Model Reduction of Multi-Dimensional and Uncertain Systems

Carolyn Beck* John Doyle† Keith Glover‡

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Abstract

We present model reduction methods with guaranteed error bounds for systems represented by a Linear Fractional Transformation (LFT) on a repeated scalar uncertainty structure. These reduction methods can be interpreted either as doing state order reduction for multi-dimensional systems, or as uncertainty simplification in the case of uncertain systems, and are based on finding solutions to a pair of Linear Matrix Inequalities (LMIs). A related necessary and sufficient condition for the exact reducibility of stable uncertain systems is also presented.

1 Introduction

The process of modelling systems and designing controllers using modern robust control methods often results in models which are high order and have large and complicated uncertainty descriptions. As a result, these models may be expensive to implement and difficult to analyze, in particular when incorporated into larger feedback systems. In response to this, we provide a systematic method for reducing both the state dimension and uncertainty descriptions with little or no resulting error, and without affecting system stability. This reduction method relies on the solution of two linear matrix inequalities (LMIs).

For system models which do not incorporate uncertainty descriptions, there exist well-known model reduction methods and associated error bounds, examples of which include the balanced model reduction method and its additive H_∞ norm error bound ([16], [9], [10], [11]); the optimal Hankel norm model reduction method and its Hankel norm error bound ([10]); and the balanced stochastic truncation model reduction method and its relative H_∞ norm error bound ([7], [22]). The balanced truncation model reduction method was extended to multi-dimensional (MD) and uncertain systems in [21]. We review this reduction method and the related error bounds, providing simplified proofs, and state new results for model reduction of uncertain systems which provide tighter bounds by a factor of two. These new results, which were first noted in [4], are based on machinery presented in [18]. A necessary and sufficient condition for exact reducibility of such systems is also stated.

*Electrical Engineering, M/S 116-81, California Institute of Technology, Pasadena, CA 91125
†Control and Dynamical Systems, M/S 116-81, California Institute of Technology, Pasadena, CA 91125
‡Department of Electrical Engineering, University of Cambridge, Cambridge, CB2 1PZ, U.K.

In this paper we are mainly concerned with uncertain systems which are modelled by a linear fractional transformation (LFT) on a repeated scalar uncertainty structure. In the most general setting, we consider uncertainty represented by arbitrary time-varying scalar operators on l_2. In a less general setting, we can view the repeated scalar uncertainty as different transform variables in a MD system, or, alternatively, as real or complex-valued parametric uncertainty. In referring to uncertain systems we will typically mean the most general case; exceptions will be noted.

Basic background material is presented in Section 2, including our notation and a review of LFT representations of uncertain systems. In Section 3, we review balanced realizations, minimality and the error bounds associated with model reduction for the 1D case. In Section 4, uncertain systems are considered: balanced realizations, stability and norms are discussed, and error bounds for balanced truncation of uncertain systems are given. A notion of reducibility is presented in Section 5 for MD and uncertain systems which is analogous to that for the 1D case. A necessary and sufficient condition for this reducibility is given.

2 Preliminaries

The notation we use is as follows: L_2 denotes the Lebesgue space of functions square integrable on the unit circle, and l_2 the space of sequences which are square summable. $\mathcal{L}(L_2)$ and $\mathcal{L}(l_2)$ represent the space of all linear time-varying operators on L_2 and l_2, respectively. For notational convenience, dimensions will not be given unless pertinent to the discussion. We represent the complex and real fields of matrices by $\mathbb{C}^{n\times m}$ and $\mathbb{R}^{n\times m}$, and the integers by $\mathbb{Z}$. The forward shift operator on l_2 is denoted by z^{-1}, and the identity matrix by I. The maximum singular value of $A \in \mathbb{C}^{n\times m}$ is denoted by $\overline{\sigma}(A)$, and A^* denotes the complex conjugate transpose. The dimensions of a matrix A are denoted $\dim(A)$.

2.1 Linear Fractional Transformations

The main focus of robust control has been to evaluate the effects of uncertainty when analyzing and designing controllers for linear systems. The source of such uncertainty might be nonlinearities in the physical system being modelled, unmodelled dynamics, disturbances, parametric uncertainties or any combination of the aforementioned. We use the LFT paradigm, represented pictorially in Figure 1 and described below, as a mathematical representation for uncertainty in system models.

We assume $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a given system realization matrix, that is, a constant matrix that describes what is known about the system, and Δ represents the system uncertainty, namely, what is unknown or poorly understood about the system. We assume Δ lies in some prescribed set, in particular, the uncertainty set $\mathbf{\Delta}$ we consider in this paper is

$$\mathbf{\Delta} = \{\mathrm{diag}\ [\delta_1 I_{n_1}, \ldots, \delta_S I_{n_S}] : \delta_i \in \mathcal{L}(L_2)\}. \tag{2.1}$$

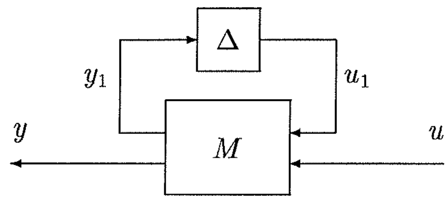

Figure 1: MD/Uncertain System

For analysis purposes, we will often consider Δ which lie in a norm-bounded subset of $\mathbf{\Delta}$, that is,

$$\mathbf{B\Delta} = \left\{\Delta \in \mathbf{\Delta} : \|\Delta\|_{l_2 \to l_2} \leq 1\right\}, \tag{2.2}$$

where $\|\cdot\|_{l_2 \to l_2}$ denotes the induced norm. The input/output mapping for this system is determined by the LFT

$$y = (\Delta \star M)u, \quad \Delta \in \mathbf{\Delta}$$

where the Redheffer star product of the system components is

$$\Delta \star M = D + C\Delta(I - A\Delta)^{-1}B,$$

whenever the inverse is well-defined. We will refer to such system models by the pair $(\mathbf{\Delta}, M)$.

One way to interpret such system models is to view the δ_i as different transform variables in a MD system ([12]). In this case, model reduction means state order reduction, and a system model which may be reduced without error is reducible or non-minimal as in the 1-D case. We would like to find reduced order models which match the full order model well at all values of the δ_i on the polydisc $|\delta_i| = 1$, $\forall i$. The results in this paper are directly relevant to this interpretation as we can guarantee error bounds of the form [21]

$$\|(\mathbf{\Delta} * M) - (\mathbf{\Delta}_r * M_r)\|_{SI2} \leq 2 \sum_{i=1}^{r} \sum_{j=k_i+1}^{t_i} \sigma_{ij} \tag{2.3}$$

and, in fact, will show that under special interpretations of the δ_i we can achieve a tighter bound than (2.3) by a factor of two (see also [13] for the 1D continuous time case). Additionally, we present a necessary and sufficient condition under which reduced order models, $(\mathbf{\Delta}_r, M_r)$, can be found such that the difference between the full and reduced order models is zero in some norm. Although the exact meaning of the notation in (2.3) has not been developed at this point, we note that in the 1D case (2.3) reduces to the standard bound for balanced truncation ([9], [10], [11]) with the error measured in the H_∞ norm.

Alternatively, we may view one of the δ_i, say δ_1 as the frequency variable ($\delta_1 = 1/z$) in an uncertain discrete-time system. The remaining δ_i are then viewed as norm-bounded perturbations. Model reduction in this context is aimed at simplifying the uncertainty description as well as reducing the state dimension and is a more subtle issue. We will henceforth refer to systems with uncertainty structures given by (2.1) as *uncertain systems*, with the implicit understanding that this includes both 1D systems with uncertainty incorporated into the model, and MD systems.

3 1D Systems

In order to more readily discuss model reduction of uncertain systems, we review Lyapunov equations, balanced realizations, minimality, and balanced truncation model reduction for 1D systems. We state the well-known upper bound for balanced truncation model reduction error originally proven by Glover [10] and by Enns [9] for continuous time systems. We provide a proof of this bound for discrete time systems which generalizes immediately to uncertain system representations. The proof was first outlined in [21], and appears here in a more precise and simple version.

We consider finite dimensional, linear time-invariant systems of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k), \end{aligned}$$

thus $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the system realization and $\Delta = z^{-1}I$. For the 1D case, we denote the system transfer function by $G(z) = \Delta \star M = D + C(zI - A)^{-1}B$, and the system operator by G. We begin by discussing contractive matrices and associated results which are of general use for both 1D and uncertain systems.

3.1 Contractive Realizations

Throughout this text we use the notion of contractive matrices, which are defined as follows.

Definition 1. *A constant matrix X is said to be contractive or a contraction if $\|X\| = \overline{\sigma}(X) \leq 1$, and strictly contractive if $\|X\| < 1$.*

If the matrix X is a realization matrix, the following lemma gives a well-known result ([5], [8]) on the relationship between the $\mathcal{H}_\infty$ norm of a transfer matrix and realizations of the transfer matrix.

Lemma 1. *Suppose $G \in \mathcal{RH}_\infty$ represents a discrete time transfer matrix, then $\|G\|_\infty \leq 1 (< 1)$ if and only if there is a realization for G, denoted by $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, such that M is contractive (strictly contractive).*

A generalized version of this lemma for uncertain systems is given in Section 4. Lemma 1 and the following lemma, which relates the contractiveness of a matrix to that of related submatrices, provide the main steps in the proof of the error bounds for balanced truncation model reduction.

Lemma 2. *Suppose* $U = \begin{bmatrix} U_{11} & U_{12} \\ Z & U_{22} \end{bmatrix}$ *and* $V = \begin{bmatrix} V_{11} & Z \\ V_{21} & V_{22} \end{bmatrix}$ *are contractive (strictly contractive). Then*

$$M := \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}U_{11} & U_{12} \\ \frac{1}{\sqrt{2}}V_{11} & Z & \frac{1}{\sqrt{2}}U_{22} \\ V_{21} & \frac{1}{\sqrt{2}}V_{22} & 0 \end{bmatrix}$$

is also contractive (strictly contractive).

Proof. The result is easily proved by dilating M to the following matrix,

$$M_d := \left[\begin{array}{ccccc} 0 & \frac{1}{\sqrt{2}}U_{11} & U_{12} & \vdots & \frac{1}{\sqrt{2}}U_{11} \\ \frac{1}{\sqrt{2}}V_{11} & Z & \frac{1}{\sqrt{2}}V_{22} & \vdots & 0 \\ V_{21} & \frac{1}{\sqrt{2}}V_{22} & 0 & \vdots & -\frac{1}{\sqrt{2}}V_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{2}}V_{11} & 0 & -\frac{1}{\sqrt{2}}U_{22} & \vdots & -Z \end{array}\right],$$

and noting that $M_d^* M_d \leq I$. □

3.2 Balanced Realizations and Reducibility

Suppose M is a 1D system realization with A stable, and Y and X are the controllability and observability gramians, respectively. That is, $Y = Y^* \geq 0$ and $X = X^* \geq 0$ satisfy the Lyapunov equations

$$AYA^* - Y + BB^* = 0 \tag{3.1}$$

$$A^*XA - X + C^*C = 0. \tag{3.2}$$

From standard Lyapunov theory, we know that the pair (A, B) is controllable if and only if $Y > 0$, and (C, A) is observable if and only if $X > 0$, in which case we say the realization is minimal, or irreducible.

Suppose the system realization, M, is transformed by the nonsingular matrix T which gives the eigenvector decomposition

$$YX = T^{-1}\Lambda T, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

Since $Y \geq 0$ and $X \geq 0$, it can be shown that YX has a real diagonal Jordan form and that $\Lambda \geq 0$. If M is a minimal realization T can always be chosen such that

$$\hat{Y} = TYT^* = \Sigma \quad \text{and} \quad \hat{P} = (T^{-1})^* P T^{-1} = \Sigma,$$

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) > 0$ and $\Sigma^2 = \Lambda$. The transformed realization, $\hat{M} = \begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}$, with controllability and observability gramians $\hat{Y} = \hat{X} = \Sigma$, is referred to as a *balanced realization.* Recall that $G(z) = \Delta \star M = \Delta \star \hat{M}$, $\forall \Delta \in \mathbf{\Delta}$.

More generally, if the realization of a 1D system is not minimal, then there exists a transformation such that the controllability and observability gramians are diagonal, and the controllable and observable subsystem is balanced. The following theorem is standard, so the proof is omitted.

Theorem 1. *For any stable system realization* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *there exists* T *such that* $\hat{M} = \begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}$ *has controllability and observability gramians given by*

$$Y = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

respectively, with Σ_1, Σ_2, Σ_3 *diagonal and positive definite.*

Since the uncontrollable and unobservable modes of any system realization are not present in the corresponding system transfer function, $G(z)$, we can truncate the associated states, corresponding to the zeros in Y and X above, and obtain a minimal realization which has both gramians equal to Σ_1. Such a system is *reducible* in that there exists a lower order realization M_r with associated G_r, such that $\|G - G_r\|_\infty = 0$, which gives us the following corollary to Theorem 1.

Corollary 1. *Every stable 1D system* $G(z)$ *has a minimal realization which is balanced.*

3.3 Discrete Balanced 1D Model Reduction

Consider a stable discrete time system $G(z)$ with the following realization

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}.$$

Suppose Y and X are two positive semi-definite symmetric matrices satisfying the following Lyapunov *inequalities*

$$AYA^* - Y + BB^* \leq 0 \tag{3.3}$$

$$A^*XA - X + C^*C \leq 0. \tag{3.4}$$

We have replaced the equalities in (3.1) and (3.2) with inequalities in (3.3) and (3.4) in order to generalize the 1D system results to uncertain systems. The significance of these inequalities in the 1D case is that while the zero-valued eigenvalues of Y or X still have corresponding uncontrollable and/or unobservable states, the converse need not be true. We can truncate states as suggested by Corollary 1 and balance X and Y exactly as before, but the resulting system may not be minimal. Subsequently, when we refer to balanced system realizations it will be in this looser sense, that is with $Y > 0$ and $X > 0$ satisfying the Lyapunov inequalities above and

$$Y = X = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

with

$$\begin{aligned} \Sigma_1 &= \operatorname{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r}) > 0 && (3.5) \\ \Sigma_2 &= \operatorname{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_n I_{s_n}) > 0, && (3.6) \end{aligned}$$

where s_i denotes the multiplicity of σ_i. Note that the σ_i are not necessarily ordered. In particular, given a balanced system realization with $\Sigma_2 = I$, we can prove the following lemma.

Lemma 3. *Given a balanced realization of the transfer matrix G with $Y = X = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & I \end{bmatrix} > 0$, satisfying the Lyapunov inequalities, then*

$$\begin{bmatrix} \Sigma_1^{-\frac{1}{2}} A_{12} & \Sigma_1^{-\frac{1}{2}} A_{11} \Sigma_1^{\frac{1}{2}} & \Sigma_1^{-\frac{1}{2}} B_1 \\ A_{22} & A_{21} \Sigma_1^{\frac{1}{2}} & B_2 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} A_{21} \Sigma_1^{-\frac{1}{2}} & A_{22} \\ \Sigma_1^{\frac{1}{2}} A_{11} \Sigma_1^{-\frac{1}{2}} & \Sigma_1^{\frac{1}{2}} A_{12} \\ C_1 \Sigma_1^{-\frac{1}{2}} & C_2 \end{bmatrix}$$

are contractive.

Proof. Rewriting equations (3.3) and (3.4) gives

$$[A \ B] \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^* \\ B^* \end{bmatrix} \leq Y \qquad (3.7)$$

and

$$[A^* \ C^*] \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} \leq X. \qquad (3.8)$$

Premultiplying and postmultiplying (3.7) by $Y^{-\frac{1}{2}}$ and (3.8) by $X^{-\frac{1}{2}}$ shows that the matrices

$$\left[Y^{-\frac{1}{2}} A Y^{\frac{1}{2}} \ Y^{-\frac{1}{2}} B\right] \quad \text{and} \quad \begin{bmatrix} X^{\frac{1}{2}} A X^{-\frac{1}{2}} \\ C X^{-\frac{1}{2}} \end{bmatrix} \qquad (3.9)$$

are contractive. Substituting $\begin{bmatrix} \Sigma_1 & 0 \\ 0 & I \end{bmatrix}$ for Y and X in (3.9), and permuting the resulting matrices gives the desired result. □

The balanced truncation model reduction results, first given for 1D discrete time systems in [11], and for MD or uncertain systems in [21], are now stated. The results are separated into a lemma stating that the truncation of a stable, balanced realization is also stable and balanced, and a theorem stating the upper error bound results. The proof for the lemma can be found in [11]. Although the following theorem is not new, a proof is included for completeness which is simpler and more precise than that given in [21]. This proof generalizes immediately to system representations which include uncertainty, and will be discussed later.

Lemma 4. *Suppose* $M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}$ *is a balanced, stable realization. Then the truncated system realization given by*

$$M_r = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$$

is also balanced and stable.

Theorem 2. *Suppose* $M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}$ *is a balanced, stable realization for* G *with* $X = Y = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} > 0$, *as defined in (3.5). Let* $M_r = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$ *denote the balanced, stable, truncated system realization for* G_r. *Then*

$$\|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

Proof. The proof of Theorem 2 relies heavily on the preceeding lemmas. We first assume that $\Sigma_2 = I$. In this case, we must show that

$$\|G - G_r\|_\infty \leq 2$$

The final result follows from scaling and applying this recursively.

By Lemma 1, it suffices to show that there exists a realization for $\frac{1}{2}(G - G_r)$ which is contractive. One realization for $\frac{1}{2}(G - G_r)$ is given by

$$M = \begin{bmatrix} A_{11} & 0 & 0 & \frac{1}{\sqrt{2}}B_1 \\ 0 & A_{11} & A_{12} & \frac{1}{\sqrt{2}}B_1 \\ 0 & A_{21} & A_{22} & \frac{1}{\sqrt{2}}B_2 \\ -\frac{1}{\sqrt{2}}C_1 & \frac{1}{\sqrt{2}}C_1 & \frac{1}{\sqrt{2}}C_2 & 0 \end{bmatrix}$$

Motivated by the results of Lemmas 2 and 3, we consider the following similarity transformation,

$$T = \begin{bmatrix} -\frac{1}{\sqrt{2}}\Sigma_1^{\frac{1}{2}} & \frac{1}{\sqrt{2}}\Sigma_1^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & I & 0 \\ \frac{1}{\sqrt{2}}\Sigma_1^{-\frac{1}{2}} & \frac{1}{\sqrt{2}}\Sigma_1^{-\frac{1}{2}}0 & 0 & \\ 0 & 0 & 0 & I \end{bmatrix},$$

giving

$$TMT^{-1} = \begin{bmatrix} \Sigma_1^{\frac{1}{2}}A_{11}\Sigma_1^{-\frac{1}{2}} & \frac{1}{\sqrt{2}}\Sigma_1^{\frac{1}{2}}A_{12} & 0 & 0 \\ \frac{1}{\sqrt{2}}A_{21}\Sigma_1^{-\frac{1}{2}} & A_{22} & \frac{1}{\sqrt{2}}A_{21}\Sigma_1^{\frac{1}{2}} & \frac{1}{\sqrt{2}}B_2 \\ 0 & \frac{1}{\sqrt{2}}\Sigma_1^{-\frac{1}{2}}A_{12} & \Sigma_1^{-\frac{1}{2}}A_{11}\Sigma_1^{\frac{1}{2}} & \Sigma_1^{-\frac{1}{2}}B_1 \\ C_1\Sigma_1^{-\frac{1}{2}} & \frac{1}{\sqrt{2}}C_2 & 0 & 0 \end{bmatrix}$$

To prove the main result, we will show that TMT^{-1} is contractive, and hence M is contractive. Note that

$$TMT^{-1} = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \hat{M}$$

where

$$\hat{M} = \begin{bmatrix} 0 & \vdots & \frac{1}{\sqrt{2}}\Sigma_1^{-\frac{1}{2}}A_{12} & \vdots & \Sigma_1^{-\frac{1}{2}}A_{11}\Sigma_1^{\frac{1}{2}} & \Sigma_1^{-\frac{1}{2}}B_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{2}}A_{21}\Sigma_1^{-\frac{1}{2}} & \vdots & A_{22} & \vdots & \frac{1}{\sqrt{2}}A_{21}\Sigma_1^{\frac{1}{2}} & \frac{1}{\sqrt{2}}B_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Sigma_1^{\frac{1}{2}}A_{11}\Sigma_1^{-\frac{1}{2}} & \vdots & \frac{1}{\sqrt{2}}\Sigma_1^{\frac{1}{2}}A_{12} & \vdots & 0 & 0 \\ C_1\Sigma_1^{-\frac{1}{2}} & \vdots & \frac{1}{\sqrt{2}}C_2 & \vdots & 0 & 0 \end{bmatrix}$$

Let

$$U_{11} = \Sigma_1^{-\frac{1}{2}}A_{12}, \quad U_{12} = [\Sigma_1^{-\frac{1}{2}}A_{11}\Sigma_1^{\frac{1}{2}} \;\; \Sigma_1^{-\frac{1}{2}}B_1], \quad U_{22} = [A_{21}\Sigma_1^{\frac{1}{2}} \;\; B_2],$$

$$V_{11} = A_{21}\Sigma_1^{-\frac{1}{2}},\ V_{21} = \begin{bmatrix} \Sigma_1^{\frac{1}{2}}A_{11}\Sigma_1^{-\frac{1}{2}} \\ C_1\Sigma_1^{-\frac{1}{2}} \end{bmatrix},\ V_{22} = \begin{bmatrix} \Sigma_1^{\frac{1}{2}}A_{12} \\ C_2 \end{bmatrix} \text{ and } Z = A_{22}.$$

Note that $U = \begin{bmatrix} U_{11} & U_{12} \\ Z & U_{22} \end{bmatrix}$ and $V = \begin{bmatrix} V_{11} & Z \\ V_{21} & V_{22} \end{bmatrix}$ are contractive by Lemma 3. Applying Lemma 2 shows that $\hat{M}$ is contractive, and therefore M is contractive. Thus $\frac{1}{2}\|G - G_r\|_\infty \leq 1$ by Lemma 1. □

For 1D system realizations with no uncertainty the role of the system Lyapunov equations and the associated controllability and observability gramians in balanced truncation model reduction and in terms of quantifying system minimality is well-defined. In the next section, we consider an extension of these concepts to system realizations which incorporate uncertainty descriptions into the model definitions.

4 System Representations with Uncertainty

We now present a generalization of the notion of reducible realizations and the balanced model reduction technique given for 1D systems to uncertain systems. We will focus on the case where the uncertainty structure is defined as in (2.1), that is, we consider $\mathbf{\Delta}$ with δ_i in the class of arbitrary linear time-varying operators on l_2.

4.1 Balanced System Realizations

Consider the system in Figure 1 with Δ and M defined as previously discussed, and A, B and C partitioned compatibly with the block structure Δ as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1S} \\ \vdots & \ddots & \vdots \\ A_{S1} & \cdots & A_{SS} \end{bmatrix}; \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_S \end{bmatrix}; \quad C = \begin{bmatrix} C_1 & \cdots & C_S \end{bmatrix}.$$

Such a system is stable when the I/O map $(\Delta \star M)$ is well-posed for every $\Delta \in \mathbf{B\Delta}$, that is, when $(I - A\Delta)$ is invertible as an operator on l_2 for every $\Delta \in \mathbf{B\Delta}$. For time-varying operators in $\mathbf{B\Delta} \subset \mathcal{L}(l_2)$, an equivalent LMI stability condition is given in [19], and is cited below. This result is an extension of the robust stability condition proven in [15] and concurrently in [20], and is proven via methods similar to those in [15]. Details can be found in [19]. The uncertainty structure $\hat{\mathbf{\Delta}}$ referred to in Theorem 3 is more general than that defined by $\mathbf{\Delta}$ in (2.1) in that full blocks are also allowed, that is,

$$\hat{\mathbf{\Delta}} = \{\mathrm{diag}\,[\Delta_\delta, \Delta_1, \ldots \Delta_F] : \Delta_\delta \in \mathbf{\Delta},\ \Delta_j \in \mathcal{L}(l_2^{m_j})\}. \tag{4.1}$$

We define the commutative matrix set for a given uncertainty set $\mathbf{\Delta}$ as $\mathcal{T} = \{T \in \mathbb{C}^{N\times N} : T\Delta = \Delta T,\ \forall \Delta \in \mathbf{\Delta}\}$. For $\hat{\mathbf{\Delta}}$ as defined above, $T \in \mathcal{T}$ if $T = \mathrm{diag}[T_1, \ldots, T_S, t_1 I_{m_1}, \ldots, t_F I_{m_F}]$, where each $T_i \in \mathbb{C}^{n_i \times n_i}$ and $t_j \in \mathbb{C}$.

Theorem 3. *Given an uncertainty set, $\hat{\mathbf{\Delta}} \subset \mathcal{L}(l_2)$, and a constant matrix A, then there exists a matrix $Y > 0$, $Y \in \mathcal{T}$ such that*

$$A^* Y A - Y < 0$$

if and only if

$$(I - A\hat{\Delta}) \text{ is invertible in } \mathcal{L}(l_2),\ \forall \hat{\Delta} \in \mathbf{B}\hat{\mathbf{\Delta}}$$

This result is used directly to develop the following notion of stability, and is related to the structured induced 2-norm for uncertain systems discussed in the following section.

Definition 2. *The uncertain system $(\mathbf{\Delta}, M)$ is stable if there exists a non-singular matrix $T \in \mathcal{T}$ such that*

$$\overline{\sigma}(TAT^{-1}) < 1$$

As in Theorem 3, stability may be expressed in LMI form by defining positive definite matrices Y and X by $Y = (TT^*)^{-1}$ and $X = T^*T$. Then the following are equivalent:

$$(i)\overline{\sigma}(TAT^{-1}) < 1 \quad (ii)AYA^* - Y < 0 \quad (iii)A^*XA - X < 0. \tag{4.2}$$

By scaling Y and X one can immediately deduce the following lemma.

Lemma 5. *$(\mathbf{\Delta}, M)$ is stable if and only if there exist $Y > 0$ and $X > 0$, both in $\mathcal{T}$ which satisfy*

$$AYA^* - Y + BB^* < 0 \text{ and } A^*XA - X + C^*C < 0. \tag{4.3}$$

As in the 1D case, we can use non-strict inequalities in (4.3). Then stability of $(\mathbf{\Delta}, M)$ implies there exist $Y \geq 0$ and $X \geq 0$ both in $\mathcal{T}$ such that

$$AYA^* - Y + BB^* \leq 0 \text{ and } A^*XA - X + C^*C \leq 0. \tag{4.4}$$

We refer to any matrices Y and X which satisfy (4.4) as *generalized gramians.* We can show that if there exist *singular* generalized gramians, then the uncertain system realization is reducible (see Section 5, [1], and [2]). As a result, in deriving the model reduction error bound, we need only consider the case of strict inequalities.

To define a notion of balanced realizations for uncertain systems, we proceed as in the 1D case by defining a similarity transformation as an invertible matrix, T, which transforms the system states, such that the realization M is transformed to $\hat{M}$ as in section 3.2. We consider similarity transformations in $\mathcal{T}$ for which it can readily be shown that

$$(\Delta \star M) = (\Delta \star \hat{M}), \quad \forall \Delta \in \mathbf{\Delta} \tag{4.5}$$

Note that when $T \in \mathcal{T}$, $\hat{Y} = TYT^*$ and $\hat{X} = (T^{-1})^* X T^{-1}$ are commutative solutions to the Lyapunov inequalites for $\hat{M}$. Thus we can define balanced realizations for uncertain systems as follows.

Definition 3. *An uncertain system realization, M, is balanced if there exist Y and X which satisfy the inequalities in (4.3) and*

$$Y = X = \Sigma = diag[\Sigma_1, \cdots, \Sigma_r] \tag{4.6}$$

where $\Sigma_i = diag[\sigma_{i1} I_{s_{i1}}, \cdots, \sigma_{it_i} I_{s_{it_i}}] > 0$; $\sigma_{i1} \geq \cdots \geq \sigma_{it_i}$ and $dim(\Sigma_i) = dim(A_{ii})$ is denoted by $n_i = \sum_{j=1}^{t_i} s_{ij}$.

For stable uncertain systems, the existence of balanced realizations is guaranteed by Lemma 5 and (4.5). Neither the balanced Y and X nor the resulting realization is unique.

Note that permutations do not affect stability or sign-definiteness of the Lyapunov inequalities. That is, let Π be any matrix such that $\Pi\, \Pi^T = I$, and suppose we have a commutative solution P to (4.3). Denote $\mathcal{A} = \Pi^T A \Pi$, $\mathcal{B} = \Pi^T B \Pi$ and $\mathcal{P} = \Pi^T P \Pi$. Then,

$$\mathcal{A}\mathcal{P}\mathcal{A}^* - \mathcal{P} + \mathcal{B}\mathcal{B}^* = \Pi^T (APA^* - P + BB^*)\Pi < 0.$$

4.2 Error Bounds: Balanced Truncation Model Reduction

In order to quantify the error resulting from reducing uncertain systems, we use the *structured induced 2-norm*, (SI2-norm), which is defined as follows

Definition 4. *The SI2-norm of a stable system $(\mathbf{\Delta}, M)$ is given by*

$$\|(\mathbf{\Delta}, M)\|_{SI2} = \sup_{\Delta \in \mathbf{B\Delta}} \|\Delta \star M\|_{\mathcal{L}_2 \to \mathcal{L}_2} \tag{4.7}$$

Note that for 1D systems, the SI2-norm is the same as the H_∞ norm.

The difference in the SI2-norm between two realizations, $(\mathbf{\Delta}_1, M_1)$ and $(\mathbf{\Delta}_2, M_2)$, is evaluated by forming the difference realization of, denoted by $(\tilde{\mathbf{\Delta}}, \tilde{M}) = (\mathbf{\Delta}_1, M_1) - (\mathbf{\Delta}_2, M_2)$, where

$$\tilde{M} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & -C_2 & D_1 - D_2 \end{bmatrix} \text{ and } \tilde{\Delta} = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \tag{4.8}$$

and computing $\left\|(\tilde{\mathbf{\Delta}}, \tilde{M})\right\|_{SI2}$.

The uncertainty structure $\mathbf{\Delta}$ as defined as in (2.1) may contain time-varying or time-invariant operators on l_2, thus the δ_i may represent system uncertainty or transform variables. If we consider uncertainty structures where all δ_i are time-varying operators on l_2, then an equivalent formulation for the SI2-norm of a system, which more readily allows for computation via recent algorithms developed for solving LMI's [6], is given in the following lemma.

Lemma 6. *The SI2-norm of an stable system $(\mathbf{\Delta}, M)$ is given by*

$$\|(\mathbf{\Delta}, M)\|_{SI2} = \inf \left\{ \gamma : \exists T \text{ such that } \overline{\sigma} \left(\begin{bmatrix} TAT^{-1} & \frac{1}{\gamma^{\frac{1}{2}}} TB \\ \frac{1}{\gamma^{\frac{1}{2}}} CT^{-1} & \frac{1}{\gamma} D \end{bmatrix} \right) < 1 \right\} \tag{4.9}$$

where $T \in \mathcal{T}$.

Proof. Let $\Delta_F \in \mathcal{L}(l_2^p)$ and $M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma^{\frac{1}{2}}} B \\ \frac{1}{\gamma^{\frac{1}{2}}} C & \frac{1}{\gamma} D \end{bmatrix}$. Then for a stable system $(\mathbf{\Delta}, M)$,

$$\inf \left\{ \gamma : \text{ there exists } T \text{ such that } \overline{\sigma} \left(\begin{bmatrix} TAT^{-1} & \frac{1}{\gamma^{\frac{1}{2}}} TB \\ \frac{1}{\gamma^{\frac{1}{2}}} CT^{-1} & \frac{1}{\gamma} D \end{bmatrix} \right) < 1 \right\} =$$

$$\inf \left\{ \gamma \mid \text{ there exists } Y > 0 : M_\gamma \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} M_\gamma^* - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} < 0 \right\}, \tag{4.10}$$

for Y and T both in $\mathcal{T}$ and $\gamma \geq 0$. Applying Theorem 3 and the stability assumption on $(\mathbf{\Delta}, M)$, we see that (4.10) is equivalent to

$$\inf \left\{ \gamma \mid \left(I - M_\gamma \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix} \right) \text{ is invertible in } \mathcal{L}(l_2) \text{ for all } \Delta_F \in \mathbf{B\Delta}_F \right\} =$$

$$\inf\left\{\gamma : \begin{bmatrix} I & 0 \\ -\frac{1}{\gamma^{\frac{1}{2}}}C\Delta(I-A\Delta)^{-1} & I \end{bmatrix} \begin{bmatrix} I - A\Delta & -\frac{1}{\gamma^{\frac{1}{2}}}B\Delta_F \\ 0 & I - (\frac{1}{\gamma}\Delta \star M)\Delta_F \end{bmatrix} \text{ invertible in } \mathcal{L}(l_2)\forall \Delta_F \in \mathbf{B\Delta}_F\right\} = \tag{4.11}$$

$$\inf\left\{\gamma \mid (I - (\frac{1}{\gamma}\Delta \star M)\Delta_F) \text{ is invertible in } \mathcal{L}(l_2) \text{ for all } \Delta_F \in \mathbf{B\Delta}_F\right\} =$$

$$\sup_{\Delta \in \mathbf{B\Delta}} \|\Delta \star M\|_{l_2 \to l_2}.$$

□

From Lemma 6, the following generalization of Lemma 1 is now obvious for the case of time-varying uncertainty.

Lemma 7. *Suppose $(\mathbf{\Delta}, M)$ represents a stable uncertain system, then $\|(\mathbf{\Delta}, M)\|_{SI2} \leq 1$ if and only if there is a realization $\hat{M}$, where $(\Delta \star M) = (\Delta \star \hat{M})$ for all $\Delta \in \mathbf{\Delta}$, such that $\hat{M}$ is contractive.*

Note that if the uncertainty structure contains time-invariant operators, or transform variables, then the expression on the right in (4.9) is an upper bound for the SI2-norm. Therefore, the existence of a contractive realization is a sufficient condition for $\|(\mathbf{\Delta}, M)\|_{SI2} \leq 1$ when the δ_i of $\mathbf{\Delta}$ are time-invariant. This sufficiency is all that is needed for the balanced truncation model reduction bounds to hold for both MD and uncertain systems.

In order to derive the model reduction error bounds for balanced uncertain systems, we partition the system matrices A, B, C and Σ in order to separate the subblocks which will be truncated. Let each block of Σ be partitioned as

$$\Sigma_i = \text{diag}[\hat{\Sigma}_{1i}, \Sigma_{2i}]$$

for $i = 1, \cdots, S$, where

$$\hat{\Sigma}_{1i} = \text{diag}[\sigma_{i1}I_{s_{i1}}, \cdots, \sigma_{ik_i}I_{s_{ik_i}}], \quad k_i \leq t_i$$

and

$$\Sigma_{2i} = \text{diag}[\sigma_{i(k_i+1)}I_{s_{i(k_i+1)}}, \cdots, \sigma_{it_i}I_{s_{it_i}}].$$

Truncate both Σ_{2i} and the corresponding parameter matrices, for example, truncate

$$A_{11} = \begin{bmatrix} \hat{A}_{11} & A_{11_{12}} \\ A_{11_{21}} & A_{11_{22}} \end{bmatrix}, \; B_1 = \begin{bmatrix} \hat{B}_1 \\ B_{1_2} \end{bmatrix} \text{ and } \begin{bmatrix} \hat{C}_1 & C_{1_2} \end{bmatrix}$$

to $\hat{A}_{11}$, $\hat{B}_1$ and $\hat{C}_1$ for each A_{ij}, B_j and C_i, $i, j = 1, \ldots, S$. The resulting truncated system is

$$\hat{M} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & D \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \cdots & \hat{A}_{1S} & \hat{B}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \hat{A}_{S1} & \cdots & \hat{A}_S & \hat{B}_S \\ \hat{C}_1 & \cdots & \hat{C}_S & D \end{bmatrix}.$$

with uncertainty structure $\hat{\Delta} = \text{diag}[\delta_1 I_{\hat{n}_1}, \cdots, \delta_S I_{\hat{n}_S}]$ where $\hat{n}_i = \sum_{j=1}^{k_i} s_{ij}$.

As in the 1D case, the truncation of a balanced stable uncertain system is still balanced and stable. We now state the model reduction error bound theorem, originally given in [21].

Theorem 4. *Suppose $(\hat{\boldsymbol{\Delta}}, \hat{M})$ is the reduced model obtained from the balanced stable system (Δ, M). Then*

$$\|(\boldsymbol{\Delta}, M) - (\hat{\boldsymbol{\Delta}}, \hat{M})\|_{SI2} \leq 2 \sum_{i=1}^{S} \sum_{j=k_i+1}^{t_i} \sigma_{ij} \tag{4.12}$$

Proof. The proof for Theorem 4, for which we provide an outline, follows from a strict generalization of the proof given for the 1D case.

By repeated permutations, scalings and truncations, we can apply the methods of Theorem 2 along with the sufficiency direction of Lemma 7 to obtain the stated bound. That is, we assume the system realization is reduced from

$$M = \begin{bmatrix} \hat{A} & A_{12} & \hat{B} \\ A_{21} & A_{22} & B_2 \\ \hat{C} & C_2 & D \end{bmatrix}$$

to $\hat{M} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & D \end{bmatrix}$, with $\Sigma = \text{diag}[\hat{\Sigma}, I]$. The corresponding uncertainty structure is reduced from $\Delta = \text{diag}[\delta_1 I_{n_1}, \cdots, \delta_S I_{n_S}]$ to $\hat{\Delta} = diag[\delta_1 I_{n_1}, \cdots, \delta_{S-1} I_{n_{S-1}}, \delta_S I_{\hat{n}_S}]$, where $\hat{n}_S = \sum_{j=1}^{(t_S-1)} s_{Sj} < n_S$, that is, only the representation submatrices corresponding to the last uncertainty variable δ_S in Δ and the last singular value σ_{St_S} in Σ_S are reduced. As in the 1D case, we assume $\sigma_{St_S} = 1$ and subsequently show that $\frac{1}{2}\left\|(\boldsymbol{\Delta}, M) - (\hat{\boldsymbol{\Delta}}, \hat{M})\right\|_{SI2} \leq 1$.

Let

$$\tilde{M} = \begin{bmatrix} \hat{A} & 0 & 0 & \frac{1}{\sqrt{2}}\hat{B} \\ 0 & \hat{A} & A_{12} & \frac{1}{\sqrt{2}}\hat{B} \\ 0 & A_{21} & A_{22} & \frac{1}{\sqrt{2}}B_2 \\ -\frac{1}{\sqrt{2}}\hat{C} & \frac{1}{\sqrt{2}}\hat{C} & \frac{1}{\sqrt{2}}C_2 & 0 \end{bmatrix}$$

and $\tilde{\boldsymbol{\Delta}} = \text{diag}[\hat{\boldsymbol{\Delta}}, \boldsymbol{\Delta}]$, with corresponding commutative matrix set $\mathcal{T}$. Using a similarity transformation $T \in \mathcal{T}$ like that in the proof of Theorem 2 gives

$$T\tilde{M}T^{-1} = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M_I$$

where

$$M_I = \begin{bmatrix} 0 & \vdots & \frac{1}{\sqrt{2}}\hat{\Sigma}^{-\frac{1}{2}}A_{12} & \vdots & \hat{\Sigma}^{-\frac{1}{2}}\hat{A}\hat{\Sigma}^{\frac{1}{2}} & \hat{\Sigma}^{-\frac{1}{2}}\hat{B} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{2}}A_{21}\hat{\Sigma}^{-\frac{1}{2}} & \vdots & A_{22} & \vdots & \frac{1}{\sqrt{2}}A_{21}\hat{\Sigma}^{\frac{1}{2}} & \frac{1}{\sqrt{2}}B_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{\Sigma}^{\frac{1}{2}}\hat{A}\hat{\Sigma}^{-\frac{1}{2}} & \vdots & \frac{1}{\sqrt{2}}\hat{\Sigma}^{\frac{1}{2}}A_{12} & \vdots & 0 & 0 \\ \hat{C}\hat{\Sigma}^{-\frac{1}{2}} & \vdots & \frac{1}{\sqrt{2}}C_2 & \vdots & 0 & 0 \end{bmatrix}.$$

Now defining U and V as in Theorem 2 and using Lemma 3 and the fact that the Lyapunov inequalities are satisfied by assumption, gives us that U and V are contractive. Applying Lemma 2 then implies that M_I is contractive, and hence $\tilde{M}$ is contractive. Finally, from Lemma 7 we have that $\frac{1}{2}\left\|(\tilde{\mathbf{\Delta}}, \tilde{M})\right\|_{SI2} \leq 1$. □

4.3 Improved Error Bounds: LMI Model Reduction

Using machinery developed by Packard [18], we can achieve a tighter model reduction bound then that given in Theorem 4, by a factor of two, for systems with uncertainty structures Δ where all $\delta_i \in \mathcal{L}(l_2)$ are assumed to be time-varying. This bound provides a direct connection to *exact* reducibility conditions for uncertain systems, that is, conditions under which model reduction can be completed with no resulting error. The tighter error bound is derived from Lemma 6, and from the following lemma. We first define some notation.

Throughout this section we refer to the following uncertainty structures:

$$\begin{aligned} \mathbf{\Delta} &= \{\text{diag } [\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_S I_{n_S}] : \delta_i \in \mathcal{L}(l_2)\}, \\ \mathbf{\Delta}_r &= \{\text{diag } [\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_S I_{r_S}] : \delta_i \in \mathcal{L}(l_2)\} \end{aligned}$$

and

$$\tilde{\mathbf{\Delta}} = \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_r \end{bmatrix} : \Delta \in \mathbf{\Delta}, \Delta_r \in \mathbf{\Delta}_r \right\},$$

with commutative matrix sets denoted by $\mathcal{T}$, $\mathcal{T}_r$ and $\tilde{\mathcal{T}}$. Note that the commutative matrix set, $\tilde{\mathcal{T}}$, for $\tilde{\mathbf{\Delta}}$ includes matrices with the following block structure:

$$T = \begin{bmatrix} \text{diag}(T_i^n) & \text{diag}(T_i^{nr}) \\ \text{diag}(T_i^{rn}) & \text{diag}(T_i^r) \end{bmatrix}$$

where $\dim(T_i^n) = n_i \times n_i$, $\dim(T_i^{nr}) = n_i \times r_i$, $\dim(T_i^{rn}) = r_i \times n_i$ and $\dim(T_i^r) = r_i \times r_i$ for all $i = 1, \ldots, S$. We denote the difference system $(\mathbf{\Delta}, M) - (\mathbf{\Delta}_r, M_r)$ by $(\tilde{\mathbf{\Delta}}, \tilde{M})$ which is formed as in (4.8). Now, using the notation of [18], we define

$$R = \begin{bmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & D \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad V = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and

$$Q = \begin{bmatrix} A_r & B_r \\ -C_r & -D_r \end{bmatrix}.$$

Note that $M_r = Q$ and $\tilde{M} = R + UQV$. Furthermore, $U_\perp = [I\ 0\ 0]^T$ and $V_\perp = [I\ 0\ 0]$, thus $\tilde{U} = [U_{\perp,1}^T\ U_{\perp,2}^T]^T = [I\ 0]^T$ and $\tilde{V} = [V_{\perp,1}\ V_{\perp,2}] = [I\ 0]$. Directly applying Lemma 5.2 and Theorem 6.3 from [18] gives the following lemma.

Lemma 8. *Suppose the realization (Δ, M) is given, with $\Delta \in \boldsymbol{\Delta}$, and R, U, V, $U_\perp$ and $V_\perp$ defined as above. Then there exist a realization M_r, an uncertainty structure $\boldsymbol{\Delta}_r$, and a matrix $Z \in \mathcal{T}$, $Z > 0$ satisfying*

$$\overline{\sigma}\left(\begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix} \tilde{M} \begin{bmatrix} Z^{-1} & 0 \\ 0 & I \end{bmatrix}\right) < 1$$

if and only if there exist $X_i^n > 0$, $Y_i^n > 0$ for $i = 1, \ldots, S$ satisfying

(a) $\tilde{U}^T \left(M \begin{bmatrix} \mathrm{diag}(X_i^n) & 0 \\ 0 & I \end{bmatrix} M^T - \begin{bmatrix} \mathrm{diag}(X_i^n) & 0 \\ 0 & I \end{bmatrix}\right) \tilde{U} < 0$

(b) $\tilde{V} \left(M^T \begin{bmatrix} \mathrm{diag}(Y_i^n) & 0 \\ 0 & I \end{bmatrix} M - \begin{bmatrix} \mathrm{diag}(Y_i^n) & 0 \\ 0 & I \end{bmatrix}\right) \tilde{V}^T < 0$

(c) $\begin{bmatrix} X_i^n & I \\ I & Y_i^n \end{bmatrix} \geq 0$

where the dimensions of $\boldsymbol{\Delta}_r$ and M_r may be computed by $r_i = \mathrm{rank}(X_i^n - Y_i^{n-1})$ for each $i = 1, \ldots, S$.

See [18] for a proof of Lemma 8. Using Lemma 8 we can prove that for any $\epsilon > 0$ a lower order realization $(\boldsymbol{\Delta}_r, M_r)$ exists such that the SI2-norm of the difference system $(\tilde{\boldsymbol{\Delta}}, \tilde{M}) = (\boldsymbol{\Delta}, M) - (\boldsymbol{\Delta}_r, M_r)$ is bounded above by ϵ if and only if solutions, X_ϵ and Y_ϵ, to the Lyapunov inequalities (4.3) exist satisfying a rank constraint. For convenience, we denote the ϵ-scaled difference system realization by

$$\tilde{M}_\epsilon = \begin{bmatrix} A & 0 & \frac{1}{\epsilon^{\frac{1}{2}}} B \\ 0 & A_r & \frac{1}{\epsilon^{\frac{1}{2}}} B_r \\ \frac{1}{\epsilon^{\frac{1}{2}}} C & -\frac{1}{\epsilon^{\frac{1}{2}}} C_r & \frac{1}{\epsilon}(D - D_r) \end{bmatrix}. \tag{4.13}$$

Theorem 5. *Given a system realization $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with uncertainty structure $\boldsymbol{\Delta}$, then there exist $M_r = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}$ and $\boldsymbol{\Delta}_r$ such that $\|(\tilde{\boldsymbol{\Delta}}, \tilde{M})\|_{SI2} \leq \epsilon$ if and only if there exists $X_\epsilon > 0$ and $Y_\epsilon > 0$, both in $\mathcal{T}$, satisfying* *(i)* $AY_\epsilon A^* - Y_\epsilon + BB^* < 0$

(ii) $A^*X_\epsilon A - X_\epsilon + C^*C < 0$

(iii) $\lambda_{min}(X_\epsilon Y_\epsilon) = \epsilon^2$, *with multiplicity* $\sum_i(n_i - r_i)$

where $\epsilon > 0$.

Proof. By Lemma 6,

$$\|(\tilde{\mathbf{\Delta}}, \tilde{M})\|_{SI2} \leq \epsilon \text{ if and only if there exists } T \in \mathcal{T} \text{ such that}$$

$$\overline{\sigma}\left(\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \tilde{M}_\epsilon \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}\right) < 1.$$

Now we need only apply Lemma 8 to $\tilde{M}_\epsilon$ and multiply out the matrices in (a) and (b). Then $\|(\tilde{\mathbf{\Delta}}, \tilde{M})\|_{SI2} \leq \epsilon$ if and only if there exist $X = \text{diag}(X_i^n) > 0$, and $Y = \text{diag}(Y_i^n) > 0$ for $i = 1, \ldots, S$ satisfying

$$AXA^* + \frac{1}{\epsilon}BB^* - X < 0, \quad A^*YA + \frac{1}{\epsilon}C^*C - Y < 0 \tag{4.14}$$

and

$$\begin{bmatrix} X_i^n & I \\ I & Y_i^n \end{bmatrix} \geq 0.$$

Multiplying the matrix inequalities in (4.14) by ϵ, and denoting $X_\epsilon = \epsilon X$ and $Y_\epsilon = \epsilon Y$ gives (i) and (ii). Additionally we have

$$\begin{bmatrix} X_{\epsilon_i}^n & \epsilon I \\ \epsilon I & Y_{\epsilon_i}^n \end{bmatrix} \geq 0, \text{ with } \text{rank}(X_{\epsilon_i}^n - (Y_{\epsilon_i}^n)^{-1}) = r_i. \tag{4.15}$$

Condition (iii) is obtained by premultiplying and postmultiplying (4.15) by $[\frac{1}{\epsilon}I \quad -(Y_{\epsilon_i}^n)^{-1}]$ and $\begin{bmatrix} \frac{1}{\epsilon}I \\ -(Y_{\epsilon_i}^n)^{-1} \end{bmatrix}$, respectively, giving

$$\frac{1}{\epsilon^2}X_{\epsilon_i}^n - (Y_{\epsilon_i}^n)^{-1} \geq 0,$$

thus $X_{\epsilon_i}^n Y_{\epsilon_i}^n \geq \epsilon^2 I$. Applying the rank condition implies $\text{rank}(X_{\epsilon_i}^n Y_{\epsilon_i}^n - \epsilon^2 I) = r_i$, thus $\lambda_{min}(X_{\epsilon_i}^n Y_{\epsilon_i}^n) = \epsilon^2$ with multiplicity r_i, for all $i = 1, \ldots, S$. Since X_ϵ and Y_ϵ are block diagonal compositions of $X_{\epsilon_i}^n$ and $Y_{\epsilon_i}^n$ the result follows. $\square$ By applying Theorem 5 recursively,

we may then show that the error bounds of Theorem 4 can be reduced by a factor of two.

When the uncertainty structure, $\mathbf{\Delta}$, contains time-invariant operators δ_i, as in the case of MD system representations, the existence of X_ϵ and Y_ϵ are sufficient to ensure $\left\|(\tilde{\mathbf{\Delta}}, \tilde{M})\right\|_{SI2} \leq \epsilon$. For 1D continuous time systems with no uncertainty, similar results have been obtained by Kavranoglu and Bettayeb [13], via an alternate method which requires simultaneously computing a pair of matrices B_0 and C_0 *augmenting* the system realization matrices B and C, and solutions X and Y to the *augmented* Lyapunov equations, such that

$$\lambda_{min}(XY) = \gamma^2 \text{ with multiplicity } n - r. \tag{4.16}$$

5 Exact Reducibility

When applying the balanced model reduction method to both 1D and uncertain systems, the system realizations which are truncated are assumed to be minimal in some sense. In the 1D case it is well understood what is meant by saying a realization is minimal, or equivalently controllable and observable, or irreducible. For the uncertain case, in order for the proof of the model reduction error bound to hold, it is assumed there exist non-singular commutative solutions, Y and X, to the Lyapunov inequalities, in which case a non-zero error bound results. Clearly, we would prefer to reduce the system realization with no error if possible, and we would like to know if we can still apply the error bounds of Theorem 4 if we have singular solutions for the Lyapunov inequalities. To obtain exact reducibility conditions which correlate to the notion of reducibility in the 1D case, we have proven the following $\epsilon = 0$ case. See [1], [2], and [3] for details.

Theorem 6. *Suppose the stable system realization* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *and uncertainty structure* $\boldsymbol{\Delta}$ *as defined in (2.1) are given. There exists a reduced realization* $M_r = \begin{bmatrix} A_r & B_r \\ C_r & D \end{bmatrix}$ *and uncertainty structure* $\boldsymbol{\Delta}_r$ *such that* $(\Delta \star M) - (\Delta_r \star M_r) = 0$, *if and only if there exists singular* $X \geq 0$ *or* $Y \geq 0$, *both in* $\mathcal{T}$, *satisfying*

(i) $AYA^* - Y + BB^* \leq 0$

or

(ii) $A^*XA - X + C^*C \leq 0$

Theorem 6 reveals that, given an uncertain (or MD) system representation, if structured singular solutions to either of a pair of LMIs can be found, then an equivalent lower order realization exists. Furthermore, if the uncertainty can be properly described by time-varying operators, then the existence of lower order realizations *requires* such singular LMI solutions. The development of computational methods for solving such LMI problems is a rapidly growing research area in the control community, and, in fact, many efficient convex optimization algorithms already exist. The fact that we would like to find singular solutions to these LMIs complicates the computational requirements, however, we can attempt to optimize over these LMI solutions to obtain the lowest amount of error possible in either balanced truncation or LMI model reduction methods.

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