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Exactness and maximal automorphic factors of unimodal interval maps

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Abstract. We study exactness and maximal automorphic factors of C^3 unimodal maps of the interval. We show that for a large class of infinitely renormalizable maps, the maximal automorphic factor is an odometer with an ergodic non-singular measure. We give conditions under which maps with absorbing Cantor sets have an irrational rotation on a circle as a maximal automorphic factor, as well as giving exact examples of this type. We also prove that every C^3 S-unimodal map with no attractor is exact with respect to Lebesgue measure. Additional results about measurable attractors in locally compact metric spaces are given.

1. *Introduction*

The notion of exactness of a non-invertible map was first introduced by Rohlin who proved, in the measure-preserving case, that exact endomorphisms have no non-trivial measurable factors with zero entropy [**21**]. He showed that some piecewise monotone and continuous interval maps described by Renyi in [**20**] were exact. In this paper we address the question of exactness for smooth maps of an interval and study the structure of many non-exact interval maps as well as give sufficient conditions for S-unimodal maps to be Lebesgue exact.

It is well known that every exact measure-preserving endomorphism has positive measure theoretic entropy. The notion of exactness of an endomorphism extends easily to non-measure-preserving, non-singular maps even though a satisfactory definition of non-singular entropy is still elusive. A non-singular map T of a measure space $(X, \mathcal{B}, \mu)$ is *exact* if the intersection $\cap_{n\geq 0} T^{-n}\mathcal{B}$, called the *tail field* of $\mathcal{B}$, contains only sets of full measure or measure zero. The maximal automorphic factor of T is its induced action on the tail field; it is the trivial map on a one-point space if and only if T is exact.

In this paper, the main dynamical systems of interest are smooth unimodal maps of the interval. Throughout, the measure of interest is one-dimensional normalized Lebesgue measure λ. We show that a variety of measurable automorphic behavior is exhibited by these maps; it is well known that the presence of a measure theoretic attractor of Lebesgue measure zero forces the map to be dissipative with respect to λ, but does not preclude the existence of conservative Lebesgue factors.

After giving a brief review of non-singular ergodic theory and preliminary definitions, we prove some results about dynamical systems with measurable attractors. We show that every ergodic non-singular dynamical system of a locally compact metric space with an attractor of measure zero admits an equivalent invariant infinite σ-finite measure.

We then turn to unimodal maps of the interval for examples. We prove that every C^3 S-unimodal map with no attractor is exact with respect to λ. Next, we consider examples with measure theoretic attractors which have been studied and classified by several authors [**2, 11, 14**]. We exhibit the easily identified maximal automorphic factor in the presence of either a stable periodic orbit or a cycle of intervals (finitely renormalizable). In the case of an infinitely renormalizable map (of bounded type), we show that it is a dissipative map with maximal automorphic factor isomorphic to an odometer. The factor measure on the odometer is non-singular but not necessarily invariant.

Finally, we discuss the case of maps which have absorbing Cantor sets. These maps were shown to exist in [**5**]. We give conditions under which the maximal automorphic factor is an irrational rotation on a circle as well as an exact example. Irrational rotation factors, as topological factors, were shown to exist in [**6**].

2. *Preliminaries*

We assume throughout this paper that $(X, \mathcal{B}, \mu)$ is a locally compact metric space with metric d, Borel σ-algebra $\mathcal{B}$ on X and μ a regular Borel probability measure on $\mathcal{B}$. Infinite measures are always assumed to be σ-finite. We assume that T is *non-singular*; i.e. $T : X \to X$ satisfies $\mu(A) = 0 \iff \mu(T^{-1}A) = 0$ for every $A \in \mathcal{B}$. We also assume that every point in X has at most countably many preimages under T. Furthermore, in all of our examples we will assume without loss of generality that T is forward non-singular as well, i.e. that $\mu(A) = 0 \iff \mu(TA) = 0$ for all measurable sets A. For example, any C^1 map of a manifold onto itself whose differential is non-vanishing except at finitely many points is forward and backward non-singular with respect to the Riemannian volume form (locally equivalent to Lebesgue measure). Let $\mathcal{B}_+ \subset \mathcal{B}$ denote the collection of measurable sets of positive measure. In order to stress the presence of both a topology and a Borel measurable structure, we will refer to $(X, \mathcal{B}, \mu, T)$ as a *non-singular dynamical system*.

Definition 2.1. Let $(X, \mathcal{B}, \mu, T)$ be a non-singular dynamical system. The non-singular dynamical system $(Y, \mathcal{C}, \nu, S)$ is a (measurable) *factor* of $(X, \mathcal{B}, \mu, T)$ if there exists a surjective measurable map $\pi : X \to Y$ such that $S \circ \pi(x) = \pi \circ T(x)$ μ-a.e., and $\nu \sim \mu \circ \pi^{-1}$.

A sub-σ-algebra $\mathcal{B}_o \subset \mathcal{B}$ is T-invariant if $T^{-1}\mathcal{B}_o \subset \mathcal{B}_o$. It is well known that every factor map gives rise to a T-invariant sub-σ-algebra, $\{\pi^{-1}C\}_{C\in\mathcal{C}} \subset \mathcal{B}$, and the converse is also true. We refer the reader to Rohlin [**21**] for details.

2.1. *Non-singular measure theory.* We review some basic definitions used in non-singular measure theory; when the map in question is neither invertible nor measure preserving, ergodic properties need careful definitions. Some equivalent definitions are no longer equivalent in this setting and others simply do not extend. (For example, it is still an open question as to what the definition of mixing should be for non-singular non-invertible maps [**1**]). We assume that $(X, \mathcal{B}, \mu, T)$ is a non-singular dynamical system, although these notions apply in more general measure theoretic settings.

A measurable set W is *(backward) wandering* if the sets $\{T^{-n}W\}_{n=0}^{\infty}$ are all disjoint. Equivalently, no point in a wandering set W ever returns to W. A measurable set V is *forward wandering* if the sets $\{T^{n}V\}_{n=0}^{\infty}$ are all disjoint. Every forward wandering set is also (backward) wandering, but the converse is not true. If T is invertible the concepts are identical. We will use the usual convention that a wandering set always refers to a backward wandering set.

The map T is *conservative* if there exist no wandering sets of positive measure. There exists a maximal set C on which T is conservative, and $C \subset T^{-1}C$. A non-conservative map is called *dissipative*; if T is not conservative on any set of positive measure, then T is *completely dissipative*, and we can write X as the (at most countable) union of wandering sets up to a set of measure zero.

The map T is *ergodic* if T has a trivial field of invariant sets or, equivalently, if any measurable set B with the property that $\mu(B \triangle T^{-1}B) = 0$ has either zero or full measure.

It follows from the definitions that T is conservative and ergodic if and only if for all sets $A, B \in \mathcal{B}_+$ there is a positive integer n such that $\mu(B \cap T^{-n}A) > 0$.

A map is *exact* if it has a trivial tail field $\cap_{n\geq 0}T^{-n}\mathcal{B} \subset \mathcal{B}$ or, equivalently, if any set B with the property $\mu(T^{-n} \circ T^{n}(B) \triangle B) = 0$ for all n has either zero or full measure. For any set $A \in \mathcal{B}_+$, we define a tail set from it by

$$\mathrm{Tail}(A) := \cup_{n\in\mathbb{N}} T^{-n} \circ T^{n}(A).$$

Denoting the tail sets (μ mod 0) by $\mathcal{T} \subset \mathcal{B}$, we have $\cap_{n\geq 0}T^{-n}\mathcal{B} = \mathcal{T}$ (μ mod 0). There is a natural factor mapping onto $\mathcal{T}$ called the exact decomposition (of T with respect to μ), and T acts as an automorphism on the factor space. We denote the factor space by $(Y, \mathcal{C}, \nu)$, and the induced automorphism by S; note that a point in Y is an atom of the measurable partition generated by the relation $x \sim w \iff T^{n}x = T^{n}w$ for some $n \in \mathbb{N}$. We call this factor the *maximal automorphic factor*; this is because if there is a factor map $\phi : X \to Z$ with induced factor automorphism R, then R is a factor of S. We remark that in general $(Y, \mathcal{C}, S, \nu)$ is a non-singular endomorphism of a Lebesgue space with no specified topology.

It is well known that any invertible ergodic non-singular transformation of a non-atomic measure space is conservative, and virtually all of the examples we consider below are ergodic, so their automorphic factors will be either conservative or atomic, or both. In the next result, we give a condition that rules out the dissipative possibility for an automorphic

factor. We recall that every invertible, dissipative, ergodic, non-singular transformation of a σ-finite space is isomorphic to $x \mapsto x+1$ on $\mathbb{Z}$ with an appropriately weighted counting measure.

LEMMA 2.1. *If $(X, \mathcal{B}, \mu, T)$ is a non-singular dynamical system, and if $\mathcal{B}_+$ contains no forward wandering sets, then every automorphic factor of T is conservative.*

Proof. The trivial factor is conservative, so we assume that T has a non-trivial automorphic factor. We will denote the projection onto the automorphic factor by π, the factor space by Y, and the induced automorphism by S. Then if S is not conservative, there exists a wandering set W of positive measure. Since S is an automorphism, the sets $\{S^{-n}W\}_{n\in\mathbb{Z}}$ are all disjoint (in Y). Then, by definition of the factor, the sets $\pi^{-1}(S^{-n}W)$ are also disjoint and equal to the (disjoint) collection of sets $\{T^{-n}(\pi^{-1}W)\}_{n\in\mathbb{Z}}$ (in X). This contradicts the hypothesis since $\pi^{-1}W$ has positive measure and is forward wandering. □

Remark. One can construct examples of ergodic dissipative maps with conservative factors. If we consider the product of a K-automorphism S with the dissipative ergodic endomorphism $R(n) = n+1$ on $\mathbb{N}$ with counting measure, then the map $T = S \times R$ is dissipative and ergodic with a conservative automorphic factor (which is S). Below we will show that such examples occur within the family of S-unimodal maps.

The following result is well known but we include it for completeness.

LEMMA 2.2. *If T is non-singular and exact, then T is totally ergodic; i.e., for each $n \in \mathbb{N}$, T^n is ergodic.*

We give a necessary and sufficient condition for exactness which will be useful in the context of interval maps.

PROPOSITION 2.1. *An ergodic non-singular endomorphism T is exact on $(X, \mathcal{B}, \mu)$ if and only if for every set $B \in \mathcal{B}_+$, $\mu(T^{-n} \circ T^{n+1}(B) \cap B) > 0$ for some $n \in \mathbb{N}$.*

Proof. Assume first that T is exact and $A \in \mathcal{B}_+$ with $\mu(T^{-n} \circ T^{n+1}(A) \cap A) = 0$ for every $n \in \mathbb{N}$. This means that $\mu(\mathrm{Tail}(TA) \cap A) = 0$, but this is impossible since $\mu(\mathrm{Tail}(TA)) = 1$ by exactness and non-singularity. To prove the other direction let $(Y, \mathcal{C}, \nu, S)$ be the maximal automorphic factor of $(X, \mathcal{B}, \mu, T)$, $\pi : X \to Y$ being the factor map. Assume by contradiction that T is not exact, i.e. $(Y, \mathcal{B}, \nu, S)$ is non-trivial. Then there exists $C \in \mathcal{C}$ such that $\nu(C) > 0$ and $C \cap S(C) = \emptyset$. Take $B := \pi^{-1}(C)$. Since the maximal automorphic factor is isomorphic to the tail field of $(X, \mathcal{B}, \mu, T)$, B satisfies $T^{-n} \circ T^n(B) = B$ (μ mod 0) for all n. The same thing is true for $T(B) = \pi^{-1} \circ S(C)$. By assumption $0 < \mu(T^{-n} \circ T^{n+1}(B) \cap B) = \mu(T^{-n} \circ T^n(T(B)) \cap B) = \mu(T(B) \cap B)$. Hence $0 < \nu(\pi(T(B) \cap B)) = \nu(S(C) \cap C)$, contradicting the choice of C. □

3. *Measurable attractors*

An important link between the topological and the measure theoretical dynamics occurs when there are attractors present. The definition of measurable attractor we give here was introduced by Milnor [**18**]. We assume that $(X, \mathcal{B}, \mu, T)$ is a non-singular dynamical system.

For any point $x \in X$, the omega limit set $\omega(x)$ is defined as $\omega(x) = \cap_n \overline{\cup_{i>n} T^i(x)}$. With our standing assumptions on X, for each $x \in X$, $\omega(x)$ is a Borel measurable set.

Definition 3.1. For a set $\mathcal{A}$, we define

$$B(\mathcal{A}) := \{x \in X : \omega(x) \neq \emptyset, \omega(x) \subset \mathcal{A}\},$$

and call it the *basin of* $\mathcal{A}$. An *attractor* is a compact subset $\mathcal{A} \subset X$ such that $\mu(B(\mathcal{A})) > 0$, and there is no proper subset $\mathcal{A}' \subset \mathcal{A}$, such that $\mu(B(\mathcal{A}')) > 0$.

Obviously, an attractor is invariant: $T(\mathcal{A}) = \mathcal{A}$. Milnor defined attractors to be closed, but as he was considering endomorphisms on compact manifolds and our space is only locally compact, we define attractors to be compact.

PROPOSITION 3.1. *If $(X, \mathcal{B}, \mu, T)$ is ergodic, there can be at most one attractor and $\mu(X \setminus B(\mathcal{A})) = 0$. Moreover, for any neighborhood U of $\mathcal{A}$ and any $x \in B(\mathcal{A})$, there exists N such that $T^n(x) \in U$ for all $n \geq N$.*

Proof. The basin of an attractor is clearly completely invariant and has positive measure, so it must have full measure.

It suffices to prove the second statement for small neighborhoods only. By continuity of T, we can take $U \supset \mathcal{A}$ so small that $T(U)$ is contained in a compact set $K \subset X$. If $x \in B(\mathcal{A})$, then there exists a sequence $\{n_i\}$ such that $T^{n_i}(x) \to \mathcal{A}$. Suppose that $T^n(x) \notin U$ infinitely often. Then there exists a sequence $\{m_i\}$, $m_i \geq n_i$, such that $T^{m_i}(x) \in U$ but $T^{m_i+1}(x) \in K \setminus U$. Because $K \setminus U$ is compact, the sequence $\{T^{m_i+1}(x)\}$ has an accumulation point $y \in \omega(x) \setminus U$, contradicting the fact that $\omega(x) \subset \mathcal{A}$. □

Without loss of generality we can assume that μ is a probability measure on X (by replacing μ by an equivalent one if necessary). Typically we are interested in the case where $\mu(\mathcal{A}) = 0$ and $\mu(B(\mathcal{A})) = 1$. In this case we can show that the map is completely dissipative.

PROPOSITION 3.2. *If $(X, \mathcal{B}, \mu, T)$ is an ergodic non-singular dynamical system with an attractor $\mathcal{A}$ satisfying $\mu(\mathcal{A}) = 0$, then T is completely dissipative.*

Proof. Suppose that there exists a set $C \in \mathcal{B}_+$ on which T is conservative. By the regularity of μ, we can find a compact set $K \subset C$, $\mu(K) > \mu(C)/2$, such that K does not intersect some neighborhood U of $\mathcal{A}$. Conservativity on K implies that μ-a.e. $x \in K$ returns to K infinitely often, but the previous lemma shows that K is in the basin of $\mathcal{A}$ μ-a.e., so μ-a.e. point enters U and stays there. Hence no such K exists. □

3.1. *Maps with attractors have σ-finite measures.* Suppose the non-singular dynamical system $(X, \mathcal{B}, \mu, T)$ has an attractor of measure zero. Even though T is completely dissipative, we show that there exists a σ-finite invariant measure equivalent to μ.

Before giving the proof of the result, we review some properties of non-singular countable-to-one maps. We assume that $(X, \mathcal{B}, \mu)$ is a Borel probability space and $T : X \to X$ is a non-singular ergodic endomorphism which is surjective and countable-to-one almost everywhere. Since T is countable-to-one, we apply a well-known result

of Rohlin [**21**] to obtain a measurable partition $\zeta = \{A_1, A_2, A_3, \dots\}$ of X into at most countably many sets, called atoms, satisfying:

(1) $\mu(A_i) > 0$ for each i;

(2) the restriction of T to each A_i, which we will write as T_i, is one-to-one;

(3) each A_i is of maximal measure in $X \setminus \bigcup_{j<i} A_j$ with respect to property (2);

(4) T_1 is one-to-one and onto X (by numbering the atoms A_i so that

$$\mu(TA_i) \geq \mu(TA_{i+1})$$

for $i \in \mathbb{N}$).

We call a partition ζ of this form a *Rohlin partition*.

The map $T_1^{-1} \circ T$ gives an obvious factor map from X onto A_1. Since T is countable-to-one, each fiber over $x \in A_1$ contains at most countably many points (these are the points $x = w_1, \dots, w_n, \dots$ such that $T(x) = T(w_j)$) and there is an atomic probability measure μ_x associated to each point x, μ_x^j, $j = 1, \dots, n, \dots$, which is just the factor decomposition of the measure μ, viewed as a measure on X over the fibers of A_1. The measures μ_x vary measurably in x.

We now turn to the main result of this section.

THEOREM 3.1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic non-singular dynamical system, where μ is a σ-finite regular Borel measure and X is a metric space such that T is countable-to-one and $T(X) = X$. Assume that X has an attractor $\mathcal{A}$ such that $\mu(\mathcal{A}) = 0$. Then there exists a σ-finite invariant measure ν which is equivalent to μ.*

Proof. As mentioned above, we can assume that $\mu(X) = 1$. We define the sets X_i, $i \geq 1$, as follows: $X_1 := X$, $X_{2i} := \{x \in T(X_{2i-1}) : d(T^n(x), \mathcal{A}) \leq 2^{-i}$ for all $n \geq 0\}$ and $X_{2i+1} := T(X_{2i})$. Obviously $T(X_i) \supset X_{i+1}$ and because $\mathcal{A}$ is an attractor, $\mu(X_i) > 0$ for all i.

Let $\hat{X}$ be the disjoint union $\sqcup_{i \geq 1} X_i$ equipped with the action

$$\hat{T}(x \in X_i) = T(x) \in \begin{cases} X_{i+1}, & \text{if } i \text{ is even,} \\ X_{i+1}, & \text{if } i \text{ is odd and } T(x) \in X_{i+1}, \\ X_i, & \text{if } i \text{ is odd and } T(x) \in X_i \setminus X_{i+1}. \end{cases}$$

If $\pi_0 : \hat{X} \to X$ is the standard projection, then $\pi_0 \circ \hat{T} = T \circ \pi_0$. We note that each X_i can be viewed as a subset of X with the restriction measure $\mu_i := \mu|_{X_i}$. Then the non-singularity and forward and backward measurability of T gives the corresponding properties for $\hat{T}$ with respect to the measure

$$\rho(B) := \sum_i \mu_i(B \cap X_i).$$

By construction, X_{2i} is backward wandering in $\hat{X}$ for each i. Moreover, for ρ-a.e. $x \in \sqcup_{j \leq 2i} X_j$, there exists a unique $n \geq 0$ such that $\hat{T}^n(x) \in X_{2i}$.

We define a sequence of measures $\hat{\mu}_n$ on $\hat{X}$ starting with $\hat{\mu}_1$. Let $\hat{\mu}_1|_{X_2} = \mu|_{X_2}$, and inductively extend the measure to $X_1 \sqcup X_2$ as follows. By our assumptions we can write $X_1 = \cup_j \hat{T}^{-j}(X_2)$, where the sets in the union are mutually disjoint and each has positive

ρ-measure. By using the ergodicity of T, we can consider this disjoint union of $X_1 := X$ as defining a measurable partition of X_1, $\mathcal{Q} = \{\hat{T}^{-1}(X_2), \hat{T}^{-2}(X_2)\dots\}$. Therefore, it is enough to define $\hat{\mu}_1|_{\hat{T}^{-j}(X_2)}$. We proceed inductively on j.

Consider any $B \subset \hat{T}^{-1}X_2$ and we suppose first that $B = \hat{T}^{-1} \circ \hat{T}(B)$ ρ-a.e., i.e. B is a $\hat{T}^{-1}\mathcal{B}$ measurable set. Then we define

$$\hat{\mu}_1(B) = \hat{\mu}_1(\hat{T}(B)).$$

In order to compensate for the fact that T, hence $\hat{T}$, is not one-to-one everywhere (and therefore there are measurable sets not of the above form), we refine $\mathcal{Q}$ by a Rohlin partition (for T on X) $\zeta = \{A_1, A_2, \dots, A_n, \dots\}$. We note that A_1 intersects every atom of $\mathcal{Q}$ in a set of positive measure since T maps A_1 onto X. Define the sets

$$Q_k^j = A_k \cap \hat{T}^{-j}X_2,$$

for each $j, k \in \mathbb{N}$. Denote for each j, k the relative size of each A_k in $\hat{T}^{-j}X_2$ by

$$\alpha_k^j = \frac{\mu(Q_k^j)}{\mu(\hat{T}^{-j}X_2)}.$$

Then for every j, $\sum_{k\geq 1}\alpha_k^j = 1$. Write $B_k := B \cap Q_k^1$ and define

$$\hat{\mu}_1(B_k) = \alpha_k^1 \cdot \hat{\mu}_1 \circ \hat{T}(B_k) \quad \text{and} \quad \hat{\mu}_1(B) = \sum_{k\geq 1}\hat{\mu}_1(B_k).$$

In particular,

$$\hat{\mu}_1(\hat{T}^{-1}X_2) = \sum_{k\geq 1}\hat{\mu}_1(Q_k^1) \leq \mu(X_2).$$

We give the inductive step. If $\hat{\mu}_1$ is defined on $\cup_{i<j}\hat{T}^{-i}X_2$, we extend it to $\hat{T}^{-j}X_2$ as follows. For $B \subset \hat{T}^{-j}X_2$, $B_k := B \cap Q_k^j$, and $\hat{T}(B_k) \subset \hat{T}^{-(j-1)}X_2$, so we define

$$\hat{\mu}_1(B_k) = \alpha_k^j \cdot \hat{\mu}_1 \circ \hat{T}(B_k).$$

Then, as before,

$$\hat{\mu}_1(B) = \sum_{k\geq 1}\hat{\mu}_1(B_k).$$

We extend the measure $\hat{\mu}_1$ to all of $\hat{X}$ by setting $\hat{\mu}_1(B) = 0$ if $B \subset \sqcup_{m>2}X_m$. Therefore, $\hat{\mu}_1(B) = \hat{\mu}_1(\hat{T}^{-1}(B))$ if $B \subset X_1 \sqcup X_2$, and $\hat{\mu}_1(B) = 0$ if $B \cap (X_1 \sqcup X_2) = \emptyset$. Note that by construction $\hat{\mu}_1(\hat{T}^{-j}(X_2)) = \mu(X_2)$ for every j, so $\hat{\mu}_1$ is infinite and σ-finite.

We now continue inductively on n and define $\hat{\mu}_{n+1}$ by

$$\hat{\mu}_{n+1} = \hat{\mu}_n \circ \hat{T}^{-1}.$$

Clearly $\hat{\mu}_{n+1} \geq \hat{\mu}_n$; furthermore, if $B \subset \sqcup_{j\leq n+1}X_j$ for any $n \geq 1$ then $\hat{\mu}_{n+1}(B) = \hat{\mu}_n(\hat{T}^{-1}(B)) = \hat{\mu}_n(B)$, so equality holds.

We claim that $\hat{X}$ admits a countable partition $\{X_k^j\}_{j,k\geq 1}$ (up to a set of ρ-measure 0), where X_k^j is a measurable subset of X_k and $\hat{\mu}_n(X_k^j) \leq \mu(X_2)$ for all $j, k \geq 1$. We then define the limit measure $\hat{\mu}$ by

$$\hat{\mu}(B) = \sum_{j,k\geq 1}\lim_{n\to\infty}\hat{\mu}_n(B \cap X_k^j).$$

Because $\hat{\mu}_n$ is invariant on $\sqcup_{k\le n} X_k$, this limit is a $\hat{T}$-invariant measure. Since $\hat{\mu}(X_k^j) \le \mu(X_2) < \infty$, $\hat{\mu}$ is clearly σ-finite. Finally, on X the desired measure is $\nu := \hat{\mu}\circ\pi_0^{-1}$, which is T-invariant and absolutely continuous with respect to μ. Moreover, $\hat{\mu}|_{X_1} \ge \hat{\mu}_1|_{X_1}$, and due to the non-singularity of T and the construction of $\hat{\mu}_1$ above, $\hat{\mu}_1|_{X_1}$, viewed as a measure on X, is equivalent to μ. Hence ν is equivalent to μ.

Ergodicity and invariance pass to any factor measure; however, we must show that ν is σ-finite since this property can fail when taking factors. Recall that $\nu(\mathcal{A}) = 0$ and $\mathcal{A} = \cap_i \cup_{j\ge i} X_j$. Therefore, if A_ε is an ε-neighborhood of $\mathcal{A}$, $\pi_0^{-1}(X \setminus A_\varepsilon)$ intersects only finitely many levels X_j. Hence $\nu|_{X\setminus A_\varepsilon}$ is σ-finite. Since this is true for all $\varepsilon > 0$ and μ is regular Borel, ν is σ-finite.

It remains to define the sets X_k^j and prove the claim. For each $k = 2i$ and $j \ge 1$, let

$$X_k^j = \hat{T}^{-j}(X_{k+2}) \cap X_k \quad \text{and} \quad X_{k+1}^j = \hat{T}^{-j}(X_{k+2}) \cap X_{k+1}.$$

Because the X_k^j are preimages of backward wandering sets, they are pairwise disjoint. We claim that

$$\sup_n \hat{\mu}_n(X_{2i}) \le \mu(X_2), \quad \text{for all } i.$$

By construction, $\hat{\mu}_n(X_2) = \mu(X_2)$ for all n. We continue by induction on i. Since $\hat{\mu}_n(\hat{T}^{-1}(B)) \ge \hat{\mu}_n(B)$, it follows that

$$\hat{\mu}_n(X_{2i+2}) \le \sum_j \hat{\mu}_n(X_{2i}^j) \le \hat{\mu}_n(X_{2i}).$$

Furthermore, $\hat{\mu}_n(X_k^j) \le \hat{\mu}_n(X_{2i+2}) \le \mu(X_2)$ for all $j, i \ge 1$ and $k = 2i$ or $2i+1$. Since $\hat{\mu}(X_k \setminus \cup_j X_k^j) = 0$ for all k, this establishes the asserted partition of $\hat{X}$. □

4. *S-unimodal maps*

We now consider a class of smooth maps of the interval as our main examples of non-singular dynamical systems. Let $f : I \to I$ be a unimodal map. By this we mean that there is a unique point c, called the *critical point*, such that f is increasing on the left and decreasing on the right of c. The iterates $f^i(c)$ will be denoted by c_i. Assuming that $c_2 < c < c_1$, we can scale f in such a way that $I = [c_2, c_1]$. For $x \in I$, let $\tau(x) = \bar{x}$ be the point such that $f^{-1} \circ f(x) = \{x, \bar{x}\}$. Note that f is two-to-one on $[c_2, \bar{c}_2] \setminus \{c\}$. Therefore it is precisely on this set that $x \ne \bar{x}$, so that τ is defined; we have $\tau \circ \tau = \mathrm{id}$ on its domain. We call a set $A \subset [c_2, \bar{c}_2] \setminus \{c\}$ *symmetric* if $\tau(A) = A$.

Throughout the paper we will assume that f is C^3 and has negative Schwarzian derivative (f is *S-unimodal*), i.e.

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \le 0$$

wherever defined. This assumption will enable us to make certain distortion estimates. We call a periodic point of period n *stable* if $|(f^n)'(p)| \le 1$, thus comprising the hyperbolic attracting and neutral case. Under the assumption of negative Schwarzian derivative, a stable periodic orbit must attract the critical point. Hence there can be only one such orbit.

The *critical order* of f is ℓ if there exists $M > 0$ such that

$$\frac{\ell|x-c|^{\ell-1}}{M} \leq |f'(x)| \leq M\ell|x-c|^{\ell-1}$$

for all $x \in I$. For some of our results it is important that f is *non-flat*, i.e. $\ell < \infty$. The critical omega limit set, $\omega(c)$, is of particular importance. If f is non-flat S-unimodal, then either $\omega(c)$ contains an interval or has Lebesgue measure zero, see [**14**].

The ergodic properties of Lebesgue measure, which we will denote by λ, for unimodal maps are well understood. We quote a result by Blokh and Lyubich [**2**].

PROPOSITION 4.1. *Suppose f is C^3 S-unimodal and has no stable periodic orbit. Then any forward invariant measurable symmetric set A (i.e. $\tau(A) = A$) with $\lambda(A) > 0$ has c as a density point.*

From this, one can easily derive that f is Lebesgue ergodic if and only if there is no stable periodic orbit. Moreover, λ-a.e. point accumulates on the critical point in this case.

THEOREM 4.1. [**2, 11, 14**] *If f is C^3 S-unimodal, then f is conservative if and only if there is no attractor. There is at most one attractor, which is of one of the following types:*

(1) *a stable periodic orbit;*
(2) *a cycle of intervals (the finitely renormalizable case);*
(3) *a Cantor set (the infinitely renormalizable case);*
(4) *an absorbing Cantor set.*

In the cases (1), (3), and (4), $\mathcal{A}$ necessarily equals $\omega(c)$, and f is completely dissipative.

A map is *renormalizable* if there exists an interval J, $c \in J \neq I$, such that $f^n(J) \subset J$ and $J, f(J), \ldots, f^{n-1}(J)$ have disjoint interiors. As a rule, we take J minimal with this property; such a J is called a *restrictive interval*. The map $f^n|J$ is again a unimodal map; it is called the renormalization of f. If there is a smallest restrictive interval J, then f is finitely renormalizable. If there are arbitrarily small restrictive intervals (of arbitrarily large period) then f is *infinitely renormalizable*. (The best known example is the Feigenbaum map; here the periods of the restrictive intervals are the powers of 2.) The attractor is a Cantor set. It attracts all points except for a nullset of first Baire category. No point in I has a dense forward orbit. By Lemma 2.2, a renormalizable map cannot be exact.

In case (4), f is not infinitely renormalizable, but $\omega(c)$ is a Cantor set such that $\omega(x) \subset \omega(c)$ for x in a full measure set of first Baire category. There is a second category set of points whose orbit lies dense in I (or if f is finitely renormalizable in a cycle of intervals). It was shown in [**5**] that so-called Fibonacci unimodal maps with sufficiently large critical orders have absorbing Cantor sets. Theorem 3.1 applied to unimodal maps yields the following result.

PROPOSITION 4.2. *If f is a non-flat C^3 S-unimodal map with an attractor of type (1), (3), or (4), then f admits a (dissipative) σ-finite invariant measure which is absolutely continuous with respect to λ.*

This result is known when $\mathcal{A}$ is a stable periodic orbit, or when $\mathcal{A}$ is an absorbing Cantor set [**16**]. To our knowledge, the result is new for infinitely renormalizable maps.

We conjecture that the result does not hold for Misiurewicz' example of a C^∞ Feigenbaum map whose attractor $\omega(c)$ has positive Lebesgue measure [**19**]. He showed that $(\omega(c), f)$ admits no σ-finite absolutely continuous invariant measure, and it seems likely that $\lambda(I \setminus \cup_n f^{-n}(\omega(c))) = 0$.

4.1. *Exact S-unimodal maps.* In this section we prove some results concerning exactness and automorphic factors for general S-unimodal maps. Let f be an S-unimodal map; we show that if f has no attractor then f is Lebesgue exact. We also demonstrate the maximal automorphic factor of f in the cases where either f has a stable periodic orbit or an attracting cycle of intervals. In later sections we deal with certain specific (dissipative) unimodal maps from the remaining two cases, i.e. where f has an attractor of type (3) or (4).

The first result is already known in the case where f admits an absolutely continuous invariant measure μ and is non-renormalizable. Ledrappier [**12**] showed that f has a Bernoulli natural extension in this case and is exact with respect to $\mu \ll \lambda$.

THEOREM 4.2. *Let f be C^3 S-unimodal. If f has no attractor, then f is Lebesgue exact.*

Our proof relies on a result by Martens and involves a property which he calls the strong Markov property [**14, 15, 17**].

PROPOSITION 4.3. (Strong Markov property) *Suppose f has no attractor of type (1), (3), or (4). Then there exist symmetric neighborhoods U and V of c, $c \in \operatorname{cl} U \subset \operatorname{int} V$, such that for λ-a.e. x the following holds: there exist integers $k_1(x) < k_2(x) < \cdots$ and nested intervals $x \in \cdots \subset I_2(x) \subset I_1(x)$ such that, for all i, $f^{k_i(x)}(x) \in U$ and $f^{k_i(x)}$ maps $I_i(x)$ monotonically onto V.*

Remark. Let L_1 and L_2 be the components of $U \setminus V$, and say $\lambda(L_1) \le \lambda(L_2)$; define $\delta := \lambda(L_1)/\lambda(V)$. Then the Koebe principle (see [**17**, Theorem IV 1.3]) gives the distortion bound $K(\delta) = ((1+\delta)/\delta)^2$:

$$\operatorname{dist}(f^{k_i(x)}, I_i(x) \cap f^{-k_i(x)}(U)) := \sup\left\{\frac{|Df^{k_i(x)}(y)|}{|Df^{k_i(x)}(z)|} : y, z \in I_i(x) \cap f^{-k_i(x)}(U)\right\} \le K(\delta). \tag{1}$$

Proof of Theorem 4.2. Let $U \subset V$ be as in Proposition 4.3 and let $\delta > 0$ and the distortion bound $K = K(\delta) \ge 1$ be as in (1).

Let p be the orientation reversing fixed point of f. Without loss of generality, we can assume that $V \subset (\bar{p}, p)$. It is well known (see, e.g., [**17**, Theorem III 4.6]) that, since f is not renormalizable, $\cup_n f^n(U) = I$. Therefore, there exists a minimal $r > 0$ such that $p \in \operatorname{int} f^r(U)$. Let $H \subset f^r(U)$ be an interval such that $f(H) \subset f^r(U)$. Let $H_1, H_2 \subset U$ be such that f^r maps H_1 and H_2 diffeomorphically onto H and $f(H)$, respectively. Let $K_0 = K_0(H_1, H_2)$ such that $\operatorname{dist}(f^{r+1}, H_1), \operatorname{dist}(f^r, H_2) \le K_0$. Take $\varepsilon := \min\{\lambda(H_1), \lambda(H_2)\}$.

Now let A be any set of positive measure. Let x be a Lebesgue density point of A such that the integers k_i and intervals $I_i(x)$ from Proposition 4.3 are well defined. Since x is a

density point, we can take i so large that $\lambda(I_i(x) \setminus A)/\lambda(I_i(x)) \le \varepsilon/3KK_0$. By the Koebe principle, $\operatorname{dist}(f^{k_i}, I_i(x) \cap f^{-k_i}(U)) \le K$. Therefore,

$$\frac{\lambda(H_1 \setminus f^{k_i}(A))}{\lambda(H_1)} \le \frac{\lambda(U \setminus f^{k_i}(A))}{\lambda(U)} \frac{\lambda(U)}{\lambda(H_1)} \le \frac{\varepsilon}{3K_0} \frac{1}{\varepsilon} \le \frac{1}{3K_0},$$

and similarly $\lambda(H_2 \setminus f^{k_i}(A))/\lambda(H_2) \le 1/3K_0$. Using the distortion bound K_0 of $f^{r+1}|H_1$ and $f^r|H_2$, we obtain

$$\frac{\lambda(f(H) \setminus f^{k_i+r+1}(A))}{\lambda(f(H))}, \frac{\lambda(f(H) \setminus \lambda(f^{k_i+r}(A)))}{\lambda(f(H))} \le \frac{1}{3}.$$

It follows that $\lambda(f^{k_i+r+1}(A) \cap f^{k_i+r}(A)) \ge \frac{1}{3}\lambda(f(H)) > 0$. By Proposition 2.1, f is exact. □

4.2. *Maps with stable periodic orbits.* We have shown that an S-unimodal map without an attractor has a trivial maximal automorphic factor. We next characterize the maximal automorphic factor of f if f has an attractor of type (1) or (2).

THEOREM 4.3.

(1) *If $(I, \mathcal{B}, \lambda, f)$ has a stable periodic orbit, then its maximal automorphic factor is isomorphic to $(\mathbb{R}, \mathcal{B}, \lambda, x \mapsto x + 1)$.*

(2) *If $(I, \mathcal{B}, \lambda, f)$ has an attractor $J \cup f(J) \cup \cdots \cup f^{n-1}(J)$ of type (2), then its maximal automorphic factor is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, i \mapsto i + 1 \bmod n)$ with counting measure.*

Proof. Suppose p is an n-periodic stable periodic point with immediate basin B. Then $\omega(x) = \operatorname{orb}(p)$ λ-a.e. If p reverses orientation, take $b \in B \setminus \{p\}$ and $U := (b, f^{2n}(b)]$. If p preserves orientation, then take points b and b' in either component of $B \setminus \{p\}$ and $U := (b, f^n(b)] \cup [f^n(b'), b')$. (If $B \setminus \{p\}$ has only one component, namely because $|(f^n)'(p)| = 1$, then take $U := (b, f^n(b)]$.) In each case U is forward wandering and fundamental in the sense that for λ-a.e. $x \in I$, there exist $i, j \ge 0$ such that $f^i(x) = f^j(y)$ for a unique $y \in U$. Define $\pi(x) = (y, i - j)$. Then it is easy to see that $\pi : I \to U \times \mathbb{Z}$ is a factor map and $(U \times \mathbb{Z}, (y, n) \mapsto (y, n + 1))$ is the maximal automorphic factor. This system is obviously isomorphic to $(\mathbb{R}, \mathcal{B}, \lambda, x \mapsto x + 1)$.

If f has an n-period restrictive interval J, then for λ-a.e. x there exists i such that $f^i(x) \in J$. The map $\pi(x) = i \bmod n$ is obviously a factor map. If $J \cup \cdots \cup f^{n-1}(J)$ is the attractor, then by Theorem 4.2, $f^n|J$ is exact. Hence $(\mathbb{Z}/n\mathbb{Z}, i \mapsto i + 1 \bmod n)$ is also the maximal automorphic factor. □

The next result states that many dissipative unimodal maps have conservative maximal automorphic factors.

PROPOSITION 4.4. *Let f be a C^3 non-flat S-unimodal map having no stable periodic orbit. Then the maximal automorphic factor of $(I, \mathcal{B}, \lambda, f)$ is conservative.*

Proof. A result by Blokh and Lyubich [**2**, §7] states that f does not admit wandering sets. (The proof was carried out for critical order $\ell = 2$, but can be generalized to any $\ell \in (1, \infty)$.) The proposition follows now from Lemma 2.1. □

4.3. *Infinitely renormalizable maps.* As mentioned earlier, a unimodal map is infinitely renormalizable if it has arbitrarily small restrictive intervals. We first review the structure of infinitely renormalizable maps in detail.

Let f denote an infinitely renormalizable S-unimodal map. Then f has a forward invariant Cantor set Ω, and the following hold.

- There exists a decreasing chain of closed subsets of I, denoted by Ω^k and satisfying
$$\cdots \Omega^{k+1} \subset \Omega^k \subset \cdots \subset \Omega^1 \subset \Omega^0 = I;$$
each Ω^k contains the critical point and each is mapped onto itself by f.
- There exists a sequence $\{p_k\}_{k\in\mathbb{N}}$, such that p_k divides p_{k+1}, such that the following hold. For each $k \in \mathbb{N}$, there exist p_k disjoint closed subintervals $\Omega_{i_1,\dots,i_k}$, $i_j \in \{0, 1\}$, which are cyclically permuted by the first $p_k - 1$ iterates of f, and such that $f^{p_k}(\Omega_{i_1,\dots,i_k}) \subset \Omega_{i_1,\dots,i_k}$; furthermore, $\Omega^k = \cup\Omega_{i_1,\dots,i_k}$.
- The critical point c always lies in the subinterval which is labelled $\Omega_{0,0,\dots,0}$; i.e. $\Omega_{0,0,\dots,0}$ is a restrictive interval, and the rest are labelled so that the action of f moves the cylinders in the usual $\{p_k\}$-odometer order (add 1 to i_0 and carry when necessary).
- The intervals are nested in the obvious way; that is,
$$\Omega_{i_1,\dots,i_{k-1},i_k} \subset \Omega_{i_1,\dots,i_{k-1}}.$$
As $f^{p_k}(\Omega_{i_1,\dots,i_k}) \subset \Omega_{i_1,\dots,i_k}$, we can rescale $f^{p_k}|\Omega_{i_1,\dots,i_k}$ to a map on the unit interval. For this reason, f is called renormalizable. The fact that we can do this for every k makes f infinitely renormalizable.
- The intersection
$$\Omega := \cap_{k\geq 1} \cup \Omega_{i_1,\dots,i_k} = \cap_{k\geq 1}\Omega^k$$
is known to be a Cantor set of Lebesgue measure zero when f is non-flat S-unimodal (however, cf. [**19**]). Furthermore, $\Omega = \omega(c)$, the ω-limit set of the critical point, and $f|\Omega$ is the $\{p_k\}$-odometer.
- The orbit of λ-a.e. point x converges to the Cantor set Ω in the sense that for every k, the orbit of x eventually lands inside Ω^k, i.e. Ω is an attractor. By $\Omega^k(x)$ we will denote the specific subinterval $\Omega_{i_1,i_2,\dots,i_k}$ containing x. Therefore, $\Omega^k(x)$ is defined for all k for λ-a.e. x.

If $p_k = 2^k$ for an infinitely renormalizable S-unimodal map of the interval, then f is commonly known as a *Feigenbaum map*. Independently, Feigenbaum [**8**] and Coullet and Tresser [**7**] discovered this pattern of renormalization for these maps.

LEMMA 4.1. *If f is S-unimodal and infinitely renormalizable, for λ-a.e. $x \in I$ there exists a unique $y \in \Omega$ such that $|f^n(x) - f^n(y)| \to 0$ as $n \to \infty$. In this case we say that x* copies $y \in \Omega$.

Proof. Using the subintervals $\Omega_{i_1,\dots,i_k}$ as a basis for the topology of the Cantor set, we have that the sequences $(i_1, \dots, i_k, \dots)$ from the corresponding basis elements give a $\{p_k\}$-adic coding for each $y \in \Omega$, and, as mentioned above, the labeling is chosen to correspond under the action of f to the usual odometer action.

We fix some $x \in I$ such that $\omega(x) \subset \Omega$. For each $k \geq 0$, there exists a smallest positive integer n_k such that $f^{n_k}(x) \in J_k = \Omega_{0,\ldots,0}$ (k zeroes). As $f^n(x) \in J_k$ only if $n = n_k + jp_k$ for some $j \geq 0$, it follows that p_k divides $n_{k+1} - n_k$. Let $i_1, \ldots, i_{k-1}$ be such that $n_k + \sum_{j=1}^{k-1} i_j p_j \equiv 0 \bmod p_k$. Then $f^{n_k}(x) \in f^{n_k}(\Omega_{i_1,\ldots,i_{k-1}})$. Furthermore, $\Omega_{i_1,\ldots,i_{k-1}} \supset \Omega_{i_1,\ldots,i_k}$ for each k. Let $y = \cap_k \Omega_{i_1,\ldots,i_k}$. We will show that x copies y and no other point.

For $\varepsilon > 0$, let k be so large that each component of Ω^k has length less than ε. Then $|f^n(x) - f^n(y)| \leq \operatorname{diam}(f^{n-n_k}(J_k)) < \varepsilon$ for all $n \geq n_k$. Since this holds for any $\varepsilon > 0$, $|f^n(x) - f^n(y)| \to 0$. On the other hand, $f|\Omega$ is distal: if $y' \in \Omega$, $y' \neq y$, then $|f^n(y) - f^n(y')|$ is bounded away from zero uniformly in n. Indeed, for k sufficiently large, y and y' are contained in different components of Ω^k, and these components remain disjoint under iteration of f. Hence y is the unique point copied by x. □

Define the map $\pi : I \to \Omega$ by $\pi(x) = y$, i.e. x is mapped to the unique point that it copies. This map is well defined except on a set of λ measure zero. Using g for the odometer action on Ω, we have $\pi \circ f(x) = g \circ \pi(x)$. It is easy to see from the dynamics of the map that the usual Borel structure generated by cylinder sets on Ω agrees with the factor measure structure (i.e. a set C is Borel in $\Omega \iff \pi^{-1}C$ is Borel in I). With respect to the factor measure $\mu(\cdot) = \lambda \circ \pi^{-1}(\cdot)$, the factor map is invertible. We take the completion of the Borel sets in Ω with respect to μ to obtain a factor Lebesgue space. Note that as a factor space, the space Ω has μ measure one (in contrast with the zero Lebesgue measure it has as a subset of I). The aim of the rest of this section is to show that $(\Omega, \pi(\mathcal{B}), \mu, g)$ is the maximal automorphic factor for a certain class of infinitely renormalizable maps.

We have defined $J_k = \Omega_{0\ldots0}$ (k zeroes) to be the kth restrictive interval; its period is p_k. For each $x \in B(\Omega)$, let

$$n_k(x) := \min\{n : f^n(x) \in J_k\}$$

and

$$m_k(x) := \min\{m : f^{p_{k-1}m}(y) \in J_k \text{ for } y = f^{n_{k-1}}(x)\}.$$

In this way (taking $p_0 = 1$) we obtain $n_k = \sum_{i=1}^{k} m_k p_{k-1}$. Basically, m_k plays the same role for $f^{p_{k-1}}|J_{k-1}$ as m_1 plays for $f|I$.

A unimodal map f is infinitely renormalizable of *bounded type* if it is infinitely renormalizable and the sequence of the quotients p_k/p_{k-1} is bounded. The geometry of the Cantor set Ω has been particularly well studied for these maps, see [**17**, Ch. VI]. For our purposes we need the following facts.

PROPOSITION 4.5.

(1) *Let f be a non-flat S-unimodal infinitely renormalizable map. Then there exists $K \geq 1$ such that for all restrictive intervals J_k the following holds: if $f^n : J \to J_k$ is a branch of the first return map to J_k, then the distortion* $\operatorname{dist}(f^n, J) \leq K$. *(In particular, $f^n : J \to J_k$ is a diffeomorphism; if $J = J_k$ then we should take $n = 0$.)*

(2) *If in addition f is renormalizable of bounded type, then there exist $\rho_- < \rho_+ < 1$ such that $\rho_-^{n+1}\lambda(J_k) \leq \lambda(\{x \in J_{k-1}; m_k(x) = n\}) \leq \rho_+^{n+1}\lambda(J_k)$.*

Proof. We only sketch the proof, since most of the details can be found in [**17**, §VI.2]. In particular, it is shown that the central gaps of the kth level, i.e. the components of

$I \setminus \Omega^k$ adjacent to the restrictive interval J_k, have size greater or equal to $\sigma|J_k|$ for some uniform constant $\sigma > 0$. (The proof in [**17**] of this fact deals with infinitely renormalizable maps of bounded type, but it remains valid for arbitrary non-flat S-unimodal infinitely renormalizable maps.) Let J be any maximal interval such that $f^n : J \to J_k$ is a diffeomorphism. Let $T \supset J$ be the maximal interval such that $f^n|T$ is a diffeomorphism. Then there exist $a < b < n$ such that $c \in \partial f^a(T), \partial f^b(T)$, and $f^n(T) = (c_{n-a}, c_{n-b}) \supset J_k$. Because $n - a$ and $n - b \neq n$, c_{n-a} and c_{n-b} lie in $\Omega^k \setminus J_k$. Hence $f^n(T)$ contains both gaps adjacent to J_k. Therefore, the components of $f^n(T) \setminus J_k$ both have size greater or equal to $\sigma|J_k|$. This is the space needed to apply the Koebe principle that yields the distortion bound given in formula (1).

For the second statement we remark that the dynamics of $f^{p_{k-1}}$ on $J_{k-1} \setminus J_k$ are hyperbolic (see, e.g., [**17**, Theorem III.5.1]). More precisely, there exist $\tau > 1$ and $C > 0$ depending only on $\lambda(J_k)/\lambda(J_{k-1})$, such that $|Df^{np_{k-1}}(x)| \geq C\tau^n$ whenever $f^{jp_{k-1}}(x) \in J_{k-1} \setminus J_k$ for $0 \leq j < n$. If f is infinitely renormalizable of bounded type, $\lambda(J_k)/\lambda(J_{k-1})$ is bounded uniformly away from zero. This implies that the assertion holds for a uniform choice of ρ_- and ρ_+. □

We now state the main result of this section.

THEOREM 4.4. *Every S-unimodal infinitely renormalizable map of bounded type has as its maximal automorphic factor an ergodic conservative non-singular adic odometer action with respect to the factor measure induced by* λ.

The idea of the proof is to show that, for λ-a.e. x, $n_k(x)/p_k$ is bounded for sufficiently many k's in a sense made precise in Proposition 4.6 below. From this it will follow that for the $\lambda \times \lambda$-a.e. pair (x, y), $|n_k(x) - n_k(y)|/p_k$ is bounded sufficiently often. Passing to fibers $I_\omega = \pi^{-1}(\omega)$, we can show that for μ-a.e. $\omega \in \Omega$ and $\lambda_\omega \times \lambda_\omega$-a.e. $(x, y) \in I_\omega \times I_\omega$, $|n_k(x) - n_k(y)|/p_k$ is bounded infinitely often. (Here λ_ω denotes the fiber measure that λ induces on I_ω.) We apply this to density points of certain tail sets and complete the proof by a distortion argument which shows that the tail sets must intersect eventually under forward iteration.

PROPOSITION 4.6. *Under the assumptions of Theorem 4.4, there exists* $N > 0$ *and a sequence* $\{k_j\}$ *(*$k_j = j^2$ *will suffice) such that for* λ*-a.e.* $x \in B(\Omega)$,

$$\liminf_{i\to\infty} \frac{1}{i}\#\{j \leq i : n_{k_j}(x) \leq Np_{k_j-1}\} \geq \frac{2}{3}.$$

Proof. Recall the constants $\rho_+ < 1$ and $K \geq 1$ from Proposition 4.5. Choose N_0 so large that

$$\frac{K}{1-\rho_+}\rho_+^{N+1} \leq K\left(\frac{\rho_+^{N+1}}{1-\rho_+} + \frac{\rho_+^{2N/3+1}}{1-\rho_+^{1/3}} + \frac{\rho_+^{N-N_0/2}}{1-\rho_+}\right) \leq \rho_+^{N/3} \tag{2}$$

for all $N \geq N_0$. We show that for any k and $N \geq N_0$,

$$P(n_k \geq Np_{k-1}) = \lambda(\{x : n_k(x) \geq Np_{k-1}\}) \leq \rho_+^{N/3}. \tag{3}$$

We will use induction on k. For $k = 1$, Proposition 4.5 and (2) immediately give

$$P(n_1 \geq Np_0) = P(m_1 \geq N) \leq \sum_{i=N}^{\infty} K\rho_+^{i+1} \leq \frac{K}{1-\rho_+}\rho_+^{N+1} \leq \rho_+^{N/3}$$

for all $N \geq N_0$. For the induction step,

$$\begin{aligned} P(n_k \geq Np_{k-1}) &\leq P(m_k \geq N) + \sum_{i=0}^{N-1} P(m_k = i \text{ and } n_{k-1} \geq (N-i)p_{k-1}) \\ &\leq \frac{K}{1-\rho_+}\rho_+^{N+1} + \sum_{i=0}^{N-N_0/2} K\rho_+^{i+1} P(n_{k-1} \geq 2(N-i)p_{k-2}) \\ &\quad + \sum_{i=N-N_0/2}^{N-1} K\rho_+^{i+1}. \end{aligned}$$

Here we have used Proposition 4.5 and the fact that $p_{k-1} \geq 2p_{k-2}$. By induction, $P(n_{k-1} \geq 2(N-i)p_{k-2}) \leq \rho_+^{2(N-i)/3}$ for $i \leq N - N_0/2$. Together with (2), this gives

$$\begin{aligned} P(n_k \geq Np_{k-1}) &\leq \frac{K}{1-\rho_+}\rho_+^{N+1} + K\sum_{i=0}^{N-N_0/2} \rho_+^{i+1}\rho_+^{2(N-i)/3} + K\sum_{i=N-N_0/2}^{N-1} \rho_+^{i+1} \\ &\leq K\left(\frac{\rho_+^{N+1}}{1-\rho_+} + \frac{\rho_+^{2N/3+1}}{1-\rho_+^{1/3}} + \frac{\rho_+^{N-N_0/2}}{1-\rho_+}\right) \leq \rho_+^{N/3}. \end{aligned}$$

This proves formula (3).

Now take $k_i = i^2$. Since $p_n \geq 2p_{n-1}$ for all $n \geq 2$, we have $p_{k_{i+1}-1} \geq 2^{2i}p_{k_i-1}$. By (3) we find that for i sufficiently large

$$P\left(n_{k_i} \geq \frac{1}{i^2}p_{k_{i+1}-1}\right) = P\left(n_{k_i} \geq \frac{1}{i^2}\frac{p_{k_{i+1}-1}}{p_{k_i-1}}p_{k_i-1}\right) \leq \rho_+^{2^{2i}/3i^2},$$

which is summable over i. The Borel–Cantelli lemma gives that the set

$$X := \left\{x : \exists j\ \forall i \geq j, n_{k_i}(x) \leq \frac{1}{i^2}p_{k_{i+1}-1}\right\} \tag{4}$$

has full measure in the basin of Ω. Write $W_i(x) := \sum_{j=k_{i-1}+1}^{k_i} m_j(x)p_{j-1}$. The random variables W_i are not independent. Nevertheless, by the arguments that proved (3) we can show that for any sequence $v_1, \ldots, v_{i-1} \in \mathbb{N}$ and any $N \geq N_0$,

$$\begin{aligned} &P(W_i \geq Np_{k_i-1} | W_j = v_j \text{ for } j < i) \\ &\quad = \lambda(\{x : W_i(x) \geq Np_{k_i-1}, W_j(x) = v_j \text{ for } j < i\}) \leq \rho_+^{N/3}. \end{aligned}$$

Next take $N_1 \geq N_0$ so large that $\rho_+^{N_1/3} \leq 3^{-3}$. By the binomial formula and Stirling's formula,

$$\begin{aligned} &P\left(\frac{1}{i}\#\{j \leq i : W_j \geq N_1 p_{k_j-1}\} \geq \frac{1}{3}\right) \\ &\quad \leq \sum_{j=\lfloor i/3 \rfloor}^{i} \binom{i}{j/3}[\rho_+^{N_1/3}]^{j/3} \leq i\binom{i}{i/3}3^{-i} \leq i2^{-2i/3}, \end{aligned}$$

which is summable in i. Therefore, the Borel–Cantelli lemma gives

$$\limsup_{i\to\infty} \frac{1}{i}\#\{j \le i : W_j(x) \ge N_1 p_{k_j-1}\} \le \frac{1}{3} \ \lambda\text{-a.e.}$$

Combining this with (4), and noting that $\sum_i 1/i^2 < 2$, we obtain

$$\limsup_{i\to\infty} \frac{1}{i}\#\{j \le i : n_{k_j} \ge (N_1+2)p_{k_j-1}\} \le \frac{1}{3} \text{ for } \lambda\text{-a.e. } x \in X.$$

This proves the proposition using $N = N_1 + 2$. □

Let N be the integer chosen in Proposition 4.6, and recall that λ_ω is the fiber measure that Lebesgue measure induces on the fiber $I_\omega = \pi^{-1}(\omega)$.

COROLLARY 4.1. *For μ-a.e. $\omega \in \Omega$ and $\lambda_\omega \times \lambda_\omega$-a.e. $(x, y) \in I_\omega \times I_\omega$, there are infinitely many values of i such that $|n_{k_i}(x) - n_{k_i}(y)| \le Np_{k_i-1}$.*

Proof. By a standard argument on fiber measures, Proposition 4.6 implies that for μ-a.e. $\omega \in \Omega$ and λ_ω-a.e. $x \in I_\omega$,

$$\limsup_{i\to\infty} \frac{1}{i}\#\{j \le i : n_{k_j} \ge Np_{k_j-1}\} \le \frac{1}{3}.$$

Hence the lower density of the set of integers i such that $n_{k_i} \le Np_{k_i-1}$ is at least $\frac{2}{3}$ within each fiber. The corollary follows immediately since $n_{k_i} \ge 0$. □

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. Let Ω be the attractor of f and let $\pi : I \to \Omega$ denote the factor map. We know that $(\Omega, \mathcal{C}, \mu, g)$ with $\mathcal{C} = \pi(\mathcal{B})$ and $\mu = \lambda \circ \pi^{-1}$ is a measurable automorphic factor of $(I, \mathcal{B}, \lambda, f)$ and that it is isomorphic to some odometer. Assume by contradiction that it is not the maximal automorphic factor. Then there exist $B, B' \in \mathcal{B}_+$ such that $\pi(B) = \pi(B') \in \mathcal{C}_+$, but $f^n(B) \cap f^n(B') = \emptyset$ for all $n \ge 0$. Take x and x' Lebesgue density points of B, respectively B'. By Corollary 4.1 we can assume that $\pi(x) = \pi(x')$, and that there is an integer N such that $|n_k(x) - n_k(x')| \le Np_{k-1} \le Np_k$ infinitely often. Let $I_k \ni x$ and $I'_k \ni x'$ be the maximal intervals such that $f^{n_k(x)}(I_k) = f^{n_k(x')}(I'_k) = J_k$.

Given $r \ge 0$, write $J_{k,r} = \{y \in J_k : m_{k+1}(y) = r\}$, so $J_{k,0} = J_{k+1}$. By Proposition 4.5, $\lambda(J_{k,r}) \ge \rho_-^{r+1}\lambda(J_k)$. Write $\delta := \rho_-^{N+1}$. Because x and x' are density points, we can take $n_k(x) \le n_k(x') = n_k(x) + rp_k$ (for some $r \le N$) so large that

$$\frac{\lambda(I_k \setminus B)}{\lambda(I_k)}, \frac{\lambda(I'_k \setminus B')}{\lambda(I'_k)} \le \frac{\delta}{3K^2}.$$

Here $K \ge 1$ is the distortion constant from Proposition 4.5. It follows that $\lambda(J_{k,0} \setminus f^{n_k(x')}(B')) \le (1/3K)\lambda(J_{k,0})$ and $\lambda(J_{k,r} \setminus f^{n_k(x)}(B)) \le (1/3K)\lambda(J_{k,r})$. Applying another rp_k iterates to $f^{n_k}(B) \cap J_{k,r}$ we find (using the same distortion bound K) that $\lambda(J_{k,0} \setminus f^{n_k(x)+rp_k}(B)) \le \frac{1}{3}\lambda(J_{k,0})$. Therefore, $f^{n_k(x')}(B') \cap f^{n_k(x')}(B) \ne \emptyset$. This contradicts the choice of B and B'. □

5. *The Fibonacci unimodal map*

The aim of this section is to prove that a Fibonacci map with a Cantor attractor has a circle rotation as maximal automorphic factor. For this we have to recall some facts from [**5, 6**].

5.1. *Factors of* $(\omega(c), f)$. Fix a unimodal map f. Let D_n be the image of the central branch of f^n (the largest monotone subinterval containing c). If $c \in D_n$, we say that n is a *cutting time*. The cutting times are denoted as

$$1 = S_0 < S_1 < S_2 < \cdots .$$

They are very important as they determine the combinatorial structure of the unimodal map completely. Obviously c_n is one endpoint of D_n. It is not hard to show that c_{n-S_k}, where S_k is the maximal cutting time less than n, is the other.

It can be shown that the difference of two subsequent cutting times is again a cutting time. Hence

$$S_k - S_{k-1} = S_{Q(k)},$$

for some integer function Q, which is called the *kneading map* [**10**]. For more details see [**3**]. We assume for the rest of the paper that

$$Q(k) \to \infty \quad \text{as } k \to \infty. \tag{5}$$

If the cutting times are the Fibonacci numbers $1, 2, 3, 5, 8, \ldots$, i.e. $Q(k) = \max\{0, k-2\}$, then f is called a *Fibonacci map*.

LEMMA 5.1. *If $Q(k) \to \infty$, then $\lambda(D_n) \to 0$ and $\omega(c)$ is a minimal Cantor set.*

Proof. See [**3**]. □

For the Fibonacci map, $(\omega(c), f)$ is isomorphic to $(\mathbb{S}^1, R_\gamma)$, where R_γ is the circle rotation over $\gamma = (\sqrt{5}-1)/2$. This was shown in [**13**]. In [**6**], the result was generalized to many other unimodal maps and group rotations. We will discuss some tools from [**6**]. First there is the S-adic transformation (where $S = \{S_k\}$ refers to the sequence of cutting times). Let

$$E := \{e \in \{0,1\}^{\mathbb{N}} : e_i = 1 \Rightarrow e_j = 0 \text{ for } Q(i+1) < j < i\},$$

endowed with product topology. On E we define T to be the addition of 1 by means of 'add and carry'; ones at entries i and $Q(i+1)$ carry to a one at entry $i+1$. If $S_k = 2^k$ (i.e. f is the Feigenbaum map), we recover the usual dyadic odometer. The set

$$E_0 := \{e \in E : \#\{i : e_i = 1\} < \infty\}$$

is the *greedy* representation of $\mathbb{N} \cup \{0\}$, see, e.g., [**9**]. Indeed, if $n \geq 0$, there is a canonical way of assigning a sequence $\langle n \rangle \in E_0$ such that $n = \sum_i \langle n \rangle_i S_i$. Take $i := \max\{k; S_k \leq n\}$ and set $\langle n \rangle_i = 1$. Repeat this process with $n - S_i$, etc. Then the restriction in the definition of E will automatically be satisfied.

LEMMA 5.2. *If $Q(k) \to \infty$, then $T : E \to E$ is continuous, $T(\langle n \rangle) = \langle n+1 \rangle$ and T is invertible, with a possible exception at $\langle 0 \rangle$.*

Let $\pi_1 : E \to \omega(c)$ be a projection defined on E_0 by $\pi_1(\langle n \rangle) = f^n(c)$, and extended to E by uniform continuity. Equivalently we can define

$$\pi_1(\langle n \rangle) = c_n \quad \text{and for } e \notin E_0, \quad \pi_1(e) = \cap_k D_{n_k}, \tag{6}$$

where $n_k = \sum_{i \le k} e_i S_i$. We have $f \circ \pi_1 = \pi_1 \circ T$. Note that π_1 need not be invertible.

For $x \in \mathbb{R}$, choose the fractional part $\mathrm{frac}(x) \in [-\frac{1}{2}, \frac{1}{2})$ such that $x - \mathrm{frac}(x)$ is an integer. If there exists $\alpha \in \mathbb{R}$ such that

$$\sum_i |\mathrm{frac}(\alpha S_i)| < \infty, \tag{7}$$

then we can define a second projection $\pi_0 : E \to \mathbb{S}^1$ by

$$\pi_0(e) := \sum_i e_i \,\mathrm{frac}(\alpha S_i) \bmod 1.$$

One can show that $\pi_0 \circ T(e) = \pi_0(e) + \alpha$ for all $e \in E$.

In particular, the Fibonacci numbers can be written as

$$S_k = \frac{5+3\sqrt{5}}{10}(1+\gamma)^k + \frac{5-3\sqrt{5}}{10}(-\gamma)^k,$$

where $\gamma = (\sqrt{5}-1)/2$. Hence there exists L such that

$$|\mathrm{frac}(\gamma S_k)| = |\mathrm{frac}((1+\gamma)S_k)| = \left|\mathrm{frac}\left(S_{k+1} + (1+2\gamma)\frac{5-3\sqrt{5}}{10}(-\gamma)^k\right)\right| \le L\gamma^k \tag{8}$$

for all k. Therefore, (7) is satisfied, and the projection π_0 is well defined.

PROPOSITION 5.1. *If $Q(k) \to \infty$ and $\sum_k |\mathrm{frac}(\alpha S_k)| < \infty$, then $\pi := \pi_0 \circ \pi_1^{-1}$ is a well-defined continuous mapping and the diagram*

$$\begin{tikzcd} & (E, T) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_0"] & \\ (\omega(c), f) \arrow[rr, "\pi"] & & (S^1, R_\alpha) \end{tikzcd}$$

commutes.

Proof. The case where S_k are the Fibonacci numbers has already been shown in [**13**]. Moreover, π is one-to-one, except on the backward orbit of c, where it is two-to-one. The general case was presented in [**6**]. □

The projection π can be shown to be one-to-one on a set of full Lebesgue measure on $\mathbb{S}^1$ for many other unimodal maps as well.

5.2. *Fibonacci maps with attractors.* The main result of [**5**] is the following.

THEOREM 5.1. *If f is a C^2 unimodal Fibonacci map with a sufficiently degenerate critical point, then $\omega(c)$ is an absorbing Cantor set.*

The idea of the proof is as follows. Let $u_1 = \bar{p}$, where p is the orientation reversing fixed point, and for $k \geq 2$,

$$u_k := \begin{cases} f^{-S_{k-1}}(u_{k-1}) \cap (u_{k-1}, c), & \text{if } c_{S_{k-1}} < c, \\ f^{-S_{k-1}}(\bar{u}_{k-1}) \cap (u_{k-1}, c), & \text{if } c_{S_{k-1}} > c. \end{cases} \tag{9}$$

It is shown in [**5**] that this is a valid definition. In fact, the points u_k are *closest-to-c prefixed* in the sense that if $(u_{k-1}, \bar{u}_{k-1}) \cap f^{-n}(p) \neq \emptyset$, then $n := \min\{i : f^i(u_k) \in \{p, \bar{p}\}\}$. $U_1 = I \setminus [u_1, \bar{u}_1]$ and $U_k = (u_{k-1}, u_k) \cup (\bar{u}_k, \bar{u}_{k-1})$ for $k \geq 2$. Define an *induced map* F by

$$F|U_k := f^{S_{k-1}} \quad \text{for all } k \geq 1.$$

By construction, for $k \geq 2$, $F(U_k) = (u_{k-3}, \bar{u}_{k-1})$ if $c_{S_{k-1}} > c$ and $F(U_k) = (u_{k-1}, \bar{u}_{k-3})$ if $c_{S_{k-1}} < c$. (Here $u_0 = u_{-1} = p$.) Hence F preserves the partition of I into sets U_k. Let

$$\chi_n(x) := k \quad \text{if } F^n(x) \in U_k.$$

As was shown in [**5**], there exists a constant K such that the distortion $\operatorname{dist}(F^n, J) \leq K$ for any $n \geq 0$ and the interval J on which F^n is continuous. The behavior of points under iteration of F is interpreted as a random walk. It is shown that, for maps of sufficiently large critical order, the expectation (with respect to Lebesgue measure)

$$\mathbb{E}(\chi_n - k | \chi_{n-1} = k) \geq \eta > 0, \tag{10}$$

where η is independent of k, of n, and of the precise path used to get to state U_k. A similar estimate can be made for the variances:

$$\operatorname{Var}(\chi_n - k | \chi_{n-1} = k) \leq V < \infty. \tag{11}$$

These estimates imply that $\chi_n(x) \to \infty$ λ-a.e., and this implies that $f^n(x) \to \omega(c)$ λ-a.e.

5.3. *The maximal automorphic factor of a Fibonacci map.* Let f be a Fibonacci map with an absorbing Cantor set. To be precise, assume that (10) and (11) hold. For $x \in I$ define

$$\beta_k(x) := \max\{n : \chi_n(x) \leq k\},$$

and let $b_k(x)$ be such that $F^{\beta_k(x)}|\{x\} = f^{b_k(x)}|\{x\}$. Because $\chi_m \to \infty$ λ-a.e., these sequences are defined λ-a.e. Recall that $\langle n \rangle$ denotes the S-adic representation of n. Define

$$\tilde{\pi}(x) := \lim_{k \to \infty} \pi_0(\langle b_k(x) \rangle),$$

whenever it exists.

LEMMA 5.3. *If $\tilde{\pi}(x)$ exists, then for every $N \geq 0$, $\tilde{\pi}(f^N(x))$ exists and $\tilde{\pi}(f^N(x)) = \tilde{\pi}(x) - N\gamma$.*

Proof. We will show that for every fixed $N \geq 0$, there exists $i, j \geq 0$ such that

$$F^i(x) = F^j(f^N(x)). \tag{12}$$

From this it follows that $b_k(f^N(x)) = b_k(x) - N$ for k sufficiently large, which proves the lemma.

Let $\{m_i\}$ and $\{n_j\}$ be such that $F^i|\{x\} = f^{m_i}$ and $F^j|\{f^N(x)\} = f^{n_j}$. Find i, j maximal such that $m_i \le N$ and $n_j + N \le m_{i+1}$. If in one of these cases equality holds, (12) is true. Let $y = F^i(x)$ and $y' = F^j(f^N(x))$. Then $y' = f^{N'}(y)$ for some $N' > 0$. Take also s and s' such that $y \in U_s$ and $y' \in U_{s'}$. In this notation, $N' < S_{s-1} < S_{s'-1} + N'$.

Recall that the points $\{u_k\}$ are the closest-to-c prefixed points, none omitted. Therefore,

$$f^n(u_k) \notin [u_k, \bar{u}_k] \quad \text{for all } n > 0$$

and

$$f^n(u_k) \notin [u_{k-1}, \bar{u}_{k-1}] \quad \text{for all } 0 < n < S_{k-1}. \tag{13}$$

Indeed, if $f^n(u_k) \in [u_k, \bar{u}_k]$, then u_k is not a closest-to-c prefixed point. If $f^n(u_k) \in (u_{k-1}, u_k)$ or $(\bar{u}_k, \bar{u}_{k-1})$, then u_k is not the first closest-to-c prefixed point after u_{k-1}. Finally, by equation (9), $f^{S_{k-1}}(u_k) \in \{u_{k-1}, \bar{u}_{k-1}\}$. Therefore, $f^n(u_k) \notin \{u_{k-1}, \bar{u}_{k-1}\}$ for $n < S_{k-1}$.

Since $f^{N'}(y) = y'$, $f^{N'}(U_s)$ intersects $U_{s'}$. We distinguish three cases.

- $f^{N'}(U_s) \subset U_{s'}$. Then $c \in f^{S_{s-1}}(U_s) \subset f^{S_{s-1}-N'}(U_{s'})$. This contradicts that $S_{s'-1} + N' > S_{s-1}$.
- $u_{s'} \in f^{N'}(U_s)$. By equation (13), $s' \le s - 2$. On the other hand, $f^{S_{s-1}-N'}(u_{s'}) \in F(U_s) \subset (u_{s-3}, \bar{u}_{s-3})$. Therefore, $s - 3 < s' - 1$. This contradicts $s' \le s - 2$.
- $u_{s'-1} \in f^{N'}(U_s)$. By equation (13), $s' - 1 \le s - 2$. On the other hand, $f^{S_{s-1}-N'}(u_{s'-1}) \in F(U_s) \subset (u_{s-3}, \bar{u}_{s-3})$. Therefore, $s - 3 < s' - 2$, contradicting $s' - 1 \le s - 2$.

These contradictions establish the proof. □

The main result of this section is that the map $\tilde{\pi}$ is defined λ-a.e., and that the circle rotation with appropriate measure algebra is the maximal automorphic factor.

THEOREM 5.2. *Let $(I, \mathcal{B}, \lambda, f)$ be a Fibonacci map satisfying (10) and (11) (and therefore f has an absorbing Cantor set). Then $\tilde{\pi} : I \to \mathbb{S}^1$ is defined λ-a.e. If $\mu := \lambda \circ \tilde{\pi}^{-1}$ and $\mathcal{C} := \tilde{\pi}(\mathcal{B})$, then the rotation $(\mathbb{S}^1, \mathcal{C}, \mu, R_\gamma^{-1})$ is the maximal automorphic factor.*

Proof. Let $V_k = \{x \in U_k : \chi_m(x) > k \text{ for all } m \ge 1\}$. Equations (10) and (11) show that a definite proportion of the set U_k never returns to $\cup_{i \le k} U_i$. Hence there exists $\eta_1 > 0$ such that $\lambda(V_k) \ge \eta_1 \lambda(U_k)$ for all $k \ge 1$. Moreover, the branches on F^m have a uniform distortion bound K. Therefore, taking $\eta_0 = \eta_1/K$, we find that the probability

$$P(F^m(x) \in V_k | \chi_m(x) = k) \ge \eta_0. \tag{14}$$

These estimates are independent of m and of the precise path used to get to state U_k. If $\beta_{k+1}(x) - \beta_k(x) > k$, then $F^i(x) \in \cup_{i>k} U_i \setminus V_i$ for $b_k(x) < i < \beta_k(x) + k$. Therefore,

$$P(\beta_{k+1}(x) - \beta_k(x) > k) \le (1 - \eta_0)^k.$$

The Borel–Cantelli lemma gives that $P\{\beta_{k+1}(x) - \beta_k(x) > k \text{ infinitely often}\} = 0$. Write $\tilde{\pi}_k(x) = \pi_0(\langle b_k(x) \rangle)$. For λ-a.e. $x \in I$ there exists k_0 such that

$$b_{k+1}(x) - b_k(x) = \sum_{i \ge 0} a_{k,i}(x) S_{k+i}, \quad \text{where } \sum_i a_{k,i}(x) \le k,$$

for all $k \geq k_0$. Therefore, (8) gives

$$|\tilde{\pi}_{k+1}(x) - \tilde{\pi}_k(x)| \leq \sum_{i \geq 0} a_{k,i} \operatorname{frac}(\gamma S_{k+i}) \leq k|\operatorname{frac}(\gamma S_k)| = Lk\gamma^k.$$

Hence $\tilde{\pi}_k(x)$ is a Cauchy sequence, converging to $\tilde{\pi}(x)$. This shows that $\tilde{\pi}$ is defined λ-a.e. and that $\mathcal{C} = \tilde{\pi}(\mathcal{B})$ and $\mu = \lambda \circ \tilde{\pi}^{-1}$ are well defined up to measure zero. The relation $\tilde{\pi} \circ f(x) = R_\gamma^{-1} \circ \tilde{\pi}(x)$ was already established in Lemma 5.3. It follows that R_γ^{-1} is non-singular with respect to μ, and that $(\mathbb{S}^1, \mathcal{C}, \mu, R_\gamma^{-1})$ is a measurable automorphic factor.

We need the following lemma to show that the automorphic factor is maximal.

LEMMA 5.4. *There exists a decreasing function $\xi : \mathbb{R}_+ \to [0, 1]$ converging to zero as $x \to \infty$ such that*

$$P(b_k \geq NS_k) := \lambda(\{x : b_k(x) \geq NS_k\}) \leq \xi(N)$$

for all k.

Proof. Using the η_0 obtained in (14), define for any $N \in \mathbb{R}^+$,

$$\xi(N) := 1 - (1 - (1 - \eta_0)^{M/2}) \left(1 - \frac{1}{\eta_0}(1 - \eta_0)^M\right),$$

with $M = (\log N - \log 10)/(\log 1/\gamma)$. Clearly $\xi(N) \to 0$ as $N \to \infty$. We will show that $\xi(N)$ gives the desired estimate.

Fix $k, N \in \mathbb{N}$ and let $x \in B(\omega(c))$ be arbitrary (provided $b_k(x)$ exists). Let $M = M(N) = (\log N - \log 10)/(\log 1/\gamma)$. First we calculate the probability that

$$\chi_j(x) > k + M \quad \text{for some } j \leq \beta_k(x). \tag{15}$$

If this occurs, then, since a point can jump back no more than two states under one iteration of F, $j < \beta_k(x) - M/2$. Since $F^i(x) \notin V_m$ for $j < i < \beta_k(x)$ and $k < m \leq k + M$ (cf. formula (14)), the probability that $\chi_j > k + M$ is less than $(1 - \eta_0)^{M/2}$.

Next we verify, using (14) again, that $P(\#\{j; \chi_j = m\} \geq t) \leq (1 - \eta_0)^{t-1}$. Indeed, because points have to avoid the set V_m $t - 1$ times to make t returns to U_m possible, it follows that

$$\begin{aligned} &P(\#\{j \leq \beta_k : \chi_j = m\} \geq 2M + k - m \text{ for some } 1 \leq m \leq k + M) \\ &\quad \leq \sum_{m=1}^{k+M} (1 - \eta_0)^{2M+k-m-1} \leq \frac{1}{\eta_0}(1 - \eta_0)^M. \end{aligned} \tag{16}$$

If neither (15) nor (16) occurs, i.e. if $\#\{j \leq \beta_k(x) : \chi_j(x) = m\} \leq 2M + k - m$ for all $1 \leq m \leq k + M$ and $\chi_j(x) \leq k + M$ for all $j \leq \beta_k(x)$, then

$$b_k(x) = \sum_{m=1}^{k+M} \#\{j \leq \beta_k(x) : \chi_j(x) = m\} S_{m-1} \leq \sum_{m=1}^{k+M} (2M + k - m) S_{m-1} \leq NS_k.$$

This happens with probability at least $(1 - (1 - \eta_0)^{M/2})(1 - \eta_0^{-1}(1 - \eta_0)^M) = 1 - \xi(N)$ which tends to one uniformly in k as $N \to \infty$. This proves the lemma. □

Returning to the proof of Theorem 5.2, we copy the arguments of Theorem 4.4. That is, we find a subsequence $\{k_i\}$ of $\mathbb{N}$ which increases fast enough so that

$$\sum_i P\left(b_{k_i} \geq \frac{1}{i^2} S_{k_{i+1}}\right) \leq \sum_i \xi\left(\frac{1}{i^2}\frac{S_{k_{i+1}}}{S_{k_i}}\right) < \infty.$$

Then by the Borel–Cantelli lemma,

$$X := \left\{x : \exists j\ \forall i \geq j, b_{k_i} \leq \frac{1}{i^2} S_{k_{i+1}}\right\}$$

has full measure in the basin of $\omega(c)$. Take N_1 so large that $\xi(N_1) \leq 3^{-3}$. A proof similar to the one of Lemma 5.4 gives $P(b_{k_i} - b_{k_{i-1}} \geq N_1 S_{k_i}) \leq \xi(N_1) \leq 3^{-3}$. Then we can derive (cf. Proposition 4.6) that $\liminf_i i^{-1}\#\{j \leq i : b_{k_j}(x) \leq N S_{k_j}\} \geq \frac{2}{3}$ for $N = N_1 + 2$ and λ-a.e. $x \in B(\omega(c))$. Given $s \in \mathbb{S}^1$, let $I_s := \tilde{\pi}^{-1}$ be the fiber over s and let λ_s be the fiber measure that λ induces on I_s. Then (cf. Corollary 4.1) for μ-a.e. s and $\lambda_s \times \lambda_s$-a.e. $(x, x') \in I_s \times I_s$, we have established that $|b_k(x) - b_k(x')| \leq N S_k$ infinitely often.

The last step in the proof is to assume by contradiction that $(\mathbb{S}^1, \mathcal{C}, \mu, R_\gamma^{-1})$ is not the maximal automorphic factor. Then there exist $B, B' \in \mathcal{B}_+$ such that $\pi(B) = \pi(B')$ and $\lambda(f^n(B) \cap f^n(B')) = 0$ for all $n \geq 0$. We choose Lebesgue density points x and x' of B and B', respectively, such that $\tilde{\pi}(x) = \tilde{\pi}(x')$. By the above arguments we can assume that $|b_k(x) - b_k(x')| \leq N S_k$ infinitely often. Take such a k; then $\langle b_k(x)\rangle_i = \langle b_k(x')\rangle_i$ for all $i > k + P$ for $P = P(N) = -2\log N/\log\gamma$. Abbreviate $\chi = \chi_{\beta_k(x)}$ and $\chi' = \chi_{\beta_k(x')}$, so $F^{\beta_k(x)}(x) \in U_\chi$ and $F^{\beta_k(x')}(x') \in U_{\chi'}$. Because $\tilde{\pi}(x) = \tilde{\pi}(x')$,

$$\langle b_k(x) + S_{\chi-1}\rangle_i = \langle b_k(x') + S_{\chi'-1}\rangle_i$$

for all $i \leq k$. By definition of β_k, both $F^{\beta_k(x)+1}(x)$ and $F^{\beta_k(x')+1}(x')$ are contained in $(u_k, \bar{u}_k)$. Let W_k be the component of $U_\chi \cap F^{-1}([u_k, \bar{u}_k])$ that contains $F^{\beta_k(x)}(x)$; similarly, let W_k' be the component of $U_{\chi'} \cap F^{-1}([u_k, \bar{u}_k])$ that contains $F^{\beta_k(x')}(x')$. Take

$$d := S_{\chi'-1} + \sum_{i=k+1}^{k+P} \langle b_k(x') + S_{\chi'-1}\rangle_i S_{i-1} \quad \text{and} \quad d' := S_{\chi-1} + \sum_{i=k+1}^{k+P} \langle b_k(x) + S_{\chi-1}\rangle_i S_{i-1}.$$

Then $D := b_k(x) + d = b_k(x') + d'$. Find the unique interval $T \subset W_k$ whose orbit is given by $\langle d\rangle$ in the following sense: let $i_0, i_1, \ldots, i_n$ (with $1 \leq n \leq P$) be the indices for which $\langle d\rangle_i = 1$. Then T is taken such that

$$F^1(T) \subset U_{i_1}, F^2(T) \subset U_{i_2}, \ldots, F^n(T) \subset U_{i_n} \quad \text{and} \quad F^{n+1}(T) = [u_{k+P}, \bar{u}_{k+P}].$$

The interval $T' \subset W_k'$ is chosen similarly. Then $F^{n+1}(T) = F^{n'+1}(T')$ (see Figure 1). Recall that the distortion of the branches of iterates of F is uniformly bounded by K. From this, one can derive that there exists $\varepsilon = \varepsilon(P, K) > 0$ such that $\lambda(T) \geq \varepsilon\lambda(W_k)$ and $\lambda(T') \geq \varepsilon\lambda(W_k')$.

We now take intervals $J \ni x$ and $J' \ni x'$ such that the maps $f^{b_k(x)} : J \to W_k$ and $f^{b_k(x')} : J' \to W_k'$ are monotone onto, and such that

$$\frac{\lambda(J \setminus B)}{\lambda(J)}, \frac{\lambda(J' \setminus B')}{\lambda(J')} \leq \frac{\varepsilon}{3K^2}.$$

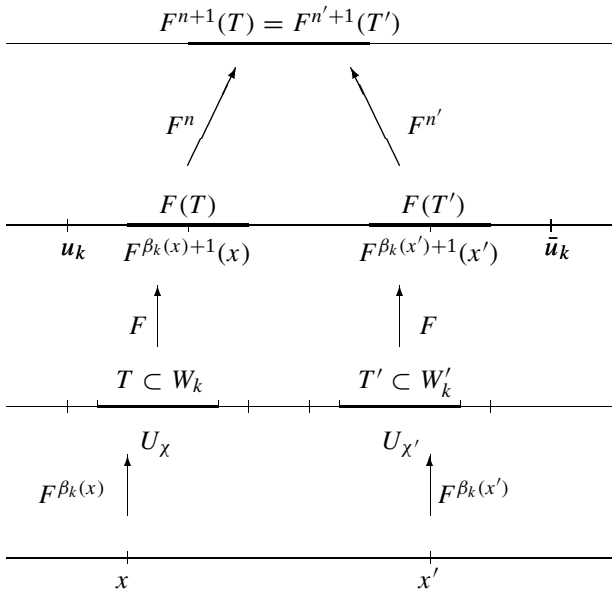

FIGURE 1. Construction of T, T', W_k and W'_k.

Since x and x' are density points, such intervals can be found for k sufficiently large. Therefore,

$$\frac{\lambda(T \setminus f^{b_k(x)}(J \cap B))}{\lambda(T)}, \frac{\lambda(T' \setminus f^{b_k(x')}(J' \cap B'))}{\lambda(T')} \leq \frac{1}{3K}.$$

Applying F^{n+1} to $f^{b_k(x)}(J) \cap T$ and $F^{n'+1}$ to $f^{b_k(x')}(J') \cap T'$, and using the distortion bound K once more, we obtain

$$\begin{aligned} 0 < \tfrac{1}{3}\lambda(F^{n+1}(T)) &\leq \lambda(F^{n+1} \circ f^{b_k(x)}(J \cap B) \cap F^{n'+1} \circ f^{b_k(x')}(J' \cap B')) \\ &\leq \lambda(f^D(B) \cap f^D(B')). \end{aligned}$$

This contradicts the choice of B and B'. □

Remark 1. We now have $\tilde{\pi} : I \to \mathbb{S}^1$ (defined λ-a.e.) such that $\tilde{\pi} \circ f = R_\gamma^{-1} \circ \tilde{\pi}$ and $\pi : \omega(c) \to \mathbb{S}^1$ such that $\pi \circ f = R_\gamma \circ \pi$. One can show that $\tilde{\pi}$ is defined on $\omega(c)$ and that $\tilde{\pi}(y) = -\pi(y)$ for all $y \in \omega(c)$. This relation plays no role in our results, so we omit the proof.

Remark 2. In view of the previous remark, $(\tilde{\pi}|\omega(c))^{-1} \circ \tilde{\pi}$ gives a factor map from I directly onto the attractor. One might expect that for λ-a.e. x there exists $y \in \omega(c)$ such that $|f^n(x) - f^n(y)| \to 0$. This is not true, in spite of the fact that for the candidate $y \in \tilde{\pi}^{-1} \circ \tilde{\pi}(x) \cap \omega(c)$ there is a sequence $\{n_i\}$ such that $|f^{n_i}(x) - c|, |f^{n_i}(y) - c|$ simultaneously tend to zero. The reason is as follows. Assume that $f^{n_i}(x) \in U_k$ is so

close to c_{S_k} or $\bar{c}_{S_k}$ that $f^{n_i+S_{k-1}}(x) \in U_{k+1}$. Then $f^{n_i+S_{k-1}+S_k-1}(x) = f^{n_i+S_{k+1}-1}(x)$ is close to $c_{S_{k+2}-1}$. Checking the kneading invariant of f shows that $c_{S_{k+2}-1}$ is close to $f^{-1}(c) \cap [c, c_1]$ if k is even and close to $f^{-1}(c) \cap [c_2, c]$ if k is odd. If at the same time $f^{n_i}(y) \in \cup_{j\geq k+2} U_j$, then $f^{n_i+S_{k+1}-1}(y)$ is close to $c_{S_{k+1}-1}$, which is close to $f^{-1}(c) \cap [c, c_1]$ if k is odd and close to $f^{-1}(c) \cap [c_2, c]$ if k is even. Therefore, $\limsup |f^n(x) - f^n(y)| \geq \operatorname{diam}(f^{-1}(c)) > 0$.

6. *A dissipative exact unimodal map*

The results of §5 can be generalized to other unimodal maps with absorbing Cantor sets. In [**4**] combinatorial conditions are given under which a unimodal map f with sufficiently large critical order has an absorbing Cantor set. The main condition on the kneading map Q is that $k - Q(k)$ is bounded. This applies to many examples from [**6**], for which $(\omega(c), \rho, f)$ (ρ being the unique invariant probability measure) is shown to be isomorphic to some circle and torus rotation with Haar measure. In particular, for the maps with kneading maps $Q(k) = \max(k - d, 0)$ for $d = 2, 3, 4$, $(\omega(c), \rho, f)$ is shown to be isomorphic to a rotation on a $(d-1)$-dimensional torus.

However, for $d \geq 5$, $(\omega(c), \rho, f)$ has no non-trivial group as factor [**6**]. In this section we show that the unimodal map f with kneading map $Q(k) = k - 5$ for $k \geq 5$ is exact on I, even in the presence of an absorbing Cantor set. The proof that f, for a sufficiently large critical order, has a Cantor attractor is similar to the proof for the Fibonacci map. The inequalities (10) and (11) can be proven. The difference from the Fibonacci map is that the leading root α of the equation $x^5 - x^4 - 1 = 0$ is not a Pisot–Vijayaraghavan number. More precisely, this equation has two roots on, two roots inside, and one root outside the unit circle. Therefore, $|\operatorname{frac}(\alpha S_i)|$ is not summable and the map π_0 cannot be defined. Related to this is the following lemma.

LEMMA 6.1. *Suppose S_k are the cutting times corresponding to the kneading map $Q(k) = \max(k-5, 0)$, i.e. $S_k = k + 1$ for $0 \leq k \leq 4$ and $S_k = S_{k-1} + S_{k-5}$ otherwise. Then for $k \geq 3$*

$$S_k = \begin{cases} S_{k-2} + S_{k-3}, & \text{if } k \equiv 1 \text{ or } 4 \bmod 6, \\ S_{k-2} + S_{k-3} + 1, & \text{if } k \equiv 2 \text{ or } 3 \bmod 6, \\ S_{k-2} + S_{k-3} - 1, & \text{if } k \equiv 0 \text{ or } 5 \bmod 6. \end{cases}$$

Proof. Straightforward by induction. □

THEOREM 6.1. *Let f be the unimodal map with kneading map $Q(k) = \max(0, k-5)$. Suppose that the critical order ℓ is so large that f has a Cantor attractor, and a fortiori, (10) and (11) hold. Then f is Lebesgue exact.*

Proof. Take A arbitrary such that $\lambda(A) > 0$. Without loss of generality, we can assume that $A \subset (u_1, \bar{u}_1)$. We will show that Proposition 2.1 applies. Because f has a Cantor attractor, and *a fortiori* $\chi_n(x) \to \infty$ λ-a.e., we can assume that $\chi_n(x) \to \infty$ for all $x \in A$. Let $x \in A$ be a density point of A, such that $f(x)$ is a density point of $f(A)$.

The proof of the existence of Cantor attractors [**4**] gives rise to the following distortion estimate: for any n and any J on which $F^n|J = f^m|J$ is continuous, we have

$\mathrm{dist}(f^m, J) \leq K$, where K depends only on ℓ. The proofs also yield that there exists $C > 0$ such that

$$\frac{1}{C\ell} \leq \frac{|u_k - u_{k+1}|}{|u_k - c|} \leq \frac{C}{\ell}$$

for all k. Let $\varepsilon = 1/6C^2\ell^2K^3 > 0$. Because x is a density point, there exists $J \ni x$ such that

$$\frac{\lambda(A \cap J')}{\lambda(J')} \geq 1 - \varepsilon$$

for any subinterval J' such that $x \in J' \subset J$. Take from now on n so large that $J_n \subset J$ whenever $J_n \ni x$ is the maximal interval on which $F^n|J_n$ is continuous.

Assume that $\chi_n(x) = k$ where $k \equiv 2, 3 \bmod 6$. Let $U \subset U_k$ be the component containing $F^n(x) = f^m(x)$. Define $\tilde{W}_1 \subset U$ to be the maximal interval such that $F(\tilde{W}_1) \subset U_k$ and $F^2(\tilde{W}_1) \subset U_{k+1}$. Similarly $\tilde{W}_2 \subset U$ will be the maximal interval such that $F(\tilde{W}_2) \subset U_{k+3}$. Since $(u_{k+5}, \bar{u}_{k+5}) \subset F(U_{k+1})$, $F(U_{k+3}) \subset (u_{k-20}, \bar{u}_{k-20})$, the overlap $F(U_{k+1}) \cap F(U_{k+3})$ satisfies $\lambda(F(U_{k+1}) \cap F(U_{k+3})) \geq \frac{1}{2}\lambda(F(U_{k+1}))$ for ℓ sufficiently large. We can find maximal intervals $W_1 \subset \tilde{W}_1$ and $W_2 \subset \tilde{W}_2$ such that $F^3(W_1) = F^2(W_2)$, and

$$\frac{\lambda(W_1)}{\lambda(U)}, \frac{\lambda(W_2)}{\lambda(U)} \geq \frac{1}{2C^2\ell^2K}.$$

Let $V_1 = f^{-m}(W_1) \cap J_n$ and $V_2 = f^{-m}(W_2) \cap J_n$. Using the distortion argument once more, we derive that

$$\frac{\lambda(V_1)}{\lambda(J_n)}, \frac{\lambda(V_2)}{\lambda(J_n)} \geq \frac{1}{2C^2\ell^2K^2}.$$

By the choice of ε,

$$\frac{\lambda(A \cap V_1)}{\lambda(V_1)}, \frac{\lambda(A \cap V_2)}{\lambda(V_2)} \geq 1 - \frac{1}{3K}.$$

Remember that $k \equiv 2, 3 \bmod 6$, so by Lemma 6.1 $S_k + S_{k+1} = S_{k+3} - 1$. Let $N := m + S_k + S_k + S_{k+1} = m - 1 + S_k + S_{k+3}$. Then

$$\lambda(f^N(A) \cap f^{N+1}(A)) \geq \tfrac{1}{3}\lambda(F(U_{k+1}) \cap F(U_{k+3})) > 0.$$

It follows that $\lambda(f^{-N} \circ f^{N+1}(A) \cap A) > 0$. This is the assumption of Proposition 2.1.

The cases $k \equiv 0, 1, 4, 5 \bmod 6$ can be dealt with in a similar way. □

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