

# Open Strings on $AdS_2$ Branes

Peter Lee, Hiroshi Ooguri, Jongwon Park, and Jonathan Tannenhauser

California Institute of Technology 452-48, Pasadena, CA 91125

peter, ooguri, jongwon, jetannen@theory.caltech.edu

## Abstract

We study the spectrum of open strings on  $AdS_2$  branes in  $AdS_3$  in an NS-NS background, using the  $SL(2, R)$  WZW model. When the brane carries no fundamental string charge, the open string spectrum is the holomorphic square root of the spectrum of closed strings in  $AdS_3$ . It contains short and long strings, and is invariant under spectral flow. When the brane carries fundamental string charge, the open string spectrum again contains short and long strings in all winding sectors. However, branes with fundamental string charge break half the spectral flow symmetry. This has different implications for short and long strings. As the fundamental string charge increases, the brane approaches the boundary of  $AdS_3$ . In this limit, the induced electric field on the worldvolume reaches its critical value, producing noncommutative open string theory on  $AdS_2$ .

# 1 Introduction

The  $SL(2, R)$  WZW model is ubiquitous in string theory. It arises in contexts ranging from the Liouville model in two-dimensional gravity [1, 2, 3, 4] to three-dimensional Einstein gravity [5] to two-dimensional black holes [6] to Neveu-Schwarz 5-branes [7] and their relation to singularities in Calabi-Yau spaces [8, 9, 10]. In addition, the  $SL(2, R)$  WZW model describes the worldsheet of a string propagating in  $AdS_3$  with a background NS-NS  $B$ -field. The application to string theory in  $AdS_3$  is of particular interest, as it opens a window onto the AdS/CFT correspondence beyond the gravity approximation. For all of these reasons, the  $SL(2, R)$  WZW model has been intensively studied for more than a decade.<sup>1</sup>

Last year, a proposal was put forward [12] and checked [11] for the structure of the Hilbert space of the model. The symmetries of the theory require that its Hilbert space decompose as a sum of irreducible representations of the current algebra  $\widehat{SL}(2, R) \times \widehat{SL}(2, R)$ . But which representations appear, and with what multiplicities? According to the proposal of [12], the Hilbert space contains discrete representations and continuous representations, as well as their images under spectral flow. It was argued in [12] that the discrete representations and their spectral flow images correspond to short strings in  $AdS_3$ , and the continuous representations and their images to long strings. In both cases, the integer  $w$  indexing the spectral flow was interpreted as the winding number of the strings about the center of  $AdS_3$ .

The analysis of [12] determined the spectrum of *closed* strings in  $AdS_3$ . We address the corresponding problem for *open* strings. More specifically, our setting is critical open bosonic string theory in  $AdS_3 \times \mathcal{M}$ , with an NS-NS background, and in the presence of a D-brane whose worldvolume fills an  $AdS_2$  subspace of  $AdS_3$  and wraps some subspace of the compact space  $\mathcal{M}$ . This  $AdS_2$  brane preserves one linear combination of the left- and right-moving current algebras. Consequently, the Hilbert space of open strings ending on the  $AdS_2$  brane decomposes as a sum of irreducible representations of a single  $\widehat{SL}(2, R)$ . Our main task will be to determine which representations appear in the spectrum.

Some intuition may be gained from the  $SU(2)$  counterpart of our problem [13, 14, 15, 16, 17, 18, 19, 20, 21]. In the  $SU(2)$  case, the D-brane worldvolumes analogous to our  $AdS_2$  branes are  $S^2$  subspaces embedded in the  $SU(2)$  group manifold  $S^3$ . These  $S^2$  branes are quantized: if the level of the WZW model is  $k$ , there are  $k+1$  possible D-brane configurations, labeled by a quantum number  $n$  taking the values  $n = -k/2, -k/2 + 1, \dots, k/2$ . If  $k$  is even, the D-brane with quantum number  $n = 0$  wraps the equatorial  $S^2$  within  $S^3$ . In general, the D-branes associated with increasing  $|n|$  wrap smaller and smaller 2-spheres. By the time  $n$  reaches  $\pm k/2$ , the  $S^2$  worldvolumes have degenerated to the north or south pole of  $S^3$ .

The Hilbert space  $\mathcal{H}_n$  of open strings ending on the  $S^2$  brane with quantum number  $n$

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<sup>1</sup>A representative sample of references is given in [11].

decomposes as

$$\mathcal{H}_n = \bigoplus_{j=0}^{k/2-|n|} D_j, \quad (1.1)$$

where  $D_j$  is the irreducible representation of the current algebra  $\widehat{SU}(2)$  whose ground states make up the spin- $j$  representation of  $SU(2)$  [16]. It is evident from (1.1) that the Hilbert space “loses” representations as  $|n|$  increases. For example, the Hilbert space of open strings ending on an equatorial  $S^2$  brane is  $\mathcal{H}_{n=0} = \bigoplus_{j=0}^{k/2} D_j$ , which is the holomorphic square root of the Hilbert space of closed strings in  $SU(2)$ , projected onto integral  $j$ . On the other hand, when  $n = \pm k/2$  and the D-brane has shrunk to the north pole or the south pole, the Hilbert space is reduced to the single representation  $\mathcal{H}_{n=\pm k/2} = D_{j=0}$ .

We seek a similarly detailed picture of the Hilbert space of  $AdS_2$  branes in  $AdS_3$ . Our method resembles that of [12]: we start by constructing *classical* open string worldsheet solutions, based on which we then conjecture the form of the *quantum* Hilbert space. To test the validity of this approach, and to introduce the tools we will need for  $SL(2, R)$  in what may be a more familiar context, we begin in section 2 with a semiclassical treatment of  $S^2$  branes in the  $SU(2)$  WZW model. The Hilbert space structure that emerges from our semiclassical methods is identical to that of (1.1), though we cannot see the quantization of the parameter  $n$ .<sup>2</sup> Our approach explains naturally in terms of the geometry of  $S^3$  why  $\widehat{SU}(2)$  representations are skimmed off the Hilbert space as  $|n|$  increases.

We then return to the  $SL(2, R)$  WZW model. Following a review in section 3 of the closed string Hilbert space, we take up the subject of  $AdS_2$  branes in sections 4 and 5. Like the  $S^2$  branes in  $SU(2)$ , the  $AdS_2$  branes in  $SL(2, R)$  are quantized. The quantum number they carry is essentially fundamental string charge. In a suitable coordinate system, each  $AdS_2$  brane is located at some fixed value  $\psi_0$  of one of the coordinates. The quantization condition is

$$\sinh \psi_0 = g_s Q, \quad (1.2)$$

where  $g_s$  is the string coupling constant and  $Q$  is the fundamental string charge carried by the brane. The condition (1.2) restricts  $\psi_0$  to a discrete (but now, neither finite nor bounded!) set of allowed values. As with the  $S^2$  branes, though, our analysis is insensitive to this quantization.

The simplest case,  $\psi_0 = 0$ , is treated in section 4. This case is the  $SL(2, R)$  analogue of the equatorial  $S^2$  branes in the  $SU(2)$  WZW model. An  $AdS_2$  brane with  $\psi_0 = 0$  is a “straight” brane cutting through the middle of  $AdS_3$ . The Dirichlet boundary condition defining such a straight brane preserves the full spectral flow symmetry of the closed string theory. Semiclassical analysis suggests that the open string Hilbert space is the holomorphic square root of the Hilbert space of closed strings in  $AdS_3$ . A one-loop Euclidean partition function calculation, described in Appendix B, confirms this conjecture.

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<sup>2</sup>A semiclassical argument for the quantization of  $n$  was given in [19].

Section 5 is devoted to branes with  $\psi_0 \neq 0$ . Exact quantitative results are unavailable here; nevertheless, we are able to arrive at a qualitative picture of the Hilbert space.

An  $AdS_2$  brane with  $\psi_0 \neq 0$  is analogous to an  $S^2$  brane with  $n \neq 0$  in the  $SU(2)$  WZW model. Varying  $\psi_0$  away from zero curves the brane towards the boundary of  $AdS_3$ . Unlike the situation for  $S^2$  branes in  $S^3$ , there is no loss of representations as  $|\psi_0|$  increases. This difference is traceable to a simple difference in the geometry of the two setups.

Introducing an  $AdS_2$  brane with  $\psi_0 \neq 0$  breaks half the spectral flow symmetry: the curved-brane Dirichlet boundary condition is preserved only if the integer  $w$  parametrizing the spectral flow is even. Nevertheless, we can construct classical short and long string solutions—and the Hilbert space contains discrete and continuous representations—of both odd and even  $w$ . There is an important difference, though, between the short and long string solutions, having to do with the action on  $AdS_3$  of PT, the spacetime parity and time-reversal symmetry. When acting on discrete representations, PT flips the parity of  $w$ . Thus it is possible to reach a discrete representation of any given value of  $w$  from a discrete representation of any other given  $w$  by a sequence of symmetry transformations: even  $w$  spectral flow and, if necessary, target space PT. Consequently, the  $\psi_0$  dependence of the density of states of the discrete representations is the same for all  $w$ . By contrast, PT maintains the parity of  $w$  when acting on continuous representations. Thus the  $\psi_0$  dependence of the density of states of the continuous representations is different for odd and even  $w$ . We highlight this difference by examining the contributions of the odd and even  $w$  sectors to the divergence structure of the one-loop Euclidean partition function.

In the limit  $\psi_0 \rightarrow \pm\infty$ , the  $AdS_2$  brane approaches the boundary of  $AdS_3$ , and the induced electric field on the brane worldvolume approaches its critical value. We therefore conjecture that the  $\psi_0 \rightarrow \infty$  limit reproduces noncommutative open string (NCOS) theory on  $AdS_2$ . At the end of section 5, we show that, in this limit, the WZW Lagrangian takes a form similar to the Lagrangian of noncommutative open strings [22, 23], appropriately modified to account for the  $AdS_2$  background. We also take some preliminary steps towards a computation of the one-loop partition function.

Section 6 contains a summary and some conclusions.

Towards the completion of this work, we received the preprint [24], which contains some overlap with certain of our results. In addition, branes in  $AdS_3$  were recently studied from a different point of view in [25].

## 2 $S^2$ Branes in the $SU(2)$ WZW Model

In this section, we study  $S^2$  branes in the  $SU(2)$  WZW model from a semiclassical point of view. Our reasons for doing so are twofold. First, the  $SU(2)$  WZW model is in several important ways similar to—and different from—the  $SL(2, R)$  WZW model which is our main focus, and it is useful to develop the ideas we will need later on in a more familiar setting.

Second, the  $SU(2)$  WZW model provides a testing ground for the semiclassical techniques that will eventually help us maneuver through the intricacies of the  $SL(2, R)$  WZW model. By examining *classical* open strings ending on  $S^2$  branes in  $S^3$ , we will be led to the picture of [16] for the structure of the *quantum* Hilbert space.

Before turning to the analysis of  $S^2$  branes in  $S^3$ , let us begin with some remarks on branes in WZW models in general [26, 18]. In free bosonic open string theory with target space coordinates  $X^a$ , the standard boundary conditions at the string endpoints may be expressed as

$$\partial_+ X^a = \pm \partial_- X^a, \quad (2.1)$$

with the plus sign indicating a Neumann condition and the minus sign a Dirichlet condition. In the free theory, the  $\partial_+ X^a$  ( $\partial_- X^a$ ) are (anti-)holomorphically conserved currents. The simplest extension to strings propagating on group manifolds replaces  $\partial_+ X^a$  ( $\partial_- X^a$ ) by the (anti-)holomorphically conserved currents  $J_R^a$  ( $J_L^a$ ). In addition, we could generalize (2.1) to the condition

$$\partial_+ X^a = R_b^a \partial_- X^b, \quad (2.2)$$

where  $R_b^a$  is a constant matrix; which directions are Neumann and which Dirichlet are then determined by the eigenvalues of  $R_b^a$ . The corresponding operation in the WZW model involves “twisting” the condition relating the left- and right-moving currents by an automorphism  $R$  of the Lie algebra [26, 18],

$$J_R^a + R_b^a J_L^b = 0. \quad (2.3)$$

The statement that  $R$  is a Lie algebra automorphism means that, for all group generators  $T^a$  and  $T^b$ ,  $[R(T^a), R(T^b)] = R([T^a, T^b])$ . The automorphism  $R$  is further required to preserve conformal invariance at the worldsheet boundary. In (2.1) we had a choice of sign; choosing the plus sign in (2.3) ensures that the boundary conditions preserve the affine symmetry algebra.<sup>3</sup>

The condition (2.3) is called a *gluing condition*, and implies the boundary conditions satisfied by the target space coordinates. The gluing condition (2.3) defines branes, whose worldvolumes within the group manifold  $G$  extend in the directions for which the boundary conditions derived from (2.3) are Neumann. The brane worldvolume containing a fixed element  $g \in G$  can be represented as the “twined” conjugacy class

$$\mathcal{W}_g^r = \{r(h)gh^{-1} : h \in G\}, \quad (2.4)$$

where  $r$  is the group automorphism induced near the identity from the Lie algebra automorphism  $R$ .<sup>4</sup>

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<sup>3</sup>We will not consider the more general brane configurations that can be obtained by relaxing this requirement.

<sup>4</sup>For  $X$  in the Lie algebra and  $t$  sufficiently small,  $r(e^{tX}) = e^{tR(X)}$ .

Now let us make this concrete for the case of branes in  $S^3$ , the group manifold of  $SU(2)$ . We write the general  $SU(2)$  group element as

$$g = \exp\left(i\vec{\psi} \cdot \vec{\sigma}\right), \quad (2.5)$$

where

$$\vec{\psi} = \left(\psi - \frac{\pi}{2}\right) (\sin \omega \cos \phi, \sin \omega \sin \phi, \cos \omega), \quad (2.6)$$

the coordinates  $(\psi, \omega, \phi)$  lie in the ranges

$$-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}, \quad 0 \leq \omega \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad (2.7)$$

and  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  is the vector of Pauli matrices. Explicitly, in this parametrization,

$$g = \begin{pmatrix} \sin \psi - i \cos \omega \cos \psi & -ie^{-i\phi} \cos \psi \sin \omega \\ -ie^{i\phi} \cos \psi \sin \omega & \sin \psi + i \cos \omega \cos \psi \end{pmatrix}. \quad (2.8)$$

The  $S^3$  metric in these coordinates is

$$ds^2 = d\psi^2 + \cos^2 \psi (d\omega^2 + \sin^2 \omega d\phi^2), \quad (2.9)$$

from which it is apparent that the surfaces of constant  $\psi$  are 2-spheres embedded in  $S^3$ .

Classical string worldsheets in  $S^3$  are given by solutions of the  $SU(2)$  WZW model. We take the worldsheet coordinates to be  $\tau$  and  $\sigma$ . For closed strings,  $\sigma$  is periodic with period  $2\pi$ ; for open strings,  $0 \leq \sigma \leq \pi$ . It is also convenient to define the light-cone combinations

$$x^\pm = \tau \pm \sigma. \quad (2.10)$$

The WZW theory possesses three right- and left-moving currents, which may be grouped into the matrices

$$J_R = k \partial_+ g g^{-1}, \quad J_L = k g^{-1} \partial_- g; \quad (2.11)$$

here  $k$  is the level of the WZW model. The WZW equations of motion state that these currents are conserved, *i.e.*,  $\partial_- J_R = \partial_+ J_L = 0$ .

Taking the Lie algebra automorphism in (2.3) to be trivial leads to the gluing condition

$$J_L = -J_R. \quad (2.12)$$

The worldvolumes of the associated branes are ordinary conjugacy classes of  $S^3$ : they are the 2-spheres given by

$$\text{Tr } g = 2 \sin \psi_0, \quad (2.13)$$

for some constant  $\psi_0 \in [-\pi/2, \pi/2]$ . The worldvolume of the  $S^2$  brane with  $\psi_0 = 0$  spans the equatorial 2-sphere of  $S^3$ ; as  $|\psi_0| \rightarrow \pi/2$ , the  $S^2$  branes degenerate to single points at the north or south pole.

It is useful to introduce a second parametrization of the  $SU(2)$  group element,

$$g = \begin{pmatrix} \sin r \sin \theta + i \cos r \sin t & -\cos r \cos t - i \cos \theta \sin r \\ \cos r \cos t - i \cos \theta \sin r & \sin r \sin \theta - i \cos r \sin t \end{pmatrix}, \quad (2.14)$$

where the coordinates  $(r, \theta, t)$  satisfy

$$0 \leq r \leq \frac{\pi}{2}, \quad 0 \leq \theta, t \leq 2\pi. \quad (2.15)$$

The  $S^3$  metric in these coordinates takes the form

$$ds^2 = \cos^2 r dt^2 + dr^2 + \sin^2 r d\theta^2. \quad (2.16)$$

One simple class of open string configurations satisfying the WZW equations of motion is given by

$$t = a\tau, \quad \theta = a\sigma + \theta_0, \quad r = r_0, \quad (2.17)$$

where  $0 \leq a \leq 1$  and  $0 \leq \theta_0 \leq \pi/2$ .<sup>5</sup> The solutions satisfy the Dirichlet condition defining the  $S^2$  brane, provided

$$\sin \theta_0 = \frac{\sin \psi_0}{\sin r_0} \quad \text{and} \quad a = 1 - \frac{2\theta_0}{\pi}. \quad (2.18)$$

It follows that

$$a \leq 1 - \frac{2|\psi_0|}{\pi}. \quad (2.19)$$

That is,  $a$  has an upper bound that decreases with increasing  $|\psi_0|$ . The geometry of the  $S^2$  branes and their attached open strings is shown in Figure 1(a).

Let us compare the result of our classical analysis with the known structure of the quantum Hilbert space of open strings ending on  $S^2$  branes. The Hilbert space is a sum of representations of  $\widehat{SU}(2)$ ; the spin  $j$  of a representation appearing in the sum is related to the parameter  $a$  of the associated classical solution by  $j = ka/2$ . Although the analysis we have just presented is not refined enough to see it,  $\psi_0$  is quantized [19] as  $\psi_0 = \frac{\pi n}{k}$ , where  $n = -k/2, -k/2 + 1, \dots, k/2$ . Thus, for given  $\psi_0$ , the bound (2.19) on  $a$  translates into a bound

$$j \leq \frac{k}{2} - |n| \quad (2.20)$$

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<sup>5</sup>This range for  $\theta_0$  assumes that  $\psi_0 > 0$ . If  $\psi_0 < 0$ , the proper range for  $\theta_0$  is  $\pi \leq \theta_0 \leq 3\pi/2$ . The bound  $0 \leq a \leq 1$  will be explained in greater detail in the analogous  $SL(2, R)$  context in section 5.1; essentially, any solution with arbitrary  $a$  can be mapped by spectral flow to a solution satisfying the bound. One important difference between the  $SL(2, R)$  and  $SU(2)$  WZW models, though, is that, in the quantum theory of the  $SL(2, R)$  WZW model, spectral flow generates new representations of the current algebra, whereas in the  $SU(2)$  WZW model, it does not.

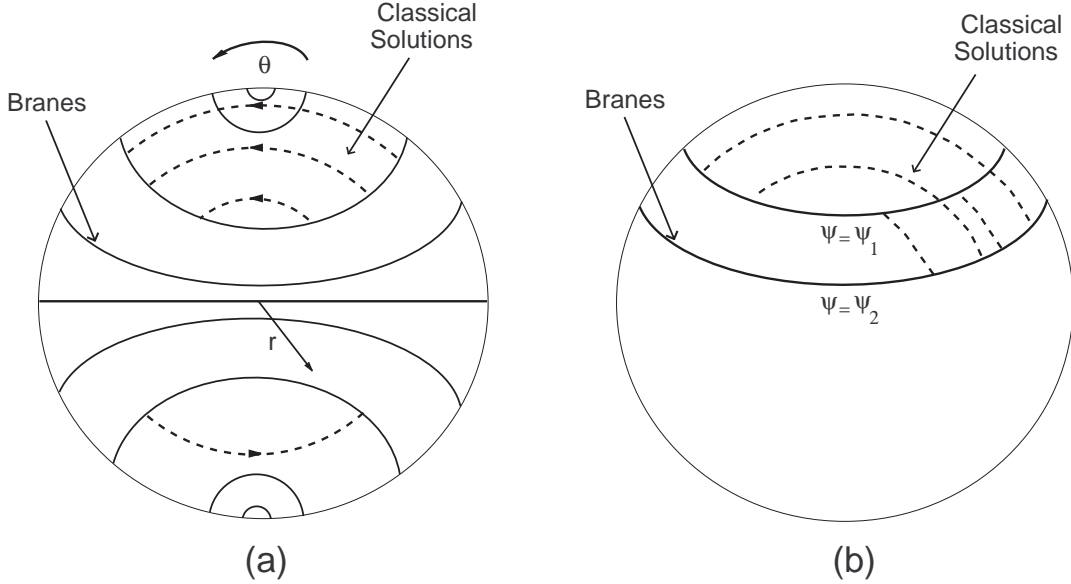


Figure 1: (a) A view at fixed  $t$  of  $S^2$  branes and open strings ending on them. (b) A view at fixed  $t$  of a system of two  $S^2$  branes with open strings.

on the spins of the allowed representations in the Hilbert space. This bound matches the conformal field theory analysis of [16], in which the Hilbert space  $\mathcal{H}_n$  of open strings ending on the brane labeled by  $n$  was shown to be

$$\mathcal{H}_n = \bigoplus_{j=0}^{k/2} N_{\alpha\alpha}^j D_j, \quad (2.21)$$

where  $\alpha = \frac{1}{2}(n + \frac{k}{2})$ ,  $D_j$  is the irreducible spin- $j$  highest weight representation of  $\widehat{SU}(2)$ , and the fusion coefficients are given by

$$N_{\alpha\beta}^j = \begin{cases} 1 & \text{if } |\alpha - \beta| \leq j \leq \min\{\alpha + \beta, k - \alpha - \beta\} \\ & \text{and } 2(j + \alpha + \beta) \equiv 0 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.22)$$

*A priori*, the sum over  $j$  is to be taken in half-integer steps (*i.e.*,  $j = 0, 1/2, \dots, k/2$ ). However, the fusion coefficient  $N_{\alpha\alpha}^j$  is nonzero only if  $j$  is an integer. The equatorial  $S^2$  brane has  $\alpha = k/4$ ; the branes at the poles have  $\alpha = 0$  and  $\alpha = k/2$ . For all  $\alpha$ , the sum cuts off at  $\min(2\alpha, k - 2\alpha)$ , which is readily seen to be equivalent to the cutoff (2.20). Thus (2.21) is identical to (1.1). Our classical methods have reproduced information about the quantum Hilbert space.



The restriction to integral  $j$  in the sum (2.21) (or (1.1)) can be simply understood, at least for equatorial  $S^2$  branes. In addition to the stringy solutions (2.17), the equatorial  $S^2$  brane also admits the particle-like geodesic solutions

$$t = a\tau, \quad r = 0. \quad (2.23)$$

Let us consider the  $k \rightarrow \infty$  limit of the theory. In this limit, if we expand around geodesic solutions like (2.23), the WZW model reduces to quantum mechanics on  $S^2$ . Its Hilbert space is therefore the space  $\mathcal{L}^2(S^2)$  of square-integrable functions on  $S^2$ . This space decomposes into spherical harmonics, which correspond to representations of integral spin only.

We conclude this section with a slight generalization. Let us consider a system of two  $S^2$  branes, located at  $\psi = \psi_1 = \pi n_1/k$  and  $\psi = \psi_2 = \pi n_2/k$ ; without loss of generality, we may assume that  $\psi_1 > 0$  and  $|\psi_1| > |\psi_2|$ , as shown in Figure 1(b). The conformal field theory analysis that led to (2.21) and the expression (2.22) for the fusion coefficients now tells us that the Hilbert space of strings stretching from the brane at  $\psi_1$  to the brane at  $\psi_2$  has the decomposition

$$\mathcal{H}_{n_1, n_2} = \bigoplus_{j=\frac{1}{2}|n_1-n_2|}^{\frac{1}{2}(k-n_1-n_2)} D_j. \quad (2.24)$$

We wish to reproduce the bound on  $j$  by semiclassical methods. We consider classical solutions of the form (2.17), subject to the Dirichlet boundary conditions

$$\sin \theta_0 \sin r_0 = \sin \psi_1, \quad (2.25)$$

$$\sin(a\pi + \theta_0) \sin r_0 = \sin \psi_2. \quad (2.26)$$

An argument like the one in the single-brane example shows that now  $a$  is bounded both above and below,

$$\frac{|\psi_1 - \psi_2|}{\pi} \leq a \leq 1 - \frac{\psi_1 + \psi_2}{\pi}. \quad (2.27)$$

The solutions saturating the inequalities are those with  $r_0 = \pi/2$ . Since  $j = ka/2$ , the bound on  $a$  translates to

$$\frac{1}{2}|n_1 - n_2| \leq j \leq \frac{1}{2}(k - n_1 - n_2), \quad (2.28)$$

which matches the conformal field theory result.

### 3 Closed Strings in $AdS_3$

Our search in sections 4 and 5 for solutions of the  $SL(2, R)$  WZW model corresponding to open strings ending on branes will be guided by the known properties of closed string solutions. In this section, we summarize the analysis of [12] of closed bosonic string theory in  $AdS_3$ . Our review has two parts. First, we survey the classical closed string solutions of

the  $SL(2, R)$  WZW model, beginning with simple geodesic solutions and building up more complicated solutions by acting with global isometries and spectral flow. Next, we sketch the structure of the quantum Hilbert space.

### 3.1 Classical Solutions

The space  $AdS_3$  is the group manifold of  $SL(2, R)$ . If we think of  $AdS_3$  (with unit anti-de Sitter radius) as the hyperboloid

$$(X^0)^2 - (X^1)^2 - (X^2)^2 + (X^3)^2 = 1 \quad (3.1)$$

embedded in  $\mathbf{R}^{2,2}$ , then a point in  $AdS_3$  is given by the  $SL(2, R)$  matrix

$$g = \begin{pmatrix} X^0 + X^1 & X^2 + X^3 \\ X^2 - X^3 & X^0 - X^1 \end{pmatrix}. \quad (3.2)$$

As usual, to avoid closed timelike curves, we actually work with the universal cover of  $AdS_3$ . We may alternatively parametrize  $g$  in terms of the global coordinates  $(\rho, \theta, t)$  defined in Appendix A,

$$g = \begin{pmatrix} \cos t \cosh \rho - \cos \theta \sinh \rho & \sin t \cosh \rho + \sin \theta \sinh \rho \\ -\sin t \cosh \rho + \sin \theta \sinh \rho & \cos t \cosh \rho + \cos \theta \sinh \rho \end{pmatrix}. \quad (3.3)$$

The  $AdS_3$  metric in these coordinates is

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2. \quad (3.4)$$

The parametrization (3.3) is the  $SL(2, R)$  counterpart of the  $SU(2)$  parametrization (2.14).

Like the  $SU(2)$  theory, the  $SL(2, R)$  theory possesses three conserved right- and left-moving currents,

$$J_R^a(x^+) = k \text{Tr} \left( T^a \partial_+ g g^{-1} \right), \quad J_L^a(x^-) = k \text{Tr} \left( T^{a*} g^{-1} \partial_- g \right) \quad (a = +, -, 3). \quad (3.5)$$

Here  $k$  is again the level of the WZW model, and the  $T^a$ , given in terms of the Pauli matrices by

$$T^3 = -\frac{i}{2} \sigma_2, \quad T^\pm = \frac{1}{2} (\sigma_3 \pm i \sigma_1), \quad (3.6)$$

form a basis of the Lie algebra of  $SL(2, R)$ . Sometimes we write the currents in the matrix form

$$J_R = k \partial_+ g g^{-1}, \quad J_L = k g^{-1} \partial_- g. \quad (3.7)$$

The general solution  $g$  of the WZW model can be factored as a product of left-moving and right-moving parts,

$$g(\sigma, \tau) = g_+(x^+) g_-(x^-), \quad (3.8)$$

but, as we have said, it is useful to begin by studying geodesic solutions, which depend only on the time coordinate  $\tau$ .

The simplest timelike geodesic solution is

$$g_0 = \begin{pmatrix} \cos \alpha\tau & \sin \alpha\tau \\ -\sin \alpha\tau & \cos \alpha\tau \end{pmatrix}, \quad (3.9)$$

describing a point particle at the center of  $AdS_3$ ,

$$t = \alpha\tau, \quad \rho = 0. \quad (3.10)$$

The most general timelike geodesic can be obtained by acting on (3.9) with the global isometry group  $SL(2, R) \times SL(2, R)$  of the WZW model. Such a solution has the form

$$g = U \begin{pmatrix} \cos \alpha\tau & \sin \alpha\tau \\ -\sin \alpha\tau & \cos \alpha\tau \end{pmatrix} V, \quad (3.11)$$

where  $U$  and  $V$  are constant  $SL(2, R)$  elements. The parameter  $\alpha$  in (3.9) and (3.11) is related through the Virasoro constraints to the conformal weight  $h$  of the sigma model on the compact space  $\mathcal{M}$ . The conserved currents of the solution (3.9) are

$$J_L^3 = J_R^3 = \frac{k\alpha}{2}, \quad J_L^\pm = J_R^\pm = 0, \quad (3.12)$$

and the energy, defined as the sum of the zero modes of  $J_L^3$  and  $J_R^3$ , is  $k\alpha$ .

The construction of spacelike geodesics is similar. The simplest spacelike geodesic solution

$$g_0 = \begin{pmatrix} e^{\alpha\tau} & 0 \\ 0 & e^{-\alpha\tau} \end{pmatrix} \quad (3.13)$$

describes a straight line through the spacelike section  $t = 0$  of  $AdS_3$ . Its currents are

$$J_L^3 = J_R^3 = 0, \quad J_L^\pm = J_R^\pm = \frac{k\alpha}{2}, \quad (3.14)$$

and its energy is zero. The most general spacelike geodesic solution is

$$g = U g_0 V, \quad (3.15)$$

where, again,  $U$  and  $V$  are constant  $SL(2, R)$  isometries.

Given a classical solution  $\tilde{g}(\sigma, \tau) = \tilde{g}_+(x^+) \tilde{g}_-(x^-)$ , we can generate a new solution  $g(\sigma, \tau) = g_+(x^+) g_-(x^-)$  by *spectral flow*, which involves setting

$$g_+ = e^{\frac{i}{2} w x^+ \sigma_2} \tilde{g}_+, \quad g_- = \tilde{g}_- e^{\frac{i}{2} w x^- \sigma_2}, \quad (3.16)$$

for some integer  $w$ . Spectral flow acts on the  $AdS_3$  global coordinates by

$$\rho \rightarrow \rho, \quad t \rightarrow t + w\tau, \quad \theta \rightarrow \theta + w\sigma, \quad (3.17)$$

and on the  $SL(2, R)$  currents by

$$J_R^3 = \tilde{J}_R^3 + \frac{kw}{2}, \quad J_R^\pm = \tilde{J}_R^\pm e^{\mp iw\alpha^+}, \quad (3.18)$$

$$J_L^3 = \tilde{J}_L^3 + \frac{kw}{2}, \quad J_L^\pm = \tilde{J}_L^\pm e^{\mp iw\alpha^-}; \quad (3.19)$$

or in terms of Fourier modes,

$$J_{R,Ln}^3 = \tilde{J}_{R,Ln}^3 + \frac{kw}{2}\delta_{n,0}, \quad J_{R,Ln}^\pm = \tilde{J}_{R,Ln\mp w}^\pm. \quad (3.20)$$

Timelike geodesics are mapped under spectral flow to short string solutions, which expand and contract periodically in global time and wind  $w$  times around the center of  $AdS_3$ . Spacelike geodesics are mapped to long string solutions, which start in the infinite global-time past as circular strings of infinite radius wound  $w$  times near the boundary of  $AdS_3$ , collapse (to a point, if there is no angular momentum) as  $t \rightarrow 0$ , and expand again as  $t \rightarrow \infty$  to wound circular strings at the boundary. After imposing the Virasoro constraints, the solutions constructed in this way have energies

$$E = \frac{kw}{2} + \frac{1}{w} \left( \mp \frac{k\alpha^2}{2} + 2h \right), \quad (3.21)$$

where the minus sign corresponds to short strings and the plus sign to long strings.

## 3.2 The Quantum Hilbert Space

We now recall the structure of the quantum Hilbert space of the  $SL(2, R)$  WZW model. The Hilbert space decomposes as the sum of *discrete* representations of the  $\widehat{SL}(2, R)$  current algebra, *continuous* representations of the current algebra, and the images of these representations under spectral flow. Let us briefly review what this means.

The zero modes  $J_0^a$  of the generators  $J^a$  of the (left- or right-moving)  $\widehat{SL}(2, R)$  current algebra form a closed subalgebra, which generates the group  $SL(2, R)$ . Among the unitary representations of this  $SL(2, R)$  are the *principal discrete* highest- and lowest-weight representations, which are realized in the Hilbert space

$$\mathcal{D}_j^\pm = \{|j; m\rangle : m = \pm j, \pm(j+1), \pm(j+2), \dots\}. \quad (3.22)$$

The states  $|j; m\rangle$  are simultaneous eigenstates of  $L_0$ , the zeroth Virasoro generator obtained from the Sugawara construction, and  $J_0^3$ , the  $SL(2, R)$  Cartan generator, with eigenvalues

$-j(j-1)$  and  $m$ . These representations are unitary if  $j > 0$ . The representations  $\hat{\mathcal{D}}_j^\pm$  of the  $\widehat{SL}(2, R)$  current algebra are constructed by considering the states in  $\mathcal{D}_j^\pm$  as primary states for the action of  $J_n^a$ . That is, the states  $|j; m\rangle$  are taken to be annihilated by the  $J_n^a$  with  $n > 0$ , while the  $J_n^a$  with  $n < 0$  are understood as creation operators, whose repeated action on the states  $|j; m\rangle$  yields states that fill out the representations  $\hat{\mathcal{D}}_j^\pm$ .

Another unitary representation of  $SL(2, R)$  is the *continuous* representation, realized in the Hilbert space

$$\mathcal{C}_j^\alpha = \{|j, \alpha; m\rangle : m = \alpha, \alpha \pm 1, \alpha \pm 2, \dots\}, \quad (3.23)$$

with  $0 \leq \alpha < 1$  and  $j = 1/2 + is$ , for real  $s$ . Again, the states in  $|j, \alpha; m\rangle$  are simultaneous eigenstates of  $L_0$  and  $J_0^3$ , with eigenvalues  $-j(j-1)$  and  $m$ . The representation  $\mathcal{C}_j^\alpha$  of  $SL(2, R)$  gives rise to a representation  $\hat{\mathcal{C}}_j^\alpha$  of the current algebra in the manner described above for the discrete representations.

As we have just noted, the representations  $\hat{\mathcal{D}}_j^\pm$  and  $\hat{\mathcal{C}}_j^\alpha$  are described by the action of the current algebra modes  $J_n^a$  on their constituent states. Spectral flow by  $w$  units alters the modes  $J_n^a$ , as noted in (3.20), and so maps the representations  $\hat{\mathcal{D}}_j^\pm$  and  $\hat{\mathcal{C}}_j^\alpha$  into new representations  $\hat{\mathcal{D}}_j^{\pm, w}$  and  $\hat{\mathcal{C}}_j^{\alpha, w}$ . The closed string Hilbert space was proposed in [12] to be the direct sum of  $\hat{\mathcal{D}}_j^{+, w} \otimes \hat{\mathcal{D}}_j^{+, w}$  and  $\hat{\mathcal{C}}_j^{\alpha, w} \otimes \hat{\mathcal{C}}_j^{\alpha, w}$ , summed over integers  $w$ . The two factors in each tensor product account for left- and right-moving states. The permissible values of  $j$  for the discrete representations are bounded by  $\frac{1}{2} < j < \frac{k-1}{2}$ ; note that, before the physical state conditions are imposed,  $j$  may be any real number in this range. The states in  $\hat{\mathcal{D}}_j^{+, w} \otimes \hat{\mathcal{D}}_j^{+, w}$  correspond to wound short strings and their excitations,<sup>6</sup> while the states in  $\hat{\mathcal{C}}_j^{\alpha, w} \otimes \hat{\mathcal{C}}_j^{\alpha, w}$  correspond to wound long strings and their excitations. The spectra of both kinds of strings were computed in [12] and were checked by an independent calculation in [11].

It is important to keep clear the distinction between the Hilbert space of the WZW model and the Hilbert space of the physical string theory. The ( $AdS_3$  part of the) latter is the subspace of the former defined by the Virasoro constraints. Spectral flow is a symmetry of the WZW model, but does not commute with the Virasoro constraints, and is therefore not realized explicitly in the physical string Hilbert space. While in our analysis of classical solutions above and in subsequent sections, we have freely borrowed and will continue to borrow the intuitions (and language) of strings, we should bear in mind that our conclusions really pertain to the Hilbert space of the WZW model before the imposition of the Virasoro constraints.

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<sup>6</sup>The representations  $\hat{\mathcal{D}}_j^{+, w}$  and  $\hat{\mathcal{D}}_{\frac{k}{2}-j}^{-, w+1}$ , may be identified. This accounts for the exclusion of  $\hat{\mathcal{D}}_j^{-, w}$  from the list of allowed representations: it would be redundant to include it.

## 4 The Straight $AdS_2$ Brane

Having reviewed closed string theory in  $AdS_3$ , we now add branes to the game. We said in section 2 that the brane worldvolumes in group manifolds can be thought of as twined conjugacy classes of the form

$$\mathcal{W}_g^r = \{r(h)gh^{-1} : h \in G\}, \quad (4.1)$$

where  $r$  is a group automorphism. In the case  $G = AdS_3$ , it was shown by Bachas and Petropoulos [27] that the only physically allowed branes are those for which  $r$  is the nontrivial outer automorphism that acts on a group element  $h$ , parametrized as a  $2 \times 2$  matrix as in (3.3), by

$$r(h) = \omega_0^{-1}h\omega_0, \quad \text{with} \quad \omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.2)$$

For this choice of  $r$ , the twined conjugacy class  $\mathcal{W}_g^r$  is equivalently characterized as the set of  $g' \in SL(2, R)$  such that  $\text{Tr}(\omega_0 g') = \text{Tr}(\omega_0 g)$ . The worldvolumes of the resulting branes are two-dimensional, since they are parametrized by arbitrary  $SL(2, R)$  group elements subject to the single condition

$$\text{Tr}(\omega_0 g') = 2 \sinh \psi_0, \quad (4.3)$$

where  $\sinh \psi_0$  is a constant whose physical meaning will become apparent shortly. The gluing conditions for the currents take the form

$$J_L = -\omega_0 J_R \omega_0. \quad (4.4)$$

Upon tracing through the procedure for extracting boundary conditions for coordinates from the gluing conditions, we indeed find the differential form of (4.3) as the Dirichlet condition for the coordinate transverse to the worldvolume, as well as the appropriate Neumann conditions for the coordinates parallel to the worldvolume.

Thinking of  $AdS_3$  as the hyperboloid (3.1) in four-dimensional space, (4.3) becomes the statement  $X^2 = \sinh \psi_0$ . Subject to this condition, (3.1) becomes

$$(X^0)^2 - (X^1)^2 + (X^3)^2 = 1 + \sinh^2 \psi_0, \quad (4.5)$$

which is the equation of two-dimensional anti-de Sitter space. Thus the worldvolume geometry of the physical branes of [27] is that of  $AdS_2$  embedded in  $AdS_3$ . Analysis of the Dirac-Born-Infeld action of these  $AdS_2$  branes shows that they consist of one D-string bound to some number of fundamental strings. The fundamental string charge  $Q$  is proportional to the constant  $\sinh \psi_0$ ; it follows that  $\psi_0$  is quantized. The quantization condition is

$$\sinh \psi_0 = g_s Q, \quad (4.6)$$

where  $g_s$  is the string coupling constant.

In dealing with  $AdS_2$  branes, it is convenient to switch to a coordinate system in which the Dirichlet condition is simple. The  $AdS_2$  coordinates  $(\psi, \omega, t)$  are defined in terms of the global coordinates by

$$\sinh \psi = \sin \theta \sinh \rho, \quad \cosh \psi \sinh \omega = -\cos \theta \sinh \rho, \quad t = t, \quad (4.7)$$

and in terms of the embedding hyperboloid coordinates by

$$X^1 = \cosh \psi \sinh \omega, \quad X^2 = \sinh \psi, \quad X^0 + iX^3 = \cosh \psi \cosh \omega e^{it}. \quad (4.8)$$

The  $AdS_3$  metric in  $AdS_2$  coordinates is

$$ds^2 = d\psi^2 + \cosh^2 \psi (-\cosh^2 \omega dt^2 + d\omega^2). \quad (4.9)$$

The  $AdS_2$  coordinates are the  $SL(2, R)$  analogue of the coordinate system given in (2.5) for  $SU(2)$ . In  $AdS_2$  coordinates, the Dirichlet condition giving the location of the  $AdS_2$  brane becomes  $\psi = \psi_0$ . Some  $AdS_2$  branes in  $AdS_3$  are shown in Figure 2.

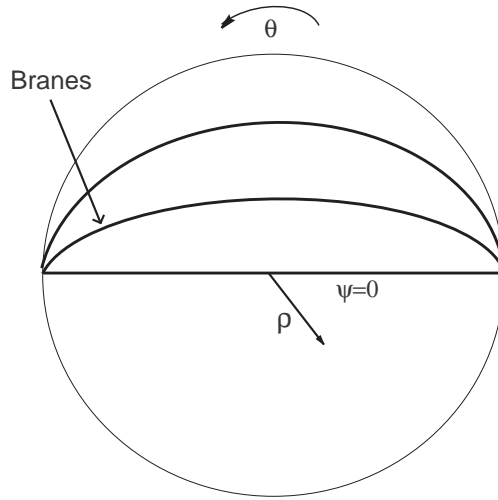


Figure 2:  $AdS_2$  branes in  $AdS_3$ . The view is of the  $(\rho, \theta)$  plane at fixed global time  $t$ . The branes are surfaces of constant  $\psi$ .

In understanding  $AdS_2$  branes and the strings that end on them, there is a crucial distinction to be made between the “straight” branes located at  $\psi = \psi_0 = 0$  and the “curved” branes located at  $\psi = \psi_0 \neq 0$ . In this section, we study straight branes, following the paradigm of section 3. First we construct classical geodesic solutions confined to the brane. Next we investigate how spectral flow generates classical string solutions that satisfy the

appropriate boundary conditions. This leads to a proposal for the open string spectrum. We conclude by describing a check of this proposal by an explicit partition function calculation modeled on that of [11].

## 4.1 Classical Solutions

Our experience with closed strings suggests that looking at geodesics might be a promising starting point for the study of open string solutions. A  $\sigma$ -independent solution that satisfies the Dirichlet condition necessarily lies entirely within the brane. The timelike and spacelike geodesics (3.9) and (3.13) obviously satisfy this requirement. In the closed string case, we built the most general timelike and spacelike geodesic solutions from these basic ones by acting with the global isometry group  $SL(2, R) \times SL(2, R)$ . In the presence of an  $AdS_2$  brane, however, the only permitted isometries are those preserving the gluing conditions

$$J_L = -\omega_0 J_R \omega_0 \quad (4.10)$$

and leaving fixed the brane worldvolume—or equivalently, preserving the Dirichlet condition

$$\text{Tr}(\omega_0 g) = 2 \sinh \psi_0. \quad (4.11)$$

The isometry  $g \rightarrow UgV$  transforms the currents according to

$$J_R \rightarrow U J_R U^{-1}, \quad J_L \rightarrow V^{-1} J_L V. \quad (4.12)$$

The conditions for this isometry to preserve (4.10) and (4.11) are therefore

$$V^{-1} J_L V = -\omega_0 U J_R U^{-1} \omega_0, \quad (4.13)$$

$$\text{Tr}(\omega_0 UgV) = \text{Tr}(\omega_0 g), \quad (4.14)$$

which are satisfied if and only if  $V = \omega_0 U^{-1} \omega_0$ . Thus the boundary conditions imposed by the  $AdS_2$  brane break the global isometry group from  $SL(2, R) \times SL(2, R)$  to a single  $SL(2, R)$ . This is natural: the two factors of  $SL(2, R)$  in closed string theory correspond to independent transformations of left- and right-moving modes, whereas the left- and right-moving modes of open strings are related by the gluing conditions at the worldsheet boundary. Our choice of gluing conditions guarantees that the  $SL(2, R)$  global symmetry is naturally promoted to an affine symmetry, just as in the closed string case.

The preceding analysis of the breaking of the isometry group holds for straight branes and curved branes alike: nothing in the argument depended on the value of  $\psi_0$ .<sup>7</sup> One difference between the straight and curved cases is that, as we have noted, straight branes contain particle-like solutions such as the ones shown in Figure 3. Curved branes, on the other hand,



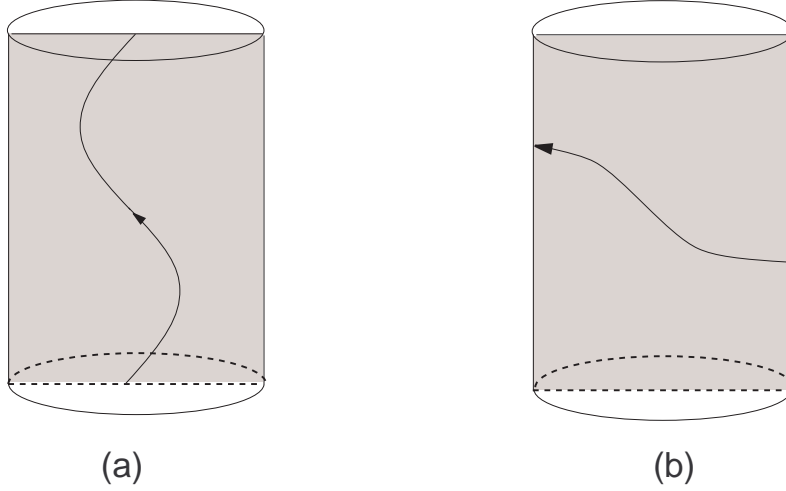


Figure 3: (a) A timelike geodesic and (b) a spacelike geodesic confined to the brane at  $\psi_0 = 0$ .

do not. When we generate the open string solutions associated with curved branes, we will need a different starting point. We will explore this in more detail in section 5.

Spectral flow generates new open string solutions from old ones. In the closed string case, the parameter  $w$  was required to be an integer, to maintain the periodicity of the closed strings. In the presence of an  $AdS_2$  brane,  $w$  must again be integer, but for a different reason: to ensure compatibility of spectral flow with the gluing conditions (4.10).

Is the Dirichlet condition compatible with spectral flow? Suppose we are given a solution satisfying the boundary condition

$$\sinh \psi \equiv \sin \theta \sinh \rho = \sinh \psi_0 \quad (4.15)$$

at  $\sigma = 0$  and  $\sigma = \pi$ . If we act with  $w$  units of spectral flow, we obtain a new solution characterized by

$$\begin{aligned} \sinh \psi_{\text{new}} &= \sin \theta \sinh \rho = \sinh \psi_0 & \text{at } \sigma = 0, \\ \sinh \psi_{\text{new}} &= \sin(\theta + \pi w) \sin \rho = \pm \sin \theta \sinh \rho = \pm \sinh \psi_0 & \text{at } \sigma = \pi, \end{aligned} \quad (4.16)$$

where the sign is plus if  $w$  is even and minus if  $w$  is odd. For a straight  $AdS_2$  brane,  $\psi_0 = 0$ , and so the Dirichlet condition imposes no added restrictions on  $w$ . On the other hand,

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<sup>7</sup>If  $\psi_0 = 0$ , the most general solution of (4.13) and (4.14) is  $V = \pm \omega_0 U^{-1} \omega_0$ , but the counting of parameters remains the same.

spectral flow is a symmetry of curved branes only for even  $w$ .<sup>8</sup> This is a key difference between straight and curved branes, and much of section 5 will be devoted to its consequences.

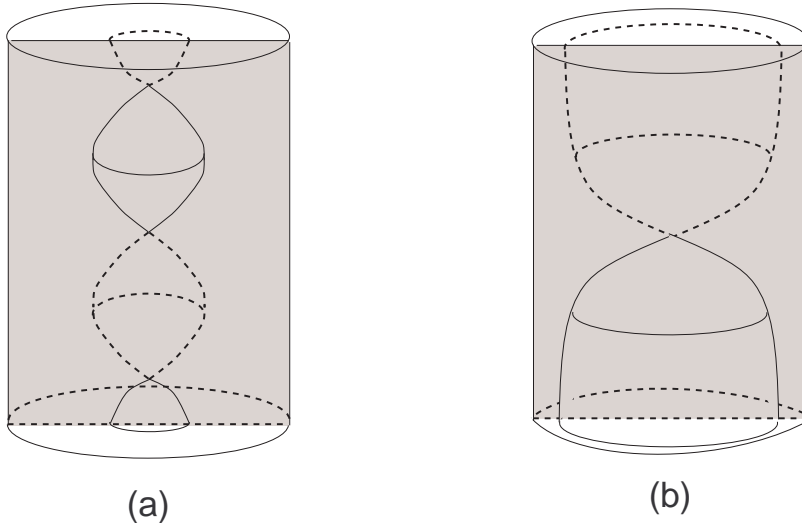


Figure 4: (a) An open short string obtained from a timelike geodesic by spectral flow with  $w = 1$ . (b) An open long string obtained from a spacelike geodesic by spectral flow with  $w = 1$ .

Spectral flow applied to timelike geodesics yields short strings. If  $w$  is odd, these are wound open strings that contract and expand periodically in  $t$ , and whose endpoints are symmetric with respect to the central axis  $\rho = 0$  of  $AdS_3$ . If  $w$  is even, the string endpoints coincide, giving wound circular strings. Spacelike geodesics are mapped by spectral flow to long strings. Examples of strings of both kinds with  $w = 1$  are shown in Figure 4. A calculation just like the one sketched in section 3 shows the spacetime energy of these solutions to be

$$E = J_0^3 = \frac{kw}{4} + \frac{1}{w} \left( \mp \frac{k\alpha^2}{4} + h \right), \quad (4.17)$$

where the minus sign is for short strings and the plus sign for long strings. The energy of these open strings is precisely half the energy (3.21) of their closed string counterparts.

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<sup>8</sup>If  $w$  is odd, then a string whose  $\sigma = 0$  endpoint lies on a brane at  $\psi = \psi_0$  will end up at  $\sigma = \pi$  with  $\psi = -\psi_0$ . This will come in handy in section 5.

## 4.2 The Quantum Hilbert Space

The classical solutions describing short and long strings ending on the straight  $AdS_2$  brane are in a sense “exactly half” of the corresponding closed string solutions: left movers and right movers are related by the gluing conditions, and the energy of the resulting strings is half that of the closed strings. As in the closed string case, discrete representations ought to be associated with short strings and continuous representations with long strings. We therefore propose that the quantum Hilbert space is the direct sum of  $\hat{\mathcal{D}}_j^{+,w}$  and  $\hat{\mathcal{C}}_j^{\alpha,w}$ , summed over all integers  $w$ , and with  $\frac{1}{2} < j < \frac{k-1}{2}$  for the discrete representations and  $j = \frac{1}{2} + is$  with  $s \in \mathbf{R}$  for the continuous representations. Our proposed open string spectrum is thus the holomorphic square root of the closed string spectrum found in [12].

In Appendix B, we verify this conjecture by an independent calculation of the spectrum. We compute the finite-temperature partition function in Euclidean  $AdS_3$  and interpret the result in terms of the free energy, summed over string states. This enables us to read off the spectrum. The result is in exact agreement with the conjecture.

## 5 The Curved $AdS_2$ Brane

We now consider open strings constrained to end on a curved  $AdS_2$  brane located at  $\psi = \psi_0 \neq 0$ ; without loss of generality, we may assume  $\psi_0 > 0$ . Our method is the same as in the closed string and straight brane cases: we begin in section 5.1 with the study of classical solutions and their symmetries, and then in section 5.2 conjecture the structure of the quantum Hilbert space. Several elements of the story are different for curved branes. First, there are no classical geodesic solutions. Second, and more seriously, spectral flow is a symmetry of the  $AdS_2$  brane only if the winding number  $w$  is even. Generating solutions with odd winding number thus calls for new tricks, which we describe in detail.

As we explain in section 5.2, the chief consequence of these differences is that, in the Hilbert space of the WZW model, the density of states of the odd winding continuous representations of  $\widehat{SL}(2, R)$  behaves differently from the density of states of the even winding continuous representations as a function of the brane position  $\psi_0$ . Both kinds of representations are present in the spectrum for all  $\psi_0$ , as is the entire set of discrete representations. This is to be contrasted with the  $SU(2)$  WZW model, whose Hilbert space in the presence of  $S^2$  branes loses representations as  $\psi_0$  increases.

In section 5.3 we present a generalization to a system with two curved  $AdS_2$  branes. In section 5.4 we study the limit  $\psi_0 \rightarrow \infty$  in which the  $AdS_2$  brane becomes highly curved. In this limit, the WZW model Lagrangian resembles that of noncommutative open string theory in  $AdS_2$ .

## 5.1 Classical Solutions

The program we followed in the last section began by constructing  $\sigma$ -independent classical solutions lying within the brane. If  $\psi_0 > 0$ , no such solutions exist. For suppose  $g = g(\tau)$  is a (non-constant) solution. It must satisfy the Dirichlet condition  $\text{Tr}(\omega_0 g) = 2 \sinh \psi_0$ , as well as the gluing condition (4.10), which for the case at hand reads

$$g^{-1} \partial_\tau g = -\omega_0 \partial_\tau g g^{-1} \omega_0, \quad (5.1)$$

as  $\partial_+ g = \partial_- g = \frac{1}{2} \partial_\tau g(\tau)$ . Multiplying on the right by  $(\partial_\tau g)^{-1}$ , inverting, and taking the trace gives  $\text{Tr}(g \omega_0) = -\text{Tr}(\omega_0 g)$ , and thus  $\text{Tr}(g \omega_0) = 0$ , in contradiction with the Dirichlet condition.

Though there are no particle-like solutions, we are led to simple string solutions by the following physical argument. Imagine starting with the timelike geodesic (3.10) in a flat  $AdS_2$  brane and then increasing  $\psi_0$  by turning on a background electric field on the brane. The timelike geodesic on the brane can be thought of as an infinitely small string, whose endpoints are equally and oppositely charged with respect to the background electric field. As  $\psi_0$  increases, the string must stretch so that its tension will balance the forces due to the electric field. This picture suggests the basic timelike string solution shown in Figure 5(a),

$$t = \alpha \tau, \quad \theta = \alpha \sigma + \theta_0, \quad \rho = \rho_0, \quad (5.2)$$

with

$$\theta_0 = \frac{\pi}{2}(1 - \alpha), \quad \sinh \rho_0 = \frac{\sinh \psi_0}{\sin \theta_0}, \quad (5.3)$$

and  $0 \leq \alpha < 1$ . At fixed time, this solution describes a string curved in a circular arc and symmetric about  $\theta = \pi/2$ . It is easily checked that (5.2) solves the equations of motion. Its currents are the same as those of (3.10), and so it obeys (4.10). The choice of  $\theta_0$  and  $\rho_0$  ensure that (4.11) is satisfied as well.<sup>9</sup>

The solution (5.2) is the  $SL(2, R)$  analogue of the  $SU(2)$  solution (2.17) for open strings ending on  $S^2$  branes. An important difference between the two is that the range of  $\alpha$  in (5.2) is  $0 \leq \alpha < 1$ , regardless of the value of  $\psi_0$ , whereas the corresponding parameter  $a$  in the  $SU(2)$  case is subject to the upper bound (2.19), which depends on  $\psi_0$ . This bound restricts the allowed representations in the  $SU(2)$  WZW model Hilbert space. We will argue in section 5.2 that the absence of such a bound for  $AdS_2$  branes implies that there is no similar restriction on the allowed representations in the  $SL(2, R)$  model Hilbert space. Our classical analysis reveals this difference between the two theories to be entirely geometric:

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<sup>9</sup>The most trivial solution imaginable is a ‘‘pointlike instanton’’: a solution with  $t$ ,  $\rho$ , and  $\theta$  all constant. It is amusing to observe that (5.2) may be obtained as the image under spectral flow by  $\alpha$  units of the pointlike instanton solution with  $t = 0$ ,  $\theta = \theta_0$ , and  $\rho = \rho_0$ . Of course,  $\alpha$  is, in general, fractional, and so the notion of spectral flow here is purely formal.

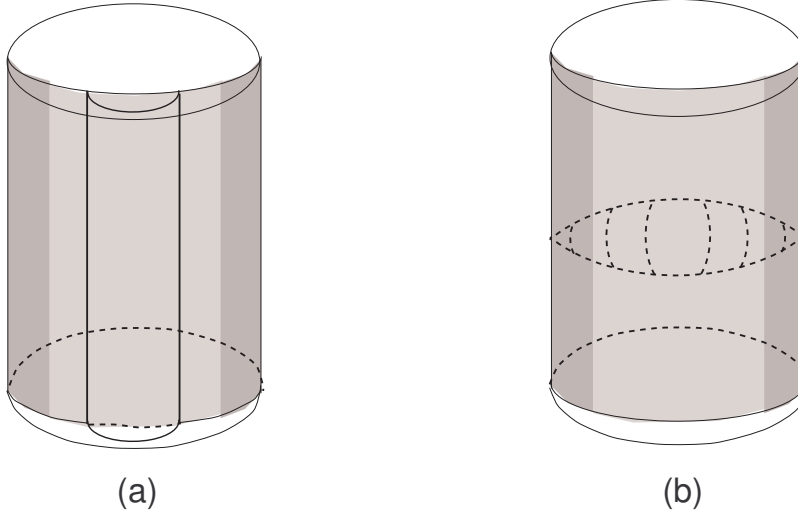


Figure 5: The basic (a) “timelike” and (b) “spacelike” string solutions ending on a curved  $AdS_2$  brane.

in the  $SL(2, R)$  WZW model, every  $AdS_2$  brane stretches from  $\theta = 0$  to  $\theta = \pi$ , while in the  $SU(2)$  case, the range of  $\theta$ , and hence of  $a$ , depends on  $\psi_0$ .

The basic spacelike solution is given in matrix form as

$$g = \begin{pmatrix} \sqrt{1 + \beta^2} e^{\alpha\tau} & \beta e^{\alpha(\sigma - \frac{\pi}{2})} \\ \beta e^{-\alpha(\sigma - \frac{\pi}{2})} & \sqrt{1 + \beta^2} e^{-\alpha\tau} \end{pmatrix}, \quad (5.4)$$

and in global coordinates as

$$\begin{aligned} \tan t &= \frac{\beta \sinh(\alpha(\sigma - \pi/2))}{\sqrt{1 + \beta^2} \cosh \alpha\tau}, \\ \tan \theta &= \frac{-\beta \cosh(\alpha(\sigma - \pi/2))}{\sqrt{1 + \beta^2} \sinh \alpha\tau}, \\ \cosh^2 \rho &= (1 + \beta^2) \cosh^2 \alpha\tau + \beta^2 \sinh^2(\alpha(\sigma - \pi/2)), \end{aligned} \quad (5.5)$$

where  $\beta = \sinh \psi_0 / \cosh \frac{\pi}{2} \alpha$ . This solution, depicted in Figure 5(b), is a stringy generalization of the spacelike geodesic (3.13).<sup>10</sup> It begins in the infinite worldsheet past  $\tau = -\infty$  at  $t = 0$

<sup>10</sup>One way to derive this solution is to assume first that  $\psi_0$  is small and perturb the spacelike geodesic (3.13) accordingly. From the lowest-order corrections to (3.13) it is possible to guess the form of (5.4), and to check that it is a valid solution even if  $\psi_0$  is not small. The basic timelike solution (5.2) may be derived from the timelike geodesic (3.9) by similar methods.

and at the edge  $\theta = 0$ ,  $\rho = \infty$  of the brane, and arrives in the infinite worldsheet future  $\tau = +\infty$  at the other edge  $\theta = \pi$ ,  $\rho = \infty$ , again at global time  $t = 0$ . Its excursion from the brane in the worldsheet interim is governed by  $\alpha$ . When  $\alpha$  is small, the string stays near the brane; as  $\alpha$  increases, it strays farther and farther away. A routine calculation shows that the currents of (5.4) are the same as those of (3.13).

Having obtained the basic timelike and spacelike solutions, our next task is to generate new solutions by acting with isometries and spectral flow. As we commented in section 4, the allowed isometries are the same in the presence of curved and straight  $AdS_2$  branes. We have already seen, though, that spectral flow is different. If  $w$  is even, spectral flow is still a symmetry of the curved brane, and we can apply it to (5.2) and (5.4) without incident. For example, spectral flow applied to (5.2) gives

$$t = (\alpha + w)\tau, \quad \theta = (\alpha + w)\sigma + \theta_0, \quad \rho = \rho_0. \quad (5.6)$$

If  $w$  is even, this solution describes a cylindrical worldsheet making  $w/2$  (not  $w$ !) complete cycles around the center of  $AdS_3$ , and whose endpoints coincide with the endpoints of the original solution (5.2).

Applying an even amount of spectral flow to (5.4) gives a long string-like solution. Its properties are most easily seen by replacing  $t \rightarrow t + w\tau$ ,  $\theta \rightarrow \theta + w\sigma$  in the coordinate description (5.5). It begins in the infinite spacetime past as a circular string of infinite radius with its endpoints coincident at  $\theta = 0$ . Next it collapses to finite radius. Its endpoints become separated in  $t$  and move in towards the center ( $\theta = \pi/2$ ) of the brane. Finally, the string re-expands towards infinite radius, where its endpoints reconverge at  $\theta = \pi$ .

How shall we generate solutions with odd  $w$ ? Our strategy will be different in the timelike and spacelike cases. The construction in the timelike case makes use of the discrete target space symmetry

$$PT : g \rightarrow \omega_0 g \omega_0. \quad (5.7)$$

Calling this symmetry PT is justified by its action on the global coordinates,

$$t \rightarrow -t, \quad \theta \rightarrow \pi - \theta, \quad (5.8)$$

which reveals it as the composition of a parity and a time-reversal transformation. The PT symmetry acts on the currents by

$$J_{R,L} \rightarrow \omega_0 J_{R,L} \omega_0, \quad (5.9)$$

and hence on their modes by

$$J_{R,Ln}^3 \rightarrow -J_{R,Ln}^3, \quad J_{R,Ln}^\pm \rightarrow -J_{R,Ln}^\mp. \quad (5.10)$$

These expressions make it clear that PT is an automorphism of the current algebra and a symmetry of the WZW model. Moreover, it preserves the gluing condition (4.10) and the Dirichlet condition (4.11).

The PT symmetry maps short strings of winding number  $w$  to short strings of winding number  $-w - 1$ . To see this in a simple example, consider the closed string or straight brane timelike geodesic (3.10). Acting with  $w$  units of spectral flow gives the solution

$$t = (\alpha + w)\tau; \tag{5.11}$$

thus, without loss of generality, we may take  $0 \leq \alpha < 1$  in (3.10), and regard a solution  $t = \alpha\tau$  with general  $\alpha$  as the image of the solution with  $0 \leq \alpha < 1$  under a suitable amount of spectral flow. Now if we apply PT to (5.11), we get a solution with

$$t = -(\alpha + w)\tau = (\alpha' + (-w - 1))\tau, \tag{5.12}$$

where  $\alpha' = 1 - \alpha$  satisfies  $0 \leq \alpha' < 1$ . By our previous logic, this is to be thought of as the image of (3.10) with parameter  $\alpha'$  under  $-w - 1$  units of spectral flow.

Our course for finding short string solutions with odd  $w$  is now clear: we simply act with PT on a solution with  $w = 0$  (*e.g.*, the image under an isometry of the basic timelike solution (5.2)) to reach the  $w = -1$  sector, and then act on the result with  $w + 1$  units of spectral flow, which is a symmetry because  $w + 1$  is even. As an example, if we implement this procedure on (5.2), we find the solution

$$t = (1 - \alpha + w)\tau, \quad \theta = (1 - \alpha + w)\sigma + \pi - \theta_0, \quad \rho = \rho_0. \tag{5.13}$$

Figure 6 shows a more complicated open short string solution with  $w = 1$ .

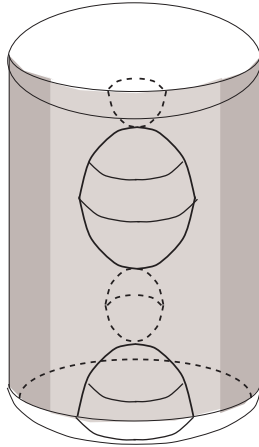


Figure 6: A classical open  $w = 1$  short string solution.

This trick fails for long string solutions. An argument like the one given above demonstrates that PT maps long string solutions with winding number  $w$  to long strings with

winding number  $-w$ , and therefore does not mix odd and even winding sectors. Instead, to construct spacelike solutions with odd  $w$ , we recall that spectral flow leaves fixed the  $\sigma = 0$  endpoint of a string ending at  $\psi = \psi_0$ , but maps the  $\sigma = \pi$  endpoint to  $\psi = -\psi_0$ . This prompts us to introduce a second  $AdS_2$  brane located at  $\psi = -\psi_0$ . If we can find an unwound spacelike string with one endpoint on the brane at  $\psi_0$  and the other endpoint on the brane at  $-\psi_0$ , then the action of spectral flow with odd  $w$  will produce a string with odd winding number, both of whose endpoints lie on the  $\psi_0$  brane.

Given (5.4), it is relatively straightforward to find unwound spacelike strings stretching from the  $\psi_0$  brane to the  $-\psi_0$  brane. There are two distinct classes of solutions, depending on the value of  $\alpha$ . Let  $\beta = \sinh \psi_0 / \sinh \frac{\pi}{2} \alpha$ . For  $|\beta| < 1$ , we have the solution

$$g = \begin{pmatrix} \sqrt{1 - \beta^2} e^{\alpha\tau} & -\beta e^{\alpha(\sigma - \frac{\pi}{2})} \\ \beta e^{-\alpha(\sigma - \frac{\pi}{2})} & \sqrt{1 - \beta^2} e^{-\alpha\tau} \end{pmatrix}. \quad (5.14)$$

For  $|\beta| > 1$ ,

$$g = \begin{pmatrix} \sqrt{\beta^2 - 1} e^{\alpha\tau} & -\beta e^{\alpha(\sigma - \frac{\pi}{2})} \\ \beta e^{-\alpha(\sigma - \frac{\pi}{2})} & -\sqrt{\beta^2 - 1} e^{-\alpha\tau} \end{pmatrix}. \quad (5.15)$$

What do these solutions look like? The two branes meet at the two lines  $\theta = 0$  and  $\theta = \pi$  on the boundary  $\rho = \infty$  of  $AdS_3$ . The solution with  $|\beta| < 1$  describes a string that begins in the infinite worldsheet past at the point  $t = 0$  on one of these lines ( $\theta = \pi$  if  $\beta$  is positive), fills out a spacelike surface between the two branes, and contracts in the infinite worldsheet future to the point  $t = 0$  on the other line. The solution with  $|\beta| > 1$  describes a string that begins at a point on one of the lines ( $\theta = \pi$  if  $\beta$  is positive), fills out a timelike surface between the two branes with minimum radius  $\cosh^{-1} \beta$ , and returns after an interval  $\pi$  of target space time to a point on the same line. The two cases are sketched in Figure 7. The solution in the borderline case  $|\beta| = 1$ ,

$$g = \begin{pmatrix} 0 & -e^{\alpha(\sigma - \frac{\pi}{2})} \\ e^{-\alpha(\sigma - \frac{\pi}{2})} & 0 \end{pmatrix}, \quad (5.16)$$

stretches between the centers  $\theta = \pi/2$  of the two branes at global time  $t = \pi/2$ .

Acting with an odd amount  $w$  of spectral flow on these strings gives long string solutions whose endpoints both lie on the brane at  $\psi = \psi_0$ . The image of (5.14) under spectral flow begins in the infinite spacetime past as a string of infinite radius whose two endpoints lie on opposite edges of the brane. With increasing  $t$ , the string contracts until, at  $t = 0$ , the endpoints cross at the center  $\theta = \pi/2$  of the brane. Afterwards, the string expands until  $t = \infty$ , when the endpoints again reach opposite edges of the brane. At  $t = \infty$ , each endpoint is at the edge opposite to the edge at which it began at  $t = -\infty$ . By contrast, in the flowed solution with  $|\beta| > 1$ , the endpoints do not have enough energy to reach the center of the



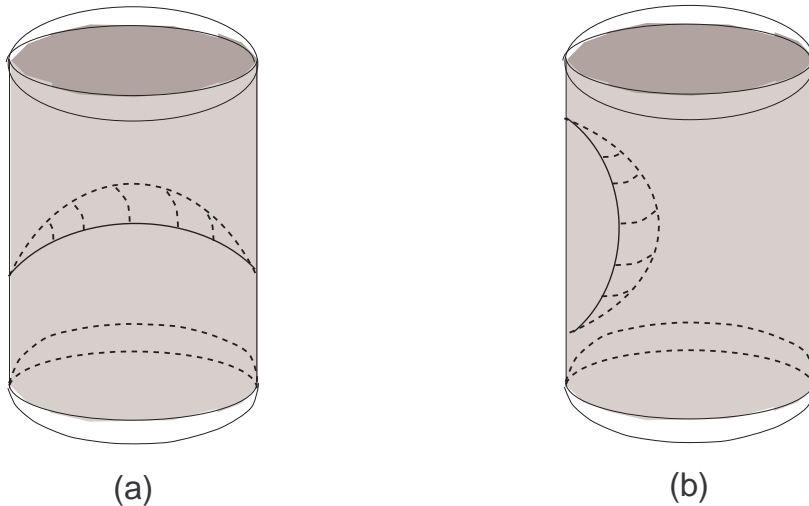


Figure 7: Strings stretched between the branes at  $\psi = +\psi_0$  and  $\psi = -\psi_0$ , with (a)  $|\beta| < 1$  and (b)  $|\beta| > 1$ .

$AdS_2$  brane, and return at  $t = \infty$  to the edge at which they began. Figure 8 depicts the long strings.

To summarize, we have constructed classical solutions for open strings ending on a curved  $AdS_2$  brane. We began with simple timelike and spacelike string solutions akin to the closed string and straight brane geodesics considered in sections 3 and 4. Acting with the isometry group  $SL(2, R)$  generated more unwound solutions. As in the straight brane case, spectral flow with even  $w$  gave us new winding solutions, but unlike the straight brane case, spectral flow with odd  $w$  was no longer a symmetry. Short string and long string solutions in odd winding sectors do exist, but to reach them, we required new techniques: target-space PT symmetry for short strings and the introduction of a brane at  $\psi = -\psi_0$  for long strings.

## 5.2 The Quantum Hilbert Space

In section 4.2 we described the structure of the Hilbert space of the  $SL(2, R)$  WZW model in the presence of an  $AdS_2$  brane at  $\psi_0 = 0$ . Now, drawing on what we have learned about classical open strings ending on curved branes, we sketch how the Hilbert space changes as  $\psi_0$  is increased above zero.

The WZW model Hilbert space at  $\psi_0 = 0$  contains discrete representations  $\hat{\mathcal{D}}_j^{+,w}$ , for  $w \in \mathbf{Z}$  and  $\frac{1}{2} < j < \frac{k-1}{2}$ . We propose that all of these representations persist in the  $\psi_0 > 0$  Hilbert space. “Discrete” is something of a misnomer in the context of the WZW model of

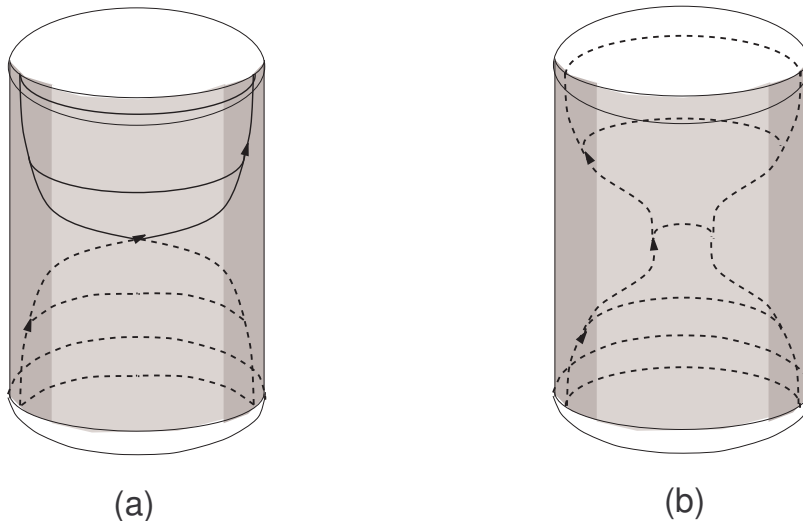


Figure 8: A classical long string in the  $w = 1$  sector, for (a)  $|\beta| < 1$  and (b)  $|\beta| > 1$ .

(the universal cover of)  $SL(2, R)$ , since the discreteness of  $j$  is enforced only by the Virasoro constraints. At the level of the WZW model Hilbert space, before the Virasoro constraints are imposed,  $j$  is not quantized, and it is meaningful to speak of the density of states defined by a measure in  $j$ -space. In the presence of a curved brane at  $\psi = \psi_0$ , this measure depends on  $\psi_0$ . We conjecture that the  $\psi_0$  dependence is the same in all winding sectors.

As in the straight brane case, the WZW model Hilbert space at nonzero  $\psi_0$  contains continuous representations  $\hat{\mathcal{C}}_{\frac{1}{2}+is}^{\alpha,w}$ , for all  $\alpha \in [0, 1)$ ,  $w \in \mathbf{Z}$ , and  $s \in \mathbf{R}$ . Once again, the density of states of these representations now depends on  $\psi_0$ . We conjecture that the  $\psi_0$  dependence is the same in all even winding sectors and in all odd winding sectors, but that the dependence in the even winding sectors is different from the dependence in the odd winding sectors.

In support of our conjectures, we note first that, for all  $\psi_0$ , we were able to construct classical short string solutions for  $\alpha$  satisfying  $0 \leq \alpha < 1$ , which is the semiclassical version of the range  $\frac{1}{2} < j < \frac{k-1}{2}$ . This, we observed, is unlike the situation in the  $SU(2)$  WZW model, where the range of  $j$  for which classical solutions exist is bounded in terms of  $\psi_0$ . In the  $SU(2)$  WZW model, the truncation of classical solutions as  $\psi_0$  increased manifested itself in the quantum theory as a loss of representations in the Hilbert space. In the  $SL(2, R)$  WZW model, there is no truncation classically, which leads us to believe that in the quantum theory, representations spanning the entire range of  $j$  are likewise present.<sup>11</sup> The measure in

<sup>11</sup>A more computational argument in support of this claim is presented in section 5.4.

the space of  $j$  may depend on  $\psi_0$ , but this is the only possible  $\psi_0$  dependence in the structure of the discrete sector of the Hilbert space.

Spectral flow by an even amount is a symmetry of the theory. It follows that the  $\psi_0$  dependence of the density of states must be the same in all even winding sectors and in all odd winding sectors. In addition, target space PT symmetry links even and odd winding short string sectors. Consequently, the  $\psi_0$  dependence of the density of discrete states must in fact be the same in all winding sectors.

The existence of classical long string solutions is strong evidence that the WZW model Hilbert space contains continuous representations. Spectral flow by an even amount is a symmetry of the long string solutions, but target space PT symmetry does not mix even and odd long string winding sectors. Therefore, the  $\psi_0$  dependence of the density of states of the continuous representations is the same in all even winding sectors and in all odd winding sectors, but the dependence in the even sectors is, in general, different from the dependence in the odd sectors.

We will presently provide further evidence that this is so by studying a certain family of *physical* short string solutions in the even and odd winding sectors. We compute the energy  $E$  of these solutions as a function of their size  $R$ . In both the odd and even winding sectors,  $E(R)$  increases monotonically at large  $R$  to a value whose functional form as a function of  $w$  is the same in all sectors. However, the dependence of  $E$  on  $R$ —and on  $\psi_0$ —is different in the two types of sectors. The existence of an upper bound for  $E(R)$  in the short string sectors implies that, above this bound, a continuous representation appears. The  $\psi_0$  dependence of  $E(R)$  at large  $R$  is different for odd and even  $w$ . This implies that the  $\psi_0$  dependence of the density of states of the emergent continuous representations is likewise different for odd and even  $w$ .

The short string solutions under consideration belong to the physical Hilbert space. To apply our conclusions to the WZW model Hilbert space, we reason *a fortiori*: since the ( $AdS_3$  part of the) physical string Hilbert space is the WZW model Hilbert space after the imposition of the Virasoro constraints, if continuous representations exist in the physical Hilbert space, surely they must exist in the WZW model Hilbert space.

We now fill in the details of this argument. To construct the even winding solutions, we begin with the basic unwound timelike solution

$$t = \alpha\tau, \quad \theta = \alpha\sigma + \theta_0, \quad \rho = \rho_0, \quad (5.17)$$

with  $0 \leq \alpha < 1$ ,  $\theta_0 = \frac{\pi}{2}(1 - \alpha)$ , and

$$\sinh \rho_0 = \frac{\sinh \psi_0}{\cos \frac{\pi\alpha}{2}}. \quad (5.18)$$

We then act with the isometry given by  $U = e^{\frac{1}{2}\rho_1\sigma_3}$  and  $V = \omega_0 U^{-1} \omega_0 = U$ , where  $\rho_1$  is a constant. Finally, we perform an even amount  $w$  of spectral flow. The currents of the

resulting solution are

$$J_R^3 = \frac{k}{2}(\alpha \cosh \rho_1 + w), \quad (5.19)$$

$$J_R^\pm = \pm \frac{ik}{2}(\alpha \sinh \rho_1 e^{\mp iw x^+}), \quad (5.20)$$

and similarly for  $J_L^a$ .

To construct the odd winding solutions, we once again introduce a second  $AdS_2$  brane at  $\psi = -\psi_0$ , and consider the unwound short string

$$t = \alpha\tau, \quad \theta = \alpha\sigma + \theta_0, \quad \rho = \rho_0, \quad (5.21)$$

with  $0 \leq \alpha < 1$ ,  $\theta_0 = \pi(1 - \frac{\alpha}{2})$ , and

$$\sinh \rho_0 = \frac{\sinh \psi_0}{\sin \frac{\pi\alpha}{2}}. \quad (5.22)$$

This is a string stretching between the two branes. Again, we act with the isometry given by  $U = e^{\frac{1}{2}\rho_1\sigma_3}$  and  $V = \omega_0 U^{-1} \omega_0 = U$ , and perform  $w$  units of spectral flow on the result. Since  $w$  now is odd, both endpoints of the resulting solution lie on the brane at  $\psi = \psi_0$ . The currents of this solution, too, are given by (5.19) and (5.20).

To obtain physical string solutions in  $AdS_3 \times \mathcal{M}$ , we must impose the Virasoro constraints  $T_{\pm\pm}^{AdS} + h = 0$ , where  $T^{AdS}$  is the energy-momentum tensor for the  $AdS_3$  modes of the string, and  $h$  is the energy-momentum tensor for  $\mathcal{M}$ , which we regard as a conformal weight for the sigma model on  $\mathcal{M}$ . The energy-momentum tensor is calculable in terms of the currents (5.19) and (5.20), and the resulting Virasoro constraint expresses  $\alpha$  in terms of  $h$  [12],

$$\alpha = \alpha_\pm = -w \cosh \rho_1 \pm \sqrt{w^2 \sinh^2 \rho_1 + \frac{4h}{k}}. \quad (5.23)$$

Choosing the branch  $\alpha = \alpha_+$ , the spacetime energy takes the form

$$E = J_0^3 = \frac{k}{2} \left( \cosh \rho_1 \sqrt{\frac{4h}{k} + w^2 \sinh^2 \rho_1} - w \sinh^2 \rho_1 \right). \quad (5.24)$$

This expression is valid for both the even and odd winding solutions.

The last ingredient we need for our argument is a precise notion of the size of the string. We define  $R$  to be the maximum value of the coordinate  $\rho(\sigma, \tau)$ . By writing even and odd winding solutions in the matrix form (3.3), it is straightforward to calculate that, for both types of solutions,

$$\cosh R = \cosh \rho_0 \cosh \rho_1. \quad (5.25)$$

In (5.24), we expressed the string energy as a function of  $\rho_1$ , the isometry parameter. We now work towards rewriting  $E$  as a function of  $R$ , in the large  $R$  limit.

As  $\rho_1 \rightarrow \infty$ ,  $\alpha = \alpha_+$  approaches 0 according to

$$\alpha = \frac{\frac{4h}{k} - w^2}{w \cosh \rho_1 + \sqrt{w^2 \sinh^2 \rho_1 + \frac{4h}{k}}} \sim \frac{1}{w} \left( \frac{4h}{k} - w^2 \right) e^{-\rho_1}. \quad (5.26)$$

It follows from (5.22) that for odd  $w$ , in this limit,

$$\sinh \rho_0 = \frac{\sinh \psi_0}{\sin \frac{\pi \alpha}{2}} \sim \frac{\sinh \psi_0}{\frac{\pi \alpha}{2}} \sim \frac{2w \sinh \psi_0}{\pi \left( \frac{4h}{k} - w^2 \right)} e^{\rho_1}, \quad (5.27)$$

and therefore

$$\cosh R = \cosh \rho_0 \cosh \rho_1 \sim \frac{2w \sinh \psi_0}{\pi \left( \frac{4h}{k} - w^2 \right)} e^{\rho_1} \cosh \rho_1, \quad (5.28)$$

so that

$$R \sim 2\rho_1 + \log \left( \frac{2w \sinh \psi_0}{\pi \left( \frac{4h}{k} - w^2 \right)} \right). \quad (5.29)$$

For even  $w$ , as  $\rho_1 \rightarrow \infty$ ,  $\rho_0 \rightarrow \psi_0$ , and so

$$\cosh R \sim \cosh \psi_0 \cosh \rho_1, \quad (5.30)$$

or

$$R \sim \rho_1 + \log \cosh \psi_0. \quad (5.31)$$

Substituting (5.29) and (5.31) into (5.24) allows us to determine  $E$  as a function of  $R$ . We find that, for odd  $w$ ,

$$E(R) = \frac{h}{w} + \frac{kw}{4} - \frac{4h - kw^2}{2\pi kw^2} e^{-R} \sinh \psi_0, \quad (5.32)$$

while for even  $w$ ,

$$E(R) = \frac{h}{w} + \frac{kw}{4} - \frac{(4h - kw^2)^2}{4kw^3} e^{-2R} \cosh^2 \psi_0, \quad (5.33)$$

plus terms respectively of order  $e^{-2R}$  and  $e^{-4R}$  for odd and even  $w$ .

What we learn from this analysis is that, in both the odd and even winding sectors,  $E(R)$  has an asymptote at  $\frac{h}{w} + \frac{kw}{4}$ . This asymptote signals the existence of continuous representations. Solutions with energies below the asymptote are bound states—short strings—trapped within a finite radius in  $AdS_3$ . By contrast, solutions with energies above the asymptote are free to escape to the boundary of  $AdS_3$ . These are the long strings, which inhabit continuous representations. This interpretation is consistent with the fact that the energy of physical long strings with winding number  $w$  is bounded below, in the semiclassical limit of large  $h$  and large  $k$ , by  $\frac{h}{w} + \frac{kw}{4}$ , the asymptotic value of  $E(R)$ . We have thus shown that continuous representations of every  $w$  exist in the physical string Hilbert space; hence *a fortiori* such representations exist in the WZW model Hilbert space.

We can take this argument one step further to confirm that the  $\psi_0 > 0$  WZW model Hilbert space not only contains continuous representations of every  $w$ , but within each winding sector contains representations  $\hat{\mathcal{C}}_{j=\frac{1}{2}+is}^{\alpha,w}$  of every  $s$ . The Virasoro constraint determines  $\alpha$  in terms of  $s$  and  $h$ . Thus, if  $s$  were somehow quantized in the WZW model Hilbert space, the physical long string spectrum at fixed  $w$  and  $h$  would be quantized as well. As we have seen, though, the physical long string spectrum at fixed  $w$  and  $h$  is continuous. Thus  $s$  must not be quantized in the WZW model Hilbert space; representations with arbitrary  $s \in \mathbf{R}$  must actually appear.

The approach of  $E(R)$  to its asymptote is different as a function of  $R$  and  $\psi_0$  in the even and odd winding sectors. This is consistent with our claim that the  $\psi_0$  dependence of the density of states in the WZW model Hilbert space is different in the odd and even winding long string sectors. An additional piece of evidence for this point comes from an analysis in the spirit of [11] and Appendix B of the divergence structure of the one-loop Euclidean partition function. The divergences in question signal the presence of continuous representations; they originate in the infinite volume factors that appear when long strings are subject to a flat potential. Accordingly, it is sufficient to consider the contribution to the functional integral of the large  $\rho$  region. In global coordinates, the WZW action at large  $\rho$  takes the form

$$S \sim \int d^2z \left( \partial\rho\bar{\partial}\rho + \frac{1}{4}e^{2\rho}|\bar{\partial}(\theta - it)|^2 + \dots \right). \quad (5.34)$$

In the one-loop calculation, the worldsheet is taken to be cylindrical and of Euclidean signature; the target space time is likewise Euclidean. The worldsheet coordinate  $z = \sigma + i\tau$  is subject to the periodicity

$$\tau \sim \tau + 2\pi t_W, \quad (5.35)$$

with  $t_W$  the worldsheet modulus. At finite temperature, the target space coordinates  $t$  and  $\theta$  describe a torus:  $\theta$  is periodic by nature, and  $t$  is periodically identified with period equal to the inverse temperature  $\beta$ ; that is,

$$\theta - it \sim \theta - it + 2\pi n + i\beta m, \quad (5.36)$$

where  $m$  and  $n$  are integers.

If  $\rho$  is assumed to be fixed at  $\rho_0$ , the equations of motion constrain  $\theta - it$  to be a harmonic map from the worldsheet to the target space. The general harmonic map from the cylinder to the torus is of the form

$$\theta - it = w\sigma + ib\tau + \theta_0, \quad (5.37)$$

where  $w$ ,  $b$ , and  $\theta_0$  are real constants. Matching the periodicities (5.35) and (5.36) of the worldsheet and target space sets  $b = \frac{\beta m}{2\pi t_W}$ . The partition function thus receives contributions from harmonic maps of the form (5.37), for all integers  $m$ . As in Appendix B.2, it is sufficient for our purposes to concentrate on the  $m = 1$  sector.

If it is to describe a legitimate open string configuration, the map (5.37) must satisfy the gluing conditions and the Dirichlet condition. A straightforward calculation shows that, in the presence of an  $AdS_2$  brane at  $\psi = \psi_0$ , the gluing conditions are satisfied if one of two conditions holds:

1.  $w = b = \frac{\beta}{2\pi t_W}$ ; or
2.  $\theta_0 = 0$  or  $\pi$ , and  $w$  is an integer.

Suppose condition 2 is fulfilled. Then the Dirichlet condition  $\sin \theta_0 \sinh \rho_0 = \sinh \psi_0$  can be satisfied only if  $\psi_0 = 0$ . In this case, there is a family of solutions of the form (5.37) with the desired properties, indexed by the continuous parameter  $\rho_0$ . The solutions (5.37) are holomorphic—that is, functions of  $z = \sigma + i\tau$ —if the worldsheet modulus takes the special value

$$t_W = \frac{\beta}{2\pi w}. \quad (5.38)$$

As explained in [11], the functional integral suffers a logarithmic divergence at this special value of  $t_W$ . The divergence is the hallmark of a continuous representation, whose density of states can be derived by properly regularizing the infinity. We have thus arrived again at a conclusion we reached in section 4: the straight brane Hilbert space contains continuous representations for all values of the winding number.

If  $\psi_0 \neq 0$ , then harmonic maps satisfying the gluing conditions must fulfill condition 1. In this case, the Dirichlet condition reads

$$\sin \theta_0 \sinh \rho_0 = \sin(w\pi + \theta_0) \sinh \rho_0 = \sinh \psi_0. \quad (5.39)$$

If  $w$  is not an integer, (5.39) determines  $\theta_0$  and  $\rho_0$  uniquely. If  $w$  is an odd integer, (5.39) has no solution. If  $w$  is an even integer, (5.39) collapses to the single condition  $\sin \theta_0 \sinh \rho_0 = \sinh \psi_0$ , which has a family of solutions indexed by a single continuous free parameter. Condition 1 trivially implies (5.38); thus the maps (5.37) are holomorphic. The functional integral thus has a divergence at  $t_W = \frac{\beta}{2\pi w}$  if  $w$  is even, but this divergence is apparently absent if  $w$  is odd. Following the logic of the last paragraph, we conclude that the WZW model Hilbert space contains continuous representations in the even winding sectors. On the other hand, this line of reasoning tells us nothing about the odd winding continuous representations. Of course, we have already independently established that continuous representations exist in all winding sectors. This argument points to a difference in the mechanism for generating continuous representations in the even and odd winding sectors, illustrating our claim that the nature of the continuous representations at nonzero  $\psi_0$ —and in particular, their density of states—depends significantly on the parity of the winding number.

### 5.3 A Two-Brane System

An interesting perspective on the Hilbert space in the presence of a curved brane at  $\psi = \psi_0$  is obtained by revisiting a device from section 5.1: the introduction of a second brane at  $\psi = -\psi_0$ . Odd spectral flow maps an open string with both endpoints on the brane at  $\psi = \psi_0$  to a string that begins at  $\sigma = 0$  on the  $\psi_0$  brane but ends at  $\sigma = \pi$  on the  $-\psi_0$  brane, and *vice versa*. The two-brane system thus preserves the full spectral flow symmetry. The Hilbert space of the enlarged system has the structure

$$\mathcal{H} = \mathcal{H}_{++} \oplus \mathcal{H}_{+-} \oplus \mathcal{H}_{-+} \oplus \mathcal{H}_{--}, \quad (5.40)$$

where, for example,  $\mathcal{H}_{+-}$  is the Hilbert space of open strings starting on the brane at  $\psi = +\psi_0$  and ending on the brane at  $\psi = -\psi_0$ , and similarly for the other summands. Clearly  $\mathcal{H}_{++} \cong \mathcal{H}_{--}$  and  $\mathcal{H}_{+-} \cong \mathcal{H}_{-+}$ .

Each summand can be further decomposed as the sum of discrete and continuous representations of  $\widehat{SL}(2, R)$ . As in the single-brane case, the symmetries of the system give clues about how the representations in various sectors of the theory are related. Spectral flow by an even amount is a symmetry of each summand individually, while spectral flow by an odd amount maps  $\mathcal{H}_{++} \leftrightarrow \mathcal{H}_{+-}$  and  $\mathcal{H}_{--} \leftrightarrow \mathcal{H}_{-+}$ . For example, the action of spectral flow with  $w = 1$  on the discrete representations of  $\mathcal{H}_{++}$  and  $\mathcal{H}_{+-}$  is given by

$$\begin{array}{ccccccccc} \cdots & \hat{\mathcal{D}}_{++ , j}^{+ , -2} & & \hat{\mathcal{D}}_{++ , j}^{+ , -1} & & \hat{\mathcal{D}}_{++ , j}^{+ , 0} & & \hat{\mathcal{D}}_{++ , j}^{+ , 1} & & \hat{\mathcal{D}}_{++ , j}^{+ , 2} & \cdots \\ & & \times & & \times & & \times & & \times & & \\ \cdots & \hat{\mathcal{D}}_{+- , j}^{+ , -2} & & \hat{\mathcal{D}}_{+- , j}^{+ , -1} & & \hat{\mathcal{D}}_{+- , j}^{+ , 0} & & \hat{\mathcal{D}}_{+- , j}^{+ , 1} & & \hat{\mathcal{D}}_{+- , j}^{+ , 2} & \cdots \end{array} . \quad (5.41)$$

Extending the reasoning of section 5.2, we can deduce that the  $\psi_0$  dependence of the density of states in the even (odd) winding discrete sectors of  $\mathcal{H}_{++}$  must be the same as the  $\psi_0$  dependence of the density of states in the odd (even) winding discrete sectors of  $\mathcal{H}_{+-}$ . A similar statement holds for  $\mathcal{H}_{--}$  and  $\mathcal{H}_{-+}$ , and for the density of states of continuous representations. Within each summand, target space PT symmetry mixes discrete representations with odd and even  $w$ , but preserves the parity of  $w$  of the continuous representations. We can hence argue further that the  $\psi_0$  dependence of the density of discrete states is the same for all winding sectors and all summands. By contrast, the  $\psi_0$  dependence of the density of states in the continuous representations is different in the odd winding sectors and the even winding sectors in each summand, and the dependence in  $\mathcal{H}_{++}$  is different from the dependence in  $\mathcal{H}_{+-}$ . One advantage of enlarging our system to include a second brane is that the entire structure of the continuous sector of the single-brane Hilbert space is encoded in the representations  $\hat{\mathcal{C}}_{++ , j}^{\alpha , 0}$ ,  $\hat{\mathcal{C}}_{+- , j}^{\alpha , 0}$  and their properties under spectral flow and the target space PT symmetry.

Just as in the  $SU(2)$  WZW model, we may further generalize to a system of  $AdS_2$  branes located at  $\psi = \psi_1$  and  $\psi = \psi_2$ . The discussion is completely parallel to what has already



been said. We begin by constructing basic unwound classical solutions stretching from one brane to the other, and employ spectral flow to generate the space of all classical solutions. The basic unwound timelike solutions are again

$$t = \alpha\tau, \quad \theta = \alpha\sigma + \theta_0, \quad \rho = \rho_0, \quad (5.42)$$

where  $0 \leq \alpha < 1$ , and  $\theta_0$  and  $\rho_0$  are chosen so as to satisfy the appropriate Dirichlet conditions. The basic spacelike solution is given in matrix form as

$$g = \begin{pmatrix} (1 + ab) \exp(\alpha\tau) & a \exp(\alpha\sigma) \\ b \exp(-\alpha\sigma) & \exp(-\alpha\tau) \end{pmatrix}, \quad (5.43)$$

where the integration constants  $a$  and  $b$  are likewise fixed to satisfy the Dirichlet boundary conditions.

Spectral flow by an even amount  $w$  is a symmetry of the system. Short string solutions in odd winding sectors may be constructed by means of the PT symmetry of the target space, as explained in section 5.1. To construct long string solutions belonging to odd winding sectors, we introduce a third  $AdS_2$  brane at  $\psi = -\psi_2$ , and follow the procedure described in section 5.1 for finding solutions stretching between the  $\psi_1$  and  $-\psi_2$  branes. Spectral flow by an odd amount  $w$  then gives open string solutions stretching between the  $\psi_1$  and  $\psi_2$  branes.

Again, these classical constructions provide us with a reasonable conjecture for the quantum Hilbert space of the two-brane system: that it contains discrete representations  $\hat{\mathcal{D}}_j^{+,w}$  for all integers  $w$  and  $j$  satisfying  $\frac{1}{2} < j < \frac{k-1}{2}$ , as well as continuous representations  $\hat{\mathcal{C}}_j^{\alpha,w}$  for all integers  $w$ , all  $\alpha$  in the range  $0 \leq \alpha < 1$ , and  $j$  of the form  $j = \frac{1}{2} + is$  for all real  $s$ .

## 5.4 The NCOS Limit

### The WZW Model Action

In the limit  $\psi_0 \rightarrow \infty$ , the  $AdS_2$  brane approaches the boundary of  $AdS_3$ . In this limit, the factor of  $\cosh^2 \psi$  in the metric (4.9) tends to suppress open string fluctuations along the  $AdS_2$  brane. However, the induced electric field on the brane also grows exponentially with  $\psi_0$ , balancing the effect of the metric. In fact, as  $\psi_0 \rightarrow \infty$ , the electric field approaches its critical value, and the theory resembles the noncommutative open strings studied in [22, 23]. Let us now make this more precise.

We work in a target space of Euclidean signature. Our first task is to find a system of coordinates that is well adapted to our problem. In the Euclidean  $AdS_2$  coordinates  $(\psi, \omega, t)$ , the  $AdS_3$  metric is

$$ds^2 = d\psi^2 + \cosh^2 \psi (\cosh^2 \omega dt^2 + d\omega^2), \quad (5.44)$$

and the invariant 2-form that describes the NS-NS background takes the form

$$\mathcal{F} = \left( \psi - \psi_0 + \frac{\sinh 2\psi}{2} \right) \cosh \omega d\omega \wedge dt. \quad (5.45)$$

It proves convenient to replace the  $AdS_2$  coordinates  $(\omega, t)$  by a complex coordinate  $\gamma$ , defined such that the  $AdS_2$  brane worldvolume is covered by the upper half plane  $\text{Im } \gamma \geq 0$ . The transformations are

$$\tanh \tau = \frac{|\gamma|^2 - 1}{|\gamma|^2 + 1}, \quad (5.46)$$

$$\sinh \omega = -\frac{\text{Re } \gamma}{\text{Im } \gamma}. \quad (5.47)$$

In these coordinates, the metric and 2-form become

$$ds^2 = d\psi^2 + \frac{\cosh^2 \psi}{(\text{Im } \gamma)^2} d\gamma d\bar{\gamma}, \quad (5.48)$$

$$\mathcal{F} = \frac{1}{2} \left( \psi - \psi_0 + \frac{\sinh 2\psi}{2} \right) \frac{d\gamma \wedge d\bar{\gamma}}{(\text{Im } \gamma)^2}. \quad (5.49)$$

Given these expressions for the metric and the 2-form, the WZW model Lagrangian takes the form

$$\mathcal{L} = \frac{k}{2\pi} \left[ 2\partial\psi\bar{\partial}\psi + \frac{\cosh^2 \psi}{(\text{Im } \gamma)^2} (\partial\gamma\bar{\partial}\bar{\gamma} + \bar{\partial}\gamma\partial\bar{\gamma}) - \left( \psi - \psi_0 + \frac{\sinh 2\psi}{2} \right) \frac{1}{(\text{Im } \gamma)^2} (\partial\gamma\bar{\partial}\bar{\gamma} - \bar{\partial}\gamma\partial\bar{\gamma}) \right]. \quad (5.50)$$

The worldsheet coordinates are  $(z, \bar{z})$ , and range over the upper half plane; the worldsheet metric is Euclidean; and the area element is  $d^2z \equiv dz d\bar{z}$ .

To take the limit  $\psi_0 \rightarrow \infty$ , we first redefine  $\psi \rightarrow \psi + \psi_0$ ; the new coordinate  $\psi$  measures deviations from the brane at  $\psi_0$ . The Lagrangian for large  $\psi_0$  is then

$$\mathcal{L} = \frac{k}{\pi} \partial\psi\bar{\partial}\psi + \frac{k}{2\pi(\text{Im } \gamma)^2} \left[ \frac{e^{2(\psi+\psi_0)}}{2} (\bar{\partial}\gamma\partial\bar{\gamma}) + \frac{1}{2} (\partial\gamma\bar{\partial}\bar{\gamma} + \bar{\partial}\gamma\partial\bar{\gamma}) - \psi (\partial\gamma\bar{\partial}\bar{\gamma} - \bar{\partial}\gamma\partial\bar{\gamma}) \right]. \quad (5.51)$$

It is useful to introduce Lagrange multipliers  $\beta$  and  $\bar{\beta}$ , giving

$$\mathcal{L} = \frac{k}{2\pi} \left[ 2\partial\psi\bar{\partial}\psi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \frac{2(\text{Im } \gamma)^2}{e^{2(\psi+\psi_0)} + 2} \beta\bar{\beta} + \frac{1}{(\text{Im } \gamma)^2} \left( \left( \frac{1}{2} - \psi \right) (\partial\gamma\bar{\partial}\bar{\gamma} - \bar{\partial}\gamma\partial\bar{\gamma}) \right) \right]. \quad (5.52)$$

We now redefine  $\psi \rightarrow \psi/2$  and take the limit  $\psi_0 \rightarrow \infty$ , with the result

$$S = \frac{k}{2\pi} \int d^2z \left( \frac{1}{2} \partial\psi\bar{\partial}\psi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + 2 \frac{\psi - 1}{(\gamma - \bar{\gamma})^2} (\partial\gamma\bar{\partial}\bar{\gamma} - \bar{\partial}\gamma\partial\bar{\gamma}) \right). \quad (5.53)$$

The action (5.53) resembles the action of noncommutative open strings in flat space. The only differences are that the metric in the  $(\gamma, \bar{\gamma})$  plane is that of  $AdS_2$ , and that there is a coupling to the extra scalar field  $\psi$ , which obeys the Dirichlet condition  $\psi = 0$  on the boundary.

## The Spectrum

Our next goal is to evaluate the Euclidean one-loop thermal partition function exactly and read off the physical string spectrum. As in Appendix B, the partition function may be written as a sum over a complete set of classical solutions and fluctuations about these solutions. Solving the WZW model equations of motion produces classical solutions, given as functions of  $(\psi, \gamma, \bar{\gamma})$ ; the fluctuations about these solutions can then be evaluated by the method of iterative Gaussians [28]. From the resulting expression for the partition function, we can determine which  $\widehat{SL}(2, R)$  representations appear in the WZW model spectrum, as discussed in Appendix B. The result is somewhat unsettling: we find that the Hilbert space contains only the discrete representation  $\widehat{\mathcal{D}}_j^{w=0}$ , for  $\frac{1}{2} < j < \frac{k-1}{2}$ .

This is clearly wrong: the existence of sectors of the Hilbert space with  $w \neq 0$  is guaranteed by the spectral flow symmetry. We understand their absence from the partition function calculation to mean that our set of classical solutions was incomplete. In particular, there seem to be solutions—short and long wound strings—that are not easily expressible in the  $(\psi, \gamma, \bar{\gamma})$  coordinates. It would be desirable to find all of the classical solutions and perform a complete computation of the one-loop free energy. Nevertheless, our computation captures part of the physical spectrum. Most important, our result implies that the spectrum contains discrete representations with  $\frac{1}{2} < j < \frac{k-1}{2}$ , which is a significant fact that cannot be seen at the semiclassical level. These bounds on  $j$  are the same as the bounds that come out of the  $\psi_0 = 0$  partition function calculation described in Appendix B. This strongly suggests that, for all  $\psi_0$ , the spectrum contains discrete representations obeying  $\frac{1}{2} < j < \frac{k-1}{2}$ .

The one-loop worldsheet is the semi-annulus in the upper half  $z$ -plane defined by the identification  $z \sim ze^{2\pi t}$ , where  $t$  is a worldsheet modulus. At finite temperature  $T$ , the  $AdS_3$  time coordinate is made periodic with period  $1/T$ ; in the coordinates we are using, this is accomplished by identifying  $\gamma \sim \gamma e^{1/T}$ . In order that the Lagrangian be single-valued, we must also identify  $\beta \sim \beta e^{-1/T}$ . These identifications mandate the relations

$$\gamma(ze^{2\pi t}) = e^{n/T}\gamma(z), \quad \beta(ze^{2\pi t}) = e^{-n/T}\beta(z), \quad (5.54)$$

for some integer  $n$ .

We begin by finding classical solutions for  $\gamma$  and  $\psi$ . The equation of motion for  $\beta$  is  $\bar{\partial}\gamma = 0$ . Classical solutions are of the form

$$\gamma_{cl}(z) = re^{i\theta} z^{\frac{n}{2\pi t T}}, \quad (5.55)$$

where  $re^{i\theta}$  is a complex constant. The range of the coordinate  $\gamma$  is  $\text{Im } \gamma \geq 0$ . This necessitates

$$t \geq \frac{|n|}{2\pi T}, \quad (5.56)$$

and also determines  $\theta$  in a manner detailed below.

Let us now expand about the classical solutions, defining fields  $\Gamma$  and  $B$  by

$$\gamma = \gamma_{cl}(1 + \Gamma), \quad \beta = \frac{B}{\gamma_{cl}}. \quad (5.57)$$

The partition function then involves functional integrals over  $B$  and  $\Gamma$ . Up to a constant, the  $SL(2, R)$ -invariant functional measure for these fields is

$$\mathcal{D} \left( \frac{e^{\psi/2} \gamma_{cl}}{\text{Im } \gamma_{cl}} \Gamma \right) \mathcal{D} \left( \frac{e^{-\psi/2} \text{Im } \gamma_{cl}}{\gamma_{cl}} B \right) \mathcal{D} \left( \frac{e^{\psi/2} \bar{\gamma}_{cl}}{\text{Im } \gamma_{cl}} \bar{\Gamma} \right) \mathcal{D} \left( \frac{e^{-\psi/2} \text{Im } \gamma_{cl}}{\bar{\gamma}_{cl}} \bar{B} \right). \quad (5.58)$$

Classically, this is equivalent to  $\mathcal{D}\Gamma \mathcal{D}B \mathcal{D}\bar{\Gamma} \mathcal{D}\bar{B}$ , but quantum mechanically, a chiral anomaly is present, which effectively shifts the Lagrangian by

$$-\frac{2}{\pi} \left[ \partial \log \left( \frac{e^{\psi/2} \gamma_{cl}}{\text{Im } \gamma_{cl}} \right) \bar{\partial} \log \left( \frac{e^{\psi/2} \bar{\gamma}_{cl}}{\text{Im } \bar{\gamma}_{cl}} \right) \right]. \quad (5.59)$$

When this term is expanded out and added to (5.53), the result is

$$\mathcal{L} = \frac{k-2}{2\pi} \left( \frac{1}{2} \partial \psi \bar{\partial} \psi + 2 \frac{\psi-1}{(\gamma_{cl} - \bar{\gamma}_{cl})^2} \partial \gamma_{cl} \bar{\partial} \bar{\gamma}_{cl} \right) + \frac{k}{2\pi} (B \bar{\partial} \Gamma + \bar{B} \partial \bar{\Gamma}) + F(\Gamma, \partial \Gamma, \bar{\Gamma}, \bar{\partial} \bar{\Gamma}), \quad (5.60)$$

where  $F$  contains terms from the Taylor expansion of the last term in (5.53). As these terms make no contribution to the partition function, we drop them in what follows. The effect of the chiral anomaly is therefore to shift the prefactor  $k$  of the terms in  $\psi$  to  $k-2$ .

Next we separate  $\psi$  into its classical and fluctuating parts, defining a new field  $\phi$  by  $\psi = \psi_{cl} + \phi$ , where  $\psi_{cl}$  satisfies the equation of motion

$$\partial \bar{\partial} \psi_{cl} = \frac{2 \partial \gamma_{cl} \bar{\partial} \bar{\gamma}_{cl}}{(\gamma_{cl} - \bar{\gamma}_{cl})^2} \quad (5.61)$$

derived from (5.60). One solution is

$$\psi_{cl} = 2 \log \text{Im } \gamma_{cl} + f(z) + \bar{f}(\bar{z}), \quad (5.62)$$

where  $f$  is an arbitrary holomorphic function. We can use the freedom of the choice of  $f$  to ensure that  $\psi_{cl}$  be single-valued and satisfy its Dirichlet boundary condition. Thus we may take

$$\psi_{cl} = \log \left[ \frac{\text{Im } \gamma_{cl}}{|\gamma_{cl}|} \right]^2 + c, \quad (5.63)$$

where the constant  $c$  is adjusted so that  $\psi_{cl} = 0$  when  $\text{Im } z = 0$ . Substituting in the form of  $\gamma_{cl}$  and letting  $z = x \in \mathbf{R}^+$ , we must have  $c = -2 \log(\sin \theta)$ . Letting  $z = -x$ , we must have  $c = -2 \log(\sin(\frac{n}{2Tt} + \theta))$ . Consistency then requires  $|\sin \theta| = |\sin(\frac{n}{2Tt} + \theta)|$ , which is satisfied (for  $n \neq 0$ ) if

$$\theta = \frac{\pi}{2} \left( m - \frac{n}{2\pi Tt} \right), \quad (5.64)$$

for some integer  $m$ . Since  $\gamma_{cl}(z)$  is constrained to lie in the upper half plane as  $z$  ranges over the worldsheet semi-annulus in the upper half  $z$ -plane, we must have  $0 \leq \arg(\gamma_{cl}(z)) = \theta + \frac{n}{2\pi T t} \arg(z) \leq \pi$ , whenever  $0 \leq \arg(z) \leq \pi$ . This is possible only if  $m = 1$ . Thus  $\theta = \frac{\pi}{2}(1 - \frac{n}{2\pi T t})$ . In summary, our classical solutions have the form

$$\gamma_{cl} = r e^{i\frac{\pi}{2}(1 - \frac{n}{2\pi T t})} z^{\frac{n}{2\pi T t}}, \quad (5.65)$$

$$\psi_{cl} = 2 \log \left| \frac{\text{Im } \gamma_{cl}}{\gamma_{cl} \cos(\frac{n}{4Tt})} \right|, \quad (5.66)$$

where  $t > \frac{|n|}{2\pi T}$  and  $r > 0$ .

Integration by parts brings the action into the form

$$S = \int d^2z \left[ \frac{k-2}{4\pi} (\partial\phi\bar{\partial}\phi + (\psi_{cl} - 2)\partial\bar{\partial}\psi_{cl}) + \frac{k}{2\pi} (B\bar{\partial}\Gamma + \bar{B}\partial\bar{\Gamma}) \right]. \quad (5.67)$$

The term involving  $\psi_{cl}$  can be easily integrated over the worldsheet semi-annulus, giving  $(k-2)n^2/8\pi T^2 t$ .

Now we evaluate the Euclidean partition function. The partition function breaks up as a sum  $\mathcal{Z} = \sum_n \mathcal{Z}_n$  over sectors indexed by the integer  $n$  that appears in the classical solutions (5.65) and (5.66). Each  $\mathcal{Z}_n$  contains an integral  $\int_{|n|/2\pi T}^{\infty} \frac{dt}{t}$  over the worldsheet modulus  $t$ ; the range of integration is set by (5.56). There is also an integral over the modulus  $r$ , but this contributes only a numerical factor. Finally, we must perform the functional integrals over  $\phi$ ,  $B$ ,  $\Gamma$ , the worldsheet ghosts, and the internal conformal field theory. Upon taking careful account of the boundary conditions for  $B$  and  $\Gamma$ , we find that the functional integral of the  $(B, \Gamma)$  system is exactly canceled by the functional determinant coming from the worldsheet ghosts. The remaining functional integral is over the free bosonic field  $\phi$ , which obeys the Dirichlet boundary condition  $\phi = 0$ . This is a standard functional determinant; the result is  $1/|\eta(it)|$ . Assembling all the pieces together, we have

$$\mathcal{Z}_n \sim \int_{|n|/2\pi T}^{\infty} \frac{dt}{t} \frac{1}{|\eta(it)|} \exp\left(-\frac{(k-2)n^2}{8\pi T^2 t}\right) q^{h - \frac{c_{int}}{24}} D(h), \quad (5.68)$$

where  $q = e^{-2\pi t}$ ,  $h$  indexes the weight in the internal conformal field theory,  $D(h)$  is the degeneracy at weight  $h$ , and  $c_{int}$  is the central charge of the internal conformal field theory.

As in [11] and Appendix B, it is sufficient for our purposes to look at the  $n = 1$  sector. The central charges  $c_{SL(2,R)}$  of the  $SL(2, R)$  conformal field theory and  $c_{int}$  of the internal conformal field theory must sum to 26; since  $c_{SL(2,R)} = 3 + \frac{6}{k-2}$ , we have  $c_{int} = 23 - \frac{6}{k-2}$ . Substituting this expression into (5.68), and expanding the factors in  $1/|\eta(it)| = e^{2\pi t/24} / \prod_{m=1}^{\infty} (1 - e^{-2\pi t m})$  as geometric sums, we may rewrite the integrand as a sum of terms containing exponentials of the form

$$\exp\left(-2\pi t(h + N - 1) - \frac{k-2}{8\pi T^2 t} - \frac{\pi t}{2(k-2)}\right), \quad (5.69)$$

where  $N$  is a non-negative integer. The dominant contribution to the partition function comes from the saddle point of the exponent,

$$t_s = \frac{k-2}{2\pi T \sqrt{1+4(k-2)(N+h-1)}}. \quad (5.70)$$

The lower bound of the  $t$  integral forces

$$\frac{1}{2\pi T} < t_s < \infty, \quad (5.71)$$

which translates into the bounds

$$0 < N+h-1 + \frac{1}{4(k-2)} < \frac{k}{4} - \frac{1}{2}. \quad (5.72)$$

This is identical to the inequality (79) of [12], with  $w=0$ . It was shown in [12] that (5.72) is equivalent to the bounds  $\frac{1}{2} < j < \frac{k-1}{2}$  on the  $SL(2, R)$  spin. By a chain of reasoning reviewed in Appendix B in the context of the corresponding straight brane calculation, we may conclude that the physical open string spectrum at  $\psi_0 = \infty$  contains the unwound discrete representation  $\hat{\mathcal{D}}_j^{+,w=0}$ , for all  $j$  obeying  $\frac{1}{2} < j < \frac{k-1}{2}$ .

As we remarked above, the appearance of the bounds  $\frac{1}{2} < j < \frac{k-1}{2}$  at  $\psi_0 = 0$  and at  $\psi_0 = \infty$  gives us cause to believe that the same bounds hold for all  $\psi_0$ . Nonetheless, this calculation is manifestly incomplete, since it misses the winding sectors. Presumably this is because our choice of coordinates is well suited to describing only a limited subset of the classical solutions. It would be interesting to find the remaining solutions and complete the calculation of the partition sum.

## 6 Summary and Discussion

We have studied the spectrum of open strings ending on  $AdS_2$  branes in  $AdS_3$  in an NS-NS background. Perturbative open string theory on an  $AdS_2$  brane is described by the  $SL(2, R)$  WZW model, subject to the boundary conditions (4.10) and (4.11), which state that the worldsheet ends on the  $AdS_2$  brane and satisfies Neumann boundary conditions in the directions parallel to the brane. The condition (4.10) also guarantees that the boundary condition preserves one copy of the  $\widehat{SL}(2, R)$  current algebra.

Our study of the open string spectrum has been modeled on the treatment of closed strings in  $AdS_3$  in [12]. The basic idea is to begin by studying classical solutions of the WZW model, beginning with the simplest solutions and building up more complicated ones by isometries and spectral flow, and, having compiled a complete catalogue of the classical solutions, to conjecture the form of the quantum Hilbert space. It was shown in [11] that this method leads to a correct proposal for the closed string WZW model Hilbert space.

We have applied this approach to open strings ending on  $AdS_2$  branes. As a warm-up and a useful point of comparison, we first looked at  $S^2$  branes in the  $SU(2)$  WZW model. It was known from conformal field theoretic analysis [16] that the Hilbert space of open strings stretched between two  $S^2$  branes at  $\psi_0 = \pi n_1/k$  and  $\psi_0 = \pi n_2/k$  is the sum of irreducible highest-weight representations of  $\widehat{SU}(2)$ , whose spin  $j$  is bounded as in (2.28). Our analysis of classical string worldsheets precisely reproduced this inequality. The only property of the quantum Hilbert space that could not be seen from our classical analysis was the quantization condition

$$2j + n_1 + n_2 + k \equiv 0 \pmod{2}. \quad (6.1)$$

We next considered strings ending on  $AdS_2$  branes in  $AdS_3$ . We began in section 4 by studying the straight brane located at  $\psi_0 = 0$ . Analysis of classical solutions led us to conjecture that the Hilbert space of the WZW model in the open string sector is the holomorphic square root of the closed string Hilbert space. In particular, the spectrum contains both short strings and long strings, and is invariant under spectral flow. We proved this conjecture in Appendix B by exactly evaluating the one-loop open string free energy, as in [11].

The situation of the brane with  $\psi_0 > 0$  is more interesting. The boundary condition (4.11) preserves only spectral flow with even  $w$ , so it was reasonable to expect differences in the  $\psi_0$  dependence of the spectra in the even and odd winding sectors. We found both short and long classical string solutions in all winding sectors, but also an important difference between the short and long strings. Short string solutions with different winding numbers can be mapped to one another, even when the difference in their winding numbers is odd, by combining spectral flow by an even amount and a PT transformation of the target space. Since both even spectral flow and the PT transformation are symmetries of the full quantum theory, we argued that the  $\psi_0$  dependence of the density of states of the short string solutions must be the same in all winding sectors.

On the other hand, we showed that the  $\psi_0$  dependence of the density of states of long string solutions is different in the odd and even winding sectors. We saw this difference explicitly by computing the energy of odd and even winding *short* strings as a function of the string size. In both cases, the energy rises to an asymptote, signaling the presence of a continuum above the asymptotic value. The rate of approach to the asymptote differs, though, in the odd and even winding sectors. Further evidence for the different behavior of long strings with odd and even winding number came from an analysis of the divergence structure of the Euclidean partition function.

In the limit  $\psi_0 \rightarrow \infty$ , the induced electric field on the worldvolume of the  $AdS_2$  brane reaches its critical value, producing noncommutative open string theory on  $AdS_2$ . In section 5.4, we calculated the worldsheet action for open strings in this limit, and obtained a result similar to that of [29] for noncommutative open strings in flat space [22, 23]. We carried

out a partial computation of the one-loop free energy in this limit, using the method of quadratures, as in [28].

Our work has focused exclusively on  $AdS_2$  branes, which preserve one copy of the  $SL(2, R)$  current algebra. There are other branes in  $AdS_3$ , some of which break the current algebra symmetry entirely [30]. It would be interesting to analyze the open string theory of these branes, as well.

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## A Coordinate Systems for $AdS_3$

The space  $AdS_3$  is defined as the hyperboloid

$$(X^0)^2 - (X^1)^2 - (X^2)^2 + (X^3)^2 = R^2, \quad (\text{A.1})$$

embedded in  $\mathbf{R}^{2,2}$ . The metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 - (dX^3)^2 \quad (\text{A.2})$$

on  $\mathbf{R}^{2,2}$  induces a metric of constant negative curvature on  $AdS_3$ . The quantity  $R$  that appears in (A.1) is the *anti-de Sitter radius*; for convenience, we set  $R = 1$ . In addition, to avoid closed timelike curves, we work not with the hyperboloid (A.1) itself, but with its universal cover.

The two coordinate systems we use most extensively are *global coordinates* and  $AdS_2$  *coordinates*. The global coordinates  $(\rho, \theta, \tau)$  are defined by

$$X^0 + iX^3 = \cosh \rho e^{it}, \quad X^1 + iX^2 = -\sinh \rho e^{-i\theta}. \quad (\text{A.3})$$

The range of the radial coordinate  $\rho$  is  $0 \leq \rho < \infty$ ; the angular coordinate  $\theta$  ranges over  $0 \leq \theta < 2\pi$ ; and the global time coordinate  $t$  may be any real number. The  $AdS_3$  metric in global coordinates is

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2. \quad (\text{A.4})$$

The  $AdS_2$  coordinates  $(\psi, \omega, t)$  are particularly well adapted to the  $AdS_2$  branes we consider in sections 4 and 5. They are defined by

$$X^1 = \cosh \psi \sinh \omega, \quad X^2 = \sinh \psi, \quad X^0 + iX^3 = \cosh \psi \cosh \omega e^{it}. \quad (\text{A.5})$$



All three  $AdS_2$  coordinates range over the entire real line. In this parametrization, the fixed  $\psi$  slices have the geometry of  $AdS_2$ . The  $AdS_3$  metric in  $AdS_2$  coordinates takes the form

$$ds^2 = d\psi^2 + \cosh^2 \psi (-\cosh^2 \omega dt^2 + d\omega^2); \quad (\text{A.6})$$

the quantity in parentheses is the metric of the  $AdS_2$  subspace at fixed  $\psi$ . The transformation between global and  $AdS_2$  coordinates is

$$\sinh \psi = \sin \theta \sinh \rho, \quad \cosh \psi \sinh \omega = -\cos \theta \sinh \rho. \quad (\text{A.7})$$

The global time  $t$  is the same in both coordinate systems.

The space  $AdS_3$  is the group manifold of the group  $SL(2, R)$ . A point in  $AdS_3$  is given by the  $SL(2, R)$  matrix

$$g = \begin{pmatrix} X^0 + X^1 & X^2 + X^3 \\ X^2 - X^3 & X^0 - X^1 \end{pmatrix}. \quad (\text{A.8})$$

In the global coordinate system,

$$g = \begin{pmatrix} \cos t \cosh \rho - \cos \theta \sinh \rho & \sin t \cosh \rho + \sin \theta \sinh \rho \\ -\sin t \cosh \rho + \sin \theta \sinh \rho & \cos t \cosh \rho + \cos \theta \sinh \rho \end{pmatrix}. \quad (\text{A.9})$$

## B A Partition Function Calculation of the Open String Spectrum

In this appendix, we explicitly verify the proposal presented in section 4.2 for the open string spectrum. First, in section B.1, we compute the worldsheet one-loop partition function  $\mathcal{Z}$  for open string theory on Euclidean  $AdS_3$  at finite temperature  $1/\beta$ . The partition function is proportional to the single-particle contribution to the spacetime free energy,  $\mathcal{Z} = -\beta F$ . The free energy, in turn, can be written as

$$F = \frac{1}{\beta} \sum_{s \in \mathcal{H}} \log(1 - e^{-\beta E_s}), \quad (\text{B.1})$$

where the sum is over states  $s$  in the physical Hilbert space  $\mathcal{H}$  of single-string states, and  $E_s$  is the energy of the state  $s$ .<sup>12</sup> By writing  $\mathcal{Z}$  in the right form, we can thus read off the spectrum of open strings in (Lorentzian)  $AdS_3$ . We show in section B.2 that the spectrum breaks up into a sum over discrete states and an integral over a continuum, with energies agreeing with the expressions found in section 4. Moreover, we compute the density of states of the continuum.

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<sup>12</sup>We work at zero chemical potential.

Our calculation is patterned on the one done in [11] for closed strings in  $AdS_3$ . Especially in section B.1, we emphasize here those features that are novel in the open string case; the reader seeking greater detail is directed to [11] and the references therein.

One point is worth clarifying at the outset. Though the free energy whose form we undertake to calculate receives contributions only from states in the physical Hilbert space of the string, our calculation is sufficient to confirm our proposal for the spectrum of the WZW model. The physical Hilbert space is the tensor product of a Hilbert space of  $AdS_3$  excitations and a Hilbert space associated with the “internal” manifold  $\mathcal{M}$ . For our purposes, we can take the spectrum of the internal conformal field theory to be arbitrary. One of the physical state conditions is  $L_0 + h = 1$ , where  $L_0$  is the zeroth Virasoro generator of the  $AdS_3$  conformal field theory, and  $h$  is a conformal weight in the conformal field theory on  $\mathcal{M}$ . This condition can be seen as parametrizing the spectrum of  $L_0$  in the WZW model. The remaining physical state conditions,  $L_n + L_n^{\mathcal{M}} = 0$ , with  $n \geq 1$ , relate the Virasoro generators in the  $AdS_3$  and internal conformal field theories. They can be solved within the tensor product of an irreducible representation of  $\widehat{SL}(2, R)$  with some subspace of the internal conformal field theory state space. Therefore, given the physical string spectrum in  $AdS_3$ , it is possible to deduce how the Hilbert space of the WZW model is decomposed into a sum of irreducible representations of  $\widehat{SL}(2, R)$ . This is why the one-loop free energy computation below, though it is carried out in the physical string Hilbert space, is nonetheless relevant to the spectrum of the WZW model.

## B.1 The One-loop Partition Function

Our first business is to write the WZW action for Euclidean  $AdS_3$  at finite temperature and the boundary conditions appropriate to a flat  $AdS_2$  brane. We define the coordinates  $(v, \bar{v}, \phi)$  on Euclidean  $AdS_3$  by

$$v = \sinh \rho e^{i\theta}, \tag{B.2}$$

$$\bar{v} = \sinh \rho e^{-i\theta}, \tag{B.3}$$

$$\phi = t - \log \cosh \rho, \tag{B.4}$$

where  $(\rho, \theta, t)$  are global coordinates. The metric in these coordinates is [28]

$$ds^2 = k \left( d\phi^2 + (dv + v d\phi)(d\bar{v} + \bar{v} d\phi) \right), \tag{B.5}$$

where  $k$  is the square of the anti-de Sitter radius, and is identified with the level of the WZW model. Euclidean  $AdS_3$  is the coset manifold  $SL(2, C)/SU(2)$ ; in the coordinates (B.2), a general element is written as

$$g = \begin{pmatrix} e^{\phi}(1 + |v|^2) & v \\ \bar{v} & e^{-\phi} \end{pmatrix}. \tag{B.6}$$

Thermal  $AdS_3$  is given by the identification

$$t \sim t + \beta, \quad (\text{B.7})$$

where  $\beta$  is the inverse temperature. In the coordinates (B.2), this translates to

$$\phi \sim \phi + \beta. \quad (\text{B.8})$$

The WZW action in the coordinates (B.2) is<sup>13</sup>

$$S = \frac{k}{\pi} \int d^2z \left( \partial\phi \bar{\partial}\phi + (\partial\bar{v} + \partial\phi\bar{v})(\bar{\partial}v + \bar{\partial}\phi v) \right). \quad (\text{B.9})$$

Throughout this calculation, we take the worldsheet to be Euclidean. In addition, we alternate between real  $(\sigma, \tau)$  and complex conjugate  $(z, \bar{z})$  worldsheet coordinates. The relation between the two sets is

$$z = \sigma + i\tau, \quad \partial \equiv \frac{\partial}{\partial z}. \quad (\text{B.10})$$

and similarly for  $\bar{z}$  and  $\bar{\partial}$ .

In the coordinates (B.2), the boundary conditions suitable for open strings ending on a straight  $AdS_2$  brane are

$$v_2 = 0, \quad (\text{B.11})$$

$$\partial_\sigma v_1 = 0, \quad (\text{B.12})$$

$$\partial_\sigma \phi = 0, \quad (\text{B.13})$$

where  $v_1$  and  $v_2$  are, respectively, the real and imaginary parts of  $v$ .

The one-loop open string partition function is obtained by considering worldsheets with the topology of a cylinder. As usual,  $0 \leq \sigma \leq \pi$ . The worldsheet is made cylindrical by imposing the periodicity

$$\tau \sim \tau + 2\pi t. \quad (\text{B.14})$$

Here (and henceforth)  $t$  is simply a modulus, and is not to be confused with the global time coordinate. The spacetime periodicity  $\phi \sim \phi + \beta$  of thermal  $AdS_3$  implies

$$\phi(\sigma, \tau + 2\pi t) = \phi(\sigma, \tau) + \beta n, \quad (\text{B.15})$$

for some integer  $n$ . If we define

$$u_n = \frac{n\tau}{4\pi t} \quad (\text{B.16})$$

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<sup>13</sup>The action given here differs from the WZW model Lagrangian given in section 5.4 by boundary terms that can be ignored only if the straight brane boundary condition (B.11) holds. If the  $AdS_2$  brane is curved, these terms must be included. The action then becomes considerably more complicated, and loses some of the special properties that make possible the partition function calculation described below.

and

$$\hat{\phi} = \phi - 2\beta u_n, \quad (\text{B.17})$$

then  $\hat{\phi}$  is periodic in  $\tau$ , *i.e.*,  $\hat{\phi}(\tau + 2\pi t) = \hat{\phi}(\tau)$ . The WZW action may be written in terms of  $\hat{\phi}$  as

$$S = \frac{k\beta^2 n^2}{8\pi t} + \frac{k}{\pi} \int d^2 z \left( |\partial \hat{\phi}|^2 + |(\partial - iu_n + \partial \hat{\phi})\bar{v}|^2 \right). \quad (\text{B.18})$$

The partition function for Euclidean  $AdS_3$  is

$$\mathcal{Z}_n(\beta) \equiv \int \mathcal{D}\hat{\phi} \mathcal{D}v \mathcal{D}\bar{v} e^{-S}, \quad (\text{B.19})$$

summed over  $n$ . The rest of this subsection is devoted to evaluating the functional integrals in (B.19).

The second term in parentheses in (B.18) couples  $\phi$ ,  $v$ , and  $\bar{v}$ . The field  $\hat{\phi}$  can be disentangled from  $v$  and  $\bar{v}$  using a standard chiral gauge transformation and the chiral anomaly formulae familiar from the closed string calculation. This procedure is valid in the open string case as well, since the string boundary conditions are left invariant by the chiral transformation. The partition function then factorizes into a functional integral over  $\hat{\phi}$  and a functional integral over  $v$  and  $\bar{v}$ , multiplied by the constant  $e^{-k\beta^2 n^2/8\pi t}$  coming from the constant term in the action. The  $\hat{\phi}$  functional integral is standard [31], and, up to normalization, yields

$$\mathcal{Z}_{\hat{\phi}} = \frac{\beta(k-2)^{1/2}}{t^{1/2}|\eta(it)|}, \quad (\text{B.20})$$

where  $\eta$  is the Dedekind eta function. The remaining functional integral may be written as

$$\mathcal{Z}_v = \int \mathcal{D}v \mathcal{D}\bar{v} e^{-S_v}, \quad (\text{B.21})$$

where

$$S_v = \frac{k}{\pi} \int d^2 z |(\partial - iu_n)\bar{v}|^2. \quad (\text{B.22})$$

Integrating  $S_v$  by parts gives

$$S_v = -\frac{k}{\pi} \left( \int_{\Sigma} d^2 z \bar{v} (\partial + iu_n) (\bar{\partial} + iu_n) v + \int_{\partial\Sigma} d\bar{z} v_1 \partial_{\sigma} v_2 - u_n \int_{\partial\Sigma} d\bar{z} v_1 v_1 \right), \quad (\text{B.23})$$

where  $\Sigma$  denotes the worldsheet cylinder. Note that the two boundary terms are pure imaginary.

Let us work on the bulk term in (B.23). We begin by expanding  $v_1$  and  $v_2$  in a complete basis of functions. The boundary conditions (B.11) and (B.12) dictate the expansions

$$v_1(\sigma, \tau) = \sum_{M \geq 0, N \in \mathbf{Z}} a_{MN} \frac{1}{\pi\sqrt{2t}} \cos M\sigma \psi_N(\tau/t), \quad (\text{B.24})$$

$$v_2(\sigma, \tau) = \sum_{M > 0, N \in \mathbf{Z}} b_{MN} \frac{1}{\pi\sqrt{2t}} \sin M\sigma \psi_N(\tau/t), \quad (\text{B.25})$$

where  $a_{MN}$  and  $b_{MN}$  are real-valued coefficients, and  $\psi_N$  is defined to be  $\cos(N\tau/t)$  for  $N \geq 0$  and  $\sin(N\tau/t)$  for  $N < 0$ . It follows that

$$v = v_1 + iv_2 = \sum_{M,N \in \mathbf{Z}} v_{MN} \frac{1}{\pi\sqrt{2t}} e^{iM\sigma} \psi_N(\tau/t), \quad (\text{B.26})$$

where

$$a_{MN} = (v_{MN} + v_{-MN})/2, \quad (\text{B.27})$$

$$b_{MN} = (v_{MN} - v_{-MN})/2. \quad (\text{B.28})$$

The advantage of this rewriting is that  $e^{iM\sigma} \psi_N(\tau/t)$  is an eigenfunction of the operator  $\Delta_v = -\frac{k}{\pi}(\partial + iu_n)(\bar{\partial} + iu_n)$  that appears in the bulk term of (B.23), with eigenvalue  $\lambda_{MN} = -((M + \hat{u}_n)^2 + (N/t)^2)$ . If we substitute the expansion (B.26) of  $v$  into (B.23), then, expressing  $S_v$  in terms of the worldsheet coordinates  $(\sigma, \tau)$ , we can immediately integrate over  $\tau$ , to obtain

$$\begin{aligned} S_v &= \frac{1}{\pi} \int_0^\pi d\sigma \sum_{M',M,N \in \mathbf{Z}} v_{M'N} v_{MN} \lambda_{MN} [\cos M'\sigma \cos M\sigma + \sin M'\sigma \sin M\sigma \\ &\quad + i(\cos M'\sigma \sin M\sigma - \sin M'\sigma \cos M\sigma)] + \text{boundary terms}, \end{aligned} \quad (\text{B.29})$$

up to a normalization factor. Since  $S_v$  is positive-definite,  $v_{MN}$  and  $\lambda_{MN}$  are real constants, and the boundary terms are pure imaginary, the imaginary part of the bulk term must cancel with the boundary terms. We are then left with

$$\begin{aligned} S_v &= \frac{1}{2} \sum_{M',M,N \in \mathbf{Z}} v_{M'N} v_{MN} \lambda_{MN} (\delta_{M',M} + \delta_{M',-M} + \delta_{M',M} - \delta_{M',-M}) \\ &= \sum_{M,N \in \mathbf{Z}} v_{MN} v_{M,N} \lambda_{M,N}. \end{aligned} \quad (\text{B.30})$$

The functional integral is a product of Gaussians, and may be evaluated by standard methods. Up to a constant,

$$\mathcal{Z}_v = \prod_{M,N \in \mathbf{Z}} \frac{1}{\sqrt{(M + 2u_n)^2 + (N/t)^2}}, \quad (\text{B.31})$$

which may be zeta-function regularized [32] to give

$$\mathcal{Z}_v^{-1} = \left| e^{-4\pi u_n^2 t} \frac{\vartheta_1(-2itu_n, it)}{\eta(it)} \right|, \quad (\text{B.32})$$

where  $\vartheta_1$  is a Jacobi theta function.

We have now obtained expressions for all of the factors entering into  $\mathcal{Z}_n(\beta; t)$ . The overall normalization of  $\mathcal{Z}_n$  is fixed in the usual way, by examining the infrared limit. Putting

everything together, we obtain

$$\begin{aligned}
\mathcal{Z}_n(\beta; t) &= \frac{1}{4\pi\sqrt{2t}} \frac{\beta(k-2)^{1/2} e^{-k\beta^2 n^2/8\pi t} e^{4\pi u_n^2 t}}{|\vartheta_1(-2itu_n, it)|} \\
&= \frac{1}{4\pi\sqrt{2t}} \frac{\beta(k-2)^{1/2}}{\sinh(\beta n/2)} \frac{e^{-(k-2)\beta^2 n^2/8\pi t} e^{\pi t/4}}{[\prod_{m=1}^{\infty} (1 - e^{-2\pi t m})(1 - e^{-2\pi t m} e^{\beta n})(1 - e^{-2\pi t m} e^{-\beta n})]}.
\end{aligned} \tag{B.33}$$

The partition function we have calculated is that of a conformal field theory with Euclidean  $AdS_3$  as its target space, but our physical open string theory contains more: we must incorporate the contributions of the  $(b, c)$  ghosts as well as those of the “internal” conformal field theory. In addition, we must integrate over the worldsheet modulus  $t$ . When this is done, the partition function becomes

$$\begin{aligned}
\mathcal{Z}(\beta) &= \frac{\beta(k-2)^{1/2}}{4\sqrt{2\pi}} \sum_{m=1}^{\infty} \int_0^{\infty} \frac{dt}{t^{3/2}} e^{2\pi t(1 - \frac{1}{4(k-2)})} \sum_h D(h) e^{-2\pi t h} \frac{e^{-(k-2)\beta^2 m^2/8\pi t}}{\sinh(\beta m/2)} \times \\
&\quad \prod_{n=1}^{\infty} \left| \frac{1 - e^{-2\pi t n}}{(1 - e^{-2\pi t n + \beta m})(1 - e^{-2\pi t n - \beta m})} \right|.
\end{aligned} \tag{B.34}$$

Here  $h$  indexes the weight in the internal conformal field theory, and  $D(h)$  is the degeneracy at weight  $h$ .

## B.2 The Spectrum

Having calculated the partition function (B.34), we now massage it into a form from which we can read off the spectrum. We noted at the beginning of this appendix that the partition function  $\mathcal{Z}$  is proportional to the the free energy

$$F = \frac{1}{\beta} \sum_{s \in \mathcal{H}} \log(1 - e^{-\beta E_s}) = \sum_{m=1}^{\infty} \sum_{s \in \mathcal{H}} \frac{1}{m\beta} e^{-m\beta E_s}. \tag{B.35}$$

The partition function is likewise a sum over  $m$  of a function of  $m\beta$ . It suffices, then, to compare the  $m = 1$  terms of the two expressions. In this subsection, we verify that  $E_s$ , extracted from the identification

$$\begin{aligned}
\sum_{s \in \mathcal{H}} \frac{1}{\beta} e^{-\beta E_s} &= \frac{\beta(k-2)^{1/2}}{4\sqrt{2\pi}} \int_0^{\infty} \frac{dt}{t^{3/2}} e^{2\pi t(1 - \frac{1}{4(k-2)})} \sum_h D(h) e^{-2\pi t h} \frac{e^{-(k-2)\beta^2/8\pi t}}{\sinh(\beta/2)} \times \\
&\quad \prod_{n=1}^{\infty} \left| \frac{1 - e^{-2\pi t n}}{(1 - e^{-2\pi t n + \beta})(1 - e^{-2\pi t n - \beta})} \right|,
\end{aligned} \tag{B.36}$$

agrees with the string spectrum proposed in section 4.

To aid us in carrying out the  $t$  integral, let us introduce a new variable  $c$ , defined by

$$e^{-(k-2)\beta^2/8\pi t} = -\frac{8\pi i}{\beta} \left(\frac{2t}{k-2}\right)^{3/2} \int_{-\infty}^{\infty} dc c e^{-\frac{8\pi t}{k-2}c^2 + 2i\beta c}. \quad (\text{B.37})$$

As explained in [11], the right-hand side of (B.34) can be expressed as a summation of terms of the form

$$\begin{aligned} & \frac{-4i}{\beta(k-2)} \int_{-\infty}^{\infty} dc c \int_{\frac{\beta}{2\pi(w+1)}}^{\frac{\beta}{2\pi w}} dt \\ & \times \exp \left[ -\beta \left( q + w + \frac{1}{2} \right) + 2ic\beta - 2\pi t \left( h + N_w + \frac{4c^2 + \frac{1}{4}}{k-2} - \frac{w(w+1)}{2} - 1 \right) \right] \\ & = \frac{-2i}{\pi\beta} \int_{-\infty}^{\infty} dc c \frac{\exp \left[ 2ic\beta - \beta \left( q + w + \frac{1}{2} \right) \right]}{-2\pi \left( h + N_w + \frac{4c^2 + \frac{1}{4}}{k-2} - \frac{w(w+1)}{2} - 1 \right)} \\ & \times \left\{ -\exp \left[ -\frac{\beta}{w} \left( h + N_w - 1 + \frac{4c^2 + \frac{1}{4}}{k-2} - \frac{w(w+1)}{2} \right) \right] \right. \\ & \left. + \exp \left[ -\frac{\beta}{w+1} \left( h + N_w - 1 + \frac{4c^2 + \frac{1}{4}}{k-2} - \frac{w(w+1)}{2} \right) \right] \right\} \end{aligned} \quad (\text{B.38})$$

where  $w$  ranges over non-negative integers. We can complete the square of the exponent in the first term (the fourth line) of (B.38) by letting  $c = s + \frac{i}{4}(k-2)w$ . Let us think of the  $c$  integral as an integration over a contour (as it happens, the real line) in the complex plane. We may then shift the contour of integration in the first term of (B.38) to  $c = s + \frac{i}{4}(k-2)w$ , and the contour of integration in the second term to  $c = s + \frac{i}{4}(k-2)(w+1)$ , where  $s$  in both cases runs over the real line. In doing so, the contour of integration crosses some poles in the integrand, and the integral picks up the residues of these poles. The residues of the poles from the first term are partially cancelled by the residues of the poles from the second. The net result of the contour shift is to pick up only the poles in the range

$$\frac{(k-2)}{4}w < \text{Im } c < \frac{(k-2)}{4}(w+1). \quad (\text{B.39})$$

Their residues are

$$\frac{1}{\beta} \exp \left[ -\beta q - \beta \left( \frac{1}{2} + w + \sqrt{\frac{1}{4} + (k-2) \left( N_w + h - 1 - \frac{1}{2}w(w+1) \right)} \right) \right]. \quad (\text{B.40})$$

The coefficient of  $-\beta$  in the exponent is supposed to be the energy of a typical state in the discrete spectrum. Considerations similar to those given in [12] for closed strings show that (4.17) (with the minus sign chosen) indeed takes the form (B.40) after the physical state conditions are imposed. Our partition function calculation thus reproduces the discrete spectrum of open strings in the physical Hilbert space.

We now turn our attention to the  $s$  integration. It is convenient to rearrange the sum in (B.38) by redefining  $w \rightarrow w - 1$  in the second term and by deforming the contours in both terms to  $c = s + \frac{i}{4}(k - 2)w$ . The result is

$$\begin{aligned} & \frac{1}{2\pi i\beta} \int_{-\infty}^{\infty} ds \left( \frac{4s}{(k-2)w} + i \right) \\ & \times \left\{ \frac{\exp \left[ -\beta q - \beta \left( \frac{kw}{4} + \frac{1}{w} \left( \frac{4s^2 + \frac{1}{4}}{k-2} + N_{w-1} + h - 1 \right) \right) \right]}{\frac{1}{4} + is - \frac{kw}{8} + \frac{1}{2w} \left( N_{w-1} + h - 1 + \frac{4s^2 + \frac{1}{4}}{k-2} \right)} \right. \\ & \left. - \frac{\exp \left[ -\beta q - \beta \left( \frac{kw}{4} + \frac{1}{w} \left( \frac{4s^2 + \frac{1}{4}}{k-2} + N_w + h - 1 \right) \right) \right]}{-\frac{1}{4} + is - \frac{kw}{8} + \frac{1}{2w} \left( N_w + h - 1 + \frac{4s^2 + \frac{1}{4}}{k-2} \right)} \right\}. \end{aligned} \quad (\text{B.41})$$

Let us consider the third line of (B.41). It can be shown [11] that summing over terms of this type yields

$$\frac{1}{2\pi i\beta} \int_{-\infty}^{\infty} ds \left( i + \frac{4s}{(k-2)w} \right) \left( 2 \log \epsilon + \left( i + \frac{4s}{w(k-2)} \right)^{-1} \frac{d}{ds} \log \Gamma \left( \frac{1}{2} - 2is - \tilde{M} \right) \right) e^{-\beta f(s)}, \quad (\text{B.42})$$

where

$$\tilde{M} = \frac{1}{w} \left( \frac{4s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right) - \frac{kw}{4}, \quad (\text{B.43})$$

$$f(s) = \frac{kw}{4} + \frac{1}{w} \left( \frac{4s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right), \quad (\text{B.44})$$

$\tilde{N} = qw + N_w$ , and  $\epsilon$  is a cutoff introduced to regularize a divergence that arises in the sum. Similarly, summing over terms in the form of the second line of (B.41) gives

$$\frac{1}{2\pi i\beta} \int_{-\infty}^{\infty} ds \left( i + \frac{4s}{(k-2)w} \right) \left( 2 \log \epsilon - \left( i + \frac{4s}{w(k-2)} \right)^{-1} \frac{d}{ds} \log \Gamma \left( \frac{1}{2} + 2is + \tilde{M} \right) \right) e^{-\beta f(s)}. \quad (\text{B.45})$$

Combining these results and making the change of variables  $s \rightarrow \frac{s}{2}$ , we find that (B.41) can be written in the form

$$\frac{2}{\beta} \int_0^{\infty} ds \rho(s) \exp[-\beta E(s)], \quad (\text{B.46})$$

where

$$\rho(s) = \frac{1}{2\pi} 2 \log \epsilon + \frac{1}{2\pi i} \frac{1}{2} \frac{d}{ds} \log \left( \frac{\Gamma(\frac{1}{2} - is + \tilde{m}) \Gamma(\frac{1}{2} - is - \tilde{m})}{\Gamma(\frac{1}{2} + is + \tilde{m}) \Gamma(\frac{1}{2} + is - \tilde{m})} \right), \quad (\text{B.47})$$



$$E(s) = \frac{k w}{4} + \frac{1}{w} \left( \frac{s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right), \quad (\text{B.48})$$

$$\tilde{m} = \frac{1}{w} \left( \frac{s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right) - \frac{k w}{4}. \quad (\text{B.49})$$

Here  $\rho(s)$  and  $E(s)$  are the density of states and energy of the long strings. The expression (B.48) is exactly what we would find if we imposed the physical state conditions on the form (4.17) given in section 4.1 for the long string energy.

Thus, by analyzing the partition function, we have reproduced our conjecture for the spectrum of the straight brane. The result is summarized by writing the free energy summand  $f(\beta)$  as

$$f(\beta) = \frac{1}{\beta} \sum D(h, \tilde{N}, w) \left[ \sum_q e^{-\beta E(q)} + \int ds \rho(s) e^{-\beta E(s)} \right], \quad (\text{B.50})$$

where  $E(q)$ ,  $E(s)$ , and  $\rho(s)$  are the discrete state energy, the continuum state energy, and the continuum density of states.

## References

- [1] A. M. Polyakov, “Quantum Geometry of Bosonic Strings,” *Phys. Lett.* **B103** (1981) 207–210.
- [2] J.-L. Gervais and A. Neveu, “The Dual Spectrum in Polyakov’s Quantization. 1.,” *Nucl. Phys.* **B199** (1982) 59.
- [3] J.-L. Gervais and A. Neveu, “Dual String Spectrum in Polyakov’s Quantization. 2. Mode Separation,” *Nucl. Phys.* **B209** (1982) 125.
- [4] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, “Fractal Structure of 2d Quantum Gravity,” *Mod. Phys. Lett.* **A3** (1988) 819.
- [5] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” *Nucl. Phys.* **B311** (1988) 46.
- [6] E. Witten, “On String Theory and Black Holes,” *Phys. Rev.* **D44** (1991) 314–324.
- [7] C. G. Callan, J. A. Harvey, and A. Strominger, “Supersymmetric String Solitons,” [hep-th/9112030](#).
- [8] S. Mukhi and C. Vafa, “Two-Dimensional Black Hole as a Topological Coset Model of  $c = 1$  String Theory,” *Nucl. Phys.* **B407** (1993) 667–705, [hep-th/9301083](#).

- [9] D. Ghoshal and C. Vafa, “ $c = 1$  String as the Topological Theory of the Conifold,” *Nucl. Phys.* **B453** (1995) 121–128, [hep-th/9506122](#).
- [10] H. Ooguri and C. Vafa, “Two-Dimensional Black Hole and Singularities of CY Manifolds,” *Nucl. Phys.* **B463** (1996) 55–72, [hep-th/9511164](#).
- [11] J. Maldacena, H. Ooguri, and J. Son, “Strings in  $AdS_3$  and the  $SL(2, R)$  WZW Model. Part 2: Euclidean Black Hole,” [hep-th/0005183](#).
- [12] J. Maldacena and H. Ooguri, “Strings in  $AdS_3$  and the  $SL(2, R)$  WZW Model. Part 1: The Spectrum,” [hep-th/0001053](#).
- [13] C. Klimčík and P. Severa, “Open Strings and D-Branes in WZNW Models,” *Nucl. Phys.* **B488** (1997) 653–676, [hep-th/9609112](#).
- [14] M. Kato and T. Okada, “D-Branes on Group Manifolds,” *Nucl. Phys.* **B499** (1997) 583–595, [hep-th/9612148](#).
- [15] A. Y. Alekseev and V. Schomerus, “D-Branes in the WZW Model,” *Phys. Rev.* **D60** (1999) 061901, [hep-th/9812193](#).
- [16] A. Y. Alekseev, A. Recknagel, and V. Schomerus, “Non-Commutative World-Volume Geometries: Branes on  $SU(2)$  and Fuzzy Spheres,” *JHEP* **09** (1999) 023, [hep-th/9908040](#).
- [17] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, “The Geometry of WZW Branes,” *J. Geom. Phys.* **34** (2000) 162–190, [hep-th/9909030](#).
- [18] S. Stanciu, “D-Branes in Group Manifolds,” *JHEP* **01** (2000) 025, [hep-th/9909163](#).
- [19] C. Bachas, M. Douglas, and C. Schweigert, “Flux Stabilization of D-Branes,” *JHEP* **05** (2000) 048, [hep-th/0003037](#).
- [20] J. Pawelczyk, “ $SU(2)$  WZW D-Branes and Their Noncommutative Geometry from DBI Action,” *JHEP* **08** (2000) 006, [hep-th/0003057](#).
- [21] A. Y. Alekseev, A. Recknagel, and V. Schomerus, “Brane Dynamics in Background Fluxes and Non-Commutative Geometry,” *JHEP* **05** (2000) 010, [hep-th/0003187](#).
- [22] R. Gopakumar, J. Maldacena, S. Minwalla, and A. Strominger, “S-Duality and Noncommutative Gauge Theory,” *JHEP* **06** (2000) 036, [hep-th/0005048](#).
- [23] N. Seiberg, L. Susskind, and N. Toumbas, “Strings in Background Electric Field, Space/Time Noncommutativity and a New Noncritical String Theory,” *JHEP* **06** (2000) 021, [hep-th/0005040](#).

- [24] P. M. Petropoulos and S. Ribault, “Some Remarks on Anti-de Sitter D-Branes,” [hep-th/0105252](#).
- [25] A. Giveon, D. Kutasov, and A. Schwimmer, “Comments on D-branes in  $AdS_3$ ,” [hep-th/0106005](#).
- [26] S. Stanciu, “D-Branes in an  $AdS_3$  Background,” *JHEP* **09** (1999) 028, [hep-th/9901122](#).
- [27] C. Bachas and M. Petropoulos, “Anti-de-Sitter D-Branes,” *JHEP* **02** (2001) 025, [hep-th/0012234](#).
- [28] K. Gawędzki, “Noncompact WZW Conformal Field Theories,” [hep-th/9110076](#).
- [29] I. R. Klebanov and J. Maldacena, “1 + 1 Dimensional NCOS and its  $U(N)$  Gauge Theory Dual,” *Int. J. Mod. Phys. A* **16** (2001) 922, *Adv. Theor. Math. Phys.* **4** (2001) 283, [hep-th/0006085](#). J. Gomis and H. Ooguri, “Non-Relativistic Closed String Theory,” [hep-th/0009181](#). U. H. Danielsson, A. Guijosa, and M. Kruczenski, “IIA/B, Wound and Wrapped,” *JHEP* **0010** (2000) 020, [hep-th/0009182](#). U. H. Danielsson, A. Guijosa, and M. Kruczenski, “Newtonian Gravitons and D-Brane Collective Coordinates in Wound String Theory,” *JHEP* **0103** (2001) 041, [hep-th/0012183](#).
- [30] J. Maldacena, G. Moore, and N. Seiberg, “Geometrical Interpretation of D-Branes in Gauged WZW Models,” [hep-th/0105038](#).
- [31] J. Polchinski, *String Theory*, vol. 1. Cambridge Univ. Press, Cambridge, 1998.
- [32] D. B. Ray and I. M. Singer, “Analytic Torsion for Complex Manifolds,” *Ann. Math.* **98** (1973) 154.