Stability and Performance Analysis of Systems Under Constraints

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Abstract

All real world control systems must deal with actuator and state constraints. Standard conic sector bounded nonlinearity stability theory provides methods for analyzing the stability and performance of systems under constraints, but it is well-known that these conditions can be very conservative. A method is developed to reduce conservatism in the analysis of constraints by representing them as nonlinear *real* parametric uncertainty.

1 Introduction

All real world control systems must deal with constraints. The control system must avoid unsafe operating regimes. In process control these constraints typically appear in the form of pressure or temperature limits. Further constraints are imposed by physical limitations—valves can only operate between fully open and fully closed, pumps and compressors have finite throughput capacity, surge tanks can only hold a certain volume.

One approach to controlling systems with constraints is to optimize the control objective on-line subject to the constraints. This approach is referred to as *model predictive control* (MPC). A quadratic program must be solved at each sampling instance, and off-the-shelf software is available for performing these calculations [10]. Model predictive control does not completely solve the constrained control problem, however. MPC is computationally too complex for many industrial processes, which is part explains why MPC is typically implemented in a supervisory mode, i.e., *on top* of the regulatory control systems. Two additional disadvantages are that some operational requirements are impossible to express through a single objective function, and the stability and performance analysis with the resulting nonlinear controller is difficult.

The traditional method for dealing with constraints was to use simple static nonlinear elements (selectors and overrides) in the control system. Despite their considerable practical importance and extensive use, there is essentially no general theory to guide the design and analysis of these selector and override schemes. Furthermore, because they modify the control system configuration dynamically, they often cause severe performance degradation such as windup and "bumps" when switching modes. Though *ad hoc* design methods have been developed for avoiding windup,

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Figure 1: Anti-windup compensation.

it has been shown that all of these techniques perform poorly (or may even lead to instability) in some situations.

A general method is needed for the design of robust constrained controllers which avoids the difficulties of model predictive control. This method should give robust controllers, be computationally simple on-line, and handle multiple performance objectives in a transparent manner. A general framework for the design of such controllers is provided by the *Anti-Windup Bumpless-Transfer* approach [4], and is illustrated by Fig. 1 for the case of actuator limitations. An additional linear compensator (R), called the anti-windup compensator, provides graceful performance degradation by modifying the error into the linear controller (K) when the constraints become active. When the constraints are inactive, the controller output equals the plant input and the anti-windup compensator does not affect the behavior of the closed loop system. This approach can be shown to be a generalization of the earlier *ad hoc* constraint-handling methods.

Note that the closed loop system involves linear systems with static memoryless nonlinearities. A necessary step in the further development of any anti-windup approach is to develop tools for analyzing stability and robustness for such systems. Campo [4] give sufficient conditions for analyzing stability and performance based on the standard conic sector bounded nonlinearity stability theory, but it is well-known that these conditions can be very conservative. This purpose of this technical report is to reduce the conservatism in these tools.

2 Background

Below we review the structured singular value which was developed by Doyle [5] for analyzing the stability and performance of linear nominal models, both to familarize the reader with the results for linear systems and to provide background needed to derive the results for nonlinear systems.

Robustness Analysis of Linear Systems In practice the model is an inaccurate representation of the true process. To account for this plant/model mismatch, the true process is represented by a *set of plants*. The term *robust* is used to indicate that some property (e.g. stability or performance) holds for a set of possible plants as defined by the uncertainty description.

The uncertainty is modeled as norm-bounded perturbations (Δ_i) on the nominal system. Through weights each perturbation is normalized to be of size one

$$\|\Delta_i\|_\infty \equiv \sup_\omega \overline{\sigma}(\Delta_i) \leq 1, \tag{1}$$

where Δ_i is complex for representing unmodeled dynamics, and real for representing parametric uncertainty. The perturbations, which may occur at different locations in the system, are collected in the block-diagonal matrix Δ_U (the U denotes uncertainty)

$$\Delta_U = \text{diag}\{\Delta_i\} \tag{2}$$

and the system is arranged to match the left block diagram in Fig. 2. The interconnection matrix M in Fig. 2 is determined by the nominal model (P), the size and nature of the uncertainty, the choice and location of disturbances and controlled variables, and the controller (K).

Performance is defined in terms of the transfer function between the disturbances $\hat{d}$ and the controlled variables $\hat{e}$ in Fig. 2. If we partition M to be compatible with Δ_U

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{3}$$

then the transfer function between disturbances and controlled variables is given by the *linear fractional transformation* (LFT)

$$F_u(M, \Delta_U) = M_{22} + M_{21}\Delta_U(I - M_{11}\Delta_U)^{-1}M_{12}. \tag{4}$$

The LFT $F_u(M, \Delta_U)$ is well-defined if and only if the inverse of $I - M_{11}\Delta_U$ exists. The subscript u on F_u is used to denote that the upper loop of M is closed by Δ_U. If the lower loop had been closed instead, then the transfer function between inputs and outputs would be the LFT $F_l(M, \Delta_U) = M_{11} + M_{12}T(I - M_{22}\Delta_U)^{-1}M_{21}$.

Nominal performance is defined in terms of the weighted H_∞-norm between disturbances and controlled variables.

Definition 2.1 *The closed loop system exhibits* nominal performance *if*

$$\|M_{22}\|_\infty \equiv \sup_\omega \overline{\sigma}(M_{22}) < 1. \tag{5}$$

For example, for rejection of disturbances at the plant output, M_{22} would be the weighted sensitivity

$$M_{22} = W_1 S W_2, \quad S = (I + PK)^{-1}. \tag{6}$$

In this case, the input weight W_2 is often equal to the disturbance model. The output weight W_1 is used to specify the frequency range over which the sensitivity function should be small and to weigh each output according to its importance. The transfer function of the plant is denoted as P and the controller is denoted as K.

Robust performance is satisfied if the performance requirements are satisfied for all plants given by the uncertainty description.

Definition 2.2 *The closed loop system exhibits* robust performance *if*

$$\|F_u(M, \Delta_U)\|_\infty \equiv \sup_\omega \overline{\sigma}(F_u(M, \Delta_U)) < 1, \qquad \forall \|\Delta_U\|_\infty \le 1. \tag{7}$$

Doyle [5] derived the *structured singular value*, μ, to test for robustness of uncertain systems. Without loss of generality we assume that each Δ_i and M is square [9]. The definition of μ is:

Definition 2.3 *Let $M \in \mathcal{C}^{n \times n}$ be a square complex matrix and define the set $\mathbf{\Delta}$ of block-diagonal perturbations by*

$$\begin{aligned} \mathbf{\Delta} \equiv \; & \{\text{diag}\{\delta_1^r I_{r_1}, \cdots, \delta_k^r I_{r_k}, \delta_{k+1}^c I_{r_{k+1}}, \cdots, \delta_m^c I_{r_m}, \Delta_{m+1}, \cdots, \Delta_l\} | \\ & \left. \delta_i^r \in \mathcal{R}, \delta_i^c \in \mathcal{C}, \Delta_i \in \mathcal{C}^{r_i \times r_i}, \sum_{i=1}^{l} r_i = n \right\}. \end{aligned} \tag{8}$$

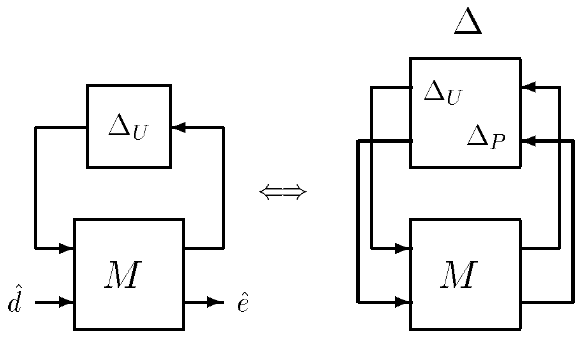

Figure 2: Robust performance and the $M - \Delta$ block structure.

Then $\mu_\Delta(M)$ (the structured singular value with respect to the uncertainty structure $\mathbf{\Delta}$) is defined as

$$\mu_\Delta(M) \equiv \begin{cases} 0 \text{ if there does not exist } \Delta \in \mathbf{\Delta} \text{ such that } \det(I - M\Delta) = 0, \\ \left[\min\limits_{\Delta \in \mathbf{\Delta}} \{\overline{\sigma}(\Delta) \,|\det(I - M\Delta) = 0\}\right]^{-1} \quad \text{otherwise.} \end{cases} \tag{9}$$

The following theorem provides a test for robust stability [5].

Theorem 2.4 (Robust Stability for Linear Systems) *The closed loop system is stable for all $\|\Delta_U\|_\infty \leq 1$ if and only if the closed loop system is nominally stable and*

$$\mu_{\Delta_U}(M_{11}(j\omega)) < 1 \qquad \forall\omega. \tag{10}$$

That robust performance can be tested via μ follows from the main loop theorem [11].

Theorem 2.5 (Main Loop Theorem) *Consider the block diagrams in Fig. 2. The following equivalence holds:*

$$\overline{\sigma}(F_u(M, \Delta_U)) < 1, \ \forall\overline{\sigma}(\Delta_U) \leq 1 \quad \Longleftrightarrow \quad \mu_\Delta(M) < 1, \tag{11}$$

where $\mathbf{\Delta} = \mathrm{diag}\{\Delta_U, \Delta_P\}$, and Δ_P is a full square matrix with dimension equal to the number of outputs (the subscript P denotes performance).

The test for robust performance then follows directly from the definition (7).

Theorem 2.6 (Robust Performance for Linear Systems) *The closed loop system exhibits robust performance if and only if the closed loop system is nominally stable and*

$$\mu_\Delta(M(j\omega)) < 1 \qquad \forall\omega, \tag{12}$$

where $\mathbf{\Delta} = \mathrm{diag}\{\Delta_U, \Delta_P\}$, and Δ_P is a full square matrix with dimension equal to the number of outputs (the subscript P denotes performance).

Multiple performance objectives can be tested similarly using block-diagonal Δ_P.

The value of $\mu_\Delta(M)$ depends on both the elements of the matrix M and the structure of the perturbation matrix Δ. Note that robust performance implies robust stability.

It is a key idea that μ is a *general* analysis tool for determining robust performance. Any system with uncertainty adequately modeled as in (1) can be put into $M - \Delta_U$ form, and robust stability and robust performance can be tested using (10) and (11). Standard programs calculate the M and Δ [1], given the transfer functions describing the system components and the location of the uncertainty and performance blocks Δ_i.

Computation of μ The value of μ is commonly calculated through upper and lower bounds. Define three subsets of $\mathcal{C}^{n\times n}$

$$\mathcal{Q} = \{\Delta \in \boldsymbol{\Delta} : -1 \leq \delta_i^r \leq 1, |\delta_i^c| = 1, \Delta_i^* \Delta_i = I_{r_i}\}, \tag{13}$$

$$\mathcal{D} = \{\text{diag}\,[D_1, \cdots, D_m, d_{m+1} I_{r_{m+1}}, \cdots, d_l I_{r_l}] : 0 < D_i = D_i^* \in \mathcal{C}^{r_i \times r_i}, 0 < d_i \in \mathcal{R}\}, \tag{14}$$

$$\mathcal{G} = \{\text{diag}\,[G_1, \cdots, G_k, O_{r_{k+1}}, \cdots, O_{r_l}] : G_i = G_i^* \in \mathcal{C}^{r_i \times r_i}\}. \tag{15}$$

Then [8]

$$\max_{Q\in\mathcal{Q}} \rho_r(QM) = \mu_\Delta(M) \leq \sqrt{\max\left\{0, \inf_{\substack{D\in\mathcal{D}\\ G\in\mathcal{G}}} \overline{\lambda}\left[\tilde{M}^*\tilde{M} + j\left(G\tilde{M} - \tilde{M}G\right)\right]\right\}}, \tag{16}$$

where $\tilde{M} \equiv DMD^{-1}$, $\overline{\lambda}(A)$ is the maximum eigenvalue of A, and $\rho_r(A) \equiv \max\{|\lambda| : \lambda$ is a real eigenvalue of $M\}$.

The leftmost maximization defined in (16) is not convex, so an algorithm which attempts to calculate the maximum may converge to a local optimum which would be a lower bound for μ. In contrast, the computation of the upper bound in (16) is convex, and so convergence is assured. However, a gap may exist between the upper bound and μ. The upper and lower bounds are almost always within a percent or so for pure complex uncertainty [11]. The gap may be larger when there are real uncertainties. Off-the-shelf software computes the upper and lower bounds for general uncertainty and usually gives a narrow gap [1, 15]. The pitfalls in attempting to calculate μ exactly is discussed by Braatz et al. [3].

The Star Product An interconnection structure related to the linear fractional transformation is the star product (see Fig. 3). Assume that two matrices Q and M are partitioned such that

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \qquad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{17}$$

and $Q_{22}M_{11}$ makes sense and is square. If $I - Q_{22}M_{11}$ is invertible, then the *star product* $Q * M$ is well-defined and is given by

$$Q * M \equiv \begin{pmatrix} F_l(Q, M_{11}) & Q_{12}(I - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1}Q_{21} & F_u(M, Q_{22}) \end{pmatrix}. \tag{18}$$

3 Conic Sector Bounded Nonlinearities

Since conic sector bounded nonlinearities are described in detail elsewhere (see, for example, [4]), here we will only illustrate the approach with an example. Fig. 4 shows a SISO saturation nonlinearity (this could due to either a state or actuator limitation—we will refer to the system component as being an actuator in what follows) covered by a conic sector. The actuator is assumed to behave linearly when the control output u is small, whereas the actuator output becomes limited when the control output becomes sufficiently large in magnitude. Two linear time invariant operators, denoted as the cone center C and the cone radius R, describe the conic sector and are shown in the figure. The purpose of covering the original nonlinearity by a conic sector is that the conic sector

Figure 4: Conic sector bounded saturation nonlinearity.

is described in terms of *linear operators*, and stability analysis for sets of nonlinearities bounded by linear operators is much more developed than stability with general nonlinearities. The standard approach [4] is then to analyze stability for all nonlinearities in the conic sector, giving a sufficient condition for stability for the original nonlinearity.

All nonlinearities in the system are covered by conic sectors, and the resulting conic sector descriptions are rearranged into the familiar leftmost block diagram in Fig. 5, where Δ has block structure as in the linear case [see (8)]. The difference in analyzing stability for this system, as opposed to linear stability analysis, is that this "uncertainty" is a nonlinear time-varying operator. The standard approach is to treat Δ as being complex, and the resulting stability condition is the optimally scaled small gain theorem [6].

Theorem 3.1 (Continuous-Time Stability with General Nonlinearities) *The leftmost system in Fig. 5 is stable for all complex perturbations* $\overline{\sigma}(\Delta) \leq 1$ *if*

1. $M(s)$ *is stable, and*
2. $\inf\limits_{D \in \mathcal{D}} \|DM(s)D^{-1}\|_\infty \leq \beta < 1$.

Though the condition is necessary and sufficient for the set of unity norm-bounded operators [13], it can be an extremely conservative stability test for the system with the original nonlinearities. One way to reduce this conservatism is to reduce the size of the set which covers the nonlinearities of interest. For example, actuator constraints are memoryless, i.e., the output of the actuator

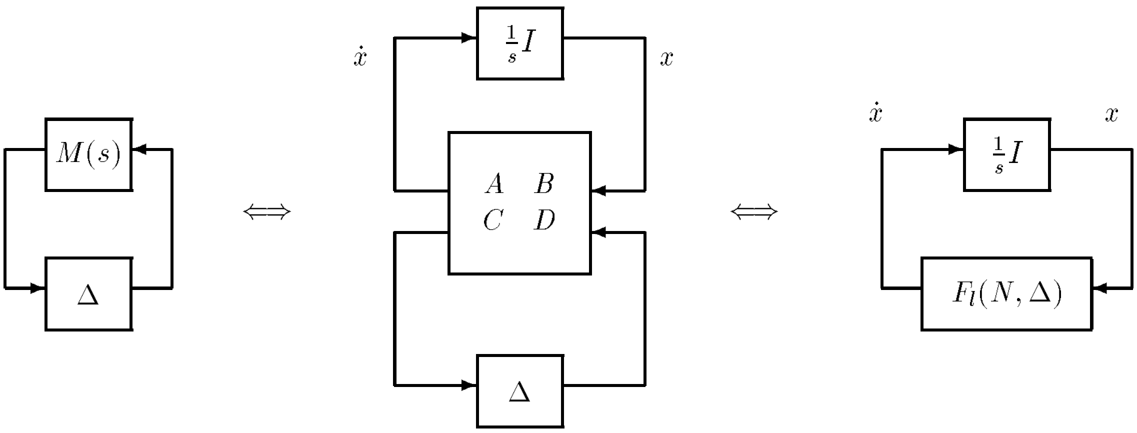

Figure 5: Equivalent block diagrams for continuous systems.

depends on its immediate input and not on past inputs. This means that the set of nonlinearities which cover the saturation nonlinearity can be taken to be *real*—this leads to Δ in Fig. 5 being real. This technical report uses this information to derive a less conservative condition for stability.

4 Stability with Memoryless Nonlinearities

Analyzing stability for discrete systems is simpler than for continuous systems, so we will first consider discrete systems and then show how to transform continuous systems into discrete.

Discrete Time Systems Consider the block diagram in Fig. 6, where the discrete nominal transfer function $M(z) = C(zI - A)^{-1}B + D$. Define the set of norm-bounded time-varying operators with the structure of $\mathbf{\Delta}$ [described in (8)]:

$$\mathbf{\Delta}^k \equiv \{\Delta(k) \in \mathbf{\Delta}, \overline{\sigma}(\Delta(k)) \leq 1, \forall k\}. \tag{19}$$

The following theorem provides a sufficient condition for stability of a discrete system with norm-bounded time-varying perturbations in terms of the upper bound of μ of a constant matrix.

Theorem 4.1 (Discrete-Time Stability with Potentially Memoryless Nonlinearities) *The equivalent systems in Fig. 6 are stable for all $\Delta \in \mathbf{\Delta}^k$ if*

(i) $M(z) = F_l(\frac{1}{z}I_n, N)$ is stable, and

(ii) $\mu^{ub}_{\tilde{\Delta}}(N) < 1$, where $\tilde{\Delta} = \begin{bmatrix} \delta^c I_n & \\ & \Delta \end{bmatrix}, \delta^c \in \mathcal{C}, \Delta \in \mathbf{\Delta}$, and $\mu^{ub}_{\tilde{\Delta}}(N)$ is the mixed upper bound [described in (16)] for $\mu_{\tilde{\Delta}}(N)$.

Proof: Consider the rightmost of the equivalent block diagrams in Fig. 6. The system is described by the difference equation

$$x_{k+1} = F_l(N, \Delta)x_k. \tag{20}$$

Assume the nominal system $M(z)$ is stable. Then a sufficient condition for stability under norm-bounded time-varying perturbations is that there exists an invertible $T \in \mathcal{C}^{n\times n}$ such that

$$\max_{\substack{\Delta \in \mathbf{\Delta} \\ \overline{\sigma}(\Delta) \leq 1}} \overline{\sigma}\left(TF_l(N, \Delta)T^{-1}\right) = \beta < 1, \tag{21}$$

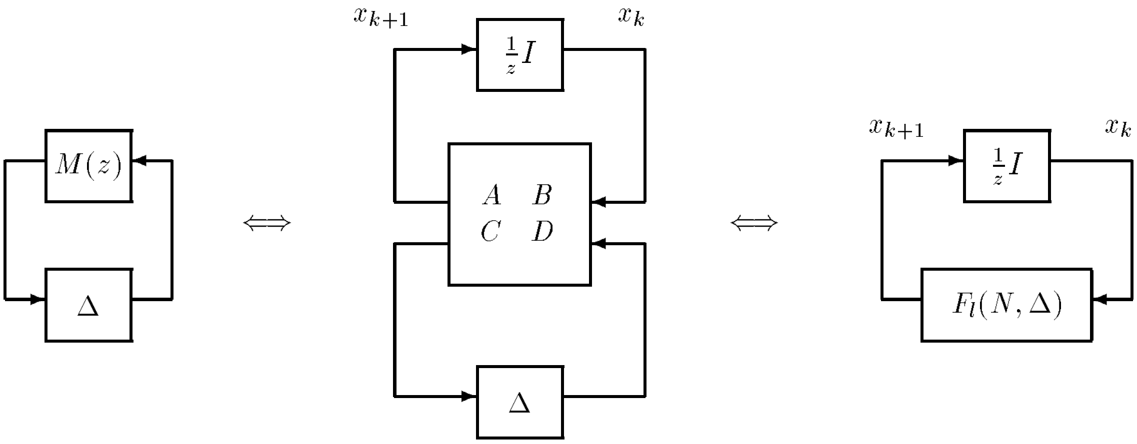

Figure 6: Equivalent block diagrams for discrete systems.

since in this case the norm of x_k obeys

$$\|x_k\| \leq \kappa(T)\beta^k \|x_0\| \tag{22}$$

where $\kappa(T)$ denotes the condition number of T.

We have from Theorem 2.5 that

$$\max_{\substack{\Delta \in \boldsymbol{\Delta} \\ \overline{\sigma}(\Delta) \leq 1}} \overline{\sigma}\left(T F_l(N,\Delta) T^{-1}\right) < 1 \iff \mu_{\hat{\Delta}}\left[\begin{pmatrix} T & \\ & I \end{pmatrix} N \begin{pmatrix} T^{-1} & \\ & I \end{pmatrix}\right] < 1 \tag{23}$$

where $\hat{\Delta} = \begin{bmatrix} \Delta_1 & \\ & \Delta \end{bmatrix}$, $\Delta_1 \in \mathcal{C}^{n\times n}$, Δ_1 full block, and $\Delta \in \boldsymbol{\Delta}$. Combining (21) and (23) gives

$$\text{Stability} \Longleftarrow \inf_{\substack{T \in \mathcal{C}^{n\times n} \\ T \text{ full}}} \mu_{\hat{\Delta}}\left[\begin{pmatrix} T & \\ & I \end{pmatrix} N \begin{pmatrix} T^{-1} & \\ & I \end{pmatrix}\right] < 1. \tag{24}$$

Calculating the minimization in the above equation is expected to be difficult, so we will replace μ with its upper bound [in (16)] to get

$$\text{Stability} \Longleftarrow \max\left\{0, \sqrt{\inf_{\substack{T \in \mathcal{C}^{n\times n} \\ T \text{ full}}} \inf_{\substack{D \in \mathcal{D} \\ G \in \mathcal{G}}} \overline{\lambda}\left[\tilde{N}^*\tilde{N} + j\left(G\tilde{N} - \tilde{N}G\right)\right]}\right\} < 1, \tag{25}$$

where

$$\tilde{N} \equiv D \begin{pmatrix} T & \\ & I \end{pmatrix} N \begin{pmatrix} T^{-1} & \\ & I \end{pmatrix} D^{-1}, \tag{26}$$

$$D = \begin{pmatrix} d_1 I & \\ & D_2 \end{pmatrix}, \tag{27}$$

$$G = \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}, \tag{28}$$

$\overline{\lambda}(A)$ is the maximum eigenvalue of A, $d_1 \in \mathcal{R}$, $D_2 \in \mathcal{D}_2$, $G_2 \in \mathcal{G}_2$, and the sets $\mathcal{D}_2$ and $\mathcal{G}_2$ are specified by the structure of $\boldsymbol{\Delta}$. Absorbing d_1 into T and noticing that the structure of $\tilde{\Delta}$ is appropriate for the new "D" and "G" scalings gives the result. QED.

Continuous Time Systems Now we will consider stability of continuous-time linear systems under norm-bounded time-varying perturbations. We will need the following lemma from [11].

Lemma 4.2 *Let $n > 0$ be an integer, $A \in \mathcal{C}^{n\times n}$, and define a matrix B by*

$$B \equiv \begin{bmatrix} I_n & \sqrt{2}I_n \\ \sqrt{2}I_n & I_n \end{bmatrix}. \tag{29}$$

Let λ_i denote the eigenvalues of A, and $\rho(A)$ denote its spectral radius. Then

$$\mathrm{Re}(\lambda_i) < 0, \forall i \iff I - A \text{ is invertible and } \rho\left(F_l(B, A)\right) < 1. \tag{30}$$

Now consider the block diagram in Fig. 5, where $M(s) = C(sI - A)^{-1}B + D$. Define the set of continuous-time norm-bounded time-varying perturbations with the structure of Δ (8)

$$\mathbf{\Delta}^t \equiv \{\Delta(t) \in \mathbf{\Delta}, \overline{\sigma}(\Delta(t)) \leq 1, \forall t\}. \tag{31}$$

Any subblock of $\mathbf{\Delta}^t$ which corresponds to a *real* subblock of $\mathbf{\Delta}$ is *memoryless.* The following theorem provides a sufficient condition for stability of a continuous linear system under norm-bounded time-varying perturbations in terms of the upper bound of μ of a constant matrix.

Theorem 4.3 (Continuous-Time Stability with Potentially Memoryless Nonlinearities) *The equivalent systems in Fig. 5 are stable for all $\Delta \in \mathbf{\Delta}^t$ if*

(i) $M(s) = F_l(\frac{1}{s}I_n, N)$ is stable, and

(ii) $\mu^{ub}_{\tilde{\Delta}}(B * N) < 1$, where $\tilde{\Delta} = \begin{bmatrix} \delta^c I_n & \\ & \Delta \end{bmatrix}, \delta^c \in \mathcal{C}, \Delta \in \mathbf{\Delta}$, and $\mu^{ub}_{\tilde{\Delta}}(B * N)$ is the mixed upper bound [in (16)] for $\mu_{\tilde{\Delta}}(B * N)$.

Proof: Consider the last of the equivalent block diagrams in Fig. 5. The system is described by the differential equation

$$\dot{x} = F_l(N, \Delta)x. \tag{32}$$

For stability of the continuous-time system, we need to test if the eigenvalues of $F_l(N, \Delta)$ are in the left half plane. The equivalence of the block diagrams in Fig. 7 follows from Lemma 4.2 with $A = F_l(N, \Delta)$. Thus we have converted the continuous stability problem to the discrete stability problem of Theorem 4.1. QED.

5 Performance with Memoryless Nonlinearities

The performance requirement for the continuous-time linear system with norm-bounded time-varying perturbations is that (see Fig. 8)

$$\overline{\sigma}\left(F_l(M(j\omega), \Delta)\right) = \beta < 1 \qquad \forall \omega, \forall \Delta \in \mathbf{\Delta}^t. \tag{33}$$

The proof of the following theorem is similar to the proof for stability above.

Theorem 5.1 (Continuous-Time Performance with Potentially Memoryless Nonlinearities) *Performance for the system in Fig. 8 is satisfied for all $\Delta \in \mathbf{\Delta}^t$ if*

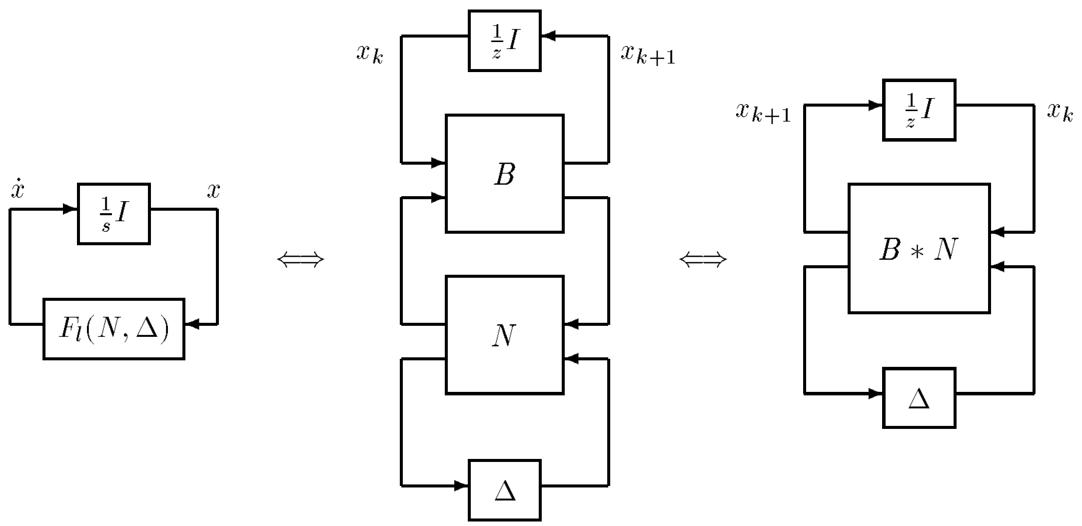

Figure 7: Transformation of the continuous stability test to the discrete stability test.

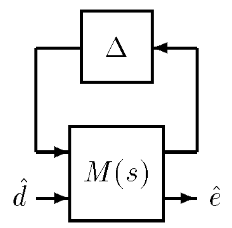

Figure 8: Continuous system with time-varying uncertainty.

(i) $M(s) = F_l(\frac{1}{s}I_n, N)$ *is stable, and*

(ii) $\mu^{ub}_{\tilde{\Delta}}(B * N) < 1$, *where* $\tilde{\Delta} = \begin{bmatrix} \delta^c I_n & & \\ & \Delta^c & \\ & & \Delta \end{bmatrix}$, $\delta^c \in \mathcal{C}$, Δ^c *a full complex block,* $\Delta \in \mathbf{\Delta}$, *and* $\mu^{ub}_{\tilde{\Delta}}(B * N)$ *is the mixed upper bound for* $\mu_{\tilde{\Delta}}(B * N)$.

The sufficient test for robust performance for discrete-time systems is similar. Scaling of the uncertainty and the performance specifications can be incorporated into the above theorems to give greater flexibility (see [11] for details). We will now consider an example which shows a substantial reduction in conservatism when taking into account the memoryless nature of the common nonlinearities encountered in process industries.

6 Example

Consider the discrete 4×4 closed loop system (N) given by the following state space matrices:

$$A = \begin{pmatrix} -1.6662 & -3.2066 & 0.2522 & 4.6348 \\ -3.5907 & -6.5803 & 0.5290 & 9.3770 \\ -10.0332 & -20.5300 & 1.7744 & 27.3046 \\ -2.7552 & -4.9830 & 0.3936 & 7.2349 \end{pmatrix}, \tag{34}$$

$$B = \begin{pmatrix} -1.0801 & -0.3601 & -0.7408 & 2.0288 \\ -2.4983 & -0.4873 & -2.0047 & 3.8125 \\ -8.2313 & -2.7842 & -7.3722 & 11.1384 \\ -1.9733 & -0.4020 & -1.5071 & 3.1886 \end{pmatrix}, \tag{35}$$

$$C = \begin{pmatrix} -6.0014 & -11.6816 & 1.0568 & 17.1462 \\ -9.0589 & -18.0075 & 1.8934 & 24.7326 \\ -5.3812 & -10.7738 & 1.1081 & 14.9160 \\ -8.5404 & -16.9704 & 1.7568 & 23.4685 \end{pmatrix}, \tag{36}$$

$$D = \begin{pmatrix} -4.5765 & -1.1854 & -3.6748 & 7.0926 \\ -7.2120 & -2.2428 & -6.2854 & 10.1094 \\ -4.2841 & -1.3153 & -3.6850 & 6.1183 \\ -6.7764 & -2.0757 & -5.8558 & 9.6079 \end{pmatrix}. \tag{37}$$

The eigenvalues of A are $\{-0.1437, 0.3945, 0.3396, 0.1724\}$, which all have magnitude less than one so $M(z)$ is nominally stable. The nonlinearity $\mathbf{\Delta}^k$ consists of four memoryless repeated scalar 1×1 blocks. If we ignore that the nonlinearity is memoryless ($\mathbf{\Delta}$ complex), then the stability margin is

$$\mu^{ub}_{\tilde{\Delta}}(B * N) = 3.57 > 1, \tag{38}$$

so stability of the closed loop system is not assured. If we take the memoryless nature of the nonlinearity into account ($\mathbf{\Delta}$ real), then the stability margin is

$$\mu^{ub}_{\tilde{\Delta}}(B * N) = 0.98 < 1, \tag{39}$$

and so stability is guaranteed. The reduction in conservatism is 264%.

Figure 9: Directionality compensation.

7 How Much Conservatism is Reduced?

Theorem 4.3 is equivalent to the standard conic sector stability test (Theorem 3.1 when the nonlinearity $\boldsymbol{\Delta}$ is complex [11]. Though Theorem 4.3 can substantially reduce the conservatism over the standard conic sector stability test by taking into account the memoryless nature of the nonlinearity, the following lemma shows that there is no reduction in conservatism when all the subblocks of Δ are independent and 1×1.

Lemma 7.1 *Theorems 4.3 is no less conservative than the optimally scaled small gain theorem (Theorem 3.1) when all the subblocks of* $\Delta \in \boldsymbol{\Delta}^k$ *are* 1×1*.*

Proof: Follows from results in [14]. QED.

The example in Section 6 showed that the conservatism can be reduced when the nonlinearity was repeated scalar. This nonlinearity is appropriate under directionality compensation which is illustrated by Fig. 9. When the control output cannot meet the constraints, the directionality compensator (which is placed immediately after the linear control and before the actuator constraints) scales back the control output *while keeping the same direction* until the control action becomes feasible. Braatz et al. [2] found that the directionality compensator performed nearly as well as model predictive control for an industrial scale adhesive coater, but with much simpler computation. A detailed discussion of the importance of directionality compensation, especially when the controller is an inverse-based design, is provided by Campo [4].

The above approach was to reduce conservatism by accounting for the memoryless nature of the nonlinearity. To reduce the conservatism of the nonlinear stability conditions by a larger margin, it is needed to remove nonlinearities such as the one shown in Fig. 10, which can have arbitrary positive or negative instantaneous slope, and arbitrarily large magnitude as the input increases. The author is currently investigating the inclusion of bounds on the slope and magnitude of the nonlinearity in the problem formulation.

Figure 10: Conic sector bounded nonlinearities.

8 Nonlinear Stability and Performance

The stability and performance tests developed in this technical report can be used to test *local* stability and performance for general nonlinear systems. The nonlinear system is linearized, and the linear part rearranged to form the nominal system M in Fig. 8. A local operating region is defined in the phase plane, and the nonlinearity as a deviation from the linear system is covered by a conic sector in this region. The theorems developed in this technical report can be used to test stability and performance for the system as long as the process stays in this operating region (for details and application to a packed bed reactor, see Doyle [7]).

In gain-scheduling, the nonlinear plant is treated as linear with time-varying parameters. The gain-scheduled controller is also linear, but dependent on the same time-varying parameters of the plant (which are assumed to be measured or estimated, see [12] for details). The tests in this technical report can be applied to analyze the *global* stability and performance for these systems, where the parameters are treated as time-varying uncertainty. Because both the controller and the plant depend on the parameters (i.e., the uncertainties are repeated), the tests can reduce conservatism by taking the real nature of the parameters into account.

9 Conclusions

Less conservative stability and performance tests are derived for memoryless nonlinearities. Though the tests cannot reduce conservatism for single-input single-output systems, the conservatism can be reduced substantially for multivariable systems with directionality compensation. The stability and performance tests developed in this technical report can also be used to test local stability and performance for general nonlinear systems, and global stability for gain-scheduled systems.

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