad \alpha_1,\alpha_2\ \text{real}. \tag{7}$$

The wave functions (6) constitute a generalization of the usual minimum-uncertainty wave packets treated in every quantum mechanics textbook, which are given by (6) with $\xi=0$. In the context of oscillators, "squeezing" is obtained when $\nu\neq 0$ in $|\mu\nu\alpha\rangle$, and ξ is related to the direction of minimum squeezing. As will be seen in the following, when $\xi>0$ the x-dependent phase in (6) leads to a narrowing of $\langle\Delta X^2(t)\rangle$ from $\langle\Delta X^2(0)\rangle$ during free evolution, in direct contrast with the well-known spreading of $\langle\Delta X^2(t)\rangle$ for minimum-uncertainty wave packets.[10] Because of this behavior, mass states (6) with $\xi>0$ will be called *contractive states.*

The first two moments of (6) are

$$\langle X\rangle\equiv\langle\mu\nu\alpha\omega|X|\mu\nu\alpha\omega\rangle=(2\hbar/m\omega)^{1/2}\alpha_1,\quad \langle P\rangle=(2\hbar m\omega)^{1/2}\alpha_2, \tag{8}$$

$$\langle\Delta X^2\rangle\equiv\langle(X-\langle X\rangle)^2\rangle=2\hbar\zeta/m\omega,\quad \langle\Delta P^2\rangle=2\hbar m\omega\eta, \tag{9}$$

$$\zeta\equiv|\mu-\nu|^2/4,\quad \eta\equiv|\mu+\nu|^2/4;\quad \zeta\eta=(1+4\xi^2)/16, \tag{10}$$

$$\langle\Delta X\Delta P\rangle=i\hbar/2-\xi\hbar,\quad \langle\Delta P\Delta X\rangle=-i\hbar/2-\xi\hbar, \tag{11}$$

$$\langle P^2/2m\rangle=\hbar\omega(\alpha_2{}^2+\eta). \tag{12}$$

The average mass energy (12) is finite when ω, α_2, and $|\nu|$ are finite. From (9)–(10) it follows that the minimum-uncertainty product $\langle\Delta X^2\rangle\langle\Delta P^2\rangle=\hbar^2/4$ is achieved if and only if $\xi=0$.

The position fluctuation for a free mass starting in an arbitrary TCS (6) is immediately obtained from (1) and (9)–(11),

$$m\langle\Delta X^2(t)\rangle/2\hbar=\zeta/\omega-\xi t+\eta\omega t^2. \tag{13}$$

If $\xi\leq 0$, $\langle\Delta X^2(t)\rangle$ increases monotonically. In contrast to this usual situation, Eq. (13) is plotted in Fig. 1 for contractive states at $t=0$ (i.e., for $\xi>0$). The minimum fluctuation $1/16\omega\eta$ can be made arbitrarily small even for fixed ω by letting η (and thus also $\langle H\rangle$) become arbitrarily large. The time t_m at this fluctuation level is $t_m=\xi/2\eta\omega$ so that $m\langle\Delta X^2(t_m)\rangle/2\hbar t_m=1/8\xi$. If $\langle\Delta X^2(t)\rangle$ is minimized with respect to ω at any given t simi-

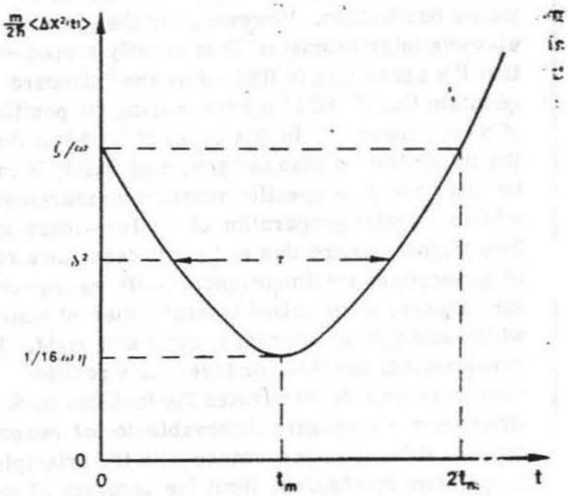

FIG. 1. The position fluctuation of a contractive state from (13); $t_m\equiv\zeta/2\eta\omega$, $t_m\to 0$ when $\zeta\to 0$.

Comment on "Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions"

In a recent Letter,[1] Yuen has considered the so-called twisted or two-photon coherent states to show that the free evolution of certain of such states (contractive states) leads to a narrowing of the position uncertainty wave packets. It is the purpose of this Comment to stress that this narrowing property has nothing to do with Yuen's coherent states and has an almost twenty-year-old history. The general criterion for narrowing of the free-motion position uncertainty has been obtained in some of the standard textbooks on quantum mechanics where the following expression has been derived[2]:

$$\langle \Delta\hat{x}^2(t)\rangle = \langle \Delta\hat{x}^2(0)\rangle + (\langle \Delta\hat{p}^2(0)\rangle/m^2)t^2 + 2t\int dx\,[x-\langle\hat{x}(0)\rangle]\,j(x). \tag{1}$$

In this equation, j is the standard quantum mechanical probability current of the initial wave function. If we assume (without loss of generality) that initially $\langle\hat{x}(0)\rangle = 0$, a narrowing of $\langle \Delta x^2(t)\rangle$ is obtained if and only if

$$\int dx\, x j(x) < 0. \tag{2}$$

There is, of course, an infinite number of states that satisfy this condition. As an example, the initial wave function

$$\psi(x,0) = f(x)\exp\left(-\frac{i|\lambda_n|}{\hbar}\frac{x^{2n}}{2n}\right) \quad n = 1,2,\ldots \tag{3}$$

with λ_n arbitrary complex numbers and with $f(x)$ a real L^2 normalizable function leads to the narrowing effect. The wave function with $n=2$ is especially instructive. For any $f(x)$ this wave function, which has a contractive phase similar to Yuen's wave packet, is not a twisted coherent state but nevertheless leads to a narrowing effect.

A general discussion on how to realize experimentally the initial wave function (3) and how to obtain the narrowing effect (including the one given by Yuen) was presented by Lamb in 1969.[3]

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[1]H. P. Yuen, Phys. Rev. Lett. 51, 719 (198[illegible]).

[2]See, for example, K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), p. 27, Eq. (41).

[3]W. E. Lamb, Jr., Phys. Today 22, No. 4, 23 (1969). See pp. 25 and 26 for details.

Yuen Responds: Contractive twisted coherent states (TCS) comprise the first explicit class of states that was shown to lead to a narrowing of the free-mass position fluctuation $\langle \Delta X^2(t)\rangle$, as far as I know. It makes little sense to say that they have nothing to do with such narrowing. Nowhere in my paper is it stated or implied that these states are the only ones leading to such narrowing, or that they are somehow essential for that purpose. In fact, when I first mentioned the possibility of such narrowing I used the eigenstate of the self-adjoint operator $X + Pt/m$ as an example, which is strictly speaking not a TCS. Among all the possible states that exhibit such narrowing, contractive TCS form a natural generalization of the usual minimum-uncertainty wave packets. In addition, the time duration of their contraction and the associated $\langle \Delta X^2(t)\rangle$ can be conveniently parametrized. Calling such states "contractive states" when other states may also contract is like calling TCS "squeezed states" when they are not the only states that exhibit squeezing.

A main objective of my paper is to give a measurement scheme that can directly monitor the positions of a free mass in a quantum-nondemolitional way. For this purpose, the states which contract are to be the ones in which a free mass would be left after a certain measurement, without additional intervention. For a discussion of this point see Caves in Ref. 8 of my paper. For contractive TCS such a measurement is the one described by $|\mu\nu\alpha\omega\rangle\langle\mu\nu\alpha\omega|$ as discussed in my paper; the possible realization of this measurement I merely stated without proof because of space limitation. Thus, contractive TCS turn out to be essential in my quantum-nondemolitional position measurement scheme. Dr. Wodkiewicz did not give a measurement which would leave the free mass in his more general contractive states. On the other hand, I would be very surprised if TCS are essential in all possible quantum-nondemolitional position measurements.

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Comment on "Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions"

Recently, Yuen has published a very interesting paper[1] in which he gives a non-QND method for beating the standard quantum limit when measuring the position of a free mass (QND stands for quantum nondemolition). The technique utilizes the so-called two-photon coherent state (TCS).

There is a difficulty with his repeated measurement scheme, however. The reason for this Comment is to call attention to this difficulty, and also to show that TCS can be used to make finite-energy QND-type measurements.

Consider the following recapitulation of Yuen's paper. At $t=0$, an arbitrary free mass state $|\psi\rangle$ is prepared into a TCS, $|\mu\nu\alpha\omega\rangle \equiv |\alpha\rangle$, by interaction of the system with a generalization of the two-meter detector of Arthurs and Kelly,[2] followed by subsequent meter reduction. This measurement can be described in the Gordon and Louisell[3] terminology (which Yuen prefers) by $|\alpha\rangle\langle\alpha|$. It is important to note that the acutal state $|\alpha\rangle$ which obtains after meter reduction is only probabilistically determined, depending on the overlap between $|\alpha\rangle$ and $|\psi\rangle$. Finally, one may also look upon this state preparation as the measurement of the non-self-adjoint operator $A(0)$, where

$$A(t)=A(x(t),p(t)) \equiv (\mu+\nu)(m\omega/2\hbar)^{1/2}x(t) + i(\mu-\nu)p(t)/(2\hbar m\omega)^{1/2}. \quad (1)$$

Thus at $t=0$ the system is found in some eigenstate $|\alpha\rangle$ of $A(0)$ with eigenvalue $\alpha=\alpha_1+i\alpha_2$.

As shown by Yuen, from the measured eigenvalue one can read off the mass's position $\langle x(0)\rangle=(2\hbar/m\omega)^{1/2}\alpha_1$ and momentum $\langle p(0)\rangle=(2\hbar m\omega)^{1/2}\alpha_2$, with uncertainties $\langle\Delta x^2(0)\rangle=2\hbar\xi/m\omega$ and $\langle\Delta p^2(0)\rangle=2\hbar m\omega\eta$, where ξ and η are functions of μ,ν. He also shows that as t goes from 0 to a time $2t_m$, the position uncertainty $\langle\Delta x^2(t)\rangle$ first decreases, and then increases to its $t=0$ value. So far, no difficulties.

However, at $t=2t_m$ Yuen calls for another measurement on the system, presumably of $A(2t_m)$. But we should not describe this measurement by $|\alpha\rangle\langle\alpha|$ as Yuen does, but as $|\alpha'\rangle\langle\alpha'|$, assuming in general that $\alpha\neq\alpha'$. This assumption is correct since one finds $[A(0),A(2t_m)]=i(\mu+\nu)^2\omega t_m \neq 0$. One can easily show by calculating $\langle\alpha|A(2t_m)|\alpha\rangle$ and the nonzero $\langle\alpha|\Delta A^2(2t_m)|\alpha\rangle$ that the system will "jump" to a range of states centered about $\alpha_1'=\alpha_1+(m\omega/2\hbar)^{1/2}\langle p(0)\rangle 2t_m/m$, $\alpha_2'=\alpha_2$. The width of the range of states and the magnitude of the effect that it has on Yuen's proposal are difficult to calculate because of the non-self-adjoint nature of $A(2t_m)$. Nevertheless this "back-action" mechanism will contaminate the measurements, and has not been included in Yuen's scheme. A complete evaluation of the extent of this difficulty will presumably involve a lengthy calculation of the "meter-interaction-and-reduction" type pioneered by Caves.

It is interesting to note that one could achieve a continuous QND-like measurement with the operator $A(0)$. This is because the operators $x(0)=x(t)-p(t)t/m$ and $p(0)=p(t)$ are separately QND. Measurement of $A(0)$ has the additional desirable property that the resulting state has finite energy, quite properly one of Yuen's motivations for examining TCS in his original proposal. However, with $x(0)$ mixing position and momentum operators, measurement of $A(0)$ could no longer be described as a position measurement, a view also taken of Yuen's scheme by Caves recently.[4]

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[1]H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983).
[2]E. Arthurs and J. L. Kelly, Jr., Bell Syst. Tech. J. **44**, 725 (1965).
[3]J. P. Gordon and W. H. Louisell, in *Physics of Quantum Electronics,* edited by P. L. Kelly *et al.* (McGraw-Hill, New York, 1966), pp. 833–840.
[4]C. M. Caves, "A Defense of the Standard Quantum Limit for Free-Mass Position" (to be published).

Yuen Reponds: I am grateful to Dr. Lynch for providing me with the opportunity to clarify certain points in connection with the standard quantum limit (SQL), quantum nondemolition (QND), and my paper. Terminology and concepts such as back action, position measurement, QND, etc., are fraught with ambiguity and imprecision, both in the QND literature and in my own paper. Before they are definitively cleared up, it is important to attend to the actual content of a result rather than its verbal representation.

Using the notations of my paper,[1] I would like to first describe more accurately my principal results as they relate to SQL and QND: (1) The previous derivation of the SQL for monitoring free-mass positions is not generally valid. In particular, a specific realizable measurement described by $|\mu\nu\alpha\omega\rangle \times \langle\mu\nu\alpha\omega|$ could leave the free mass in a contractive two-photon coherent state (TCS). The SQL can then be broken to an arbitrary degree in a second sufficiently accurate position measurement. Note that the SQL would *not* be broken by the same $|\mu\nu\alpha\omega\rangle\langle\mu\nu\alpha\omega|$ measurement. As explained later, another one with $|\Psi^M\rangle$ having a smaller associated position fluctuation $\langle\Psi^M|\Delta X^2|\Psi^M\rangle$ is required, for example, $|\mu'\nu'\alpha\omega\rangle\langle\mu'\nu'\alpha\omega|$ with $|\mu'-\nu'|^2 < |\mu-\nu|^2$ performed at $t\sim t_m$. (2) The $|\mu\nu\alpha\omega\rangle \times \langle\mu\nu\alpha\omega|$ measurement can be used to monitor the free-mass positions in an arbitrary sequence of measurement, with a limitation $\hbar/m$ on the ratio of the position resolution to the time lapse between two measurements. No such limitation exists for the measurement $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$, of which no realization is known, however. If such measurement is indeed realizable, there can be no quantum limit of any kind on position monitoring.

Dr. Lynch's main point[2] appears to result from a combination of both his confusion about approximate simultaneous measurements and my overly condensed presentation as a Letter. [There are also a number of misprints in my paper, some of which have been corrected in the erratum.[1] I have since found four more: "this work" should read "these works" in the second sentence of the paper; $1+i\xi$ should read $1+i2\xi$ in Eq. (6); $\zeta/2\eta\omega$ should read $\xi/2\eta\omega$ in the figure caption; and $\hbar\xi/m\omega$ should read $4\hbar\xi/\eta\omega$ in line 12 of the second column of p. 721.] The measurement described by $|\mu\nu\alpha\omega\rangle \times \langle\mu\nu\alpha\omega|$, the $A(0)$ measurement in Lynch's terminology, is an approximate simultaneous measurement of position and momentum: α being a variable whose real and imaginary parts provide the *measurement readings* corresponding to the position and momentum estimates. The state after a measurement with reading α is just $|\mu\nu\alpha\omega\rangle$; it is *not* probabilistically determined. In the case of point (2) above corresponding to that discussed by Lynch, the same $|\mu\nu\alpha\omega\rangle\langle\mu\nu\alpha\omega|$ measurement is made at $t=0$ and $t=2t_m$ while the mass state has evolved. It is important to note that $\langle\Delta X^2\rangle$ gives only the state contribution to the position fluctuation in a measurement; it is the fluctuation observed in a perfect or "exact position measurement." In an "approximate measurement" there would be additional fluctuation from the measurement itself. In my paper, both the state and measurement contributions to the position fluctuation have been included through the resolution factor $4\hbar\xi/m\omega$ instead of $2\hbar\xi/m\omega$. This resolution value is obtained from the probability $|\langle\mu\nu\alpha'\omega|\mu\nu\alpha\omega\rangle|^2$; it makes no sense to set $\alpha'=\alpha$. [It turns out that this doubling of the position uncertainty exactly cancels out the factor of 2 advantage of Eq. (15) compared to the SQL. This explains why a second measurement with $|\psi^M\rangle$ having lower $\langle\Delta X^2\rangle$ is required for bleaching the SQL.] The momentum reading needs never be made; it has no effect on the position fluctuation during the sequence of measurements. Thus, whatever "back action" there is has already been accounted for.

The $|\mu\nu\alpha\omega\rangle\langle\mu\nu\alpha\omega|$ measurement without the α_2 reading is emphatically a position measurement on all grounds: physical, formal, and the purpose of such measurements. The α_1 reading indicates the free-mass position before and after measurements within prescribed uncertainties. It is mathematically equivalent to an exact position measurement in the presence of meter-reading fluctuation, as far as the measurement probability is concerned. There is no reason to call an expression a "quantum limit" if it does not cover this kind of approximate position measurements which serve the purpose of monitoring the mass positions. The SQL is meant to apply to *all* conceivable measurements.

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[1]H. P. Yuen, Phys. Rev. Lett. **51**, 719, 1603(E) (1983).

[2]R. Lynch, preceding Comment [Phys. Rev. Lett. **52**, 1729 (1984)].

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Repeated Contractive-State Position Measurements and the Standard Quantum Limit

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It is shown that if the "standard quantum limit" is taken in a predictive sense, then a repeated measurement scheme involving contractive states, recently proposed by Yuen, does not break this limit.

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Recently Yuen[1] has proposed a scheme to beat the "standard quantum limit" (SQL) on free-mass position monitoring by means of contractive states. There have been several unpublished responses[2] to Yuen's proposal which seek to defend the SQL on general grounds.

A Comment[3] I wrote takes a different view. A qualitative point made was that even assuming the validity of the framework adopted by Yuen, he has not fully considered the impact of measurements in his scheme. It is the purpose of this paper to flesh out the arguments of that Comment, and to show that if the SQL is taken in a predictive sense, Yuen's proposal fails to beat this limit precisely because of such measurement corrections.

Before turning to the detailed discussion it is worthwhile reviewing the reasoning which leads to the SQL. Suppose at $t=0$ one places a free mass approximately at the origin, with the intent of measuring its subsequent position in time (to see if a weak force is acting on it, for example.) How often, and how closely, should one monitor the particle's position? If it is decided to make a measurement every t seconds, one must make t short enough to counter any possible spreading of the wave function. On the other hand, each measurement to precision Δx produces a variance of momentum Δp (by means of the uncertainty principle), which then feeds back into the uncertainty of the position, $\Delta x(t)$, at the time of the next measurement. An analysis[4] of this "back action" leads to the SQL, $\Delta x(t) \geqslant (\hbar t/m)^{1/2}$.

Yuen seeks to beat the SQL by means of "contractive states," and a measurement formalism based on the work of Gordon and Louisell.[5] The contractive states are the so-called "two-photon coherent states" (TCS), $|\mu\nu\alpha\omega\rangle$. For a full discussion of these states the reader is referred to Yuen's original paper[1] and the references therein. Here I simply recall that this state may be taken to represent a free particle of mass m, whose expectation values of position and momentum are $\langle x\rangle = (2\hbar/m\omega)^{1/2}\,\mathrm{Re}(\alpha)$, $\langle p\rangle = (2\hbar m\omega)^{1/2} \times \mathrm{Im}(\alpha)$, with variances $\langle \Delta x^2\rangle = 2\hbar\zeta/m\omega$, $\langle \Delta p^2\rangle = 2\hbar m\omega\eta$. Here ω is an arbitrary parameter, and $\zeta = |\mu-\nu|^2/4$, $\eta = |\mu+\nu|^2/4$, subject to $|\mu|^2 - |\nu|^2 = 1$.[6]

As Yuen has shown, for values of the parameter $\xi \equiv \mathrm{Im}(\mu^*\nu) > 0$, the $|\mu\nu\alpha\omega\rangle$ states are *contractive*, that is, the initial position variance $\langle \Delta x^2\rangle$ narrows under free evolution for a time $t_m \equiv \xi/2\eta\omega$. This result is cleverly exploited by Yuen to avoid spreading of the wave function. The idea then is to make a sharp position measurement at time $t=t_m$ when the position uncertainty is a minimum, while leaving the system in a $|\mu\nu\alpha\omega\rangle$ state after the measurement, ready to undergo another contraction.

In the Gordon-Louisell terminology such a measurement is described by the projection operator, $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$.[7] This notation is somewhat abstract—in fact, it is not clear that such measurements are physically possible, in the sense of a Hamiltonian realization, for example. Nevertheless, if one assumes the existence such measurements, and if one considers a system initially in the state $|\psi\rangle$, then according to the Gordon-Louisell theory a $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$ measurement yields the value α' and the corresponding state $|\mu\nu\alpha'\omega\rangle$ after measurement, with probability density $|\langle\mu'\nu'\alpha'\omega'|\psi\rangle|^2/\pi$.

Consider the following sequence of events: The system starts in the state $|\mu\nu\alpha\omega\rangle$ at $t=0$, and evolves freely until time t; at that time $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$ measurement is made. What are the statistics of such a measurement? For our purposes, it is physically more transparent to be able to label the states $|\mu\nu\alpha\omega\rangle$ resulting from the measurement by their expectation value of momentum and position, $\langle x'\rangle$ and $\langle p'\rangle$, respectively, than by the values of $\mathrm{Re}(\alpha')=(m\omega/2\hbar)^{1/2}\langle x'\rangle$ and $\mathrm{Im}(\alpha')=\langle p'\rangle/(2\hbar m\omega)^{1/2}$. In terms of these variables the probability density $P(\langle x'\rangle,\langle p'\rangle;\langle x\rangle,\langle p\rangle|t)$ for a measurement at time t to yield a state whose position and momentum expectation values are $\langle x'\rangle$ and $\langle p'\rangle$, respectively, given that the state had position and momentum expectation values $\langle x\rangle$ and $\langle p\rangle$, respectively, at $t=0$, is then

$$P(\langle x'\rangle,\langle p'\rangle;\langle x\rangle,\langle p\rangle|t)=|\langle\mu'\nu'\alpha'\omega'|\psi(t)\rangle|^2/h=|\langle\mu'\nu'\alpha'\omega'|\exp(-i\hat{H}t/\hbar)|\mu\nu\alpha\omega\rangle|^2/h, \quad (1)$$

where $\hat{H}=\hat{p}^2/2m$. The factor of $1/h$ results from the Jacobian of the transformation such that $d^2\alpha'\rightarrow d\langle x'\rangle d\langle p'\rangle/2\hbar$.

With this probability function it is found after some calculation[8] that the measurement at time t produces the following expected values and variances:

$$\langle\langle x'\rangle\rangle=\int d\langle x'\rangle d\langle p'\rangle\langle x'\rangle P=\langle x\rangle+\langle p\rangle t/m, \quad (2)$$

$$\langle\Delta\langle x'\rangle^2\rangle=\int d\langle x'\rangle d\langle p'\rangle\langle x'\rangle^2P-\langle\langle x'\rangle\rangle^2=2\hbar\zeta'/m\omega'+(2\hbar/m)(\zeta/\omega-\xi t+\eta\omega t^2), \quad (3)$$

$$\langle\langle p'\rangle\rangle=\int d\langle x'\rangle d\langle p'\rangle\langle p'\rangle P=\langle p\rangle, \quad (4)$$

$$\langle\Delta\langle p'\rangle^2\rangle=\int d\langle x'\rangle d\langle p'\rangle\langle p'\rangle^2P-\langle\langle p'\rangle\rangle^2=2\hbar m\omega'\eta'+2\hbar m\omega\eta, \quad (5)$$

$$\langle\langle x'\rangle\langle p'\rangle\rangle=\langle\langle p'\rangle\langle x'\rangle\rangle=\int d\langle x'\rangle d\langle p'\rangle\langle x'\rangle\langle p'\rangle P=-(\xi+\xi')\hbar+2\hbar\eta\omega t+\langle p\rangle\langle x\rangle+\langle p\rangle^2t/m. \quad (6)$$

Here ζ',η', and ξ' are defined exactly as the corresponding ζ,η, and ξ with all quantities primed.

It is worth noticing two features of Eqs. (2)–(6). The first is that since $\langle\Delta\langle x'\rangle^2\rangle$ and $\langle\Delta\langle p'\rangle^2\rangle$ in (3) and (5) are in general nonzero, we conclude that the measurement "demolishes" the initial state $|\mu\nu\alpha\omega\rangle$. That is, if we have an ensemble of systems all prepared in the state $|\mu\nu\alpha\omega\rangle$ at $t=0$, and perform the measurement at time t, over the ensemble we will obtain values of $\langle x'\rangle$ and $\langle p'\rangle$ in a range given by (3) and (5), centered on (2) and (4).

We also note that if we choose the measurement time to be $t=t_m$, then as is shown in Ref. 1, the second term on the right-hand side of (3) is a minimum. Then by appropriate choice of the values of ζ',ω' we can make the magnitude of the first term as small as we wish, leading to an overall $\langle\Delta\langle x'\rangle^2\rangle$ which can beat the SQL. As explained above, this is in essence the scheme espoused by Yuen.

But does this tell the whole story? One difficulty is that this claim is being made on the basis of a single measurement.[9] The effects of back action, however, are seen on the *second* measurement. Furthermore, (3) and (5) *do* reveal the presence of back action. For example, if ζ' is chosen to be small in (3) in order to define the particle's position sharply, then the spread of momentum values in (5) becomes large, since ζ',η' satisfy an uncertainty-type relation, $\zeta'\eta'=(1+4\xi'^2)/16\geq\frac{1}{16}$.

The back action in this case differs from that which obtains when one measures the particle's position only. Here we are making simultaneous approximate position and momentum measurements, and even though the latter's value is disturbed by the measurement, one obtains the disturbed value as a result of the measurement. With the measured values of $\langle x'\rangle$ and $\langle p'\rangle$ in hand one is then able to predict the position $\langle x''\rangle$ which will be found at the *next* measurement at $t=2t_m$ by means of (2), to arbitrary sharpness, as discussed above.

However back action *does* prevent one from being able to predict the value of $\langle x''\rangle$ to better than the SQL *prior* to making the two measurements. To see this consider the probability density $\mathcal{P}$ that measurements at times t, $2t$ will yield values (and the corresponding states) $\langle x'\rangle,\langle p'\rangle$ and $\langle x''\rangle,\langle p''\rangle$, respectively. We have

$$\mathcal{P}=P(\langle x'\rangle,\langle p'\rangle;\langle x\rangle,\langle p\rangle|t)P(\langle x''\rangle,\langle p''\rangle;\langle x'\rangle,\langle p'\rangle|t). \quad (7)$$

The expected outcome of the second position measurement is

$$\langle\langle x''\rangle\rangle=\int d\langle x'\rangle d\langle p'\rangle d\langle x''\rangle d\langle p''\rangle\langle x''\rangle\mathcal{P}=\langle x\rangle+2\langle p\rangle t/m. \quad (8)$$

The second line follows easily from the first if one does the $\langle x''\rangle,\langle p''\rangle$ integrals first, using Eqs. (2)–(6). The variance of the measured value is found after some calculation to be

$$\langle \Delta \langle x'' \rangle^2 \rangle = \int d\langle x' \rangle \, d\langle p' \rangle \, d\langle x'' \rangle \, d\langle p'' \rangle \, \langle x'' \rangle^2 \mathcal{P} - \langle \langle x'' \rangle \rangle^2 = (2\hbar/m)\Gamma(t), \tag{9}$$

where $\Gamma(t) = \Gamma_1(t) + \Gamma_2(t)$ and

$$\Gamma_1(t) \equiv 2\zeta'/\omega' - \xi' t + \eta' \omega' t^2, \tag{10a}$$

$$\Gamma_2(t) \equiv 2\{\zeta/\omega - \xi t + \eta t^2\} - \xi t + 3\eta\omega t^2. \tag{10b}$$

If we now take the measuring time to be t_m, $2t_m$, and minimize the value of $\langle \Delta \langle x'' \rangle^2 \rangle$ with respect to the parameters $\zeta, \eta, \omega, \zeta', \eta', \omega'$, subject to the constraint $\zeta\eta = (1+4\xi^2)/16$ (and, as noted above, the corresponding primed equation), we find after some calculation

$$\begin{aligned} \langle \Delta \langle x'' \rangle^2 \rangle &\geq \tfrac{1}{2}(1+\sqrt{2})(\hbar/m)(2t_m) \quad (t = t_m) \\ &\sim 1.2(\hbar/m)(2t_m), \end{aligned} \tag{11}$$

i.e., 1.2 times the SQL.

Besides being an interesting result in itself, this predictive limit implies a real shortcoming of Yuen's proposal, at least when position monitoring is used for weak-force detection. Because the position of the particle jumps around in a random way at each measurement, one must decide at the end of each measurement interval whether the measured position indicates the presence or not of a force. Since one is unable to predict the particle's position after a number of measurements to better than the SQL, this then limits the ability of the scheme to detect the accumulated effect of a weak force over a long time base to this limit.

In summary, then, one may say that the use of contractive states in repeated position measurements is subject to the SQL in a predictive sense, which makes the technique less than straightforward compared to, say, quantum nondemolition[4] techniques. The reason is that $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$ measurements demolish the state. Then in the repeated measurement of the position of the particle the resulting back action causes the measured position values to jump around in a random way. This underlines the prime importance given to avoiding state demolition in quantum nondemolition theory.

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[1]H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983).

[2]C. M. Caves, to be published; B. L. Schumaker, to be published. A Comment has also appeared by K. Wódkiewicz [Phys. Rev. Lett. **52**, 787 (1984)] which is not relevant to the issues discussed here.

[3]R. Lynch, Phys. Rev. Lett. **52**, 1729 (1984).

[4]See, for example, C. M. Caves *et al.*, Rev. Mod. Phys. **52**, 341 (1980), especially p. 359.

[5]J. P. Gordon and W. H. Louisell, in *Physics of Quantum Electronics*, edited by P. L. Kelley *et al.* (McGraw Hill, New York, 1966), pp. 833–840.

[6]W. G. Unruh (private communication) has pointed out that a contractive state may be viewed simply as a minimum-uncertainty wave packet propagated backwards in time. Thus, given a minimum-uncertainty wave packet $\psi(x)$ at $t=0$, with position and momentum uncertainties satisfying $\Delta x \Delta p = \hbar/2$, Yuen's contractive state $\langle x|\mu\nu\alpha\omega\rangle$ is then $\exp(i\hat{H}t_m/\hbar)\psi(x)$. Hence the discussion in this paper could be carried out in the context of these more familiar wave functions. Alternatively the treatment could be presented in terms of the also somewhat simpler notation of "squeezed states." See C. M. Caves, Phys. Rev. D **23**, 1963 (1981), particularly pp. 1965 f. However, in the spirit of Yuen's paper, I will stick to his notation. Calculations are in the Schrödinger picture.

[7]Here I only consider the repeated $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$ measurement schemes, which are prominently featured in Yuen's original paper (Ref. 1). A new point raised by Yuen in his author's reply [Phys. Rev. Lett. **52**, 1730 (1984)] to my Comment, that a single $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega|$ measurement at $t \sim t_m$ could break the SQL, is not particularly surprising. It has long been realized that special state preparation and detection could rival quantum nondemolition measurements. See C. M. Caves, in *Quantum Optics, Experimental Gravitation and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983), p. 605; W. G. Unruh, *ibid.*, pp. 637–645.

[8]The calculation may be accomplished in a straightforward (and probably inelegant) way by taking explicit wave functions for $|\mu\nu\alpha\omega\rangle$, $|\mu'\nu'\alpha'\omega'\rangle$ as in (6) of Ref. 1. The time dependence can then be included by a Green's function. See E. Merzbacher, in *Quantum Mechanics, 2nd Ed.* (Wiley, New York, 1970), p. 163. The details are tedious and not particularly enlightening, so are not given here.

[9]In view of Eqs. (4) and (6), the $|\mu\nu\alpha\omega\rangle$ at $t=0$ cannot be, in general, the outcome of an initial $|\mu\nu\alpha\omega\rangle\langle\mu'\nu'\alpha\omega'|$ measurement without state selection, a possibility we specifically exclude—see Ref. 7.

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Defense of the Standard Quantum Limit for Free-Mass Position

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Measurements of the position x of a free mass m are thought to be governed by the standard quantum limit (SQL): In two successive measurements of x spaced a time τ apart, the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$. Yuen has suggested that there might be ways to beat the SQL. Here I give an improved formulation of the SQL, and I argue for, but do not prove, its validity.

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Conventional wisdom[1,2] holds that in two successive measurements of the position x of a free mass m, the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$, where τ is the time between measurements. This limit is called the *standard quantum limit* (SQL) *for monitoring the position of a free mass.*

The standard "textbook" argument for the SQL runs as follows. Suppose that the first measurement of x at $t=0$ leaves the free mass with position uncertainty $\Delta x(0)$. This first measurement disturbs the momentum p and leaves a momentum uncertainty $\Delta p(0) \geqslant \hbar/2\Delta x(0)$. By the time τ of the second measurement the variance of x (squared uncertainty) increases to

$$(\Delta x)^2(\tau) = (\Delta x)^2(0) + [(\Delta p)^2(0)/m^2]\tau^2 \geqslant 2\Delta x(0)\Delta p(0)\tau/m \geqslant \hbar\tau/m. \tag{1}$$

The standard argument views the SQL as a straightforward consequence of the position-momentum uncertainty principle $\Delta x(0)\Delta p(0) \geqslant \frac{1}{2}\hbar$.

Yuen[3] has pointed out a serious flaw in the standard argument. Between the two measurements the free mass undergoes unitary evolution. In the Heisenberg picture the position operator $\hat{x}$ evolves as

$$\hat{x}(t) = \hat{x}(0) + \hat{p}(0)t/m. \tag{2}$$

Thus the variance of x at time τ is given not by Eq. (1), but by

$$(\Delta x)^2(\tau) = (\Delta x)^2(0) + \frac{(\Delta p)^2(0)}{m^2}\tau^2 + \frac{\langle \hat{x}(0)\hat{p}(0) + \hat{p}(0)\hat{x}(0)\rangle - 2\langle \hat{x}(0)\rangle\langle \hat{p}(0)\rangle}{m}\tau. \tag{3}$$

The standard argument assumes implicitly that the last term in Eq. (3) is zero or positive. Yuen's point[3] is that some measurements of x leave the free mass in a state for which this term is negative. He calls such states *contractive states* because the variance of x decreases with time, at least for a while. As a result, the uncertainty $\Delta x(\tau)$ can be smaller than the SQL. Yuen[3,4] concludes that there are measurements of x that beat the SQL. My conclusion is different: The flaw lies in the standard argument, not in the SQL. In this Letter I give a new, heuristic argument for the SQL, formulate an improved statement of the SQL, and analyze a measurement model that supports the heuristic argument.

The heuristic argument is based on including the effect of the imperfect resolution σ of one's measuring apparatus. If the free mass is in a position eigenstate at the time of a measurement of x, then σ is defined to be the uncertainty in the result; thus, roughly speaking, the measuring apparatus can resolve positions that are more than σ apart. I assume that the measuring apparatus is coupled linearly to x, so that in general the variance of a measurement of x is the sum of σ^2 and the variance of x at the time of the measurement. Consider now two measurements of x at times $t=0$ and $t=\tau$, made with identical measuring apparatuses. In the absence of *a priori* knowledge, the result of the first measurement is completely unpredictable. Nonetheless, the first measurement does yield a value for x. Since this value does not tell one the position before the measurement, it is hard to see what one could mean by a "measurement of x with resolution σ" unless one means that the measurement determines the position immediately after the measurement to be within roughly a distance σ of the measured value.[5] Therefore, I assume that just after the first measurement, the free mass has position uncertainty $\Delta x(0) \leqslant \sigma$; this assumption implies that an *immediate* repetition of the same measurement would yield the same result within approximately the resolution σ. The variance of the second measurement ($t=\tau$) is given by

$$\Delta_2^2 = \sigma^2 + (\Delta x)^2(\tau) \geqslant (\Delta x)^2(0) + (\Delta x)^2(\tau) \geqslant 2\Delta x(0)\Delta x(\tau) \geqslant \hbar\tau/m. \tag{4}$$

According to this argument, *the SQL is a consequence of the uncertainty principle*

$$\Delta x(0)\Delta x(\tau) \geqslant \tfrac{1}{2}|\langle[\hat{x}(0),\hat{x}(\tau)]\rangle| = \hbar\tau/2m, \tag{5}$$

provided that $\sigma^2 \geqslant (\Delta x)^2(0)$. Contractive states do not vitiate this argument; even if the free-mass state after the first measurement is a contractive state such that $\Delta x(\tau) < (\hbar\tau/m)^{1/2}$, the SQL is valid.

Yuen uses a measurement model developed by Gordon and Louisell,[6] which includes the measuring-apparatus resolution. How then can he contend that it is possible to violate the SQL? The answer lies in the assumption $\sigma \geqslant \Delta x(0)$, which links the uncertainty Δ_2 in the second measurement to the position uncertainty $\Delta x(0)$ just after the first measurement. An easy way to circumvent this link, pointed out by Yuen,[4] is to use measuring apparatuses which are not identical. The first measurement, performed with an apparatus of poor resolution $\sigma_1 \geqslant \Delta x(0) >> (\hbar\tau/2m)^{1/2}$, is designed to leave the free mass in a contractive state such that $\Delta x(\tau) << (\hbar\tau/2m)^{1/2}$; the second measurement, performed with an apparatus of good resolution $\sigma_2 << (\hbar\tau/2m)^{1/2}$, has uncertainty $\Delta_2 = [\sigma_2^2 + (\Delta x)^2 \times(\tau)]^{1/2} << (\hbar\tau/m)^{1/2}$, which violates the SQL. Two such measurements should be regarded as a single measurement process, because in a sequence of measurements one would repeat the entire process, not the individual measurements separately. The first measurement is a preparation procedure for the second; it puts the free mass in a state that becomes a near eigenstate of position at the time of the second measurement. It is obvious that a measurement of x can have arbitrarily small uncertainty if one is allowed an arbitrary prior preparation procedure. Although the possibility of two such measurements may be important, more important is to sharpen the formulation of the SQL—to rule out this case to which the SQL clearly cannot apply.

With the preceding discussion in mind, I formulate the SQL as follows: *Let a free mass m undergo unitary evolution during the time τ between two measurements of its position x, made with identical measuring apparatuses; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$.* For this formulation to be true in general, the uncertainty in the second measurement must be understood to be an average uncertainty, averaged over the possible results of the first measurement; the averaging procedure is made explicit in the model considered below [see discussion preceding Eq. (14)]. This improved formulation of the SQL is still an important restriction, because it applies to a class of real experiments.[2] In these experiments one has available a particular technique for measuring x, which is used to make a sequence of measurements on a single free mass. The objective is to detect some external agent (e.g., a force) that disturbs x. The relevant question is how small a disturbance can be detected or, equivalently, how well one can predict the result of each measurement in the absence of the disturbance. The improved SQL addresses precisely this question. Notice that the improved SQL explicitly disallows any tinkering with the free mass during the interval between measurements; the free mass must evolve unitarily with no state preparation and no modification of its Hamiltonian.

Yuen would not agree even with the improved version of the SQL, because he believes that there are measurements that violate the assumption $\sigma \geqslant \Delta x(0)$.[3,4] Specifically, he suggests that there are ways to measure x which have good resolution $\sigma << (\hbar\tau/2m)^{1/2}$, but which leave the free mass in a contractive state with $\Delta x(0) >> (\hbar\tau/2m)^{1/2} >> \sigma$ such that $\Delta x(\tau) << (\hbar\tau/2m)^{1/2}$. Although contractive states are essential to this scheme, they are not enough to invalidate the SQL; also required are measurements of resolution σ that do not determine the position just after the measurement to be within σ of the measured value. Yuen states his suggestion in the notation of the Gordon-Louisell[6] formalism, which can describe formally measurements with $\Delta x(0) \geqslant \sigma$. The existence of this formal description, however, does not guarantee that such measurements can be realized. Gordon and Louisell simply assume the existence of certain measurements [Eqs. (22) and (23) of Ref. 6] without demonstrating that all such measurements can be realized. The measurements suggested by Yuen are among those for which no realization is known.[3,4]

I turn now to a simple model of measurements of x. Within the model one can demonstrate the validity of the SQL, but the model by no means provides a general proof. The model is, however, sufficiently general that it clarifies the meaning of the SQL and indicates where one might seek violations. I work in the Schrödinger picture; operators are denoted by caret.

The first task is to model the initial measurement of x; the model that I employ is like that of Arthurs and Kelly.[7] The free mass is coupled to a "meter," a one-dimensional system with coordinate Q and momentum P, which can be regarded as the first stage of a macroscopic measuring apparatus. The coupling is turned on from $t=-\tilde{\tau}$ to $t=0$ ($\tilde{\tau} << \tau$), it is described by an interaction Hamiltonian $K\hat{x}\hat{P}$ (K is a coupling constant), and it is treated in the impulse approximation (the coupling is so strong that the free Hamiltonians of the free mass and the meter can be neglected). The coupling correlates Q with x. When the interaction is turned off at $t=0$, one "reads out" a value for Q, from which one infers a value for x. The readout of Q can be viewed as an ideal measurement of Q made by the subsequent stages of the measuring apparatus.

At $t=-\tilde{\tau}$, just before the coupling is turned on, the free-mass wave function is $\psi(x)$, and the meter is prepared in a state with wave function $\Phi(Q)$. The total wave function is $\Psi_0(x,Q)=\psi(x)\Phi(Q)$; expectation values and variances with respect to $\Psi_0(x,Q)$ are distinguished by a subscript 0. For simplicity I assume that $\langle\hat{Q}\rangle_0=\langle\hat{P}\rangle_0=0$. At the end of the interaction time ($t=0$) the total wave function becomes

$$\Psi(x,Q)=\psi(x)\Phi(Q-x) \tag{6}$$

(units such that $K\tilde{\tau}=1$); expectation values and variances with respect to $\Psi(x,Q)$ are distinguished by having no subscript. The expectation value of Q at $t=0$ is $\langle\hat{Q}\rangle=\langle\hat{x}\rangle_0$. Thus the result of the first measurement—the inferred value of x—is the value $\overline{Q}$ obtained in the readout of Q. The expected result is $\langle\hat{Q}\rangle=\langle\hat{x}\rangle_0$, and the variance of the measurement is the variance of Q at $t=0$:

$$\Delta_1^2=(\Delta Q)^2=\sigma^2+(\Delta x)_0^2. \tag{7}$$

Here σ is the resolution of the meter, defined by $\sigma^2\equiv(\Delta Q)_0^2=\int dQ\, Q^2|\Phi(Q)|^2$. Notice that the variance (7) has the form assumed in Eq. (4)—a consequence of using an interaction Hamiltonian $K\hat{x}\hat{P}$ that is linear in $\hat{x}$.

The free-mass wave function $\psi(x|\overline{Q})$ just after the first measurement ($t=0$) is obtained (up to normalization) by evaluating $\Psi(x,Q)$ at $Q=\overline{Q}$:

$$\psi(x|\overline{Q})=\Psi(x,\overline{Q})/[P(\overline{Q})]^{1/2}=\psi(x)\Phi(\overline{Q}-x)/[P(\overline{Q})]^{1/2}, \tag{8}$$

$$P(Q)\equiv\int dx\,|\Psi(x,Q)|^2=\int dx\,|\psi(x)|^2|\Phi(Q-x)|^2. \tag{9}$$

Notice that $P(\overline{Q})$ is the probability distribution to obtain the value $\overline{Q}$ as the result of the first measurement. Expectation values and variances with respect to $\psi(x|\overline{Q})$ are distinguished by a subscript $\overline{Q}$.

During the time τ until the second measurement the free mass evolves unitarily. The second measurement is described and analyzed in exactly the same way as the first (assumption of identical measuring apparatuses). The expected result is the expectation value of x at time τ, which can be written as

$$\langle\hat{x}(\tau)\rangle_{\overline{Q}}=\int dx\,\psi^*(x|\overline{Q})[x+(\hbar\tau/im)(\partial/\partial x)]\psi(x|\overline{Q}), \tag{10}$$

$$\hat{x}(\tau)\equiv\hat{x}+\hat{p}\tau/m. \tag{11}$$

The result $\overline{Q}$ of the first measurement is known, and the meter wave function $\Phi(Q)$ is under one's control, but the free-mass wave function $\psi(x)$ before the first measurement is presumably not known. Nonetheless, I assume knowledge of $\psi(x)$ so that $\langle\hat{x}(\tau)\rangle_{\overline{Q}}$ can be calculated exactly. Then the unpredictability of the second measurement is characterized by its variance

$$\Delta_{2,\overline{Q}}^2=\sigma^2+[\Delta x(\tau)]_{\overline{Q}}^2, \tag{12}$$

$$[\Delta x(\tau)]_{\overline{Q}}^2=\int dx\,\psi^*(x|\overline{Q})[x+(\hbar\tau/im)(\partial/\partial x)-\langle\hat{x}(\tau)\rangle_{\overline{Q}}]^2\psi(x|\overline{Q}). \tag{13}$$

A simple case, corresponding to the argument leading to Eq. (4), occurs when one has almost no *a priori* knowledge about x before the first measurement—i.e., when $\psi(x)=|\psi(x)|e^{i\theta(x)}$ is such that $|\psi(x)|$ varies slowly on the scale set by σ. Then one finds that $\psi(x|\overline{Q})=\Phi(\overline{Q}-x)e^{i\theta(x)}$, which implies $\langle\hat{x}\rangle_{\overline{Q}}=\overline{Q}$ and $(\Delta x)_{\overline{Q}}^2=\sigma^2$;

the variance (12) of the second measurement becomes

$$\Delta^2_{2,\bar{Q}} = (\Delta x)^2_{\bar{Q}} + [\Delta x(\tau)]^2_{\bar{Q}} \geqslant |\langle[\hat{x},\hat{x}(\tau)]\rangle_{\bar{Q}}| = \hbar\tau/m.$$

An arbitrary $\psi(x)$ requires more care. It is possible to find $\psi(x)$ and $\Phi(Q)$ such that $\Delta^2_{2,\bar{Q}} < \hbar\tau/m$ for some values of $\bar{Q}$. Since one cannot control the outcome of the first measurement, a reasonable way to characterize the unpredictability of the second measurement is to average $\Delta^2_{2,\bar{Q}}$ over all values of $\bar{Q}$, weighting each value by its probability $P(\bar{Q})$[8]:

$$\Delta^2_2 = \int d\bar{Q}\, P(\bar{Q})\Delta^2_{2,\bar{Q}} = \sigma^2 + \langle[\hat{x}(\tau) - \langle\hat{x}(\tau)\rangle_{\hat{Q}}]^2\rangle, \tag{14}$$

$$\langle[\hat{x}(\tau) - \langle\hat{x}(\tau)\rangle_{\hat{Q}}]^2\rangle = \int dx\, d\bar{Q}\, \Psi^*(x,\bar{Q})[x + (\hbar\tau/im)(\partial/\partial x) - \langle\hat{x}(\tau)\rangle_{\bar{Q}}]^2\Psi(x,\bar{Q}). \tag{15}$$

The notation $\langle\hat{x}(\tau)\rangle_{\hat{Q}}$ emphasizes that $\langle\hat{x}(\tau)\rangle_{\hat{Q}}$ is a function of the operator $\hat{Q}$. The average variance of x just after the first measurement satisfies

$$\int d\bar{Q}\, P(\bar{Q})(\Delta x)^2_{\bar{Q}} = \langle(\hat{x} - \langle\hat{x}\rangle_{\hat{Q}})^2\rangle = \sigma^2 - \langle(\hat{Q} - \langle\hat{x}\rangle_{\hat{Q}})^2\rangle \leqslant \sigma^2, \tag{16}$$

$$\langle(\hat{x} - \langle\hat{x}\rangle_{\hat{Q}})^2\rangle = \int dx\, d\bar{Q}\,(x - \langle\hat{x}\rangle_{\bar{Q}})^2|\Psi(x,\bar{Q})|^2. \tag{17}$$

Equation (16) is the analog of the assumption $\sigma \geqslant \Delta x(0)$ in the heuristic argument leading to Eq. (4). By noting that

$$\langle\langle\hat{x}\rangle_{\hat{Q}}\rangle = \int d\bar{Q}\, P(\bar{Q})\langle\hat{x}\rangle_{\bar{Q}} = \langle\hat{x}\rangle, \quad \langle\langle\hat{x}(\tau)\rangle_{\hat{Q}}\rangle = \int d\bar{Q}\, P(\bar{Q})\langle\hat{x}(\tau)\rangle_{\bar{Q}} = \langle\hat{x}(\tau)\rangle,$$

one can write the inequality

$$\begin{aligned}\Delta^2_2 &\geqslant [\Delta(x - \langle\hat{x}\rangle_Q)]^2 + [\Delta(x(\tau) - \langle\hat{x}(\tau)\rangle_Q)]^2 \\ &\geqslant |\langle[\hat{x} - \langle\hat{x}\rangle_{\hat{Q}}, \hat{x}(\tau) - \langle\hat{x}(\tau)\rangle_{\hat{Q}}]\rangle| = |\langle[\hat{x},\hat{x}(\tau)]\rangle| = \hbar\tau/m.\end{aligned} \tag{18}$$

Thus the average variance obeys the SQL.

Both the heuristic argument and the model just considered require an essential assumption—that the measuring apparatus is coupled linearly to x. Within the context of a linear coupling, the model is quite general, since it allows the meter to be prepared in any state. Linear coupling does apply to the specific case Yuen describes in Refs. 3 and 4, which involves Gaussian free-mass contractive states that he calls "twisted coherent states"; any nonlinear coupling to x would destroy the Gaussian character of these states. To seek violations of the SQL, one should consider nonlinear couplings—e.g., $Kf(\hat{x})\hat{P}$. A word of caution: The interaction $Kf(\hat{x})\hat{P}$ describes directly measurements of the quantity $y = f(x)$; interpreting and analyzing such measurements as measurements of x is difficult.

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[8]When both $|\psi(x)|^2$ and $|\Phi(Q)|^2$ are Gaussian functions, Eq. (8) shows that $(\Delta x)^2_{\bar{Q}} = [(\Delta x)_0^{-2} + \sigma^{-2}]^{-1} \leqslant \sigma^2$; thus, in this case, the SQL follows directly from the argument leading to Eq. (4), with no need for any averaging.

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Measurement Breaking the Standard Quantum Limit for Free-Mass Position

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An explicit interaction-Hamiltonian realization of a measurement of the free-mass position with the following properties is given: (1) The probability distribution of the readouts is exactly the same as the free-mass position distribution just before the measurement. (2) The measurement leaves the free mass in a contractive state just after the measurement. It is shown that this measurement breaks the standard quantum limit for the free-mass position in the sense sharpened by the recent controversy.

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For monitoring the position of a free mass such as the gravitational-wave interferometer,[1] it is usually supposed[2,3] that the predictability of the results is limited by the so-called standard quantum limit (SQL). In the recent controversy,[4-8] started with Yuen's proposal[4] of a measurement which beats the SQL, the meaning of the SQL has been much clarified and yet no one has given a general proof nor a counterexample for the SQL. Recently, Ni[9] succeeded in constructing a repeated-measurement scheme to monitor the free-mass position to an arbitrary accuracy. However, it is open whether this scheme beats the SQL in the sense sharpened by the recent controversy. In particular, the following problem remains open: Can we realize a high-precision measurement which leaves the free mass in a contractive state?

In the present paper, I shall give a model of measurement of a free-mass position which breaks the SQL in its most serious formulation. An explicit form of the system-meter interaction Hamiltonian will be given and it will be shown that if the meter is prepared in an appropriate contractive state[4] then the measurement leaves the free mass in a contractive state and the uncertainty of the prediction for the next identical measurement decreases in a given duration to a desired extent. Thus Yuen's original proposal[4] is fully realized. This result will open a new way to an arbitrarily accurate non-quantum-nondemolition monitoring for gravitational wave detection and other related fields such as optical communications.

The precise formulation of the SQL is given by Caves[8] as follows: Let a free mass m undergo unitary evolution during the time τ between two measurements of its position x, made with identical measuring apparatus; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$ in average over all the first readout values. Caves[8] showed that the SQL holds for a specific model of a position measurement due to von Neumann[10] and he also gave the following heuristic argument for the validity of the SQL. His point is the notion of the imperfect resolution σ of one's measuring apparatus. His argument runs as follows: *The first assumption* is that the variance of the measurement of x is the sum of σ^2 and the variance of x at the time of the measurement; this is the case when the measuring apparatus is coupled linearly to x. *The second assumption* is that just after the first measurement, the free mass has position uncertainty $\Delta x(0) \leq \sigma$. Under these conditions, he derived the SQL from the uncertainty relation $\Delta x(0)\Delta x(\tau) \geq \hbar\tau/2m$.

However, his definition of the resolution of a measurement is ambiguous. In fact, he used three different definitions in his paper: (1) the uncertainty in the result, (2) the position uncertainty after the measurement, and (3) the uncertainty of the meter before the measurement. These three notions are essentially different, although they are the same for von Neumann's model. I

Does a Conservation Law Limit Position Measurements?

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The demonstrations of Wigner and others, that observables which do not commute with additive conserved quantities cannot be measured precisely, are reexamined. A proposed new formulation of the claim is shown to be valid for observables with a continuous spectrum whenever the conserved quantities are bounded. However, a countermodel is constructed, and it suggests that the position can be measured as precisely as the momentum even though the measuring interaction conserves the total linear momentum.

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Recently, there has been considerable interest in the analysis of fundamental quantum limits on measurements of unquantized quantities, such as positions of masses and amplitudes of harmonic oscillators, for applications, in particular, to optical communications [1] and gravitational wave detection [2]. A major achievement in this area is that we have breached the two types of quantum limits posed previously, the so-called standard quantum limit for amplitude measurements on harmonic oscillators [3] and the so-called standard quantum limit for monitoring of free-mass positions [4,5]. However, it has long been claimed by several authors [6–10] that *observables which do not commute with additive conserved quantities cannot be measured precisely.* This limit, which will be called the *conservation-law-induced quantum limit* (CQL) for measurements, implies that the conservation law of the linear momentum limits the accuracy of position measurements. Although implications of the CQL in measurements of the spin components have been examined in detail [9], those in position measurements have not been discussed seriously. An obvious difficulty for discussions about the limits on position measurements lies in the fact that the position observable has a continuous spectrum and that any observable with a continuous spectrum cannot be measured with absolute precision, whether it commutes with the additive conserved quantities or not. Thus, if one would claim the CQL for position measurements in a physically meaningful way, the claim would imply that the accuracy of position measurements has an apparent limitation compared with the accuracy of momentum measurements. In this Letter the validity of the CQL is examined from this point of view and it is shown that the CQL is *not* generally valid for position measurements.

We shall first give a rigorous statement of the CQL. Suppose that an observable (self-adjoint operator) A of a quantum system, called an *object*, represented by a Hilbert space $\mathcal{H}_1$, is actually measurable by a measuring instrument. Then we can describe the interaction between the object and the instrument by quantum mechanics in principle. Let $\mathcal{H}_2$ be the Hilbert space of the instrument system. The interaction is supposed to be turned on during a finite time interval from time $t=0$ to $t=\tau$ and represented by a unitary operator U on $\mathcal{H}_1\otimes\mathcal{H}_2$. Just after the interaction is turned off the object is separated from the instrument and the observer measures an observable B of the instrument to get the outcome of this measurement. In the Heisenberg picture, we can write $A(0)=A\otimes 1$, $B(0)=1\otimes B$, $A(\tau)=U^\dagger(A\otimes 1)U$, and $B(\tau)=U^\dagger(1\otimes B)U$. By saying that this measurement is an *exact* measurement of the observable A it is meant that the measurement satisfies (i) the statistical formula for the probability distribution of the outcome of a measurement (Ref. [11], pp. 200 and 201) and (ii) the repeatability hypothesis: *If an observable is measured twice in succession in the same individual system, then we get the same value each time* (Ref. [11], p. 335).

When does the measuring interaction U give an exact measurement of A? A simple but general condition sufficient for it is as follows: *There is some self-adjoint operator N, called the noise operator, in $\mathcal{H}_2$ for which U, A, and B satisfy the relations*

$$U^\dagger(1\otimes B)U=A\otimes 1+1\otimes N, \tag{1}$$

$$U^\dagger(A\otimes 1)U=A\otimes 1. \tag{2}$$

Let ψ be the initial state of the object and ξ the initial state of the instrument. Note that we can assume without any loss of generality that 0 is in the spectrum of the noise operator N; otherwise, replace B in Eq. (1) by $B-\lambda 1$ for any λ in the spectrum of N. In our formulation, a B measurement at time τ gives the outcome of an A measurement at time 0; i.e., a $B(\tau)$ measurement gives the outcome of an $A(0)$ measurement in the initial Heisenberg state $\psi\otimes\xi$. Then, if we prepare the instrument in the eigenstate of N for the eigenvalue 0, i.e., $N\xi=0$, it follows easily from Eq. (1) that the outcome of the $B(\tau)$ measurement has the same probability distribution as the $A(0)$ measurement. As to the repeatability hypothesis, the first measurement is the $A(0)$ measurement and the second measurement is an A measurement at the time just after the object system is separated from the first measuring instrument, so that the latter is just an $A(\tau)$ measurement. On the other hand, Eq. (2) assures that the $A(0)$ measurement and the $A(\tau)$ measurement give the same value. Thus a measuring interaction satisfying Eqs. (1) and (2) gives an exact measurement of an observable A. Indeed, the conditions are fulfilled in the conventional approach to measurements of discrete observables by a suitable relabeling of eigenvalues, where the unitary U is given by the relation $U(\varphi_m\otimes\xi_n)=\varphi_m$

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Quantum Limits in Interferometric Measurements.

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Abstract. – Quantum noise limits the sensitivity of interferometric measurements. It is generally admitted that it leads to an ultimate sensitivity, the «standard quantum limit». Using a semi-classical analysis of quantum noise, we show that a judicious use of squeezed states allows one in principle to push the sensitivity beyond this limit. This general method could be applied to large-scale interferometers for gravitational wave detection.

Quantum noise ultimately limits the sensitivity in interferometric detection of gravitational waves [1-3]. A gravitational wave is detected as a phase difference between the optical lengths of the two arms. It seems accepted that there exists a «standard quantum limit» (SQL), equivalent to an ultimate detectable length variation:

$$(\Delta z)_{\mathrm{SQL}} = \sqrt{\hbar\tau/M}\,, \tag{1}$$

where M is the mass of the mirrors and τ the measurement time [4]. The SQL can be derived by considering that the positions $z(t)$ and $z(t+\tau)$, which are noncommuting observables, are measured [5]. This interpretation of SQL has given rise to a long controversy [6].

Alternatively, the SQL can be understood by considering the quantum noise as a sum of two contributions. Photon counting noise corresponds to fluctuations of the number of photons detected in the two output ports, while radiation pressure noise stems from the random motion of the mirrors which is sensitive to the fluctuations of the numbers of photons in each arm. The sum of these two contributions leads to an optimal sensitivity given by expression (1). This limit is reached for very large laser power which is not presently achievable.

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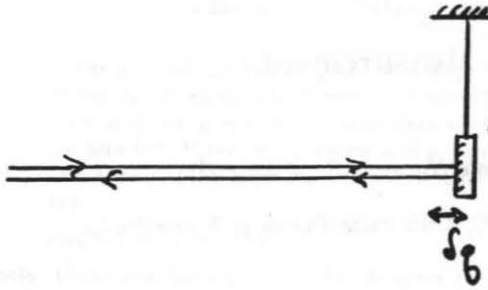
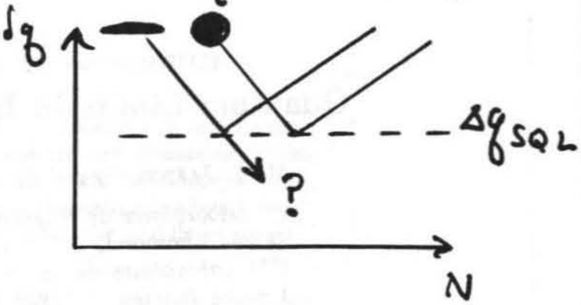
"Practical" Resolution $\left\{\begin{array}{l}\text{Unruh}\\ \text{Jaekel \& Reynaud}\end{array}\right.$





At Δq_{SQL} –

- Instead of  , try  !

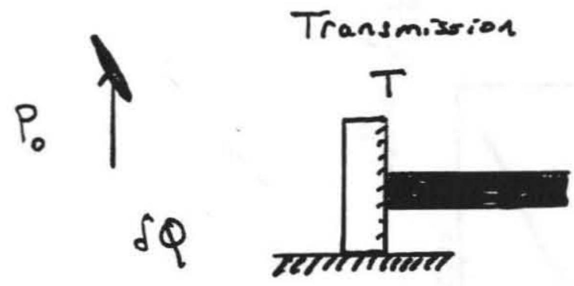
$$<\begin{pmatrix}\Delta\,\text{phase}\\ \Delta X_-\end{pmatrix}\cdot\begin{pmatrix}\Delta\,\text{amplitude}\\ \Delta X_+\end{pmatrix}> \;\rightarrow\; \text{Correlated}, < 0.$$

- Recall previous "derivation" –

$$<\left[\Delta q(0) + \frac{\Delta p(0)}{m}\tau\right]^2> \;\longrightarrow\; <\Delta q(0)\Delta p(0)> \neq 0 \quad < 0$$

- Prospects ?

(Overlay)



→ Use a resonant cavity

(1) $\delta\varphi \simeq \frac{4\pi}{\lambda}\delta q \cdot \frac{1}{T}$

$T = 1.6\times10^{-6}$

$R = 1 - T = 0.9999984$

(2) $P_o \rightarrow P_o T^2$

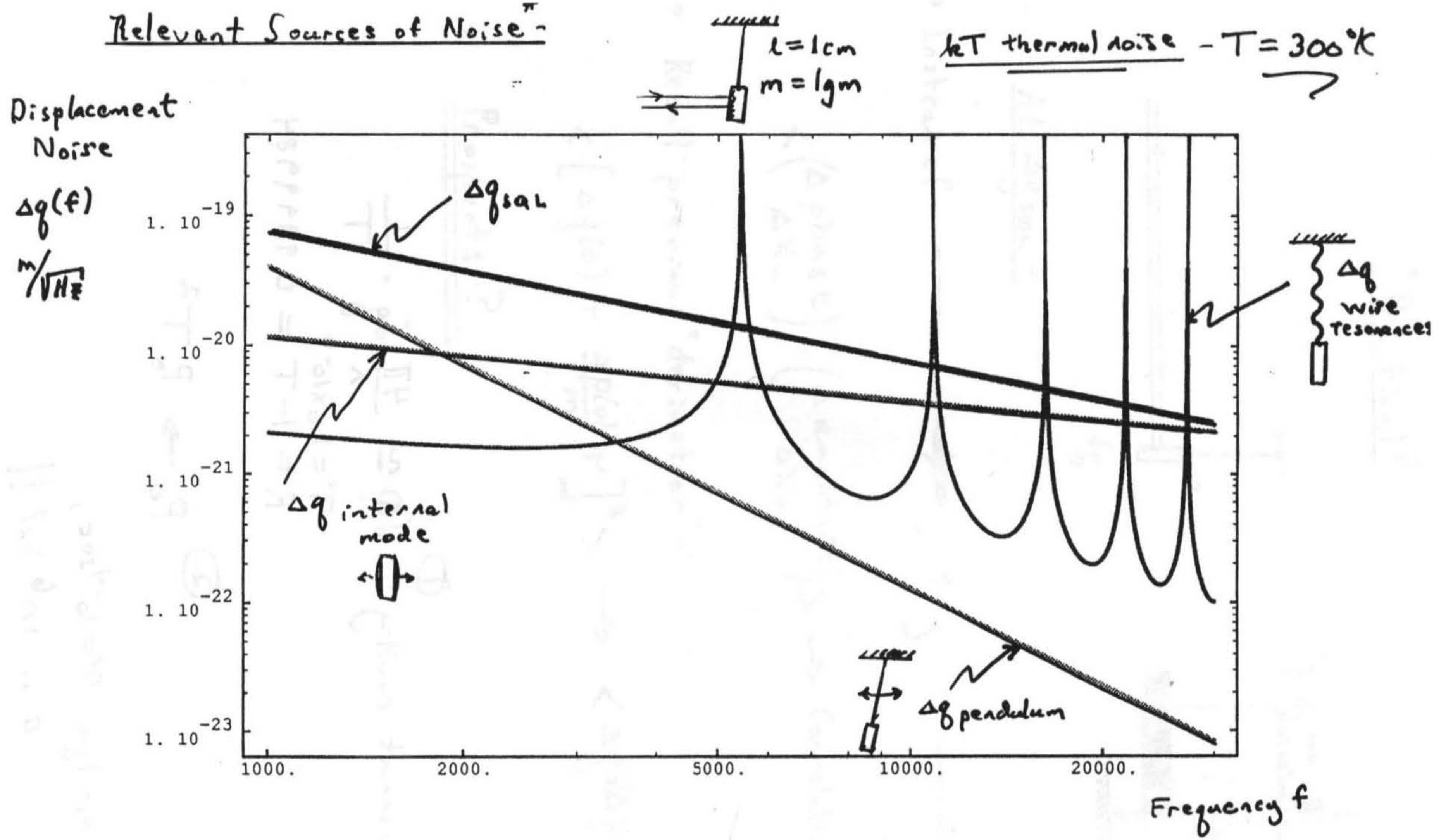
—•—

Note: For $m = 1\,gm$, $\tau = 10^{-4}\,sec$,

$P_o \sim 10^6\,W$!!

Relevant Sources of Noise*-

kT thermal noise - $T = 300°K$



$Q_{pendulum} = 10^6$ (Braginsky - 10^8)

$Q_{wire} = 10^6$

$Q_{internal} = 10^7$ (" $- 10^7$ at 1 MHz)

* P. Saulson, Phys. Rev. D42, 2437 (1990).