# Constrained Stabilization of Discrete-Time Systems

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**Abstract**

Based on the growth rate of the set of states reachable with unit-energy inputs, we show that a discrete-time controllable linear system is globally controllable to the origin with constrained inputs if and only if all its eigenvalues lie in the closed unit disk. These results imply that the constrained Infinite-Horizon Model Predictive Control algorithm is globally stabilizing for a sufficiently large number of control moves if and only if the controlled system is controllable and all its eigenvalues lie in the closed unit disk.

In the second part of the paper, we propose an implementable Model Predictive Control algorithm and show that with this scheme a discrete-time linear system with $n$ poles on the unit disk (with any multiplicity) can be globally stabilized if the number of control moves is larger than $n$. For pure integrator systems, this condition is also necessary. Moreover, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input.

## 1 Introduction

Most practical control schemes have to deal with bounds on the control inputs; these bounds may arise from physical constraints on the input, e.g. actuator saturation. Various approaches have been employed to study the problem of controlling systems with bounded inputs: e.g. optimal control (Mayne and Michalska (1990), Tsirukis and Morari (1992), and Yang and Polak (1993)), smooth nonlinear control (Sontag (1984), Sontag and Sussmann (1990), Sontag and Yang (1991), and Teel (1992)), and semi-global stabilization (Lin and Saberi (1993)). In this paper, we use Model Predictive Control (MPC) (also called Moving Horizon Control and Receding Horizon Control). MPC has become a powerful feedback strategy to control systems with constraints because of its ability to handle the constraints in an optimal fashion.

Under the MPC scheme, the control input at any time instant is obtained by solving a quadratic program (that is, minimizing an objective that is quadratic in the optimization variables, subject to linear constraints). Recently, Rawlings and Muske (1993) showed that MPC with an infinite output horizon can globally stabilize linear discrete-time systems *provided* that the quadratic program is feasible. However, the following question was not answered: under what condition is the quadratic program feasible?

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It was shown by Sontag and Sussmann (1990) and Tsirukis and Morari (1992) that for strictly unstable systems (that is, systems with some poles outside the unit disk), there *always* exist initial conditions for which the quadratic program is infeasible. Conversely (see Tsirukis and Morari (1992)), for stabilizable systems with poles in the closed unit disk, given any initial condition, the quadratic program is *always* feasible, *provided* that there are enough optimization variables. In the first part of this paper, we prove a stronger version of this result: Based on the growth rate of the set of states reachable with unit-energy inputs, we show that a discrete-time controllable linear system is globally controllable to the origin with unit-energy inputs if and only if all its eigenvalues lie in the closed unit disk. Then we show that the constrained Infinite-Horizon MPC algorithm is globally stabilizing for a sufficiently large number of control moves (optimization variables).

However, the number of control moves needed for feasibility of the quadratic program depends on the initial condition; it is generally difficult to determine *a priori* and can be arbitrarily large. Furthermore, in practice an unmeasured disturbance could still cause the quadratic program to become infeasible and an even larger number of control moves may have to be chosen. Therefore, this strategy is *not* easily implementable. In the second part of this paper, we propose an implementable MPC algorithm and show that with this scheme a discrete-time linear system with $n$ poles on the unit disk (with any multiplicity) can be globally stabilized if the number of control moves is larger than $n$. For the specific case of a chain of $n$ integrators, this condition is also necessary. Furthermore, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input. Thus, the first part of the paper may be viewed as a theoretical existence-type result for the MPC scheme, while the second part provides an algorithmic, implementable scheme.

The first part of the paper is organized as follows. In Section 2, we show that the singular values of the ellipsoidal set of states reachable in $N$ steps with unit-energy inputs for a discrete-time $n$-integrator system grow as $\{O(N^{2n-1}), O(N^{2n-3}), \ldots, O(N)\}$. In Section 3, we show that this implies that a discrete-time controllable linear system is globally controllable to the origin if and only if all its eigenvalues lie in the closed unit disk. In Section 4, we show that the Infinite-Horizon Model Predictive Control (IH-MPC) scheme is globally stable if and only if all the eigenvalues of the controlled system lie in the closed unit disk.

The second part of the paper opens with some preliminaries in Section 5. An implementable MPC algorithm is then proposed. In Section 6, we give both necessary and sufficient conditions on the number of control moves for global asymptotic stability in the presence of asymptotically constant disturbances entering at the plant input. Two examples are presented in Section 7. Section 8 concludes the paper. For notational simplicity, all the results in the second part of the paper are proven for Single-Input Single-Output (SISO) systems; we briefly discuss the extension of the results to Multi-Input Multi-Output (MIMO) systems.

# Part I: Constrained stabilizability of discrete-time systems

## 2 Reachable set for a multiple-integrator system

Consider the discrete-time integrator chain

$$x(k+1) = J_n x(k) + e_{\text{last}} u(k), \tag{1}$$

where $J_n$ is a Jordan block of size $n$ with eigenvalue 1:

$$J_n = \begin{bmatrix} 1 & 1 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix},$$

and $e_{\text{last}}$ is the last Euclidean basis vector, that is, $e_{\text{last}} = [0 \ \cdots \ 0 \ 1]^T$. The size of $e_{\text{last}}$ will be determined from context (of course, here $e_{\text{last}} \in \Re^n$).

The set of states reachable with unit-energy inputs in $N$ steps for system (1) is

$$\mathcal{R}_N \triangleq \left\{ z \;\middle|\; x(0) = 0,\ x(N) = z,\ x(\cdot) \text{ satisfies (1) and } \sum_{k=0}^{N-1} u(k)^2 \leq 1 \right\}. \tag{2}$$

Of course, $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots \subseteq \mathcal{R}_N$. Moreover, it is well-known (see Callier and Desoer (1991), for example) that $\mathcal{R}_N$ is the ellipsoid

$$\left\{ z \;\middle|\; z^T W_{n,N}^{-1} z \leq 1 \right\},$$

where $W_{n,N} = \sum_{k=0}^{N-1} J_n^k e_{\text{last}} e_{\text{last}}^T (J_n^T)^k$. We will refer to $W_{n,N}$ as the $N$-step reachability Gramian of the pair $(J_n, e_{\text{last}})$. We denote the $i$th singular value of $W_{n,N}$ by $\sigma_i(W_{n,N})$, and state the following result:

**Theorem 1** *The $n$ singular values of $W_{n,N}$,* viz. $\{\sigma_1(W_{n,N}), \sigma_2(W_{n,N}), \ldots, \sigma_n(W_{n,N})\}$ *are*

$$\{O(N^{2n-1}), O(N^{2n-3}), \ldots, O(N)\}$$

*in $N$. Moreover, the corresponding singular vectors of $W_{n,N}$ converge to the standard Euclidean basis of $\Re^n$* $\{[1\ 0 \cdots 0], [0\ 1 \cdots 0], \ldots, [0\ 0 \cdots 1]\}$.

*Proof.* We first note that

$$J_n^k e_{\text{last}} = \left[ \begin{pmatrix} k \\ n-1 \end{pmatrix} \begin{pmatrix} k \\ n-2 \end{pmatrix} \cdots \begin{pmatrix} k \\ n-n \end{pmatrix} \right]^T,$$

where

$$\begin{pmatrix} m \\ n \end{pmatrix} \triangleq \begin{cases} \dfrac{m!}{n!(m-n)!} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $W_N$ equals

$$\sum_{k=0}^{N-1} \begin{bmatrix} \begin{pmatrix} k \\ n-1 \end{pmatrix}\begin{pmatrix} k \\ n-1 \end{pmatrix} & \begin{pmatrix} k \\ n-1 \end{pmatrix}\begin{pmatrix} k \\ n-2 \end{pmatrix} & \cdots & \begin{pmatrix} k \\ n-1 \end{pmatrix}\begin{pmatrix} k \\ n-n \end{pmatrix} \\ \begin{pmatrix} k \\ n-2 \end{pmatrix}\begin{pmatrix} k \\ n-1 \end{pmatrix} & \begin{pmatrix} k \\ n-2 \end{pmatrix}\begin{pmatrix} k \\ n-2 \end{pmatrix} & \cdots & \begin{pmatrix} k \\ n-2 \end{pmatrix}\begin{pmatrix} k \\ n-n \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} k \\ n-n \end{pmatrix}\begin{pmatrix} k \\ n-1 \end{pmatrix} & \begin{pmatrix} k \\ n-n \end{pmatrix}\begin{pmatrix} k \\ n-2 \end{pmatrix} & \cdots & \begin{pmatrix} k \\ n-n \end{pmatrix}\begin{pmatrix} k \\ n-n \end{pmatrix} \end{bmatrix}.$$

In the sequel, given matrices $A$ and $B$ that depend on $k$, we will say "$A(k) \approx B(k)$ for large $k$" to mean that $\lim_{k\to\infty} A_{ij}(k)/B_{ij}(k) = 1$. Then, since

$$\begin{pmatrix} k \\ n-j \end{pmatrix} = \frac{k(k-1)\cdots(k-n+j+1)}{(n-j)!},$$

we have

$$\begin{pmatrix} k \\ n-j \end{pmatrix} \approx \frac{k^{(n-j)}}{(n-j)!}$$

for large $k$ and

$$\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-i \end{pmatrix} \begin{pmatrix} k \\ n-j \end{pmatrix} \approx \frac{N^{(2n-i-j+1)}}{(n-i)!\,(n-j)!\,(2n-i-j+1)} \tag{3}$$

for large $N$.

Therefore, we conclude that for large $N$, $W_N$ is

$$\begin{bmatrix} O(N^{2n-1}) & O(N^{2n-2}) & \dots & O(N^{2n-n}) \\ O(N^{2n-2}) & O(N^{2n-3}) & \dots & O(N^{2n-n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ O(N^{2n-n}) & O(N^{2n-n-1}) & \dots & O(N^{1}) \end{bmatrix}. \tag{4}$$

Intuition suggests that this fact means that the largest singular value $\sigma_1(W_{n,N})$ grows as $O(N^{2n-1})$ and the corresponding left and right singular vectors tend to $e_1 = [1 \;\; 0 \cdots 0]$, that the second singular value $\sigma_2(W_{n,N})$ grows as $O(N^{2n-3})$ and the corresponding left and right singular vectors tend to $e_2 = [0 \;\; 1 \cdots 0]$, etc. Let us now prove this.

We start by writing $W_{n,N}$ as

$$W_{n,N} = \begin{bmatrix} \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2 & \left( \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}} \right)^T \\ \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}} & W_{n-1,N} \end{bmatrix}.$$

Applying a congruence on $W_{n,N}$ with

$$Q = \begin{bmatrix} 1 & 0 \\ -\dfrac{\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}}}{\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2} & I \end{bmatrix},$$

we get

$$QW_NQ^T = \begin{bmatrix} \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2 & 0 \\ 0 & W_{n-1,N} - \dfrac{\left( \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}} \right) \left( \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}} \right)^T}{\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2} \end{bmatrix}$$

Using routine algebraic manipulations, it can be shown that

$$W_{n-1,N} - \frac{\left( \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}} \right) \left( \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} J_{n-1} e_{\mathrm{last}} \right)^T}{\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2}$$

is approximately

$$\begin{bmatrix} \frac{1}{2n-2} & & & \\ & \frac{2}{2n-3} & & \\ & & \ddots & \\ & & & \frac{n-1}{n} \end{bmatrix} W_{n-1,N} \begin{bmatrix} \frac{1}{2n-2} & & & \\ & \frac{2}{2n-3} & & \\ & & \ddots & \\ & & & \frac{n-1}{n} \end{bmatrix}$$

for large $N$.

We now observe that the congruence matrix $Q \to I$ as $N \to \infty$, implying that the maximum singular value $\sigma_1(W_{n,N}) \approx \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2$ for large $N$, and that the corresponding singular vector converges to the first Euclidean basis vector $e_1$. Applying the block diagonalization technique recursively to $W_{n-1,N}$, $\ldots W_{1,N}$, we conclude that the $i$th singular value of $W_{n,N}$

$$\sigma_i(W_{n,N}) \approx \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-i \end{pmatrix}^2 \Big/ \begin{pmatrix} 2n-i \\ i-1 \end{pmatrix}^2$$

for large $N$, and the singular vectors tend to the the standard Euclidean basis of $\Re^n$, i.e.,

$$\{[1 \;\; 0 \cdots 0], [0 \;\; 1 \cdots 0], \ldots, [0 \;\; 0 \cdots 1]\}.$$

Using (3), we may finally write

$$\sigma_i(W_{n,N}) \approx \left( \frac{(i-1)!\,(2n-2i+1)!}{(n-i)!\,(2n-i)!} \right)^2 \frac{1}{(2n-2i+1)} N^{2n-2i+1},$$

for large $N$, which concludes the proof. $\square$

Figure 1 illustrates Theorem 1 for $n = 4$.

**Corollary 1** *Consider the system*

$$x(k+1) = J_n x(k) + Bu(k), \tag{5}$$

*where $B \in \Re^{n \times p}$ has a nonzero last row (so that the system is controllable). Theorem 1 holds for system (5) as well.*

*Proof.* Let $b_1^T, b_2^T, \ldots, b_n^T$ be the rows of $B$, so that $B^T = [b_1 \; b_2 \; \cdots \; b_n]$. Then,

$$\tilde{W}_{n,N} = \sum_{k=0}^{N-1} J_n^k \begin{bmatrix} b_1^T b_1 & b_1^T b_2 & \cdots & b_1^T b_n \\ b_2^T b_1 & b_2^T b_2 & \cdots & b_2^T b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^T b_1 & b_n^T b_2 & \cdots & b_n^T b_n \end{bmatrix} (J_n^T)^k = \sum_{1 \le i,j \le n} b_i^T b_j \sum_{k=0}^{N-1} J_n^k e_i e_j (J_n^T)^k$$

Since

$$J_n^k e_j = \left[ \begin{pmatrix} k \\ j-1 \end{pmatrix} \; \begin{pmatrix} k \\ j-2 \end{pmatrix} \; \cdots \begin{pmatrix} k \\ j-n \end{pmatrix} \right]^T, \quad j = 1, 2, \ldots, n,$$

we have

$$\sum_{1 \le i,j \le n} b_i^T b_j \sum_{k=0}^{N-1} J_n^k e_i e_j (J_n^T)^k \approx b_n^T b_n \sum_{k=0}^{N-1} J_n^k e_n e_n (J_n^T)^k$$

for large $N$, and the claim made in the corollary follows (recall that $b_n \neq 0$). $\square$

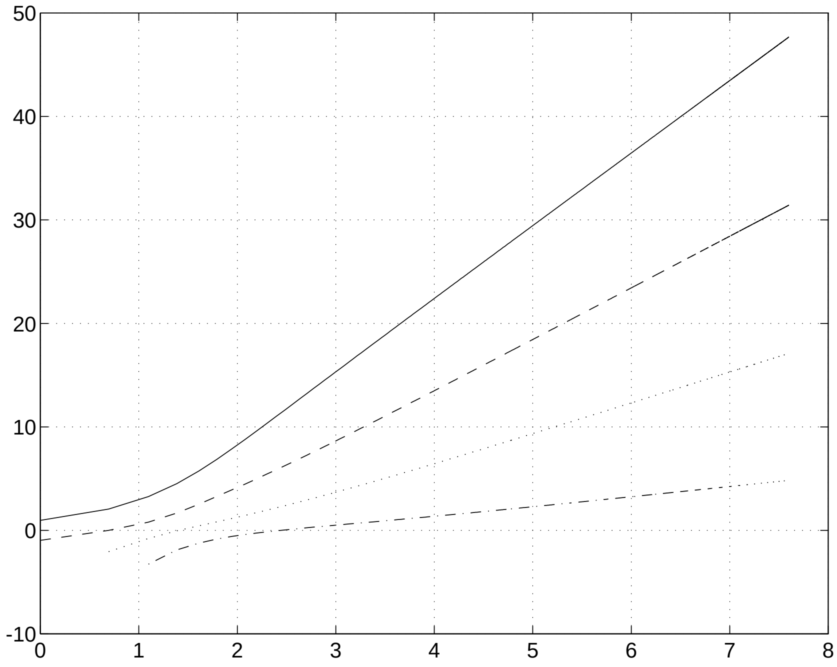


Figure 1: Logarithms of singular values of $W_{4,N}$ versus $N$

Next, consider the system

$$x(k+1) = TJ_nT^{-1}x(k) + TBu(k), \tag{6}$$

where $B \in \Re^{n\times p}$ has a nonzero last row. Let the $N$-step reachability Gramian of the pair $(TJ_nT^{-1}, TB)$ be denoted by $\overline{W}_{n,N}$. We then have the following theorem.

**Theorem 2** *Let $T = QR$ be the QR-factorization of $T$, i.e., $Q$ is orthogonal and $R$ is upper triangular with positive diagonal entries. Then, the singular values of $\overline{W}_{n,N}$ grow as*

$$\{O(N^{2n-1}), O(N^{2n-3}), \ldots, O(N)\}.$$

*Moreover, the matrix whose columns comprise the singular vectors of $\overline{W}_{n,N}$ converges to $Q$.*

*Proof.* The $N$-step reachability Gramian $\overline{W}_{n,N}$ of the pair $(TJ_nT^{-1}, TB)$ equals $T\tilde{W}_{n,N}T^T$, where $\tilde{W}_{n,N}$ is the $N$-step reachability Gramian of the pair $(J_n, B)$. Then $\overline{W}_{n,N} = QR\tilde{W}_{n,N}R^TQ^T$. A direct calculation shows that

$$R\tilde{W}_{n,N}R^T \approx \begin{bmatrix} R_{11} & & & \\ & R_{22} & & \\ & & \ddots & \\ & & & R_{nn} \end{bmatrix} \left( \sum_{k=0}^{N-1} J_n^k e_{\mathrm{last}} e_{\mathrm{last}}^T (J_n^T)^k \right) \begin{bmatrix} R_{11} & & & \\ & R_{22} & & \\ & & \ddots & \\ & & & R_{nn} \end{bmatrix}$$

for large $N$, where $R_{ii}$ is the $i$th diagonal element of $R$. (This is a direct consequence of the fact that $R$ is upper-triangular.) This completes the proof. $\square$

**Corollary 2** *The above results extend immediately to the case when the eigenvalue of the Jordan block is not unity, but equals $re^{j\theta}$ for some $\theta \in [0, 2\pi]$ and some $r > 1$. (In this case, $W_{n,N}$ is defined to be $\sum_{k=0}^{N-1} J_n^k e_{\mathrm{last}} e_{\mathrm{last}}^T (J_n^*)^k$.) Then the singular values of $W_{n,N}$ grow as*

$$\{O(r^{2N}N^{2n-1}), O(r^{2N}N^{2n-3}), \ldots, O(r^{2N}N)\}$$

*with $N$.*

*Proof.* Let $J_n^{(\lambda)}$ be a Jordan block of size $n$ with eigenvalue $\lambda = re^{j\theta}$. It is easy to show that $J_n^{(\lambda)}$ is similar to $\lambda J_n$. This fact, combined with Theorems 1 and 2 immediately yields the desired conclusion. $\square$

**Corollary 3** *Let*

$$A = \begin{bmatrix} J^{(1)} & & & \\ & J^{(2)} & & \\ & & \ddots & \\ & & & J^{(m)} \end{bmatrix},$$

*where $J^{(i)}$ is a Jordan block of size $\nu_i$ and eigenvalue $\lambda_i = e^{j\theta_i}$ for $i = 1, \dots, m$ with $(A, B)$ being controllable. Then the minimum eigenvalue of the $N$-step reachability Gramian of the pair $(A, B)$ is $O(N)$.*

*Proof.* The proof is very similar to the proof of Theorem 1. For simplicity of exposition, we will demonstrate the proof for the special case when the size of each Jordan block is two (i.e., $\nu_i = 2$ for all $i$), and when $B_i = e_{\text{last}}$. The proof for the general case should be readily apparent.

We first perform a similarity transformation so that

$$A = \begin{bmatrix} \lambda_1 J_{\nu_1} & & & \\ & \lambda_2 J_{\nu_2} & & \\ & & \ddots & \\ & & & \lambda_m J_{\nu_m} \end{bmatrix},$$

and

$$B = \begin{bmatrix} \lambda_1 e_{\text{last}} \\ \lambda_2 e_{\text{last}} \\ \vdots \\ \lambda_m e_{\text{last}} \end{bmatrix}.$$

(With some abuse of notation, we will use $A$ and $B$ to denote the state-space matrices in the new coordinate systems as well, in order to avoid proliferation of symbols.)

We follow this with another similarity transformation (in fact, a simple permutation similarity) so that

$$A = \begin{bmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 \\ \Lambda\,\mathbf{1} \end{bmatrix},$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ and $\mathbf{1}$ is a vector of length $m$ with each component unity.

In this new coordinates, the $N$-step reachability Gramian $W_N$ satisfies

$$W_N \approx \begin{bmatrix} \sum_{k=0}^{N-1} k^2 I & \sum_{k=0}^{N-1} kI \\ \sum_{k=0}^{N-1} kI & \sum_{k=0}^{N-1} I \end{bmatrix}$$

for large $N$. Using the block diagonalization technique in the proof of Theorem 1, it is straightforward to show that $m$ singular values of $W_N$ are $O(N^3)$ and the remaining $m$ singular values of $W_N$ are $O(N)$. $\square$

## 3 Controllability to the origin with bounded inputs

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k). \tag{7}$$

Since we may always perform a state coordinate transformation that puts $A$ in its Jordan form, we may assume, without loss of generality that

$$A = \begin{bmatrix} J^{(1)} & & & \\ & J^{(2)} & & \\ & & \ddots & \\ & & & J^{(m)} \end{bmatrix},$$

where $J^{(i)}$ is a Jordan block of size $\nu_i$ and eigenvalue $\lambda_i$ for $i = 1, \ldots, m$. For future reference, we partition $B$ and $x$ conformally as

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

We now consider the problem of controlling the state of system (7) to the origin with unit-energy inputs:

$$\text{Given } x(0), \text{ find } u \text{ with } \|u\|_2 \leq 1 \text{ such that } \lim_{k\to\infty} x(k) = 0. \tag{8}$$

We will show that a necessary and sufficient condition for this is that $A$ has all its eigenvalues in the closed unit disk, that is $\rho(A) \leq 1$. Indeed, we will show that for every $x(0) \in \Re^n$, there exists $N$ such that the following problem is feasible, if and only if $\rho(A) \leq 1$:

$$\text{Given } x(0), \text{ find } u \text{ and } N \text{ with } \left( \sum_{k=0}^{N-1} |u(k)|^2 \right)^{1/2} \leq 1 \text{ such that } x(N) = 0. \tag{9}$$

First let us assume that $\rho(A) \leq 1$. Indeed, we may as well assume that all the eigenvalues of $A$ are on the unit circle: Any eigenvalue in the open unit disk is a *stable* eigenvalue, and the projection of the initial condition $x(0)$ on the eigenspace of this eigenvalue decays to zero exponentially, with zero input, and therefore we may "ignore" these eigenvalues. (If there is no eigenvalue on the unit circle, then the problem is trivially solved with zero input!)

The condition $x(N) = 0$ yields

$$0 = \begin{bmatrix} B \; AB \; \cdots \; A^{N-1}B \end{bmatrix} \begin{bmatrix} u(0)^T \; u(1)^T \; \cdots \; u(N-1)^T \end{bmatrix}^T + A^N x(0).$$

Then, we need

$$x(0) = -\begin{bmatrix} A^{-1}B \; A^{-2}B \; \cdots \; A^{-N}B \end{bmatrix} \begin{bmatrix} u(N-1)^T \; u(N-2)^T \cdots u(0)^T \end{bmatrix}^T.$$

In other words, $x(0)$ must be reachable for the system

$$\tilde{x}(k+1) = A^{-1}\tilde{x}(k) - A^{-1}Bu(k)$$

with unit-energy $u$, over $N$ time steps. Since every eigenvalue of $A^{-1}$ is of the form $e^{j\theta}$ for some $\theta \in [0, 2\pi]$, it follows from Corollary 3 that this is so. Thus sufficiency of the condition $\rho(A) \leq 1$ is proved.

Conversely, let $\rho(A) > 1$. Without loss of generality, say $|\lambda_1| > 1$. Then it is quite easy to show that for every initial condition of the form $x(0) = [z_1^T \;\; 0]^T$ with $z_1^T W_c^{-1} z_1 > 1$, problem (9) is infeasible, where $W_c$ is given as the unique solution to the Lyapunov equation

$$W_c - J^{(1)} W_c (J^{(1)})^T + B_1 B_1^T = 0.$$

Thus, we have the following theorem.

**Theorem 3** *For every $z \in \Re^n$, there exists $N$ such that the system (7) is controllable from $z$ to 0 in $N$ time steps with unit-energy inputs if and only if $\rho(A) \leq 1$.*

*Remark.* Since the set of reachable states grows linearly with the energy bound on the input, we note that the above claims hold for any *arbitrarily small* bound on the energy, not necessarily unity.

Often, the following variation on problem (9) is of interest:

$$\text{Given } x(0), \text{ find } u \text{ and } N \text{ with } \|u(k)\|_2^2 \leq 1, \ k = 0, 1, \ldots, N-1, \text{ such that } x(N) = 0. \tag{10}$$

This problem concerns the controllability to the origin from $x(0)$ with *unit-peak* inputs, in contrast to the unit-energy inputs considered earlier.

It may be shown that problem (10) is feasible if and only if all the eigenvalues of $A$ are in the closed unit disk. It follows immediately that the latter condition is sufficient for problem (10) to be feasible: the set of unit-peak inputs contains the set of unit-energy inputs.

The proof of necessity can be outlined as follows. Suppose that one of the eigenvalues of $A$ is outside the unit circle. At the sampling time $k$, the value of the state has two contributions, one from the initial condition ($x(0)$) and the other from the controls ($u$) up to the sampling time $k-1$. For sufficiently large $k$, the contribution from the initial condition behaves as $\beta e^{\lambda k}$ where $\lambda > 0$ and $\beta$ is a constant that depends on the initial condition and can be made *arbitrarily large* for some initial condition. The contribution from the control input at the sampling time $i < k$ behaves as $\gamma e^{\lambda(k-i)}$. Since the control input is bounded, $\gamma$ is bounded. Simple calculations show that the total contribution from the controls up to the sampling time $k-1$ is bounded by $\bar{\gamma} e^{\lambda k}$ where $\bar{\gamma}$ is constant. Thus if we chose an initial condition such that $|\beta| > \bar{\gamma}$, then the output will grow unbounded regardless of control actions. Therefore, there are initial conditions that cannot be controlled to the origin, even with unit-peak inputs, if the controlled system has eigenvalues outside the unit disk. In other words, Theorem 3 may be extended to the case of unit-peak inputs:

**Theorem 4** *For every $x(0) \in \Re^n$, there exists $N$ such that the system (7) is controllable from $x(0)$ to 0 over $N$ time steps with unit-peak inputs if and only if $\rho(A) \leq 1$.*

*Remark.* As before, the claim in Theorem 4 holds for any *arbitrarily small* bound on the peak, not necessarily unity.

# 4 Stability of constrained MPC schemes

The Infinite-Horizon Model-Predictive-Control (IH-MPC) scheme refers to the control input design for the stabilizable system

$$x(k+1) = Ax(k) + Bu(k).$$

At every time $k$, the optimal input $u(k)$ equals the first element $v(0)$ of the sequence $\{v(0), v(1), \ldots, v(N-1)\}$ which is the minimizer of the optimization problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=0}^{\infty} \left( z(i)^T R z(i) + v(i)^T S v(i) \right) \\ \text{subject to} & z(i+1) = Az(i) + Bv(i), \quad z(0) = x(k) \\ & |v(i)|_\infty \leq 1, \ i = 0, 1, \ldots, N-1 \\ & v(i) = 0, \ i \geq N \end{array} \tag{11}$$

where $R = R^T > 0$ and $S = S^T > 0$. $N$ is referred to as the "input horizon". We will denote by $J(x(k))$ the optimal value of the objective function in problem (11).

An important question associated with the IH-MPC scheme is that of stability: *Given $x(0)$, does the above scheme always lead to a control $u$ that steers the state to zero?*

We may break the answer to this question into two parts: First, we require $J(x(k)) < \infty$ for each $k$. If this condition is satisfied, we may then ask if the overall strategy—that of implementing as input only the first element of the minimizer at each step—is stable.

Obviously, $J(x(k)) < \infty$ for all $x(k) \in \Re^n$ if and only if for every $x(k)$, the projection of $z(N)$ on the eigenspace of $A$ corresponding to the unstable (that is, with magnitude that is not less than one) eigenvalues is zero. The results of Section 3 immediately give us the following: for every $x(k)$, there exists a value of $N$ such that $J(x(k)) < \infty$ if and only if $(A, B)$ is stabilizable and all the eigenvalues of $A$ are in the closed unit disk.

Next, let us consider the stability of the moving horizon strategy. First, if $J(x(k)) < \infty$ for some $k$, then $J(x(k+1)) < \infty$. Indeed $J(\cdot)$ serves as a Lyapunov function that proves the stability of the horizon strategy. This can be seen as follows. Assuming $J(x(k)) < \infty$, let $\{v(0), v(1), \ldots, v(N-1)\}$ be the minimizer of problem (11). Then, recalling that $u(k) = v(0)$, we conclude that for problem (11) at time $k+1$, the input $\{v(1), v(2), \ldots, v(N-1), 0\}$ leads to a finite objective that equals $J(x(k)) - \big(x(k)^T Rx(k) + u(k)^T Su(k)\big)$. Thus, if $J(x(k)) < \infty$, then $J(x(k+1)) < \infty$. Also,

$$J(x(k+1)) + x(k)^T Rx(k) + u(k)^T Su(k) \leq J(x(k)),$$

which yields

$$J(x(k)) + \sum_{i=0}^{k-1} \left[x(i)^T Rx(i) + u(i)^T Su(i)\right] \leq J(x(0)) < \infty,$$

for all $k > 0$, which, in turn, implies that $x(k) \to 0$ as $k \to \infty$. The above discussion is summarized in the following theorem.

**Theorem 5** *The closed loop system with the IH-MPC is globally asymptotically stable for a sufficiently large finite $N$ if and only if $(A, B)$ is stabilizable and $\rho(A) \leq 1$.*

Thus, given $x(0)$, we conclude that the IH-MPC scheme is stabilizing for some horizon $N$ if and only if $(A, B)$ is stabilizable and $\rho(A) \leq 1$.

# Part II: An implementable MPC algorithm

## 5 Preliminaries

**Notation and Assumptions** $k \geq 0$ and $N \geq 0$ denote the sampling time and the number of control moves, respectively. $y(k+i|k)$ denotes the output at the sampling time $k+i$ predicted at the sampling time $k$. $u(k+i|k)$ is the input at the sampling time $k+i$ calculated at the sampling time $k$. $r$ is the setpoint which is assumed to be constant. The input is constrained between $u^{min} < 0$ and $u^{max} > 0$. At each sampling time, $N$ control moves are calculated and only the first one is implemented. $u(k+N+i|k) = u(k+N-1|k), i \geq 0$, is assumed.

In Part I, we showed that the IH-MPC scheme globally asymptotically stabilizes a controllable system with poles inside the unit circle *provided* that the number of control moves ($N$) is sufficiently large. However, $N$ depends on the initial condition; thus, it is generally difficult to determine *a priori* and can be arbitrarily large. Furthermore, in practice an unmeasured disturbance could still cause the quadratic program to become infeasible and an even larger number of control moves may have to be chosen. Therefore, the strategy is *not* easily implementable. In this part, we propose an implementable MPC algorithm and show that with this scheme a discrete-time linear system with $n$ poles on the unit disk (with any multiplicity) can be globally stabilized if the number of control moves is larger than $n$. For the specific case of a chain of $n$ integrators, this condition is also necessary. Furthermore, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input. For notational simplicity, all the results in this part of the paper are proven for Single-Input Single-Output (SISO) systems. We discuss the extension of the results to Multi-Input Multi-Output (MIMO) systems.

### 5.1 Systems

The system which we will consider here is linear time-invariant discrete-time with poles on the unit circle and can be represented generally as follows.

$$\left[(1-q^{-1})^{n_0}(1+q^{-1})^{n_1}\prod_{i=2}^{n_a}(1+2a_iq^{-1}+q^{-2})^{n_i}\right]y(k)=\left[\sum_{i=1}^{n_b}b_iq^{-i}\right]u(k) \tag{12}$$

where $n_i, i=0,1,\cdots,n_a$, and $n_b$ are integers, $q^{-1}$ is the backward-shift operator, and $|a_i|<1, i\geq 2$. The term $(1-q^{-1})^{n_0}$ represents $n_0$ integrators, $(1+q^{-1})^{n_1}$ $n_1$ poles at $-1$ and $(1+2a_iq^{-1}+q^{-2})^{n_i}$ $n_i$ pairs of complex conjugate poles at $-a_i\pm\sqrt{a_i^2-1}$. Assume that the left-hand and right-hand polynomials of (12) do not have any common roots. Define

$$\begin{aligned} n &= n_0+n_1+2\sum_{i=2}^{n_a}n_i \\ n_{\max} &= \max_{0\leq i\leq n_a} n_i \\ n_{\text{modes}} &= \min(n_0,1)+\min(n_1,1)+2\sum_{j=2}^{n_a}\min(n_j,1) \end{aligned}$$

Here $n$ is the total number of poles on the unit disk, $n_{\max}$ is the largest multiplicity, $n_{\text{modes}}$ is the total number of poles on the unit disk not counting multiplicity. The unforced response, *i.e.* $u(k)=0,\ \forall\ k\geq 0$, is

$$y(k)=\sum_{i=1}^{n_{\max}}P_i(k)Q_i\quad\forall\ k\geq n_b \tag{13}$$

where $P_i(k)=[1\ \cos(\pi k)\ \sin(\omega_2k)\ \cos(\omega_2k)\ \cdots\ \sin(\omega_{n_a}k)\ \cos(\omega_{n_a}k)]k^{i-1}$, [1] $\omega_j=\arccos(-a_j)\in(0,\pi), j\geq 2$, and $Q_i$ is a constant column vector that depends on the initial condition $y_0=[y(-n+n_b)\ \cdots\ y(n_b-1)]$. Let $Q=\begin{bmatrix}Q_1\\ \vdots\\ Q_{n_{\max}}\end{bmatrix}\in\Re^{n\times 1}$ and $P(k)=[P_1(k)\ \cdots\ P_{n_{\max}}(k)]\in\Re^{1\times n}$, then we have $y(k)=P(k)Q$.

**Example 1** Consider the system

$$(1+q^{-1})(1-q^{-1}+q^{-2})^2y(k)=u(k-1)$$

with initial condition

$$y_0=[y(-4)\ y(-3)\ y(-2)\ y(-1)\ y(0)].$$

Then $n=5$, $n_{\text{modes}}=3$, $n_{\max}=2, \omega_2=\arccos(0.5)=\frac{\pi}{3}, P_1(k)=[\cos(\pi k)\ \sin(\omega_2k)\ \cos(\omega_2k)]$, and $P_2(k)=[\sin(\omega_2k)k\ \cos(\omega_2k)k]$. $Q$ can be calculated using the relationship

$$y_0=[P(-4)^T\ \cdots\ P(0)^T]^TF\equiv D_0F.$$

Notice that $D_k$ is not singular for all $k$. Otherwise, there would be some coefficients that do not depend on the initial condition.

[1] Since $n_i$ is not necessarily equal to $n_{\max}$ for all $0\leq i\leq n_a$, $P_i$ may not contain every term shown here. For example, if $n_0=0$, then $P_i(k)$ does not contain the constant term 1 for all $i\geq 1$.

## 5.2 Objective function

Consider the infinite-horizon objective function:

$$\Phi(k,0)=\sum_{i=1}^{\infty}|r-y(k+i|k)|^2+\Gamma_u\sum_{i=0}^{N-1}|\Delta u(k+i|k)|^2 \qquad 0\le\Gamma_u<\infty \tag{14}$$

where $\Delta u(k)=u(k)-u(k-1)$. This objective function differs from the one in Part I where $u$ is penalized instead of $\Delta u$. In Part I, we were interested in global stabilization to the origin. Here, we want the output to track some setpoint. Penalizing $\Delta u$ instead of $u$ provides the integral control which is necessary for offset-free tracking. Since the system (12) contains poles on the unit disk and the input is constrained, it may not be possible to bring the steady-state to the setpoint with $N$ control moves for some initial conditions. Then the value of the objective function is infinite. This motivates the following *modified* objective function.

$$\Phi(k,\alpha)=\lim_{p\to\infty}\frac{1}{p^{\beta(\alpha)}}\left[\sum_{i=1}^{p}|r-y(k+i|k)|^2+\Gamma_u\sum_{i=0}^{N-1}|\Delta u(k+i|k)|^2\right] \tag{15}$$

where $\beta(\alpha)=\max(2\alpha-1,0)$ and $\alpha$ is the smallest nonnegative integer such that the optimal value of the objective function is finite.

*Remark.* The poles inside the unit disk do not affect $\Phi(k,\alpha)$, $\alpha\ge 1$. This is because $\sum_{i=1}^{\infty}|y_s(k+i|k)|^2$, where $y_s$ denotes the output contribution from the poles inside the unit disk, is finite.

*Remark.* The modified objective function (15) can be extended directly to handle MIMO systems as follows.

$$\Phi(k,\alpha)=\lim_{p\to\infty}\frac{1}{p^{\beta(\alpha)}}\left[\sum_{i=1}^{p}|r-y(k+i|k)|_2^2+\Gamma_u\sum_{i=0}^{N-1}|\Delta u(k+i|k)|_2^2\right] \tag{16}$$

where $|r-y|_2^2=\sum_{i=1}^{n_y}(r_i-y_i)^2$, $|\Delta u|_2^2=\sum_{i=1}^{n_u}\Delta u_i^2$, and $y_i$ and $\Delta u_i$ are the $i^{th}$ output and $i^{th}$ input, respectively. $\beta(\alpha)=\max(2\alpha-1,0)$ and $\alpha$ is the smallest nonnegative integer such that the optimal value of the objective function is finite.

## 5.3 Control design

that $\Phi(k,\alpha)$ is minimized where $\alpha$ is the smallest integer such that the optimal value of $\Phi(k,\alpha)$ is finite. The value of $\alpha$ can be determined as follows: since the optimal output grows at most as $k^{n_{max}}$, $J(k,n_{max}+i)=0,\ \forall\ i\ge 1$. Starting with the initial guess $n_{max}$ for $\alpha$, we reduce the value of $\alpha$ by one until $J(k,\alpha)>0$. Then the optimal control moves are generated by *Implementable MPC Controller.*

**Definition 1 Implementable MPC Controller:** *At sampling time $k$, the control moves $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k),\cdots,u(k+m-1|k)\}$ which is the minimizer of the optimization problem*

$$\begin{aligned}
&J(k,\alpha)=\min_{U_k}\Phi(k,\alpha)\\
&\text{subject to}\quad\left\{\begin{array}{ll}
u^{min}\le u(k+i|k)\le u^{max}, & i=0,\cdots,N-1\\
u(k+N+i|k)=u(k+N-1|k), & i\ge 0\\
\Phi(k,\alpha+i)=0, & i=1,\cdots,n_{max}-\alpha
\end{array}\right.
\end{aligned} \tag{17}$$

*where $U_k=[u(k|k)\ \cdots\ u(k+N-1|k)]^T$.*

Notice that $\Phi(k,\alpha+i)=0, i=1,\cdots,n_{max}-\alpha$, is necessary to ensure that $\alpha$ is the smallest integer for which the optimal value of the objective function is finite. *Remark.* In the absence of disturbances, the value of $\alpha$ does not increase with time. The value of $\alpha$ at time $k$ can be determined by starting with the value at time $k-1$ as the initial guess. However, in practice, because of disturbances and/or model/plant mismatch, the value of $\alpha$ at each sampling time must be determined by starting with the initial guess $n_{max}$. The infinite-horizon minimization problem is converted into a finite-dimensional optimization via the following lemma.

**Lemma 1** *Suppose* $r = 0$. *Assume that at sampling time* $k$, *the coefficients* $Q$ *is calculated by treating* $k + N + n_b - 1$ *as the initial time. Clearly,* $Q$ *depends on* $U_k$. *Then* $J(k,\alpha)$ *is finite if and only if* $Q_i = 0,\ i \geq \alpha+1$. *Moreover, if* $\alpha \neq 0$ *and* $Q_i = 0,\ i \geq \alpha+1$, *then* $J(k,\alpha) = \min\limits_{U_k} Q_\alpha^T W_\alpha Q_\alpha$ *where* $W_\alpha = \frac{1}{2\alpha-1}\mathrm{diag}\{1,1,\frac{1}{2},\cdots,\frac{1}{2}\}$.

*Proof.* If $Q_{\alpha+1} \neq 0$, then the output grows as $\mathcal{O}(k^\alpha)$. $\lim\limits_{p\to\infty} \frac{1}{p^\beta}\sum\limits_{k=1}^{p} |\mathcal{O}(k^\alpha)|^2$ clearly approaches infinity for all $\alpha \geq 0$. If $Q_i = 0, \forall\ i \geq 1$, then $J(k,0)$ is clearly finite. The sufficiency for $\alpha \geq 1$ follows by establishing the second part of the lemma which we do now.

Since the horizon is infinite, the term $P_\alpha(k)Q_\alpha$ in the output which grows as $\mathcal{O}(k^{\alpha-1})$ dominates. The second term in the objective function also vanishes. WLOG, assume that $k$ is chosen such that $u(k) = 0, k \geq 0$.[2] Then by Equation (13), we have

$$\begin{aligned}
\Phi(k,\alpha) &= \lim_{p\to\infty} \frac{1}{p^{2\alpha-1}} \sum_{k=1}^{p} [P_\alpha(k)Q_\alpha]^2 \\
&= \lim_{p\to\infty} \frac{1}{p^{2\alpha-1}} \sum_{k=1}^{p} Q_\alpha^T P_\alpha^T(k) P_\alpha(k) Q_\alpha \\
&= Q_\alpha^T W Q_\alpha
\end{aligned}$$

where

$$\begin{aligned}
W &= \lim_{p\to\infty} \frac{1}{p^{2\alpha-1}} \sum_{k=1}^{p} P_\alpha^T(k) P_\alpha(k) \\
&= \lim_{p\to\infty} \frac{1}{p^{2\alpha-1}} \sum_{k=1}^{p} \begin{bmatrix} 1 \\ \cos(\pi k) \\ \sin(\omega_2 k) \\ \cos(\omega_2 k) \\ \vdots \\ \cos(\omega_{n_a} k) \end{bmatrix} [1\ \ \cos(\pi k)\ \ \sin(\omega_2 k)\ \ \cos(\omega_2 k)\ \ \cdots\ \ \cos(\omega_{n_a} k)]\, k^{2\alpha-2} \\
&= \lim_{p\to\infty} \frac{1}{p^{2\alpha-1}} \sum_{k=1}^{p} \begin{bmatrix} 1 & \cos(\pi k) & \cdots & \cos(\omega_{n_a} k) \\ \cos(\pi k) & \cos(\pi k)^2 & \cdots & \cos(\pi k)\cos(\omega_{n_a} k) \\ & & \vdots & \\ \cos(\omega_{n_a} k) & \cos(\pi k)\cos(\omega_{n_a} k) & \cdots & \cos(\omega_{n_a} k)^2 \end{bmatrix} k^{2\alpha-2} \\
&= \frac{1}{2\alpha-1}\mathrm{diag}\{1,1,\frac{1}{2},\cdots,\frac{1}{2}\}
\end{aligned}$$

The last equality follows from the following integrals.

$$\begin{aligned}
\int_1^p k^{2\alpha-2}\sin(\omega_1 k)\cos(\omega_2 k)dk &\sim O(p^{2\alpha-2}) \quad \text{for large } p \\
\int_1^p k^{2\alpha-2}\sin(\omega_1 k)\sin(\omega_2 k)dk &\sim \begin{cases} O(p^{2\alpha-1}) & \text{if } \omega_1 = \omega_2 \\ O(p^{2\alpha-2}) & \text{if } \omega_1 \neq \omega_2 \end{cases} \quad \text{for large } p
\end{aligned}$$

[2] In the presence of the disturbance $w$ entering at the plant input, $u(k) + w = 0$.

$$\int_1^p k^{2\alpha-2}\cos(\omega_1 k)\cos(\omega_2 k)dk \quad \sim \quad \begin{cases} O(p^{2\alpha-1}) & \text{if } \omega_1 = \omega_2 \\ \mathcal{O}(p^{2\alpha-2}) & \text{if } \omega_1 \neq \omega_2 \end{cases} \qquad \text{for large } p$$

$\square$

*Remark.* If $r \neq 0$ is such that the steady-state input is strictly within the constraints, the lemma still holds. By change of variables, the desired output becomes the origin and $Q_\alpha$ must be determined using the values for the new variables.

*Remark.* $W_\alpha$ may not contain every term shown. For example, for the system considered in Example 1, $W_1 = \text{diag}\{1, \frac{1}{2}, \frac{1}{2}\}$ and $W_2 = \frac{1}{3}\text{diag}\{\frac{1}{2}, \frac{1}{2}\}$. If we used the $L_2$-norm ($\int_0^p |y(k+t|k)|^2 dt$) instead of the $l_2$-norm ($\sum_{i=1}^{p} |y(k+i|k)|^2$), then $W_\alpha = \frac{1}{2\alpha-1}\text{diag}\{1, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\}$.

*Remark.* One difficulty may arise in extending this lemma to MIMO systems. The order of growth for each output may be different. For example, one output may grow as $O(k^2)$ while another one may grow as $O(k^4)$. Therefore, different values of $\alpha$ may have to be used for each output. A general approach is under development.

For $\alpha \geq 1$, the solution to this optimization problem may *not* be unique. If this is the case, we assume that the unique solution is such that

$$\left\| \begin{bmatrix} u(k) \\ U_{k+1} \end{bmatrix} - \begin{bmatrix} U_k \\ u(k+N|k) \end{bmatrix} \right\|_2^2$$

is minimized over all feasible control moves for which the objective function has the optimal value.

# 6 Stability of the implementable MPC algorithm

The following theorem establishes a necessary condition and a sufficient condition on $N$ such that the closed-loop system is globally asymptotically stable. The proof of this theorem is lengthy and can be found in the Appendix.

**Theorem 6** *Suppose that a disturbance $w$ enters at the plant input and that the disturbance has the following properties:*

1. *$w(k) \to \bar{w}$ as $k \to \infty$ and $-\bar{w}$ is strictly within the input limits,* i.e. $u^{min} - u_r^{\text{ss}} < -\bar{w} < u^{\max} - u_r^{\text{ss}}$ *where $u_r^{\text{ss}}$ is the steady-state input resulting from the setpoint change $r$.*
2. *For any $\epsilon > 0$, there exists a finite $T$ such that $|w(k+1) - \bar{w}| < \epsilon \ \forall \ k \geq T$.*

*The future disturbance is estimated by assuming that it is a step. Then the closed-loop system with* Implementable MPC Controller *is globally asymptotically stable,* i.e. *$y(k) \to r$ as $k \to \infty$, if $N \geq n+1$ and only if $N \geq n - n_{\text{modes}} + 2$ where $n$ is the total number of poles (with any multiplicity) on the unit disk.*

*Proof.* See Appendix. $\square$

For pure integrator systems, $n_{\text{modes}} = 1$ and the following corollary is immediate.

**Corollary 4** *Under the conditions of Theorem 6, the closed-loop system with* Implementable MPC Controller *is globally asymptotically stable if and only if $N \geq n+1$ for pure integrator systems.*

In the absence of the disturbance, we have the following corollary.

**Corollary 5** *In the absence of the disturbance,* $J(k, \alpha) = 0 \; \forall \; \alpha \geq 1$ *for sufficiently large finite* $N$.

This corollary implies that for a sufficiently large number of control moves, the original objective function (14) is finite. Thus this result parallels those in Section 4 of this paper and those in the paper by Tsirukis and Morari (1992).

Up to now we have assumed that the state is measured. When the state is estimated, we can treat the estimation error as a disturbance. If the system is observable and the state is estimated with an asymptotic observer, then the estimation error approaches zero asymptotically. Thus, Theorem 6 holds as well.

**Theorem 7** *Assume that the system is observable and that the state is estimated with an asymptotic observer. Under the conditions of Theorem 6, the overall system with* Implementable MPC controller *and the asymptotic observer is globally asymptotically stable.*

*Remark.* Theorem 7 holds as well if the system is merely detectable.

# 7 Examples

We have shown, that with $N$ properly chosen, MPC globally asymptotically stabilizes *any* constrained stabilizable system with poles on the closed unit disk. Example 2 compares the closed loop responses for the IH-MPC scheme presented in Part I with other design methods. Example 3 illustrates how to choose $N$ for *Implementable MPC Controller* to reach the best compromise between performance and computational complexity.

**Example 2 (Tsirukis and Morari (1992))** Consider the following system from Sontag and Yang (1991)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 + u \end{aligned}$$

where $u$ must satisfy the constraint $|u| \leq 1$. The system has four poles on the imaginary axis $(-j, -j, j, j)$.

The system was discretized with a sampling time of 0.1 to apply the MPC algorithm. The initial condition is $x_0 = [1\ 0.5\ 0.5\ 1]^T$. The weights are $R = I$ and $S = 10$. The input horizon is $N = 50$. Figure 2 depicts the time-evolution of state $x_1$ for the controller from Sontag and Yang (1991) and the MPC controller. The behavior of the other three states is similar. The corresponding control actions are shown in Figure 3. Although both controllers stabilize the system, the difference in performance is striking. In all fairness, we should point out that the controller was designed by Sontag and Yang (1991) to ensure stability and that they made no attempt to achieve good performance.

**Example 3 (Sussmann, Sontag and Yang (1992))** Consider the following triple-integrator system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{18}$$

As shown by Teel (1992), no linear controller can globally stabilize this system. We discretize the system with a sampling time of 0.1. The initial condition is $x(0) = [3\ \text{-}1\ 3]^T$ and the control input is constrained between the saturation limits $\pm 1$. The "sufficiently large number of control moves" for this initial condition

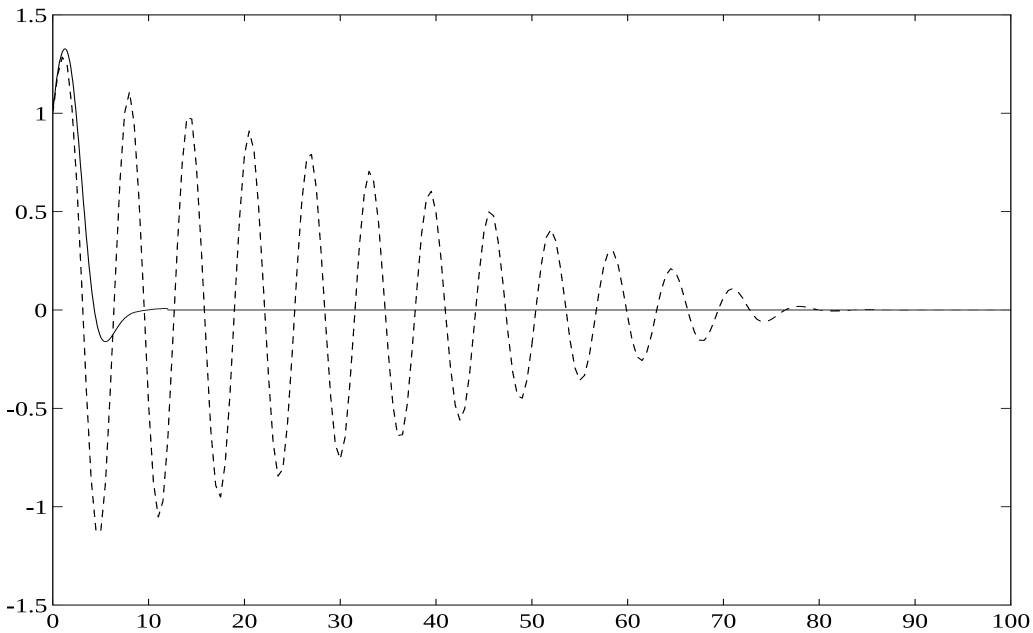


Figure 2: Time-evolution of $x_1$ for Example 1 (solid – MPC; dash – from Sontag and Yang 1991)

is approximately 150. Figure 4 shows the responses for $N = 4, 10, 20, 40$ and 60 along with the response for the nonlinear controller designed in Sussmann et al. (1992). The input weight is $\Gamma_u = 0$. As we can see, the performance improves as the number of control moves ($N$) increases. However, the amount of computation increases *dramatically*. Thus a trade-off between performance and computation arises. Although Theorem 6 states that $N = 4$ is sufficient to globally stabilize this system, $N$ should be chosen to reach the best compromise between performance and computation.

# 8 Conclusions

Based on the growth rate of the set of states reachable with unit-energy inputs, we have shown that a discrete-time controllable linear system is globally controllable to the origin with bounded inputs if and only if all its poles are in the closed unit disk. Using these results, we show that systems with poles in the closed unit disk can be globally stabilized using IH-MPC for a sufficiently large number of control moves.

However, since it is difficult to determine *a priori* what the "sufficiently large number of control moves" is, the IH-MPC scheme is not easily implementable. To overcome this problem, we proposed an implementable MPC algorithm. We showed that with this scheme global asymptotic stability can be guaranteed for systems with $n$ poles on the unit disk if the number of control moves is larger than $n$. For pure integrator systems, this condition is also necessary. Furthermore, global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant.

# 9 Acknowledgments

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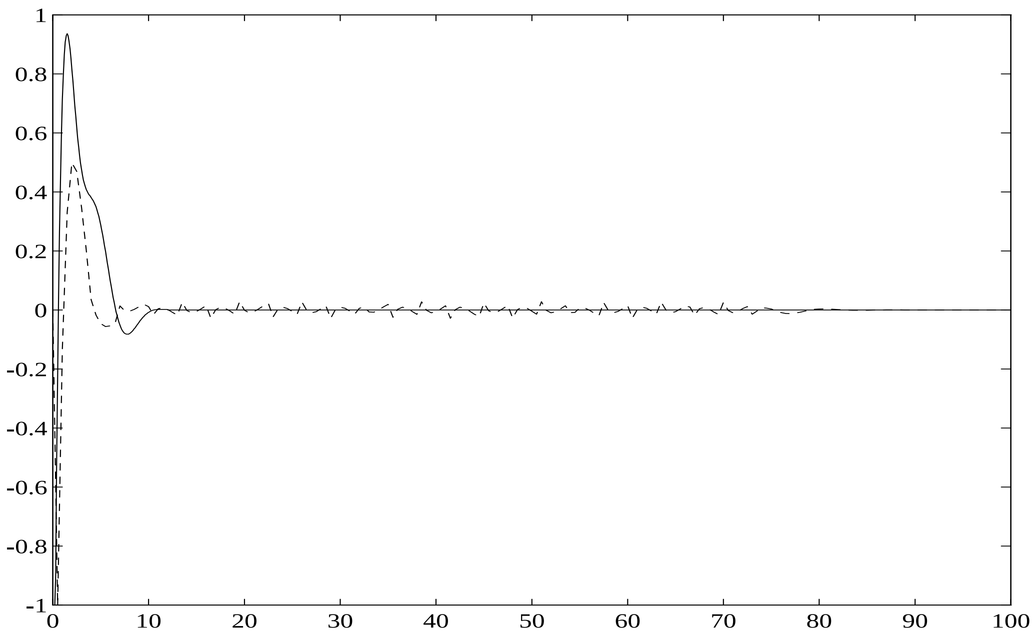


Figure 3: Time-evolution of control action for Example 1 (solid – MPC; dash – from Sontag and Yang 1991)

# 10 Appendix – Proof of Theorem 6

Before we prove Theorem 6, let us first establish some preliminary results.

**Claim 1** *Let $V \in \Re^{m \times m}$ be a unitary matrix. $z_2^{opt} = \arg\min_{z_2} z_2^T z_2$ subject to*

$$0 \geq x^{min} \leq V \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leq x^{\max} \geq 0 \tag{19}$$

*where $z_2 \in \Re^{m_2}$, $m > m_2$, $x^{min} \in \Re^m$ and $x^{\max} \in \Re^m$. There exists a positive constant $\lambda$ such that*

$$z_1^T z_1 \geq \lambda (z_2^{opt})^T z_2^{opt} \quad \text{for all feasible } z_1{}^3$$

*Proof.* If $z_2^{opt} = 0$, the claim clearly holds. Assume that $z_2^{opt} \neq 0$. Then the optimal solution must occur on the boundary. The feasible region formed by the constraints (19) has $m2^{m-1}$ edges (or lines). Each edge is represented by

$$V_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = x$$

where $V_1$ consists of $m-1$ rows of $V$ and $x$ consists corresponding rows from either $x^{min}$ or $x^{\max}$. After eliminating $m-2$ variables (only one variable in $z_1$ and one variable in $z_2$ remain), we obtain

$$\mu_i z_1(i) + \nu_j z_2(j) = c_{ij} \quad i = 1, \cdots, m - m_2 \quad \text{and} \quad j = 1, \cdots, m_2$$

If $\mu_i = 0$, then any change in $z_1(i)$ does not affect $z_2(j)$ and $z_2(j)^{opt} = 0$ since it is feasible. Let $\lambda$ be the smallest value of $\min_{i,j,\mu_i \neq 0} \frac{|\mu_i|}{|\nu_j|}$ over all edges. We have $\lambda (z_2^{opt})^T z_2^{opt} \leq z_1^T z_1$ for all edges where the optimal

[3] $z_2^{opt}$ clearly depends on $z_1$.

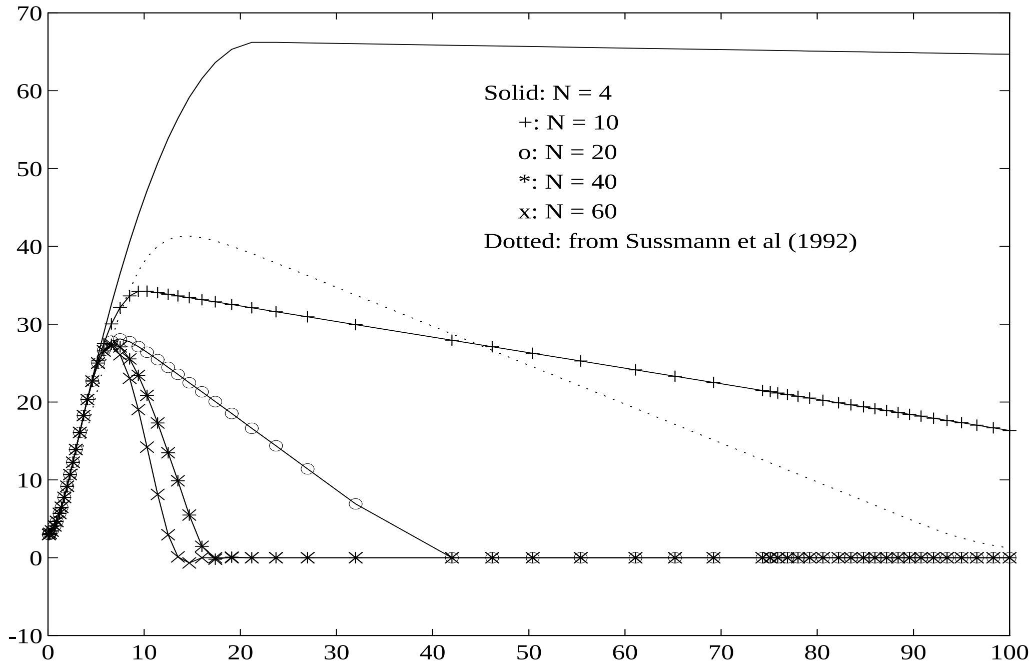


Figure 4: Output Responses for Example 2

solution lies. If the optimal solution does not occur on any edge, then some of the constraints are not satisfied as equalities and the value of $(z_2^{opt})^T z_2^{opt}$ must be smaller. Thus, we have

$$z_1^T z_1 \geq \lambda (z_2^{opt})^T z_2^{opt} \quad \text{for all feasible } z_1$$

where $\lambda$ is a positive constant. □

**Claim 2** *Let $X$ be a closed convex set. Suppose the point $x_0$ lies outside $X$. Then there is a plane that strictly separates $X$ from $x_0$.*

*Proof.* See, for example, Luenberger (1968). □

**Claim 3** *Let $J = \min_{x \in X} (x_0 - x)^T W (x_0 - x)$ where $W > 0$, $X$ is a closed convex set and $0 \in X$. Suppose that $x^{opt}$ is the optimal solution. Then $J \leq x_0^T W x_0 - (x^{opt})^T W x^{opt}$.*

*Proof.* WLOG, assume that $W$ is the identity matrix, *i.e.* $J = (x_0 - x^{opt})^T (x_0 - x^{opt})$. If $x_0 \in X$, *i.e.* $x^{opt} = x_0$, then $J = 0$ and the claim clearly holds. Suppose $x_0$ lies outside $X$. By Claim 2, there is a plane that strictly separates $X$ from $x_0$. Let $P$ be the separating plane that is orthogonal to the line passing through the points $x_0$ and $x^{opt}$ and contains the point $x^{opt}$. Since the origin belongs to the set $X$, there exists another plane $P'$ which contains the origin and is parallel to $P$. Let the intersection of the plane $P'$ and the line passing through the points $x_0$ and $x^{opt}$ be $y$. Since $x_0, x^{opt}$ and $y$ form one line and $x^{opt}$ is between $x_0$ and $y$, $(x_0 - x^{opt})^T (x^{opt} - y) \geq 0$. Since both the origin and $y$ belong to $P'$ and the line passing through the points $x_0, x^{opt}$ and $y$ is perpendicular to $P'$, $(x_0 - y)^T (y - 0) = 0$ and $(x^{opt} - y)^T (y - 0) = 0$, *i.e.* $x_0^T y = y^T y$ and $(x^{opt})^T y = y^T y$. We have

$$\begin{aligned} (x_0 - y)^T (x_0 - y) + y^T y = x_0^T x_0 + 2y^T y - 2x_0^T y &= x_0^T x_0 \\ (x^{opt} - y)^T (x^{opt} - y) + y^T y &= (x^{opt})^T x^{opt} \end{aligned}$$

Thus,

$$\begin{aligned} x_0^T x_0 - (x^{opt})^T x^{opt} &= (x_0 - y)^T(x_0 - y) - (x^{opt} - y)^T(x^{opt} - y) \\ &= (x_0 - x^{opt} + x^{opt} - y)^T(x_0 - x^{opt} + x^{opt} - y) - (x^{opt} - y)^T(x^{opt} - y) \\ &= (x_0 - x^{opt})^T(x_0 - x^{opt}) + 2(x_0 - x^{opt})^T(x^{opt} - y) \\ &\geq (x_0 - x^{opt})^T(x_0 - x^{opt}) \\ &= J \end{aligned}$$

$\Rightarrow J \leq x_0^T x_0 - (x^{opt})^T x^{opt}$. □

The following claim is a generalization of the previous claim.

**Claim 4** *Let* $J = \min_{x \in X}(a_0 + Ex)^T W(a_0 + Ex)$ *where* $X = \{x : x \in \Re^m, Gx = 0, 0 \leq x^{min} \leq x \leq x^{\max} \geq 0\}$, $W \in \Re^{n \times n} > 0$, *and* $m \geq n$. $\begin{bmatrix} E \\ G \end{bmatrix}$ *has full row rank. If the solution is not unique, the optimal solution* $(x^{opt})$ *is determined as* $\arg\min x^T x$ *over all feasible solutions for which* $J$ *has the optimal value. Then there exists a positive constant* $\gamma$ *such that* $J \leq x_0^T W x_0 - \gamma (x^{opt})^T x^{opt}$.

*Proof.* Let $\begin{bmatrix} E \\ G \end{bmatrix} = \begin{bmatrix} U_E \\ U_G \end{bmatrix} \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^T$ where $\begin{bmatrix} U_E \\ U_G \end{bmatrix}$ and $V^T$ are unitary matrices and $\Sigma$ contains all the singular values. Since $\begin{bmatrix} E \\ G \end{bmatrix}$ has full row rank, $\Sigma > 0$. Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V^T x$. The optimization problem becomes

$$J = \min_{z_1}(a_0 + U_E \Sigma z_1)^T W(a_0 + U_E \Sigma z_1) \tag{20}$$

subject to

$$\begin{cases} U_G \Sigma z_1 = 0 \\ 0 \geq x^{min} \leq V \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leq x^{\max} \geq 0 \end{cases}$$

For any given $z_1^{opt}$, $z_2^{opt} = \arg\min_{z_2} z_2^T z_2$ subject to $0 \geq x^{min} \leq V \begin{bmatrix} z_1^{opt} \\ z_2 \end{bmatrix} \leq x^{\max} \geq 0$ and $z_1^{opt}$ is such that the constraints are feasible. By Claim 1, there exists a positive constant $\lambda$ such that $(z_1^{opt})^T z_1^{opt} \geq \lambda (z_2^{opt})^T z_2^{opt}$. This together with the fact $(x^{opt})^T x^{opt} = (z_1^{opt})^T z_1 + (z_2^{opt})^T z_2$ (since $V$ is unitary) gives

$$|z_1^{opt}|_2^2 \geq \frac{1}{1 + \frac{1}{\lambda}} |x^{opt}|_2^2$$

Thus, $|Ex^{opt}|_2^2 = |U_E \Sigma z_1^{opt}|_2^2 = \left| \begin{bmatrix} U_E \\ U_G \end{bmatrix} \Sigma z_1^{opt} \right|_2^2 = |\Sigma z_1^{opt}|_2^2 \geq \bar{\lambda} |x^{opt}|_2^2$ where $\bar{\lambda} = \underline{\sigma}(\Sigma) \frac{1}{1 + \frac{1}{\lambda}}$ and $\underline{\sigma}(\Sigma) > 0$ is the smallest singular value of $\Sigma$.

$$\begin{aligned} J &= \min_{x \in X}(a_0 + Ex)^T W(a_0 + Ex) \\ &= \min_{x \in X}(-a_0 - Ex)^T W(-a_0 - Ex) \\ &\leq (-a_0)^T W(-a_0) - (Ex^{opt})^T W(Ex^{opt}) \quad \text{(by Claim 3)} \\ &= a_0^T W a_0 - (Ex^{opt})^T W(Ex^{opt}) \\ &\leq a_0^T W a_0 - \underline{\sigma}(W)(Ex^{opt})^T(Ex^{opt}) \\ &\leq a_0^T W a_0 - \gamma (x^{opt})^T x^{opt} \end{aligned}$$

where $\gamma = \underline{\sigma}(W)\bar{\lambda}$ and $\underline{\sigma}(W) > 0$ is the smallest singular value of $W$. □

*Remark.* As one can see, the optimal solution of $J = \min_{x \in X}(a_0 + Ex)^T W(a_0 + Ex)$ may not be unique. If we do not determine the unique optimal solution as $\arg\min x^T x$ over all feasible solutions for which $J$ has the optimal value, then this claim does not hold in general. Now we are ready to prove Theorem 6.

*Proof.* WLOG, assume that $u^{min} + \delta \leq -w(k) \leq u^{\max} - \delta \;\forall\; k \geq 0$, where $\delta > 0$ is constant, and $|w(k+1) - w(k)| \leq \epsilon \;\forall\; k \geq 0$.[4] The future disturbance is estimated by assuming that it is step-like, *i.e.* $\hat{w}(k+i|k) = w(k-1) \;\forall\; i \geq 0$ where $\hat{w}$ denotes the estimate of $w$. Thus $u(k+N-1|k) + \hat{w}(k+N-1|k) = 0$ is always feasible, *i.e.* $\Phi(k, n_{\max}) = 0 \;\forall\; k \geq 0$ is always feasible. Only $N-1$ control moves are used to minimize the objective function. Let $Q(j|i)$ be the coefficients calculated at time $j$ with reference time at $i$, *i.e.* $i$ is treated as the initial time (0). We have

$$\begin{bmatrix} P(-n+1) \\ P(-n+2) \\ \vdots \\ P(0) \end{bmatrix} Q(k|k+N+n_b-1) = \begin{bmatrix} y(k+N-n+n_b-1|k) \\ \vdots \\ y(k+N+n_b-2|k) \\ y(k+N+n_b-1|k) \end{bmatrix} =$$

$$C \begin{bmatrix} y(k) \\ \vdots \\ y(k-n+1) \end{bmatrix} + D \begin{bmatrix} u(k+N-n+2-n_b) - w(k+N-n+2-n_b) \\ \vdots \\ u(k|k) - w(k-1) \\ u(k+1|k) - w(k-1) \\ \vdots \\ u(k+N-2|k) - w(k-1) \\ u(k+N-1|k) - w(k-1) = 0 \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} P(-n+1) \\ P(-n+2) \\ \vdots \\ P(0) \end{bmatrix} Q(k+1|k+N+n_b-1) = \begin{bmatrix} y(k+N-n+n_b-1|k+1) \\ \vdots \\ y(k+N+n_b-2|k+1) \\ y(k+N+n_b-1|k+1) \end{bmatrix} =$$

$$C \begin{bmatrix} y(k) \\ \vdots \\ y(k-n+1) \end{bmatrix} + D \begin{bmatrix} u(k+N-n+2-n_b) - w(k+N-n+2-n_b) \\ \vdots \\ u(k) - w(k) \\ u(k+1|k+1) - w(k) \\ \vdots \\ u(k+N-1|k+1) - w(k) \end{bmatrix} \tag{22}$$

Subtraction of the above two equations and a few lines of algebra give

$$Q(k+1|k+N+1) = Q(k|k+N+1) + F\Delta v_{k+1} + G(w(k) - w(k-1)) \tag{23}$$

where $\Delta v_{k+1} = [u(k+1|k+1) \;\cdots\; u(k+N-1|k+1)]^T - [u(k+1|k) \;\cdots\; u(k+N-1|k)]^T$. Or equivalently,

$$Q_\alpha(k+1|k+N+1) = Q_\alpha(k|k+N+1) + F_\alpha \Delta v_{k+1} + G_\alpha(w(k) - w(k-1)), \quad \alpha = 1, \cdots, n_{\max} \tag{24}$$

where $F = \begin{bmatrix} F_1 \\ \vdots \\ F_{n_{\max}} \end{bmatrix}$ and $G$ is defined similarly. *Remark.* Notice that $Q(k+1|k+N+2)$ may not be necessarily equal to $Q(k+1|k+N+1)$. However, by Corollary 1, $Q_i(k+1|k+N+2) = 0 \;\forall\; i \geq \alpha$ and $Q_\alpha(k+1|k+N+2)^T W_\alpha Q_\alpha(k+1|k+N+2) = Q_\alpha(k+1|k+N+1)^T W_\alpha Q_\alpha(k+1|k+N+1)$ if and only if $Q_i(k+1|k+N+1) = 0 \;\forall\; i \geq \alpha$,

[4] By assumptions on the disturbance, this is always possible by appropriately defining the initial time.

The optimization problem, with slight abuse of notations, becomes the following:

$$J = \min_{\Delta v_{k+1}} \left[Q_\alpha + F_\alpha \Delta v_{k+1}\right]^T W_\alpha \left[Q_\alpha + F_\alpha \Delta v_{k+1}\right] \tag{25}$$

subject to

$$\left\{ \begin{array}{l} F_{\alpha+i}\Delta v_{k+1} = G_\alpha(w(k) - w(k-1)) = O(\epsilon) \ \ \forall\ i = 1,\cdots,n_{\max} - \alpha \\ u(k+N|k+1) - u(k+N-1|k) = -w(k) + w(k-1) \\ u^{min} \le u(k+i|k+1) \le u^{\max}\ \forall\ i = 1,\cdots,m \end{array} \right. \tag{26}$$

The following claim is obvious.

**Claim 5** *The matrix consisting of the last $n$ columns of $F$ is nonsingular if $N \ge n+1$.*

*Proof.* Since the system is controllable, we can transfer any initial state to an arbitrary state with at most $n$ control moves if the controls are unconstrained. Since the last control move is such that $u(k+N-1|k) + w(k-1) = 0$, we can take the coefficients from any initial condition to any arbitrary values with $n+1$ control moves. Therefore, the matrix consisting of the last $n$ columns of $F$ must be nonsingular if $N \ge n+1$. $\square$

The proof is completed with the following two claims.

**Claim 6** *If $w(k) - w(k-1) = 0\ \forall\ k \ge 1$, then*

$$J(k+n+1,\alpha) \le \max(J(k,\alpha) - \eta(\alpha), 0)\ \forall\ \alpha \ge 1$$

*where $\eta(\alpha)$ is a positive constant that depends on $\alpha$ if $N \ge n+1$ and only if $N \ge n - n_{\text{modes}} + 2$.*

*Proof.* ($\Rightarrow$) $N \ge n+1$.

**Case 1:** Suppose $|\Delta v_{k+i}|_\infty \le \beta,\ \forall\ i = 1,\cdots,n$ and let $\beta = \frac{\min(|u^{min} - w(k)|, |u^{\max} - w(k)|)}{n+1} \ge \frac{\delta}{n+1} > 0$. We have $|u(k+N+i|k+n) - u(k+N+i|k)| \le \beta n\ \forall\ i = -1,\cdots,n-1$. Since $u(k+N+i|k) = u(k+N-1|k) = -w(k-1)\ \forall\ i \ge 0$, $|u(k+N+i|k+n) + w(k-1)| \le \beta n = \frac{n}{n+1}\delta\ \forall\ i \ge -1$. This together with the fact $u^{min} + \delta \le -w(k-1) \le u^{\max} - \delta$ gives $\min(u(k+N+i|k+n) - u^{min}, u^{\max} - u(k+N+i|k+n)) \ge \beta\ \forall\ i = -1,\cdots,n-1$. Thus at the sampling time $k+n+1$, the last $n+1$ elements of $\Delta v_{k+n+1}$, denoted by $\Delta v'$, can be varied within $\pm\beta$, *i.e.* $-\beta \ge v^{min} \le \Delta v' \le v^{\max} \ge \beta$. Assume that the first $N-n-1$ elements of $\Delta v_{k+n+1}$ are zeros. Then we have

$$J(k+n+1,\alpha) \le \min_{\Delta v'} \left[Q_\alpha + H_1 \Delta v'\right]^T W_\alpha \left[Q_\alpha + H_1 \Delta v'\right]$$

subject to

$$\left\{ \begin{array}{c} H_2 \Delta v' = 0 \\ -\beta \ge v^{min} \le \Delta v' \le v^{\max} \ge \beta \end{array} \right.$$

where $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ is the last $n$ columns of $\begin{bmatrix} F_{\alpha+1} \\ \vdots \\ F_{n_{\max}} \end{bmatrix}$. Notice that the inequality follows from the assumption that the first $N-n-1$ elements of $\Delta v_{k+n+1}$ are zeros. By Claim 5, $H$ must have full row rank. Then there exists a positive constant (it can be taken, for example, as the largest radius of balls centered at the origin within the set) $\eta(\alpha)$ such that $J(k+n+1,\alpha) \le \max(J(k,\alpha) - \eta(\alpha), 0)$.

**Case 2:** If $|\Delta v_{k+i}|_\infty \ge \beta$ for some $i \in \{1,\cdots,n\}$, then by Claim 4, $J(k+n,\alpha) \le J(k,\alpha) - \gamma\beta^2$. This completes the proof for the *if* part.

($\Leftarrow$) If $N \le n - n_{\text{modes}} + 2$, then for $\alpha = 1$, $\begin{bmatrix} F_2 \\ \vdots \\ F_{n_{\max}} \end{bmatrix}$ has more columns than rows and the only solution, if feasible, is $\Delta v_{k+1} = 0$ for some initial conditions. Thus no degree of freedom is left to minimize $J(k,1)$. For some initial conditions, $J(k,1)$ cannot be reduced to zero. $\square$

**Claim 7** *For sufficiently large $k$, there exists an integer $o$, $2(n+1) \geq o \geq n+1$ such that*

$$J(k+o,\alpha) \leq \max(J(k,\alpha) - \eta'(\alpha), 0) \ \forall \ \alpha \geq 1$$

*where $\eta'(\alpha) > 0$ if $N \geq n+1$.*

*Proof.* Because of the disturbance, the constraints (26) may not be feasible at the sampling time $k+1$ even though they are feasible at the sampling time $k$. We want to show, however, that for sufficiently large $k$, or equivalently for sufficiently small $\epsilon$, there exists an integer $1 \leq l \leq n+1$ such that the constraints are feasible at the sampling time $k+l$. Suppose that the constraints are not feasible for all $l \leq n$; otherwise, we are done. By Claim 4, $\Delta v_{k+i} \sim O(\epsilon) \ \forall \ i = 1, \cdots, n$. Since there exists a positive constant $\delta$ such that $u^{min} + \delta \leq -w(k+i) \leq u^{\max} - \delta \ \forall \ i \geq 0$, for sufficiently small $\epsilon$, following the similar arguments as in the proof of Claim 6, the last $n+1$ elements of $\Delta v_{k+n+1}$, denoted by $\Delta v'$, are allowed to vary within $\pm\beta$ where $\beta > 0$ is as defined in the proof of Claim 6, *i.e.* $-\beta \geq x^{min} \leq \Delta v' \leq x^{\max} \geq \beta$. Thus $\begin{bmatrix} F_{\alpha+1} \\ \cdots \\ F_{n_{\max}} \end{bmatrix}$ subject to the constraints $-\beta \geq x^{min} \leq \Delta v' \leq x^{\max} \geq \beta$ covers a ball centered at the origin with radius of $\rho$. For sufficiently small $\epsilon$, $\begin{bmatrix} F_{\alpha+1} \\ \cdots \\ F_{n_{\max}} \end{bmatrix} = O(\epsilon)$ must be feasible. Therefore, for sufficiently small $\epsilon$, there exists an integer $1 \leq l \leq n+1$ such that the constraints are feasible at the sampling time $k+l$.

Suppose that at the sampling time $o$, where $2(n+1) \geq o \geq n+1$, the constraints are feasible. By Claim 4, the control moves in making the constraints feasible are $\mathcal{O}(\epsilon)$. Therefore, the effect of the control moves on $J(k+o,\alpha)$ is $O(\epsilon)$. This combined with the previous claim gives

$$J(k+o,\alpha) \leq \max(J(k,\alpha) - \eta(\alpha) + \mathcal{O}(\epsilon), 0)$$

Thus for sufficiently small $\epsilon$, we have

$$J(k+o,\alpha) \leq \max(J(k,\alpha) - \eta'(\alpha), 0)$$

where $\eta'(\alpha) = \eta(\alpha) - O(\epsilon) > 0$. □

Thus, for sufficiently large $k$, $|r - y(k+N+i|k)| \sim O(\epsilon)$ and the output approaches $r$ asymptotically. This completes the proof of Theorem 6. □

# References


Callier, F. M. and Desoer, C. A. (1991). *Linear System Theory*, Springer Texts in Electrical Engineering, Springer-Verlag.

Lin, Z. and Saberi, A. (1993). Semi-global exponential stabilization of linear systems subject to 'input saturation' via linear feedbacks, *Systems and Control Letters* **21**(3): 225–239.

Luenberger, D. G. (1968). *Optimization by Vector Space Methods*, John Wiley & Sons, Inc., NY, NY.

Mayne, D. and Michalska, H. (1990). An implementable receding horizon controller for the stabilization of nonlinear systems, *Conf. on Decision and Control*, pp. 3396–3397.

Rawlings, J. and Muske, K. R. (1993). The stability of constrained receding horizon control, *IEEE Transactions on Automatic Control* **38**: 1512–1516.

Sontag, E. (1984). An algebraic approach to bounded controllability of linear systems, *International Journal of Control* **39**: 181–188.

Sontag, E. and Sussmann, H. J. (1990). Nonlinear output feedback design for linear systems with saturating controls, *Conf. on Decision and Control*, pp. 3414–3416.

Sontag, E. and Yang, Y. (1991). Global stabilization of linear systems with bounded controls, *Technical Report Report SYCON-91-09*, Rutgers University.

Sussmann, H. J., Sontag, E. D. and Yang, Y. (1992). A general result on the stabilization of linear systems using bounded controls, *Technical Report Report SYCON-91-XX*, Rutgers University.

Teel, A. (1992). Global stabilization and restricted tracking for multiple integrators with bounded controls, *Systems & Control Letters* **18**: 165–171.

Tsirukis, A. and Morari, M. (1992). Controller design with actuator constraints, *Conf. on Decision and Control*, pp. 2623–2628.

Yang, T. and Polak, E. (1993). Moving horizon control of linear systems with input saturation and plant uncertainty, parts 1 and 2, *International Journal of Control* **58**(3): 613–663.