Stabilization of Uncertain Linear Systems: An LFT Approach*

Wei-Min Lu†, Kemin Zhou‡, and John C. Doyle§

Abstract

This paper develops machinery for control of uncertain linear systems described in terms of linear fractional transformations (LFTs) on transform variables and uncertainty blocks, with primary focus on stabilization and controller parametrization. This machinery directly generalizes familiar state-space techniques. The notion of $\mathcal{Q}$-stability is defined as a natural type of robust stability, and output feedback stabilizability is characterized in terms of $\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability, which in turn are related to full information and full control problems. Computation is in terms of convex linear matrix inequalities (LMIs), the controllers have a separation structure, and the parametrization of all stabilizing controllers is characterized as a LFT on a stable free parameter.

1 Introduction

Linear fractional transformations (LFTs) have come to play an important role in control system design [36], [12], [14], [28], [7], [31], [44]. In this paper, we develop machinery for **linear fractional uncertain systems**, represented as an LFT of a constant matrix on a block structure which includes the transform variables and uncertainty blocks. This LFT machinery is a direct generalization of the now standard state-space machinery for linear system analysis and synthesis. One of the advantages of the use of LFT representations is that it facilitates manipulation using state-space-like machinery, and computation involves constant matrix manipulation. The LFT machinery provides a uniform framework for realization, analysis, and synthesis for uncertain systems, and even streamlines some of the standard linear systems results [4].

We focus on synthesis of stabilizing controllers for systems with LFT representations. The notion of $\mathcal{Q}$-stability is employed for such systems[1]; it is a natural generalization of the conventional notions of stability for linear systems, robust stability for linear fractional uncertain systems, and robust performance of linear fractional uncertain systems. In particular, for a linear fractional uncertain system with norm-bounded **linear time varying** (LTV) or nonlinear uncertainty, $\mathcal{Q}$-stability is a necessary and sufficient test for robust stability and robust performance, and $\mathcal{Q}$-stability, $\mathcal{Q}$-stabilizability, and $\mathcal{Q}$-detectability can be characterized using **linear matrix inequalities** (LMIs), which result in computationally appealing convex optimization problems.

*This paper was published on *IEEE Transactions on Automatic Control*, vol.41(1), pp.50–65, 1996.

†W.-M. Lu is with the Department of Electrical Engineering 116-81, California Institute of Technology, Pasadena, CA 91125, wmlu@cds.caltech.edu.

‡K. Zhou is with the Department of Electrical and Computer Engineering, Louisiana State University, Baton Rouge, LA 70803.

§J. Doyle is with the Department of Electrical Engineering and Department of Control and Dynamical Systems 116-81, California Institute of Technology, Pasadena, CA 91125.

[1] Another stability notion, μ-stability, is considered in [25].

We first consider the the stabilization problem in several special cases, i.e., the full information, partial information, and constant output feedback. The construction for the **output feedback** (OF) synthesis problem is then achieved via a separation argument and two special problems: **full information** (FI) and **full control** (FC). Our approach is similar to that in [13], and directly generalizes standard stabilization results and state-space methods, while streamlining much of the development. In particular, in the FI case, it is shown that the solvability is equivalent to $\mathcal{Q}$-stabilizability and is characterized by the positive definite solutions of an LMI, and the controllers can be chosen as constant feedbacks. Dual results hold for the FC case, and $\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability are necessary and sufficient for robust stabilization by output feedback.

The OF controllers are represented by LFTs on the same block structures as the plants, and all stabilizing controllers are parametrized as an LFT on a free stable parameter. An appealing feature of this approach is that the necessity portion of the controller parametrization relies heavily on elegant LFT machinery [36], [19], [44] and avoids the need for coprime factorizations (see [43], [12], [30]). In a further generalization it is noted that all that is required for the separation principle to hold for the LFT systems is that the stability is invariant under certain system transformations and cascade interconnections. The machinery used here can also be generalized to deal with controller parametrization for nonlinear systems [26].

There are several interpretations for control schemes that are LFTs on the same block structure as the plant. In the multidimensional system case, the block structure represents transform variables, and the resulting controller then provides dynamic feedback [16], [20]; for an uncertain linear system with structured parametric or dynamical perturbations, which can be viewed as a **linear parameter varying** (LPV) or linear dynamic varying system [39], [41], [37], [5], [2], the resulting controllers, which are dynamical and depend on the perturbations, therefore may be thought of as gain scheduled or dynamically scheduled [42],[31]; the application implications of such control schemes are obvious when the uncertainties can be measured or identified on line [42],[37].

The control problems of linear systems whose coefficients depend on unknown parameters is also addressed in the context of other algebraic structures [24], [23], [5], [25], and the parameter-dependent, or gain-scheduled, control solutions are characterized [23], [5], [25]. One of the major differences between the proposed LFT approach and others are that the arbitrary perturbations, including LTV and nonlinear uncertainties, can also be treated by the the LFT approach, where the $\mathcal{Q}$-stability exactly captures this feature. In addition, the proposed LFT framework provides a systematic approach for gain-scheduled control design, where control solutions can be analytically constructed by solving the corresponding LMIs independent of the parameters (see [28], [33], [27], [31]), and therefore the conventional *ad hoc* point-wise controller design and curve-fitting procedures (see [39]) are avoided; moreover, the type of scheduling that results from this LFT approach guarantees the global stability and global performances even when the uncertainty varies arbitrarily fast and thus avoids any potential hazards arising from conventional scheduling [41], [39]. On the other hand, one disadvantage of the proposed approach, which emphasizes $\mathcal{Q}$-stabilization, is that unless the uncertainties are arbitrarily time-varying operators, the stability analysis on which the synthesis is based is potentially conservative [32]. This conservativeness may be most troublesome when the parameters are slowly time-varying, since neither the slow variation nor the parametric nature of the uncertainty is exploited. Thus the methods in this paper should be viewed as a possible aid in conventional gain scheduling design rather than a replacement.

This paper is based on the conference paper [28], where the main results were presented. Some relevant analysis results, in particular the LMI characterizations, were also obtained in [33] in the parallel context of robust performance synthesis for linear fractional uncertain systems. Though robust stability and robust performance can be uniformly treated as $\mathcal{Q}$-stability, the essential difference between the results in this paper and those in [33] lies in their approaches taken for synthesis problems; the approach using separation arguments for the stability synthesis problem in this paper can not be naively extended to the performance synthesis problem because of the block dependent nature of the designed controllers. Many extensions and generalizations have been done since the appearance of these two conference papers [34], [27], [31], [21]. More recently, the LMI treatment has been extended to cope with robust performance synthesis for continuous-time systems [17], [2], [22]. A fairly complete review of LMIs and their roles in control theory is given in [8], also a software toolbox for the numerical solutions to LMIs is available [18].

This paper is presented in an axiomatic fashion in the order: Section II.1, Section III.1, Section IV, and Section V. The remaining structure is as follows. In the Appendix, some background material about LFTs, μ, and matrix dilation is reviewed. In Section II, we first present a general framework for linear systems with LFT descriptions, $\mathcal{Q}$-stability is defined and characterized, and the robust stability and robust performance for linear fractional uncertain systems are examined to give some motivations for the stability notions; the implication of the stability notions in multidimensional systems is given in Section VII.2. In Section III, the properties of stabilizability and detectability for LFT systems are examined, in particular, $\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability are characterized in terms of LMIs. Some simple cases for the stabilization of linear fractional uncertain systems, including full information feedback, partial information feedback, and constant output feedback, are examined. In Sections IV and V, the stabilization problem is considered; in Section IV, the stabilization problems are stated, and the relevant system structural properties are examined; Section V is devoted to the solutions of the general output-feedback problems, the special FI, DF, FC, and OE problems are first treated to develop machinery; the stabilizing controller and controller parametrization are constructed via separation arguments.

The notations used in this paper is quite standard. $\mathbf{Z}^+$ denotes the set of non-negative integers. For a matrix $M \in \mathbf{R}^{n\times m}(\mathbf{C}^{n\times m})$, M^T denotes the transpose of M and M^* the conjugate transpose of M; $\bar{\sigma}(M)$ denotes the largest singular value of M and $\rho(M)$ the spectral radius of M. The block diagonal matrix is denoted with specified diagonal blocks as $\Delta := \mathrm{Diag}[\Delta_1, \cdots, \Delta_r]$. The upper and lower LFTs of M on N are denoted as $\mathcal{F}_u(M,N)$ and $\mathcal{F}_l(M,N)$, respectively; and the Redheffer star product of M and N is $\mathcal{S}(M,N)$.

2 Linear Fractional Uncertain Systems and Stability

A control implication for **linear fractional transformation** (LFT) is the feedback structure. In this section, an LFT framework is proposed for the description of uncertain linear systems where the perturbations enter the systems in feedback fashions. We will consider the general linear systems with LFT descriptions, and their implications for linear fractional uncertain systems are revealed afterwards.

2.1 Systems with LFT Descriptions

Let's consider a class of systems each of which can be represented as an (upper) LFT on some block structure Δ, i.e.,

$$G = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta) \tag{1}$$

with $(A,B,C,D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$. We will refer to this class of linear systems as **LFT systems**, and (1) its transfer function. The block structure Δ is defined as follows,

$$\Delta := \mathrm{Diag}[\lambda I_{n_0}, \delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f], \tag{2}$$

which may include repeated full blocks and has dimension $n \times n$; the block structure may have several interpretations. We define a **scaling matrix set** with respect to the block structure Δ for the system as follows,

$$\mathcal{D} := \{D \in \mathbf{C}^{n\times n} \text{ is nonsingular} : D\Delta = \Delta D \text{ for all } \Delta \in \mathbf{C}^{n\times n} \text{ with structure (2)}\}. \tag{3}$$

By analogy with standard terminology, we can also give the representation (1) a state-space interpretation. As in the conventional one-dimensional systems, (non-singular) state variable transformations are useful in the analysis and synthesis of LFT systems. However, not all transformations are allowed in this setting; if we think of the system has "state" vector x, then the admissible state variable transformations $x \mapsto Tx$ is therefore specified as follows:

Definition 2.1 (Admissible Transformation) *Consider the LFT system (1) with block structure Δ defined by (2) and the corresponding scaling matrix set $\mathcal{D}$ by (3). If $T \in \mathcal{D}$, then the following system transformation:*

$$\mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta) \mapsto \mathcal{F}_u(\begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}, \Delta)$$

is said to be admissible.

It is remarked that the transfer function after the transformation does not change. We will later show that many other properties of LFT systems are also invariant under admissible transformations.

With the block structure Δ defined in (2) and the corresponding scaling matrix set (3), the μ and $\mathcal{Q}$ values of the matrix A are well defined. Next, we give some stability notions in terms of these two notions.

Definition 2.2 (μ-Stability and $\mathcal{Q}$-Stability) *Consider the LFT system (1) (or matrix A). It is μ-stable (with respect to Δ) if $\mu_\Delta(A) < 1$. It is $\mathcal{Q}$-stable (with respect to Δ) if $\mathcal{Q}_\Delta(A) < 1$, i.e., there is a $D \in \mathcal{D}$ such that $\bar{\sigma}(DAD^{-1}) < 1$.*

$\mathcal{Q}$-stability is a sufficient test for μ-stability, but μ-stability doesn't imply $\mathcal{Q}$-stability in general. These two stability notions are equivalent if and only if $\mu_\Delta(A) = \mathcal{Q}_\Delta(A)$. The implications of $\mathcal{Q}$-stability and μ-stability are given in the next subsection. In the following, we only consider $\mathcal{Q}$-stability, the μ-stability is discussed in [25]. It is noted the $\mathcal{Q}$-stability can be given the following Lyapunov characterization [14], [32].

Lemma 2.3 *System A with block structure Δ is $\mathcal{Q}$-stable if and only if there exists a $P \in \mathcal{D}$ with $P = P^* > 0$ such that*

$$APA^T - P < 0 \tag{4}$$

where the matrix set $\mathcal{D}$ is defined as in (3).

The following structural properties of LFT systems follows immediately from the properties of $\mathcal{Q}$ and a little algebra, so the proofs are omitted.

Theorem 2.4 *The $\mathcal{Q}$-stability of LFT systems is invariant under the admissible (system) transformations.*

Theorem 2.5 *Let A_1 and A_2 be two system matrices with respect to the block structures Δ_1 and Δ_2, respectively. Then*

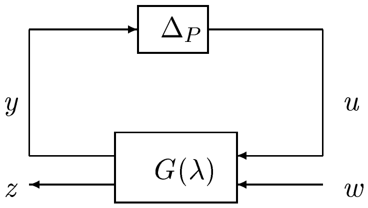
(i) *if the system matrix* $\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ *with any compatibly dimensioned matrices A_{12} and A_{21} is $\mathcal{Q}$-stable with respect to the block structure $\Delta := \mathrm{Diag}[\Delta_1, \Delta_2]$, where Δ_1 and Δ_2 are independent, then A_1 and A_2 are also $\mathcal{Q}$-stable with respect to structures Δ_1 and Δ_2, respectively.*

(ii) *the system matrix* $\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ *with any compatibly dimensioned matrix A_{12} is $\mathcal{Q}$-stable with respect to the block structure $\Delta := \mathrm{Diag}[\Delta_1, \Delta_2]$ if and only if A_1 and A_2 are also $\mathcal{Q}$-stable with respect to structures Δ_1 and Δ_2, respectively.*

Note that the statement **(i)** holds in the case where the structures Δ_1 and Δ_2 are independent, while **(ii)** holds even when Δ_1 and Δ_2 depend on each other. **(ii)** implies that a cascade system is $\mathcal{Q}$-stable if and only if each subsystem is $\mathcal{Q}$-stable. The definitions of μ-stability and $\mathcal{Q}$-stability are generalizations of the notions of conventional stability (see Section 7.2), robust stability, and robust performance [32, 25]. In the next section, we will further reveal those facts for $\mathcal{Q}$-stability.

2.2 Robust Stability of Linear Fractional Uncertain Systems

A large class of uncertain linear systems can be described in terms of LFTs on some specified structures. Consider an uncertain system, where the uncertainty enters the systems in a feedback fashion as illustrated as follows,



where G is the nominal linear discrete-time system, Δ_P is the uncertainty which belongs to a designated uncertainty set. Both G and Δ_P are causal, and the interconnection for the uncertain system is well-posed for each admissible uncertainty; w is an input vector, and z is an output vector. Thus, the transfer function of the uncertain system from w to z for each uncertainty Δ_P is represented in terms of the LFT formula as

$$\mathcal{F}_u(G(\lambda), \Delta_P). \tag{5}$$

Suppose that the uncertainty structure has the following form

$$\Delta_P := \mathrm{Diag}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f], \tag{6}$$

where no blocks are repeated. Define a scaling matrix set corresponding to the uncertainty structure Δ_P as

$$\mathcal{D}_P := \{\mathrm{Diag}[D_1, \cdots, D_s, d_1 I, \cdots, d_f I] \in \mathbf{C}^{m\times m} \text{ nonsingular} : D_i \in \mathbf{C}^{r_i\times r_i}, d_j \in \mathbf{C}\} \tag{7}$$

where the dimensions of each block are the same as those of the corresponding block in Δ_P.

Note that the transfer function (from $\begin{bmatrix} u \\ w \end{bmatrix}$ to $\begin{bmatrix} y \\ z \end{bmatrix}$) of the nominal system with dimension n_0 can be described in terms of the LFT formula:

$$G(\lambda) = \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{12}(\lambda) & G_{22}(\lambda) \end{bmatrix} =: \mathcal{F}_u(M, \lambda I_{n_0}), \quad M := \left[\begin{array}{c:cc} M_{11} & M_{12} & M_{13} \\ \hdashline M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array}\right], \tag{8}$$

where λ is the transform variable, i.e., the delay operator. Therefore, the transfer function in (5) is

$$\mathcal{F}_u(G(\lambda), \Delta_P) = \mathcal{F}_u(\mathcal{F}_u(M, \lambda I_{n_0}), \Delta_P) = \mathcal{F}_u(M, \begin{bmatrix} \lambda I_{n_0} & 0 \\ 0 & \Delta_P \end{bmatrix}).$$

Thus, the uncertain system is described as an LFT system:

$$\mathcal{F}_u(M, \Delta) =: \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta), \tag{9}$$

where

$$A := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad B := \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix}, \quad C := \begin{bmatrix} M_{31} & M_{32} \end{bmatrix}, \quad D := M_{33}, \tag{10}$$

and the block structure with dimension $(n_0 + m) \times (n_0 + m)$:

$$\Delta := \begin{bmatrix} \lambda I_{n_0} & 0 \\ 0 & \Delta_P \end{bmatrix} \tag{11}$$

Let $n := n_0 + m$, define a scaling matrix set corresponding to the block structure Δ as follows,

$$\mathcal{D} := \{\mathrm{Diag}[D_1, D_2] \in \mathbf{C}^{n\times n} : D_1 \in \mathbf{C}^{n_o\times n_o} \text{ is nonsingular}, D_2 \in \mathcal{D}_P\}. \tag{12}$$

Suppose the nominal system is internally stable, i.e., $\rho(M_{11}) < 1$. It is known that if the uncertainty Δ_P is linear time-invariant and bounded by 1, then the notion of μ-stability captures the robust stability and robust performances of the corresponding linear fractional uncertain systems [25]. Next, our emphasis is on dealing with the uncertain structure $\boldsymbol{\Delta}_P$ which includes time-varying uncertainty; we will consider the robust stability and performance of the linear fractional uncertain system given by (9) and establish the relations to the $\mathcal{Q}$-stability.

Let $\ell^{s\times t}$ denote the set of linear time-varying causal operators: $\ell_2^t(\mathbf{Z}^+) \to \ell_2^s(\mathbf{Z}^+)$. Consider the linear fractional uncertain system (9); the permissible uncertainty set is defined as

$$\mathbf{B}\boldsymbol{\Delta}_P^{LTV} := \{\Delta_P = \mathrm{Diag}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] \in \ell^{m\times m} : \|\Delta_P\|_{\ell_2} \le 1\}. \tag{13}$$

where $\|\bullet\|_{\ell_2}$ is the ℓ_2-induced norm of an LTV operator.

The transfer function of system (9) is $\mathcal{F}_u(G(\lambda), \Delta_P)$ with $\Delta_P \in \mathbf{B}\boldsymbol{\Delta}_P^{LTV}$. The system (9) is said to be **robustly stable** if the system is asymptotically stable for each fixed $\Delta_P \in \mathbf{B}\boldsymbol{\Delta}_P^{LTV}$.

It has **robust performance** if the system is stable and has ℓ_2-induced norm less than 1 for each $\Delta_P \in \mathbf{B}\mathbf{\Delta}_P^{LTV}$. The robust stability and robust performance problems have been extensively studied (see [29], [40], [35]). The analysis is reduced to the gain-analysis of some scaled systems. In fact, let $G(\lambda)$ be partitioned conformably with the block structure Δ_P as in (8). The following results are relatively new, but are now well-known (cf. [40], [29], [35], [9]).

Lemma 2.6 *Consider the uncertain system (9) with $\Delta_P \in \mathbf{B}\mathbf{\Delta}_P^{LTV}$. Then*

(i) *It is robustly stable if and only if the nominal system is stable, i.e., $\rho(M_{11}) < 1$, and there exists a positive definite matrix $D_P \in \mathcal{D}_P$ such that*

$$\sup_{|\lambda|\leq 1} \bar{\sigma}\left(D_P G_{11}(\lambda) D_P^{-1}\right) < 1.$$

(ii) *It has robust performance if and only if the nominal system is stable, and there exists a positive definite matrix $D_Q \in \mathcal{D}_P$ such that*

$$\sup_{|\lambda|\leq 1} \bar{\sigma}\left(\begin{bmatrix} D_Q & 0 \\ 0 & I \end{bmatrix} G(\lambda) \begin{bmatrix} D_Q^{-1} & 0 \\ 0 & I \end{bmatrix}\right) < 1.$$

It is noted that the above results also hold for nonlinear uncertainty [9]. We further have the following LMI characterizations.

Theorem 2.7 *Consider the uncertain system (9) with $\Delta_P \in \mathbf{B}\mathbf{\Delta}_P^{LTV}$. Then*

(i) *It is robustly stable if and only if there exists a positive definite matrix $P \in \mathcal{D}$ such that*

$$APA^T - P < 0.$$

(ii) *It has robust performance if and only if there exists a positive definite matrix $Q \in \mathcal{D}$ such that*

$$M \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} M^T - \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} < 0.$$

It is known from the discussion in the previous section that the LMI conditions of robust stability and robust performance for a linear fractional uncertain system are exactly the conditions for $\mathcal{Q}$-stability for some LFT systems. Therefore, from Theorem 2.7 and the definition of $\mathcal{Q}$-stability, we can conclude that the Q-stability is an abstraction of robust stability and robust performance of a linear fractional uncertain system under LTV perturbations.

3 Stabilization of LFT Systems: Stabilizability and Detectability

In this section, we will first examine the stabilizability and detectability of LFT systems in the context of $\mathcal{Q}$-stability, then we will consider their robust control implications for the linear fractional uncertain systems.

3.1 Stabilizability and Detectability

Consider an LFT system with block structure Δ defined by (2):

$$G = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta). \tag{14}$$

where y is the measurement output, and u is control input. Suppose $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$ and assume further that B and C are of full column and row ranks, respectively, i.e., $\text{Rank}(B) = p \leq n$ and $\text{Rank}(C) = q \leq n$. The scaling matrix set $\mathcal{D}$ with respect to Δ is given by (3).

The notions of stabilizability and detectability of the system (14) play important roles in the stabilization problem; they are defined in terms of the following two special structures of (14), respectively,

$$G_{SF} = \mathcal{F}_u(\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \Delta), \qquad G_{OI} = \mathcal{F}_u(\begin{bmatrix} A & I \\ C & 0 \end{bmatrix}, \Delta).$$

Definition 3.1 (Stabilizability) *The system (14) is $\mathcal{Q}$-stabilizable if there exists a controller for the corresponding system G_{SF}:*

$$K = \mathcal{F}_u(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \Delta_0)$$

such that the closed loop system is $\mathcal{Q}$-stable with respect to the induced block structure.

Definition 3.2 (Detectability) *The system (14) is $\mathcal{Q}$-detectable if there exists a controller for the corresponding system G_{OI}:*

$$K = \mathcal{F}_u(\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \Delta_0)$$

such that the closed loop system is $\mathcal{Q}$-stable with respect to the induced block structure.

It is known that for a one-dimensional system, the stabilizability is equivalent to the statement that the system can be stabilized by a constant state-feedback. Is this property still true for a general LFT system? The answer is positive for the $\mathcal{Q}$-case. But first, we shall examine when the LFT system can be $\mathcal{Q}$-stabilized by a constant state-feedback.

Suppose the LFT system is $\mathcal{Q}$-stabilized by a constant state-feedback matrix $F \in \mathbf{R}^{p\times n}$, i.e., $\mathcal{Q}_\Delta(A + BF) < 1$, then by the Lyapunov characterization of $\mathcal{Q}$-stability, there exists a $P \in \mathcal{D}$ with $P = P^T > 0$ such that

$$(A + BF)P(A + BF)^T - P < 0.$$

If $\text{Rank}(B) = p < n$ we can find a $B_\perp \in \mathbf{R}^{n\times(n-p)}$ such that $B^T B_\perp = 0$ and $\text{Rank}(B_\perp) = n - p$, then we have

$$B_\perp^T(A + BF)P(A + BF)^T B_\perp - B_\perp^T P B_\perp < 0,$$

i.e.,

$$B_\perp^T A P A^T B_\perp - B_\perp^T P B_\perp < 0.$$

So the solvability of the last LMI is necessary for the system to be constant-state-feedback $\mathcal{Q}$-stabilizable. Surprisingly, this condition is also sufficient as stated by the following proposition.

Proposition 3.3 *Consider the LFT system (14) with block structure* Δ*, and* $\mathrm{Rank}(B) = p < n$*. Let* $B_\perp \in \mathbf{R}^{n\times(n-p)}$ *be such that* $B^T B_\perp = 0$ *and* $\begin{bmatrix} B & B_\perp \end{bmatrix}$ *be invertible. There exists a constant state feedback* F *such that* $A + BF$ *is* $\mathcal{Q}$*-stable with respect to the block structure* Δ *if and only if there exists a matrix* $P \in \mathcal{D}$ *with* $P = P^T > 0$ *such that*

$$B_\perp^T APA^T B_\perp - B_\perp^T PB_\perp < 0. \tag{15}$$

Moreover, if $P \in \mathcal{D}$ *with* $P = P^T > 0$ *satisfies the above inequality, then a* $\mathcal{Q}$*-stabilizing constant state feedback matrix can be chosen as*

$$F = -(B^T P^{-1} B)^{-1} B^T P^{-1} A. \tag{16}$$

Proof. By the definition of $\mathcal{Q}$-stability, there exists a constant feedback F such that the closed loop system matrix $A + BF$ is $\mathcal{Q}$-stable with respect to the block structure Δ if and only if there exists a $D \in \mathcal{D}$, such that

$$1 > \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(D(A + BF)D^{-1}) = \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(DAD^{-1} + DBFD^{-1}).$$

Let

$$V_\perp^T = (B_\perp^T (D^T D)^{-1} B_\perp)^{-\frac{1}{2}} B_\perp^T D^{-1},$$

then it is easy to check that $V_\perp^T V_\perp = I$ and $V_\perp^T(DB) = 0$. By Lemma 7.8, we have

$$1 > \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(D(A + BF)D^{-1}) = \bar{\sigma}(V_\perp^T DAD^{-1}),$$

i.e.,

$$(V_\perp^T DAD^{-1})(V_\perp^T DAD^{-1})^T < I.$$

Take $P = (D^T D)^{-1}$, then $P \in \mathcal{D}$ and $P = P^T > 0$; hence, we have

$$(B_\perp^T PB_\perp)^{-\frac{1}{2}} B_\perp^T APA^T B_\perp (B_\perp^T PB_\perp)^{-\frac{1}{2}} - I < 0,$$

or

$$B_\perp^T APA^T B_\perp - B_\perp^T PB_\perp < 0.$$

Moreover, if some $P \in \mathcal{D}$ with $P = P^T > 0$ satisfies the above inequality, then we can construct a constant state feedback matrix F via Lemma 7.8 such that $A + BF$ is $\mathcal{Q}$-stable.

In fact, let

$$V_0^T = (B^T(D^T D)B)^{-1/2} B^T D^T.$$

Then $[V_0, V_\perp]$ is unitary; thus $FD^{-1} = -(V_0^T DB)^{-1} V_0^T DAD^{-1}$, or

$$F = -(V_0^T DB)^{-1} V_0^T DA = -(B^T P^{-1} B)^{-1} B^T P^{-1} A.$$

□

The following theorem gives another LMI characterization for $\mathcal{Q}$-stabilizability.

Theorem 3.4 *Consider the LFT system (14) with block structure* Δ*. Then there exists a constant state feedback* F *such that* $A + BF$ *is* $\mathcal{Q}$*-stable if and only if there exists a matrix* $P \in \mathcal{D}$ *with* $P = P^T > 0$ *such that*

$$APA^T - P - BB^T < 0. \tag{17}$$

Moreover, if $P \in \mathcal{D}$ *with* $P = P^T > 0$ *satisfies the above inequality, then a* $\mathcal{Q}$*-stabilizing constant state feedback matrix can be chosen as*

$$F = -(B^T P^{-1} B)^{-1} B^T P^{-1} A. \tag{18}$$

Proof. If B is square and of full rank, then the result is straightforward. We thus consider the case where $\mathrm{Rank}(B) = p < n$. We only need to show that the two LMI characterizations (15) and (17) are equivalent. The implication (17) $\Rightarrow$ (15) follows by observing that

$$B_\perp^T(APA^T - P - BB^T)B_\perp = B_\perp^T(APA^T - P)B_\perp.$$

To show the reverse implication, suppose there exists a positive definite matrix $Q \in \mathbf{F}^{n\times n}$ such that

$$B_\perp^T(AQA^T - Q)B_\perp < 0,$$

Note that $(B_\perp^T(AQA^T - Q)B)(B^TB)^{-2}(B^T(AQA^T - P)B_\perp)$ is a symmetric matrix, then there exist $\alpha > 0$ sufficiently small such that

$$B_\perp^T(AQA^T - Q)B_\perp + \alpha(B_\perp^T(AQA^T - Q)B)(B^TB)^{-2}(B^T(AQA^T - Q)B_\perp) < 0.$$

Let $P = \alpha Q$ which is positive definite, then the above inequality implies

$$\begin{bmatrix} B_\perp^T(APA^T - P)B_\perp & B_\perp^T(APA^T - P)B \\ B^T(APA^T - P)B_\perp & -(B^TB)^2 \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} B_\perp^T \\ B^T \end{bmatrix}(APA^T - P - BB^T)\begin{bmatrix} B_\perp & B \end{bmatrix} < 0. \tag{19}$$

which implies (17). $\square$

Using the above result we can easily get the following theorem.

Theorem 3.5 *The system G is $\mathcal{Q}$-stabilizable if and only if there exists a constant feedback matrix F such that $A + BF$ is $\mathcal{Q}$-stable with respect to the same block structure.*

Proof. The sufficiency is obvious. As for the necessity, assume that the system can be $\mathcal{Q}$-stabilized by a controller $K = \mathcal{F}_u(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \Delta_0)$ where Δ_0 depends on the system block structure Δ. By the interconnection properties of LFTs, this statement is equivalent to the fact that the augmented system,

$$G_a = \mathcal{F}_u\left(\left[\begin{array}{cc:cc} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ \hdashline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right], \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}\right),$$

is $\mathcal{Q}$-stabilized by the constant feedback $\begin{bmatrix} F_{22} & F_{21} \\ F_{12} & F_{11} \end{bmatrix}$ with respect to the block structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$. Denote the scaling matrix set with respect to Δ_N by $\mathcal{D}_N$, then by the above proposition, there exists a positive definite matrix $P_N =: \begin{bmatrix} P & P_1 \\ P_1^T & P_0 \end{bmatrix} \in \mathcal{D}_N$ such that

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P_N \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^T - P_N - \begin{bmatrix} BB^T & 0 \\ 0 & I \end{bmatrix} < 0,$$

which implies

$$APA^T - P - BB^T < 0.$$

Thus the above LMI has a solution $P > 0$. The conclusion follows via the previous theorem if we can verify that $P \in \mathcal{D}$. Indeed, since $P_N \in \mathcal{D}_N$, $P_N\Delta_N = \Delta_N P_N$, i.e.,

$$\begin{bmatrix} P & P_1 \\ P_1^T & P_0 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix} \begin{bmatrix} P & P_1 \\ P_1^T & P_0 \end{bmatrix} \tag{20}$$

for all complex matrices Δ and Δ_0 with the given structures; in particular, (20) implies $P\Delta = \Delta P$. By the definition of the the scaling matrix set $\mathcal{D}$ (3), we have $P \in \mathcal{D}$. □

The dual notion $\mathcal{Q}$-detectability can be similarly characterized in terms of LMIs by some dual arguments. We have the following theorem.

Theorem 3.6 *The given system $G(\Delta)$ is $\mathcal{Q}$-detectable if and only if there exists a matrix $P \in \mathcal{D}$ with $P = P^T > 0$ such that*

$$A^T PA - P - C^T C < 0. \tag{21}$$

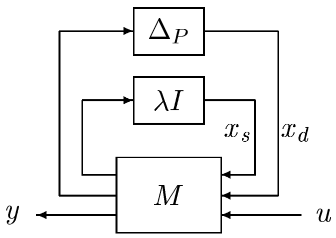
In addition, there exists a constant output injection matrix L such that $A + LC$ is $\mathcal{Q}$-stable with respect to the same frequency structure, and such an output injection matrix L can be taken as

$$L = -AP^{-1}C^T(CP^{-1}C^T)^{-1}.$$

for some $P \in \mathcal{D}$ with $P = P^T > 0$ satisfying the LMI (21).

3.2 Control of Linear Fractional Uncertain Systems: Constant Feedback

In the last subsection, we considered the general notions of stabilizability for LFT systems. It was further shown that the stabilizability leads to constant feedback stabilization. In the following, we will further explore the implications of the notions of $\mathcal{Q}$-stabilizability, and discuss several simple stabilization problems for the linear fractional uncertain systems. The system considered is represented as an LFT of a constant matrix on some block structure and is illustrated by the following diagram.



The uncertainty Δ_P is allowed to be structured and LTV, and varies in the uncertainty set $\mathbf{B\Delta}_P^{LTV}$ as defined in (13). x_s is the real state of the system and x_d is the signals from the uncertainty (and external disturbances if performance problem is considered). The transfer function is

$$G = \mathcal{F}_u(M, \Delta) =: \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta), \tag{22}$$

where $(A, B, C, D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n} \times \mathbf{R}^{q\times p}$, A is partitioned conformably with the block structure $\Delta := \mathrm{Diag}[\lambda I_{n_0}, \Delta_P]$ as $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, and B and C are of full column and row ranks, respectively; to avoid triviality, it is assumed that $\mathrm{Rank}(B) = p < n$ and $\mathrm{Rank}(C) = q < n$. The scaling matrix set $\mathcal{D}$ with respect to Δ is given by (3).

Constant Full Information Feedback

Suppose the full information $x := \begin{bmatrix} x_s \\ x_d \end{bmatrix}$ is available; the following statement directly follows from the discussions in the last subsection.

Theorem 3.7 *Consider the linear fractional uncertain system (22), There exists a constant full information feedback such that the system is robustly stable if and only if there exists a positive definite solution in $P \in \mathcal{D}$ such that any of the following LMIs is satisfied,*

$$B_\perp^T(APA^T - P)B_\perp < 0,$$

$$APA^T - P - BB^T < 0;$$

and such a constant feedback is given by

$$u = -(B^T P^{-1} B)^{-1} B^T A P^{-1} x.$$

Therefore, the full information problem is solvable if and only if the related LFT system is $\mathcal{Q}$-stabilizable. In the following, we will consider other two cases, i.e., only the real state of the system x_s is available to the controller, and the measured output y is available to the controller.

Constant State Feedback

Theorem 3.8 *Consider system (22), there exists a constant state feedback $u = Kx_s$ such that the closed loop system is robustly stable if and only if there exists a positive definite matrix $P =$ $\mathrm{Diag}[P_1, P_2] \in \mathcal{D}$ satisfying the following two LMIs:*

$$B_\perp^T(APA^T - P)B_\perp < 0, \tag{23}$$

$$A_2 P_2 A_2^T - P < 0. \tag{24}$$

Proof. By the definition of $\mathcal{Q}$-stabilizability, the closed loop system $A + B \begin{bmatrix} K & 0 \end{bmatrix}$ is robustly stable iff there exists $D \in \mathcal{D}$ such that

$$\inf_{K \in \mathbf{R}^{p \times n_0}} \bar{\sigma}(D(A + B \begin{bmatrix} K & 0 \end{bmatrix})D^{-1}) < 1. \tag{25}$$

Let $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$, then the above inequality is equivalent to

$$\inf_{K \in \mathbf{R}^{p \times n_0}} \bar{\sigma}(DAD^{-1} + DB \begin{bmatrix} KD_1^{-1} & 0 \end{bmatrix})$$

Let $V_\perp^T = (B_\perp^T(D^TD)^{-1}B_\perp)^{-\frac{1}{2}} B_\perp^T D^{-1}$ and $U_\perp^T = \begin{bmatrix} 0 & I \end{bmatrix}$. It is easy to check that $V_\perp^T V_\perp = I$ and $V_\perp^T(DB) = 0$, and $U_\perp^T U_\perp = I$ and $\begin{bmatrix} KD_1^{-1} & 0 \end{bmatrix} U_\perp^T = 0$. By Proposition 7.7, we have that (25) holds if and only if

$$\max\{\bar{\sigma}(V_\perp^T DAD^{-1}), \bar{\sigma}(DA_2D_2^{-1})\} < 1. \tag{26}$$

Take

$$P = (D^T D)^{-1} = \begin{bmatrix} (D_1^T D_1)^{-1} & 0 \\ 0 & (D_2^T D_2)^{-1} \end{bmatrix} =: \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

then $P \in \mathcal{D}$ and $P = P^T > 0$. The condition (26) can be rewritten as follows,

$$\bar{\sigma}(V_\perp^T DAD^{-1}) < 1 \quad \Leftrightarrow \quad B_\perp^T (APA^T - P) B_\perp < 0,$$

and

$$\bar{\sigma}(DA_2D_2^{-1}) < 1 \quad \Leftrightarrow \quad A_2P_2A_2^T - P < 0.$$

Thus, (25) holds iff we can find some matrix $P = P^T > 0$ satisfying LMIs (23) and (24). □

In the above theorem, such a constant state-feedback matrix K is explicitly given in [21] by the use of Parrott's theorem (Lemma 7.6); also the condition (23) can be replaced by the following LMI:

$$APA^T - P - BB^T < 0,$$

which, as well as (23), implies $\mathcal{Q}$-stabilizability.

Constant Output Feedback

The following theorem gives one constant output feedback solution; it can be proved similarly to the above theorem by using the Parrott's Theorem (Proposition 7.7).

Theorem 3.9 *Consider the system (9). Assume that $B_\perp \in \mathbf{R}^{n\times(n-p)}$ is such that $B^T B_\perp = 0$ and $\begin{bmatrix} B & B_\perp \end{bmatrix}$ is invertible, and $C_\perp \in \mathbf{R}^{(n-q)\times n}$ is such that $C_\perp C^T = 0$ and $\begin{bmatrix} C \\ C_\perp \end{bmatrix}$ is invertible. Then there exists an admissible constant output-feedback controller if and only if there exists a positive definite matrix $X \in \mathcal{D}$ such that the following two matrix inequalities hold:*

$$B_\perp^T AXA^T B_\perp - B_\perp^T X B_\perp < 0$$

$$C_\perp A^T X^{-1} A C_\perp^T - C_\perp X^{-1} C_\perp^T < 0.$$

Unfortunately, the above LMIs are not convex in X and may not even have connected solution sets. Thus it is not clear that the search over X is easier than the original problem of directly searching over controller gains.

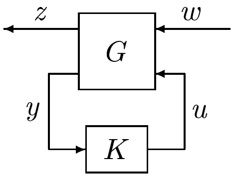
It is a routine adaptation of Parrott's theorem to get a parametrization of all constant $\mathcal{Q}$-stabilizing output-feedback matrices. The above matrix inequality characterizations imply that the system is $\mathcal{Q}$-stabilizable and $\mathcal{Q}$-detectable. The above treatment in each case is suitable not only for the robust stabilization problem, but also for robust performance synthesis problem by the standard trick of adding the additional performance block into Δ_P [15].

4 Control of LFT Systems: Problem Statement and Special Structures

From now on, the stabilization problem for general LFT systems are investigated. The implication of the results for robust stabilization of linear fractional uncertain systems is obvious. In this section, we will examine the basic structures of the problems.

4.1 Problem Statement

Consider the control LFT system with standard block diagram



where G is the plant with two sets of inputs: the exogenous inputs w and the control inputs u, and with two sets of outputs: the measured outputs y and the outputs z measuring the behavior of the system. The control problem is to design a feedback controller K such that the resulting closed loop system has some prescribed properties.

Suppose that the plant G with block structure Δ is defined as follows,

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \mathcal{F}_u(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \Delta), \tag{27}$$

where all matrices are real and have compatible dimensions with the related physical variables, the block structure is defined in (2) and the corresponding scaling matrix set is defined in (3). In addition, let the controller K with block structure Δ_0 be defined as

$$K = \mathcal{F}_u(\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}, \Delta_0). \tag{28}$$

In the following, it is assumed that the the feedback system is well-posed. The block structure Δ_0 of the controller is dependent on Δ. In particular, as we will see, for the output-feedback $\mathcal{Q}$-stabilization problem to be considered, the controller may have the same dependence on the block structure as the plant, i.e., $\Delta_0 = \Delta$. This type of controllers for uncertain linear systems can be given a “gain scheduling” or “dynamic scheduling” interpretation; in the linear multidimensional system case, this means that dynamical feedback controllers are allowed (see Section 7.2).

Since we are solely focusing on $\mathcal{Q}$-stabilization problem, we will say that a feedback controller K is **admissible** if it has the same dependence on the block structure as the plant and $\mathcal{Q}$-stabilizes (27), i.e., $\mathcal{F}_l(G, K)$ is $\mathcal{Q}$-stable with respect to the induced block structure $\mathrm{Diag}[\Delta, \Delta]$. Then the admissible controller set is denoted as $\mathcal{K}$. For convenience, this general synthesis problem is called the **output feedback** (OF) problem. The following two synthesis problems are considered in this paper:

- **(Stabilization)** Find a dynamical output feedback $K \in \mathcal{K}$ which $\mathcal{Q}$-stabilizes (27).
- **(Parametrization)** Characterize all controllers $K \in \mathcal{K}$ that $\mathcal{Q}$-stabilize (27), or more specifically, find an LFT system J which has the same dependence on the block structure as the plant such that $\mathcal{K} = \{\mathcal{F}_l(J, Q) : Q \text{ is a } \mathcal{Q}\text{-stable LFT system}\}$.

Note that G is $\mathcal{Q}$-stabilized by K if and only if G_{22} is $\mathcal{Q}$-stabilized by K. Thus, the input w and output z do not affect the final stabilization results. They are preserved in the following treatment only for technical reasons and to facilitate comparison with related problems where they are present. Next, we will examine some basic structural property for control LFT systems; we will use the following notation to represent an LFT system,

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \mathcal{F}_u(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right], \Delta),$$

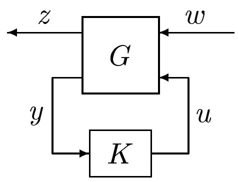
provided that the block structure Δ is clear from context.

4.2 Duality, Equivalence, and Special Problems

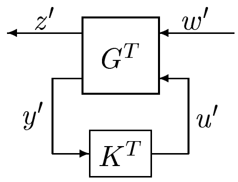
Duality of LFT Systems

Duality plays an important role in the analysis and synthesis of linear systems. We now examine this notion further in the LFT setting employed in this paper. Algebraically, the **dual system** of an LFT system G with real parameters is defined as the system represented by the transpose G^T of G with the same structure. It is noted that if $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, then its dual system is $G^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$.

Next, we consider a feedback system with block diagram



where the plant G and the controller K are represented as LFTs on some real matrices with respect to the same block structure Δ. The dual structure of the above feedback system is shown as follows



whose plant and controller are the dual objects of G and K, i.e., G^T and K^T, respectively. It is routine to verify that $\mathcal{F}_l(G^T, K^T) = [\mathcal{F}_l(G, K)]^T$. In addition, K $\mathcal{Q}$-stabilizes G with respect to the induced block structure $\Delta_N = \mathrm{Diag}[\Delta, \Delta]$ if and only if K^T $\mathcal{Q}$-stabilizes G^T with respect to the block structure Δ_N. Whence, as far as stabilization or other synthesis problems are concerned, we can obtain the results for system G^T from those of its dual object G if available.

Equivalence of LFT Systems

System equivalence notion will also play an important role in the following. Two systems are said to be **equivalent** if every achievable closed loop map of one system can be achieved by the other through some controller. More concretely, two LFT systems G_1 and G_2 with the same block structure are equivalent, if for all possible K_1, there exists an suitable K_2 depending on the same block structure as K_1, such that $\mathcal{F}_l(G_1, K_1) = \mathcal{F}_l(G_2, K_2)$; and also for all possible K_2, there is an suitable K_1 depending on the same block structure as K_2, such that $\mathcal{F}_l(G_1, K_1) = \mathcal{F}_l(G_2, K_2)$.

Special Structures

We consider four special structures which are related to the general OF problem whose associated plant G with block structure Δ is given as in (27):

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{29}$$

- **Full information** (FI) problem. The corresponding plant has a structure as

$$G_{FI} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right], \tag{30}$$

  where the "state" and the disturbance are directly measured.

- **Full control** (FC) problem. The associated plant is given by

$$G_{FC} = \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right], \tag{31}$$

  where two independent parts of the control input directly affect both the "state" and regulated output.

- **Disturbance feedforward** (DF) problem. The corresponding plant has a structure as

$$G_{DF} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array}\right], \tag{32}$$

  where the disturbance enters the measurement directly.

- **Output estimation** (OE) problem. The corresponding plant has the following structure.

$$G_{OE} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array}\right], \tag{33}$$

  where the control input enter the regulated output directly.

Note that all of these special systems have the same block structures as G. However, the parameters in the special structures do not necessarily refer to the same parameters of the above OF structure G. They are said to be special cases of the OF problem only in their structures. The reader is referred to [13] for motivations of different problems. Structurally, FI and FC problems are dual, so are DF and OE problems. More precisely, G_{FI}^T has an FC structure, G_{FC}^T has an FI structure, and so on. In addition, FI and DF, and FC and OE are equivalent structures, respectively; this point will be made precise in the next section.

5 Stabilization and Controller Parametrization of LFT Systems

In this section, we will show through the construction of solutions that the $\mathcal{Q}$-stabilizability and $\mathcal{Q}$-detectability are necessary and sufficient conditions for the output-feedback $\mathcal{Q}$-stabilization problem to be solvable.

5.1 Solutions to Stabilization Problems

In this subsection, we give the main results about the stabilization of LFT systems.

Theorem 5.1 (Stabilization) *Given a system G (27). There exists an admissible controller, i.e., $\mathcal{K} \neq \emptyset$, if and only if there exist two positive definite matrices $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ such that the following two LMIs hold:*

$$AXA^T - X - B_2B_2^T < 0, \tag{34}$$

$$A^TYA - Y - C_2^TC_2 < 0. \tag{35}$$

Moreover, when the conditions hold, such a controller is given by

$$K = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right] \tag{36}$$

with the same block structure Δ as the plant, where

$$F = -(B_2^TX^{-1}B_2)^{-1}B_2^TX^{-1}A, \qquad L = -AY^{-1}C_2^T(C_2Y^{-1}C_2^T)^{-1}. \tag{37}$$

The LMI conditions (34) and (35) in Theorem 5.1 imply that the system is $\mathcal{Q}$-stabilizable and $\mathcal{Q}$-detectable.

Theorem 5.2 (Controller Parametrization) *Assume that the conditions in Theorem 5.1 are satisfied, then the admissible controller set for system (27) can be characterized by*

$$\mathcal{K} = \{\mathcal{F}_l(J, Q) : Q \textit{ is } \mathcal{Q}\textit{-stable}\}, \tag{38}$$

where

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right]$$

with same block structure Δ as the plant.

The next two subsections are mainly devoted to the constructive proofs of the above two main theorems. The necessity follows from Theorems 3.9 and 3.4; we will only consider the sufficiency in the following. We follow [13] to present a state-space-like approach to this problem without using any idea from coprime factorization, but using LFT machinery. We first reduce the OF problem into the simpler FI and OE problems, then solve the output feedback problem by a separation argument.

5.2 Stabilization Problems for Special Systems

In this subsection, we will develop machinery leading to a constructive proofs of the main theorems, and consider the $\mathcal{Q}$-stabilization problems for special problems.

Admissible Controllers for FI and FC Systems

We first examine the dual structures: FI and FC. Consider the plant G_{FI} and (30) and G_{FC} (31) with the same block structure Δ; we immediately have the following statements using Theorem 2.5.

Proposition 5.3 *Let F be a constant matrix such that $A + B_2F$ is $\mathcal{Q}$-stable. Then a class of admissible controllers for the FI system can be parametrized as*

$$K_{FI} = \begin{bmatrix} F & Q \end{bmatrix}$$

with $\mathcal{Q}$-stable Q.

Proposition 5.4 *Let L be a constant matrix such that $A + LC_2$ is $\mathcal{Q}$-stable. Then a class of admissible controllers for FC can be parametrized as*

$$K_{FC} = \begin{bmatrix} L \\ Q \end{bmatrix}$$

with any $\mathcal{Q}$-stable Q.

Equivalence Relations between Special Problems

The equivalence relations between DF and FI, and OE and FC problems are examined in this subsection. We will construct the controller parametrizations for DF and OE problem through the equivalence relations. The different structures G_{FI}, G_{FC}, G_{DF}, and G_{OE} are given as in the preceding section. We first have the following observation about DF and FI problems.

Proposition 5.5 *Consider FI and DF structures as given in the last section. Then*

(i) $G_{DF} = \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & I \end{bmatrix} G_{FI}.$

(ii) $G_{FI} = \mathcal{S}(G_{DF}, P_{DF})$, *where $\mathcal{S}$ denotes the* **Redheffer star product** *and*

$$P_{DF} = \left[\begin{array}{c|cc} A - B_1C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ \begin{bmatrix} I \\ -C_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]. \tag{39}$$

Proof. **(i)** is easy, we only prove **(ii)**. Consider system $\mathcal{S}(G_{DF}, P_{DF})$, let x and $\hat{x}$ denote the "states" of G_{DF} and P_{DF}, respectively; conduct a state transformation

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \mapsto \begin{bmatrix} x - \hat{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix},$$

which is admissible. The resulting interconnected system is as follows

$$\left[\begin{array}{cc|cc} A-B_1C_2 & 0 & 0 & 0 \\ B_1C_2 & A & B_1 & B_2 \\ \hline C_1 & C_1 & D_{11} & D_{12} \\ \begin{bmatrix} 0 \\ C_2 \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

with respect to the block structure Diag$[\Delta, \Delta]$. The resulting transfer matrix is exactly G_{FI}, as claimed. □

The following theorem follows the above observation immediately:

Theorem 5.6 **(i)** *$K_{FI} := K_{DF}\left[\begin{array}{cc} C_2 & I \end{array}\right]$ $\mathcal{Q}$-stabilizes G_{FI} if K_{DF} $\mathcal{Q}$-stabilizes G_{DF}. Furthermore,*

$$\mathcal{F}_l(G_{FI}, K_{DF}\left[\begin{array}{cc} C_2 & I \end{array}\right]) = \mathcal{F}_l(G_{DF}, K_{DF}).$$

(ii) *Suppose that $A - B_1C_2$ is $\mathcal{Q}$-stable. Then $K_{DF} := \mathcal{F}_l(P_{DF}, K_{FI})$ $\mathcal{Q}$-stabilizes G_{DF} if K_{FI} $\mathcal{Q}$-stabilizes G_{FI}. Furthermore, $\mathcal{F}_l(G_{DF}, \mathcal{F}_l(P_{DF}, K_{FI})) = \mathcal{F}_l(G_{FI}, K_{FI})$.*

Proof. **(i)** it is easy. As for **(ii)**, note that by Proposition 5.5, we have

$$\mathcal{F}_l(G_{FI}, K_{FI}) = \mathcal{F}_l(\mathcal{S}(G_{FI}, P_{DF}), K_{FI}) = \mathcal{F}_l(G_{DF}, \mathcal{F}_l(P_{DF}, K_{FI})),$$

the $\mathcal{Q}$-stability of the latter is confirmed by Theorem 2.5 (ii), because of the structure of the closed loop system in terms of the previous observation, the $\mathcal{Q}$-stability of $A - B_1C_2$, and the choice of K_{FI}. □

This theorem shows that if $A - B_1C_2$ is $\mathcal{Q}$-stable, then problems FI and DF are equivalent.

Consider the DF system defined in the last section. Let K_{DF} be an admissible controller. We next parametrize K_{DF}. By Theorem 5.6, $K_{FI} = K_{DF}\left[\begin{array}{cc} C_2 & I \end{array}\right]$ $\mathcal{Q}$-stabilizes the corresponding G_{FI}. Then by Proposition 5.3, given F such that $A + B_2F$ is $\mathcal{Q}$-stable, there exists Q which is $\mathcal{Q}$-stable such that $K_{FI} = \left[\begin{array}{cc} F & Q \end{array}\right]$ $\mathcal{Q}$-stabilizes G_{FI}. Furthermore, $\mathcal{F}_l(J_{DF}, Q) = \mathcal{F}_l(P_{DF}, K'_{FI})$ where

$$J_{DF} = \left[\begin{array}{c|cc} A + B_2F - B_1C_2 & B_1 & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]. \tag{40}$$

Hence by Theorem 5.6, $K_{DF} := \mathcal{F}_l(J_{DF}, Q)$ stabilizes G_{DF} for any $\mathcal{Q}$-stable Q.

Actually, the above constructed parametrization characterizes all admissible controllers (not just a class of them) for the DF system.

Proposition 5.7 *Consider the DF system. Suppose it is $\mathcal{Q}$-stabilizable and $A - B_1C_2$ is $\mathcal{Q}$-stable. Then all admissible controllers for the DF problem can be characterized by $K_{DF} = \mathcal{F}_l(J_{DF}, Q_0)$ with $\mathcal{Q}$-stable Q_0, where J_{DF} is given in (40).*

Proof. From the above construction, it is known that the controllers expressed in the given LFT formula do $\mathcal{Q}$-stabilize G_{DF}. Let K_{DF} be any admissible controller for G_{DF}, then $Q := \mathcal{F}_l(\hat{J}_{DF}, K_{DF})$ is $\mathcal{Q}$-stable where

$$\hat{J}_{DF} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right],$$

since $\hat{J}_{DF}$ and G_{DF} have the same parameters (C_2, A, B_2) which are the only parameters affecting the stabilization results. We now claim that $K_{DF} = \mathcal{F}_l(J_{DF}, Q)$.

In fact, $\mathcal{F}_l(J_{DF}, Q) = \mathcal{F}_l(J_{DF}, \mathcal{F}_l(\hat{J}_{DF}, K_{DF})) =: \mathcal{F}_l(J_{tmp}, K_{DF})$, where

$$J_{tmp} = \left[\begin{array}{cc|cc} A - B_1C_2 + B_2F & -B_2F & B_1 & B_2 \\ -B_1C_2 & A & B_1 & B_2 \\ \hline F & -F & 0 & I \\ -C_2 & C_2 & I & 0 \end{array}\right].$$

By conducting an admissible transformation, we have

$$J_{tmp} = \left[\begin{array}{cc|cc} A - B_1C_2 & -B_2F & B_1 & B_2 \\ 0 & A + B_2F & 0 & 0 \\ \hline 0 & -F & 0 & I \\ 0 & C_2 & I & 0 \end{array}\right] = \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array}\right].$$

Hence, $\mathcal{F}_l(J_{DF}, Q) = \mathcal{F}_l(J_{tmp}, K_{DF}) = K_{DF}$. This shows that any admissible controller can be expressed in the form of $\mathcal{F}_l(J_{DF}, Q)$ for some $\mathcal{Q}$-stable Q. □

Dually, we can also conclude that the structures FC and OE are equivalent provided $A - B_2C_1$ is $\mathcal{Q}$-stable. Furthermore, we have the following results about the structures FC and OE.

Theorem 5.8 **(i)** $K_{FC} := \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}$ *$\mathcal{Q}$-stabilizes G_{FC} if K_{OE} $\mathcal{Q}$-stabilizes G_{OE}. Furthermore,*

$$\mathcal{F}_l(G_{FC}, \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}) = \mathcal{F}_l(G_{OE}, K_{OE}).$$

(ii) *Suppose that $A - B_2C_1$ is $\mathcal{Q}$-stable. Then $K_{OE} := \mathcal{F}_l(P_{OE}, K_{FC})$ $\mathcal{Q}$-stabilizes G_{OE} if K_{FC} $\mathcal{Q}$-stabilizes G_{FC}. Furthermore, $\mathcal{F}_l(G_{OE}, \mathcal{F}_l(P_{OE}, K_{FC})) = \mathcal{F}_l(G_{FC}, K_{FC})$.*

Proposition 5.9 *Assume that $A - B_2C_1$ is $\mathcal{Q}$-stable. All admissible controllers for the OE problem can be characterized as $\mathcal{F}_l(J_{OE}, Q_0)$ with any $\mathcal{Q}$-stable Q_0, where J_{OE} is defined as*

$$J_{OE} = \left[\begin{array}{c|cc} A - B_2C_1 + LC_2 & L & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

with L such that $A + LC_2$ is $\mathcal{Q}$-stable.

5.3 Stabilization by Output Feedback

In this subsection, we will prove the main results based on the results provided in the last subsection. We only prove Theorem 5.2; Theorem 5.1 is obtained by letting $Q = 0$. The construction essentially involves reducing the OF problem to the simpler FI and FC problem. Consider System G (27) with the block structure Δ:

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

Controller Construction

We shall assume $D_{22} = 0$ without loss of generality. In fact, if $D_{22} \neq 0$, then the mapping

$$\hat{K} = K(I - D_{22}K)^{-1} = \mathcal{F}_l(\begin{bmatrix} 0 & I \\ I & D_{22} \end{bmatrix}, K)$$

is well defined by the assumption that the closed-loop system is well-posed. Define

$$\hat{G} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

then $\mathcal{F}_l(G, K) = \mathcal{F}_l(\hat{G}, \hat{K})$. Thus, if $\hat{K}$ is designed for the above structure, and K can be obtained from Lemma 7.1 as

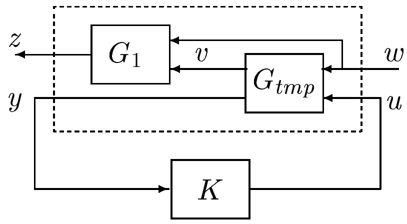
$$K = \mathcal{F}_u(\begin{bmatrix} -D_{22} & I \\ I & 0 \end{bmatrix}, \hat{K}) = \mathcal{F}_l(\begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix}, \hat{K}). \tag{41}$$

This justifies the simplification.

Next, we construct the controllers for OF problem with $D_{22} = 0$. Let x denote the state of the system G. Since (A, B_2) is $\mathcal{Q}$-stabilizable, there is a constant matrix F such that $A + B_2F$ is $\mathcal{Q}$-stable. Note that $\begin{bmatrix} F & 0 \end{bmatrix}$ is actually a special FI stabilizing controller. Let

$$v = u - Fx,$$

then the system can be broken into two subsystems G_1 and G_{tmp} as shown pictorially below



with

$$G_1 = \left[\begin{array}{c|cc} A + B_2F & B_1 & B_2 \\ \hline C_1 + D_{12}F & D_{11} & D_{12} \end{array}\right]$$

which is $\mathcal{Q}$-stable, and

$$G_{tmp} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -F & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right].$$

Both have the same block structure Δ as the original plant. Since G_1 is $\mathcal{Q}$-stable, by Theorem 2.5 K $\mathcal{Q}$-stabilizes G if and only if K $\mathcal{Q}$-stabilizes G_{tmp}. Note that G_{tmp} is of OE structure. Let L be such that $A + LC_2$ is $\mathcal{Q}$-stable. Since $A + B_2F$ is $\mathcal{Q}$-stable by construction, by Theorem 5.8 all controllers $\mathcal{Q}$-stabilizing G_{tmp} are given by

$$K = \mathcal{F}_l(J, Q)$$

where

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$

In particular, if $Q = 0$, then

$$K = \left[\begin{array}{c|c} A + B_2F + LC_2 & -L \\ \hline F & 0 \end{array}\right].$$

When D_{22} is arbitrary, the conclusions in the main theorems follow by the transformation (41).

Uniqueness of Parametrization

Let us further examine Theorem 5.2. The theorem says that any admissible controller K can be characterized as an LFT of a $\mathcal{Q}$-stable parameter matrix Q, i.e., $K = \mathcal{F}_l(J, Q)$. In fact, such a Q can be uniquely determined by K. To state this precisely, recall from the inversion formulas for LFTs in Lemma 7.1 that we can solve the equation $K = \mathcal{F}_l(J, Q)$ to uniquely give

$$Q = \mathcal{F}_u(J^{-1}, K) = \mathcal{F}_l(\hat{J}, K)$$

where a little algebra shows that

$$J^{-1} = \left[\begin{array}{c|cc} A & B_2 & L \\ \hline C_2 & D_{22} & I \\ -F & I & 0 \end{array}\right]$$

and

$$\hat{J} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} J^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array}\right] =: \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}.$$

Note that Q is stable if and only if K stabilizes $\hat{J}_{22}$. But $\hat{J}_{22} = G_{22}$, so Q is stable if and only if K stabilizes G, as desired. We summarize this observation as follows.

Theorem 5.10 *Any admissible controller K can be characterized as an LFT of a $\mathcal{Q}$-stable parameter matrix Q, i.e., $K = \mathcal{F}_l(J, Q)$ with Q realized by*

$$Q := \mathcal{F}_l(\hat{J}, K)$$

where

$$\hat{J} = \left[\begin{array}{c|cc} A & L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & D_{22} \end{array}\right]$$

and the realization for K is $\mathcal{Q}$-stabilizable and $\mathcal{Q}$-detectable. Moreover, this characterization is unique for a given pair F and L satisfying the requirements stated in Theorem 5.1.

5.4 A Separation Principle for LFT Systems

Observer-Based Controller

The above construction was conducted by reducing the synthesis of OF problem to the independent synthesis of FI and OE problems. This reduction is based on the separation property. And it also leads to a separation structure for the resulting closed loop system.

Let $\hat{x}$ be the "state variables" for the central controller which is represented as follows:

$$\Theta : \begin{cases} \Delta^{-1}\hat{x} = A\hat{x} + B_2 u - L(y - C_2\hat{x}) \\ u = F\hat{x} \end{cases} \tag{42}$$

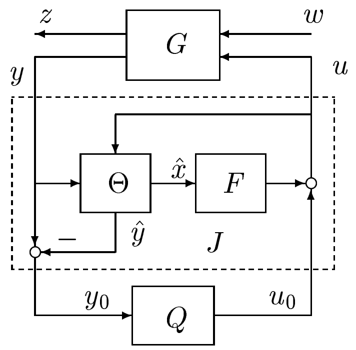
which has an observer structure. The corresponding closed loop system, under the admissible transformation $T := \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$, has the following transformation:

$$\left[\begin{array}{cc|c} A & B_2F & B_1 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} \\ \hline C_1 & D_{12}F & D_{11} \end{array}\right] \mapsto \left[\begin{array}{cc|c} A + B_2F & B_2F & B_1 \\ 0 & A + LC_2 & B_1 - LD_{21} \\ \hline C_1 - D_{12}F & D_{12}F & D_{11} \end{array}\right],$$

i.e., the transformed system is decoupled into two separated $\mathcal{Q}$-stable subsystems: the state-feedback system and the output-injection system; hence, it is also $\mathcal{Q}$-stable with respect to the new block structure $\Delta_N = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$ by Theorem 2.5, so is the original closed-loop system as desired.

Structure of Parametrized Closed Loop Systems

The central controller for the controller parametrization (with $Q = 0$) is an observer-based controller, and the observer Θ is given by (42) where $\hat{x}$ is the estimate of the state x of the original system. Therefore, a parametrized controller has a separation structure, and the closed loop system is structured by the following diagram.



Next, consider the closed loop map which is also parametrized as

$$T_{zw} = \mathcal{F}_l(G, K) = \mathcal{F}_l(G, \mathcal{F}_l(J, Q)) =: \mathcal{F}_l(T, Q).$$

We have the following theorem about the structures of the closed loop maps, whose proof is straightforward and is omitted.

Theorem 5.11 *Let the state feedback and output injection matrices F and L be chosen as in the previous theorem. Then the closed loop maps are parametrized as $T_{zw} = \mathcal{F}_l(T, Q)$ with*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} := \left[\begin{array}{cc|cc} A+BF & -BF & B_1 & B \\ 0 & A+LC & B_1+LD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C & D_{21} & 0 \end{array}\right] \tag{43}$$

and Q is stable. Moreover, T_{zw} is affine in Q, i.e.,

$$T_{zw} = T_{11} + T_{12} Q T_{21}. \tag{44}$$

6 Concluding Remarks

We have developed machinery for analysis, stabilization, and controller parametrization for linear fractional uncertain systems. All of the manipulations have been conducted in the LFT framework, and have based on the naturally defined $\mathcal{Q}$-stability notion for the LFT systems; the analysis and synthesis results are characterized in terms of LMIs. A separation principle is confirmed for the stabilization of linear fractional uncertain systems. It is noted that most of the results, including the separation theory, also hold in the μ-stability case via simple change of notation [25]. It is also remarked that although the stability notion also captures the robust performances, the separation argument can not be naively carried out for the performance synthesis problem. This is the essential difference between the approach used here and that in [33], [31].

It is observed that the separation property discussed in this paper holds in greater generality than for just the robust stabilization problem; all that is required for the separation proof is that the notion of stability satisfies two requirements: 1) stability invariance under a sufficiently rich set of similarity transformations, as in Theorem 2.4, and 2) a certain structural property as given in Theorem 2.5. It would clearly be possible to develop a more abstract axiomatic stabilization theory using these 2 properties.

Acknowledgements

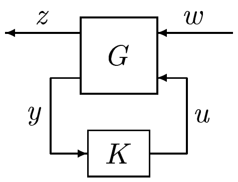
The authors would like to thank Prof. A. Packard at UC Berkeley for helpful discussion. They also gratefully acknowledge helpful comments on the previous versions of this paper from C. Beck, B. Bodenheimer, Y. Huang, J. Morris, F. Paganini, J. Tierno, as well as the anonymous reviewers. Support for this work was provided by NSF, AFOSR, and ONR.

7 Appendix

7.1 LFTs and μ

Linear Fractional Transformations

The LFT formula arises naturally when we describe a well-posed feedback system as shown by the following block diagram.



The resulting input/output relation can be represented as $z = \mathcal{F}_l(G, K)w$, where $\mathcal{F}_l(G, K)$ is said to be the (lower) **linear fractional transformation** (LFT) of G on K. More explicitly, suppose G is partitioned conformably as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{45}$$

Then

$$\mathcal{F}_l(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{46}$$

provided the inverse is well defined [36]. Similarly, the (upper) LFT on Δ, which corresponds to the feedback Δ around upper loop, is defined as

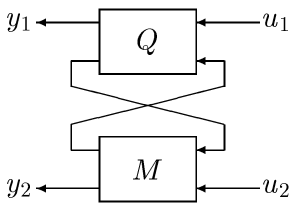
$$\mathcal{F}_u(G, \Delta) = G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}. \tag{47}$$

The following observation is about the inversion property of an LFT [19].

Lemma 7.1 *Suppose G is partitioned as in (45). Let $P = \mathcal{F}_l(G, K)$. If G, G_{12} and G_{21} are square and invertible, and $\det(G_{11} - P) \neq 0$, then $K = \mathcal{F}_u(G^{-1}, P)$.*

Redheffer Star Products

Consider the following well-posed feedback configuration:



Then the **Redheffer star product** of Q and M [36], denoted as $\mathcal{S}(Q,M)$, is the resulting transfer function, i.e., $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathcal{S}(Q,M) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. If Q and M are suitably partitioned as follows,

$$Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

then $(I - Q_{22}M_{11})^{-1}$ is well-defined, and

$$\mathcal{S}(Q,M) := \begin{bmatrix} \mathcal{F}_l(Q, M_{11}) & Q_{12}(I - M_{11}Q_{22})^{-1} M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1} Q_{21} & \mathcal{F}_u(M, Q_{22}) \end{bmatrix}. \tag{48}$$

Note that for any compatibly dimensioned matrix K, we have

$$\mathcal{F}_l(\mathcal{S}(Q,M), K) = \mathcal{F}_l(Q, \mathcal{F}_l(M,K))$$

provided that the related LFTs are well-defined.

Structured Singular Values

Consider a matrix $M \in \mathbf{C}^{n\times n}$ and an underlying block structure Δ:

$$\Delta := \mathrm{Diag}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f], \tag{49}$$

Corresponding to the this block structure, define a set:

$$\boldsymbol{\Delta} := \{\mathrm{Diag}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] : \delta_i \in \mathbf{C}, \Delta_j \in \mathbf{C}^{m_j\times m_j}\} \subset \mathbf{C}^{n\times n}. \tag{50}$$

We first have the following definition structured singular value [11].

Definition 7.2 *The* **structured singular value** $\mu_\Delta(M)$ *of a matrix* M *with respect to structure* $\boldsymbol{\Delta}$ *is defined as*

$$\mu_\Delta(M) := \sup_{\Delta\in\boldsymbol{\Delta}} \{\frac{1}{\bar{\sigma}} : \det[I - \Delta M] = 0\}, \tag{51}$$

unless no $\Delta \in \boldsymbol{\Delta}$ *makes* $I - \Delta M$ *singular, in which case* $\mu_\Delta(M) := 0$.

From the definition, it follows immediately that $\mu_\Delta(M) = \rho(M)$ if $\boldsymbol{\Delta} = \{\delta I : \delta \in \mathbf{C}\}$ and $\mu_\Delta(M) = \bar{\sigma}(M)$ if $\boldsymbol{\Delta} = \mathbf{C}^{n\times n}$.

The **scaling matrix set** $\mathcal{D}$ with respect to the block structure is defined as

$$\mathcal{D} = \{D \in \mathbf{C}^{n\times n} \text{ nonsingular} : D\Delta = \Delta D, \Delta \in \boldsymbol{\Delta}\}. \tag{52}$$

Definition 7.3 *The* $\mathcal{Q}$**-value** *of* M *with respect to structure* $\boldsymbol{\Delta}$ *is defined as*

$$\mathcal{Q}_\Delta(M) := \inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}). \tag{53}$$

It is known that $\mathcal{Q}_\Delta(M)$ is an upper bound of $\mu_\Delta(M)$ [11], i.e., $\mu_\Delta(M) \le \inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1})$. The upper bound is achieved for the following special block structures:

1) $\boldsymbol{\Delta} = \{\mathrm{Diag}[\delta I_r, \Delta] : \delta \in \mathbf{C}, \Delta \in \mathbf{C}^{(n-r)\times(n-r)}\}$.
2) $\boldsymbol{\Delta} = \{\mathrm{Diag}[\Delta_1, \cdots, \Delta_f] : \Delta_i \in \mathbf{C}^{m_i\times m_i}\} \in \mathbf{C}^{n\times n}$, $f \le 3$, where no blocks are repeated.

Remark 7.4 *Both* $\mu_\Delta(M)$ *and* $\mathcal{Q}_\Delta(M)$ *are continuous functions of* M *for fixed block structure* $\boldsymbol{\Delta}$.

To end this review, we state the following result known as **Main-Loop Theorem** in [32].

Lemma 7.5 *Given a matrix* $M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ *partitioned conformably with the block structure* $\Delta = \mathrm{Diag}[\Delta_1, \Delta_2]$, *then* $\mu_\Delta(M) < 1$ *if and only if*

$$\mu_{\Delta_2}(M_{22}) < 1, \qquad \max_{\bar{\sigma}(\Delta_2)\leq 1} \mu_{\Delta_1}(\mathcal{F}_l(M, \Delta_2)) < 1.$$

7.2 Multidimensional Systems Represented by LFTs

Consider a 2-dimensional **linear shift-invariant** (LSI) system of order (n_1, n_2) described by the Roesser state-space equation [38]:

$$\begin{cases} x_1(k_1+1, k_2) &= A_{11}x_1(k_1, k_2) + A_{12}x_2(k_1, k_2) + B_1u(k_1, k_2) \\ x_2(k_1, k_2+1) &= A_{21}x_1(k_1, k_2) + A_{22}x_2(k_1, k_2) + B_2u(k_1, k_2) \\ y(k_1, k_2) &= C_1x_1(k_1, k_2) + C_2x_2(k_1, k_2) + Du(k_1, k_2) \end{cases} \tag{54}$$

where $x_1(k_1, k_2) \in \mathbf{R}^{n_1}$ and $x_2(k_1, k_2) \in \mathbf{R}^{n_2}$ denote the state vectors, $u(k_1, k_2) \in \mathbf{R}^p$ the input vector, and $y(k_1, k_2) \in \mathbf{R}^q$ the output vector. Define

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

and $\Delta = \mathrm{Diag}[\lambda_1 I_{n_1}, \lambda_2 I_{n_2}]$, where $\lambda_i I$ can be interpreted as a backward shift operator. The transfer matrix for this system with zero initial conditions is

$$G = D + C\Delta(I - \Delta A)^{-1}B = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta), \tag{55}$$

i.e., this system is represented as an LFT with respect to block structure Δ. More generally, an N-dimensional discrete LSI system with order $(n_1, \cdots, n_N)$ can also be represented in terms of an LFT as (55) with respect to block structure

$$\Delta = \mathrm{Diag}[\lambda_1 I_{n_1}, \cdots, \lambda_N I_{n_N}]. \tag{56}$$

Define $\bar{U}^N := \{(\lambda_1, \cdots, \lambda_N) : \lambda_i \in \mathbf{C}, \mid \lambda_i \mid \leq 1\}$. It is known that the N-dimensional system with system matrix A defined above is internally stable if and only if $T(\lambda_1, \cdots, \lambda_N) := \mathrm{Det}[I - \Delta A] \neq 0$ in $\bar{U}^N$ [6, 1]. Equivalently, $\mu_\Delta(A) < 1$; in addition, the system is stable if there exists $P = \mathrm{Diag}[P_1, \cdots, P_N]$ which is positive definite and $P_i \in \mathbf{R}^{n_i \times n_i}, i = 1, \cdots, N$, such that the following Lyapunov inequality holds [1],

$$APA^T - P < 0.$$

Note that the above Lyapunov condition is equivalent to the $\mathcal{Q}$-stability.

7.3 Parrott's Theorem

The following lemma is known as Parrott's Theorem [10], [12].

Lemma 7.6 *Suppose* $(B, C, A) \in \mathbf{R}^{n_1 \times m_2} \times \mathbf{R}^{n_2 \times m_1} \times \mathbf{R}^{n_2 \times m_2}$, *then*

$$\gamma_0 := \inf_{X \in \mathbf{R}^{n_1 \times m_1}} \bar{\sigma}\left(\begin{bmatrix} X & B \\ C & A \end{bmatrix}\right) = \max\left\{\bar{\sigma}\left(\begin{bmatrix} C & A \end{bmatrix}\right),\ \bar{\sigma}\left(\begin{bmatrix} B \\ A \end{bmatrix}\right)\right\}.$$

Let $\gamma > \gamma_0$, *then those matrices* $X \in \mathbf{R}^{n_1 \times m_1}$ *such that* $\bar{\sigma}\left(\begin{bmatrix} X & B \\ C & A \end{bmatrix}\right) < \gamma$ *are exactly those of the form*

$$X = -YA^T Z + \gamma(I - YY^T)^{1/2} W (I - Z^T Z)^{1/2},$$

where $Y = B(\gamma^2 I - A^T A)^{-1/2}$, $Z = (\gamma^2 I - AA^T)^{-1/2} C$, *and* W *is an arbitrary contraction:* $\bar{\sigma}(W) < 1$.

As a corollary to the above lemma, the following result plays the key role in this paper.

Proposition 7.7 *Consider the triple* $(A, B, C) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times p} \times \mathbf{R}^{q \times n}$ *with* $\mathrm{Rank}(B) = p < n$ *and* $\mathrm{Rank}(C) = q < n$. *Let* $B_\perp \in \mathbf{R}^{n \times (n-p)}$ *and* $B_0 \in \mathbf{R}^{p \times n}$ *be such that* $B_\perp^T B = 0$ *and* $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ *is unitary, and let* $C_\perp \in \mathbf{R}^{(n-q) \times n}$ *and* $C_0 \in \mathbf{R}^{n \times q}$ *be such that* $C_\perp C^T = 0$ *and* $\begin{bmatrix} C_0 \\ C_\perp \end{bmatrix}$ *is unitary. Then*

$$\inf_{F \in \mathbf{R}^{p \times q}} \bar{\sigma}(A + BFC) = \max\{\bar{\sigma}(B_\perp^T A), \bar{\sigma}(AC_\perp^T)\}.$$

Proof. Notice that both $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ and $\begin{bmatrix} C_0 & C_\perp \end{bmatrix}$ are unitary matrix. Then

$$\bar{\sigma}(A + BFC) = \bar{\sigma}(\begin{bmatrix} B_0 & B_\perp \end{bmatrix}^T (A + BFC) \begin{bmatrix} C_0 & C_\perp \end{bmatrix}$$

$$= \bar{\sigma}\left(\begin{bmatrix} B_0^T A C_0^T + B_0^T B F C C_0^T & B_0^T A C_\perp^T \\ B_\perp^T A C_0^T & B_\perp^T A C_\perp^T \end{bmatrix}\right) = \max\{\bar{\sigma}(B_\perp^T A), \bar{\sigma}(AC_\perp^T)\}$$

which follows the preceding lemma. □

Given $\gamma > \max\{\bar{\sigma}(B_\perp^T A), \bar{\sigma}(AC_\perp^T)\}$, it is a routine adaptation of Parrott's theorem to get a parametrization of all matrices F such that $\bar{\sigma}(A + BFC) < \gamma$. A special case of the above proposition is stated as follows.

Lemma 7.8 *Assume* $(A, B) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times p}$ *and* $\mathrm{Rank}(B) = p < n$. *Let* $B_\perp \in \mathbf{R}^{n \times (n-p)}$ *and* $B_0 \in \mathbf{R}^{p \times n}$ *be such that* $B_\perp^T B = 0$ *and* $\begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ *is unitary. Then*

$$\inf_{F \in \mathbf{R}^{p \times n}} \bar{\sigma}(A + BF) = \bar{\sigma}(B_\perp^T A)$$

and the infimum is attained by $F = -(B_0^T B)^{-1} B_0^T A$.

Proof. Since $U := \begin{bmatrix} B_0 & B_\perp \end{bmatrix}$ is unitary,

$$\inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(A+BF) = \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}(U^T(A+BF)) = \inf_{F\in\mathbf{R}^{p\times n}} \bar{\sigma}\left(\begin{bmatrix} B_0^TA + B_0^TBF \\ B_\perp^TA \end{bmatrix}\right) = \bar{\sigma}(B_\perp^TA).$$

Moreover the infimum is attained if $B_0^TA + B_0^TBF = 0$ or $F = -(B_0^TB)^{-1}B_0^TA$. $\square$

The matrix B_0 in Lemma 7.8 can be chosen as $B_0 = B(B^TB)^{-1/2}$, in which case, $F = -(B^TB)^{-1}B^TA$.

References

[1] B.D.O. Anderson, P. Agathoklis, E.I. Jury, and M. Mansour, Stability and the Matrix Lyapunov Equation for Discrete 2-Dimensional Systems, *IEEE Trans. C.A.S.*, Vol.CAS-33, pp.261–266, 1986.

[2] P. Apkarian and P. Gahinet, A Convex Characterization of Parameter-Dependent $\mathcal{H}_\infty$ Controllers, *IEEE Trans. A.C.*, Vol.40, pp.853–864, 1995.

[3] C.L. Beck, The Computation Issue of LMIs, *Proc. of 1991 IEEE CDC*, Brighton, England, pp.1259–1260, 1991.

[4] C.L. Beck, W.-M. Lu, F. Paganini, R. D'Andrea, and J.C. Doyle, A State-Space Theory of Uncertain Systems, *1996 IFAC World Congress* (Submitted), San Francisco, CA, July 1996.

[5] G. Becker and A. Packard, Robust Performance of Linear Parametrically Varying Systems Using Parametrically Dependent Linear Feedback, *Syst. Contr. Lett.*, Vol.23, pp.205–215, 1994.

[6] N.K. Bose, *Applied Multidimensional Systems Theory*, NY: Van Nostrand Reinhold Co., 1982.

[7] S.P. Boyd, and C. Barratt, *Linear Controller Design: Limits of Performance*, Englewood Cliffs, NJ: Prentice-Hall, 1991.

[8] S.P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

[9] M.A. Dahleh and I.J. Diaz-Bobillo, *Control of Uncertain Systems: A Linear Programming Approach*, Prentice-Hall, 1994.

[10] C. Davis, W.M. Kahan, and H.F. Weinberger, Norm-Preserving Dilations and Their Applications to Optimal Error Bounds, *SIAM J. Numer. Anal.*, Vol.19, pp.445–469, 1982.

[11] J.C. Doyle, Analysis of Feedback Systems with Structured Uncertainties, *IEE Proceedings*, Part D, Vol.133, pp.45–56, 1982.

[12] J.C. Doyle, Lecture Notes in Advances in Multivariable Control, *ONR/Honeywell Workshop*, Minneapolis, MN, 1984.

[13] J.C. Doyle, K. Glover, P. Khargonekar, and B. Francis, State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems, *IEEE Trans. A.C.*, Vol.AC-34, pp.831–847, 1989.

[14] J.C. Doyle, A. Packard, and K. Zhou, Review of LFTs, LMIs and μ, *Proc. of 1991 IEEE CDC*, Brighton, England, pp.1227–1232, 1991.

[15] J.C. Doyle, J.E. Wall, and G. Stein (1982), Performance and Robustness Analysis for Structured Uncertainty, *Proc. 21st IEEE CDC*, pp.629–636.

[16] R. Eising, Realization and Stabilization of 2-D Systems, *IEEE Trans. A.C.*, Vol. AC-23, pp.793–799, 1987.

[17] P. Gahinet and P. Apkarian, A Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control, *Int. J. of Robust and Nonlinear Control*, Vol.4, pp.421–448, 1994.

[18] P. Gahinet, A. Nemirovskii, A.J. Laub, and M. Chilali, The LMI Control Toolbox, *Proc. 1994 IEEE CDC*, Orlando, FL, pp.2038–2041, 1994.

[19] K. Glover, A Tutorial on Hankel-Norm Approximation, In *From Date to Model* (J.C.Willems, ed.), NY: Springer-Verlag, 1989.

[20] J.P. Guiver and N.K. Bose, Causal and Weakly Causal 2-D Filters with Applications in Stabilization, in *Multidimensional Systems Theory* (N.K.Bose, ed.), D.Reidel Publishing Co., 1985.

[21] Y. Huang and W.-M. Lu, An LFT Approach to Autopilot Design for Missiles, *Proc. 1995 ACC*, Seattle, WA, June 1995.

[22] T. Iwasaki and R.E. Skelton, A Complete Solution to the General $\mathcal{H}_\infty$ Control Problem: LMI Existence Conditions and State Space Formulas, *Proc. 1993 ACC*, San Francisco, CA, pp.605 – 609, 1993.

[23] E.W. Kamen and P.P. Khargonekar, On the Control of Linear Systems Whose Coefficients Are Functions of Parameters, *IEEE Trans. A.C.*, Vol. AC-29, pp. 25 –33, 1984.

[24] P.P. Khargonekar and E.D. Sontag, On the Relation Between Stable Matrix Factorizations and Regulable Realizations of Linear Systems Over Rings, *IEEE Trans. A.C.*, Vol. AC-27, pp. 627–638, 1982.

[25] W.-M. Lu, *Control of Uncertain Systems: State-Space Characterizations*, Ph.D. Thesis, California Institute of Technology, 1995.

[26] W.-M. Lu, A State-Space Approach to Parametrization of Stabilizing Controllers for Nonlinear Systems, *IEEE Trans. A.C.*, Vol.40, pp.1576 – 1588, 1995.

[27] W.-M. Lu and J.C. Doyle, $\mathcal{H}_\infty$-Control of LFT Systems: An LMI Approach, *Proc. of 1992 IEEE CDC*, Tucson, AZ, pp.1997–2001, 1992.

[28] W.-M. Lu, K. Zhou, and J.C. Doyle, Stabilization of LFT Systems, *Proc. of 1991 IEEE CDC*, Brighton, England, pp.1239–1244, 1991.

[29] A. Megretski, Necessary and Sufficient Conditions of Stability: A Multiloop Generalization of Circle Criterion, *IEEE Trans. A.C.*, Vol.38, pp.753–756, 1993.

[30] C.N. Nett, C.A. Jacobson, and N.J. Balas, A Connection between State-Space and Doubly Coprime Fractional Representations, *IEEE Trans. A.C.*, Vol.AC-29, pp.831–832, 1984.

[31] A. Packard, Gain-Scheduling via Linear Fractional Transformations, *Systems and Control Letters*, Vol.22, pp.79 –92, 1994.

[32] A. Packard and J.C. Doyle, The Complex Structured Singular Value, *Automatica*, Vol.29, pp.71–109, 1993.

[33] A. Packard, K. Zhou, P. Pandey, and G. Becker, A Collection of Robust Control Problems Leading to LMIs, *Proc. of 1991 IEEE CDC*, Brighton, England, pp.1245–1250, 1991.

[34] A. Packard, K. Zhou, P. Pandey, J. Leonhardson, and G. Balas, Optimal, Constant I/O Similarity Scaling for Full-Information and State-Feedback Problems, *Systems and Control Letters*, Vol.19, pp.271–280, 1992.

[35] F. Paganini and J.C. Doyle, Analysis of Implicitly Defined Systems, *Proc. 1994 IEEE CDC*, Orlando, FL, pp.3673–3678, 1994.

[36] R.M. Redheffer, On a Certain Linear Fractional Transformation, *J. Math. Phys.*, Vol.39, pp.269–286, 1960.

[37] R.T. Reichert, Dynamic Scheduling of Modern-Robust-Control Autopilot Designs for Missiles, *IEEE Control System Magazine*, Vol.12, No.5, pp.35–42, 1992.

[38] R.P. Roesser, A Discrete State-Space Model for Linear Image Processing, *IEEE Trans. A.C.*. Vol.AC-20, pp.1–10, 1975.

[39] W.J. Rugh, Analytical Framework for Gain Scheduling, *IEEE Control System Magazine*, Vol.11, No.1, pp.79–84, 1991.

[40] J.S. Shamma, Robust Stability with Time-Varying Structured Uncertainty, *IEEE Trans. A.C.*, Vol.39, pp.714–724, 1994.

[41] J.S. Shamma and M. Athans, Gain Scheduling: Potential Hazards and Possible Remedies, *IEEE Control System Magazine*, Vol.12, No.3, pp.101–107, 1992.

[42] G. Stein, Adaptive Flight Control: A Pragmatic View, *Applications of Adaptive Control* (K.S. Narendra and R.V. Monopoli, eds), New York: Academic Press, 1980.

[43] D.C. Youla, H.A. Jabr, and J.J. Bongiorno, Modern Wiener-Hopf Design of Optimal Controllers: The Multivariable Case, *IEEE Trans. A.C.*, Vol.AC-21, pp.319–338, 1976.

[44] K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control*, Upper Saddle River, NJ: Prentice-Hall, 1995.