LECTURE 2: RANDOM PROCESSES

Lecture by Kip S. Thorne

Assigned Reading:

D. Pages 5-1 through 5-24 of "Chapter 5. Random Processes" from the textbook manuscript *Applications of Classical Physics* by Roger Blandford and Kip Thorne.

Suggested Supplementary Reading:

a. L. A. Wainstein and V. D. Zubakov, *Extraction of Signals from Noise* (Prentice Hall, London, 1962; Dover, New York, 1970). [This wonderful book—a sort of biblical primer on the subject—is long since out of print. Kip will put his personal xerox copy on reserve in Millikan Library for a few weeks, along with the library's only copy.]

Two Suggested Problems from Blandford and Thorne's "Chapter 5, Random Processes":

5.1 *Bandwidths of a finite-Fourier-transform filter and an averaging filter* [page 5-21]

5.2 *Wiener's Optimal Filter* [page 5-22]. This is an especially important exercise, since the optimal filter underlies much of the data analysis to be done in LIGO.

Lecture 2
Random Processes

by Kip S. Thorne, 1 April 1994

This lecture actually consumed only half of the 90 minutes on 1 April; the completion of Lecture 1 consumed the other half.

This lecture was largely just a blackboard presentation of the key issues in Reference D [pages 5-1 through 5-24 of "Chapter 5. Random Processes" from the textbook manuscript *Applications of Classical Physics* by Roger Blandford and Kip Thorne]. Since that reference is included in Volume II, we here present, as a record of Lecture 2, only the scrawled notes from which Kip lectured.

1. Example of RP ~~Rat~~ $I_{PD}(t)$; $h(t) = C \cdot I_{PD}(t)$; $x(t)$ test mass
~~$\Delta h(t) = \ldots$~~ [Mean Removed]

2. ~~Sp~~ Noise spectrum — $\boxed{\text{VG show}}$ — what means?
~~Show~~ ↑ $h(f) \equiv \sqrt{S_h(f)}$

Like FT: $$\tilde{h}'(f) = \int_{-\infty}^{+\infty} h(t)\, e^{i 2\pi f t}\, dt$$

... no ... diverges; complex

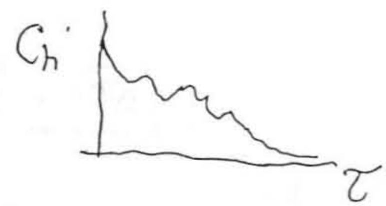
Try $$\lim_{T\to\infty} \frac{1}{T}\left|\int_{-T/2}^{+T/2} h(t)\, e^{i2\pi f t}\, dt\right|^2$$ Same f & −f
→ fold over

$$\to \quad G_h(f) = S_h(f) \equiv [\tilde{h}(f)]^2 = \lim_{T\to\infty} \frac{2}{T}\left|\int_{-T/2}^{T/2} h(t)\, e^{i2\pi f t}\, dt\right|^2$$

[For EM wave $G_E(f) = \frac{4\pi}{c}\frac{dE}{dA\,dt\,df}$] Units: strain/$\sqrt{Hz}$

3. Correlation function
$$C_h(\tau) \equiv \mathcal{E}\ \overline{h(t)\,h(t+\tau)} = \lim_{T\to\infty}\frac{1}{T}\int_{T/2}^{T/2} h(t)\,h(t+\tau)\, dt$$

4. Wiener-Khintchine Thm



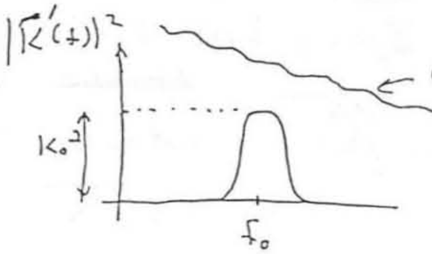
$$C_h(\tau) = \int_0^\infty G_h(f)\cos 2\pi f\tau\, df$$

$$G_h(f) = 4\int_0^\infty C_h(\tau)\cos 2\pi f\tau\, df$$

5. Variance:
$$C_h(0) = \int_0^\infty G_h(f)\, df = \overline{[h(t)]^2} = \sigma_h^2$$
— but might not converge @ low f.

6. Filtering
a. $$H(t) \equiv \int_{-\infty}^{+\infty} K(t-t')\, h(t')\, dt'$$
b. If were ~~not~~ confined to finite time: $\tilde{H}'(f) = \tilde{K}'(f)\, \tilde{h}'(f)$
c. For RP $$G_H(f) = \underbrace{|\tilde{K}'(f)|^2}_{\text{converges}}\, G_h(f)$$

7. Band Pass Filter



$$\sigma_H^2(f) = \int |\tilde{K}'(f)|^2 \underbrace{G_h(f)}_{\text{const}}\, df \simeq G_h(f_0) \underbrace{\int |\tilde{K}'(f)|^2\, df}_{K_0^2\, \Delta f}$$

$$\sigma_H^2(f) = \underbrace{\left[G_h(f_0)\cdot \Delta f\right]}_{\text{rms fluctuations of } h \text{ in bandwidth } f_0} \cdot K_0^2$$

a. Simple example of Bandpass filter:

$$H(t) = \int_{t-\Delta t}^{t} \cos\left[2\pi f_0 (t-t')\right] y(t')\, dt'$$

$$\boxed{\Delta f = 1/\Delta t}$$

8. Example: A line spike in spectrum ... large rms fluctuations near that f.

9. Wiener Optimal Filter:

a. $h(t) = A s(t) + n(t)$ .... want to find out if $s(t)$ there and if so how strong, A
(A unknown)

Best way should be to cross correlate: $\int h(t)\cdot s(t)\, dt$

– Better: Suppress frequencies where detector is noisy .... $\tilde{S}_F^*(f) = \dfrac{\tilde{S}'(f)}{G_h(f)}$ -- then cross correlate:

$$W = \int h(t)\, S_F(t)\, dt = \int \frac{\tilde{h}'(f)\, \tilde{S}'^*(f)\, df}{G_h(f)}$$

b. Signal contribution to W is S; noise N is random @ some mean $\bar{N}$: $W = S + N$.

$$\boxed{\frac{S}{\bar{N}} = 4 \int_0^\infty \frac{|\tilde{S}(f)|^2}{G_h(f)}\, df}$$

11. Knowing spectrum does ==not== tell us $P(N)$, the probability distn of N. But ==if== we know it is Gaussian, then

$$\cancel{P(N) = \frac{1}{\sqrt{2\pi \bar{N}}} \exp\left[\frac{(N-\bar{N})^2}{2\bar{N}} \right]}$$

$$\boxed{P(N) = \frac{1}{\sqrt{2\pi \bar{N}^2}} \exp\left[-\frac{N^2}{2\bar{N}^2} \right]}$$

~~Key issue in LIGO is~~

12. If instrument clean, ~~then~~ enough [not likely], noise is superposition of tiny influences of many different things. Central limit. thm ⇒ Gaussian

13. Central issue will be: Gaussian or not?