Technical Report CIT-CDS 03-006

Flocking with Obstacle Avoidance *

Reza Olfati Saber †
e-mail: olfati@cds.caltech.edu

February 15, 2003

Abstract

In this paper, we provide a dynamic graph theoretical framework for flocking in presence of multiple obstacles. In particular, we give formal definitions of nets and flocks as spatially induced graphs. We provide models of nets and flocks and discuss the realization/embedding issues related to structural nets and flocks. This allows task representation and execution for a network of agents called α-agents. We also consider flocking in the presence of multiple obstacles. This task is achieved by introducing two other types of agents called β-agents and γ-agents. This framework enables us to address split/rejoin and squeezing maneuvers for nets/flocks of dynamic agents that communicate with each other. The problems arising from switching topology of these networks of mobile agents make the analysis and design of the decision-making protocols for such networks rather challenging. We provide simulation results that demonstrate the effectiveness of our theoretical and computational tools.

1 Introduction

A special behavior of large number of interacting dynamic agents called "flocking" has attracted many researchers from diverse fields of scientific and engineering disciplines. The term "flocking" in English means "moving together in large numbers". This behavior exists in the nature in the form of flocking of birds, schooling of fish, and swarming of bacteria [19].

Reynolds introduced three *ad-hoc* protocols for autonomous agents moving in a 3-D space called "boids" [17]. The combination of these three protocols led to creation of the first animation of flocking in 1987. In [17], the society of boids is viewed as a distributed system. This is precisely the point of view in the present paper. No analysis of the proposed protocols or definition of flocking is given in [17]. Later, the work of Reynolds motivated a group of scientists to simulate and analyze one of the three protocols of Reynolds for attitude alignment in Vicsek *et al.* [20]. A similar attitude alignment problem was recently investigated by Jadbabaie *et al.* [9]. That work is motivated by the work of Vicsek *et al.*. Similar to the work of Vicsek *et al.*, in [9], no connections are established between the first two flocking protocols of Reynolds and the connectivity of the

*This research is supported in part by AFOSR under the grant F49620–01–1–0361 and by DARPA under the grant F33615–98–C–3613.

†Postdoctoral scholar, Division of Engineering and Applied Science, Control and Dynamical Systems, California Institute of Technology, Pasadena, CA, 91125.

network of mobile agents. The important contribution of the work of Jadbabaie *et al.* is that connectivity in all times is not needed and connectivity of the network on average is sufficient for alignment of the agents. In Liu *et al.* [10], stability analysis of swarms with fixed interconnection topology is studied. Furthermore, Gazi and Passino [6] use social potentials to create cohesion in swarms.

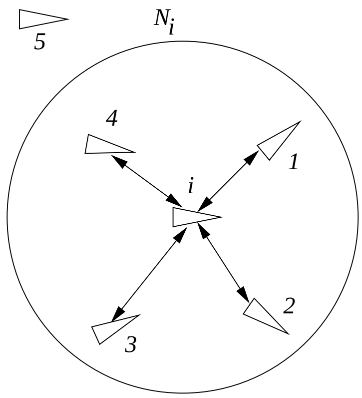
Obstacle avoidance for single-vehicle systems has a long history in robotics [18]. This is certainly not the case for multi-vehicle obstacle avoidance. Obstacle avoidance for multi-agent systems using gyroscopic forces with connections to flocking is recently discussed by Chang *et al.* in [3]. This method only uses local information and relies on the work of Chang and Marsden [2]. The use of gyroscopic forces is an alternative approach to obstacle avoidance using centralized constructions of potential functions by Rimon and Kodischek in [18].

In this paper, our main goal is design and analysis of distributed algorithms (or protocols) for cooperative decision-making in networks of mobile agents. Flocking is an example of cooperative behavior in a complex system that is formed by the communication and interaction among large number of agents. Flocking in presence of multiple environmental obstacles or adversarial agents, leads to solving decision-making problems in networks of dynamic agents with *switching topology* [11]. Particularly, this is the case in missions that require low-altitude flight of unmanned air vehicles (UAVs). Moreover, in a competitive team-on-team game with two teams A and B, some of the players of the team B might block the view or communication between the members of team A. In this work, we consider flocking in presence of multiple fixed obstacles. These environmental obstacles can be viewed as immobile players of team B. The role of the obstacles (or players of team B) is to break the communication links between the agents in a flock (i.e. members of team A called α-agents).

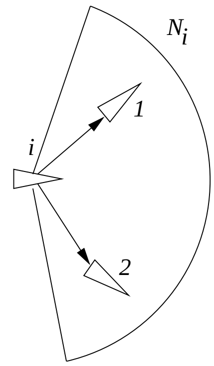
In general, groups of agents that move in large numbers might not be able to pass through tight spaces together while maintaining safe inter-agent distances. This phenomenon is also known as "escape panic" [7]. Escape panic claims the lives of so many people in disasters that occur in crowed enclosed places with few exits (see [7] and Figure 8).

The main contributions of this work is to introduce a theoretical framework for flocking with multiple obstacle avoidance that relies on a combination of dynamical systems, mechanics, and graph theory. In particular, formal definitions of a "flock" and the behavior of "flocking" are presented. The problem of performing flocking is reduced to solving two separate problems: the structural stabilization of a flock and the tracking problem for the flock as a whole. This is formally stated as a flocking separation principle. The problem of flocking in presence of multiple obstacles is also discussed. The solution to the obstacle avoidance problem uses the same theoretical framework as structural stabilization of flocks. As a by-product, we manage to perform *split, rejoin, and squeezing maneuvers* for flocks of mobile agents. The issues regarding solving decision-making problems with limited (or local) communication in a network with switching topology are discussed [11]. It turns out that a number of open problems are left that need to be addressed in the future.

An outline of the paper is as follows: in Section 2 nets and flocks are defined as spatially induced graphs. In Section 3, the notion of structural nets are introduced as less restrictive forms of multi-agent formation graphs. In Section 4, structural energy of nets and flocks is defined as the Hamiltonian of a dynamic graph. In Section 5, the issues regrading the differentiability of structural energy of nets and flocks with switching topology is discussed. In Section 6, two types of protocols for maintaining a given distance between two α-agents are given. In Section 7, the



(a)



(b)

Figure 1: (a) A spherical neighborhood (or shell) and (b) a conic neighborhood.

connections between Reynolds three rules of flocking and the interaction protocol between two α-agents are clarified. In Section 8, the results on flocking using dissipation of structural energy of nets are presented. In Section 9, quasi-realization of nets and the notion of the defect factor of conformations of structural nets are discussed. A separation principle for flocking is given in Section 10. In Section 11, obstacle avoidance and the definition and behavior of β and γ agents are stated. Finally, in Section 12, concluding remarks are made.

2 Nets and Flocks: Spatially Induced Graphs

In this section, we use several basic notions from graph theory. Further information on graph theory is available in [1, 5, 8]. A graph is denoted by $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ as the set of nodes and $\mathcal{E}$ as the set of edges of the graph. The order of graph is the number of nodes of the graph $n = |G| = |\mathcal{V}|$. An edge is denoted by ij, (i, j) or (v_i, v_j) with $i, j \in \mathbb{N}$ as the node indices and $v_i, v_j \in \mathcal{V}$.

Let $q_i \in \mathbb{R}^d$ (e.g. $d = 2, 3$) with $i \in \mathbb{N}$ denote the position of the ith node v_i. Define $q = \text{col}(q_i) \in \mathbb{R}^{nd}$ where $n = |\mathcal{V}|$ is the number of nodes. A *spherical neighborhood* (or *shell*) of radius $r_i \geq 0$ around q_i is defined as

$$B(q_i, r_i) := \{x \in \mathbb{R}^d : \|x - q_i\| \leq r_i\}. \tag{1}$$

Denote $\mathbf{r} = \text{col}(r_i)$. We refer to the pair $(q, \mathbf{r})$ as a *cluster* with *configuration* q and vector of radii $\mathbf{r}$. We define a *spatial adjacency matrix* $\mathcal{A}(q) = [a_{ij}(q)]$ induced by a cluster q as follows

$$a_{ij}(q) = \begin{cases} 1, & \text{if } q_j \in B(q_i, r_i), j \neq i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The spatial adjacency matrix $\mathcal{A}(q)$ defines a *spatially induced graph* $G(q)$. We call $G(q)$ a *net*, i.e. a net is a graph that is spatially induced by a cluster. Figure 1(a) shows an example of a node with a spherical neighborhood and the set of neighbors N_i defined as

$$N_i = N_i(q) := \{j : a_{ij}(q) > 0\} \tag{3}$$

In general, a net is a directed graph (or digraph). This is because if $r_i > r_j$, then $j \in N_i$ does not necessarily imply $i \in N_j$. Notice that the graph $G(q)$ is undirected (i.e. $a_{ij}(q) = a_{ji}(q)$ for all i, j) for all q if and only if $r_i = r_j$ for all i, j. This means that if all the nodes have reached an agreement regarding the radius of their shells, then the induced net by cluster $(q, \mathbf{r})$ with $\mathbf{r} = (r, \dots, r)^T$ is an undirected net.

If $r_i = r$ for all i, we denote a cluster by (q, r) and call it a *uniform cluster*. Furthermore, the graph $G(q)$ that is induced by a uniform cluster is called a *uniform net*. All uniform nets that are induced by clusters with spherical neighborhoods are undirected. However, this is not the case for clusters with conic neighborhoods as shown in Figure 1(b).

Consider a cluster with nodes that have a position $x_i \in \mathbb{R}^2$ and an attitude (or heading angle) $\theta_i \in \mathbb{R}$. The configuration of the node i can be written as $q_i = \mathrm{col}(x_i^T, \theta_i) \in \mathbb{R}^3$. A *conic neighborhood* of node i is defined as

$$C(x_i, \theta_i, r_i, \varphi_i) := \{(x, \theta) \in \mathbb{R}^2 \times \mathbb{R} : \|x - x_i\| \leq r_i, |\theta - \theta_i| \leq \varphi_i\}. \tag{4}$$

The use of conic neighborhoods for flocking is due to Reynolds [17]. A cluster with nodes that have conic neighborhoods can be represented as $(q, \mathbf{r}, \vec{\varphi})$ where $\mathbf{r} = \mathrm{col}(r_i)$ and $\vec{\varphi} = \mathrm{col}(\varphi_i)$. Similarly, a uniform cluster with conic shells is represented by (q, r, φ_0) where $r_i = r$ and $\varphi_i = \varphi_0$ for all i. The spatial adjacency matrix for a uniform cluster with conic shells can be written as

$$a_{ij}(q) = \begin{cases} 1, & \|x_j - x_i\| \leq r, |\theta_j - \theta_i| \leq \varphi_0, j \neq i \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Remark 1. In general, for agents with configurations $(x_i, R_i) \in SE(d)$, a conic region in $\mathbb{R}^d$ $(d \geq 2)$ can be defined as

$$S_i = \{(x, R) \in \mathbb{R}^d \times SO(d) : \|x - x_i\| \leq r_i, \varphi(R, R_i) \leq \varphi_i\}, \tag{6}$$

where $0 \leq \varphi_i < 2$ and $\varphi(R, R_i) = \frac{1}{2} Trace(I - R^T R_i)$ is a distance-type function on the Lie group $SO(d)$.

In general, a uniform net induced by a uniform cluster with conic shells is a directed graph. This makes the motion planning analysis related to nets/flocks induced by clusters with conic shells rather challenging.

Definition 1. (flock) A *flock* is a weakly connected net.

Note that a digraph is called weakly connected if there exists a path that connects any two distinct nodes of the graph irrespective of the direction of the edges that constitute the path, i.e. for any two nodes $i, j, i \neq j$, there exists a set of indices $i_1, i_2, \dots, i_m$ with $i_1 = i$ and $i_m = j$ that defines a path

$$\pi_{i,j} = \{(i_1, i_2), (i_2, i_3), \dots, (i_{m-1}, i_m)\}$$

such that for all $k \in \{1, \dots, m-1\}$, $\mathcal{E}(G)$ contains (i_k, i_{k+1}) or (i_{k+1}, i_k).

Suppose a net $\mathcal{N} = G(q)$ consists of m weakly connected components, then $\mathcal{N}$ contains m flocks $F_1(q), F_2(q), \dots, F_m(q)$. We refer to $|\mathcal{N}|$ (i.e. the order of a net/graph) as the *population* of a net. The density of a net is defined as follows

$$\delta_0 = \delta_0(\mathcal{N}) := \frac{\max_k |\mathcal{F}_k|}{\sum_{k=1}^{m} |\mathcal{F}_k|} \tag{7}$$

Apparently, a net of density 1 is called a flock. We call a net with a relatively high density satisfying $0 < 1 - \delta_0 \ll 1$ a *quasi-flock.*

Throughout this paper, we assume that all the nets are uniform. For a uniform net, the elements of the adjacency matrix $\mathcal{A}(q)$ can be rewritten as

$$a_{ij}(q) = \begin{cases} 1, & \|q_j - q_i\| \leq r, j \neq i \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Note that $a_{ii}(q) = 0$. Let us define the *in-degree* and *out-degree* of node v_i as

$$\begin{array}{rcl} \deg_{in}(v_i) & = & \sum_{j \in N_i} a_{ji} = \sum_j a_{ji}(q) \\ \deg_{out}(v_i) & = & \sum_{j \in N_i} a_{ij} = \sum_j a_{ij}(q) \end{array} \tag{9}$$

The *spatial degree matrix* $\Delta(q) = [\Delta_{ij}(q)]$ is diagonal matrix with diagonal elements

$$\Delta_{ii}(q) = \deg_{out}(q) \tag{10}$$

The *spatial Laplacian matrix* $L(q)$ of the net $G(q)$ is defined as

$$L(q) = \Delta(q) - \mathcal{A}(q) \tag{11}$$

In general, if $q(t)$ changes in time, both the net $G(q(t))$ and its Laplacian $L(q(t))$ (possibly) change in time. This creates a *network with switching topology* [15]. We call a net (or graph) *balanced* if and only if $\deg_{in}(v_i) = \deg_{out}(v_i)$ for all nodes $v_i \in \mathcal{V}$. Let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$, then the following result summarizes the properties of a balanced net $G(q)$.

Proposition 1. *Let $\mathcal{N} = G(q)$ be a directed net. Then, the following properties hold:*

i) If $\mathcal{N}$ is uniform, then $\mathcal{N}$ is a balanced net.

ii) $\mathcal{N} = G(q)$ is balanced if and only if $\mathbf{1}^T L(q) = 0$ for all clusters $(q, \mathbf{r})$.

Proof. All uniform nets are undirected and all undirected graphs are balanced. This proves part i). The condition $\mathbf{1}^T L(q) = 0$ means that the column sum of $L(q)$ is zero for all columns. On the other hand, the ith column sum of $L(q)$ is equal to $\deg_{in}(v_i) - \deg_{out}(v_i)$. Thus, $\mathbf{1}^T L(q) = 0$ if and only if $\deg_{in}(v_i) - \deg_{out}(v_i) 0$ for all nodes v_i, i.e. $G(q)$ is balanced. □

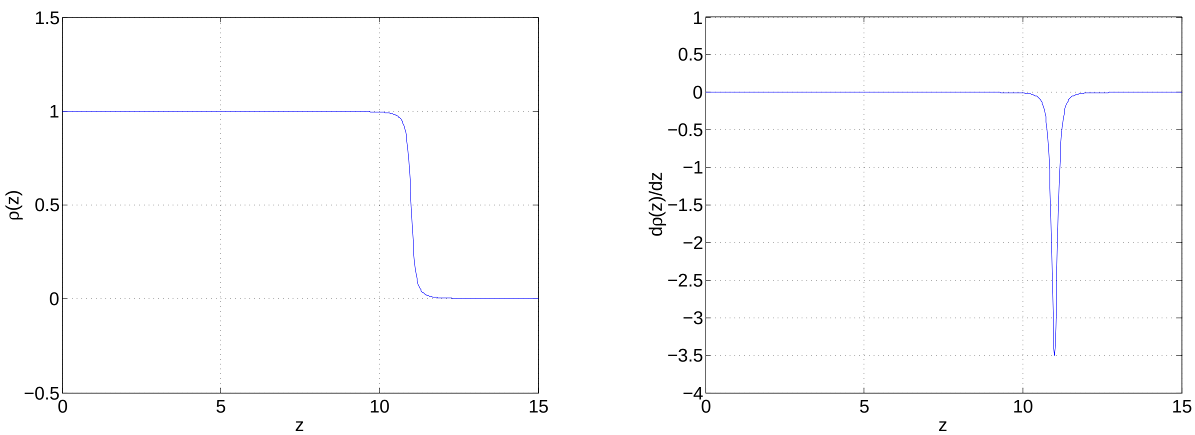
The result in Proposition 1 can be used for attitude alignment in a flock of agents or any other consensus problem for agents with dynamics

$$\dot{\theta}_i = \omega_i \tag{12}$$

using the *agreement protocol*

$$\omega_i = \sum_{j \in N_i(q)} (\theta_j - \theta_i) \tag{13}$$

with nonsmooth (or smooth) adjacency matrix $\mathcal{A}(q)$. Further information on agreement in networks with switching information flow is available in [15].



(a) (b)

Figure 2: (a) A bump function $\rho(z)$ and (b) $\rho'(z)$.

Now, we show that even for a small net with $n = 2$ nodes, the graph Laplacian $L(q)$ is a discontinuous function of q for nets. The following two adjacency matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

correspond to the cases where $a_{12} = 0$ and $a_{12} = 1$, respectively. The Laplacians associated with A_1 and A_2 are given by

$$L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15}$$

Clearly, all four elements of $L(q)$ are discontinuous functions of q and the jumps occur at points q satisfying

$$\|q_2 - q_1\| = r.$$

This is certainly not a problem if one does not try to differentiate $L(q(t))$ as a function of time.

In the following, we explain a smoothing process for $L(q)$. Consider a smooth *bump function* $\rho(z) : \mathbb{R} \to [0,1]$ satisfying the following properties:

$$\rho(z) = \begin{cases} 1, & z \le k_0 \\ 0, & z \ge 1 \\ \in (0,1) & \text{otherwise.} \end{cases}, \; \rho'(z) = \frac{d\rho(z)}{dz} = \begin{cases} 0, & z \le k_0 \\ 0, & z \ge 1 \\ \le 0 & \text{otherwise.} \end{cases} \tag{16}$$

where $0 < k_0 < 1$. We assume that all the bump functions in this work satisfy $|\rho'(z)| \le L_\rho$ (i.e. have uniformly bounded derivative). An example of a bump function $\rho(z/r)$ with $k_0 = \frac{5}{6}$ and $r = 12$ is shown in Figure 2. Clearly, for all $z \le k_0 r$, $\tilde{\rho}(z) = \rho(z/r) = 1$.

For a uniform cluster (q, r) with spherical shells a *smooth adjacency matrix* $\mathcal{A}(q) = [a_{ij}(q)]$ by its elements

$$a_{ij}(q) := \rho(\|q_j - q_i\|/r) \tag{17}$$

Similarly, for a uniform cluster (q, r, φ_0) with conic shells, the elements of the adjacency matrix of the directed net $G(q)$ are defined as

$$a_{ij}(q) := \rho(\|x_j - x_i\|/r)\rho(|\theta_j - \theta_i|/\varphi_0) \tag{18}$$

Thus, for a cluster of $n = 2$ nodes with spherical shells, a smooth Laplacian of the net induced by this net is given by

$$L(q) = \rho(\|q_j - q_i\|/r) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{19}$$

The following lemma guarantees that the calculations needed for applying the protocols presented in this paper can be performed in a distributed manner.

Lemma 1. *For all the nodes $j \notin N_i$ that are not among the neighbors of node i, $a_{ij}(q) \equiv 0$ and $\nabla a_{ij}(q) \equiv 0$.*

Proof. By direct calculation. □

3 Structural Nets/Flocks and their Realizations

Let us refer to all the nodes of a net/flock as *α-agents*. Roughly speaking, the objective of an α-agent is to maintain a distance d_α between itself and another α-agent provided that $0 < d_\alpha < r$. In the presence of obstacles, an α-agent tries to avoid collision to the obstacles and meanwhile try to maintain a distance d_α between itself and other neighboring α-agents. However, in many situations including *split/rejoin maneuver* and *squeezing maneuver* these two objectives of an α-agent are conflicting goals. Thus, an α-agent needs to prioritize the tasks assigned to it and give the highest priority to avoiding collision to obstacles. Later, we formalize what we mean by task prioritization. First, we assume that temporarily there are no obstacles and formalize the task of keeping a distance d_α from other neighboring α-agents.

Remark 2. For future use, we need to distinguish between the agents in a net and other types of agents that are due to the presence of obstacles or adversarial agents in an environment.

A net $G(q)$ of α-agents is called an *α-net*. We refer to the pair (d_α, r) with $k_0 = d_\alpha/r < 1$ as an *indefinite structural α-net*. A *definite structural α-net* is a triplet (n, d_α, r) where n in an integer.

Note 1. Let (q, r) be a uniform cluster. By calling $G(q)$ a "net", we mean both $G(q)$ as a "graph" $G = (\mathcal{V}, \mathcal{E}) = G(q)$ and $G(q)$ as a"framework" (G, q) (i.e. a graph together with the coordinates of its nodes) whichever makes sense in the context.

We say a net $G(q)$ is a *realization* (or *embedding*) of the indefinite structural net (d_α, r) in $\mathbb{R}^2$ if and only if the following condition is satisfied

$$\|q_j - q_i\| = d_\alpha, \forall j \in N_i \tag{20}$$

for all the α-agents $v_i \in \mathcal{V}$. Furthermore, if $G(q)$ is a net with n nodes, then $G(q)$ is referred to as a realization (or embedding) of a definite structural net (n, d_α, r).

It is rather trivial to provide a realization of a definite (or an indefinite) structural net. A polygon with n vertices and equilateral sides of length d_α specifies a cluster that induces a cycle of length n as a (generic) realization of a definite structural net (n, d_α, r).

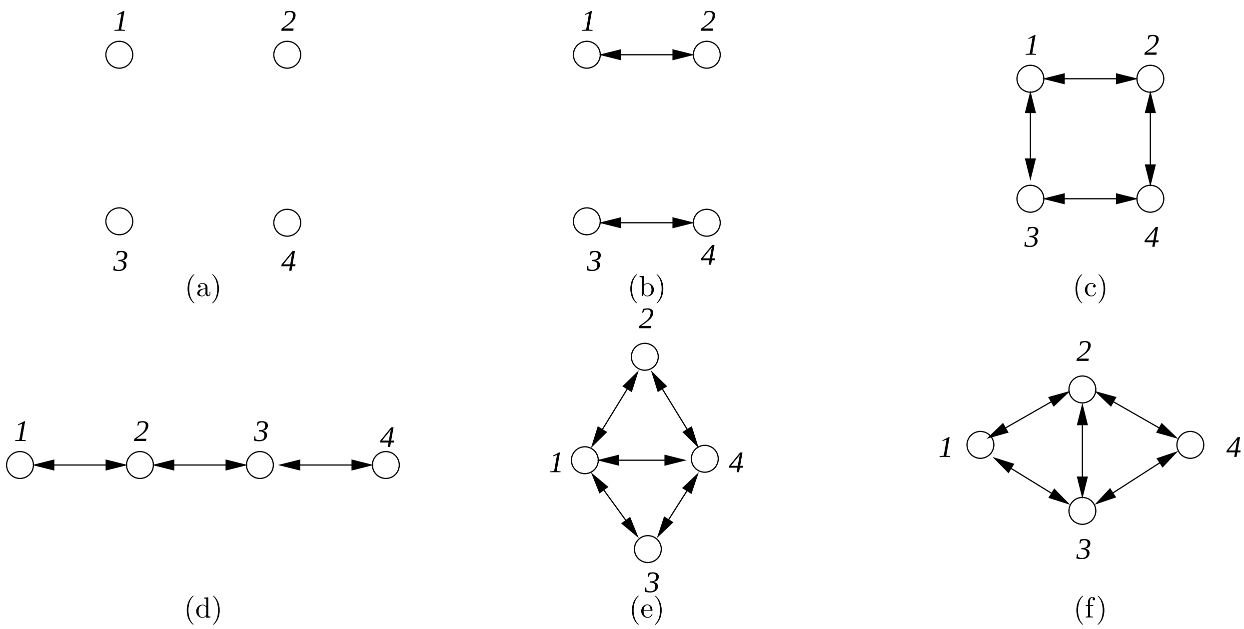


Figure 3: Different realizations of a definite structural α-net (n, d_α, r) with $n = 4$ nodes satisfying $d_\alpha < r < \sqrt{2}d$:(a) an α-net (or quasi α-flock) with density $\delta_0 = \frac{1}{4}$, (b) a quasi α-flock with density $\delta_0 = \frac{2}{4}$ consisting of two flocks, (c) an α-flock with 4 edges, (d) a chained-form α-flock or conformation with 3 edges, (e) a rigid conformation with 5 edges, and (f) a rigid α-flock with 5 edges (but different set of edges than part (e)).

Let $(\tilde{q}, r)$ be a uniform cluster that is obtained from cluster (q, r) via permutation of the 2 blocks of q. We call $\tilde{\mathcal{N}} = G(\tilde{q})$ a *permutation* of the net $\mathcal{N} = G(q)$.

Lemma 2. *Let (n, d_α, r) be a structural net with realization $\mathcal{N} = G(q)$. Then, any permutation $\tilde{\mathcal{N}}$ of $\mathcal{N}$ is also a realization of (n, d_α, r).*

Proof. The proof follows from the definition. □

Following the result of Lemma 2, let $[G(q)]$ denote the similarity class of all graphs that are obtained via permutation of the nodes of $G(q)$. We call $[G(q)]$ a *conformation* of the structural net (n, d_α, r).

Note 2. All the properties and names of a net is carried over to a flock if the net is weakly connected. For example, an *α-flock* is a weakly connected α-net.

Example 1. In Figure 3, six different realizations of a structural α-net (n, d_α, r) with $n = 4$ nodes satisfying the condition $1 < r/d_\alpha < \sqrt{2}$ are shown. In Figure 3(a), all nodes are mutually too far from each other to form *links* (i.e. undirected edges). In Figure 3(b), two pairs of nodes $(1, 2)$ and $(3, 4)$ are close enough to form links but the end points of these two links are too far from each other. In Figure 3(c), a cycle is formed via the 4 links. However, since the distance between the pairs of nodes $(1, 4)$ and $(2, 3)$ is equal to $\sqrt{2}\, d_\alpha > r$, no links are formed between these two pairs. In Figure 3(d) a *chain* (i.e. a path of length n going through all nodes) is shown as the realization of (n, d_α, r). All the nodes that are not connected in this chain are at least $2d_\alpha > r$ apart from each other. Figure 3(e) shows a flexing of the non-rigid framework in Figure 3(c) where nodes 1 and 4 get close enough to each other to form a link (see [14] for the definition of a flexing and graph rigidity). Similarly, Figure 3(f) shows another flexing of the non-rigid framework in Figure 3(c) so that nodes 2 and 3 get closer to each other and form a link. Both realizations in Figures 3(e) and (f) belong to the same conformation of this structural net.

From Figure 3, it is evident that the generic realizations (or conformations) of structural nets are not unique. This is the opposite of the local uniqueness property of the realizations of rigid structured graphs [14]. Here is a problem for researchers interested in combinatorics:

Problem 1. *Calculate $f(n, k)$, the number of conformations of a definite structural α-net (n, d_α, r) with $r = kd_\alpha$ for $n \geq 2$ and $k > 1$.*

Lemma 3. *$f(n, k) \leq 2^{\frac{n(n-1)}{2}}$ where $f(n, k)$ is defined in Problem 1.*

Proof. Let $[G(q)]$ be a conformation of the structural net in the question. In general, $[G(q)]$ has a subset of the edges of a complete graph on n nodes. □

Remark 3. One can show that for $1 < k < \sqrt{2}$, $f(n, k)$ takes the following values $1, 2, 4, 9$, respectively, for $n = 1, 2, 3, 4$. Clearly, $2^{\frac{n(n-1)}{2}}$ is a conservative upper bound on $f(n, k)$ which is by no means tight for $n > 2$.

4 Structural Energy of Nets and Flocks

Consider α-agents with the following dynamics

$$\text{agent dynamics:} \begin{cases} \dot{q}_i &= p_i, \\ \dot{p}_i &= u_i. \end{cases} \tag{21}$$

Let us define the position/velocity of an α-net as the average position/velocity of all the α-agents in the net, i.e.

$$\bar{q} = Ave(q),\ \bar{p} = Ave(p) \tag{22}$$

with $Ave(x) := \frac{1}{n}(\sum_{i=1}^n x_i)$. Defining $\bar{u} = Ave(u)$, the translational dynamics of the net can be expressed as

$$\text{translational dynamics:} \begin{cases} \dot{\bar{q}} &= \bar{p}, \\ \dot{\bar{p}} &= \bar{u}, \end{cases} \tag{23}$$

with $\bar{q}, \bar{p}, \bar{u} \in \mathbb{R}^2$. Let $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$ denote the vector of ones and $\otimes$ denote the Kronecker product of two matrices defined by

$$A \otimes B = [a_{ij} B].$$

This means the ijth block of $A \otimes B$ is $a_{ij}B$. Define the *relative position, velocity, and control* of agent i as

$$\tilde{q}_i = q_i - \bar{q},\ \tilde{p}_i = p_i - \bar{p},\ \tilde{u}_i = u_i - \bar{u} \quad \forall i \tag{24}$$

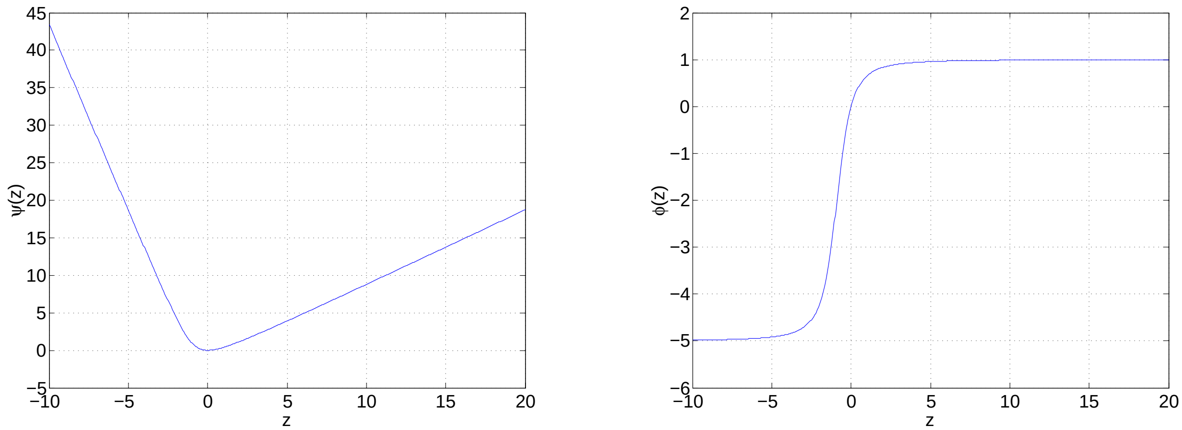


Figure 4: The potential and action functions with parameters $a = 1$ and $b = 5$: (a) $\psi_\alpha(z)$ and (b) $\phi_\alpha(z)$.

Then, one can write

$$\begin{array}{rcl} \tilde{q} &=& q - \mathbf{1} \otimes \bar{q}, \\ \tilde{p} &=& p - \mathbf{1} \otimes \bar{p}, \\ \tilde{u} &=& u - \mathbf{1} \otimes \bar{u}, \end{array} \tag{25}$$

and define the *relative dynamics* of the net as

$$\left\{ \begin{array}{rcl} \dot{\tilde{q}} &=& \tilde{p}, \\ \dot{\tilde{p}} &=& \tilde{u}. \end{array} \right. \tag{26}$$

Following the general idea of construction of potential/cost functions for *formation graphs* in [13, 14], we define the following Hamiltonian as the *structural energy* of an α-net (or α-flock):

$$H_s(q, \tilde{p}) = V(q) + K_r(\tilde{p}) \tag{27}$$

where *potential energy* $V(q)$ and *relative kinetic energy* $K_r(\tilde{p})$ of the α-net are defined as the following

$$\begin{array}{rcl} V(q) &=& \frac{1}{2} \sum_{i=1}^n V_i(q), \\ V_i(q) &=& \sum_{j \in N_i} \psi_\alpha(\|q_j - q_i\| - d_\alpha), \\ K_r(\tilde{p}) &=& \frac{1}{2} \sum_{i=1}^n \|\tilde{p}_i\|^2. \end{array} \tag{28}$$

The potential and action functions $\psi_\alpha(z)$ and $\phi_\alpha(z)$ are defined as follows

$$\begin{array}{rcl} \psi_\alpha(z) &=& \left(\frac{a+b}{2}\right)\left(\sqrt{1+(z+c)^2} - \sqrt{1+c^2}\right) + \left(\frac{a-b}{2}\right) z, \\ \phi_\alpha(z) &=& \left(\frac{a+b}{2}\right) \dfrac{z+c}{\sqrt{1+(z+c)^2}} + \left(\frac{a-b}{2}\right), \end{array} \tag{29}$$

where $b > a > 0$ and $c = |a-b|/2\sqrt{ab} > 0$. Notice that $\phi_\alpha(z) = d\psi_\alpha(z)/dz$ is a uniformly bounded sigmoidal function. These functions are plotted in Figure 4. If a node has no neighbors, or $N_i = \emptyset$, we set $V_i(q) := 0$.

The following result shows that reducing $H_s(q, \tilde{p})$ to zero is meaningful for the purpose of "flocking" (i.e. "to gather or move in large numbers" *Longman Dictionary*).

Proposition 2. *(zero structural energy) Given the definition of the structural energy $H_s(q, \tilde{p})$ of a net in* (27)*, the following statements hold:*

i) $V(q) = 0$ *if and only if the net* $G(q)$ *is a realization of the structural net* (d_α, r)*.*

ii) If $K_r(\tilde{p}) = 0$ *for all* $t \geq t_0$*, then the distance between any two* α*-agents remains constant for all* $t \geq t_0$*. Moreover, the converse holds if a) the net is an undirected flock, b) no two agents ever collide, and c) there exists no two agents with different velocities such that* $p_j - p_i$ *is orthogonal to* $q_j - q_i$*.*

iii) If $K_r(\tilde{p}) = 0$ *for all* $t \geq t_0$*, then the topology of the net* $G(q(t))$ *remains invariant for all* $t \geq t_0$*.*

Proof. Please, see Section A.1 in the Appendix. □

We summarize the results of Proposition 2 in the following:

Proposition 3. *(zero structural energy) Suppose that the structural energy* $H_s(q(t), \tilde{p}(t))$ *is zero for all* $t \geq t_0$*. Then, the following statements holds:*

i) the net $G(q(t))$ *has an invariant topology over* $[t_0, \infty)$ *that is a realization of the structural net* (d_α, r)*.*

ii) the distance between any two arbitrary α*-agents in the* α*-net remains constant for all* $t \geq t_0$*.*

iii) the velocity of all α*-agents are equal.*

Proof. Parts i) and ii) follow from Proposition 2. Part iii) is due to the fact that $H_s(q, \tilde{p}) \implies K_r(\tilde{p}) = 0 \implies \tilde{p}_i = 0$ for all i. Thus, $p_i = \bar{p}$ for all i and the velocities of all nodes are equal. □

Motivated by Proposition 3, we define "flocking" as follows.

Definition 2. (flocking) Given the protocol $u = k(q, p)$, we say a *dynamic net* $(G(q), q, p, u)$ is *structurally asymptotically stable* if and only if both of the following conditions hold:

i) There exists a constant $C > 0$ such that $H_s(q(t), \tilde{p}(t)) \leq C$ for all $t \geq 0$.

ii) $\lim_{t \to \infty} H_s(q(t), \tilde{p}(t)) = 0$, i.e. for all $\epsilon > 0$, there exits $T = T(\epsilon) > 0$ such that

$$H_s(q(t), \tilde{p}(t)) < \epsilon$$

for all $t > T$.

5 Smooth and Nonsmooth Structural Energies

In general, for nets with nonsmooth spatial adjacency matrices $\mathcal{A}(q)$, the Hamiltonian $H_s(q, p)$ is a discontinuous function of (q, p). The Hamiltonian associated with an α-net $G(q)$ can be expressed as

$$H_s(q, \tilde{p}) = \sum_{i,j,i<j} a_{ij}(q) \psi_\alpha(\|q_j - q_i\| - d_\alpha) + \frac{1}{2} \sum_{i=1}^{n} \|\tilde{p}_i\|^2 \tag{30}$$

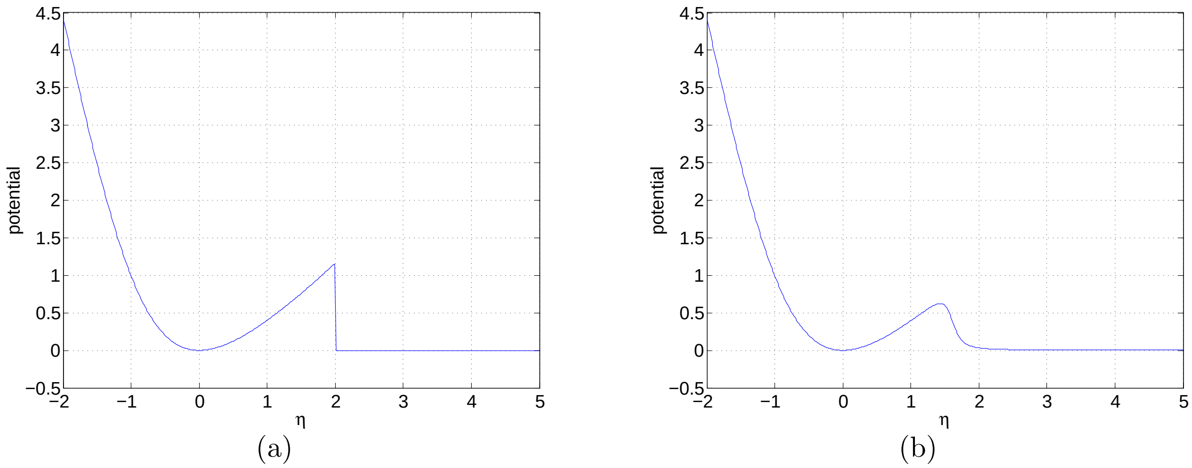


Figure 5: The pair-wise potential function between two α-agents: (a) the nonsmooth potential and (b) the smooth potential.

Suppose that a link $\{(i,j),(j,i)\}$ is created in a net and no other links are created or lost. Then, there will be a positive jump in the energy before and after the creation of this link that is given by

$$\Delta H_s = H_s^+(q,\tilde{p}) - H_s^-(q,\tilde{p}) = V^+(q) - V^-(q) = \psi_\alpha(r - d_\alpha) =: h_0 > 0 \tag{31}$$

where $H_s^{\pm}(q,\tilde{p})$ denotes $H(q(t),\tilde{p}(t))|_{t=t_0^{\pm}}$ and $\|q_j(t_0) - q_i(t_0)\| = r$, i.e. the link is created at time $t = t_0$. Similarly, $V^{\pm}(q) = V(q(t_0^{\pm}))$. The loss of a link $\{(i,j),(j,i)\}$ causes a negative jump in energy that is equal to the constant $h_0 > 0$. This is not the case for a net with a smooth adjacency matrix $\mathcal{A}(q) = [a_{ij}(q)]$ where $a_{ij}(q) = \rho(\|q_j - q_i\|/r)$.

Example 2. The potential function of a net with two α-agents is shown in Figure 5(a). For two agents, the potential function takes the form

$$V(q) = a_{12}(q)\psi_\alpha(\|q_2 - q_1\| - d_\alpha). \tag{32}$$

Define a scalar *edge deviation variable*

$$\eta = \|q_2 - q_1\| - d_\alpha \tag{33}$$

and consider the potential energy as a function of η. The condition $\|q_2 - q_1\| \leq r$ reduces to $\eta \leq r - d_\alpha$. Figure 5(a) shows the plot of the potential between a pair of α-agents as a function of η. The choice of parameters are $d_\alpha = 10$, $r = 1.2d_\alpha$, $a = 1$, $b = 5$, and $c = |a - b|/2\sqrt{ab}$. The discontinuous jump occurs at $\eta = r - d_\alpha = 2$. Later, we explain the smoothing process of a nonsmooth potential function to obtain differentiable pair-wise potentials as shown in Figure 5(b).

6 Protocol for (α, α) Interactions

A dynamic net is a net in which each node is a dynamical system. Consider the following (α,α) *protocol* for the interaction between an α-agent and all of its neighboring α-agents (i.e. an (α,α)

interaction) in a dynamic α-net with a nonsmooth adjacency matrix $\mathcal{A}(q)$:

$$u_i^{(\alpha,\alpha,1)} = \sum_{j \in N_i} \phi_\alpha(\|q_j - q_i\| - d_\alpha)\mathbf{n}_{ij} + c_1(p_j - p_i), \quad c_1 > 0 \tag{34}$$

where $\mathbf{n}_{ij} = (q_j - q_i)/\|q_j - q_i\|$ is a unit vector along the line connecting node i to node j.

Remark 4. Protocol (34) is very similar to the protocol given by the author(s) in [13] with a minor difference in the damping terms and the fact that all distances in a rigid formation are not necessarily equal.

For a dynamic α-net with a smooth adjacency matrix $\mathcal{A}(q)$, the protocol for an (α, α) interaction is as follows

$$u_i^{(\alpha,\alpha,2)} = \sum_{j \in N_i} [a_{ij}(q)\phi_\alpha(\|q_j - q_i\| - d_\alpha) + \frac{1}{r}\rho'(\frac{\|q_j - q_i\|}{r})\psi_\alpha(\|q_j - q_i\| - d_\alpha)]\mathbf{n}_{ij} + c_1(p_j - p_i) \tag{35}$$

Consider an approximation of protocol (35) given by

$$\hat{u}_i^{(\alpha,\alpha,2)} = \sum_{j \in N_i} a_{ij}(q)\phi_\alpha(\|q_j - q_i\| - d_\alpha)\mathbf{n}_{ij} + c_1(p_j - p_i) \tag{36}$$

which is the same as protocol (34) as $k_0 \to 1^-$ for the bump function $\rho(z)$. Then, assuming $\|\rho'(z)\| \leq L_\rho$, uniformly in z, the approximation error can be bounded as follows

$$\sum_{i=1}^{n} \|u_i^{(\alpha,\alpha,2)} - \hat{u}_i^{(\alpha,\alpha,2)}\| \leq \frac{2L_\rho}{r}V(q). \tag{37}$$

Thus, the approximation error remains relatively small if the net has a relatively low structural potential.

7 Reynolds Rules of Flocking and the (α, α) Protocol

According to the first two protocols of Reynolds in [17], an agent moves towards the center of mass (CM) of its nearest neighbors if it is too far from them and avoids going towards the CM of its nearest neighbors if it is getting too close to them. In the following, we show that both of these rules follow as special cases of protocol (34) with $c_1 = 0$. Moreover, there is a special case that has not been accounted for by any of the three flocking rules of Reynolds in [17]. Finally, the third rule of Reynolds for attitude alignment is the same as a consensus protocol given in equation (13) (for more information, please see [16]).

Let us define the weights between agent i and its neighbors $j \in N_i$ as

$$w_{ij}(q) = \frac{\phi_\alpha(\|q_j - q_i\| - d_\alpha)}{\|q_j - q_i\|}, \quad j \in N_i \tag{38}$$

with the property that $w_{ij}(q) > 0$, if agents i and j are more than d_α apart, and $w_{ij}(q) \leq 0$ otherwise. Particularly, if these two agents get too close to each other (or $\epsilon = \|q_j - q_i\| \ll 1$) agent

i gets pushed away by agent j with a force proportional to $\phi_\alpha(d_\alpha - \|q_j - q_i\|) \approx \phi_\alpha(d_\alpha) \gg 1$ along $\mathbf{n}_{ji}$. Let us rewrite the approximate $u_i^{(\alpha,\alpha,1)}$ with zero damping terms (or $c_1 = 0$) as

$$\hat{u}_i = \sum_{j \in N_i} w_{ij}(q)(q_j - q_i) = \sum_{j \in N_i} w_{ij}(q) q_j - (\sum_{j \in N_i} w_{ij}) q_i \tag{39}$$

Set $S_i(q) = \sum_{j \in N_i} w_{ij}(q)$ and define the vector of *weighted average of the position of the neighbors of agent* i as

$$q_i^{ave} = \frac{1}{S_i(q)} \left(\sum_{j \in N_i} w_{ij}(q) q_j \right) \tag{40}$$

whenever $S_i(q) \neq 0$. We call $S_i(q)$ the *neighbors vote* which quantifies how all spatial neighbors of agents i "think" whether agent i is close, far or neutral with respect to its neighbors. There are three possible cases:

i) $S_i(q) > 0$: In this case the neighbors vote is positive and

$$\hat{u}_i = S_i(q)(q_i^{ave} - q_i) \propto (q_i^{ave} - q_i) \tag{41}$$

implies that agent i moves towards q_i^{ave} (because $S_i(q) > 0$).

ii) $S_i(q) < 0$: In this case the neighbors vote is negative (or the neighbors all think that agent i is far from them). Based on

$$\hat{u}_i = S_i(q)(q_i^{ave} - q_i) \propto -(q_i^{ave} - q_i) \tag{42}$$

we conclude that in this case agent i moves away from q_i^{ave}.

iii) $S_i(q) = 0$: In this case the neighbors total vote is zero, i.e. the neighbors are divided into three groups: a) neighbors $N_i^+ = \{j \in N_i : w_{ij}(q) > 0\}$ whose votes are positive and their total vote adds up to $S_i^+(q) = \sum_{j \in N_i^+} w_{ij}(q)$, b) neighbors $N_i^- = \{j \in N_i : w_{ij}(q) > 0\}$ whose votes are negative and their total vote adds up to $S_i^-(q) = \sum_{j \in N_i^-} w_{ij}(q)$, and c) neighbors $N_i^0 = \{j \in N_i : w_{ij}(q) = 0\}$ whose votes are zero (i.e. they are exactly d_α away from agent i) and their total vote adds up to zero, because of $S_i^0(q) = \sum_{j \in N_i^0} w_{ij}(q) = 0$. Notice that $S_i^-(q) = -S_i^+(q)$ due to $S_i(q) = S_i^-(q) + S_i^0(q) + S_i^+(q) = 0$. Let us define the following two weighted average quantities

$$q_i^{ave(+)} = \frac{1}{S_i^+(q)} \left(\sum_{j \in N_i^+} w_{ij}(q) q_j \right), \; q_i^{ave(-)} = \frac{1}{S_i^-(q)} \left(\sum_{j \in N_i^-} w_{ij}(q) q_j \right) \tag{43}$$

In this case, we get

$$\begin{aligned} \hat{u}_i &= S_i^+(q)(q_i^{ave(+)} - q_i) + S_i^-(q)(q_i^{ave(-)} - q_i) \\ &= S_i^+(q)[(q_i^{ave(+)} - q_i) - (q_i^{ave(-)} - q_i)] \\ &= S_i^+(q)(q_i^{ave(+)} - q_i^{ave(-)}) \end{aligned} \tag{44}$$

which means

$$\hat{u}_i \propto (q_i^{ave(+)} - q_i^{ave(-)}). \tag{45}$$

Thus, the protocol for case iii) is as follows: agent i ignores all agents that had neutral (or zero) votes and moves parallel to $q_i^{ave(+)} - q_i^{ave(-)}$, i.e. the difference between the weighted averages of all who voted positive and all who voted negative. For the case that there are two groups that have voted equally in opposite directions, surprisingly, protocol (45) cannot be obtained from any of the three flocking rules of Reynolds in [17].

In conclusion, *the first two flocking rules of Reynolds are "hidden" in the* (α, α) *protocol* in (34). Note that there is a "minor glitch" in Reynolds rules where the damping terms do not exist (due to $c_1 = 0$). The *third flocking rule of Reynolds for alignment is the same as a linear consensus protocol* with no time-delay [15, 16].

8 Flocking by Dissipation of Structural Energy

Here are two results on dissipation of nonsmooth and smooth structural energy of nets for the purpose of flocking:

Proposition 4. *Consider a uniform* α*-net with protocol* (34)*. Let* $t_0, t_1, t_2, \dots$ *be an increasing sequence of switching times of the topology of the net* $G(q(t))$ *over the interval* $[t_0, \infty)$ *so that at* $t = t_{k+1}, k \geq 0$ *at least one link is created or lost in the undirected net* $G_k = G(q(t_k))$*. Let* $H_s^{(k)}(q, \tilde{p})$ *be the structural energy of the net* G_k *with discontinuous adjacency elements* $a_{ij}(q) \in \{0, 1\}$*. Then,* $H_s^{(k)}(q, \tilde{p})$ *is a weak Lyapunov function for the closed-loop net dynamics over the interval* $[t_k, t_{k+1})$*, i.e.* $H_s^{(k)}(q, \tilde{p}) \leq 0$ *for all* $t \in [t_k, t_{k+1})$*. Moreover, if the switching sequence* $t_0, t_1, , \dots, t_m$ *is finite and* $G(q(t_m))$ *is a flock, then asymptotically all* α*-agents asymptotically move with the same velocity and their inter-agent distances are preserved.*

Proof. Please, see Section A.2 in the Appendix. □

Remark 5. At this point, the authors are unaware of the quantitative sufficient conditions that guarantee the switching time sequence in Proposition 4 remains finite. Our observation from experimental results demonstrates that under the conditions in Proposition 4, the switching time sequence is always finite due to the fact that $H_s^{(k)}(q(t_k), \tilde{p}(t_k))$ is a decreasing sequence. The complete analysis of this case including finding appropriate conditions that guarantee the finiteness of the switching times is the subject of ongoing research.

Proposition 5. *Consider an* α*-net with a smooth structural energy* $H_s(q, \tilde{p})$*. Given the protocol in* (35)*,* $H_s(q, \tilde{p})$ *is a weak Lyapunov function for the closed-loop net dynamics, i.e.* $\dot{H}(q, \tilde{p}) \leq 0$ *for all* $t \geq t_0$*. Furthermore, if there exists a time* $T > t_0$ *such that the net* $G(q(t))$ *is a flock for all* $t \geq T$*, then asymptotically all the* α*-agents move with the same velocity and their inter-agent distances are preserved.*

Proof. The proof is rather similar to the proof of Proposition 4 and will not be repeated. Based on protocol (35), we have

$$\dot{H}_s(q, \tilde{p}) = -\frac{c_1}{2} \sum_{(i,j) \in \mathcal{E}_{G(q)}} \|\tilde{p}_j - \tilde{p}_i\|^2$$

If the graph (or net) $G(q(t))$ induced by $(q(t), r)$ is connected for all $t > T$. Then, $\tilde{p}_j = \tilde{p}_i$, for all the edges (i, j) of the net $G(q(t))$. This implies $\tilde{p}_i = 0$ for all i. Therefore, asymptotically all α-agents move with equal velocities and the inter-agent distances are preserved. □

9 Quasi-Realizations and Defect Factors

We define the *defect function* (or *factor*) associated with a net $G(q)$ with structural potential $V(q)$ as follows:

$$\mu = \mu(G(q)) = V(q) + w\|\nabla V(q)\|^2 \geq 0 \tag{46}$$

where $w > 0$ is a constant weight. We also define the *normalized defect factor* of $G(q)$ as

$$\mu_n = \mu_n(G(q)) = \frac{1}{|\mathcal{E}_{G(q)}|h_0}\big(V(q) + w\|\nabla V(q)\|^2\big) \tag{47}$$

where $|\mathcal{E}_{G(q)}|$ denotes the total number of edges in the net $G(q)$ and $h_0 = \psi_\alpha(r - d_\alpha)$. From a computational point of view, the normalized defect factor is more meaningful. Since, for a large-scale net with many agents, μ_n stays relatively small if a small subgraph of the net is not a realization of the structural net (d_α, r) but the rest of the net is a valid realization of (d_α, r).

Any α-net $G(q)$ that is a realization of a structural net (d_α, r), has a zero defect factor. This is due to the fact that for any realization of (d_α, r), $V(q) = 0$ and $\nabla V(q) = 0$. However, in presence of external forces, α-flocks (or α-nets) do not usually converge to a realization of a structural net (d_α, r). Instead, flocks converge to what is rather "close" to a realization of (d_α, r). To quantify the quality of similarity of the limiting formation of a flock to a realization of (d_α, r), we need to define the notion of quasi-realizations of a structural net (d_α, r) and measure its "quality" (in terms of satisfying all algebraic inter-agent distance-based constraints imposed by (d_α, r)) using the defect factor.

Definition 3. (quasi-realizations) We say $G(q)$ induced by the cluster (q, r) is a *quasi-realization* (or *quasi-embedding*) of the structural net (d_α, r) with the *defect factor* $\mu = \mu(G(q))$.

In both limiting cases in Propositions 4 and 5, the flock converges to a formation (i.e. framework $(G(q), q)$) in which asymptotically the inter-agent distances are preserved and the defect factor of $G(q)$ is a constant $\mu = \mu^*$. If this constant defect factor μ^* is zero, then the flock converges to a conformation that is a realization of the structural net (d_α, r). Otherwise, it converges to a quasi-realization of the structural net (d_α, r) that is not a flexing of any realization of (d_α, r) (recall that flexings preserve edge length of frameworks [14]).

10 A Separation Principle for Flocking

One notices that in both (α, α)-interaction protocols (defined in equations (34) and (35)) satisfy the property

$$\sum_{i=1}^{n} u_i^{(\alpha,\alpha,k)} = 0, \quad k = 1, 2.$$

This guarantees the invariance of $\bar{p} = (\sum_{i=1}^{n} p_i)/n$ in time. Motivated by this invariance property, assume that each α-agent uses the following protocol

$$u_i = u_i^\alpha + \bar{u}, \tag{48}$$

with the property that $\sum_i u_i^\alpha = 0$. Let $H_s(q, \tilde{p})$ be the smooth structural energy associated with the net $G(q)$. We show that $\dot{H}_s(q, \tilde{p})$ does not depend on the choice of $\bar{u}$.

Following the line of construction of tracking (or navigation) integrated cost functions in [12], we define the *translational energy* of the net (or group) $G(q)$ as

$$H_{tr}(\bar{q}, \bar{p}) = V_{tr}(\bar{q}) + \frac{1}{2}\|\bar{p} - p_d\|^2 \tag{49}$$

where $\bar{q}, \bar{p} \in \mathbb{R}^2$, $V_{tr}(\bar{q}) = \sqrt{1 + \|\bar{q} - q_d\|^2} - 1$, and (q_d, p_d) denotes a *desired destination.* If $p_d = 0$, we call this destination a *sink.* If $V_{tr}(q) \equiv 0$, then p_d is called a *desired group velocity.* Our objective is to combine H_s and H_{tr} to perform both structural stabilization and tracking. In the following, the term CLF stands for "control Lyapunov function".

Proposition 6. *(flocking separation principle) Let $H(q, p) = H_s(q, \tilde{p}) + H_{tr}(\bar{q}, \bar{p})$ where $H_s(q, \tilde{p})$ and $H_{tr}(\bar{q}, \bar{p})$ are smooth structural energy and tracking energy of the dynamic net $(G(q), q, p, u)$ with protocol* (48)*, respectively. Then, the following separation principles hold:*

i) $\dot{H}_s(q, \tilde{p})$ does not depend on the choice of $\bar{u}$ and $\dot{H}_s(q, \tilde{p})$ is a weak CLF for the dynamic net, i.e. there exists a protocol u^α such that $\dot{H}_s(q, \tilde{p}) \leq 0, \forall t \geq 0$.

ii) $\dot{H}_{tr}(\bar{q}, \bar{p})$ does not depend on the choice of u^α and $\dot{H}_{tr}(\bar{q}, \bar{p})$ is a weak CLF for the average dynamics of the net for both cases of a desired sink and group velocity, i.e. there exists a protocol $\bar{u}$ such that $\dot{H}_{tr}(\bar{q}, \bar{p}) \leq 0, \forall t \geq 0$.

Furthermore, $H(q, p)$ is a weak CLF for the dynamic net.

Proof. Please, see Section A.3 in the Appendix. □

Remark 6. The proof of proposition 6 contains all the necessary information regarding the tracking control design for flocking.

Let $\bar{u} = k_{tr}(\bar{q}, \bar{p})$ denote the translational controller of the net. Each α-agent can calculate u_i^α in a distributed manner. But calculation of $k_{tr}(\bar{q}, \bar{p})$ (given in the proof of Proposition 6) requires the knowledge of $\bar{q}$ and $\bar{p}$ which are not immediately available to each agent. Either this information can be communicated to the agent via a coordinator, or all agents need to solve average-consensus problems [16] in a distributed fashion. The former approach is clearly a centralized algorithm which is highly undesirable for flocking due to its high communication cost. The second approach is feasible if the net is connected (i.e. if the net is a flock). In the following, we propose a third approach that is distributed and does not require the connectivity of the net.

Suppose each agent uses the following protocol

$$u_i = u_i^\alpha + f_i, \; f_i = -q_i - c_2 p_i, \quad c_2 > 0 \tag{50}$$

to solve flocking in the presence of a sink at the origin $(q_d, p_d) = (0, 0)$. Notice that f_i can be calculated by each agent without any need for communication with other agents. One can rewrite protocol (50) as

$$u_i = u_i^\alpha - \tilde{q}_i - c_2 \tilde{p}_i + \bar{u}, \tag{51}$$

with the PD controller $\bar{u} = -\bar{q} - c_2\bar{p}$. In lack of u^α (i.e. if no edges ever exist between any two agents), $u_i = -\tilde{q}_i - c_2\tilde{p}_i$ and thus $H_a(\tilde{q}, \tilde{p})$ defined by

$$H_a(\tilde{q}, \tilde{p}) = \|\tilde{q}\|^2 + \|\tilde{p}\|^2 \tag{52}$$

is a valid Lyapunov function for the system and all agents will converge to the center of mass (CM) of the net at $\tilde{q} = 0$, i.e. $(\tilde{q}, \tilde{p}) = 0$ is globally asymptotically stable. However, if all agents converge to the origin, after some finite time $T > 0$, all agents enter a closed ball $B(\bar{q}, r_0)$ where $0 < r_0 < r/2$. Thus, any two agents become neighbor of each other and the induced net $G(q)$ is flock and a complete undirected graph).

On the other hand, in presence of the (α, α)-interaction forces (or u^α), convergence of all the agents to the origin is in contradiction with reduction of the structural energy of the net. Since when all agents coincide $V(q)$ takes its global maximum. In other words, in presence of both u_i^α and f_i, the center of mass of the flock exponentially converges to zero and all agents "try to converge to the CM of the net" and "keep a distance d_α from their neighbors". The tasks in quotes are conflicting as described earlier.

Based on simulation result, we observe that the net asymptotically converges to a flock with its CM at the origin. After some finite time $T > 0$, the topology of this flock remains invariant. At this point, we are unable to prove that our observation formally holds. For more information, see the simulation results in Section 11.

11 Obstacle Avoidance and Notions of β-Agents and γ-Agents

In this section, we present our approach to multiple fixed obstacle avoidance for a net/flock of α-agents. We postpone stating any formal results regarding multiple obstacle avoidance by groups of agents to upcoming papers. But we provide the main protocol that combines the results from the preceding sections on (α, α)-interaction protocols and the flocking separation principle with obstacle avoidance.

For a net of α-agents we consider the task of moving with a desired group velocity $p_d \neq 0$ along the desired direction $\mathbf{n}_d = p_d/\|p_d\|$ that is a unit vector while avoiding collision to finite number of fixed obstacles. The main assumption on the obstacles is that they are convex and compact sets and their boundaries are closed differentiable Jordan curves in $\mathbb{R}^2$. For the sake of simplicity of representation and calculations, we only treat the case in which there exist m spherical obstacles (or closed balls) $O_k = B(b_k, l_k)$ for $k = 1, \dots, m$. We define the distance between an α-agent i and O_k as $d(q_i, O_k) = \min_{x \in O_k} \|x - q_i\|$ and define the projection of q_i on the boundary of O_k as

$$\hat{q}_i^k = \text{argmin}_{x \in O_k} \|x - q_i\| \tag{53}$$

The existence and uniqueness of $\hat{q}_i^k$ is due to convexity and compactness properties of O_k.

We refer to an agent with position $\hat{q}_i^k$ as a *β-agent* provided that $d(q_i, O_k) \leq r_0$ (Here, we assume $r_0 = r/2$). In other words, the projection of an α-agent on an obstacle O_k is called a *β-agent*. For flat obstacles like walls, this projection technique is previously used by Helbing *et al.* [7] and Desai *et al.* [4]. An example of a β-agent is agent 2 in Figure 6.

Whenever an α-agent exists in a neighborhood of O_k, we refer to O_k as an *active obstacle*. Otherwise, we call O_k an *inactive obstacle*. We define a *bipartite graph* K_{n_β, n_β} with n_β edges that

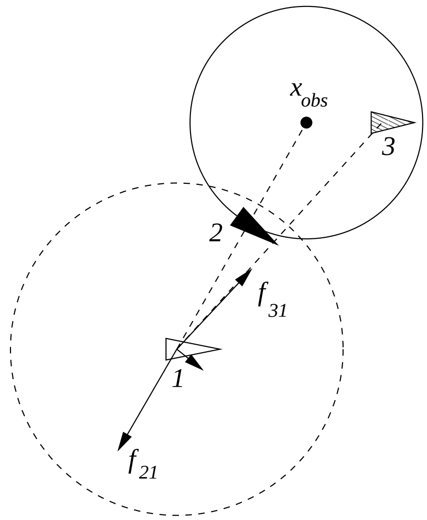


Figure 6: The interaction between an α-agent (agent 1) and the effect of an obstacle represented by a β-agent (agent 2) and a γ-agent (agent 3).

connect each α-agent to its corresponding β-agent. The total number of β-agents is denoted by n_β. If all the obstacle are inactive, or $n_\beta = 0$, we set $K_{n_\beta,n_\beta} = (\emptyset, \emptyset)$.

In Figure 6, agent 3 is called a γ-agent. A *γ-agent* is an agent with a fixed position at $x_k = b_k + \lambda_d \mathbf{n}_d$ where $\lambda_d \in (0, 1]$ is a constant. An α-agent views a β-agent as a repelling point on the obstacle and treats a γ-agent as another α-agent that only exists if the corresponding β-agent exists. Both β and γ agents associated with an α-agent v_i adjacent to an obstacle O_k "disappear" as soon as no points on the boundary of O_k belongs to the shell of node v_i. In other words, the existence of β-agents and γ-agents is conditional.

The presence of a γ-agent is necessary to steer an α-agent around an obstacle. Otherwise, an α-agent v_i might stay behind or near a distance r_0 from an obstacle for a relatively long time due to "peer panic" by other α-agents v_j that are on the way of agent v_i and hold v_i back. The role of a γ-agent is crucial in both *split/rejoin maneuver* and *squeezing maneuver*. The latter one occurs during the *escape panic* phenomenon [7], i.e. cases where many agents (or vehicles) need to pass through a narrow pathway between two obstacles (or mountains). In the case of performing a squeezing maneuver between two nearby obstacles, both obstacles might become active with respect to a single α-agent. In other words, there could be multiple β and γ agents that correspond to a given α-agent.

Let (q_i, p_i), $(\hat{q}_i^k, \hat{p}_i^k)$, and $(x_i, 0)$ denote the pairs of (position,velocity) associated with an α-agent, β-agent, and γ-agent, respectively. Then the protocol used by the α-agent can be expressed as follows:

$$u_i^{O_k} = c_3 \rho(\|\hat{q}_i^k - q_i\|/r_0)[\phi_\beta(\hat{q}_i^k - q_i\| - d_\beta)\mathbf{n}(\hat{q}_i^k - q_i) + \phi_\gamma(\|x_i - q_i\| - d_\gamma)\mathbf{n}(x_i - q_i)] + c_4(\hat{p}_i^k - p_i) \quad (54)$$

where $c_3, c_4 > 0$, $\mathbf{n}(z) = z/\|z\|$ for $z \neq 0$, $\phi_\beta(z)$ is a repelling force, and $\phi_\gamma(z) = \phi_\alpha(z)$. The velocity of a β-agent $\hat{p}_i^k$ on the boundary of obstacle O_k can be calculated as follows. The position of the β-agent $\hat{q}_i^k$ can be expressed as

$$\hat{q}_i^k = s q_i + (1 - s) b_k \quad (55)$$

with $s = l_k/(l_k + \|\hat{q}_i - q_i\|)$. According to the assumption $\dot{b}_k = 0$, we have

$$\hat{p}_i^k = sp_i - l_k(\dot{s}/s)\mathbf{n}(\hat{q}_i^k - q_i) \tag{56}$$

where

$$\dot{s} = \frac{l_k[p_i^T \cdot \mathbf{n}(\hat{q}_i^k - q_i)]}{(l_k + \|\hat{q}_i - q_i\|)^2}. \tag{57}$$

Remark 7. Apparently, the assumption $\dot{b}_k$ plays no crucial role in the derivation of $\hat{p}_i^k$ and can be eliminated. This creates the possibility of dealing with the case of multiple moving obstacle avoidance.

By setting $c_3 \gg 1$, one can make obstacle avoidance the task with the highest priority. The second priority for an α-agent can be given to reaching a desired group velocity. Finally, the third priority can be given to keeping a distance d_α from other α-agents. The overall protocol used by an α-agent is given by

$$u_i = u_i^{(\alpha,\alpha,2)} + \sum_{O_k \text{ active for } i} u_i^{O_k} + u_{tr} \tag{58}$$

where the first term contains all (α, α) interaction forces, the second term contains all (α, β) and (α, γ) interaction forces, and u_{tr} is the translational controller.

Now, we present the simulation results for a group of $n = 100$ agents that use protocol (58). The first task is to reach a desired group velocity $p_d = (0, 10)^T$ and maintain an inter-agent distance of $d_\alpha = 7$ with $r = 1.2d_\alpha$ while avoiding $m = 6$ obstacles that their locations and radii are given by the $3 \times m$ matrix

$$M_{obs} = \begin{bmatrix} 100 & 120 & 150 & 160 & 200 & 200 \\ 20 & 40 & 40 & 0 & -5 & 50 \\ 10 & 2 & 5 & 3 & 20 & 20 \end{bmatrix}. \tag{59}$$

Each row of the obstacle matrix M_{obs} is a vector $(b_i, l_i)^T \in \mathbb{R}^2 \times \mathbb{R}_{>0}$ that contains the position and radius of the ith obstacle for $i = 1, \ldots, 6$.

We use a set of $n = 100$ random initial positions and zero initial velocities as the initial condition of the net dynamics. The snapshots of flocking for the first task are shown in Figures 7 and 8. The only difference in these two figures is that the edges of the net are drawn in Figure 8 and omitted in Figure 7. Apparently, the location of obstacles are chosen such that split/rejoin maneuvers have to be performed upon reaching all obstacles. Furthermore, notice that O_5 and O_6 are within a distance $d_{56} = 15$ from each other. Thus, it is impossible for 3 α-agents to pass between these two obstacles at the same time while their positions projected along p_d are equal. This is due to the fact that $2d_\alpha + 2r_0 > 3d_\alpha = 21 > 15 = d_{56}$. Thus, the portion of the net that is vertically between O_5 and O_6 needs to *squeeze* through the space between the two obstacles. This squeezing maneuver can be seen in Figures 7 and 8. We define the heading angle of each agent as the angle of the agent's velocity p_i, if $p_i \neq 0$, and zero otherwise. Based on Figure 8, there are very few agents that are not connected to the most populated flock and the density of the limiting flock is approximately equal to $\delta_0 = 0.97$ (only 3 agents out of 100 agents are not part of the main flock). Clearly, all agents (approximately) move with the same velocity and heading angles.

Again, consider a group of $n = 100$ α-agents start from random initial positions with zero initial velocities. The second task is to perform flocking in presence of a sink at $(q_d, p_d) = (0, 0)$ and in lack of any obstacles. Each agent receives the value of q_d once at time $t = 0$ (the beginning of the task) and from that point on every calculation can be done in a distributed manner. The α-agents need to maintain an inter-agent distance equal to $d_\alpha = 7$ with $r = 1.2d_\alpha$. In this case, we expect that a net that is initially disconnected, after going through a finite number of switching events, asymptotically converges to a flock with an invariant topology.

In general, the limiting conformation is a quasi-realization of the structural net (d_α, r). Our simulation results for this case are shown in Figure 9. Apparently, the limiting conformation of the net in this case is relatively close to an embedding of (d_α, r), i.e. it has a relatively low normalized defect factor. The final conformation in this case happens to be a planar graph that dominantly consists of equilateral triangular faces. The net becomes connected after 4 seconds. It is worth mentioning that the limiting flock is a rigid graph.

12 Conclusion

In this work, we provided a graph theoretical framework that enables modeling the flocking of dynamic agents in presence of multiple obstacles. We presented formal definitions of nets and flocks as graphs that are spatially induced by a set of node configurations (i.e. clusters). The realization (or embedding) issues of structural nets and flocks were discussed. This discussion led to task representation and execution for a network of agents called α-agents. The primary α-agent was to maintain a certain distance from other α-agents in their spatial neighborhood (or shell). Flocking was defined as achieving both structural stabilization and navigational tracking. We showed that the first two flocking rules of Reynolds follow from the special cases of a single protocol called the (α, α) protocol. The third rule of Reynolds is the same as a simple consensus protocol. In addition, we discussed certain situations that are not accounted for by Reynolds three flocking rules.

We also discussed flocking in the presence of multiple fixed obstacles. To perform this task, two other types of agents called β-agents and γ-agents were introduced. These agents are located on the boundary and inside of an obstacle. The existence of β and γ agents is contingent to the presence of an α-agent in a neighborhood of the corresponding obstacle. This framework enables us to address split/rejoin and squeezing maneuvers for nets and flocks of dynamic agents that communicate with each other. The presence of obstacles might force the members of a net (or flock) to split into more flocks and lead to loss of communication links. The loss of existing links might lead to disconnectivity of the network and change of the topology of the network. In general, flocking in presence of obstacles leads to solving decision making problems for agents with *limited communication* in a network with *switching topology.* Analysis of the protocols for this case is rather challenging and a number of problems including conformation of a connected network from an initially disconnected mobile network remain open. We provided simulation results that demonstrate flocking in presence of six obstacles and conformation of connected networks. The simulation results were consistent with the predictions suggested by the theoretical results in our work.

Acknowledgments

The first author would like to thank Emilio Frazzoli for the discussions on split/rejoin maneuver and the use of randomized algorithms for obstacle avoidance. This research is supported in part by AFOSR under the grant F49620–01–1–0361 and by DARPA under the grant F33615–98–C–3613.

References

[1] B. Bollobás. *Modern Graph Theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, 1998.

[2] D. E. Chang and J. Marsden. Gyroscopic Forces and Collision Avoidance. *the Conference in Honor of A. J. Krener's 60th Birthday*, 2002.

[3] D. E. Chang, S. Shadden, J. Marsden, and R. Olfati Saber. Collision Avoidance for Multiple Agent Systems. *submitted to the IEEE Conf. on Decision and Control*, February 2003.

[4] J. P. Desai, J. P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE Trans. on Robotics and Automation*, 17(6), December 2002.

[5] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, 2000.

[6] V. Gazi and K.M. Passino. Stability analysis of swarms. *IEEE Trans. on Automatic Control (to appear)*, May 2003.

[7] D. Helbing, I. Farkas, and T. Vicsek. Simulating dynamical features of escape panic. *Nature*, 407:487–490, 2000.

[8] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1987.

[9] A. Jadbabaie, J. Lin, and S. A. Morse. Coordination of groups of mobile agents using nearest neighbor rules. *To appear in the IEEE Trans. on Automatic Control*, 2002.

[10] Y. Liu, K.M. Passino, and M. M. Polycarpou. Stability analysis of M-dimensional asynchronous swarms with afixed communication topology. *IEEE Trans. On Automatic Control*, 48(1):76–95, January 2003.

[11] R. Olfati Saber. Agreement Problems in Networks with Directed Graphs and Switching Topology. Technical Report 2003–005, California Institute of Technology, Control and Dynamical Systems, Pasadena, California, February 2003.

[12] R. Olfati Saber, W. B. Dunbar, and R. M. Murray. Cooperative control of multi-vehicle systems using cost graphs and optimization. *Proc. of the American Control Conference (to appear)*, June 2003.

[13] R. Olfati-Saber and R. M. Murray. Distibuted cooperative control of multiple vehicle formations using structural potential functions. *The 15th IFAC World Congress*, June 2002, `http://www.cds.caltech.edu/~olfati/papers/ ifac02/ifac02_ros_rmm.html`.

[14] R. Olfati-Saber and R. M. Murray. Graph Rigidity and Distributed Formation Stabilization of Multi-Vehicle Systems. *Proceedings of the IEEE Int. Conference on Decision and Control*, Dec. 2002, http://www.cds.caltech.edu/~olfati/papers/cdc02/cdc02a.html.

[15] R. Olfati Saber and R. M. Murray. Agreement Problems in Networks with Directed Graphs and Switching Topology. *Submitted to the IEEE Conference on Decision and Control*, December 2003.

[16] R. Olfati Saber and R. M. Murray. Consensus Protocols for Networks of Dynamic Agents. *Proc. of the American Control Conference*, June 2003.

[17] C. W. Reynolds. Flocks, herds, and schools: a distributed behavioral model. *Computer Graphics (ACM SIGGRAPH '87 Conference Proceedings)*, 21(4):25–34, July 1987.

[18] E. Rimon and D. E. Koditschek. Exact robot navigation using artificial potential functions. *IEEE Trans. on Robotics and Automation*, pages 501–518, 1992.

[19] J. Toner and Y. Tu. Flocks, herds, and schools: A quantitative theory of flocking. *Physical Review E*, 58(4):4828–4858, October 1998.

[20] T. Vicsek, A. Cziroók, E. Ben-Jacob, and O. Cohen, I. Shochet. Novel type of phase transition in a system of self-deriven particles. *Physical Review Letters*, 75(6):1226–1229, August, 1995.

A Appendix

In this appendix, we present the proof of some of the propositions.

A.1 Proof of Proposition 2

Part i) We have

$$V(q) = 0 \iff \psi_\alpha(\|q_j - q_i\| - d_\alpha) = 0, \forall j \in N_i, \forall i \iff \|q_j - q_i\| = d_\alpha \forall j \in N_i, \forall i$$

which means q is a realization of the structural net (d_α, r).
Part ii) $K_r(\tilde{p}) = 0 \iff \tilde{p}_i = 0$ for all i. This means that for any two nodes i and j, the distance $\|q_j - q_i\|$ is a constant because

$$\frac{d}{dt}\|q_j - q_i\|^2 = (p_j - p_i)^T(q_j - q_i) = (\tilde{p}_j - \tilde{p}_i)^T(q_j - q_i) = 0.$$

This property holds regardless of whether these two nodes are neighboring nodes or not. To prove the converse, suppose that the net is an undirected flock (i.e. the graph $G(q)$ is connected) and the distance between any two nodes is constant but $K_r(\tilde{p}) \neq 0$. Thus, there exists a node i^* with $\tilde{p}_{i^*} \neq 0$. On the other hand, the distance between any other node $j \neq i^*$ and i^* remains constant for all $t \geq t_0$. Thus, we have

$$\frac{d}{dt}\|q_j - q_{i^*}\|^2 = (p_j - p_{i^*})^T(q_j - q_{i^*}) = (\tilde{p}_j - \tilde{p}_{i^*})^T(q_j - q_{i^*}) = 0.$$

But if $(p_j - p_{i^*}) = (\tilde{p}_j - \tilde{p}_{i^*}) \neq 0$, then $p_j - p_{i^*}$ cannot be orthogonal to $(q_j - q_{i^*})$ based on part c) of ii). Also, $q_j \neq q_{i^*}$ based on part b) of ii). Therefore, $\tilde{p}_j = \tilde{p}_{i^*}$ for all $j \neq i^*$. In other words, the relative velocities of all agents are equal and the same as $\tilde{p}_i^*$. By definition, $\sum_{i=1}^n \tilde{p}_i = 0$. This implies $n\tilde{p}_{i^*} = 0$ or $\tilde{p}_{i^*} = 0$ which contradicts the assumption that $\tilde{p}_{i^*} \neq 0$ and the result follows.
Part iii) This follows from part ii) and the fact that $G(q(t))$ remains invariant for all $t \geq t_0$ if the distance between any two arbitrary nodes remains invariant for all $t \geq t_0$. □

A.2 Proof of Proposition 4

Given $u_i = u_i^{(\alpha,\alpha,1)}$, we have $\sum_i u_i = 0$ and thus $\bar{p} = (\sum_{i=1}^n p_i)/n$ is an invariant quantity, i.e. $\dot{\bar{p}} = 0$. Define $\tilde{p}_i = p_i - \bar{p}$ and notice that $\sum \tilde{p}_i = 0$ and $\dot{\tilde{p}}_i = \dot{p}_i = u_i$. Considering that the topology of the net is invariant for all $t \in [t_k, t_{k+1})$, the (nonsmooth) structural energy of the net can be expressed as

$$H_s^{(k)}(q, \tilde{p}) = V(q) + K(\tilde{p}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i} \psi_\alpha(\|q_j - q_i\| - d_\alpha) + \frac{1}{2}\sum_{i=1}^{n} \|\tilde{p}_i\|^2 \tag{60}$$

where $N_i = N_i(q(t_k))$ is invariant in time over $[t_k, t_{k+1})$ and thus $H_s^{(k)}(q, \tilde{p})$ is differentiable with respect to $(q, \tilde{p})$. This is because no new links are created in the net and no existing links are lost (i.e. no energy jumps exist). Furthermore, if $G(q)$ is a flock, then by definition $G(q)$ a connected undirected graph satisfying $\text{rank}(L(q)) = n - 1$ with $n = |G(q)|$ where $L(q)$ is the Laplacian associated with the flock $G(q)$. By differentiating $H_s^{(k)}(q, \tilde{p})$ with respect to time, we get

$$\begin{array}{rcl}
\dot{H}_s^{(k)}(q, \tilde{p}) & = & \sum_{i=1}^n (\nabla_{q_i} V(q) p_i + u_i^T \tilde{p}_i) \\
& = & c_1 \sum_{i=1}^n \sum_{j\in N_i} \tilde{p}_i^T (p_j - p_i) \\
& = & c_1 \sum_{i=1}^n \sum_{j\in N_i} \tilde{p}_i^T (\tilde{p}_j - \tilde{p}_i) \\
& = & \frac{c_1}{2} \sum_{(i,j)\in\mathcal{E}_{G(q)}} \tilde{p}_i^T (\tilde{p}_j - \tilde{p}_i) + \tilde{p}_j^T (\tilde{p}_i - \tilde{p}_j) \\
& = & -\frac{c_1}{2} \sum_{(i,j)\in\mathcal{E}_{G(q)}} (\tilde{p}_j - \tilde{p}_i)^T (\tilde{p}_j - \tilde{p}_i) \\
& = & -\frac{c_1}{2} \sum_{(i,j)\in\mathcal{E}_{G(q)}} \|\tilde{p}_j - \tilde{p}_i\|^2 \leq 0
\end{array} \tag{61}$$

which means $H_s^{(k)}(q, \tilde{p})$ is a weak Lyapunov function. Now, suppose the topology of the net does not change after $t = t_m$. This means $H_s^{(m)}(q, \tilde{p})$ is a weak Lyapunov function for all $t \geq t_m$. On the other hand, $\dot{H}_s^{(m)} = 0 \iff \tilde{p}_i = \tilde{p}_j$ for all edges (i, j) of the net $G(q(t_m))$. But $G_m(q(t_m))$ is a flock and thus connected. Therefore, $\tilde{p}_i = \tilde{p}_j$ for all nodes $i, j, i \neq j$. Since $\sum_i \tilde{p}_i = 0$ and all the p_i's are equal, one concludes that $\tilde{p}_i = 0$ for all i. In other words, $\dot{H}_s^{(m)} = 0$ implies $p_i = \bar{p} = Ave(p(t_0))$ for all i. Since all relative inter-agent velocities are zero, given $\dot{H}_s^{(m)} = 0$, we have $u^{(\alpha,\alpha,1)} = -\nabla V(q) = 0$. In other words, based on LaSalle's invariance principle, $(q(t), p(t))$ asymptotically converges to a relative equilibrium (q^*, p^*) with $p^* = \mathbf{1} \otimes \bar{p}$ and q^* in the set of local minima of $V(q)$. Since $\tilde{p}_i = 0$ for all nodes, we get

$$\frac{d}{dt}\|q_j^* - q_i^*\|^2 = (p_j^* - p_i^*)^T (q_j^* - q_i^*) = (\tilde{p}_j^* - \tilde{p}_i^*)^T (q_j^* - q_i^*) = 0,$$

and the length of all existing edges are asymptotically invariant. □

A.3 Proof of Proposition 6

Notice that $\dot{\bar{q}} = \bar{p}$ and $\dot{\bar{p}} = \bar{u}$. To prove i), we explicitly calculate $\dot{H}_s(q, \tilde{p})$ with $\dot{H}_s(q, \tilde{p}) = V(q) + \frac{1}{2}\sum_i \|\tilde{p}_i\|^2$ as follows

$$\dot{H}_s(q, \tilde{p}) = \nabla V(q) \cdot p + \sum_i \tilde{p}_i(u_i - \bar{u}) = \nabla V(q) \cdot p + \sum_i \tilde{p}_i u_i^\alpha \leq 0, \forall t \geq 0 \tag{62}$$

The last inequality holds with $u^\alpha = u^{(\alpha,\alpha,2)}$ based on Proposition 5. Similarly, given $H_{tr}(\bar{q}, \bar{p})$ with $p_d = 0$ for the case of a desired sink, we obtain

$$\dot{H}_{tr}(\bar{q}, \bar{p}) = \nabla V_{tr}(\bar{q}) \cdot \bar{p} + \bar{p}^T \cdot \bar{u} = -c_2 \bar{p}^T \phi(\bar{p}) \leq 0, \quad c_2 > 0 \tag{63}$$

with the bounded state feedback

$$\bar{u} = k_{tr}^{(1)}(\bar{q}, \bar{p}) := -\nabla V_{tr}(\bar{q}) - c_2\phi(\bar{p}) = -\phi(\bar{q} - q_d) - c_2\phi(\bar{p}), \quad c_2 > 0 \tag{64}$$

where $\phi(z) = z/\sqrt{1 + \|z\|^2}$ is a uniformly bounded function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $z^T\phi(z) > 0$ for all $z \neq 0$. Clearly, $\dot{H}_{tr}(\bar{q}, \bar{p})$ does not depend on the choice of u^α provided that $\sum_i u_i^\alpha = 0$. For the case of a desired group velocity $p_d \neq 0$ and $V_{tr}(q) \equiv 0$, we have $H_{tr}(\bar{q}, \bar{p}) = \frac{1}{2}\|p - p_d\|^2$ and thus

$$\dot{H}_{tr}(\bar{q}, \bar{p}) = \nabla V_{tr}(\bar{q}) \cdot \bar{p} + (\bar{p} - p_d)^T \cdot \bar{u} = -c_2(\bar{p} - p_d)^T \phi(\bar{p} - p_d) < 0, \quad \forall \bar{p} \neq p_d \tag{65}$$

with the bounded velocity feedback

$$\bar{u} = k_{tr}^{(2)}(\bar{p}) := -c_2\phi(\bar{p} - p_d), \quad c_2 > 0 \tag{66}$$

Notice that $\|k_{tr}^{(1)}(\bar{q}, \bar{p})\| \leq 1 + c_2$ and $\|k_{tr}^{(2)}(\bar{p})\| \leq c_2$. Finally, based on parts i) and ii), according to

$$\dot{H}(q, p) = \dot{H}_s(q, \tilde{p}) + \dot{H}_{tr}(\bar{q}, \bar{p}) \leq 0$$

given that $u^\alpha = u^{(\alpha,\alpha,2))}$ and $\bar{u} = k_{tr}^j$ with $j = 1, 2$, $H(q(t), p(t))$ can be rendered monotonically non-increasing for all $t \geq 0$. □

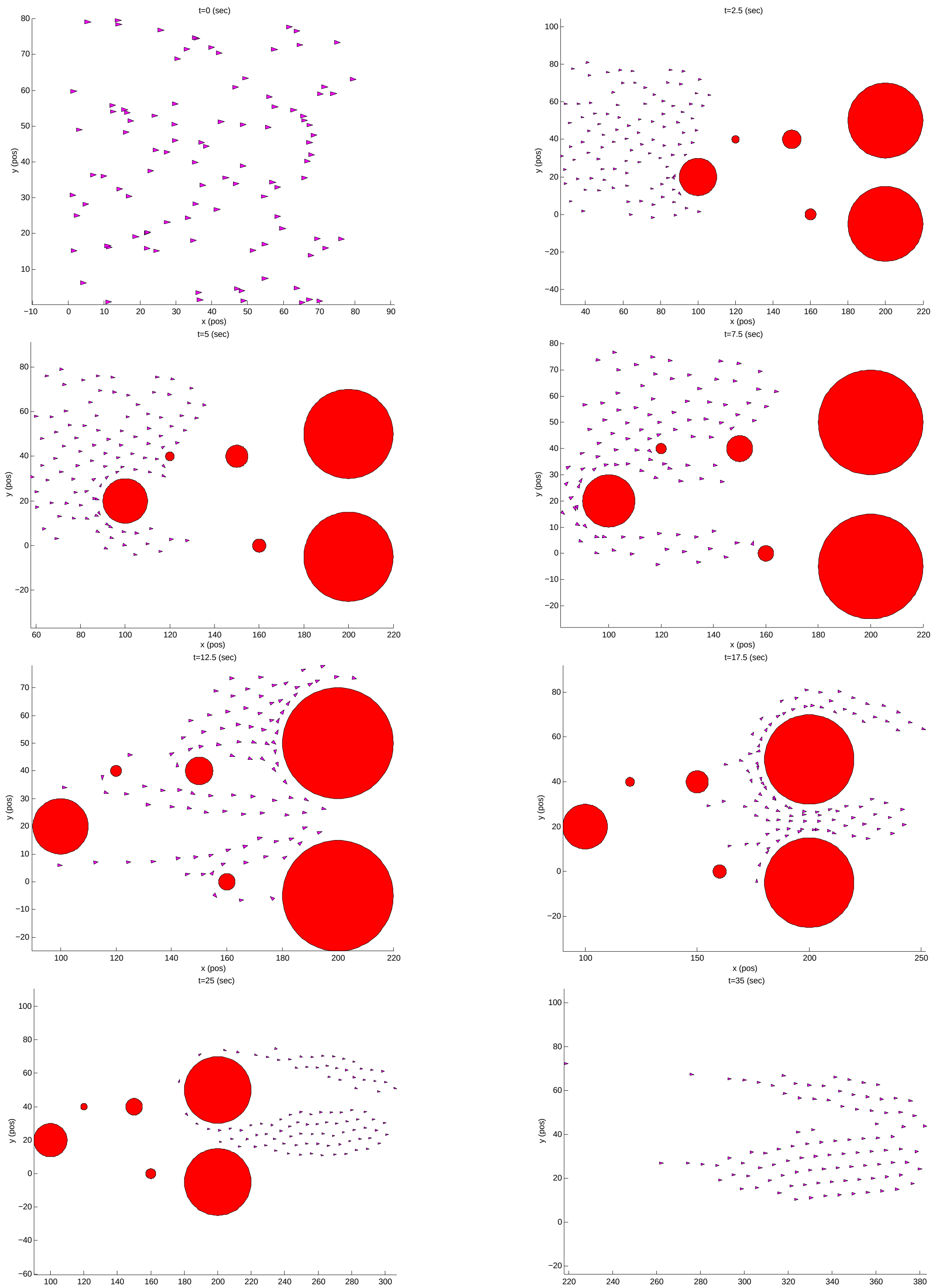


Figure 7: Consecutive snapshots of flocking for a cluster of $n = 100$ agents in presence of $m = 6$ obstacles and split/rejoin/squeezing maneuvers.

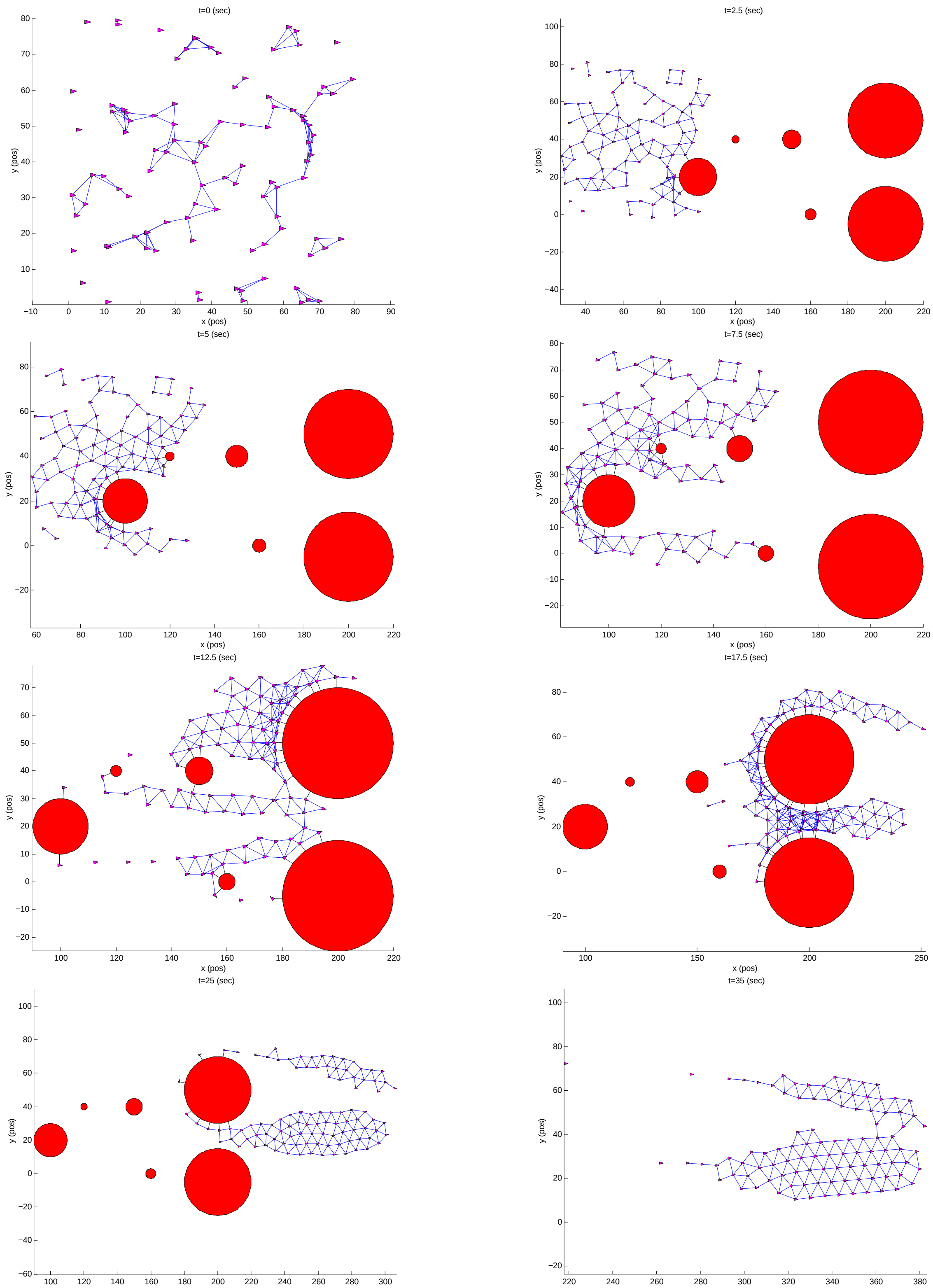


Figure 8: Consecutive snapshots of flocking for a net of $n = 100$ agents in presence of $m = 6$ obstacles and split/rejoin/squeezing maneuvers.

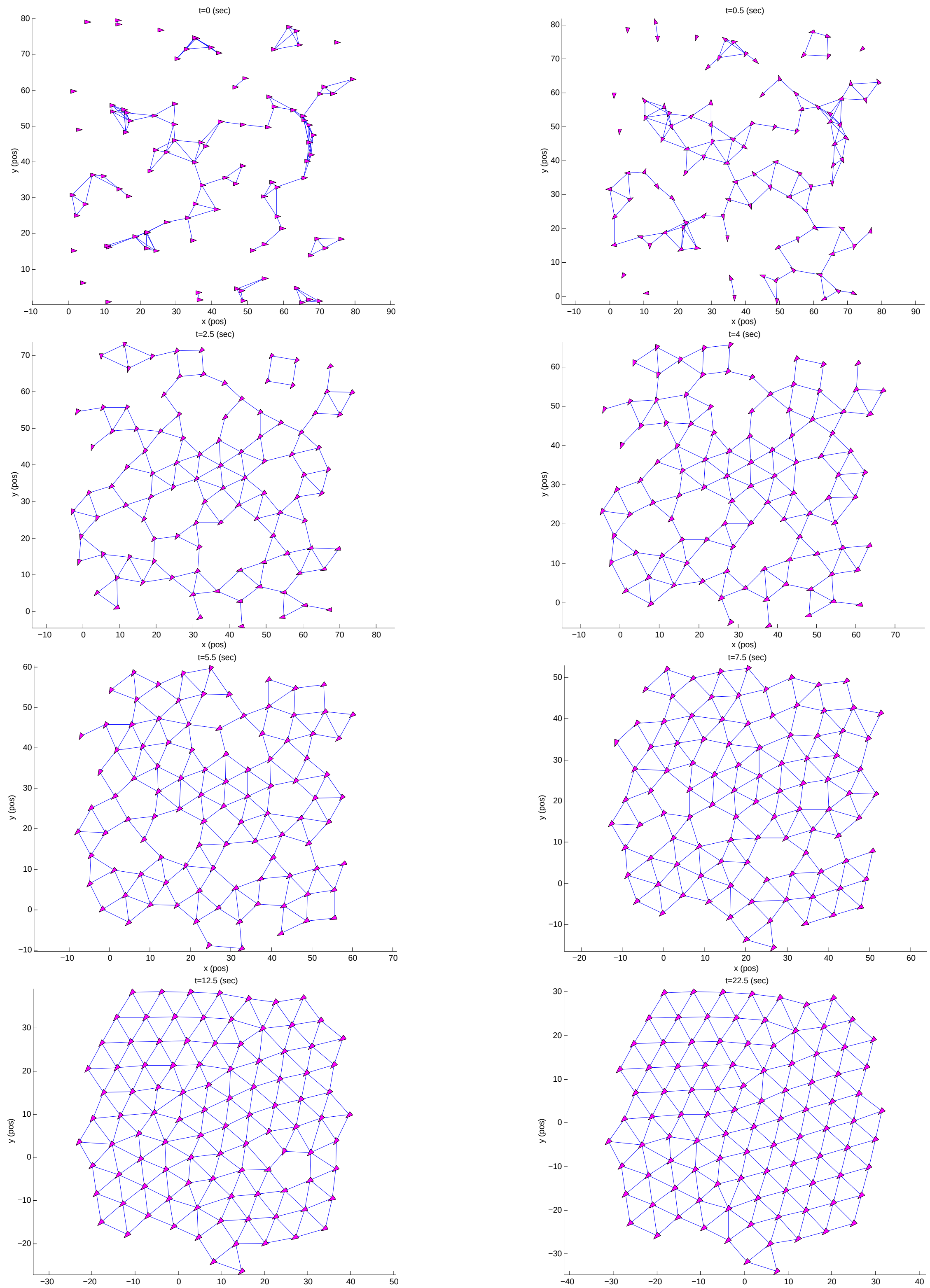


Figure 9: Consecutive snapshots of conformation of a flock from a net for a cluster of $n = 100$ agents in presence of a sink at $(q_d, p_d) = (0, 0)$.