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"Uncertain Behavior"

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Abstract

The invited session on behaviors, modeling, and robust control reflects an emerging view that these apparently disjoint subjects have important connections that can have a major impact on both theoretical and practical aspects of control. The behavioral setting provides a convenient framework for connecting modeling and robust control, and is being extended to make deeper contacts with more mainstream control.

1 Why Behave?

In an attempt to capture the essence of the behavioral paradigm, one would do quite well with the following: the absence of a preconceived notion that dynamical systems must be described as input-output maps. Invariably, the majority of texts that deal with dynamical systems take the de facto stance that systems are inherently input-output maps. A brief excursion to the bookshelf yielded the following:

From Signals and Systems, [13]:

> A *system* can be viewed as any process that results in the transformation of signals. Thus, a system has an input signal and an output signal which is related to the input through the system transformation.

and from Signals and Linear Systems, [8]:

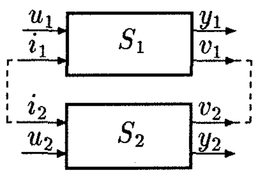
> A system is a mathematical model or abstraction of a physical process that relates inputs or external forces to the output or response of the system.

It is not necessary in the behavioral setting to consider systems as operators that transform inputs into outputs. Rather, all variables are considered a priori on an equal footing, and the input-output framework is developed as a special case which in many situations can actually be deduced from the original model.

It may seem that the above is simply a generalization for its own sake, and adopted solely to allow for greater levels of abstraction. This is a reasonable interpretation if we view control in its most narrow sense: designing controllers for a fixed plant whose models and specifications have been given to us. If, however, we want to view control more broadly, which we need to do if we ever want to make any real impact in technology, then we must make more contact with modeling, identification, and system design, not just control design. As we shall see, it is in this broader context that adopting the behavioral framework is not just convenient, but rather quite fundamental.

The predominant reason for not describing systems as input-output operators is that in practice it is often not clear which of the variables should be regarded as inputs and which as outputs. Examples of this are situations in which the system is an interconnection of several subsystems. Such an interconnection may induce constraints such that variables which could have been considered as inputs or as outputs in a subsystem can no longer be labeled as such in the interconnected system.

Thus it would seem that imposing an input-output structure to a system, or component, would require an a priori knowledge of how that component will be used. As a simple example, consider two electrical networks, described as input-output operators:



It is desired to connect a node together, ie., setting $v_1 = v_2$ and $i_1 = i_2$. If the two networks are defined as above, where i_1 and i_2 are inputs, and v_1 and v_2 are outputs, there is no clear way to represent this interconnection. The problem is, of course, with the choice of inputs and outputs. If System 2 were represented with v_2 as an input and i_2 as an output, a simple expression for the above interconnection could then be derived.

It might also be the case that an input-output structure is not required: should a resistor in an electrical circuit be considered as a voltage to current operator, or a current to voltage operator? As long as the relation $v - Ri = 0$ is satisfied, the answer to this question is quite irrelevant. It would thus seem that the effort undertaken by the design engineer to provide input-output system representations might not be warranted.

In addition, behavioral equations arise naturally when modeling physical systems from first principles; it is almost always the case that components are modeled in terms of mass, momentum, or energy balances or physical laws, which are inherently of the form $f(\vec{w}) = 0$, where f is an operator, and $\vec{w}$ a vector of variables.

Another reason for adopting the behavioral paradigm is that this approach unifies first principles modeling and interconnection. Both are mathematically equivalent, since both consist of combining constraint equations. Tools developed for interconnection can then be used for modeling purposes, and vice-versa.

One might then ask why we need to bother with input-output representations at all. We need only consider control systems to supply the needed response. Unquestionably, the majority of control algorithms are implemented today with computers. These devices are inherently of an input-output nature, and thus require the system to which they interface to have a well-defined input-output structure. This is usually implemented via subsystems known as *actuators* and *sensors.*

How the behavioral methodology fits in with modeling and control can be nicely illustrated by an example:

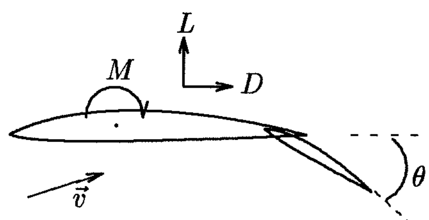


Figure 1: Elevator Control

It is required to model the relationship between an airplane's elevator position, denoted θ, and other relevant flight variables, such as airplane velocity, lift, drag, etc., denoted $\mathbf{v}$. In practice, the elevator position is not controlled directly. Rather, an actuator positions the elevator via a control signal u (for example, a voltage from a digital to analog converter).

A simplistic approach to the modeling problem would be to first find an expression for $\mathbf{v}$ as an operator on θ, and then to model the dynamics of the actuator as an operator mapping u to θ:

$$\begin{aligned} \mathbf{v} &= V(\theta) \\ \theta &= A(u) \end{aligned} \tag{1}$$

The modeling problem would then be solved, with $\mathbf{v} = V \circ A(u)$. A major problem with this approach is that it totally ignores the interdependence between the actuator and the flight variables. This is not an issue if the actuator has large control authority over the elevator. In other words, if it is impervious to operating conditions, or is buffered. In high performance systems however, where weight and size constraints preclude the use of arbitrarily powerful actuators, the assumption of being buffered is no longer valid, and we can no longer cascade transfer functions together.

What must be recognized is that the actuator positions the elevator by applying forces and torques, denoted $\mathbf{f}$, to the elevator mechanism. $\mathbf{f}$ is a function of the elevator position θ and the control signal u. θ is then a function of the applied forces and torques $\mathbf{f}$ and the flight variables $\mathbf{v}$. Thus we need the following relations:

$$\begin{aligned} \mathbf{v} &= V(\theta) \\ \theta &= G(\mathbf{v}, \mathbf{f}) \\ \mathbf{f} &= T(\theta, u) \end{aligned} \tag{2}$$

which is depicted in Figure 2.

The resulting feedback system can then be manipulated to give $\mathbf{v}$ as a function of u. Our claim is that the behavioral approach is much more suited to the above task.

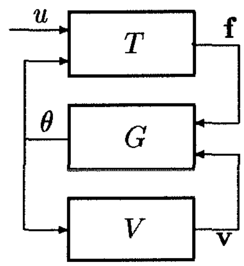


Figure 2: Interconnected System

It will most surely be the case that the relationship between $\mathbf{v}$ and θ is more naturally expressed as an implicit operator. The same applies to G and T. Hence equations (2) would have been originally derived from relations of the type:

$$\begin{aligned} V_b(\mathbf{v},\theta) &= 0 \\ G_b(\mathbf{v},\mathbf{f},\theta) &= 0 \\ T_b(\mathbf{f},\theta,u) &= 0 \end{aligned} \tag{3}$$

Massaging (3) to get input-output maps (2) requires an unwarranted effort, since equations (3) can be manipulated directly to express $\mathbf{v}$ as a function of u. This approach has several advantages. The first, as was previously mentioned, is that the equations are much simpler to deal with in behavioral form. Secondly, the input-output maps used to derive the map from u to $\mathbf{v}$ might not be proper (where proper is taken to mean that no derivatives of the input appear at the output), even though the final map is. This would preclude the use of most state space approaches. In the behavioral approach, this is not an issue, except perhaps at the final step if an input-output map for the purpose of control is desired. In that case, we would be hesitant of using a signal as a control input which has its derivatives appearing at the output. Thus the issue of properness should only be considered when we are ready to control the system. Finally, it is a simple matter to add more equations to reflect further modeling. For example, the control algorithm will not have $\mathbf{v}$ at its disposal, but rather a signal $\mathbf{y}$, which is related to $\mathbf{v}$, and perhaps other variables, by means of a relation $S_b(\mathbf{v},\mathbf{y},...) = 0$ (which could be interpreted as the sensor equations).

Hopefully we have provided some motivation as to why we should adopt a behavioral approach when modeling and interconnecting systems. As will be illustrated, there is strong motivation to incorporate uncertainty into our system description. Before pursuing this line of thought, however, it is important that the reader be familiar with the behavioral paradigm. What follows is a brief introduction to the behavioral approach which highlights the material most relevant to modeling and control.

2 Layman's guide to Behavioralism

2.1 Introduction

The idea that a system should not be necessarily described as an input-output map, but rather as a family of trajectories, is the basic starting point of the Behavioral framework. There exist sources, especially in the circuit theory literature, which sense the awkwardness of the input-output setting. It is only recently, however, that a detailed, self-contained exposition of systems described as families of trajectories has been pursued. Willems in [24] lays down the foundation for what is now termed the *Behavioral Approach* to the study of dynamical systems.

2.2 Dynamical Systems

A natural starting point is the definition of a dynamical system:

Definition 1 *A* dynamical system Σ *is a triple* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ *with* $\mathbb{T} \subseteq \mathbb{R}$ *the* time axis, $\mathbb{W}$ *the* signal space, *and* $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the* behavior.

Discrete time dynamical systems are characterized by $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}^+$, while continuous time dynamical systems are characterized by $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}^+$.

Thus, a dynamical system is defined by $\mathbb{T}$, the time instants of interest, $\mathbb{W}$, the space in which the time signals which the system produces take their values, and $\mathcal{B}$, a family of $\mathbb{W}$-valued time trajectories. The sets $\mathbb{T}$ and $\mathbb{W}$ define the setting, while $\mathcal{B}$ formalizes the laws which govern the system. According to the dynamical model Σ, time signals in $\mathcal{B}$ can in principle occur, while those outside $\mathcal{B}$ cannot. This definition of a dynamical system as a family of trajectories without reference to input-output maps or relations is simple and succinct, and presents a general starting point for the study of numerous questions in the theory of dynamical systems. We will, however, for the purposes of this paper, only consider systems which are finite-dimensional linear time invariant (FDLTI), unless otherwise stated. The characterization of these systems can be found in [26].

2.3 Representation

There are many ways to capture the behavior of a dynamical system. One way is to consider all solutions w of

$$R(\sigma)w = 0 \tag{4}$$

where $R(\sigma)$ is a matrix polynomial function in σ. σ can be either the differentiation operator or the time shift operator, depending on the nature of $\mathbb{T}$. The above system of equations is referred to as an *autoregressive* (AR) system representation.

At times it may be simpler and more natural to represent a dynamical system with variables in addition to the ones that constitute a system's behavior (which are referred to as *manifest variables*). These variables, referred to as *latent variables*, may facilitate the modeling of a system. Latent variables form an integral part of a theory of modeling. They provide a way of formalizing models which contain auxiliary variables. These will almost always be present in models obtained from first principles. An illustrative example is the modeling of an electronic circuit. It will usually be advantageous to introduce extra voltages and currents when writing down loop or node equations.

An important class of latent variables are *state variables.* An example of where they arise is system descriptions of the form

$$E\sigma x + Fx + Gw = 0 \tag{5}$$

where E, F, and G are constant matrices. x is the *state*, and w is the vector of manifest variables. The above system representation is referred to as a *dual pencil representation*, which is extensively studied in [10]. It is a fact that every state representation can be written in dual pencil form. Another type of state representation is the so called *output nulling* representation:

$$\begin{aligned} \sigma x &= Ax + Bw \\ 0 &= Cx + Dw \end{aligned} \tag{6}$$

whose properties are explored in [23].

The reader is referred to [20] for a comprehensive list of system representations.

2.4 Controllability

Two important concepts in conventional input-output realization theory have undoubtedly been those of controllability and observability. It is a well known fact that every transfer function and every convolution can be represented by a minimal state-space system and that minimality is equivalent to controllability and observability. Thus the two concepts are properties of the realization, and are not intrinsic to the relation between inputs and outputs of the system being modeled.

In the behavioral framework, controllability is defined as to be a property of the system, independent of the realization:

Definition 2 *Let $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$, be a time-invariant dynamical system. Σ is said to be* controllable *if for all w_1, $w_2 \in \mathcal{B}$ there exists a $t_0 \in \mathbb{T}$, $t_0 \geq 0$, and a $w : \mathbb{T} \cap [0, t_0] \to \mathbb{W}$ such that $\acute{w} \in \mathcal{B}$, with $\acute{w} : \mathbb{T} \to \mathbb{W}$ defined by*

$$\acute{w}(t) = \begin{cases} w_1(t) & \text{for } t < 0 \\ w(t) & \text{for } 0 \leq t \leq t_0 \\ w_2(t - t_0) & \text{for } t > t_0 \end{cases} \tag{7}$$

Informally, a dynamical system is controllable if any allowable past trajectory can be patched up to any allowable future trajectory in some finite time and by some allowable trajectory. A simple example of an uncontrollable dynamical system is

$$\dot{x} = 0 \tag{8}$$

Any constant is a solution to the above equation. It is clear that if $x = a$ and $x = b$ are two trajectories, there is no way to join them together with a trajectory that satisfies (8), ie., with another constant. In fact, the above is an example of an *autonomous* system, in which the past of a trajectory determines its future completely. It is a fact that every dynamical system can be decomposed into controllable and autonomous parts.

2.5 Inputs and Outputs

Although the behavioral framework is general in that it allows all system variables to be considered on an equal footing, it is important to incorporate inputs and outputs in this setting. The natural question that arises then is what exactly is meant by an input to a dynamical system and what is meant by an output.

There are two basic criteria that a valid partition of the manifest variables into inputs and outputs must pass. The first is that the input should, in combination with the laws of the system and the initial conditions, determine the output. This property is termed *processing.* The second is that the input should be allowed to vary freely over its signal space. Thus the input itself cannot be explained by the model.

It is a fact that every dynamical system allows an input-output state space representation of the form

$$\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{9}$$

Thus not only can signals always be partitioned into inputs and outputs, there exists a way to do so such that y does not anticipate u in discrete time systems, or that derivatives of u do not appear explicitly in y in continuous time systems. In fact, there may be systems for which there exists more than one way to partition the variables so that a proper transfer function results. A simple example is any input-output state space representation with an invertible D matrix.

2.6 Interconnection

If Σ_1 and Σ_2 are two dynamical systems, their interconnection can be simply considered as imposing the laws of both Σ_1 and Σ_2. Formally the *interconnection* of $\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_1)$ and $\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_2)$ is denoted by $\Sigma_1 \wedge \Sigma_2$ and defined as

$$\Sigma_1 \wedge \Sigma_2 := (\mathbb{T}, \mathbb{W}, \mathcal{B}_1 \cap \mathcal{B}_2) \tag{10}$$

Thus interconnection can be interpreted as the intersection of behaviors, or as combining constraint equations.

If the laws of the two systems are independent, an interconnection may be *singular* or *regular.* Singular interconnections arise when the interconnection forces an algebraic constraint between states in the systems. A simple example is the parallel connection of two capacitors. Before connection, a state is required for the voltage across each capacitor. After connecting them together, there exists an algebraic constraint on the states that requires them to be equal for all time.

We can think of interconnection in two contexts. First, when interconnection is simply an artifice of our modeling process, where we have broken the system into subsystems. Second, when a physical interconnection is established at a particular time. When interconnecting for modeling, a singular interconnection is simply a flag that our states are constrained and therefore we might want to simplify the model. When connecting two system at a particular instant in time, however, a singular interconnection would require that the states be matched in advance, otherwise we will have a transient phenomenon which is not modeled and is potentially damaging; in the simple parallel capacitors example, the residual charges on the two capacitors must be compatible, otherwise a large current may flow at the time of contact. Singular interconnections are said to impose *compliance constraints* on the systems to be connected. Regular interconnections are then said to be *compliance free*.

There is an interesting interpretation of *feedback* in terms of regular and singular interconnections. The interconnection of two systems, Σ_1 and Σ_2, can be depicted as follows:

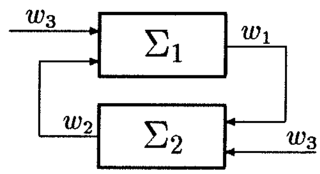


Figure 3: Interconnection Interpretation

where $w = (w_1, w_2, w_3)$ makes up the signal space of both Σ_1 and Σ_2. If the interconnection is regular, there will always exist a partition of w into w_1, w_2, and w_3 such that i) the closed loop transfer function from w_3 to w_1 and w_2 is proper, and ii) Σ_1 and Σ_2 as depicted in Figure 3 are also proper. For singular interconnections, if (i) is achieved, (ii) cannot be. Usual feedback interconnections considered in system theory are regular, while PID control laws with a differentiating action are examples of singular feedback interconnections. The concept of interconnection is explored in detail in [25].

2.7 Modeling from Data

While the behavioral framework is natural when constructing models from first principles, it is also well suited to black box techniques, where the model is obtained directly from the observed data.

An important concept in behavioral system identification is that of the *Most Powerful Unfalsified Model* (MPUM) for a given set of data. The MPUM may be characterized as the model which explains a given set of observations and as little else as possible. Thus it can be considered as the model which imposes the most constraints. This captures the main idea in modeling; the more the model forbids and the easier it is to falsify, the better it is. For FDLTI dynamical systems, the MPUM will have at most as many inputs as all other unfalsified models, and at most as many state variables as all other unfalsified models with the same number of inputs.

3 Modeling Uncertainty

3.1 Introduction

The title of this section seems somewhat contradictory. We think of (mathematical) models as our means of analyzing physical reality, and uncertainty is the gap that is left between models and reality, the part of reality that is not accounted for in our model. Apparently then, it makes no sense to model uncertainty; uncertainty is always what remains after modeling is finished.

The previous point of view seems to lead to the following attitude towards models and reality: derive the best models you can, to the utmost detail, then work with them as though they correspond to reality, and hope for the best; by definition nothing can be done about the remaining uncertainty after our best model is obtained. However, this attitude has its shortcomings because of the following:

- A very detailed model is often hard to obtain.
- There is a principle of "parsimony" that makes us have little faith on very detailed models: we distrust our own ability to discover very complex or small phenomena in physical reality.
- Most importantly, if a model is too complicated, it is as hard to deal with it as with reality itself, so the whole purpose of modeling is defeated.

This leads us to the prevailing attitude towards modeling: we must pursue models up to the extent we feel confident about their predictions while remaining reasonably simple, and deliberately leave the rest as uncertainty. As an example of this, in classical time-series analysis we fit a linear model to observations, stop at a certain order (for simplicity and to avoid "overfitting") and explain the rest as a random disturbance (a form of uncertainty).

In mentioning confidence towards our models a subjective element is introduced; the desire for more objective characterizations necessitates a description of the amount and structure of the remaining uncertainty. This is the meaning of modeling (or better, describing) uncertainty.

So descriptions of uncertainty come in as quality tags for our models; a mathematical description of uncertainty (such as a random process, or a norm bounded operator) does not attempt to describe the real mismatch between models and reality, which is more of an epistemological question. It is merely a tool to "measure" our lack of confidence in the model, as simply as possible, so that the user can have a quantitative *estimate* of the reliability of the model and the relative weight of different sources of error.

In control engineering uncertainty plays a major role. In fact, the main reason to introduce feedback in a system is to reduce the effects of uncertainty in the system behavior. Only by explicitly considering uncertainty descriptions can an engineer measure the quality of the resulting system and evaluate the inherent performance limitations. Ideally, the control engineer would want a simple, unified language in which uncertainty is described.

The purpose of this section is to review the descriptions of uncertainty that usually appear in the area of control systems, and briefly describe attempts made in the robust control literature to give a unified framework to the various descriptions of uncertainty.

3.2 A survey of uncertainty descriptions.

Traditionally, one obtains models by two means: analysis from "first principles" or "black-box" system identification. In the latter a model is sought in one shot from experimental data, whereas in first principles constructions one reduces a system to smaller subsystems of which good models already exist, derived by someone else by methods that involve, eventually, experimental data. These "first principles" carry the weight of having already gained acceptance by the scientific community and having been tried in many situations, and may be derived from experiments under more constrained conditions. They don't, however, escape from the inevitable fact that they are models too, and thus must be accompanied by uncertainty.

Independently of the method used to derive the model, the result is a *nominal* mathematical model and descriptions of the remaining uncertainty which can be broadly classified in the following four classes:

- Parametric uncertainty
- Unmodeled nonlinearities.
- Unmodeled dynamics.
- Disturbances.

We will describe in what follows (with no attempt at being exhaustive) how these typically arise in modeling.

3.2.1 Uncertainty in system identification

A method based on frequency response is considered first. Figure 4 shows points in a Nyquist plot obtained empirically (by exciting the system with sinusoids, for example), and a corresponding model $G(j\omega)$ (which is usually chosen to be low with regard to parsimony) which approximates the data.

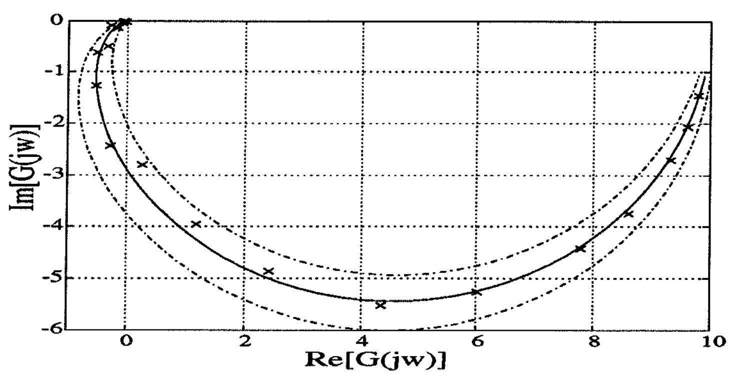


Figure 4: Identification using the frequency response.

A convenient way to express the remaining uncertainty in this model is to cover the points by a region of the form $G(j\omega)(1+W(j\omega)\Delta(j\omega))$, where $\Delta(j\omega)$ can be considered to be *unmodeled dynamics* , i.e. an unspecified dynamical system normalized such that $|\Delta(j\omega)| \leq 1$, and $W(j\omega)$ a weight function chosen to give the band in the figure. This defines a family of transfer functions, of which one could have accounted for the data.

Another very popular method in conventional system identification is to explain the data in a discrete time experiment by the expression

$$y = G(q,\theta)u + H(q,\theta)n$$

where q is the shift operator, θ is a vector of parameters, G and H are parameterized transfer functions, and n is assumed to be noise. Here there are two ways of specifying uncertainty. One is by the characteristics of the noise, the other by the specification of the parameter values.

The noise variable is an example of using a *disturbance* to describe model uncertainty. Choices here are to use stochastic models (n is a trajectory of a random process), popular in the system identification literature, or deterministic models (n is an arbitrary signal with some norm bound, or additional constraints), which are usually preferred in the robust control literature.

On the other hand, the characterization of θ is an example of *parametric uncertainty* . This can be specified, for example, as an interval in which θ must lie (which can be obtained as a statistical confidence interval).

For a thorough treatment of system identification, the reader is referred to [11].

3.2.2 Uncertainty in modeling from first principles

To make the discussion concrete, we will illustrate the concepts of first principles modeling by considering the circuit below, which consists of a transistor and two resistors. Despite the apparent simplicity of this system, we will soon discover that all four uncertainty descriptions outlined earlier enter our model.

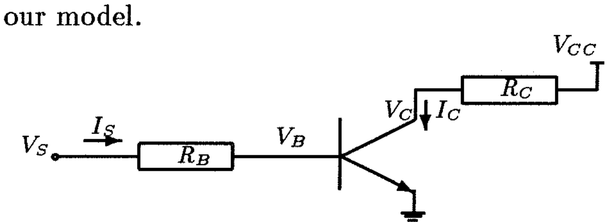


Figure 5: Illustrative example: a simple circuit

The starting point is to write models for the components: we will use the usual linear model $v = Ri$ for the resistors; V_{CC} is a DC (constant) voltage source. Concentrating on the transistor, there exist satisfactory nonlinear models at low frequencies; we will use the following:

$$\begin{aligned} I_B &= I_0 \left(e^{\frac{V_B}{V_T}} - 1 \right) \\ I_C &= \beta I_B \end{aligned} \tag{11}$$

Here I_B is the current in the base of the transistor (nominally the same as I_S), while I_0, V_T and β are constants. An important observation is that (11) is already a simplified version of the more general static equations, and that it is a good approximation only in the so-called "active" operating region.

The next stage is to simplify and combine the previous component models to obtain a model for the full system.

A common procedure to obtain tractable results is to linearize the transistor equations about an operating point in the active region. I_{B0}, V_{B0}, etc. will denote the values of the variables at the operating point, while lowercase symbols will denote the increments of the variables (i.e., $i_B = I_B - I_{B0}$, etc.). Linearizing the first equation results in $v_B = ri_B$, where $r = \left(\frac{dV_B}{dI_B}\right)_0$, an approximation of the nonlinear static map. Then the entire circuit may be modeled as a linear system as shown in the block diagram of Figure 6.

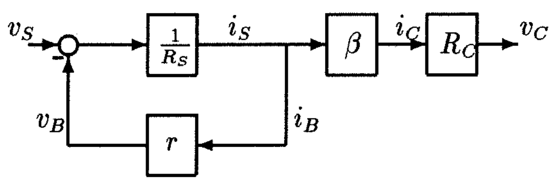


Figure 6: Nominal linear model for the circuit

We shall now analyze this model with the aim of describing the remaining uncertainty, considering different effects in 4 steps.

Step 1: Nonlinearities

If we wish to have bounds on the error involved in the linearization, we can employ a "conic sector" description of the nonlinearity by writing

$$|v_B - ri_B| \leq k_n |i_B| \tag{12}$$

This procedure is depicted in Figure 7 below. Equation (12) implies that the characteristic falls in the cone drawn in Figure 7 (this of course is valid for a limited range of v_B, i_B). The

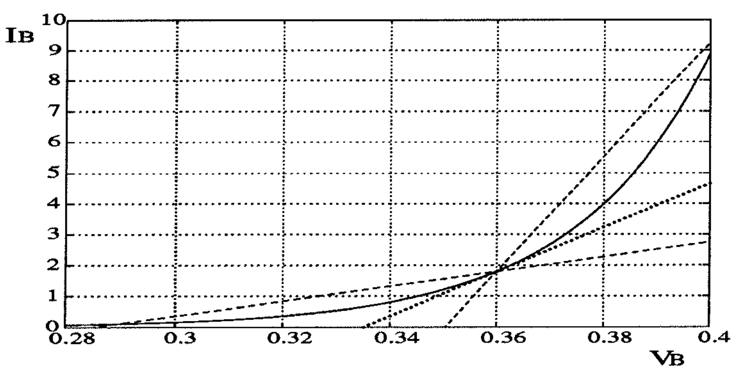


Figure 7: Linearization and conic sector bounds

previous bound is a static constraint, but we can generalize this by writing $v_B = ri_B + k_n\delta_n(i_B)$, where δ_n is an unknown nonlinear operator such that $\|\delta_n\| \leq 1$. This constraint is weaker, as δ_n need not be memoryless (it can still be given a "conic" interpretation, see [32]), but we might think of it "covering" dynamic effects which are not described in our static equations.

Step 2: Parametric Uncertainty

Our previous block diagram includes many parameters, whose values will not be known precisely, so an uncertainty description must define the possible range of variation. In principle this applies to all parameters (e.g. resistor values), but for simplicity we shall just consider the β of the transistor, which typically has large variations due to the manufacturing process and operating temperature variations. In fact, one of the main reasons for using feedback in the design of electronic amplifiers is to reduce these variations in the gain. A typical model for these variations is $\beta = \beta_0 + k_\beta \delta_\beta$, where δ_β is normalized to vary in the interval $[-1, 1]$ and k_β is a constant weight.

Step 3: Unmodeled Dynamics

We have chosen as a starting point static models for our components, which appear to be reasonably good at low frequencies. At high enough frequencies, however, simple models such as $v = Ri$ fail to describe the variation of the electromagnetic fields in a complex geometry; numerous distributed effects appear, and "complete" models are unthinkable. Usually some "lumped" components are added which provide a better approximation for intermediate frequencies, though of course these models also break down at some point. We will analyze a common approximation for the transistor, which entails adding a "parasitic capacitance" in the input portion of the linearized model, as shown in Figure 8.

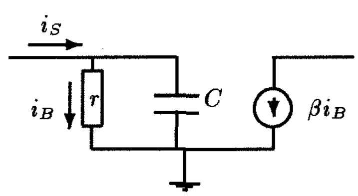


Figure 8: Linear model for the transistor at higher frequency

Solving for current i_B in terms of i_S results in $i_B(j\omega) = \frac{i_S(j\omega)}{1+\tau j\omega}$, where $\tau = rC$. If a better model for the circuit is desired, these extra dynamics could have been incorporated in the nominal model to begin with. In many cases, however, it is more convenient to work with the original (simpler) model, and "cover" this effect with uncertainty. The uncertain relation

$$i_B(j\omega) = i_S(j\omega)\left(1 + W(j\omega)\delta_u(j\omega)\right), \quad |\delta_u(j\omega)| \leq 1 \tag{13}$$

with $W(j\omega) = \frac{\tau j\omega}{1+\tau j\omega}$, contains the previous one when $\delta_u(j\omega) = -1$ (We could also increase the weight $W(j\omega)$ somewhat to have some extra tolerance). The advantage of this approach is that effects that are not explained even by our more complex model might still be covered by the uncertainty. For examples and motivation of these descriptions, refer to [5].

An important observation is that we have dealt separately with two issues in steps 1 and 3. Nonlinearity was only considered in the *static* model, and unmodeled dynamics in the linearized model. One could think of "covering" both effects at once, but this would require a first principles model that dealt with both issues, which is usually not readily available. This highlights another important fact about models in physical systems: we often only have partial, not entirely compatible models and engineering judgement is used to decide how to derive conclusions from them.

Step 4: Disturbances

The previous analysis assumed that the system is "at rest" when $v_S = 0$, but other unmodeled effects will result in nonzero values for the (incremental) variables; we will consider two here.

The first is thermal noise; all electromagnetic devices exhibit this phenomenon, attributed to vibrations of charged particles in the material. The result can be modeled as a disturbance voltage source in the circuit, which condenses the total effect of these fluctuations. As to the characteristics of this voltage variable, the most commonly used model is a white, gaussian random process. In the example, the thermal noise effect in the transistor will be included by adding an input noise variable v_n to the voltage v_B.

A second source of disturbances is more deterministic in nature. The constant voltage source V_{CC} is typically implemented by a rectification of an alternating current; the result is not perfect and a certain "ripple" voltage remains, typically periodic. This effect can be included as an additive voltage v_d in the transistor output.

Uncertain model

The final model which incorporates all the sources of uncertainty described above is depicted in block diagram form in Figure 9.

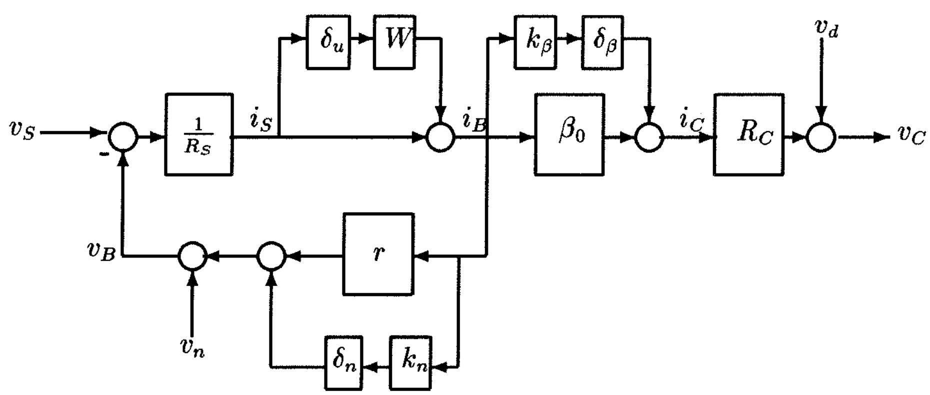


Figure 9: Model with uncertainty

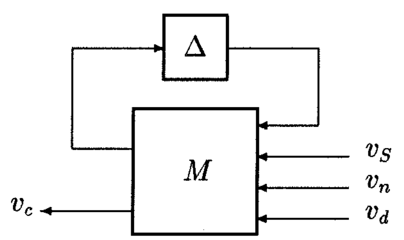
We have outlined above the various descriptions of uncertainty that arise naturally in modeling. We hope to have conveyed a humble outlook on mathematical models and their relationship with reality: far from being exact "physical laws", mathematical models for control engineering are rough approximations to help us deal with very complex problems.

This realization is the starting point for robust control: the understanding that to have an impact on real problems, the role of control theory is not to design "optimal" controllers for ideal mathematical objects, but to provide tools with which engineers can consider issues of robustness of control systems under the inevitable sources of uncertainty.

This brings us to a fundamental challenge: to unify the descriptions of uncertainty in a simple common mathematical language, understandable by the practicing control engineer.

3.3 A unified framework for uncertainty

The previous decades have shown intensive research in this direction. We will concentrate in what follows in one proposed paradigm for uncertainty descriptions which appears to have value as a unifying scheme: the linear fractional transformation (LFT) approach. We have already virtually adopted this approach by obtaining a diagram with uncertainty blocks such as Figure 9 (this attests to the naturalness of LFTs). By isolating the uncertainty blocks we can redraw our block diagram as in Figure 10.



where $\Delta = \boxed{\begin{matrix} \delta_\beta & 0 & 0 \\ 0 & \delta_u & 0 \\ 0 & 0 & \delta_n \end{matrix}}$

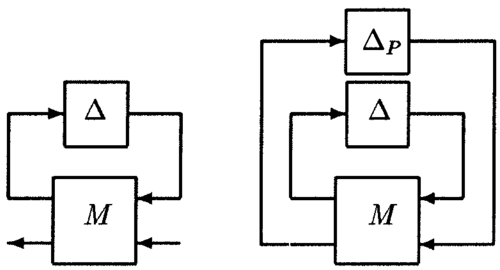
Figure 10: LFT representation of uncertain system

System matrix M captures the information of the nominal model as well as how the uncertainty blocks enter the system; the disturbances are inputs to M. Δ is a *structured* uncertainty operator: it is block diagonal, with blocks that are not all the same. In fact, δ_β is constrained to be a real constant, δ_u a dynamical, linear time invariant block, and δ_n a nonlinear operator.

The above provides a simple setup in which to pose robustness issues: robust stability of an uncertain system can be stated as the question of whether there exists a perturbation Δ, $||\Delta|| \leq 1$, that causes the closed loop system to be unstable.

The next step is to specify performance objectives in this setup. This can be done by redefining outputs of our system as signals which must be made "small" (e.g., errors between command signals and outputs we wish to control). The inputs will be the disturbances and command signals, and we can state the performance objective as making the gain between inputs and outputs small. The measure of gain (i.e. the system norm, typical examples of which are the $\mathcal{H}_2$ $\mathcal{H}_\infty$, and $\mathcal{L}_1$ norms) will depend on the characterization and choice of norms in the signals (for a description of how these norms arise, see [4]).

Robust performance can then be stated as the property of having small system gain from inputs to outputs for the worst case in the uncertainty block. In this respect, particularly attractive as performance measures are those in which the objective function is an induced norm (worst-case gain on a ball of signals in some space). The reason is that in this case performance can be restated as a stability problem via a small gain theorem artifice: in Figure 11, the problem of ensuring that system a) is robustly stable and that (for instance) the $\mathcal{L}_2$-induced norm from inputs to outputs is less than 1 can be restated as the robust stability of system b), where Δ_P is an arbitrary operator of $\mathcal{L}_2$-induced norm less than 1.



a) *b)*

Figure 11: Robust Performance restated as robust stability

Finally, controllers can be easily included in this setup; if a controller is interconnected to a block diagram as in Figure 9, the result is that matrix M in Figure 10 is itself an LFT on a controller K. The synthesis problem in robust control is to design a controller such that the closed loop satisfies robust performance conditions.

A considerable amount of research effort has gone into developing analysis and synthesis techniques which allow for rich uncertainty structures. A (by no means exhaustive) selection of references which adhere to the particular point of view in this section is [6],[14],[31] and [4].

3.4 Conclusion

We have reviewed the issue of uncertainty in system models for control, surveying different sources of error and usual simple ways of describing them. At the component level, descriptions are somewhat crude and simplicity is obtained at the price of some conservatism, but simplicity is essential for these results to be easily transferred to technology. The LFT framework appears to be a simple setup where issues of interconnection can be addressed, and uncertainty in subsystems can be transferred to a structured uncertainty in the total system without adding further conservatism.

There is, however, one incomplete aspect to the framework as described above: it is implicitly assumed that components are input-output operators subject to cascade and feedback interconnections. As argued previously in this paper, behavioral equations are more natural models at the component level, and interconnections are better performed in this environment. This leads to the natural next step, which is to extend the issues of uncertainty and the LFT framework to the behavioral setup.

4 What Lies Ahead

From the previous discussions, it should be clear that behavioral equations are a natural way to describe dynamical systems, especially when considering issues of modeling and interconnection. A substantial body of research has laid out the theoretical foundations of the behavioral paradigm. The purpose of this session is to bridge the gap between this theory and the more mainstream issues of control technology, in particular addressing the issues of uncertainty, which are a crucial part of any control strategy.

An attempt at combining behavioral equations and model uncertainty has been made by the authors in [3] by expressing behavioral systems as kernels of operators obtained by an LFT between a constant matrix and an uncertainty structure:

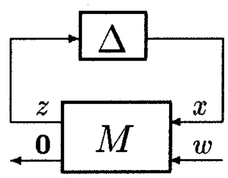


Figure 12: Kernel Representation of Uncertain Dynamical System

The above leads to the definition of an *uncertain dynamical system.* Methods of interconnecting models of these systems are then developed. Furthermore, a method for obtaining input-output maps from behavioral descriptions of uncertain dynamical systems is outlined, which makes possible the use of existing robust control methods. The authors believe that [3] establishes a foundation for subsequent research in the field of uncertain behaviors.

An important issue in system modeling and controller design is that of model reduction. Model reduction is performed when it is desired to reduce the complexity of a model, while at the same time guaranteeing error bounds on the reduced model. Advantages of working with reduced order models include better numerical properties and faster algorithms. Almost all existing model reduction procedures take the input-output standpoint. The ones that don't are strictly one dimensional in nature. In [2], however, a method of model reducing uncertain behavioral models of the type introduced in [3] is presented. Machinery for gap-metric model reduction and multidimensional model reduction using Linear Matrix Inequalities is extended to these models. The result is a systematic method for reducing the complexity of uncertain components in hierarchically developed models, which exhibit behaviors "similar" to the original models.

So far, the emphasis has been on using the behavioral framework for modeling and interconnection purposes, without any mention of the actual control design. It has been tacitly assumed that existing control methods could be used to this end. We are forced to ask ourselves, however, whether control design can be performed directly at the behavioral level. In [30], a formulation of the classical LQ-problem is given which completely fits into the behavioral setting. Similar extensions to the H_∞ and L_1 problems are being sought. Once the solutions to these problems have been realized, the next task will be to extend them to uncertain behavioral models.

Another attractive feature of the kernel representation as shown in Figure 12 is presented in [16]. As pointed out in the previous section, models for disturbances can either be stochastic or deterministic, the latter being more easy to combine with other sources of uncertainty by means of a performance block. This, however, can be done only for disturbances defined by norm constraints. In [16], deterministic characterizations of *white* signals are presented, and it is shown that they can be converted to an uncertain behavioral equation of the type depicted in Figure 12. This stimulates a direction of future research, which consists of extending robustness analysis and synthesis techniques to uncertain behaviors; this would give solutions to the white noise rejection problem subject to plant uncertainty.

In another line of work, the potential benefits of the behavioral framework for black box modeling, already considered in [28] and [1], are exploited in [18]; a model of restricted complexity is obtained from a vector time series, where the error is measured by the l_2-distance between the time series and the system behavior.

In order for theoretical developments in the behavioral field to have an impact in technology, software tools for the practicing engineer must be made available. In existing software, we are forced to manipulate component descriptions into input-output format. The advantages of the behavioral paradigm for modeling and interconnection can be fully exploited by software that deals with kernel representations of components. Such a software tool is Omola, presented in [12] together with the corresponding simulation tool, Omsim, which is based on solvers for the differential-algebraic equations (DAEs) which are characteristic of kernel representations.

It is the hope of the authors that the importance of connecting the seemingly disparate subjects of behaviors, modeling, and robust control has been demonstrated. The research described above is a start at exploring this connection. The next step is to address the various open questions which have been posed throughout this discussion, in particular the extension of existing control design techniques to behavioral systems.

A major challenge in the field of control theory (in the broadest sense), is the development of an environment in which modeling, robust control design, and simulation can interact naturally; where an engineer can develop models from first principles or by system identification, interconnect them, model reduce them, perform controller design and simulate, and still maintain coherent descriptions at all levels. It appears that the behavioral paradigm will play an important role in the achievement of this goal.

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