The Rank One Mixed μ Problem and "Kharitonov-Type" Analysis

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Abstract

The general mixed μ problem has been shown to be NP hard, so that the exact solution of the general problem is computationally intractable, except for small problems. In this paper we consider not the general problem, but a particular special case of this problem, the rank one mixed μ problem. We show that for this case the mixed μ problem is equivalent to its upper bound (which is convex), and it can in fact be computed easily (and exactly). This special case is shown to be equivalent to the so called "affine parameter variation" problem (for a polynomial with perturbed coefficients) which has been examined in detail in the literature, and for which several celebrated "Kharitonov-type" results have been proven.

1 Introduction

It is now known that the general mixed μ problem is NP hard, and this strongly suggests that the exact solution of the general problem is computationally intractable, except for small problems [3]. In this paper we consider not the general problem, but a particular special case of this problem, the rank one mixed μ problem. The rank one mixed μ problem is that of computing $\mu_{\mathcal{K}}(M)$ for $M = uv^*$ with $u, v \in \mathbb{C}^n$.

Note that imposing the condition that M be a dyad is a strong restriction, and this will limit the applicability of this analysis to real engineering applications. The reason for our interest in this particular problem is that it turns out that a special case of the rank one mixed μ problem is equivalent to the so called "affine parameter variation" problem (for a polynomial with perturbed coefficients) which has been examined in detail in the literature, and for which several celebrated "Kharitonov-type" results have been proven. These results provide exact robust stability tests for such problems, with respect to real parametric uncertainty (see [12] and the references therein).

It will be seen that this special μ problem does indeed avoid the NP hardness issues of the general problem. We will show that for such problems μ equals its upper bound, which is a convex problem. In fact it is not even necessary to solve a convex optimization problem, and we are able to obtain a complete solution to the rank one problem, in terms of quantities which are easily computed. This solution also provides us with the means to examine the properties of the rank one problem, and arrive at a number of interesting results (for example an "edge" result holds for rank one mixed μ problems).

2 Notation and Preliminaries

The notation used here is fairly standard and is essentially taken from [6] and [15]. For any square complex matrix M we denote the complex conjugate transpose by M^*. The largest singular value and the structured singular value are denoted by $\overline{\sigma}(M)$ and $\mu_{\mathcal{K}}(M)$ respectively. The spectral radius is denoted $\rho(M)$ and $\rho_R(M) = max\{|\lambda| : \lambda \textit{ is a real eigenvalue of } M\}$, with $\rho_R(M) = 0$ if M has no real eigenvalues. For a Hermitian matrix M, then $\overline{\lambda}(M)$ and $\lambda_{min}(M)$ denote the largest and smallest (real) eigenvalues respectively. For any complex vector x, then x^* denotes the complex conjugate transpose and $|x|$ the Euclidean norm. We denote the $k \times k$ identity matrix and zero matrix by I_k and O_k respectively.

The definition of μ is dependent upon the underlying block structure of the uncertainties, which is defined as follows. Suppose we have a matrix $M \in \mathbb{C}^{n \times n}$ and three non-negative integers m_r, m_c, and m_C (with $m := m_r + m_c + m_C \leq n$) which specify the number of uncertainty blocks of each type. Then the block structure $\mathcal{K}(m_r, m_c, m_C)$ is an m-tuple of positive integers

$$\mathcal{K} = (k_1, \ldots, k_{m_r}, k_{m_r+1}, \ldots, k_{m_r+m_c}, k_{m_r+m_c+1}, \ldots, k_m) \tag{1}$$

This m-tuple specifies the dimensions of the perturbation blocks, and we require $\sum_{i=1}^{m} k_i = n$ in order that these dimensions are compatible with M. This determines the set of allowable perturbations, namely define

$$\begin{aligned} X_{\mathcal{K}} = \{\Delta = block\ diag(\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \ldots, \Delta_{m_C}^C) : \\ \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \Delta_i^C \in \mathbb{C}^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}}\} \end{aligned} \tag{2}$$

Note that $X_{\mathcal{K}} \in \mathbb{C}^{n \times n}$ and that this block structure is sufficiently general to allow for (any combination of) repeated real scalars, repeated complex scalars, and full complex blocks. The purely complex case corresponds to $m_r = 0$, and the purely real case to $m_c = m_C = 0$.

Note also that all the results which follow are easily generalized to the case where the full complex blocks need not be square, and the blocks may come in any order. We make these restrictions in (2) purely for notational convenience.

Definition 1 ([5]) *The structured singular value, $\mu_{\mathcal{K}}(M)$, of a matrix $M \in \mathbb{C}^{n \times n}$ with respect to a block structure $\mathcal{K}(m_r, m_c, m_C)$ is defined as*

$$\mu_{\mathcal{K}}(M) = \left(\min_{\Delta \in X_{\mathcal{K}}} \{\overline{\sigma}(\Delta) : det(I - \Delta M) = 0\} \right)^{-1} \tag{3}$$

with $\mu_{\mathcal{K}}(M) = 0$ if no $\Delta \in X_{\mathcal{K}}$ solves $det(I - \Delta M) = 0$.

In order to develop the relevant theory we need to define some sets of block diagonal scaling matrices (which, like μ itself, are dependent on the underlying block structure).

$$\mathbb{B}X_{\mathcal{K}} = \{\Delta \in X_{\mathcal{K}} : \overline{\sigma}(\Delta) \leq 1\} \tag{4}$$

$$\mathcal{Q}_{\mathcal{K}} = \{\Delta \in X_{\mathcal{K}} : \delta_i^r \in [-1\ 1], \delta_i^{c*}\delta_i^c = 1, \Delta_i^{C*}\Delta_i^C = I_{k_{m_r+m_c+i}}\} \tag{5}$$

$$\begin{aligned} \hat{\mathcal{D}}_{\mathcal{K}} = \{block\ diag(e^{\mathbf{j}\theta_1} D_1, \ldots, e^{\mathbf{j}\theta_{m_r}} D_{m_r}, D_{m_r+1}, \ldots, D_{m_r+m_c}, d_1 I_{k_{m_r+m_c+1}}, \ldots, d_{m_C} I_{k_m}) : \\ \theta_i \in [-\frac{\pi}{2}\ \frac{\pi}{2}], 0 < D_i = D_i^* \in \mathbb{C}^{k_i \times k_i}, 0 < d_i \in \mathbb{R}\} \end{aligned} \tag{6}$$

$$\tilde{\mathcal{D}}_{\mathcal{K}} = \{block\ diag(D_1, \ldots, D_{m_r+m_c}, d_1 I_{k_{m_r+m_c+1}}, \ldots, d_{m_C} I_{k_m}) : \\ D_i = D_i^* \in \mathbb{C}^{k_i \times k_i}, d_i \in \mathbb{R}\} \tag{7}$$

$$\mathcal{D}_{\mathcal{K}} = \{D \in \tilde{\mathcal{D}}_{\mathcal{K}} : D > 0\} = \{D \in \hat{\mathcal{D}}_{\mathcal{K}} : \theta_i = 0, i = 1, \ldots, m_r\} \tag{8}$$

$$\mathcal{G}_{\mathcal{K}} = \{block\ diag(G_1, \ldots, G_{m_r}, O_{k_{m_r+1}}, \ldots, O_{k_m}) : G_i = G_i^* \in \mathbb{C}^{k_i \times k_i}\} \tag{9}$$

We introduce one further piece of notation. Suppose $M \in \mathbb{C}^{n \times n}$ has an eigenvalue λ with right and left eigenvectors x and y respectively. Then partition x and y compatibly with the block structure as

$$x = \begin{bmatrix} x_{r_1} \\ \vdots \\ x_{r_{m_r}} \\ x_{c_1} \\ \vdots \\ x_{c_{m_c}} \\ x_{C_1} \\ \vdots \\ x_{C_{m_C}} \end{bmatrix}, \qquad y = \begin{bmatrix} y_{r_1} \\ \vdots \\ y_{r_{m_r}} \\ y_{c_1} \\ \vdots \\ y_{c_{m_c}} \\ y_{C_1} \\ \vdots \\ y_{C_{m_C}} \end{bmatrix} \tag{10}$$

where $x_{r_i}, y_{r_i} \in \mathbb{C}^{k_i}$, $x_{c_i}, y_{c_i} \in \mathbb{C}^{k_{m_r+i}}$, $x_{C_i}, y_{C_i} \in \mathbb{C}^{k_{m_r+m_c+i}}$. These will be referred to as the "block components" of x and y, and we define the "non-degeneracy" assumption to be that for every i (in the appropriate set), $|y_{r_i}^* x_{r_i}| \neq 0, |y_{c_i}^* x_{c_i}| \neq 0, |y_{C_i}||x_{C_i}| \neq 0$.

3 "Kharitonov-Type" Analysis

Before embarking on a study of the rank one mixed μ problem, we first place this problem in context, by considering the "affine parameter variation" problem, for a polynomial with perturbed coefficients. This formulation of the "affine parameter variation" problem is fairly standard, and is taken from [10].

Consider a real monic polynomial in the complex variable s, whose coefficients are affine functions of a real vector of uncertainties, $k \in \mathbb{R}^m$

$$p(s,k) = s^n + a_1(k)s^{n-1} + a_2(k)s^{n-2} + \ \ldots \ + a_n(k) \tag{11}$$

where $a_i(k)$ for $i = 1, \ldots, n$ are affine functions of k, i. e., there exists $F \in \mathbb{R}^{n \times m}$ and $g \in \mathbb{R}^m$ such that

$$a(k) \doteq [a_1(k) \quad a_2(k) \quad \ldots \quad a_n(k)]^T = Fk + g \tag{12}$$

Thus we can rewrite this set of polynomials as

$$p(s,k) = s^n + [s^{n-1} \quad s^{n-2} \quad \ldots \quad 1](Fk + g) \tag{13}$$

Since this polynomial will typically be the closed loop characteristic polynomial of some uncertain system, we will say that it is stable if it has all its roots in the open left half plane. We assume nominal stability, i. e., $p(s,0)$ is stable. Thus in order to check robust stability we can show by simple continuity arguments that it suffices to check that the polynomial has no roots on the imaginary axis for any k. Assume in the following analysis that we are considering points on the imaginary axis, i. e., $s = \mathbf{j}w$ where $w \in \mathbb{R}$. Note that since we have nominal stability $p(s,0) \neq 0$ for all $s = \mathbf{j}w$. Thus we have

$$p(s,0) = s^n + [s^{n-1} \quad s^{n-2} \quad \dots \quad 1]g \;\neq\; 0 \tag{14}$$

Now note that we have a root of the uncertain polynomial on the imaginary axis iff for some $s = \mathbf{j}w$ and some $k \in \mathbb{R}^m$, $p(s,k) = 0$. This can be reformulated as

$$\begin{aligned} p(s,k) = 0 \quad &\longleftrightarrow \quad [s^{n-1} \quad s^{n-2} \quad \dots \quad 1]Fk = -s^n - [s^{n-1} \quad s^{n-2} \quad \dots \quad 1]g \\ &\longleftrightarrow \quad v^*k = 1 \\ &\longleftrightarrow \quad 1 - v^*k = 0 \end{aligned} \tag{15}$$

where $v \in \mathbb{C}^m$ is given by

$$\begin{aligned} v &= \left(\frac{1}{-s^n - [s^{n-1} \quad s^{n-2} \quad \dots \quad 1]g}\right) F^T [s^{n-1} \quad s^{n-2} \quad \dots \quad 1]^* \\ &= \left(\frac{-1}{p(s,0)}\right) F^T [s^{n-1} \quad s^{n-2} \quad \dots \quad 1]^* \end{aligned} \tag{16}$$

But now define the quantities $\Delta \in \mathbb{R}^{m \times m}$ and $u \in \mathbb{R}^m$ by

$$\Delta \doteq diag(k_1, \dots, k_m) \tag{17}$$

$$u \doteq (1 \quad \dots \quad 1)^T \tag{18}$$

and we have that $\Delta u = k$. Thus an equivalent condition for the existence of an imaginary axis root of the uncertain polynomial is given by

$$\begin{aligned} 1 - v^*k = 0 \quad &\longleftrightarrow \quad 1 - v^*\Delta u = 0 \\ &\longleftrightarrow \quad det(I_m - \Delta u v^*) = 0 \\ &\longleftrightarrow \quad det(I_m - \Delta M) = 0 \end{aligned} \tag{19}$$

where $M \in \mathbb{C}^{m \times m}$ is the dyad $M = uv^*$. Checking this condition is exactly a rank one μ problem and thus we see that the "affine parameter variation" problem, for a polynomial with perturbed coefficients, is a special case of the rank one mixed μ problem (with only real perturbations).

It is possible to consider a number of different stability problems arising from this set-up, by allowing for different stability regions, and different norms to measure the size of k. We will only be interested in the case where stability is associated with all the roots in the open left half plane, and we use $|k|_\infty \doteq \max_{i \leq m} |k_i|$ to measure the size of k. In this case we find that we can use the standard definition of μ, and we are required to compute the peak value across frequency of μ, for a transfer matrix which is rank one. The treatment of other norms/regions is discussed in [4].

A number of different stability results have been presented for this type of problem. One of the strongest motivations for pursuing these problems was provided by Kharitonov's celebrated result for "interval polynomials" [7]. This is a special case of the above setting where one further restricts the uncertainty description to be of the form

$$a_i(k) = a_i + b_i k_i \quad \texttt{for} \quad i = 1, \dots, m \tag{20}$$

where $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$ and $m = n$. Thus the coefficients of the polynomial are independent of each other, and known only to lie within certain intervals. For this problem it was shown in [7] that one need only check four specific polynomials to establish stability of the whole family. This is clearly a polynomial time computation, and we have restricted the problem sufficiently to beat the NP-hardness of the general problem. In doing so however we have placed quite severe restrictions on the allowable problem class, and so the applicability of the result to engineering problems is rather limited.

If we consider the "affine parameter" case, then it was shown in [1] that it suffices to check stability of the edges of the parameter space, i. e., we may take every element of k except one to be at an extremal value of it's allowed range. Note that this requires checking a combinatoric number of edges, so that even this increase in the generality of the problem produces dramatic increases in computation. Exact results for this type of problem typically involve checking the vertices or edges of some polytope in the parameter space, and hence involve exponential growth in computation (see [12]). If one is prepared to allow a frequency search then this exponential growth can be avoided (see [10]). This can also be seen from the framework we will develop here, since these problems can all be tackled as rank one μ problems, which we will see can be easily solved.

Thus we will find that it is possible to develop exact robust stability tests, in the μ or polynomial frameworks, for this type of problem. Of course we must note once more that the applicability of this rank one μ analysis is rather limited, and the fact that the general problem is NP hard strongly suggests that results for this case cannot be usefully extended to the general case. This is the reason why the "Kharitonov-type" analysis methods do not extend to the more general "multilinear" or "polynomial" cases (which correspond to more general μ problems), and one is forced to use approximate and/or iterative methods (see [11, 16] for example).

4 Equivalence with the Upper Bound

This section is devoted to proving the main result of this paper, which is that for a rank one μ problem, the upper bound always achieves μ, regardless of the block structure. A preliminary version of this result was proven in [14], where additional assumptions were imposed to make the proof fairly simple. Here we will not make any such assumptions on the problem, and it turns out that this makes the proof substantially more difficult. First we recall the mixed μ upper bound from [6], stated here in a slightly different form.

Theorem 1 ([6]) *For any matrix $M \in \mathbb{C}^{n \times n}$, and any compatible block structure $\mathcal{K}$, suppose α_* is the result of the minimization problem*

$$\alpha_* = \inf_{\substack{D \in \mathcal{D}_{\mathcal{K}} \\ G \in \mathcal{G}_{\mathcal{K}}}} \left[\min_{\alpha \in \mathbb{R}} \{ \alpha : (M^* D M + j(GM - M^* G) - \alpha D) \leq 0 \} \right] \tag{21}$$

then an upper bound for μ is given by

$$\mu_{\mathcal{K}}(M) \leq \sqrt{\max(0, \alpha_*)} \tag{22}$$

Note that this upper bound is a convex minimization problem so that we can compute the global minimum, and algorithms for this computation have been developed (see [17, 2]).

Theorem 2 *Suppose we have a rank one matrix* $M \in \mathbb{C}^{n\times n}$*, then for any block structure,* $\mathcal{K}$*,* $\mu_{\mathcal{K}}(M)$ *equals its upper bound from theorem 1.*

This theorem gives us the means to compute rank one μ problems exactly, since it says that it is equivalent to consider the upper bound problem, which is convex (in fact we will see later that it is not even necessary to resort to convex programming methods to solve this problem). Before tackling the proof of this theorem, we need a few preliminary results.

Lemma 1 *Suppose we have matrices* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{n\times n}$*, with* $A \leq 0$*. Define* $\mathcal{S} = \{x \in \mathbb{C}^n : x^*Ax = 0, |x| \neq 0\}$*. Then we have that* $A + tB < 0$ *for sufficiently small* $0 < t \in \mathbb{R}$ *iff either* $\mathcal{S} = \emptyset$*, or* $x^*Bx < 0$ *for all* $x \in \mathcal{S}$*.*

Proof: $(\rightarrow)$ Since we have that for all $x \neq 0$

$$x^*(A+tB)x = x^*Ax + tx^*Bx < 0$$

with $t > 0$, then if for any $x \neq 0$ we have $x^*Ax = 0$, for that x we must also have $x^*Bx < 0$.

$(\leftarrow)$ If $\mathcal{S} = \emptyset$ then $A < 0$, and so $A + tB < 0$ for sufficiently small $t > 0$ by a simple continuity argument. Suppose instead that $\mathcal{S} \neq \emptyset$, but $x^*Bx < 0$ for all $x \in \mathcal{S}$. Define $\hat{\mathcal{S}} = \mathcal{S} \cap \mathcal{B}$ where $\mathcal{B} = \{x : |x| = 1\}$. Then $\hat{\mathcal{S}} \subset \mathcal{S}$ is compact and so by continuity there exists a set $\mathcal{V} \supset \hat{\mathcal{S}}$, which is open in $\mathcal{B}$, with $x^*Bx < 0$ for all $x \in \mathcal{V}$. Thus we have

$$x^*(A+tB)x = x^*Ax + tx^*Bx < 0$$

for all $x \in \mathcal{V}$, for *any* $t > 0$. If $\mathcal{B}\backslash\mathcal{V} = \emptyset$ we are done immediately, so assume $\mathcal{B}\backslash\mathcal{V} \neq \emptyset$. Now $\mathcal{B}\backslash\mathcal{V}$ is compact and so both x^*Ax and x^*Bx achieve maxima on $\mathcal{B}\backslash\mathcal{V}$. Suppose we have

$$\max_{x\in\mathcal{B}\backslash\mathcal{V}} x^*Ax = -\alpha$$
$$\max_{x\in\mathcal{B}\backslash\mathcal{V}} x^*Bx = \beta$$

with $\alpha > 0$ since $\mathcal{B}\backslash\mathcal{V} \cap \hat{\mathcal{S}} = \emptyset$. If $\beta \leq 0$ then $x^*(A+tB)x < 0$ for all $x \in \mathcal{B}\backslash\mathcal{V}$ for *any* $t > 0$ and we are done, so assume $\beta > 0$. Then we have

$$x^*(A+tB)x = x^*Ax + tx^*Bx \leq -\alpha + t\beta < 0 \quad \texttt{for} \quad t < \frac{\alpha}{\beta}$$

Thus we have $x^*(A+tB)x < 0$ on $\mathcal{B}\backslash\mathcal{V}$ for sufficiently small $t > 0$. Combining this with our earlier result we have that for sufficiently small $t > 0$, $x^*(A+tB)x < 0$ for all $x \in \mathcal{B}$, or in other words $A + tB < 0$. □

Lemma 2 *Suppose we have a rank one matrix* $M = uv^*$*, with* $u, v \in \mathbb{C}^n$*, and a block structure* $\mathcal{K}$*. Then there exists a sequence of matrices* $D^j \in \mathcal{D}_{\mathcal{K}}$ *such that the following is true:*

1. *Define* $u^j \doteq D^j u$*,* $v^j \doteq D^{j-1} v$ *and the following limits exist:*

$$\begin{aligned} \lim_{j\to\infty} u^j &= \overline{u} \\ \lim_{j\to\infty} v^j &= \overline{v} \end{aligned} \tag{23}$$

2. *Partition $u, v, \overline{u}, \overline{v}$ compatibly with the block structure as in (10). Then we have that the following is true:*

$$\begin{aligned} \overline{v}^*_{r_i}\overline{u}_{r_i} &= v^*_{r_i}u_{r_i} \quad \texttt{for} \quad i = 1,\ldots,m_r \\ \overline{v}^*_{c_i}\overline{u}_{c_i} &= v^*_{c_i}u_{c_i} \quad \texttt{for} \quad i = 1,\ldots,m_c \\ \overline{v}^*_{C_i}\Delta\overline{u}_{C_i} &= v^*_{C_i}\Delta u_{C_i} \quad \texttt{for} \quad i = 1,\ldots,m_C \quad \texttt{for any } \Delta \in \mathbb{C}^{k_{m_r+m_c+i}\times k_{m_r+m_c+i}} \end{aligned} \tag{24}$$

3. *With this notation we also have that*

$$\begin{aligned} |\overline{v}^*_{r_i}\overline{u}_{r_i}| = |\overline{v}_{r_i}|^2 &= |\overline{u}_{r_i}|^2 \quad \texttt{for} \quad i = 1,\ldots,m_r \\ |\overline{v}^*_{c_i}\overline{u}_{c_i}| = |\overline{v}_{c_i}|^2 &= |\overline{u}_{c_i}|^2 \quad \texttt{for} \quad i = 1,\ldots,m_c \\ |v_{C_i}||u_{C_i}| = |\overline{v}_{C_i}||\overline{u}_{C_i}| = |\overline{v}_{C_i}|^2 &= |\overline{u}_{C_i}|^2 \quad \texttt{for} \quad i = 1,\ldots,m_C \end{aligned} \tag{25}$$

4. *Define $M^j \doteq u^j v^{j*}$ and the following limit exists*

$$\lim_{j\to\infty} M^j = \lim_{j\to\infty} D^j M D^{j-1} = \overline{M} \tag{26}$$

5. *We have $\overline{M} = \overline{u}\,\overline{v}^*$, with $\mu_{\mathcal{K}}(\overline{M}) = \mu_{\mathcal{K}}(M)$.*

Proof: Consider first the repeated real scalar blocks, i.e. , u_{r_i}, v_{r_i} for $i = 1,\ldots,m_r$. Suppose first that for some i in this range we have that $|v^*_{r_i}u_{r_i}| \neq 0$. Then we have

$$|v^*_{r_i}u_{r_i}| = \gamma e^{\mathbf{j}\theta}$$

for some $\gamma, \theta \in \mathbb{R}$ with $\gamma > 0$. Thus we have that

$$v^*_{r_i}(e^{-\mathbf{j}\theta}u_{r_i}) = \gamma > 0$$

and so by lemma 4.2 from [9] we can conclude that there exists $D_i = D_i^* > 0$ such that

$$v_{r_i} = e^{-\mathbf{j}\theta}D_i^2 u_{r_i}$$

Choose $D_i^j = D_i$ for all j and so for this block we have for all j

$$|v^j_{r_i}| = |D_i^{-1}v_{r_i}| = |e^{-\mathbf{j}\theta}D_i u_{r_i}| = |D_i u_{r_i}| = |u^j_{r_i}|$$

and furthermore

$$|v^*_{r_i}u_{r_i}| = |v^*_{r_i}e^{-\mathbf{j}\theta}u_{r_i}| = |v^*_{r_i}D_i^{-2}v_{r_i}| = |D_i^{-1}v_{r_i}|^2 = |v^j_{r_i}|^2$$

Finally note that

$$v^{j*}_{r_i}u^j_{r_i} = v^*_{r_i}D_i^{-1}D_i u_{r_i} = v^*_{r_i}u_{r_i}$$

For this block then properties 1, 2, 3 all hold.

Now suppose that for some $i \le m_r$ we have $|v^*_{r_i}u_{r_i}| = 0$, or in other words v_{r_i} orthogonal to u_{r_i} (if we have $|v_{r_i}| = 0$ and/or $|u_{r_i}| = 0$ then see the treatment for the full complex blocks later). Thus we can choose a Hermitian positive definite matrix D_i^j with u_{r_i} as an eigenvector corresponding to the smallest eigenvalue of D_i^j, and v_{r_i} as an eigenvector corresponding to the largest eigenvalue. Now simply choose such D_i^j with

$$\begin{array}{lcl} \overline{\lambda}(D_i^j) \uparrow \infty & \texttt{as} & j \uparrow \infty \\ \lambda_{min}(D_i^j) \downarrow 0 & \texttt{as} & j \uparrow \infty \end{array} \tag{27}$$

and we have that

$$\begin{array}{rcccl} \lim_{j\to\infty} |v_{r_i}^j| & = & \lim_{j\to\infty} |D_i^{j-1} v_{r_i}| & = & 0 \\ \lim_{j\to\infty} |u_{r_i}^j| & = & \lim_{j\to\infty} |D_i^j u_{r_i}| & = & 0 \end{array} \tag{28}$$

so that properties $1, 2, 3$ hold once more. The treatment for the repeated complex scalar blocks (i.e., u_{c_i}, v_{c_i} for $i = 1, \ldots, m_c$) is identical.

Now consider the full complex blocks, i.e., u_{C_i}, v_{C_i} for $i = 1, \ldots, m_C$. First consider a block where (for some i in the appropriate range) $|v_{C_i}||u_{C_i}| \neq 0$. Choose the scalar d_i as

$$d_i = \sqrt{\frac{|v_{C_i}|}{|u_{C_i}|}} \tag{29}$$

and choose $D_i^j = d_i I_{k_{m_r+m_c+i}}$ for all j. Then for this block we have that

$$\begin{array}{l} |u_{C_i}^j|^2 = |d_i u_{C_i}|^2 = d_i^2 |u_{C_i}|^2 = \dfrac{|v_{C_i}|}{|u_{C_i}|} |u_{C_i}|^2 = |u_{C_i}||v_{C_i}| \\ |v_{C_i}^j|^2 = |\dfrac{1}{d_i} v_{C_i}|^2 = \dfrac{1}{d_i^2} |v_{C_i}|^2 = \dfrac{|u_{C_i}|}{|v_{C_i}|} |v_{C_i}|^2 = |u_{C_i}||v_{C_i}| \end{array} \tag{30}$$

and furthermore for any $\Delta \in \mathbb{C}^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}}$ we have

$$v_{C_i}^* \Delta u_{C_i} = \left(\frac{1}{d_i} v_{C_i}\right)^* \Delta(d_i u_{C_i}) = v_{C_i}^{j*} \Delta u_{C_i}^j$$

Note that this implies

$$|v_{C_i}^j||u_{C_i}^j| = |v_{C_i}||u_{C_i}|$$

so that properties $1, 2, 3$ all hold for this block.

Suppose now that for some i we have $|v_{C_i}||u_{C_i}| = 0$, and hence $|u_{C_i}| = 0$ and/or $|v_{C_i}| = 0$. If both are zero simply choose $D_i^j = I_{k_{m_r+m_c+i}}$ for all j, and it is easy to check that properties $1, 2, 3$ all (trivially) hold. If not then choose $D_i^j = d_i^j I_{k_{m_r+m_c+i}}$ where d_i^j is chosen to satisfy

$$\begin{array}{lclll} d_i^j \uparrow \infty & \texttt{as} & j \uparrow \infty & \texttt{if} & |u_{C_i}| = 0 \\ d_i^j \downarrow 0 & \texttt{as} & j \uparrow \infty & \texttt{if} & |v_{C_i}| = 0 \end{array} \tag{31}$$

In either case we have that

$$\begin{array}{rcccl} \lim_{j\to\infty} |v_{C_i}^j| & = & \lim_{j\to\infty} |\frac{1}{d_i^j} v_{C_i}| & = & 0 \\ \lim_{j\to\infty} |u_{C_i}^j| & = & \lim_{j\to\infty} |d_i^j u_{C_i}| & = & 0 \end{array} \tag{32}$$

so that properties $1,2,3$ hold once more. Thus stacking up the blocks we have constructed we obtain our $D^j \in \mathcal{D}_\mathcal{K}$ satisfying properties $1,2,3$.

Now note that

$$M^j \doteq u^j v^{j*} = D^j u v^* D^{j-1} = D^j M D^{j-1}$$

is a product of convergent sequences, u^j and v^j, and hence converges, so we have property 4. Furthermore by standard properties of limits we have that $\overline{M} = \overline{u}\,\overline{v}^*$. Note finally that employing properties $1,2,3,4$ we obtain that for any $\Delta \in X_\mathcal{K}$

$$\begin{aligned}
det(I_n - \Delta\overline{M}) &= det(I_n - \Delta\overline{u}\,\overline{v}^*) \\
&= 1 - \overline{v}^*\Delta\overline{u} \\
&= 1 - \sum_{i=1}^{m_r} \delta_i^r \overline{v}_{r_i}^* \,\overline{u}_{r_i} - \sum_{i=1}^{m_c} \delta_i^c \overline{v}_{c_i}^* \,\overline{u}_{c_i} - \sum_{i=1}^{m_C} \overline{v}_{C_i}^* \Delta_i^C \overline{u}_{C_i} \\
&= 1 - \sum_{i=1}^{m_r} \delta_i^r v_{r_i}^* \, u_{r_i} - \sum_{i=1}^{m_c} \delta_i^c v_{c_i}^* \, u_{c_i} - \sum_{i=1}^{m_C} v_{C_i}^* \Delta_i^C u_{C_i} \\
&= det(I_n - \Delta M)
\end{aligned} \tag{33}$$

and hence $\mu_\mathcal{K}(\overline{M}) = \mu_\mathcal{K}(M)$, which is property 5. $\square$

Remarks: Note the the sequence of matrices $D^j \in \mathcal{D}_\mathcal{K}$ satisfies the following:

$$\begin{aligned}
\lim_{j\to\infty} \overline{\sigma}(D^j M D^{j-1}) &= \overline{\sigma}(\overline{M}) = |\overline{u}||\overline{v}| = |\overline{u}|^2 \\
&= \sum_{i=1}^{m_r} |\overline{u}_{r_i}|^2 + \sum_{i=1}^{m_c} |\overline{u}_{c_i}|^2 + \sum_{i=1}^{m_C} |\overline{u}_{C_i}|^2 \\
&= \sum_{i=1}^{m_r} |\overline{v}_{r_i}^* \,\overline{u}_{r_i}| + \sum_{i=1}^{m_c} |\overline{v}_{c_i}^* \,\overline{u}_{c_i}| + \sum_{i=1}^{m_C} |\overline{v}_{C_i}||\overline{u}_{C_i}| \\
&= \sum_{i=1}^{m_r} |v_{r_i}^* \, u_{r_i}| + \sum_{i=1}^{m_c} |v_{c_i}^* \, u_{c_i}| + \sum_{i=1}^{m_C} |v_{C_i}||u_{C_i}|
\end{aligned} \tag{34}$$

This final expression is exactly μ for the associated complex μ problem. Therefore this lemma proves that complex μ equals its upper bound for rank one matrices (which is well known), and in the process we explicitly constructed the sequence of scaling matrices that does the job. Thus the sequence $D^j \in \mathcal{D}_\mathcal{K}$ is exactly the optimal scalings from the upper bound of the associated complex μ problem.

Note that this lemma provides us with a μ invariant transformation from M to $\overline{M}$. The point of carrying out this transformation is that property 3 implies that the vectors $\overline{u}, \overline{v}$ of the dyad $\overline{M}$ are perfectly balanced in the sense that each sub block of $\overline{u}$ has the same norm as the corresponding sub block of $\overline{v}$. Consequently we have that for each sub block of $\overline{u}$ and $\overline{v}$ we either satisfy the non degeneracy assumptions, or the corresponding sub blocks of $\overline{u}$ and $\overline{v}$ are *both* identically zero.

Theorem 3 *Suppose we have $M = uv^*$ with $u, v \in \mathbb{C}^n$, and a compatible block structure $\mathcal{K}$, with $\mu_\mathcal{K}(M) > 0$. Partition u, v with respect to this block structure, and assume that for each sub block of u and v we either satisfy the non degeneracy assumptions, or the corresponding sub blocks of u*

and v are both identically zero. Then there exist matrices $Q \in \mathcal{Q}_\mathcal{K}$ and $\hat{Q} \in \mathcal{Q}_\mathcal{K}$ with

$$\hat{Q}_{ij} = \begin{cases} 1 & \texttt{if } i = j \le m_r \texttt{ and } Q_{ij} = 0 \\ Q_{ij} & \texttt{otherwise} \end{cases} \tag{35}$$

together with matrices $D_R \in \tilde{\mathcal{D}}_\mathcal{K}, D_R \ge 0$ and $D_I \in \mathcal{G}_\mathcal{K}$, and a real scalar $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$ such that

$$v = e^{\mathbf{j}\psi}\left(D_R Q + \mathbf{j} D_I \hat{Q}\right) u \tag{36}$$

with $0 < v^* Q u = \mu_\mathcal{K}(M)$.

Proof: The proof of this theorem is essentially an application of the same machinery used to prove theorem 2 in [15]. First suppose that $\mu_\mathcal{K}(M) = 1$. Then by theorem 1 in [15] there exists $Q \in \mathcal{Q}_\mathcal{K}$ with $\rho_R(QM) = 1$. Since we may absorb a factor of $\pm$ into Q we may assume that the eigenvalue achieving $\rho_R(QM)$ is positive. But note that this implies

$$0 = det(I_n - QM) = det(I_n - Quv^*) = 1 - v^* Q u \tag{37}$$

and hence $0 < v^* Q u = 1 = \mu_\mathcal{K}(M)$.

Since M is rank one, so is QM, and hence QM has at most one non-zero eigenvalue (not repeated). Thus the eigenvalue at one is distinct and furthermore we have that

$$\begin{aligned} QM(Qu) = &\ Quv^*Qu &= (Qu) \\ v^*QM = &\ v^*Quv^* &= v^* \end{aligned} \tag{38}$$

Since Qu and v are both non-zero this implies that they are the right and left eigenvectors of QM corresponding to the unity eigenvalue, and furthermore they are normalized with $v^*(Qu) = 1$. Since this eigenvalue is distinct we can differentiate it, and applying the machinery from [15] to $\hat{M} \doteq QM$ we can derive that the following relations hold

$$\begin{aligned} Re(e^{j\psi} q_i^r v_{r_i}^* u_{r_i}) \ge 0 \quad &, \quad i = 1, \ldots, m_r \\ Re(e^{j\psi} q_i^r v_{r_i}^* u_{r_i}) = 0 \quad &, \quad \texttt{if } i \le m_r \texttt{ and } |q_i^r| < 1 \\ e^{j\psi} q_i^c v_{c_i}^* u_{c_i} \in [0\ \infty) \quad &, \quad i = 1, \ldots, m_c \\ Re(e^{j\psi} v_{C_i}^* G_i^C Q_i^C u_{C_i}) \le 0 \quad &, \quad \text{for all } G_i^C \in \mathbb{C}^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}} \\ & \quad\quad \text{with } G_i^C + G_i^{C*} \le 0, \ \ i = 1, \ldots, m_C \end{aligned} \tag{39}$$

for some $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$. Now we may apply lemmas 4.1 and 4.2 from [9] to each sub block to obtain

$$\begin{aligned} v_{r_i} = e^{j\psi} e^{j\theta_i} q_i^r D_i u_{r_i} \quad &, \quad 0 < D_i = D_i^*, \ \ \theta_i \in [-\frac{\pi}{2}\ \frac{\pi}{2}], \ \ i \le m_r \texttt{ and } |q_i^r| = 1 \\ v_{r_i} = e^{j\psi} e^{j\theta_i} q_i^r D_i u_{r_i} \quad &, \quad 0 < D_i = D_i^*, \ \ \theta_i = \pm\frac{\pi}{2}, \ \ i \le m_r \texttt{ and } 0 < |q_i^r| < 1 \\ v_{r_i} = e^{j\psi} e^{j\theta_i} D_i u_{r_i} \quad &, \quad 0 < D_i = D_i^*, \ \ \theta_i = \pm\frac{\pi}{2}, \ \ i \le m_r \texttt{ and } |q_i^r| = 0 \\ v_{c_i} = e^{j\psi} q_i^c D_{m_r+i} u_{c_i} \quad &, \quad 0 < D_{m_r+i} = D_{m_r+i}^*, \ \ i = 1, \ldots, m_c \\ v_{C_i} = e^{j\psi} d_i Q_i^C u_{C_i} \quad &, \quad 0 < d_i \in \mathbb{R}, \ \ i = 1, \ldots, m_C. \end{aligned} \tag{40}$$

Note that in order to apply these lemmas we need to assume that the non-degeneracy assumptions are satisfied for that sub block. However we have assumed at the outset that u, v either satisfy the non-degeneracy assumption for a given sub block, or have both sub blocks of u and v identically zero, in which case the above relationships hold trivially. Applying this argument to v and Qu we get that the above relationships hold for *every* block. The only case where this argument breaks down is for the repeated real scalar blocks with $q_i^r = 0$. For these blocks however we can show that the above relationships hold by a simple geometric argument.

All that remains now is to define the appropriate quantities. Define $\hat{Q} \in \mathcal{Q}_{\mathcal{K}}$ directly from $Q \in \mathcal{Q}_{\mathcal{K}}$ via (35). For $i \leq m_r$ note that

$$e^{j\theta_i} = \cos(\theta_i) + j\sin(\theta_i)$$

so that we may split each scaling matrix as

$$e^{j\theta_i} D_i = \cos(\theta_i) D_i + j\sin(\theta_i) D_i \doteq D_{R_i} + jD_{I_i} \quad \texttt{for} \quad i = 1, \ldots, m_r \tag{41}$$

For the complex blocks we simply define

$$\begin{aligned} D_{R_{m_r+i}} &= D_{m_r+i} \quad \texttt{for} \quad i = 1, \ldots, m_c \\ d_{R_i} &= d_i \quad \texttt{for} \quad i = 1, \ldots, m_C \end{aligned} \tag{42}$$

Now stack these definitions up to define $D_R \in \tilde{\mathcal{D}}_{\mathcal{K}}$ and $D_I \in \mathcal{G}_{\mathcal{K}}$. Since for each θ_i we have that $\theta_i \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$, it follows that $\cos(\theta_i) \geq 0$ and hence $D_R \geq 0$. It is now easy to verify that with these definitions the relationships in (40) may be written in matrix form as $v = e^{\mathbf{j}\psi}\left(D_R Q + \mathbf{j} D_I \hat{Q}\right) u$. This proves the result for $\mu_{\mathcal{K}}(M) = 1$. The result for $\mu_{\mathcal{K}}(M) > 0$ follows immediately from this by simply scaling u so that $\mu_{\mathcal{K}}(M) = 1$, applying the result for $\mu_{\mathcal{K}}(M) = 1$, and then reabsorbing the scaling back into D_R and D_I. □

Remarks: Although it appears at first sight as a rather unmotivated mathematical abstraction, we will see that in fact this alignment condition between v and u is the key to the equivalence between μ and it's upper bound. Note also that we have such an alignment for any $Q \in \mathcal{Q}_{\mathcal{K}}$ achieving a *local* maximum of $\rho_R(QM)$ over $Q \in \mathbb{B}X_{\mathcal{K}}$ with $\rho_R(QM) > 0$. This follows since we we derived the alignment condition simply from stationarity conditions, and did not use the fact that $\mu_{\mathcal{K}}(M)$ is the *global* maximum at all.

Note that the conditions on u and v assumed in theorem 3 are exactly those guaranteed for the transformed vectors in lemma 2. Thus by first transforming the dyad as in lemma 2 we can show that (for the transformed dyad) we *always* have an alignment condition as in theorem 3, *without* requiring any type of non-degeneracy assumptions (except that $\mu_{\mathcal{K}}(M) > 0$).

We are now in a position to combine these results to prove the main result.

Proof of Theorem 2: We will consider separately the cases $\mu_{\mathcal{K}}(M) > 0$ and $\mu_{\mathcal{K}}(M) = 0$. First suppose that $\mu_{\mathcal{K}}(M) > 0$ and for the moment further assume that in fact $\mu_{\mathcal{K}}(M) = 1$. Carry out the transformation of lemma 2 to define $\overline{u}, \overline{v}, \overline{M}$ with $\mu_{\mathcal{K}}(\overline{M}) = 1$. Now $\overline{u}, \overline{v}, \overline{M}$ satisfy the assumptions of theorem 3, so we may apply this theorem to conclude that we have matrices $Q \in \mathcal{Q}_{\mathcal{K}}$ and $\hat{Q} \in \mathcal{Q}_{\mathcal{K}}$ with $\hat{Q}$ as in (35), together with matrices $D_R \in \tilde{\mathcal{D}}_{\mathcal{K}}, D_R \geq 0$ and $D_I \in \mathcal{G}_{\mathcal{K}}$, and a real scalar $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$ such that

$$\overline{v} = e^{\mathbf{j}\psi}\left(D_R Q + \mathbf{j} D_I \hat{Q}\right) \overline{u} \tag{43}$$

with $\overline{v}^*Q\overline{u} = 1$. Note that this implies

$$\overline{v}^*Q\overline{u} = e^{-\mathbf{j}\psi}\overline{u}^*(Q^*D_R - \mathbf{j}\hat{Q}^*D_I)Qu = 1$$

and hence

$$\overline{u}^*Q^*D_RQ\overline{u} - \mathbf{j}\overline{u}^*\hat{Q}^*D_IQ\overline{u} = e^{\mathbf{j}\psi} = \cos(\psi) + \mathbf{j}\sin(\psi) \tag{44}$$

Now we have immediately that $0 \leq \overline{u}^*Q^*D_RQ\overline{u} \in \mathbb{R}$ and furthermore

$$\overline{u}^*\hat{Q}^*D_IQ\overline{u} = \overline{u}^*\hat{Q}^*QD_I\overline{u} = \overline{u}^*Q^*QD_I\overline{u} = \overline{u}^*Q^*D_IQ\overline{u}$$

so that $\overline{u}^*\hat{Q}^*D_IQ\overline{u} \in \mathbb{R}$ as well, and hence by comparing real and imaginary parts in (44)

$$\cos(\psi) = \overline{u}^*Q^*D_RQ\overline{u} = \overline{u}^*Q^*QD_R\overline{u} = \overline{u}^*D_R\overline{u} \tag{45}$$

Define $\hat{G} \in \mathcal{G}_{\mathcal{K}}$ as $\hat{G} \doteq -D_I\hat{Q}$, and substituting D_R and $\hat{G}$ into the upper bound expression we obtain

$$\begin{aligned}
& \overline{M}^*D_R\overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^*\hat{G}) - D_R \\
= \; & \overline{v}\,\overline{u}^*D_R\overline{u}\,\overline{v}^* + \mathbf{j}(-D_I\hat{Q}\overline{u}\,\overline{v}^* + \overline{v}\,\overline{u}^*D_I\hat{Q}) - D_R \\
= \; & -\overline{v}\,\overline{v}^*(\overline{u}^*D_R\overline{u}) - D_R + D_RQ\overline{u}\,\overline{v}^* + \overline{v}\,\overline{u}^*Q^*D_R
\end{aligned} \tag{46}$$

where we have made use of the substitution $-\mathbf{j}D_I\hat{Q}\overline{u} = -e^{-\mathbf{j}\psi}\overline{v} + D_RQ\overline{u}$ and (45). Thus for any $\eta \in \mathbb{C}^n$ we have that

$$\begin{aligned}
& \eta^*\left(\overline{M}^*D_R\overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^*\hat{G}) - D_R\right)\eta \\
= \; & 2Re\left((\eta^*D_RQ\overline{u})(\overline{v}^*\eta)\right) - (\overline{u}^*D_R\overline{u})|\overline{v}^*\eta|^2 - \eta^*D_R\eta \\
= \; & -\left((\overline{v}^*\eta)Q\overline{u} - \eta\right)^*D_R\left((\overline{v}^*\eta)Q\overline{u} - \eta\right) \;\leq\; 0
\end{aligned} \tag{47}$$

Now note that we get equality (to zero) in (47) iff

$$(\overline{v}^*\eta)Q\overline{u} - \eta = -x \tag{48}$$

where $x \in Ker(D_R)$, because $D_R \geq 0$. But this implies

$$\overline{v}^*\eta = (\overline{v}^*Q\overline{u})(\overline{v}^*\eta) + \overline{v}^*x = \overline{v}^*\eta + \overline{v}^*x$$

and hence $\overline{v}^*x = 0$. From this it is easy to check that

$$\eta = (\overline{v}^*\eta)Q\overline{u} + x \qquad \longleftrightarrow \qquad \eta = \alpha Q\overline{u} + x \ \texttt{for some} \ \alpha \in \mathbb{C}.$$

To summarize, we have shown that

$$\eta^*\left(\overline{M}^*D_R\overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^*\hat{G}) - D_R\right)\eta \;\leq\; 0 \tag{49}$$

with equality iff η is of the form

$$\eta = \alpha Q\overline{u} + x \tag{50}$$

for some $\alpha \in \mathbb{C}$ and $x \in \mathbb{C}^n$ with $D_R x = 0$ and $\overline{v}^* x = 0$. Now define $\hat{D} \doteq D_R + tI_n$, where $0 < t \in \mathbb{R}$ will be chosen later, and note that for $t > 0$, $\hat{D} \in \mathcal{D}_{\mathcal{K}}$. Now choose *any* $\epsilon > 0$ and we have that

$$\begin{aligned} \overline{M}^* \hat{D} \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - (1+\epsilon)\hat{D} &= \left(\overline{M}^* D_R \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - D_R - \epsilon D_R\right) \\ &+ t\left(\overline{M}^* \overline{M} - (1+\epsilon) I_n\right) \\ &\doteq A + tB. \end{aligned} \tag{51}$$

We wish to apply lemma 1 to (51). Note that since we have

$$\begin{aligned} \overline{M}^* D_R \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - D_R &\le 0 \\ -\epsilon D_R &\le 0 \end{aligned}$$

we immediately have that $A \le 0$, and furthermore $\eta^* A \eta = 0$ iff

$$\begin{aligned} \eta^* \left(\overline{M}^* D_R \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - D_R\right) \eta &= 0 \\ \eta^* D_R \eta &= 0 \end{aligned}$$

From our earlier results this implies

$$\eta = \alpha Q \overline{u} + x$$

with $D_R x = 0$, $\overline{v}^* x = 0$, and $\eta^* D_R \eta = 0$. But since $D_R \ge 0$ this implies that $\eta \in Ker(D_R)$, and hence $\alpha Q \overline{u} \in Ker(D_R)$. Now note that this gives

$$0 = (\alpha Q \overline{u})^* D_R (\alpha Q \overline{u}) = |\alpha|^2 \overline{u}^* Q^* D_R Q \overline{u} = |\alpha|^2 \cos(\psi)$$

Since $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$ we have $\cos(\psi) > 0$, and so this expression vanishes iff $\alpha = 0$. Thus we obtain $\eta = x$ and so we have

$$\eta^* A \eta = 0 \quad \longleftrightarrow \quad D_R \eta = 0 \ \ \texttt{and} \ \ \overline{v}^* \eta = 0$$

Now note that for such η we have

$$\begin{aligned} \eta^* B \eta &= \eta^* \left(\overline{M}^* \overline{M} - (1+\epsilon) I_n\right) \eta \\ &= \eta^* \left(\overline{v}\, \overline{u}^* \overline{u}\, \overline{v}^* - (1+\epsilon) I_n\right) \eta \\ &= -(1+\epsilon)|\eta|^2 \ < \ 0 \ \texttt{for} \ \eta \ne 0 \end{aligned} \tag{52}$$

Thus applying lemma 1 we have that for sufficiently small $t > 0$, $A + tB < 0$ and hence choosing such a t we obtain

$$\overline{M}^* \hat{D} \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - (1+\epsilon)\hat{D} \ < \ 0 \tag{53}$$

with $\hat{D} \in \mathcal{D}_{\mathcal{K}}$ and $\hat{G} \in \mathcal{G}_{\mathcal{K}}$. It now remains to unwrap the transformation from $\overline{M}$, to recover the result for M. Recall from lemma 2 that we have a sequence $D^j \in \mathcal{D}_{\mathcal{K}}$ such that

$$\lim_{j \to \infty} D^j M D^{j-1} = \overline{M}$$

Since this implies that we may choose $D^j \in \mathcal{D}_{\mathcal{K}}$ so that $D^j M D^{j-1}$ is arbitrarily close to $\overline{M}$, by continuity we may choose $D^j \in \mathcal{D}_{\mathcal{K}}$ so that

$$(D^j M D^{j-1})^* \hat{D} (D^j M D^{j-1}) + \mathbf{j}(\hat{G}(D^j M D^{j-1}) - (D^j M D^{j-1})^* \hat{G}) - (1+\epsilon)\hat{D} \ < \ 0 \tag{54}$$

Since $D^j > 0$ we may multiply both sides of this expression by D^j without affecting the definiteness to yield

$$M^*(D^j\hat{D}D^j)M + \mathbf{j}((D^j\hat{G}D^j)M - M^*(D^j\hat{G}D^j) - (1+\epsilon)(D^j\hat{D}D^j) \;<\; 0 \tag{55}$$

so that defining $D \doteq D^j\hat{D}D^j$ and $G \doteq D^j\hat{G}D^j$ we have

$$M^*DM + \mathbf{j}(GM - M^*G) - (1+\epsilon)D \;<\; 0 \tag{56}$$

with $D \in \mathcal{D}_{\mathcal{K}}$ and $G \in \mathcal{G}_{\mathcal{K}}$. But this shows that the upper bound achieves $\sqrt{1+\epsilon}$ where $\epsilon > 0$ was arbitrary. Thus by taking the infimum we find that the upper bound gives 1, which by assumption equals $\mu_{\mathcal{K}}(M)$. This proves the result for $\mu_{\mathcal{K}}(M) = 1$. The extension to $\mu_{\mathcal{K}}(M) > 0$ is easy. Suppose $\mu_{\mathcal{K}}(M) = \beta > 0$. Then $\mu_{\mathcal{K}}(\frac{M}{\beta}) = 1$ so we may apply this result to obtain, for any $\hat{\epsilon} > 0$, $\hat{D} \in \mathcal{D}_{\mathcal{K}}$ and $\hat{G} \in \mathcal{G}_{\mathcal{K}}$ such that

$$\frac{M^*}{\beta}\hat{D}\frac{M}{\beta} + \mathbf{j}(\hat{G}\frac{M}{\beta} - \frac{M^*}{\beta}\hat{G}) - (1+\hat{\epsilon})\hat{D} \;<\; 0. \tag{57}$$

Now given *any* $\epsilon > 0$ choose $\hat{\epsilon} = (1+\frac{\epsilon}{\beta})^2 - 1$, and choose $\hat{D}, \hat{G}$ as above. Then this expression gives

$$M^*\hat{D}M + \mathbf{j}((\beta\hat{G})M - M^*(\beta\hat{G})) - (\beta+\epsilon)^2\hat{D} < 0$$

so that defining $D \doteq \hat{D}$ and $G \doteq \beta\hat{G}$ we obtain

$$M^*DM + \mathbf{j}(GM - M^*G) - (\beta+\epsilon)^2 D \;<\; 0$$

with $D \in \mathcal{D}_{\mathcal{K}}$ and $G \in \mathcal{G}_{\mathcal{K}}$, and hence by the same reasoning as before this implies that the upper bound achieves β, which by assumption equals $\mu_{\mathcal{K}}(M)$. This proves the result for $\mu_{\mathcal{K}}(M) > 0$.

The only case we have left to deal with is when $\mu_{\mathcal{K}}(M) = 0$. This case is not covered by the previous analysis since in this case there is no destabilizing perturbation ($Q \in \mathcal{Q}_{\mathcal{K}}$). Start first by employing the transformation to $\overline{M}$ in lemma 2. If $\overline{\sigma}(\overline{M}) = 0$ we are done, so assume not, i.e., $|\overline{u}||\overline{v}| \neq 0$. Note immediately that all the complex blocks must be zero,

$$\begin{array}{c} |\overline{u}_{c_i}| = |\overline{v}_{c_i}| = 0 \quad \texttt{for} \quad i = 1,\ldots,m_c \\ |\overline{u}_{C_i}| = |\overline{v}_{C_i}| = 0 \quad \texttt{for} \quad i = 1,\ldots,m_C \end{array} \tag{58}$$

else we could find $Q \in \mathcal{Q}_{\mathcal{K}}$ with $\rho_R(QM) > 0$ simply by choosing all the real perturbations to be zero, and appropriate complex perturbations. Furthermore it is easy to check via a simple geometric argument that for every non-zero repeated real scalar sub block, the quantity $\frac{\overline{v}_{r_i}^*\overline{u}_{r_i}}{|\overline{v}_{r_i}^*\overline{u}_{r_i}|}$ must be the same modulo $\pm$, (for $i = 1,\ldots,m_r$), and this quantity cannot be purely real. If this were not so then once again we could find $Q \in \mathcal{Q}_{\mathcal{K}}$ with $\rho_R(QM) > 0$. These conditions are equivalently expressed as

$$\overline{v}_{r_i}^*\overline{u}_{r_i} = \gamma_i e^{-\mathbf{j}\psi}e^{-\mathbf{j}\theta_i} \;\texttt{ for each }\; i \leq m_r \;\texttt{ with }\; |\overline{v}_{r_i}^*\overline{u}_{r_i}| \neq 0 \tag{59}$$

where for each i we have $\theta_i = \pm\frac{\pi}{2}$, $0 < \gamma_i \in \mathbb{R}$, and $\psi \in (-\frac{\pi}{2}\;\frac{\pi}{2})$. Thus applying lemma 4.2 from [9] to each of these blocks we have

$$\overline{v}_{r_i} = e^{\mathbf{j}\psi}e^{\mathbf{j}\theta_i}D_i\overline{u}_{r_i} \;\texttt{ for each }\; i \leq m_r \;\texttt{ with }\; |\overline{v}_{r_i}^*\overline{u}_{r_i}| \neq 0 \tag{60}$$

where for each i we have $0 < D_i = D_i^*$. Stacking these relationships up we obtain $D_I \in \mathcal{G}_\mathcal{K}$ such that

$$\overline{v} = \mathbf{j}e^{\mathbf{j}\psi} D_I \overline{u} \tag{61}$$

Note that for the zero sub blocks of $\overline{u}$ and $\overline{v}$ this relationship holds trivially, and for the non-zero sub blocks it follows from (60). Now choose $D_R = 0_n$ and $\hat{G} = -D_I$. Note that $D_R \in \mathcal{D}_\mathcal{K}, D_R \geq 0$, $\hat{G} \in \mathcal{G}_\mathcal{K}$, and for *any* $0 < \epsilon \in \mathbb{R}$ we have

$$\begin{aligned}
& \overline{M}^* D_R \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - \epsilon D_R \\
= \; & \mathbf{j}(-D_I \overline{u}\,\overline{v}^* + \overline{v}\,\overline{u}^* D_I) \\
= \; & -2\cos(\psi)\overline{v}\,\overline{v}^* \quad \leq \quad 0
\end{aligned} \tag{62}$$

where we have used the substitution $\mathbf{j}D_I\overline{u} = e^{-\mathbf{j}\psi}\overline{v}$. Now define $\hat{D} \doteq D_R + tI_n = tI_n$, where $0 < t \in \mathbb{R}$ will be chosen later, and note that for $t > 0$, $\hat{D} \in \mathcal{D}_\mathcal{K}$. Then we have that

$$\begin{aligned}
\overline{M}^* \hat{D} \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - \epsilon \hat{D} \;&=\; \left(\overline{M}^* D_R \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - \epsilon D_R\right) \\
&\quad + \; t\left(\overline{M}^*\overline{M} - \epsilon I_n\right) \\
&\doteq \; A + tB.
\end{aligned} \tag{63}$$

By our earlier results $A \leq 0$ and furthermore we have

$$\eta^* A \eta = -2\cos(\psi)|\overline{v}^*\eta|^2 \quad \leq \quad 0 \tag{64}$$

with equality iff $\overline{v}^*\eta = 0$ (since $\cos(\psi) > 0$). For such η we have that

$$\begin{aligned}
\eta^* B \eta \;&=\; \eta^* \left(\overline{M}^*\overline{M} - \epsilon I_n\right)\eta \\
&=\; \eta^* \left(\overline{v}\,\overline{u}^*\overline{u}\,\overline{v}^* - \epsilon I_n\right)\eta \\
&=\; -\epsilon|\eta|^2 \quad < \quad 0 \;\; \texttt{for} \;\; \eta \neq 0
\end{aligned} \tag{65}$$

Thus applying lemma 1 we obtain that $A + tB < 0$ for sufficiently small $t > 0$. Choosing such a t we have that

$$\overline{M}^* \hat{D} \overline{M} + \mathbf{j}(\hat{G}\overline{M} - \overline{M}^* \hat{G}) - \epsilon \hat{D} \quad < \quad 0 \tag{66}$$

with $\hat{D} \in \mathcal{D}_\mathcal{K}$ and $\hat{G} \in \mathcal{G}_\mathcal{K}$. Now we can unwrap the transformation in lemma 2 exactly as before to obtain $D \in \mathcal{D}_\mathcal{K}$ and $G \in \mathcal{G}_\mathcal{K}$ satisfying

$$M^* D M + \mathbf{j}(GM - M^* G) - \epsilon D \quad < \quad 0. \tag{67}$$

But this shows that the upper bound achieves $\sqrt{\epsilon}$ where $\epsilon > 0$ was arbitrary. Thus by taking the infimum we have that the upper bound achieves 0 which by assumption equals $\mu_\mathcal{K}(M)$. This concludes the $\mu_\mathcal{K}(M) = 0$ case, and combining this with our earlier result for $\mu_\mathcal{K}(M) > 0$ the theorem is proved. $\square$

5 Additional Properties

Note that this proof is substantially more involved than the one given in [14], where additional simplifying assumptions were made. Note however that this is a constructive proof, and so it actually gives us quite a bit more information about the rank one problem than we obtained with the earlier result. We are able not only to say that the upper bound achieves μ, but to explicitly construct the D, G scaling matrices that do the job. This allows us to examine the properties of these scaling matrices as a function of the problem data, and to arrive at several interesting conclusions.

First note that the construction of the optimal sequence of D, G scaling matrices was based on employing the "μ-values" construction from [14]. Note that theorem 3 holds for any of the "μ-values" (see the remarks following theorem 3). To be more specific this means that given any $Q \in \mathcal{Q}_{\mathcal{K}}$ achieving a *local* maximum over $Q \in \mathbb{B}X_{\mathcal{K}}$ of $\rho_R(QM)$, with $\rho_R(QM) = \beta > 0$ then we can employ the machinery of lemma 2, and theorems 2 and 3, to construct (a sequence of) $D \in \mathcal{D}_{\mathcal{K}}, G \in \mathcal{G}_{\mathcal{K}}$ verifying that $\mu_{\mathcal{K}}(M) \leq \beta$. But since we already have $\beta = \rho_R(QM) \leq \mu_{\mathcal{K}}(M)$ this gives $\mu_{\mathcal{K}}(M) = \beta$. Thus we find that for the rank one problem any non-zero local maximum of $\rho_R(QM)$ (as defined above) is global. We state this as a theorem.

Theorem 4 *Suppose we have $M = uv^*$, with $u, v \in \mathbb{C}^n$, and a compatible block structure $\mathcal{K}$. Further suppose we have $Q \in \mathcal{Q}_{\mathcal{K}}$ such that $\rho_R(QM) = \beta > 0$ is a local maximum of $\rho_R(QM)$ over $Q \in \mathbb{B}X_{\mathcal{K}}$. Then $\beta = \mu_{\mathcal{K}}(M)$.*

This offers further insight into why the rank one problem is easy. For the general problem we do not have any such guarantees about local maxima, and in fact one can easily construct problems with local maxima that are not global.

In fact we can characterize the solution to the rank one μ problem in terms of this alignment condition.

Theorem 5 *Suppose we have $M = uv^*$, with $u, v \in \mathbb{C}^n$ satisfying the non-degeneracy assumptions, and a compatible block structure $\mathcal{K}$. Further suppose we have $Q \in \mathcal{Q}_{\mathcal{K}}$ with $q_i^r \neq 0$ for $i = 1, \ldots, m_r$ such that $\rho_R(QM) = \beta > 0$. Then we have $\beta = \mu_{\mathcal{K}}(M)$ iff there exists $D \in \hat{\mathcal{D}}_{\mathcal{K}}$ with $\theta_i = \pm\frac{\pi}{2}$ for $|q_i^r| < 1$ and $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$ such that*

$$v = e^{\mathbf{j}\psi} DQu \tag{68}$$

Proof: ($\rightarrow$) Note that as in the proof of theorem 2 we can show that Qu and v are the right and left eigenvectors of QM corresponding to the distinct eigenvalue β, with $v^*Qu = \beta$. Since $\beta = \mu_{\mathcal{K}}(M)$ we are at a local maximum by assumption and the result follows from theorem 2 of [15].

($\leftarrow$) Again we have that Qu and v are the right and left eigenvectors of QM corresponding to the distinct eigenvalue β, with $v^*Qu = \beta$. But now for each $i \leq m_r$ we have

$$e^{\mathbf{j}\theta_i} = \cos(\theta_i) + \mathbf{j}\sin(\theta_i)$$

so that defining $D_R \in \mathcal{D}_{\mathcal{K}}, D_R \geq 0$ and $D_I \in \mathcal{G}_{\mathcal{K}}$ by

$$\begin{aligned} D_{R_i} = \cos(\theta_i)D_i \ \texttt{and} \ D_{I_i} = \sin(\theta_i)D_i \quad & \texttt{for} \quad i = 1, \ldots, m_r \\ D_{R_{m_r+i}} = D_{m_r+i} \quad & \texttt{for} \quad i = 1, \ldots, m_c \\ d_{R_{m_r+m_c+i}} = d_{m_r+m_c+i} \quad & \texttt{for} \quad i = 1, \ldots, m_C \end{aligned}$$

we have $v = e^{\mathbf{j}\psi}(D_R + \mathbf{j}D_I)Qu$ as in theorem 3. From this alignment we can construct (a sequence of) $\hat{D} \in \mathcal{D}_{\mathcal{K}}, \hat{G} \in \mathcal{G}_{\mathcal{K}}$ showing that $\mu_{\mathcal{K}}(M) \leq \beta$ (as in theorem 2) and hence $\beta = \mu_{\mathcal{K}}(M)$. □

Roughly speaking this theorem says that Q achieves μ iff it aligns v and Qu (as above). This is employed in [13] to develop a lower bound power iteration to compute a lower bound for $\mu_{\mathcal{K}}(M)$ for *general* mixed μ problems. Note that for this theorem we added some technical assumptions. These can once again be dealt with via the machinery of lemma 2, but since the main use of this theorem is for the lower bound power iteration in [13] we do not bother with this added complication.

Recall that we remarked earlier that we can solve for the D, G scaling matrices in the upper bound *without* resorting to actually solving the associated convex optimization problem. In order to do this however we need to know the value of $\mu_{\mathcal{K}}(M)$, and the associated destabilizing perturbation $Q \in \mathcal{Q}_{\mathcal{K}}$ (if there is one). In the following section we will see that in fact we can obtain both of these quantities in closed form, so that we have a complete solution to the rank one μ problem in closed form. Thus we can easily compute all the relevant quantities *without* ever having to resort to numerical solution of an optimization problem.

6 A Graphical Interpretation

It is interesting to consider a graphical interpretation of the rank one mixed μ problem, in the complex plane. Suppose that we have $M = uv^*$, with $u, v \in \mathbb{C}^n$, and some compatible block structure $\mathcal{K}$. Then note that for any $\Delta \in \mathbb{B}X_{\mathcal{K}}$ and $0 < \alpha \in \mathbb{R}$ we have that

$$\begin{aligned}
\mathtt{det}(I_n - \frac{\Delta M}{\alpha}) = 0 \quad &\longleftrightarrow \quad \mathtt{det}(\alpha I_n - \Delta M) = 0 \\
&\longleftrightarrow \quad v^* \Delta u = \alpha \\
&\longleftrightarrow \quad \sum_{i=1}^{m_r} \delta_i^r v_{r_i}^* u_{r_i} + \sum_{i=1}^{m_c} \delta_i^c v_{c_i}^* u_{c_i} + \sum_{i=1}^{m_C} v_{C_i}^* \Delta_i^C u_{C_i} = \alpha
\end{aligned} \tag{69}$$

This equation forms the basis of our graphical interpretation. If we think of the components, $\delta_i^r v_{r_i}^* u_{r_i}$, $\delta_i^c v_{c_i}^* u_{c_i}$, $v_{C_i}^* \Delta_i^C u_{C_i}$, as vectors in the complex plane, then the rank one mixed μ problem simply amounts to choosing δ_i^r, δ_i^c, Δ_i^C so that these vectors add up to a positive real number, which is as large as possible. Note that it is now evident that we do not affect the problem if we throw away the degenerate blocks (with $|v_{r_i}^* u_{r_i}| = 0$, or $|v_{c_i}^* u_{c_i}| = 0$, or $|v_{C_i}||u_{C_i}| = 0$).

First of all it is clear that, since the complex perturbations can have arbitrary phase, we will always wish to choose $|\delta_i^c| = 1$ and $\overline{\sigma}(\Delta_i^C) = 1$ so that these vectors have their maximum length, namely

$$\begin{aligned}
\delta_i^c v_{c_i}^* u_{c_i} &= e^{\mathbf{j}\psi_{c_i}} |v_{c_i}^* u_{c_i}| \\
v_{C_i}^* \Delta_i^C u_{C_i} &= e^{\mathbf{j}\psi_{C_i}} |v_{C_i}||u_{C_i}|
\end{aligned} \tag{70}$$

Furthermore it is also clear from the geometry of the problem that in fact we will wish to align all these vectors, so as to make one vector of maximal length whose phase we are free to choose. Thus we may take $\psi_{c_i} = \psi$ and $\psi_{C_i} = \psi$ (with appropriate ranges for i), and so we have

$$\sum_{i=1}^{m_c} \delta_i^c v_{c_i}^* u_{c_i} + \sum_{i=1}^{m_C} v_{C_i}^* \Delta_i^C u_{C_i} = e^{\mathbf{j}\psi} \left(\sum_{i=1}^{m_c} |v_{c_i}^* u_{c_i}| + \sum_{i=1}^{m_C} |v_{C_i}||u_{C_i}| \right) \tag{71}$$

where we have one free parameter, ψ, left to choose for the complex blocks. Note that it is easy to choose δ_i^c and Δ_i^C so that we satisfy (71). With this observation denote $0 \leq L_C \in \mathbb{R}$ as

$$L_C = \sum_{i=1}^{m_c} |v_{c_i}^* u_{c_i}| + \sum_{i=1}^{m_C} |v_{C_i}||u_{C_i}| \tag{72}$$

and the rank one mixed μ problem reduces to choosing real numbers $\delta_i^r \in [-1 \;\; 1]$ and $\psi \in [-\pi \;\; \pi]$ so as to maximize $0 < \alpha \in \mathbb{R}$ where

$$\sum_{i=1}^{m_r} \delta_i^r v_{r_i}^* u_{r_i} + e^{\mathbf{j}\psi} L_C = \alpha \tag{73}$$

The solution to this problem can be obtained geometrically, by thinking of (73) as a vector sum in the complex plane. See figure 1 for an example illustration with three real blocks (and any number of complex blocks). Note that having chosen the values of δ_i^r then it is easy to choose ψ so as

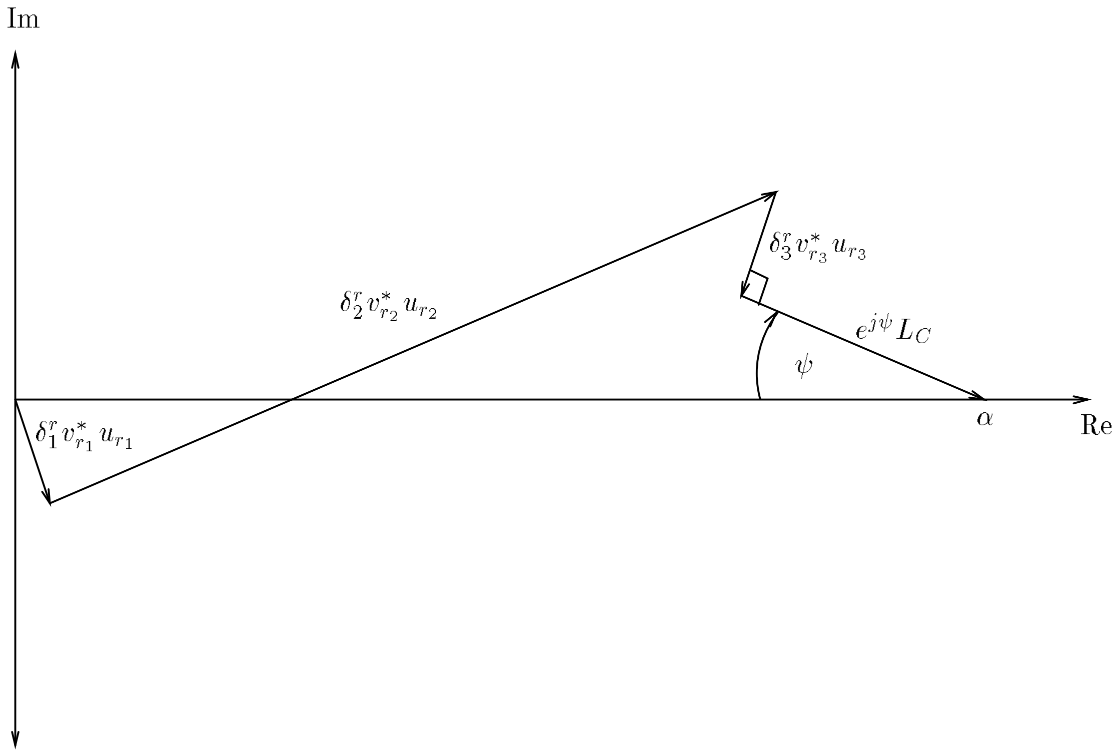

Figure 1: Graphical solution of the rank one mixed μ problem

to maximize α, or alternately to verify that no ψ exists to make the summation in (73) add up to a real number. The real question then is how to choose the values of δ_i^r. Consider the following algorithm (some of the key ideas for this approach are from Newlin [8]).

Algorithm 1 (Rank One μ Solution)

1 Choose starting values for the real perturbations as $\delta_i^r = \texttt{Sgn}(\texttt{Re}(v_{r_i}^* u_{r_i}))$*. Then for all* δ_i^r *we have* $\texttt{Re}(\delta_i^r v_{r_i}^* u_{r_i}) \geq 0$*. Now compute*

$$S = \texttt{Sgn}\left(\texttt{Imag}\left(\sum_{i=1}^{m_r} \delta_i^r v_{r_i}^* u_{r_i}\right)\right)$$

2 Rank all the components $\delta_i^r v_{r_i}^* u_{r_i}$ *by argument, so that the highest rank is assigned to the greatest value of* $\texttt{Arg}(S\delta_i^r v_{r_i}^* u_{r_i})$*.*

3 Consider the highest rank component which has not yet been looked at. Compute δ_{opt}*, which is the optimal value of this* δ_i^r*, for* $\delta_{opt} \in \mathbb{R}$ *unrestricted in sign or magnitude, and all the other real perturbations fixed.*

4 If $\texttt{Sgn}(\delta_{opt}) = -\texttt{Sgn}(\delta_i^r)$ *and* $|\delta_{opt}| > 1$ *and not all the components have been looked at, then set* $F = 1$*, else set* $F = 0$*. Now reassign* δ_i^r *with* $\max[-1 \;\; \min[1 \;\; \delta_{opt}]]$*.*

5 If $F = 1$ *then go to step 3.*

6 For these values of δ_i^r *compute the optimal value of* ψ*, and hence* $\mu_{\mathcal{K}}(M)$ *and the destabilizing perturbation* $Q \in \mathcal{Q}_{\mathcal{K}}$ *(if there is one).*

This algorithm guarantees to compute μ exactly for a rank one mixed μ problem, together with an optimal destabilizing perturbation $Q \in \mathcal{Q}_{\mathcal{K}}$ (if there is one). The reasoning behind this algorithm is simple if one thinks of the problem geometrically. The objective is to make the summation

$$\sum_{i=1}^{m_r} \delta_i^r v_{r_i}^* u_{r_i} + e^{\mathbf{j}\psi} L_C$$

add up to a real number which is as large as possible. First of all one chooses the perturbations, δ_i^r, so that the real parts of each component, $\delta_i^r v_{r_i}^* u_{r_i}$, are nonnegative. Then consider the summation

$$\sum_{i=1}^{m_r} \delta_i^r v_{r_i}^* u_{r_i} \tag{74}$$

to which we must add the complex component, $e^{\mathbf{j}\psi} L_C$, so as to make it add up to a real number. Suppose the imaginary part of the summation in (74) is nonnegative (symmetric arguments apply for the other case). Then we consider the component $\delta_i^r v_{r_i}^* u_{r_i}$ with the largest argument. This component is ranked the "worst" component in the summation, in the sense that it contributes the most positive imaginary part (which we have too much of) for a given positive real part (which we want). Then we compute the optimal value, δ_{opt}, for this parameter, δ_i^r (with all the other real perturbations fixed), and reassign δ_i^r with the value of δ_{opt}, clipped to the interval $[-1 \;\; 1]$. If δ_{opt} is not both of opposite sign to the original value of δ_i^r, and greater than one, then it says that you could not improve the summation by further reducing the imaginary contribution from this component, $\delta_i^r v_{r_i}^* u_{r_i}$. But this component had the "worst" ratio of imaginary to real contribution, so you could not improve by changing any other component, and hence you are done. If the above condition is not met then you could get further improvement with this component so you check the next rank component, until you meet the condition or you have checked them all. In this way you proceed with at most a linear search over the real parameters to obtain the optimal values for all the real perturbations, δ_i^r. Given these, then it is easy to compute the remainder of the solution.

It is easy to verify that the computation of δ_{opt} in step 3 and ψ in step 6 boil down to just simple trigonometry. It can also be verified that we always have $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$ and denoting by θ_i the angle between $e^{\mathbf{j}\psi} L_C$ and $\delta_i^r v_{r_i}^* u_{r_i}$, then $\theta_i \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$. Furthermore if $|\delta_i^r| < 1$ then $\theta_i = \pm\frac{\pi}{2}$. But now if we examine the vector summation in figure 1 then we see that this solution is exactly the alignment condition derived in theorem 3. From the geometric viewpoint it is now clear why this alignment must hold at the maximum of $\rho_R(QM)$.

Note that algorithm 1 requires at most a search over the real parameters, which grows linearly with m_r. All the computations required can be performed via simple trigonometry, so that this *algorithm* is really a shorthand notation for the closed form solution to the rank one mixed μ problem (which would otherwise be cumbersome to write, involving a "max" over m_r possibilities). Thus we have a closed form solution, with trivial computational requirements, for both $\mu_{\mathcal{K}}(M)$ and the associated $Q \in \mathcal{Q}_{\mathcal{K}}$.

The fact that the graphical solution to the rank one μ problem is so simple, really lets us see what is going on with the problem. As an illustration of this we immediately obtain the following theorem.

Theorem 6 *Suppose we have $M = uv^*$, with $u, v \in \mathbb{C}^n$, and a compatible block structure $\mathcal{K}$. Then in computing $\mu_{\mathcal{K}}(M)$ it suffices to consider perturbations $\Delta \in X_{\mathcal{K}}$ with at most one of the real variables, δ_i^r having $|\delta_i^r| < 1$.*

Proof: Note that algorithm 1 starts out by assigning all the real variables at extremal values. It can be seen from steps 4 and 5 that the algorithm quits if ever any variable is reassigned internally. Since algorithm 1 guarantees to find $\mu_{\mathcal{K}}(M)$ we find that an optimal destabilizing perturbation can be found with at most one real variable internal. $\square$

This is the mixed μ counterpart of the well known "edge theorem" [1] for the "affine parameter variation" case for a polynomial with perturbed coefficients (see section 3). Note that the result holds for pure real or mixed μ problems. Once again the reason for this result is clear when we look at the problem geometrically: if we have more than one variable internal then we can always increase one of them (in magnitude), and compensate with the other so that the summation (73) stays real and does not decrease. We simply do this until all but one (or none) of the real variables is at its extremal value.

In fact if one considers the geometry of the problem, then it is possible to state a slightly stronger version of this "edge result": aside from cases where we have real degenerate blocks (with $|v_{r_i}^* u_{r_i}| = 0$), or real blocks with the same phase modulo $\pm$ (i. e., $\mathtt{Arg}(v_{r_i}^* u_{r_i}) = \mathtt{Arg}(v_{r_j}^* u_{r_j})$ or $\mathtt{Arg}(v_{r_j}^* u_{r_j}) + \pi$ for $i \neq j$), then the *only* optimal destabilizing perturbations are on the edges.

Note that an exact expression for $\mu_{\mathcal{K}}(M)$ with M rank one was also obtained in [4]). The authors were then able to take this result and solve several problems from the literature, noting that these problems can be treated as special cases of rank one μ problems.

Thinking about the rank one μ problem graphically makes it easy to construct examples with a particular value for $\mu_{\mathcal{K}}(M)$, and particular properties for the alignment condition at the maximum of $\rho_R(QM)$. To conclude we present a series of such examples which illustrate certain facts about

μ and rank one problems. These facts may not be obviously true (or false) from the definition of μ, but are immediately clear when one considers the graphical interpretation for the corresponding rank one example.

Fact 1: *For a rank one mixed μ problem, it is not generic that the worst case perturbation is on a vertex.* Consider the following example:

$$M = \begin{pmatrix} \mathbf{j} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad \Delta = \texttt{diag}(\delta^r, \delta^c)$$

with $\delta^r \in \mathbb{R}$ and $\delta^c \in \mathbb{C}$. Then it is easy to see that the worst case perturbation is $\delta^r = 0, \delta^c = 1$. This has δ^r internal, and this property holds for small perturbations to the problem.

Fact 2: *For a rank one mixed μ problem, it is not generic that the worst case perturbation is not on a vertex.* Consider the following example:

$$M = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad \Delta = \texttt{diag}(\delta^r, \delta^c)$$

with $\delta^r \in \mathbb{R}$ and $\delta^c \in \mathbb{C}$. Then it is easy to see that the worst case perturbation is $\delta^r = 1, \delta^c = 1$. This has δ^r at a vertex, and this property holds for small perturbations to the problem.

Fact 3: *For a rank one pure real μ problem, it is generic that the worst case perturbation is not on a vertex.* This follows by noting that given any summation as in (74), then we can perturb the components by an arbitrarily small amount so that the summation cannot be made purely real with every $\delta_i^r = \pm 1$.

Fact 4: *For a rank one pure real or mixed μ problem, with at least two uncertainty blocks, it is generic that $\mu_{\mathcal{K}}(M) > 0$.* This follows by noting that for any problem with at least two blocks we can perturb the components by an arbitrarily small amount so that the summation (73) can be made real and positive.

Fact 5: *One can have μ problems, where the worst case perturbation has all the real variables internal, or even zero.* This follows from the example given in fact 1.

7 Conclusion

It has been shown that the "affine parameter variation" problem for a polynomial with perturbed coefficients can be recast as a rank one mixed μ problem. This setting forms the basis for a number of "Kharitonov-type" exact robust stability tests with respect to real parametric uncertainty. This rank one mixed μ problem has been shown to be equivalent to its upper bound, which is a convex problem. This enables exact computation in the μ framework as well, and in fact a closed form solution to the rank one μ problem was obtained with trivial computational requirements.

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