# Analysis of Implicit Uncertain Systems

## Part II: Constant Matrix Problems and Application to Robust $\mathcal{H}_2$ Analysis

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**Abstract**

This paper introduces an implicit framework for the analysis of uncertain systems, of which the general properties were described in Part I. In Part II, the theory is specialized to problems which admit a finite dimensional formulation. A constant matrix version of implicit analysis is presented, leading to a generalization of the structured singular value $\mu$ as the stability measure; upper bounds are developed and analyzed in detail. An application of this framework results in a practical method for robust $\mathcal{H}_2$ analysis: computing robust performance in the presence of norm-bounded perturbations and white-noise disturbances.

# 1 Introduction

Part I of this paper introduced a framework for analysis of uncertain systems in implicit form, combining the behavioral approach to system theory [24], the Linear Fractional Transformation (LFT) paradigm for uncertainty descriptions [14], and the Integral Quadratic Constraint (IQC [26, 11]) formulation. We summarize the main ideas of this formulation. The same notational conventions apply.

An implicit system is described by equations of the form $Gw = 0$, where $w$ is a vector of signal variables, and $G$ is an operator in signal space. In this paper we only consider linear

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systems (i.e., with $G$ linear) in discrete time. The concept of stability in this formulation is characterized by the left invertibility of the operator $G$. Two versions were considered: "$l_2$-stability", meaning that $G$ is left invertible as an operator on $l_2$; "stability", meaning that $G$ has a causal, finite gain left inverse in the corresponding extended space $l_{2e}$.

Uncertain systems are characterized by a parameterized equation operator $G(\Delta)$, where $\Delta$ is a structured uncertainty operator of the form

$$\Delta = diag\,[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}] \tag{1}$$

Robust ($l_2$) stability means that the $G$ is ($l_2$) stable for all structured $\Delta$ in the normalized ball of uncertainty $\mathbf{B_\Delta}$. A parameterization $G(\Delta)$ of Linear Fractional Transformation (LFT) form was considered; it was shown how the general case reduces to the canonical version

$$\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} z = 0 \tag{2}$$

In addition to including the standard theory as a special case, it was shown that the implicit formulation allows for the representation of IQCs, which can be used to describe properties of signals, components, or mathematical restrictions in a problem under consideration.

In Part II we take (2) as a starting point, and move closer to the computational aspects of implicit analysis questions by considering problems which can be cast in a finite dimensional setting, in terms of constant matrices. Different cases which result in constant matrix analysis are reviewed in Section 2. These include some special instances of (2), as well as a problem from an entirely different origin, related to model validation [22, 13].

The constant matrix formulation leads naturally to an extended version of the structured singular value $\mu$ [5, 14], which is introduced in Section 3. The issue of upper bounds to the structured singular value for implicit systems is extensively considered; as in the standard case, these bounds are attractive computationally since they reduce to a convex feasibility problem. Conditions under which this bound is exact for the implicit case are discussed.

In Section 4, the case of state space implicit descriptions is addressed, and the analysis conditions are related to the case of linear time-varying (LTV) structured perturbations, already considered in Part I.

The paper concludes with an application of the machinery to robust performance tests in the presence of structured uncertainty and white noise disturbances, which has been referred to as the Robust $\mathcal{H}_2$ performance problem [15, 23]. The main idea is to consider *deterministic* descriptions [16, 17] of white signals obtained by constraints in $l_2$ space. The method described in Part I allows for these constraints to be included in an implicit analysis problem of the type (2). Some examples are included for illustration.

Preliminary versions of these results were presented in the conference papers [18, 19]. The proofs are collected in the Appendix.

# 2 Motivation for Constant Matrix Analysis

In many important cases, robustness analysis can be conducted in a *constant matrix* representation. In the implicit framework, these have the form

$$\begin{bmatrix} I - \Delta_0 A \\ C \end{bmatrix} z = 0, \quad \Delta_0 \in \mathbf{\Delta_0} \subset \mathbb{C}^{n \times n} \tag{3}$$

where $A \in \mathbb{C}^{n \times n}$, $C \in \mathbb{C}^{p \times n}$, and the structure $\mathbf{\Delta_0}$ is still of the form (1), but with blocks which are constant, complex matrices rather than dynamical operators. $\mathbf{\Delta_0}$ could also have blocks restricted to be *real* as in [27], which can be used to capture real parametric uncertainty. By analogy with the dynamic case, we will say that the implicit system (3) is *stable* if

$$Ker \begin{bmatrix} I - \Delta_0 A \\ C \end{bmatrix} = 0 \;\; \forall \Delta_0 \in \mathbf{B_{\Delta_0}} \tag{4}$$

Condition (4) strongly resembles the PBH test in standard system theory. In fact, for the special case $\Delta_0 = \delta I$, stability is equivalent to detectability of the pair (C,A). Further connections will be mentioned in section 3.

Different problems which lead to a constant matrix formulation are now reviewed.

## 2.1 LTI uncertainty

Analogously to the standard input-output case, if the perturbations $\Delta$ are linear time-invariant (LTI), then robustness analysis can be reduced to a constant matrix test across frequency (over the unit circle or the unit disk). For a finite dimensional LTI map $M$, $M(e^{j\omega})$ will denote the frequency response, and for causal $M$, $M(\xi)$, $\xi \in \mathbb{C}$ will denote the $Z$-transform $M(\xi) = \sum_t M(t)\xi^t$.

**Proposition 1** *Consider the implicit system* (2)*, where $A$, $C$, $\Delta$ are finite dimensional LTI maps in $\mathcal{L}(l_2)$. Let the constant matrix $\Delta_0$ have the same spatial structure as $\Delta$. Then system* (2) *is robustly $l_2$-stable if and only if*

$$Ker \begin{bmatrix} I - \Delta_0 A(e^{j\omega}) \\ C(e^{j\omega}) \end{bmatrix} = 0 \quad \forall \Delta_0 \in \mathbf{B}_{\mathbf{\Delta}0}, \quad \forall \omega \in [-\pi, \pi], \tag{5}$$

**Proposition 2** *Consider the implicit system* (2)*, where $A$, $C$, $\Delta$ are causal finite dimensional LTI maps in $\mathcal{L}(l_{2e})$. Let the constant matrix $\Delta_0$ have the same spatial structure as $\Delta$. Then system* (2) *is robustly stable if and only if*

$$Ker \begin{bmatrix} I - \Delta_0 A(\xi) \\ C(\xi) \end{bmatrix} = 0 \quad \forall \Delta_0 \in \mathbf{B}_{\mathbf{\Delta}0}, \quad \forall |\xi| \le 1 \tag{6}$$

### 2.2 State space representations

Another standard method to obtain a constant matrix formulation is by writing a state-space realization for the discrete-time (causal) maps $A$, $C$; consider the joint realization

$$\begin{bmatrix} A \\ C \end{bmatrix} = \left[\begin{array}{c|c} A_x & A_{xz} \\ \hline A_{zx} & A_z \\ C_x & C_z \end{array}\right] \tag{7}$$

The state equations are of the form $x = \lambda(A_x x + A_{xz} z)$, with $\lambda$ the delay operator, and $x$ the state. By adding these equations in implicit form to (2) we obtain

$$\begin{aligned} \left(I - \begin{bmatrix} \lambda I & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} A_x & A_{xz} \\ A_{zx} & A_z \end{bmatrix}\right) \begin{bmatrix} x \\ z \end{bmatrix} &= 0 \\ \begin{bmatrix} C_x & C_z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} &= 0 \end{aligned} \tag{8}$$

which will be represented by

$$\begin{bmatrix} I - \Delta_S A_S \\ C_S \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = 0 \tag{9}$$

with $A_S$, $C_S$ constant matrices, and the augmented delay-uncertainty operator $\Delta_S \triangleq diag[\lambda,\ \Delta]$. For LTI uncertainty $\Delta$, (9) can in turn be reduced to a test in terms of constant complex perturbations $\Delta_{S_0}$; constant matrix conditions in state space can also be given for the LTV case. These issues will be discussed in Section 4.

## 2.3 Model Validation as Implicit Analysis

Substantial attention in recent years has focused in establishing closer connections between robust control and system identification. In this category fall the results of [22, 13], where a model validation problem is posed as a generalization of the structured singular value $\mu$. In particular, [22] considers a $\mu$ problem constrained to a subspace.

This is precisely the type of extension provided by an implicit representation as (3), as noted in [7]. A simple example is presented here to illustrate this point. Consider the validation of a linear regression model of the form

$$y = M\theta + d \tag{10}$$

with $\|\theta\| \leq K$, $\|d\| \leq \gamma$. In this equation, $y$ is a given vector in $\mathbb{R}^n$, and $M$ a given matrix in $\mathbb{R}^{n \times m}$, both related to experimental data. The validation problem is to determine whether there exist vectors $\theta$ and $d$ in the allowed class, satisfying (10).

The basic observation is that the size constraints on $\theta$ and $d$ can be captured by implicit uncertain equations $\theta = \Delta_\theta K$ and $d = \Delta_d \gamma$, where $\Delta_\theta$, $\Delta_d$ are respectively $m \times 1$, $n \times 1$ matrices of norm bounded by 1. Figure 1 jointly represents equations (10) and the constraints by means of an auxiliary "input" $w = 1$. The existence of nontrivial solutions for these equations is equivalent to the validation of the model (the constraint $w = 1$ can be obtained by normalization). This is a stability question in an implicit LFT system, and can be readily reduced to the form (3) by the method described in Part I. For a more complete discussion of these issues in a more general setting we refer to [7].

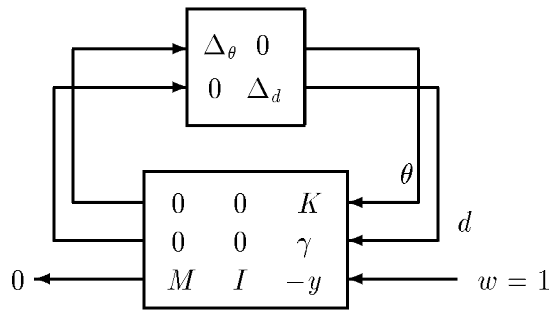


Figure 1: Validation of a linear regression model

## 3 A Structured Singular Value for Implicit Systems

Standard robust stability analysis for constant matrices is provided by the structured singular value $\mu$ [5, 14]. In the constrained case of (3), the natural extension is given by the following:

**Definition 1** *The structured singular value $\mu_{\mathbf{\Delta_0}}(C, A)$ of the implicit system* (3) *is defined as follows:*

*If* $Ker \begin{bmatrix} I - \Delta_0 A \\ C \end{bmatrix} = 0 \ \forall \Delta_0 \in \mathbf{\Delta_0}$, *define* $\mu_{\mathbf{\Delta_0}}(C, A) \triangleq 0$, *otherwise*

$$\mu_{\mathbf{\Delta_0}}(C, A) \triangleq \left( min \left\{ \bar{\sigma}\left(\Delta_0\right) : \Delta_0 \in \mathbf{\Delta_0}, Ker \begin{bmatrix} I - \Delta_0 A \\ C \end{bmatrix} \neq 0 \right\} \right)^{-1} \tag{11}$$

A restatement of this definition is to say that (3) is stable if and only if $\mu_{\mathbf{\Delta_0}}(C, A) < 1$.

Equivalently, Definition 1 translates the analysis problem to the computation of the function $\mu_{\mathbf{\Delta_0}}(C, A)$; as in the standard case, exact computation is in general hard and one must rely on upper and lower bounds. We will only comment briefly here on the lower bound problem, and develop in detail the upper bound theory.

The lower bounds for the standard unconstrained case (no $C$ equations) are based on the fact that $\mu_{\mathbf{\Delta}}(A) = \max_{\Delta \in \mathbf{B_\Delta}}(\rho(\Delta A))$, where $\rho(\cdot)$ denotes spectral radius. Algorithms which resemble the power iteration for spectral radius have been developed [14, 27], which have good performance on typical problems.

For the constrained case, only eigenvalues with eigenvectors in the kernel of $C$ are relevant. In the following, it will be convenient to parameterize this kernel by a matrix $C_\perp^*$, whose columns form a basis for the kernel of $C$. This leads to

$$Ker\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} = 0 \Longleftrightarrow \; Ker[C_\perp^* - \Delta A C_\perp^*] = 0 \tag{12}$$

Denoting $\rho^g(M,N) = \max\{|\beta| : \beta M - N \text{is singular}\}$ ( maximum modulus of a generalized eigenvalue of $M$, $N$), we have

$$\mu_{\mathbf{\Delta}}(C,A) = \max_{\Delta \in \mathbf{B_\Delta}} \rho^g(C_\perp, \Delta A C_\perp) \tag{13}$$

These observations will presumably lead to an extension of the standard $\mu$ lower bound; some difficulties arise, however: generalized eigenvalues do not always exist, and also the maximum in (13) need not occur on the boundary. Some initial work is documented in [7].

We will now consider the upper bounds for this version of the structured singular value. The following theory strongly parallels that of [14] for the standard case. We will use the same notation $\mathbf{Y}$ as in Part I for the real vector space of hermitian matrices which commute with the elements in $\mathbf{\Delta_0}$, which have the form $Y = \operatorname{diag}[Y_1, \ldots, Y_L,\, y_{L+1}I_{m_1}, \ldots, y_{L+F}I_{m_F}]$. We recall that $\langle Y, \bar{Y}\rangle = \sum_{i=1}^{L} \operatorname{tr}(Y_i\bar{Y}_i) + \sum_{j=1}^{F} y_{L+j}\bar{y}_{L+j}$ defines an inner product in $\mathbf{Y}$. Also, $\mathbf{X}$ and $\bar{\mathbf{X}}$ are, respectively, the convex subsets of $\mathbf{Y}$ of positive and nonnegative scalings.

**Lemma 3** *For fixed $\beta > 0$, the following are equivalent:*

$$(i) \;\; \exists X \in \mathbf{X} : \;\; A^*XA - \beta^2 X - C^*C \; < \; 0 \tag{14}$$

$$(ii) \; \exists X \in \mathbf{X} : \; C_\perp(A^*XA - \beta^2 X)C_\perp^* \; < \; 0 \tag{15}$$

The previous conditions are both Linear Matrix Inequalities (LMIs, [2]) (strictly speaking, $(i)$ is affine rather than linear). Testing whether an LMI is satisfied is a convex feasibility problem, for which interior point methods are available [2, 9]. While version $(i)$ is more directly related to robustness analysis tests, $(ii)$ is of lower dimensionality and therefore preferable from a computational point of view. We define the upper bound for $\mu$,

$$\hat{\mu}_{\mathbf{\Delta_o}}(C, A) = \inf\{\beta > 0 : \text{ (14) is satisfied}\} \tag{16}$$

The fact that $\mu_{\mathbf{\Delta_o}}(C, A) \leq \hat{\mu}_{\mathbf{\Delta_o}}(C, A)$ is a consequence of Theorem 4 below. We first mention the following remarks:

- Combining the upper bound with Propositions 1 and 2 provides tractable sufficient conditions for robust ($l_2$) stability in the case of LTI perturbations; for example,

$$\exists X(\omega) \in \mathbf{X} : A(e^{j\omega})^* X(\omega) A(e^{j\omega}) - X(\omega) - C(e^{j\omega})^* C(e^{j\omega}) < 0 \ \ \forall \omega \in [-\pi, \pi] \tag{17}$$

  an LMI across frequency, guarantees robust $l_2$-stability of (2) under LTI perturbations.

- LMI (15) for $\beta = 1$ has appeared in previous work [10] on stabilization of input-output LFT systems, where it characterizes the so-called $Q$-detectability of the pair ($A$,$C$) (this reinforces the connection with the PBH test mentioned earlier). It is shown in [10] that it is equivalent to the existence of an output injection matrix $L$ such that $\inf_X \bar{\sigma}\left(X(A + LC)X^{-1}\right) < 1$.

- If the structure includes real $\delta I$ blocks (corresponding to parametric uncertainty), the upper bound can be improved in the same manner as the standard case (see [27]).

For the analysis of the upper bound we introduce a static version of the $\nabla$ set defined in Part I. For $\zeta \in \mathbb{C}^n$, let

$$\begin{aligned}
&\Phi_i^0(\zeta) = (A\zeta)_i (A\zeta)_i^* - \zeta_i \zeta_i^* \qquad i = 1 \ldots L \\
&\sigma_{L+j}^0(\zeta) = (A\zeta)_{L+j}^* (A\zeta)_{L+j} - \zeta_{L+j}^* \zeta_{L+j} \qquad j = 1 \ldots F \\
&\Lambda^0(\zeta) = \operatorname{diag}\left[\Phi_1^0(\zeta), \ldots, \Phi_L^0(\zeta),\, \sigma_{L+1}^0(\zeta) I_{m_1}, \ldots, \sigma_{L+F}^0(\zeta) I_{m_F}\right]
\end{aligned} \tag{18}$$

Define $\nabla^0 = \{\Lambda^0(\zeta) : C\zeta = 0,\ |\zeta| = 1\} \subset \mathbf{Y}$; $co(\nabla^0)$ denotes its convex hull.

**Theorem 4** *1.* $\mu_{\Delta_0}(C, A) < 1$ *if and only if* $\nabla^0 \cap \bar{\mathbf{X}} = \phi$.

*2. The following are equivalent:*

$$\begin{aligned} &(i) \quad \hat{\mu}_{\mathbf{\Delta_0}}(C, A) < 1 \\ &(ii) \quad \exists X \in \mathbf{X} : A^* X A - X - C^* C < 0 \\ &(iii) \quad co(\nabla^0) \cap \bar{\mathbf{X}} = \phi \end{aligned} \tag{19}$$

In this constant matrix case, the upper bound will be strict in general; equivalently, LMI (19) is not a necessary test for stability. Referring to Part I, convexity of the $\nabla$ sets played an essential role in the necessity results, and this does not hold in the static context: $\nabla^0$ is not in general convex (so $\nabla^0 \cap \bar{\mathbf{X}} = \phi$ does not imply $co(\nabla^0) \cap \bar{\mathbf{X}} = \phi$).

In a similar manner as in the standard case [14], we now pose the question as to which special $\mathbf{\Delta}$ structures give equality of $\mu$ and $\hat{\mu}$.

**Definition 2** *The structure* $\mathbf{\Delta}$ *is* $\mu$*-simple in the implicit case if* $\mu_{\mathbf{\Delta}}(C, A) = \hat{\mu}_{\mathbf{\Delta}}(C, A)$ *for* any *matrices* $A$*,* $C$*.*

**Theorem 5** *The following structures are* $\mu$*-simple in the implicit case.*

*(i)* $\mathbf{\Delta} = \{\delta I : \delta \in \mathbb{C}\}$

*(ii)* $\mathbf{\Delta} = \{diag[\Delta_1, \dots \Delta_F] : \Delta_i \in \mathbb{R}^{m_i \times m_i}\}$*, with* $F \leq 2$*, for* $A, C$ *real.*

*(iii)* $\mathbf{\Delta} = \{diag[\Delta_1, \dots, \Delta_F] : \Delta_i \in \mathbb{C}^{m_i \times m_i}\}$*, with* $F \leq 3$*.*

In reference to structures with only full blocks, the situation is analogous to the standard case of [14]: the bound is exact for a maximum of 3 complex full blocks or 2 real full blocks. The only notable difference in the implicit case is the fact that the structure $\mathbf{\Delta} = \{diag[\delta_1 I, \Delta_2]\}$ is no longer $\mu$-simple, as shown in the following example.

**Example 1** *Let* $\Delta = \mathrm{diag}\,[\delta_1 I_2, \Delta_2]$, $\delta_1 \in \mathbb{C}$, $\Delta_2 \in \mathbb{C}^{2\times 2}$

$$C_\perp^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 4 \end{bmatrix}; \quad AC_\perp^* = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 3 & 2 \\ 2 & 2 \end{bmatrix}$$

*The top half of* $C_\perp^* - \Delta A C_\perp^*$ *is* $\begin{bmatrix} 1-2\delta_1 & 0 \\ 0 & 1-3\delta_1 \end{bmatrix}$, *so the kernel is nontrivial only for* $\delta_1 = 1/2$ *or* $\delta_1 = 1/3$. *In the first case, the kernel must be the span of* $[1,\ 0]'$ *therefore* $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \Delta_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. *This can be achieved with* $\bar{\sigma}(\Delta_2)$ *of at least* $4/\sqrt{13}$.

*A similar argument with* $\delta_1 = 1/3$, *shows that for a nontrivial kernel,* $\bar{\sigma}(\Delta_2) \geq \sqrt{2}$. *The first perturbation is smaller so* $\mu_{\mathbf{\Delta}}(C, A) = \sqrt{13}/4 < 1$.

*For the LMI, write* $X = \mathrm{diag}\,[X_0, I_2]$, *with* $X_0 = \begin{bmatrix} x & y \\ y^* & z \end{bmatrix}$. *Some algebra gives*

$$C_\perp(A^* X A - X)C_\perp^* = \begin{bmatrix} 3(x-1) & 5(y+2) \\ 5(y^*+2) & 8(z-1) \end{bmatrix} \tag{20}$$

*For (20) to be negative definite, and* $X > 0$, *we must have*

$$0 < x < 1; \quad 0 < z < 1; \quad |y|^2 < xz; \quad |y+2|^2 < \frac{24}{25}(1-x)(1-z) \tag{21}$$

*This implies* $|y| < 1$, $|y+2| < 1$ *which is impossible, so there is no solution to LMI (15) with* $\beta = 1$. *Consequently,* $\mu_{\mathbf{\Delta}}(C, A) < \hat{\mu}_{\mathbf{\Delta}}(C, A)$.

To conclude this section, we relate the LMI test (19) to the results of Part I. If $A$ and $C$ remain constant but allowed to operate on $l_2$ signals, and $\Delta_0$ is substituted by an arbitrary structured operator on $l_2$, Theorem 1, Part I implies that (19) is a necessary test for robust stability. This amounts to an infinite horizon augmentation of the constant matrix problem (3); for this case where $A$ and $C$ are constant, a finite horizon augmentation suffices:

**Theorem 6** *Let* $A$,$C$, *be constant matrices, operating on finite horizon signals* $z = \{z_k\}_{k=1}^d \in (\mathbb{C}^n)^d$. *Let* $\mathbf{\Delta}$ *be the class of structured (as in (1)) operators in* $(\mathbb{C}^n)^d$. *If* $d \geq dim(\mathbf{Y})$, *then*

$$Ker \begin{bmatrix} I - \Delta A \\ C \end{bmatrix} = 0 \ \ \forall \Delta \in \mathbf{B}_{\mathbf{\Delta}} \Longleftrightarrow \text{(19) holds} \tag{22}$$

The previous finite horizon augmentation can also be rewritten in matrix form, by defining matrices which are $d$ times larger than $A$, $C$, and $\Delta_0$, obtained by adequate repetition. A similar result (for standard $\mu$-analysis) has been obtained using very different methods in [1], where the size of the augmentation is $n$ rather than $dim(\mathbf{Y})$ (these two are the same when $\boldsymbol{\Delta}$ consists of only scalar blocks).

In comparison, in the case where $A$ and $C$ have unbounded memory, an infinite horizon augmentation (to LTV operators on $l_2$ as in Part I) is required.

# 4 Analysis of state space systems

In this section we consider the state space representation of (9), where $A_S$,$C_S$ are constant matrices, and the structure $\Delta_S = \text{diag}[\lambda I \ \Delta]$ has a "special" first block, given by the delay operator. As in Proposition 2, we will be dealing with causal operators $\Delta$ and the corresponding notion of robust stability.

## 4.1 LTI Uncertainty

If the uncertainty $\Delta$ is LTI, robustness analysis reduces once more to a constant matrix problem, where the corresponding constant matrix structure $\boldsymbol{\Delta}_{\mathbf{S_0}}$ has a $\delta I$ first block.

**Proposition 7** *System (9), with $\Delta$ a structured LTI operator in $\mathcal{L}_C(l_{2e})$, is robustly stable if and only if $\mu_{\boldsymbol{\Delta}_{\mathbf{S_0}}}(C_S, A_S) < 1$.*

The upper bound for $\mu$ will provide a computationally tractable sufficient condition for robust stability of the form

$$A_S^* X_S A_S - X_S - C_S^* C_S < 0 \tag{23}$$

where $X_S = diag[X_0, X]$ is defined to commute with $\Delta_S$: $X_0$ is a positive square matrix of dimension equal to the number of states, and $X \in \mathbf{X}$. This condition is in general conservative, even in the case where $\Delta$ is unstructured (one full block); this is a consequence of the fact that in the implicit case, the structure $\Delta_{S_0} = diag[\delta I, \Delta]$ is not $\mu$-simple, as shown by Example 1.

### 4.2 LTV Uncertainty

Since (23) is a constant scales test it can be related to analysis with LTV perturbations, as considered in Part I. We have the following:

**Proposition 8** *If* (23) *holds,* (9) *is stable for all* $\Delta_S = diag[\lambda, \Delta]$, $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ *where* $\mathbf{\Delta}$ *is the set of arbitrary causal bounded LTV operators* ($\mathbf{\Delta} \subset \mathcal{L}_C(l_{2e})$).

It is not clear in general whether the converse of Proposition 8 holds, as it does in the unconstrained case (no $C_S$); the results of Part I do not apply directly since here the first block of $\Delta_S$ is constrained to be the delay.

A special case where the converse holds is when the constraints $C_S$ do not involve the state variables $x$: consider the state space implicit system of Figure 2 a), which corresponds to the case $C_x = 0$ in (8). The corresponding system (2), depicted in Figure 2 b), has *static* $C$ constraints. As was shown in Part I, a general robust performance problem with a finite number of IQCs in the disturbance variables can be cast in this special form.

**Theorem 9** *In reference to Figure 2, let* $\Delta$ *vary in the class of structured, otherwise arbitrary causal operators in* $\mathcal{L}_C(l_{2e})$. *The following are equivalent:*

$(i)$ (23) *holds with* $A_S = \begin{bmatrix} A_x & A_{xz} \\ A_{zx} & A_z \end{bmatrix}$, $C_S = \begin{bmatrix} 0 & C \end{bmatrix}$

$(ii)$ *The implicit system* (9) *of Figure 2 (a) is robustly stable.*

$(iii)$ $\rho(A_x) < 1$, *and the implicit system* (2) *of Figure 2 (b) is robustly stable.*

$(iv)$ $\rho(A_x) < 1$, *and* $\exists X \in \mathbf{X} : A(e^{j\omega})^* X A(e^{j\omega}) - X - C^*C < 0 \;\; \forall \omega \in [-\pi, \pi]$ (24)

For the case of LTV perturbations considered above, the same $X$ (constant scaling) must hold across frequency, which is a stronger condition than the one given in (17) for the LTI case. Solving (24) in a set of frequency points gives a *coupled* LMI problem. The "state-space LMI" (23) gives $X$ in just one LMI, but must handle an extra full block in the scaling $X_S$.

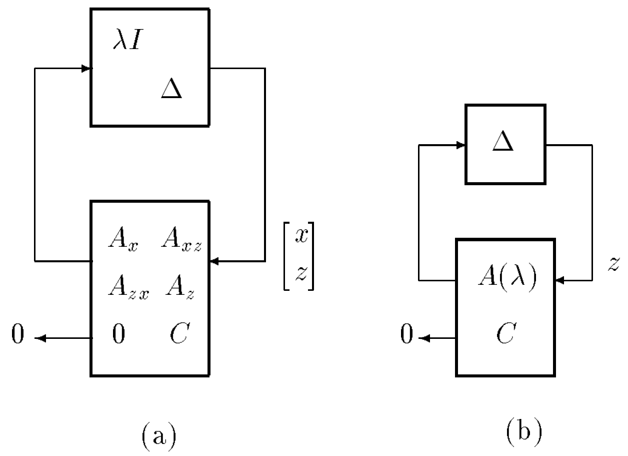


Figure 2: Two formulations of the robust stability problem

**Remark:** Condition $(iv)$ can be rewritten (see the proof of the theorem) in the form

$$\left\| X^{\frac{1}{2}} A C_{\perp}^{*} (C_{\perp} X C_{\perp}^{*})^{-\frac{1}{2}} \right\|_{\infty} < 1 \tag{25}$$

which is especially adequate for synthesis methods extending the so-called $D-K$ iteration for $\mu$-synthesis. In this case $A$ is a function of a controller $K$, and an $\mathcal{H}_\infty$ synthesis step is alternated with an analysis fit of $X$. These issues are addressed in [4].

# 5 Application to Robust $\mathcal{H}_2$ Analysis

In this section we apply the implicit analysis framework developed in this paper to the problem of analyzing white noise rejection properties of an uncertain system.

For an LTI input-output system in the absence of uncertainty, the relevant measure is the $\mathcal{H}_2$ norm of the system, which arises, for example, as the expected output power when the input is a stationary random process with flat power spectrum.

The stochastic paradigm is less attractive, however, when analyzing systems subject to additional sources of uncertainty (parameters, unmodeled dynamics) which are usually expressed more naturally in a deterministic setting. Research addressing this "Robust $\mathcal{H}_2$"

problem [15, 23] has faced the difficulty of analyzing the average effect of the disturbance together with the worst-case effect of the uncertainty. This is the main reason for which other deterministically motivated performance measures ($\mathcal{H}_\infty$, $\mathcal{L}_1$) have gained popularity in the robust control literature.

For many disturbances arising in applications, however, a white noise model is more appropriate than, for instance, the class of arbitrary bounded power signals considered in the $\mathcal{H}_\infty$ problem, where the worst-case signals (sinusoids) are often very unrealistic.

The following treatment is based on imposing *deterministic* constraints on the disturbance inputs by considering set characterizations of white noise [16, 17]. These constraints can be included in an implicit analysis problem of the form (2), and analyzed within this framework.

### 5.1 Set descriptions for white noise

This section shows how deterministic descriptions of white noise can be fit into the implicit analysis framework. We will first consider the case of scalar signals. In [17], sets of signals with a parameterized degree of "whiteness" are defined as

$$W_{\gamma,T} = \{w \in l_2, |r_w(\tau)| \leq \gamma r_w(0) \;\; \tau = 1 \ldots T\} \tag{26}$$

where $r_w(\tau) = \langle w, \lambda^\tau w \rangle$ is the autocorrelation of a scalar signal $w(t)$. Restricting an input signal to such a class is a deterministic method to rule out highly correlated signals (e.g., sinusoids) which are deemed unrealistic in a particular problem, and which dramatically affect the system gain. In fact, the worst-case gain of an input-output LTI system $H$ under signals in $W_{\gamma,T}$, denoted $\|H\|_{W_{\gamma,T}}$, satisfies

$$\|H\|_2^2 \leq \|H\|_{W_{\gamma,T}}^2 \leq \|H\|_2^2 + \gamma \sum_{\tau=-T}^{T} |r_h(\tau)| + \sum_{|t|>T} |r_h(t)| \tag{27}$$

where $r_h(\tau)$ are the autocorrelations of the system impulse response. So for small $\gamma$ (one could use $\gamma = 0$) and large $T$, the worst case induced norm of the system under the autocorrelation constraints approximates the $\mathcal{H}_2$ norm. Alternative descriptions in the frequency domain are also considered in [17], by restricting signals to have equal energy in a number of frequency bands, thereby enforcing an approximately flat spectrum. The relationship between these descriptions and the alternative stochastic paradigm is analyzed in [16, 17].

The advantage of a description such as (26) based on a finite number of quadratic constraints (as opposed to the ideal specification that $r_w(\tau)$ be the delta function) was already pointed out in [11]. In our setting, it corresponds to the fact that the resulting sets $W_{\gamma,T}$ can be represented by the behavior of an implicit uncertain system, as was shown for general IQCs in Part I.

We include the following derivation to motivate the multivariable case. Consider the constraints in (26), for real signals:

$$\pm\ r_w(\tau) \leq \gamma\ r_w(0) \quad \tau = 1\dots T \tag{28}$$

Simple manipulations reduce (28) to

$$2(1-\gamma)\|w\|^2 \leq \|w \pm \lambda^\tau w\|^2 \quad \tau = 1\dots T \tag{29}$$

For a fixed $\tau$ and, for instance, the plus sign in (29), let $P_+^\tau = \sqrt{2(1-\gamma)}$, $Q_+^\tau = 1 + \lambda^\tau$. The corresponding constraint $\left\|P_+^\tau w\right\|^2 \leq \left\|Q_+^\tau w\right\|^2$ is equivalent by Lemma 1, Part I to

$$(P_+^\tau - \delta_+^\tau Q_+^\tau)w = 0 \tag{30}$$

for some contractive operator $\delta_+^\tau$. The same procedure can be repeated for the minus sign in (29), and for $\tau = 1\dots T$. The constraints (30) can then be jointly represented by $(P - \Delta_C Q)w = 0$, where $P = [P_+^1, P_-^1 \dots P_-^T]'$, $Q = [Q_+^1, Q_-^1 \dots Q_-^T]'$, and $\Delta_C = diag[\delta_+^1, \delta_-^1 \dots \delta_-^T]$. The set of white signals $W_{\gamma,T}$ has been represented in the form

$$W_{\gamma,T} = \bigcup_{\Delta_C \in \mathbf{B_\Delta}} Ker(P - \Delta_C Q) \tag{31}$$

The previous construction can be extended to the multivariable case, by considering the autocorrelation *matrix* of a vector valued signal $w \in l_2^m$, $R_w(\tau) = \sum_{t=-\infty}^{\infty} w(t+\tau)w(t)^*$. For $w$ to be white, $R_w(\tau)$ must be 0 for $\tau \neq 0$, and $R_w(0)$ must be a multiple of the identity matrix. These matrix conditions could be reduced, entry by entry, to a number of scalar constraints, and treated as before.

For $\tau \neq 0$, and using the complex field, a simpler method is given by operator-valued $\delta I$ blocks in the uncertainty [1]. Let $\tau \neq 0$ be fixed, and define $P^\tau = \sqrt{2}I$, $Q^\tau_\pm = (1 \pm \lambda^\tau)I$, $\bar{Q}^\tau_\pm = (1 \pm j\lambda^\tau)I$. Consider the following four implicit equations, where each $\delta$ is an arbitrary operator on $l_2$.

$$(P^\tau - \delta^\tau_\pm I \; Q^\tau_\pm)w = 0 \tag{32}$$

$$(P^\tau - \bar{\delta}^\tau_\pm I \; \bar{Q}^\tau_\pm)w = 0 \tag{33}$$

By use of Lemma 2, Part I, (32-33) are equivalent to

$$2\|\eta^* w\|^2 \;\leq\; \|\eta^*(w \pm \lambda^\tau w)\|^2 \;\; \forall \eta \in \mathbb{C}^m \tag{34}$$

$$2\|\eta^* w\|^2 \;\leq\; \|\eta^*(w \pm j\lambda^\tau w)\|^2 \;\; \forall \eta \in \mathbb{C}^m \tag{35}$$

which in turn reduce to $\langle \eta^* w, \eta^* \lambda^\tau w\rangle = 0 \;\; \forall \eta \in \mathbb{C}^m \Longleftrightarrow R_w(\tau) = 0$.

Comparing with the IQC formulation, the constraints (32-33) for multivariable white noise correspond to matrix-valued IQCs, as mentioned in Part I.

### 5.2 Robust $\mathcal{H}_2$ Performance Analysis

The framework will now be applied to a problem of white noise rejection analysis in the presence of uncertainty. For this purpose, we return to the general setup of Section 3.4, Part I, which for convenience is represented in Figure 3. We are given an uncertain input-output system given as an LFT between an LTI map $H(\lambda)$ and a structured uncertainty operator $\Delta_u$. The question is to test whether the worst-case $l_2$ gain from $d$ to $y$ in the presence of uncertainty $\Delta_u$ is less than $\beta$, when the input signal $d$ is forced to satisfy "whiteness" constraints of the form discussed in section 5.1. These constraints are represented by a map $P - \Delta_C Q$, on the left in the picture, where without loss of generality $P$ can be chosen to be static. The same formulation allows for arbitrary IQCs applied to $d$. The "performance IQC" $\|y\| \geq \beta \, \|d\|$ is represented by the block $\Delta_P$.

[1] This provides a motivation for this type of blocks

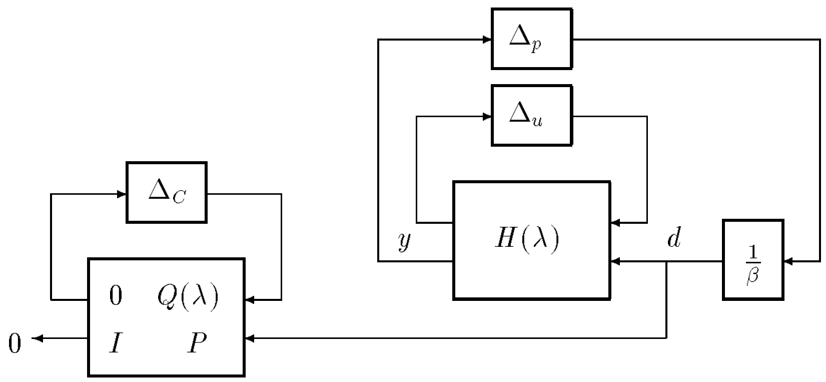


Figure 3: Robust $\mathcal{H}_2$ Analysis

The reduced representation (2) for this problem was already obtained in Part I. Adding the scaling $\beta$ gives

$$\Delta = \begin{bmatrix} \Delta_u & 0 & 0 \\ 0 & \Delta_P & 0 \\ 0 & 0 & \Delta_C \end{bmatrix}, \quad A = \begin{bmatrix} H_{11} & \frac{1}{\beta}H_{12} & 0 \\ H_{21} & \frac{1}{\beta}H_{22} & 0 \\ 0 & \frac{1}{\beta}Q & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \frac{1}{\beta}P & -I \end{bmatrix}, \quad z = \begin{bmatrix} z_u \\ z_p \\ z_c \end{bmatrix} \tag{36}$$

In this case, the blocks $\Delta_P$ and $\Delta_C$ are already in the class of structured LTV operators. We will analyze the case where $\Delta_u$ is also structured LTV, which may give a conservative answer if it includes parametric or LTI uncertainty. Since $C$ is static, the robust stability test is given by (24), or equivalently (from Theorem 9) by the state-space version (23), which is obtained by writing state-space realizations

$$H(\lambda) = \left[\begin{array}{c|cc} A_H & B_{H1} & B_{H2} \\ \hline C_{H1} & D_{H11} & D_{H12} \\ C_{H2} & D_{H21} & D_{H22} \end{array}\right] \qquad Q(\lambda) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$$

The corresponding equations (9) have $\Delta_S = diag[\lambda I, \lambda I, \Delta_u, \Delta_P, \Delta_C]$,

$$A_S = \begin{bmatrix} A_H & 0 & B_{H1} & \frac{1}{\beta}B_{H2} & 0 \\ 0 & A_Q & 0 & \frac{1}{\beta}B_Q & 0 \\ C_{H1} & 0 & D_{H11} & \frac{1}{\beta}D_{H12} & 0 \\ C_{H2} & 0 & D_{H12} & \frac{1}{\beta}D_{H22} & 0 \\ 0 & C_Q & 0 & \frac{1}{\beta}D_Q & 0 \end{bmatrix} \tag{37}$$

$$C_S = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\beta}P & -I \end{bmatrix} \tag{38}$$

Let $\beta_{opt}$ be the infimum of the values of the parameter $\beta$ such that LMI (23) is feasible; this is a measure of the worst-case gain under uncertainty $\Delta_u$ and autocorrelation constraints. Asymptotically, as the number of constraints increases, the process converges *down* to a robust $\mathcal{H}_2$ performance measure, so that a finite number of constraints always gives an upper bound.

## 5.3 Examples

We will present two simple examples to demonstrate the machinery, applied to problems involving the $\mathcal{H}_2$ norm.

### 5.3.1 An example without uncertainty

The first example consists of calculating the $\mathcal{H}_2$ norm of the transfer function $H(\lambda) = \frac{1}{\lambda - 2}$ using this approach. There are of course exact ways to compute the $\mathcal{H}_2$ norm, which give a result of $1/\sqrt{3} = 0.577$; this example is included for verification purposes.

The process described above was performed with a number $T$ of autocorrelation constraints (for $\gamma = 0$). The feasibility of LMI (23) was checked using the software package LMI-Lab [9]. Figure 4 depicts $\beta_{opt}$ as a function of $T$. Starting at $T = 0$ with the unconstrained ($\mathcal{H}_\infty$) norm which is 1, $\beta_{opt}$ asymptotically converges to the $\mathcal{H}_2$ norm, as expected, at a rate consistent with bound (27).

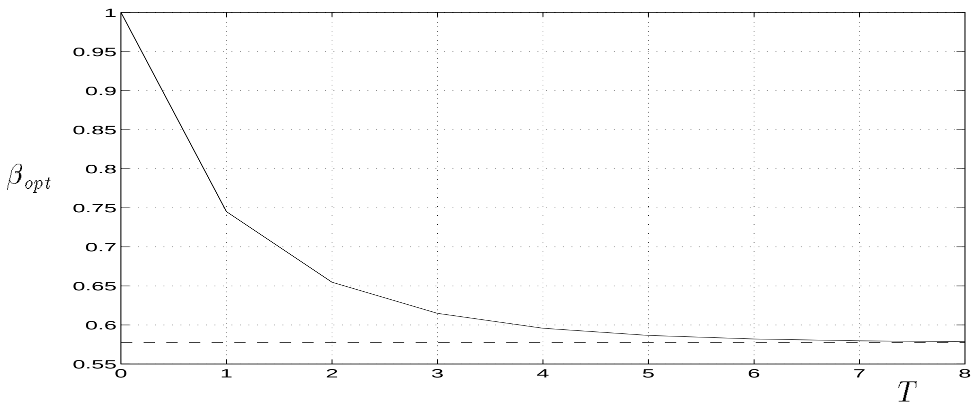


Figure 4: Approximation to the $\mathcal{H}_2$ norm

### 5.3.2 Robust $\mathcal{H}_2$ example

We consider the standard SISO feedback system of Figure 5, where the plant $P$ is subject to multiplicative uncertainty. We wish to analyze sensitivity of the tracking error $e$ to a *white noise* disturbance appearing in $d$ (which could be due, for example, to sensor noise). The map from $d$ to $e$ (sensitivity function) of the uncertain system is given by

$$S = \frac{1}{1 + PK(1 + W\Delta_u)} = \frac{S_0}{1 + WT_0\Delta_u} \tag{39}$$

where $S_0 = \frac{1}{1+PK}$ and $T_0 = \frac{PK}{1+PK}$ are the nominal sensitivity and complementary sensitivity functions.

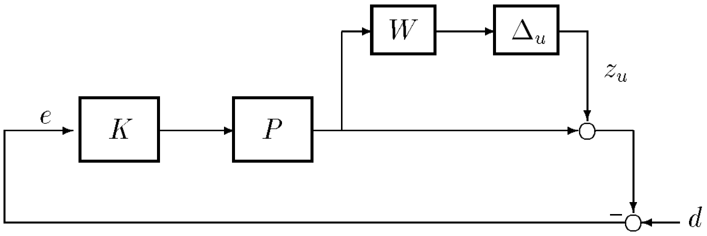


Figure 5: Rejection of sensor noise

**LTI uncertainty**:

If the perturbation $\Delta_u$ is assumed be LTI, for this simple case the worst-case $\mathcal{H}_2$ norm can be computed in the frequency domain; this will allow us to evaluate the results obtained from the analysis framework. Assuming that $\|WT_0\|_\infty < 1$, we have

$$\max_{|\Delta_u(e^{j\omega})| \leq 1} |S(e^{j\omega})| = \frac{S_0(e^{j\omega})}{1 - |WT_0(e^{j\omega})|} \tag{40}$$

This fact has been typically used (see [6]) to show that the worst case $\mathcal{H}_\infty$ norm of the system is given by $\left\| \frac{S_0}{1-|WT_0|} \right\|_\infty$. Here we will use it to obtain a worst case $\mathcal{H}_2$ norm of $\left\| \frac{S_0}{1-|WT_0|} \right\|_2$; for this we allow $\Delta_u$ to be a noncausal ($\mathcal{L}_\infty$) operator, which can achieve bound (40) for every frequency.

We now choose $K = 2$, $P = \frac{1}{1-3.3\lambda+\lambda^2}$, and $W = 0.25$. These values were chosen so that the uncertainty affects the sensitivity in a significant way; this is exhibited in Figure 6, where the lower curve indicates the nominal sensitivity function, and the upper curve the worst-case sensitivity from (40). We obtain the values

$$\left\| \frac{S_0}{1 - |WT_0|} \right\|_\infty = 6.50 \qquad (41)$$

$$\left\| \frac{S_0}{1 - |WT_0|} \right\|_2 = 2.39 \qquad (42)$$

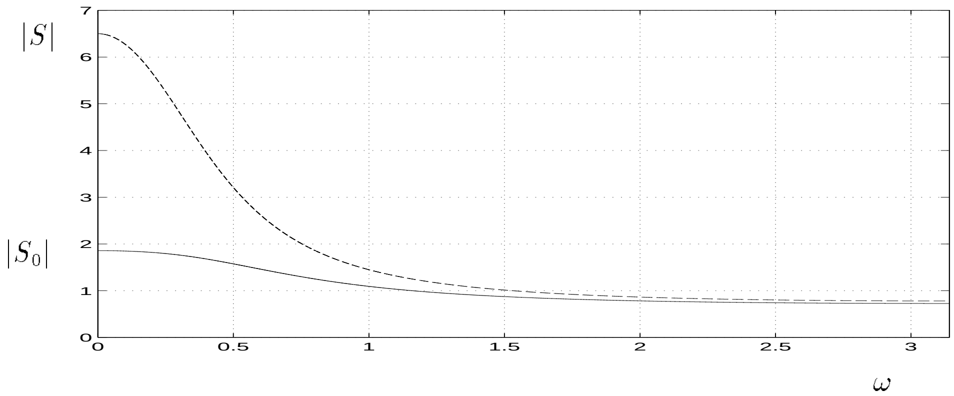

Figure 6: Nominal and worst-case sensitivity functions (magnitude)

**LTV uncertainty**

Exact analysis for $\Delta_u$ an arbitrary LTV operator can be obtained from the procedure described in Section 5.2. Figure (7) shows the corresponding plot of $\beta_{opt}$ (obtained using LMI-Lab) as a function of the number of correlation constraints $T$.

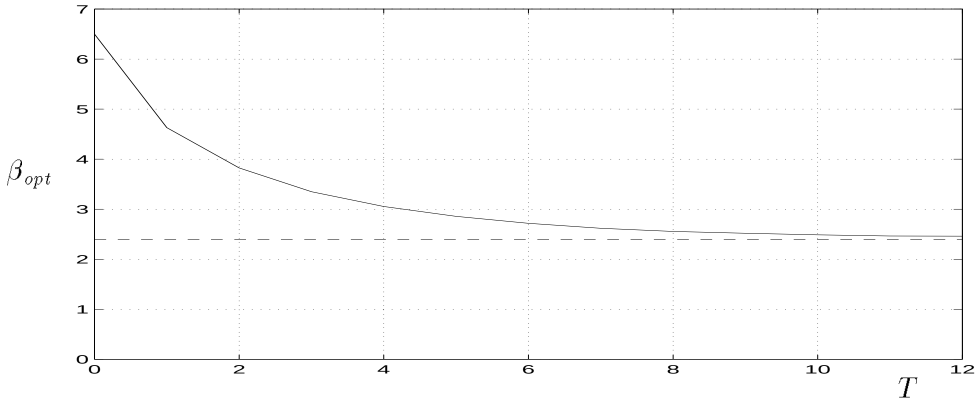

Figure 7: Induced norm of uncertain system

For $T = 0$ (no constraints) we retrieve the value from (41) for the worst case $\mathcal{H}_\infty$ norm (it is well known that in this unstructured case, the worst-case perturbation is LTI). As $T$ increases, we approach the worst-case gain under white noise signals. The asymptotic value also appears to coincide in this case, with the value (42) obtained from LTI uncertainty, which is plotted for comparison [2].

Although for this case the frequency domain method is much simpler, it does not generalize to multivariable systems or to structured uncertainty. The procedure based on the implicit framework applies in principle to any case, although for this method to be practical in large problems, improvements in the efficiency of LMI solvers are required.

## 6 Conclusions

Implicit representations have been shown to be an attractive general framework where various forms of system uncertainty, performance requirements and signal constraints can be expressed. A robustness analysis theory has been developed which includes what is available for the standard input/output setting, and enhances its domain of applications to include overconstrained problems, exemplified in this paper by robust $\mathcal{H}_2$ performance analysis.

The computational properties of this extension are similar to those of the standard case. Conditions obtained in terms of LMIs lead in principle to tractable computation, but the size of the problems is also a concern. In this respect, tests with LTV uncertainty yield either coupled LMIs across frequency such as (24), or a large full block of multipliers for the state space version (23). Additional research is required on practical methods for these large problems, and also for the case of mixed LTI/LTV uncertainty, which arises naturally in this setting (for example, when $\Delta_u$ in (36) is LTI). Here the coupling is not easily avoided; some initial work on this problem is reported in [20].

One of the main reasons to adopt the LFT framework is that it allows for the consideration of highly structured (e.g. real parametric) uncertainty which is not captured by IQCs. For

[2] This is not a general fact; other examples (see [18]) exhibit a gap between LTI and LTV uncertainty.

these cases the exact analysis conditions in terms of $\mu$ will have increased computational complexity, as in the standard case where they are known to be NP hard [3]. Although this implies unacceptable computation time in the worst case, $\mu$ lower bounds [14, 27] have proven to be efficient on "typical" problems. Their extension to the implicit framework is a direction of future research.

If successful, this extension can have an enormous impact in problems involving data, such as the model validation problem mentioned in Section 2.3, and the corresponding system identification problem (see [7]). The implicit LFT framework would then appear as the natural setting for unifying modeling, analysis, model validation and system identification, under a common set of mathematical and computational tools.

Another natural extension of the results in this paper is the question of synthesis of controllers for robust performance in this setting. In [4], it is shown how standard "D-K iteration" methods for $\mu$-synthesis extend to this formulation, and allow in particular for design of controllers for Robust $\mathcal{H}_2$ performance.

## Acknowledgements

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## Appendix: Proofs

### Propositions 1 and 2

The proof follows easily from Proposition 5 in Part I. We sketch the argument for the case of Proposition 1, the other one is analogous. For any finite dimensional LTI $\Delta$, $\Delta(e^{j\omega}) \in \mathbf{B}_{\boldsymbol{\Delta}0}$ for every $\omega$; if condition (5) holds,

$$\begin{bmatrix} I - \Delta(e^{j\omega})A(e^{j\omega}) \\ C(e^{j\omega}) \end{bmatrix} \tag{43}$$

is full column rank for all $\omega \in [-\pi, \pi]$, therefore $l_2$ stability follows from Prop. 5, Part I. Conversely if (5) fails at some $\omega_0$, $\Delta_0$, it is easy to construct an LTI perturbation $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ such that $\Delta(e^{j\omega_0}) = \Delta_0$, which violates $l_2$ stability from Prop. 5, Part I.

## Lemma 3

Since $CC_\perp^* = 0$, $(i)$ implies $(ii)$. If $(ii)$ holds, there exists $X \in \mathbf{X}$ such that $A^*XA - X < 0$ on the kernel of $C$. By continuity, there exists $\epsilon > 0$ such that $\langle (A^*XA - X)v, v\rangle < 0$ for all $v$, $\|v\| = 1$, $\|Cv\|^2 \leq \epsilon$.

Now choose $\eta > 0$ such that $\lambda_{max}(A^*\eta XA - \eta X) < \epsilon$.

This gives $\langle (A^*\eta XA - \eta X - C^*C)v, v\rangle < 0$ for all $v \neq 0$, so $\eta X$ solves $(i)$.

## Theorem 4

1. Assume $\mu_{\mathbf{\Delta}_0}(C, A) \geq 1$. Then there exists $\Delta_0 \in \mathbf{B}_{\mathbf{\Delta}_0}$ such that (3) has a nontrivial kernel; let $\zeta$ of norm 1 be in the kernel. Then $C\zeta = 0$, $\Delta_0 A\zeta = \zeta$; focusing on the blocks of $\Delta_0$, we obtain

$$\begin{array}{lclcll} \delta_i I(A\zeta)_i = \zeta_i & \Rightarrow & (A\zeta)_i(A\zeta)_i^* \geq \zeta_i\zeta_i^* & \Rightarrow & \Phi_i^0(\zeta) \geq 0 & i = 1 \ldots L \\ \Delta_{L+j}(A\zeta)_{L+j} = \zeta_{L+j} & \Rightarrow & \|(A\zeta)_{L+j}\| \geq \|\zeta_{L+j}\| & \Rightarrow & \sigma_{L+j}^0(\zeta) \geq 0 & j = 1 \ldots F \end{array} \quad (44)$$

   Therefore the matrix $\Lambda^0(\zeta)$ is in $\nabla^0 \cap \bar{\mathbf{X}}$. The converse follows similarly.

2. The equivalence of $(i)$ and $(ii)$ is obvious from the definition of $\hat{\mu}_{\mathbf{\Delta}0}(C, A)$.

   Let $X > 0$ solve $(ii)$. For any $\zeta \in \mathbb{C}^n$, $C\zeta = 0$, some algebra shows that

$$\langle X, \Lambda^0(\zeta)\rangle = \sum_{i=1}^{L} \mathrm{tr}(X_i\Phi_i^0(\zeta)) + \sum_{j=1}^{F} x_{L+j}\sigma_{L+j}^0(\zeta) = \zeta^*(A^*XA - X - C^*C)\zeta < 0 \quad (45)$$

   Also, $\langle X, Y\rangle \geq 0$ for all $Y \in \bar{\mathbf{X}}$. Therefore the hyperplane $\langle X, Y\rangle = 0$ in $\mathbf{Y}$ separates the sets $\nabla^0$ and $\bar{\mathbf{X}}$, which implies their respective convex hulls $co(\nabla^0)$ and $\bar{\mathbf{X}}$ are disjoint, proving $(iii)$. Conversely, if $co(\nabla^0)$, $\bar{\mathbf{X}}$ are disjoint, a separating hyperplane can be found leading back to $(ii)$.

## A Lemma from Convex Analysis

**Lemma 10** *Let $K \subset V$, where $V$ is a $d$ dimensional real vector space. Every point in $co(K)$ is a convex combination of at most $d+1$ points in $K$; for $K$ compact, every point in the boundary of $co(K)$ is a convex combination of at most $d$ points in $K$.*

The first statement is a classical result from Caratheodory (see [21]), which implies that for every $v \in co(K)$, there exists a *simplex* of the form

$$S(v_1, \ldots, v_{d+1}) = \left\{ \sum_{k=1}^{d+1} \alpha_k v_k : \alpha_k \geq 0, \sum_{k=1}^{d+1} \alpha_k = 1 \right\}$$

with vertices $v_k \in K$, which contains $v$. If the $v_k$ are in a lower dimensional hyperplane, then $d$ points will suffice. If not, then every point in $S(v_1, \ldots, v_{d+1})$ corresponding to $\alpha_k > 0 \ \forall k$ will be *interior* to $S(v_1, \ldots, v_{d+1}) \subset co(K)$. Therefore for points $v$ in the *boundary* of $co(K)$, one of the $\alpha_k$'s must be 0 and a convex combination of $d$ points will suffice.

## Theorem 5

(**i**) In the case $\Delta = \delta I$, if $\mu_{\mathbf{\Delta}}(C, A) < 1$ then $(C, A)$ is detectable in the usual system theoretic sense, so there exists an output injection $L$ such that $\rho(A + LC) < 1$. From Lyapunov theory this implies there exists $X > 0$ such that

$$(A + LC)^* X (A + LC) - X < 0 \tag{46}$$

Multiplying on the left and right by $C_\perp$, $C_\perp^*$ gives $C_\perp(A^* X A - X)C_\perp^* < 0$ which implies $\hat{\mu}_{\mathbf{\Delta}}(C, A) < 1$.

(**ii**) The only nontrivial case is $F = 2$. Let $A$, $C$, $\Delta = diag[\Delta_1, \Delta_2]$ be *real* matrices. To analyze this case we must consider a real version of the $\nabla^0$ set, of the same form as (18) but with $\zeta \in \mathbb{R}^n$. Consider the $n \times r$ matrix $\Gamma = C_\perp^*$ parameterizing the kernel of $C$ ($\zeta = \Gamma v$), and assume it is isometric. Then $\nabla^0$ can be rewritten as

$$\nabla^0 = \{\Lambda(v) = [\sigma_1(v), \sigma_2(v)], v \in \mathbb{R}^r, \|v\| = 1\} \tag{47}$$

where $\sigma_j(v) = v' H_j v$, and $H_j = (A\Gamma)'_j (A\Gamma)_j - (\Gamma)_j (\Gamma)_j$ is a real, symmetric matrix for $j = 1, 2$. To prove that this structure is $\mu$-simple is equivalent, by Theorem 4 to the fact that

$$\nabla^0 \cap \bar{\mathbf{X}} = \emptyset \Longrightarrow co(\nabla^0) \cap \bar{\mathbf{X}} = \emptyset \tag{48}$$

for any $A$, $C$, or equivalently for any symmetric $H_1$, $H_2$. In this case $\bar{\mathbf{X}} = (\mathbb{R}^+)^2$, the closed first quadrant in $\mathbb{R}^2$. We have therefore restated the problem as a geometric condition on the range of two real quadratic forms. In this notation, (48) is equivalent to an "S-procedure losslessness" theorem from Yakubovich [25]; since this literature is not easily accessed we include a proof which is based on some modifications to the parallel results of [5].

Let $P = \Lambda(v_P)$, $Q = \Lambda(v_Q)$, be two distinct points in $\nabla^0$ ($v_P$,$v_Q \in \mathbb{R}^r$, $\|v_P\| = \|v_Q\| = 1$). Define

$$M = \begin{bmatrix} v'_P \\ v'_Q \end{bmatrix} \begin{bmatrix} v_P & v_Q \end{bmatrix} > 0 \tag{49}$$

$$\hat{H}_i = M^{-\frac{1}{2}} \begin{bmatrix} v'_P \\ v'_Q \end{bmatrix} H_i \begin{bmatrix} v_P & v_Q \end{bmatrix} M^{-\frac{1}{2}}, \quad i = 1, 2 \tag{50}$$

$$E = \left\{ [\ \eta' \hat{H}_1 \eta\ ,\ \eta' \hat{H}_2 \eta\ ], \eta \in \mathbb{R}^2, \|\eta\| = 1 \right\} \tag{51}$$

Then

- $E \subset \nabla^0$. This follows from the fact that if $\|\eta\| = 1$, $\left\| \begin{bmatrix} v_P & v_Q \end{bmatrix} M^{-\frac{1}{2}} \eta \right\| = 1$ from (49).

- $P, Q \in E$. For $P$ set $\eta_P = M^{\frac{1}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which verifies $\|\eta_P\|^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \|v_P\|^2 = 1$, and analogously for $Q$.

- $E$ is an ellipse in $\mathbb{R}^2$ (which may degenerate to a segment).

  Parameterize $\eta = (cos(\theta), sin(\theta))$, $\theta \in [-\pi, \pi]$. If $\hat{H}_j = \begin{bmatrix} a_j & b_j \\ b_j & c_j \end{bmatrix}$, then

  $$\eta' \hat{H}_j \eta = \frac{a_j + c_j}{2} + \begin{bmatrix} \frac{a_j - c_j}{2} & b_j \end{bmatrix} \begin{bmatrix} cos(2\theta) \\ sin(2\theta) \end{bmatrix} \quad j = 1, 2 \tag{52}$$

  This implies that $E$ is the image of the unit circle by an affine map, an ellipse.

We have shown that given two points in $\nabla^0$, there exists an ellipse $E \subset \nabla^0$ through those points. Now we return to (48). If $co(\nabla^0) \cap (\mathbb{R}^+)^2 \neq \emptyset$, since $\nabla^0$ is bounded and $(\mathbb{R}^+)^2$ is a cone, there exists a point in the *boundary* of $co(\nabla^0)$ which falls in the first quadrant. Using Lemma 10 there exist two points $P$,$Q$ in $\nabla^0$ such that the segment $PQ$ intersects the first quadrant. But then the corresponding ellipse $E$ will intersect $(\mathbb{R}^+)^2$; (the geometric picture is given in Figure 8(ii)). This implies $\nabla^0 \cap (\mathbb{R}^+)^2 \neq \emptyset$.

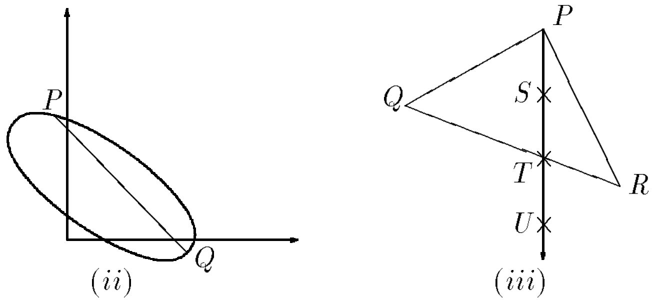


Figure 8: Illustration to the proofs

**(iii)**. We consider the case $F = 3$, the others follow similarly. The same procedure as in $(ii)$ yields

$$\nabla^0 = \{\Lambda(v) = [\sigma_1(v), \sigma_2(v), \sigma_3(v)] \in \mathbb{R}^3, v \in \mathbb{C}^r, \|v\| = 1\} \tag{53}$$

where $\sigma_j(v) = v^* H_j v$, and $H_j$ are complex hermitian forms in $\mathbb{C}^r$. Similarly, we must show the geometric result

$$\nabla^0 \cap (\mathbb{R}^+)^3 = \emptyset \Longrightarrow co(\nabla^0) \cap (\mathbb{R}^+)^3 = \emptyset \tag{54}$$

for any $H_1$, $H_2$, $H_3$. Once again, this result appears in the "S-procedure" formulation [8]. The following proof is based on [5]. In particular, it is shown in [5] (analogously to (52)) that for the case $r = 2$, the set $\nabla^0$ is the image of the unit sphere in $\mathbb{R}^3$ by an affine map $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. This gives an ellipsoid $E$ (with no interior) in $\mathbb{R}^3$, which could also degenerate to a projection of such an ellipsoid in a lower dimensional subspace.

Given two distinct points $P = \Lambda(v_P)$, $Q = \Lambda(v_Q)$ in $\nabla^0$, an analogous construction as the one given in (49-51) (with analogous proof) shows that there is such an ellipsoid $E \subset \nabla^0$

through the two points.

Assume now that $co(\nabla^0) \cap (\mathbb{R}^+)^3 \neq \emptyset$. Picking a point in the the boundary of $co(\nabla^0)$, Lemma 10 implies that there are 3 points $P$,$Q$,$R$ in $\nabla^0$ such that some convex combination $S = \alpha P + \beta Q + \gamma R$ falls in $(\mathbb{R}^+)^3$. Geometrically, the triangle $PQR$ intersects the positive "octant" at $S$.

**Claim:** $S$ lies in a segment between 2 points in $\nabla^0$.

This is obvious if $P$,$Q$, $R$ are aligned or if any of $\alpha$,$\beta$, $\gamma$ is 0. If not, consider the following reasoning, illustrated in Figure 8. Write

$$S = \alpha P + \beta Q + \gamma R = \alpha P + (\beta + \gamma)\frac{1}{\beta + \gamma}(\beta Q + \gamma R) = \alpha P + (\beta + \gamma)T \tag{55}$$

where $T$ lies in the segment $QR$. Now consider the ellipsoid $E \subset \nabla^0$ through $Q$ and $R$. If it degenerates to 1 or 2 dimensions, then $T \in E \subset \nabla^0$ and the claim is proved. If not, $T$ is interior to the ellipsoid $E$. The half line starting at $P$, through $T$ must "exit" the ellipsoid a point $U \in E \subset \nabla^0$ such that $T$ is in the segment $PU$. Therefore $S$ in the segment $PU$, and $P, U \in \nabla^0$, proving the claim.

To finish the proof, we have found two points in $\nabla^0$ such that the segment between them intersects $(\mathbb{R}^+)^3$ . The corresponding ellipsoid $E \subset \nabla^0$ between these points must clearly also intersect $(\mathbb{R}^+)^3$. Therefore $\nabla^0 \cap (\mathbb{R}^+)^3 \neq \emptyset$.

## Theorem 6

Sufficiency of condition (19) is a consequence of the more general result given in Part I; the same argument as exhibited in the proof of Theorem 1 applies to this finite horizon setup.

For the necessity, assume (19) does not hold. Therefore $co(\nabla^0) \cap \bar{\mathbf{X}} \neq \emptyset$, and we can choose a point in the boundary of $co(\nabla^0)$, which belongs to $\bar{\mathbf{X}}$. Since $\nabla^0$ is in a $d$ dimensional real vector space $\mathbf{Y}$, Lemma 10 implies that there exists $d$ points in $\nabla^0$ whose convex combination

$$\sum_{k=1}^{d} \alpha_k \Lambda(\zeta_k) = X_0 \geq 0, \tag{56}$$

with $\zeta_k \in \mathbb{C}^n$, $C\zeta_k = 0$, $\alpha_k \geq 0$, and $\sum_k \alpha_k = 1$. Define $z = \{z_k\}_{k=1}^d$, $z_k = \sqrt{\alpha_k}\zeta_k$. Since $A$, $C$ are static, we find that $Cz = 0$, and

$$\sum_{k=1}^{d}(Az_k)_i(Az_k)_i^* - (z_k)_i(z_k)_i^* \geq 0 \qquad i = 1 \ldots L \tag{57}$$

$$\|(Az)_{L+j}\|^2 \geq \|(z)_{L+j}\|^2 \qquad j = 1 \ldots F \tag{58}$$

Now we apply Lemmas 1, 2 from Part I to conclude there exists $\Delta \in \mathbf{B_\Delta}$, structured operator in $(\mathbb{C}^n)^d$, such that $\Delta A z = z$. Therefore $\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} z = 0$ violating the hypothesis.

## Propositions 7 and 8

Proposition 7 follows by the same arguments as Proposition 2.

Proposition 8: Since $C_S$ is static, Theorem 12, Part I is in force and LMI (23) implies that $\begin{bmatrix} I - \Delta_S A \\ C \end{bmatrix}$ has a left inverse in $\mathcal{L}_C(l_{2e})$ for any structured $\Delta_S \in \mathcal{L}_C(l_{2e})$, $\|\Delta_S\| \leq 1$, thus in particular for $\Delta_S = diag[\lambda I, \Delta]$, with $\Delta \in \mathcal{L}_C(l_{2e})$, $\|\Delta\| \leq 1$.

## Theorem 9

$(i) \Rightarrow (ii)$ This is a special case of Proposition 8.

$(ii) \Rightarrow (iii)$ From $(ii)$, $\begin{bmatrix} I - \Delta_S A_S \\ C_S \end{bmatrix}$ has a left inverse in $\mathcal{L}_C(l_{2e})$ (causal, finite gain) for every $\Delta_S = diag[\lambda I, \Delta]$, $\Delta \in \mathbf{B_\Delta}$. Setting $\Delta_S = diag[\lambda I, 0]$ implies that $(I - \lambda A_x)$ has a left inverse in $\mathcal{L}_C(l_{2e})$, so $\rho(A_x) < 1$. Now for any fixed $\Delta_S$, the identity

$$\begin{bmatrix} I - \Delta_S A_S \\ C_S \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -\Delta A_{zx}(I - \lambda A_x)^{-1} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I - \lambda A_x & 0 \\ 0 & I - \Delta A(\lambda) \\ 0 & C \end{bmatrix} \begin{bmatrix} I & -(I - \lambda A_x)^{-1}\lambda A_{xz} \\ 0 & I \end{bmatrix} \tag{59}$$

implies that the second term in the right hand side of (59) has a left inverse in $\mathcal{L}_C(l_{2e})$. So $\begin{bmatrix} I - \Delta A \\ C \end{bmatrix}$ has a left inverse in $\mathcal{L}_C(l_{2e})$.

$(iii) \Rightarrow (iv)$ This is a direct application of Theorem 11, Part I.

$(iv) \Rightarrow (i)$ Let the columns of $C_\perp^*$ form a basis for the kernel of $C$; $(iv)$ leads to

$$C_\perp A(e^{j\omega})^* X A(e^{j\omega}) C_\perp^* < C_\perp X C_\perp^* \quad \forall \omega \in [-\pi, \pi] \tag{60}$$

Since $C_\perp X C_\perp^* > 0$, pre and post multiplication by $(C_\perp X C_\perp^*)^{-\frac{1}{2}}$ gives $\|\bar{A}(\lambda)\|_\infty < 1$, where $\bar{A}(\lambda) = X^{\frac{1}{2}} A C_\perp^* (C_\perp X C_\perp^*)^{-\frac{1}{2}}$, as remarked in (25). We will now change notation, redefining $C_\perp^*(C_\perp X C_\perp^*)^{-\frac{1}{2}}$ as $C_\perp^*$, since it still spans the same column space. So $C_\perp X C_\perp^* = I$, and $\bar{A}(\lambda) = X^{\frac{1}{2}} A C_\perp^*$, which has state space realization

$$\left[\begin{array}{c|c} \bar{A}_x & \bar{A}_{xz} \\ \hline \bar{A}_{zx} & \bar{A}_z \end{array}\right] = \left[\begin{array}{c|c} A_x & A_{xz} C_\perp^* \\ \hline X^{\frac{1}{2}} A_{zx} & X^{\frac{1}{2}} A_z C_\perp^* \end{array}\right] \tag{61}$$

It is well known that $\rho(\bar{A}_x) < 1$, $\|\bar{A}(\lambda)\|_\infty < 1$ implies the existence of a solution $X_0 > 0$ to

$$\begin{bmatrix} \bar{A}_x & \bar{A}_{xz} \\ \bar{A}_{zx} & \bar{A}_z \end{bmatrix}^* \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_x & \bar{A}_{xz} \\ \bar{A}_{zx} & \bar{A}_z \end{bmatrix} - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < 0 \tag{62}$$

Substituting the expressions for $\bar{A}_x, \bar{A}_{xz}, \bar{A}_{zx}, \bar{A}_z$ , and using $C_\perp X C_\perp^* = I$, (62) leads to

$$\begin{bmatrix} I & 0 \\ 0 & C_\perp \end{bmatrix}^* \left( A_S^* \begin{bmatrix} X_0 & 0 \\ 0 & X \end{bmatrix} A_S - \begin{bmatrix} X_0 & 0 \\ 0 & X \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & C_\perp^* \end{bmatrix} < 0$$

Since $C_S = [0 \;\; C]$, $C_{S\perp} = \begin{bmatrix} I & 0 \\ 0 & C_\perp \end{bmatrix}$, so setting $X_S = \mathrm{diag}[X_0 \; X]$ gives

$$C_{S\perp}(A_S^* X_S A_S - X_S) C_{S\perp}^* < 0 \tag{63}$$

which implies $(i)$ from Lemma 3.

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