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"Exponential Stabilization of Driftless Nonlinear Control Systems"

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Exponential Stabilization of Driftless Nonlinear Control Systems

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Abstract

This dissertation lays the foundation for practical exponential stabilization of driftless control systems. Driftless systems have the form,

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m, \quad x \in \mathbb{R}^n.$$

Such systems arise when modeling mechanical systems with nonholonomic constraints. In engineering applications it is often required to maintain the mechanical system around a desired configuration. This task is treated as a stabilization problem where the desired configuration is made an asymptotically stable equilibrium point. The control design is carried out on an approximate system. The approximation process yields a nilpotent set of input vector fields which, in a special coordinate system, are homogeneous with respect to a non-standard dilation. Even though the approximation can be given a coordinate-free interpretation, the homogeneous structure is useful to exploit: the feedbacks are required to be homogeneous functions and thus preserve the homogeneous structure in the closed-loop system. The stability achieved is called *ρ-exponential stability.* The closed-loop system is stable and the equilibrium point is exponentially attractive. This extended notion of exponential stability is required since the feedback, and hence the closed-loop system, is not Lipschitz. However, it is shown that the convergence rate of a Lipschitz closed-loop driftless system cannot be bounded by an exponential envelope.

The synthesis methods generate feedbacks which are smooth on $\mathbb{R}^n \setminus \{0\}$. The solutions of the closed-loop system are proven to be unique in this case. In addition, the control inputs for many driftless systems are velocities. For this class of systems it is more appropriate for the control law to specify actuator forces instead of velocities. We have extended the kinematic velocity controllers to controllers which command forces and still ρ-exponentially stabilize the system.

Perhaps the ultimate justification of the methods proposed in this thesis are the experimental results. The experiments demonstrate the superior convergence performance of the ρ-exponential stabilizers versus traditional smooth feedbacks. The experiments also highlight the importance of transformation conditioning in the feedbacks. Other design issues, such as scaling the measured states to eliminate hunting, are discussed. The methods in this thesis bring the practical control of strongly nonlinear systems one step closer.

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List of Symbols

Symbol	: Description and page when applicable
$\mathrm{ad}_X^j Y$	: Recursive Lie bracket on X
co	: Convex hull
Δ_λ	: Dilation with parameter λ, 8
D^+	: Upper right Dini derivative
X_E	: Euler vector field, 12
ψ	: Flow of differential equation, 8
∂F	: Generalized Jacobian of map F, 53
S_Δ	: Homogeneous sphere, 12
$[\cdot, \cdot]$	: Lie bracket of vector fields
$L_X f$	: Lie derivative of function f with respect to the vector field X
$\lVert \cdot \rVert_2$	: Euclidean norm
f_*	: Tangent space map of function
ρ	: Homogeneous norm, 9

Chapter 1

Introduction

This thesis studies the problem of locally exponentially stabilizing analytic driftless control systems. Driftless systems have the form

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m \quad x \in \mathbb{R}^n, \tag{1.1}$$

where the control inputs u_i are real valued and the X_i are analytic "input" vector fields. A diverse set of mechanical systems may be modeled as driftless control systems. The special form of the model is often the result of *nonholonomic* constraints that the kinematic variables of the system must satisfy. A mobile robot with wheels that roll without slipping is an example of a system with nonholonomic constraints [24], [32]. Dextrous manipulation with multi-fingered robotic hands is another application where driftless control systems arise from nonholonomic constraints [27], [26], [35]. Reorientation of rigid bodies with zero angular momentum through internal motion may be studied as a driftless control system. In this case, the angular momentum constraint enforces a nonholonomic-like constraint on the system model [25], [21], [12]. Finally, nonholonomic actuators are studied in [5].

A problem of practical interest is how to transfer the system to some desired final state. The change in state may be affected by two different approaches. The first approach is an open-loop strategy. This involves defining the control inputs as functions of time so that the initial state of the model (1.1) is transferred to the desired final state. Initial efforts to control driftless systems were directed at open-loop path planning. The literature in this area is large. A recent paper containing a comprehensive reference list is [32]. However, the limitations of an open-loop methodology restrict its use in physical systems: without feedback the system performance is degraded by modeling errors and external disturbances. In other words, small errors in the initial state measurement or the model resulted in poor performance (the performance being measured by the deviation from the desired final condition for the physical system). The second strategy then, is to feed back the state of system. Thus, feedback stabilization is a process in which the desired final state is made an asymptotically stable equilibrium point by proper choice of the control inputs. The feedback should impart some measure of robustness to the modeling errors and measurement errors noted above. This thesis concentrates solely on the feedback problem and assumes that the controller has access to the measured state in real-time. A well known result by Brockett [4] implies that driftless systems cannot be asymptotically stabilized about any desired point with continuous autonomous feedback. Appendix A contains a precise statement of Brockett's theorem.

Several research groups have derived discontinuous feedbacks. Brockett's condition does not apply when the closed-loop system is not continuous. Bloch et al. [3] derive piecewise analytic feedbacks to stabilize Chaplygin systems. Their controllers have the advantage of returning the system to the desired state in finite time. However, the control action is of bang-bang type. Since the control inputs are velocities for many driftless systems, such a control is not physically realizable. Canudas de Wit and Sørdalen develop piecewise smooth controllers for a set of low dimensional physical examples [9]. However in several of their examples the desired equilibrium point is not stable even though it is attractive.

The primary advantage of continuous control laws is the fact that problems of chattering and infinitely fast switching are not an issue. Samson demonstrated that continuous *time-periodic* feedbacks could stabilize a nonholonomic cart [40]. This result motivated much research into continuous time-varying feedbacks for stabilizing general driftless systems. Coron showed that for a large class of driftless systems there exists a smooth time-periodic feedback that renders the desired equilibrium point globally asymptotically stable [7]. Coron's result is an existence result and does not provide a constructive procedure for obtaining the feedback. Pomet was able to adapt the ideas in Coron's proof to provide an algorithm for deriving time-periodic smooth feedbacks for a more restrictive class of driftless systems [37]. Teel et al. [44] gave explicit expressions for time-peiodic smooth control laws which asymptotically stabilized the special "chained-form" driftless systems.

While these algorithms are useful for understanding the structure of driftless systems, the rates of convergence cannot be bounded by an exponential envelope. The authors in [33] showed that slower than exponential rates are always obtained with C^1 feedbacks. This thesis extends this result to include all Lipschitz feedbacks. Improvements in the convergence rates are desirable in order to make the algorithms more practical and applicable to real world applications. Sørdalen [38] and Canudas de Wit and Sørdalen [10] consider the problem of exponential stabilization with a slightly modified notion of exponential stability. Their methods rely on piecewise analytic feedbacks and are not continuous functions of the state. Another existence result by Coron [8] states that controllable driftless systems may be stabilized to the origin in finite time by a continuous time-periodic feedback which is smooth on $\mathbb{R}^n \setminus \{0\}$. Coron's work is germane to the results in this dissertation and are reviewed more thoroughly in Appendix A.

The work of Hermes is perhaps closest in spirit to the approach presented in this thesis. Hermes' paper [17] relies on homogeneous approximations of the control system and generalizes the notion of the linear regulator to a homogeneous nonlinear regulator for the approximate system. The systems he considers are two-dimensional small-time locally controllable systems and certain three-dimensional systems. This class of two-dimensional systems automatically satisfy Brockett's condition as does the three-dimensional example. Although Brockett's condition fails for driftless systems the homogeneous approximations still play a very important role.

This thesis is concerned with the the exponential stabilization of driftless analytic control system with time-periodic continuous feedback. The contributions are:

i) **Explicit construction of ρ-exponentially stabilizing feedbacks.** Two methods are presented for deriving exponentially stabilizing feedbacks for a large class of driftless systems. One method is an extension of Pomet's algorithm to the framework presented in this thesis. The other method specifies sufficient conditions for a smooth stabilizer to be rescaled into an exponential stabilizer. The latter method is attractive from an implementation point of view since it requires only slightly more computation than the smooth control law from which it was derived.

ii) **Proof that non-Lipschitz feedback is necessary for exponential stabilization.** The stabilizing feedbacks are degree one homogeneous functions which are not Lipschitz since the dilation is a nonstandard one. The non-Lipschitz character of the feedbacks is shown to be a necessary feature of the control law if exponential stability of the driftless system is desired.

iii) **Analysis results for homogeneous differential equations.** Several analysis results are also proven for homogeneous systems. For example, the feedbacks derived from the synthesis methods result in unique solutions of the closed-loop system. This fact is not automatic since the closed-loop vector field is not Lipschitz. In addition, an averaging theorem for degree zero homogeneous systems is proven. This extends the usual stability results for C^2 systems to degree zero vector fields.

iv) **Extension of kinematic controllers to allow torque inputs.** The control outputs are often velocities in driftless models. It is shown that servo motors may be used to command torques instead of velocities for these systems while maintaining exponential rates of convergence. Furthermore, the sensitivity of the control signal to sensor noise is exacerbated by the non-Lipschitz nature of the feedbacks. Low pass filtering of the state variables may be used to smooth the input into the controller. The effect of inserting a low pass filter into the loop is quantified with a singular perturbation result for homogeneous degree zero systems.

v) **Experimental verification.** The theory is experimentally tested on a mobile robot. Comparisons are made with controllers derived by other means. The superiority of the exponential stabilizers is clearly demonstrated.

The thesis is organized as follows. Chapter 2 introduces the background necessary to understand the results in the thesis. In particular the definitions and properties of homogeneous functions and homogeneous vector fields are reviewed. These concepts are central to understanding how the feedbacks exponentially stabilize the system. Every set of controllable vector fields may be locally approximated by a controllable nilpotent set of vector fields. Furthermore, in special local coordinates the approximating vector fields are homogeneous with respect to a dilation associated with the growth of the Lie algebra of the vector fields. This approximation theory is also central to the exponential stabilization problem and is briefly covered. Chapter 2 ends with a review of converse Lyapunov results for homogeneous systems. Homogeneous degree zero systems are of particular interest and it is shown how the current converse results which exist for autonomous homogeneous systems extend only to the time-periodic homogeneous degree zero case, a counterexample being given for situation when the degree of the vector field is different from zero. The converse theorems imply a simple stability result for perturbed systems.

Chapter 3 opens with a proof of uniqueness of solutions of homogeneous degree zero vector fields which are locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ (the origin is assumed to be an equilibrium point). The requirement that closed-loop solutions of the control system be unique is of practical importance: numerical simulations are often the only way to assess the performance of a nonlinear system and uniqueness of solutions guarantees continuity of the flow with respect to the initial condition. Finally, an averaging theorem for homogeneous degree zero systems is proven. The averaging result is not required for the subsequent analysis however it is of interest in its own right since it extends the stability results of C^2 dynamical systems to a class of non-Lipschitz vector fields.

In Chapter 4 the formalism of Chapter 2 and the analysis results of Chapter 3 are combined to obtain time-periodic, continuous, exponentially stabilizing feedbacks for a large class of analytic driftless systems. The non-Lipschitz property of the feedbacks is shown to be a necessary ingredient for the exponential stabilization of driftless systems. Moreover the closed-loop system solutions are unique. The sensitivity of the closed loop system in the vicinity of the origin to sensor noise is mitigated by filtering the measured state variables. The singular perturbation results are used to demonstrate that exponential stability is still maintained after the introduction of low-pass filtering.

The control inputs of driftless models often correspond to velocities in the physical system. It is unreasonable to insist that the velocities may be specified exactly since the motion in a mechanical system is realized by application of forces and torques. With this in mind, the "kinematic" velocity controllers are extended to torque controllers for the system augmented with a set of integrators to model the actuator dynamics.

Chapter 5 presents the results obtained with the *nonholomobile*, an experimental mobile robot constructed at Caltech. The objective of the experiments is to compare the performance of the exponential stabilizers derived using the theory in this dissertation to the more traditional smooth feedbacks proposed by other researchers. The experiments also verify that torques may still be commanded with a non-Lipschitz velocity controller and that the driftless kinematic models for robot systems are suitable for control design. Chapter 6 concludes with some open problems and other areas of importance for the stabilization of driftless control systems.

Appendix A reviews the controllability properties of driftless systems and the implications of Brockett's necessary condition for driftless systems.

Appendix B presents an algorithm, implemented in Mathematica, for computing the local diffeomorphism necessary to place the driftless system into the coordinates where the nilpotent homogeneous approximation are the leading order terms in the input vector fields.

Appendix C contains a proof that a control law used throughout the dissertation to illustrate certain concepts is ρ-exponentially stabilizing.

Chapter 2

Introduction to Homogeneous Systems

The approach to exponential stabilization in this thesis relies on the notion of homogeneous functions, homogeneous vector fields and homogeneous approximations of sets of vector fields. The most familiar definition of the homogeneous property scales each coordinate function by the same amount. However, nonisotropic scalings may also be defined. An expanded definition of homogeneous functions and vector fields, where the coordinates are scaled by different factors, is reviewed below. The usefulness of these definitions becomes apparent when it is recalled how a set of analytic vector fields which generate a full rank Lie algebra (interpreted here to be the input vector fields of a controllable driftless system) may be approximated by a nilpotent set which, in special coordinates, is homogeneous. A slightly modified notion of exponential stability, called ρ-exponential stability, is defined. This definition allows for non-Lipschitz dependence on initial conditions in the case where the equilibrium point is exponentially attractive and uniformly stable and reduces to the usual definition of exponential stability when there exists a linearization at the equilibrium point. Homogeneous approximations of vector fields are discussed in the references [13, 18, 1, 42]. Applications which utilize the homogeneous form of the approximating control system may be found in [17, 20]. These papers consider two-dimensional small time locally controllable systems and certain three-dimensional systems.

Finally, a converse Lyapunov theorem is reviewed for time-periodic homogeneous vector fields with asymptotically stable equilibrium point. The Lyapunov results are used to show that higher-order perturbations (in the sense defined below) do not locally affect the stability of the equilibrium point. Stability theorems for homogeneous systems where first proven by Hermes without the use of Lyapunov functions [16]. His results where extended by Rosier who proved a general converse Lyapunov for autonomous homogeneous systems [39].

To establish some notation, functions will be denoted by lower case letters and vector fields by capital letters. We will occasionally abuse notation and define the differential equation $\dot{x} = X(t,x)$ in local coordinates on $\mathbb{R}^n$ associated with the vector field X (more properly the *direction field* in the nonautonomous case). The flow of a differential equation is denoted ψ where $\psi(t, t_0, x_0)$ is the solution, at time t, which passes through the point x_0 at time t_0. When it is necessary to distinguish between flows of vector fields a subscript will be used; i.e. ψ_X is the flow of X, ψ_Y is the flow of Y, etc.

2.1 Definitions and Properties of Homogeneous Systems

2.1.1 Definitions

This section reviews dilations and homogeneous vector fields. A dilation $\Delta_\lambda{}^r : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is defined with respect to a fixed choice of coordinates $x = (x_1, x_2, \ldots, x_n)$ on $\mathbb{R}^n$ by assigning n positive rationals $r = (r_1 = 1 \leq r_2 \leq \cdots \leq r_n)$ and positive real parameter $\lambda > 0$ such that

$$\Delta_\lambda^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n), \quad \lambda > 0.$$

We usually write Δ_λ in place of Δ_λ^r.

Definition 2.1 A continuous function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $l \geq 0$ *with respect to* Δ_λ, denoted $f \in H_l$, if $f(t, \Delta_\lambda x) = \lambda^l f(t,x)$.

Definition 2.2 A continuous vector field $X(t,x) = \sum a_i(t,x)\partial/\partial x_i$ on $\mathbb{R} \times \mathbb{R}^n$ is *homogeneous of degree* $m \leq r_n$ *with respect to* Δ_λ if $a_i \in H_{r_i - m}$.

The variable t represents explicit time dependence and is never scaled in our applications.

Definition 2.3 A continuous map from $\mathbb{R}^n$ to $\mathbb{R}$, $x \mapsto \rho(x)$, is called a *homogeneous norm* with respect to the dilation $\Delta_\lambda$when

1. $\rho(x) \geq 0, \quad \rho(x) = 0 \iff x = 0,$
2. $\rho(\Delta_\lambda x) = \lambda \rho(x) \quad \forall \lambda > 0.$

For example, a homogeneous norm which is smooth on $\mathbb{R}^n \setminus \{0\}$ may always be defined as,

$$\rho(x) = |x_1^{c/r_1} + x_2^{c/r_2} + \cdots + x_n^{c/r_n}|^{1/c}, \tag{2.1}$$

where c is some positive integer *evenly* divisible by r_i. We are primarily interested in the convergence of time dependent functions using a homogeneous norm as a measure of their size. When a vector field is homogeneous it is most natural to use a corresponding homogeneous norm as the metric. The usual vector p-norms are homogeneous with respect to the standard dilation ($r_i = 1$).

2.1.2 Properties of homogeneous functions

In the sequel we will define continuous homogeneous functions which are differentiable everywhere except the origin. We state some properties of these functions.

Property 2.4 Suppose $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and differentiable with respect to x on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree m with respect to the dilation Δ_λ. Then $\frac{\partial}{\partial x_i}(f)(t,x)$ is a homogeneous function of degree $m - r_i$ with respect to Δ_λ. If $m - r_i > 0$ then we define $\frac{\partial}{\partial x_i}(f)(t,0) = 0$ in order to make the new function continuous with respect to x on $\mathbb{R}^n$.

The following property shows how the magnitude of homogeneous functions may be estimated with the homogeneous norm.

Property 2.5 If $f(t,x)$ is degree m (possibly < 0) and continuous with respect to x on $\mathbb{R}^n \setminus \{0\}$ and continuous with respect to t then there exists a continuous function $M_1 : \mathbb{R} \to \mathbb{R}$ such that

$$|f(t,x)| \leq M_1(t)\rho^m(x).$$

When $f(x,t)$ is continuously differentiable with respect to x on $\mathbb{R}^n \setminus \{0\}$ then

$$\left|\frac{\partial f}{\partial x_i}\right| \leq M_2(t)\rho^{m-r_i}(x) \quad i = 1, \ldots, n,$$

where $M_2(\cdot)$ is continuous. When M_1 or M_2 is bounded we define $\overline{M}_i = \sup_t M_i(t)$.

Proof: Define $M_1(t) = \max_{\rho(y)=1} |f(t,y)|$ and $y = \Delta_{1/\rho(x)}x$ so

$$\begin{aligned} |f(x,t)| &= |f(t, \Delta_{\rho(x)}y)| \\ &= \rho^m(x)|f(t,y)| \\ &\leq M_1(t)\rho^m(x). \end{aligned}$$

If $m < 0$ then f is unbounded in every neighborhood of the origin. However M_1 is still defined since f is continuous on the homogeneous sphere $\rho(x) = 1$. For the differentials we define $M_2(t) = \max_i \max_{\rho(y)=1} |\partial f/\partial x_i(t,y)|$ and apply the same scaling as above. ■

Property 2.6 Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be homogeneous of degree $m > 0$ with respect to Δ_λ and continuous in all arguments. Let $g : \mathbb{R} \times \mathbb{R}^n \setminus 0 \to \mathbb{R}$ be continuous and homogeneous of degree $l > -m$ (in particular, g may be unbounded with respect to x in every neighborhood of the origin in $\mathbb{R}^n$), then the function h defined by,

$$h(t,x) = \begin{cases} f(t,x)g(t,x) & x \in \mathbb{R}^n \setminus \{0\} \\ 0 & x = 0 \end{cases}$$

is homogeneous of degree $m + l$ and continuous on $\mathbb{R}^n \times \mathbb{R}$.

Proof: Set $M(t) = \max_{\rho(x)=1} |(fg)(t,x)|$, where ρ is a homogeneous norm. Choose $\epsilon > 0$, fix t, and compute $\lambda > 0$ such that $\lambda^{m+l}M(t) = \epsilon$. Define the set

$$U_t = \{x \in \mathbb{R}^n | 0 < \rho(x) \leq \lambda\}.$$

For $x \in U_t$,

$$\begin{aligned}
|f(t,x)g(t,x)| &= \left(\frac{\rho(x)}{\lambda}\right)^{m+l} |fg(t, \Delta_{\frac{\lambda}{\rho(x)}} x)| \\
&= \left(\frac{\rho(x)}{\lambda}\right)^{m+l} \lambda^{m+l} |fg(t, \Delta_{\frac{1}{\lambda}} \Delta_{\frac{\lambda}{\rho(x)}} x)| \\
&\leq \left(\frac{\rho(x)}{\lambda}\right)^{m+l} \lambda^{m+l} M(t) \\
&= \left(\frac{\rho(x)}{\lambda}\right)^{m+l} \epsilon \\
&< \epsilon \quad \text{since} \quad x \in U_t.
\end{aligned}$$

Thus $\lim_{x \to 0} |fg(t,x)| = 0$ for all t. ■

The preceding lemmas are useful when defining a new function as the Lie derivative of a homogeneous function with respect to a homogeneous vector field. Suppose $X : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous homogeneous vector field of degree l and f is a continuous homogeneous function of degree m differentiable on $\mathbb{R}^n \setminus \{0\}$. $L_X f$ is a homogeneous function of degree $m - l$. If m is greater than l then the new function is continuous on $\mathbb{R}^n$ if it is defined to be zero at the origin.

Property 2.7 If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous positive definite homogeneous degree l function, differentiable on $\mathbb{R}^n \setminus \{0\}$, then $\nabla f \neq 0$ for all $x \neq 0$.

Proof: Suppose $\nabla f(\overline{x}) = 0$ for some $\overline{x} \neq 0$. Let $\gamma(t) = \Delta_{1-t}\overline{x}$, $t \in [0,1)$, be a parameterized path with non-zero velocity. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) &= \nabla f(\gamma(t)) \cdot \gamma'(t) \\
&= \nabla f(\Delta_{1-t}\overline{x}) \cdot \gamma'(t) \\
&= \nabla f(\overline{x}) \cdot \mathrm{diag}\left((1-t)^{l-r_i}\right) \cdot \gamma'(t) \\
&= 0 \quad \forall \quad t \in [0,1).
\end{aligned}$$

Thus $f(\gamma(t)) = 0$, since $f(0) = 0$ and f is continuous, so f cannot be positive definite. ■

Definition 2.8 The *ρ-homogeneous unit* $(n-1)$ *sphere* is defined as the set

$$S_\Delta^{n-1} = \{x | \rho(x) = 1\},$$

where ρ is a homogeneous norm.

Definition 2.9 The *Euler vector field corresponding to a dilation* Δ_λ is defined as,

$$X_E(x) = \sum r_i x_i \frac{\partial}{\partial x_i}.$$

Thus the images of trajectories of the system $\dot{x} = X_E(x)$ are the Δ-homogeneous rays obtained by scaling the points on the sphere S_Δ^{n-1} with the dilation.

2.1.3 Stability definitions

The fundamental definitions of stability are reviewed below. They are contrasted to a slightly modified definition of exponential stability. The point $x = 0$ is taken to be an equilibrium point of the differential equation $\dot{x} = X(t, x)$. The trajectory of this differential equation passing through (t_0, x_0) is denoted $\psi(t, t_0, x_0)$.

Definition 2.10 The equilibrium point $x = 0$ is *uniformly stable* if for all $\epsilon > 0$ there exists a $\delta > 0$, which may be chosen independent of t_0, such that $\|x_0\| < \epsilon \implies \|\psi(t, t_0, x_0)\| < \delta$ for all $t > t_0$.

Definition 2.11 The equilibrium point $x = 0$ is *uniformly asymptotically stable* if it is uniformly stable and in addition for all $\epsilon, \delta > 0$ there exists $T \geq 0$, independent of t_0, such that $\|x_0\| < \epsilon \implies \|\psi(t, t_0, x_0)\| < \delta \;\; \forall t > T + t_0$.

The usual definition of exponential stability is recalled to contrast it with a modified definition used in this dissertation.

Definition 2.12 The equilibrium point $x = 0$ is *locally exponentially stable* if there exist constants $\alpha, \beta > 0$ and a neighborhood U of the origin such that the trajectories of the system are bounded by

$$\|\psi(t, t_0, x_0)\|_2 \leq \beta \|x_0\|_2 e^{-\alpha(t-t_0)} \quad \forall t \geq t_0, \forall x_0 \in U.$$

$\|\cdot\|_2$ is the Euclidean norm.

The Euclidean norm is not crucial: $\|\cdot\|_2$ may be replaced by any other vector p-norm. The concept of exponential stability of a vector field is now defined in the context of a homogeneous norm. This definition was introduced by Kawski [20].

Definition 2.13 The equilibrium point $x = 0$ is *locally exponentially stable with respect to the homogeneous norm* $\rho(\cdot)$ if there exist two constants $\alpha, \beta > 0$ and a neighborhood of the origin U such that

$$\rho(\psi(t, t_0, x_0)) \leq \beta \rho(x_0) e^{-\alpha(t-t_0)} \quad \forall t \geq t_0, \forall x_0 \in U.$$

This stability type is denoted *ρ-exponential stability* to distinguish it from the prior definition.

This notion of stability is important when considering vector fields which are homogeneous with respect to a dilation. The convergence of trajectories is naturally studied using the corresponding homogeneous norm. This definition is not equivalent to the usual definition of exponential stability except when the dilation is the standard dilation ($r_i = 1$). This is evident from the following bounds on the Euclidean norm in terms of the smooth homogeneous norm (2.1) on the unit cube $\mathcal{C} = \{x : |x_i| < 1, i = 1, \ldots, n\}$ (recall $c \geq 2$ in Definition 2.1). The lower bound is,

$$\begin{aligned}
\rho^c(x) &= \sum_i x_i^{c/r_i} \leq \sum_i x_i^2 = \|x\|_2^2 \quad x \in \mathcal{C} \\
&\implies \rho^{c/2}(x) \leq \|x\|_2.
\end{aligned}$$

An upper bound is computed to be,

$$\begin{aligned}
\rho^c(x) &= \sum_i x_i^{c/r_i} \\
&\geq \sum_i x_i^{c/r_1} \quad \text{for } x \in \mathcal{C} \\
&\geq M\|x\|_2^{c/r_1} \quad \text{where } M = \min_{\|x\|_2=1} \sum_i x_i^{c/r_1}.
\end{aligned}$$

Both bounds yield,

$$\rho^{c/2}(x) \leq \|x\|_2 \leq \frac{1}{M^{r_1/c}} \rho^{r_1}(x) \quad x \in \mathcal{C}.$$

Hence, the solutions of a ρ-exponentially stable system also satisfy

$$\|\psi(t, t_0, x_0)\|_2 \leq \frac{\beta}{M^{r_1/c}} \|x_0\|_2^{\frac{2r_1}{c}} e^{-r_1\alpha(t-t_0)}. \tag{2.2}$$

Thus, each state may be bounded by a decaying exponential envelope except that the size of the envelope does not scale linearly in the initial condition as in the usual definition of exponential stability. Furthermore, ρ-exponential stability allows for non-Lipschitz dependence on the initial conditions. To illustrate how this non-Lipschitz dependence on the initial condition is often necessary, consider the following two-dimensional system,

$$\begin{aligned}
\dot{x}_1 &= -x_1 + \gamma\sqrt{|x_2|} \\
\dot{x}_2 &= -x_2.
\end{aligned}$$

The equations are degree zero with respect to the dilation $\Delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2)$ and, by computing explicit solutions, the system is ρ-exponentially stable with the homogeneous norm $\rho(x) = |x_1^4 + x_2^2|^{1/4}$ i.e. $c = 4$ in equation 2.1. In addition, it can be shown that there exist initial conditions arbitrarily close to the origin such that

$$\sup_t |x_1(t)| \geq \frac{|\gamma|}{2}\sqrt{|x_2(0)|}.$$

The bound in equation (2.2) reflects this behavior since the exponent on the Euclidean norm of the initial condition is $2r_1/c = 1/2$. It is in this sense that the bound (2.2) is tight.

2.1.4 Properties of homogeneous degree zero vector fields.

Some useful facts concerning degree zero vector fields are reviewed in this section. The notion of a symmetry of a vector field is first introduced. The *differential* of a map $f : M \to N$, where M and N are manifolds, is denoted f_*.

Definition 2.14 Suppose the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. A vector field $X(t,x)$ is said to be *invariant* under f if $f_*X(t,x) = X(t, f(x))$ for all $x \in \mathbb{R}^n$. f is called a *symmetry* of X.

There is a more general definition which subsumes invariance as a special case.

Definition 2.15 Let X_M and X_N be vector fields on smooth manifolds M and N, respectively, with $\dim M > \dim N$. Suppose $g : M \to N$ is a smooth map. The vector fields are said to be *g-related* if they satisfy

$$g_* X_M = X_N \circ g.$$

A vector field which is invariant with respect to a *one-parameter group* of symmetries, denoted G, is often π-related to a vector field on the quotient manifold $N = M/G$ where $\pi : M \to N$ is the projection operator defined by identifying all points in M which differ by an element of G. A homogeneous degree zero vector field $X(t,x)$ is invariant with respect to the dilation,

$$(\Delta_\lambda)_* X(t,x) = X(t, \Delta_\lambda x) \quad \lambda > 0.$$

If we set $M = \mathbb{R}^n \setminus \{0\}$ then the quotient space M/Δ_λ becomes the homogeneous sphere S^{n-1}_Δ naturally embedded in $\mathbb{R}^n$. The projection operator $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}_\Delta$ is given by

$$\pi(x) = \left(\frac{x_1}{\rho^{r_1}(x)}, \ldots, \frac{x_n}{\rho^{r_n}(x)} \right).$$

Explicit computations are carried out below to determine the components of the induced vector field on S^{n-1}_Δ. The vector field $X(t,x)$ may be written as $X(t,x) = \sum_i a_i(t,x)\partial/\partial x_i$, where $a_i(t, \Delta_\lambda x) = \lambda^{r_i} a_i(t,x)$ since X is degree zero. The corresponding differential equation is $\dot{x}_i = a_i(t,x), i = 1, \ldots, n$. The induced vector field, denoted $\tilde{X}$, is determined by differentiating the coordinate functions of the projection operator,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{x_i}{\rho^{r_i}(x)}\right) &= \frac{\dot{x}_i}{\rho^{r_i}(x)} - x_i \sum_{k=1}^{n} \frac{r_i}{r_k} x_k^{c/r_k - 1} \frac{1}{\rho^{r_i + c}(x)} a_k(t,x) \\
&= a_i(t, \Delta_{1/\rho(x)} x) - \frac{x_i}{\rho^{r_i}(x)} \sum_{k=1}^{n} \frac{r_i}{r_k} \left(\frac{x_k}{\rho^{r_k}(x)}\right)^{c/r_k - 1} a_k(t, \Delta_{1/\rho(x)} x) \\
&= a_i(t, \pi(x)) - \pi_i(x) \sum_{k=1}^{n} \frac{r_i}{r_k} (\pi_k(x))^{c/r_k - 1} a_k(t, \pi(x)), \quad i = 1, \dots, n
\end{aligned}
$$

where $\pi_i(x)$ denotes the i^{th} component of π. The vector field

$$\tilde{X}(t,y) = \sum_i \tilde{a}_i(t,y) \partial/\partial y_i$$

leaves the sphere S_Δ^{n-1} invariant since the Lie derivative of the function $g : S_\Delta^{n-1} \to \mathbb{R} : g(y) = \rho(y)$ with respect to $\tilde{X}$ is zero. Thus, X and $\tilde{X}$ are π-related, i.e., $\pi_* X(t,x) = \tilde{X}(t, \pi(x))$. The flow of $\tilde{X}$ may be computed by solving the set of differential equations $\dot{y}_i = \tilde{a}_i(t,y), i = 1, \dots, n$ with initial conditions on S_Δ^{n-1}. The solutions of the original vector field X are recovered from $x_i(t) = \rho^{r_i}(t) y(t)$. Thus the differential equation specifying $\rho(t)$ is required. This equation is obtained by differentiating $\rho(x(t))$ with respect to t,

$$
\begin{aligned}
\frac{d}{dt}\rho(x) &= \frac{d}{dt}\left(\sum_{k=1}^{n} x_k^{c/r_k}\right)^{1/c} && (2.3) \\
&= \frac{1}{c\rho^{c-1}(x)} \left(\sum_{k=1}^{n} \frac{c}{r_k} x_k^{c/r_k - 1} \dot{x}_k\right) && (2.4) \\
&= \rho(x) \sum_{k=1}^{n} \frac{1}{r_k} (\pi_k(x))^{c/r_k - 1} a_k(t, \pi(x)) && (2.5) \\
&= \left(\sum_{k=1}^{n} \frac{1}{r_k} y_k^{c/r_k - 1} a_k(t,y)\right) \rho(x). && (2.6)
\end{aligned}
$$

This scalar equation is linear in ρ with a time-varying coefficient which depends only on the solution of the sphere equation i.e. $\dot{\rho} = Q(t,y)\rho$, where $Q(t,y) = \sum_{k=1}^{n} \frac{1}{r_k} y_k^{c/r_k - 1} a_k(t,y)$. Thus $\rho(t)$ may be computed by quadratures after $y(t)$ has been determined. The usefulness of this reduction procedure for degree zero systems lies not in solving the equations but rather the connection it makes between uniform asymptotic stability and ρ-exponential stability.

Lemma 2.16 *If $X(t,x)$ is a homogeneous degree zero vector field, then local uniform asymptotic stability is equivalent to global exponential stability with respect to the homogeneous norm $\rho(x)$.*

Proof: Hahn [14] deals with the case in which the dilation is the standard dilation. His proof extends to the case with the nonstandard dilation. The key observation is that uniform asymptotic stability implies that the integral of the coefficient in the $\dot{\rho}$ equation (2.6) has the following bound

$$\int_{t_0}^{t} Q(t, y(t)) \leq K_1 - K_2(t - t_0) \quad K_1 \in \mathbb{R}, K_2 > 0,$$

where K_1 and K_2 are independent of t_0. This aspect of the proof is worked out in detail in Hahn [14]. The bound implies that $\rho \to 0$ exponentially. In other words, $x = 0$ is ρ-exponentially stable.

Global stability follows from the fact that the equation is degree zero. Suppose ψ_i represents the i^{th} component of a solution $\psi(t, t_0, x_0)$. A trajectory scaled with Δ_λ satisfies the original differential equation,

$$\begin{aligned} \frac{d}{dt}\lambda^{r_i}\psi_i(t, t_0, x_0) &= \lambda^{r_i}\frac{d}{dt}\psi_i(t, t_0, x_0) \\ &= \lambda^{r_i} a_i(t, \psi(t, t_0, x_0)) \quad i = 1, \ldots, n. \end{aligned}$$

Since the initial condition of the scaled solution is $\Delta_\lambda x_0$ then $\psi(t, t_0, \Delta_\lambda x_0) = \Delta_\lambda \psi(t, t_0, x_0)$. Thus a trajectory with arbitrary initial condition has a "local" analog which may be obtained via the dilation. ∎

The following example illustrates these properties on a linear system.

Example 2.17 Consider the linear system $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$. This system is invariant with respect to the standard dilation $\Delta_\lambda x = \lambda x$ since $(\Delta_\lambda)_* Ax = \lambda Ax = A\lambda x = A\Delta_\lambda x$. A convenient homogeneous norm to use is the Euclidean norm $\|\cdot\|_2$. Hence, the quotient manifold is the sphere $\|x\|_2 = 1$ embedded in $\mathbb{R}^n$. The projection onto the sphere is $\pi : \mathbb{R}^n \to S^{n-1}, y = \pi(x) = x/\|x\|_2$. The vector field defined on the sphere is computed to be

$$\dot{y} = Ay - \langle y, Ay \rangle y, \tag{2.7}$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. The corresponding equation for $\rho(x(t))$ is $\dot{\rho} = \langle y, Ay \rangle \rho$. If v is an eigenvector of A corresponding to the eigenvalue σ, then the point $\tilde{v} = \pi(v) \in S^{n-1}$ is an equilibrium point of the sphere equations (2.7) since

$$\begin{aligned} \dot{y} &= A\tilde{v} - \langle \tilde{v}, A\tilde{v} \rangle \tilde{v} \\ &= \sigma\tilde{v} - \sigma\langle \tilde{v}, \tilde{v} \rangle \tilde{v} \\ &= 0. \end{aligned}$$

The ρ equation becomes $\dot{\rho} = \sigma\rho$ with solution

$$\begin{aligned} \rho(t) &= \exp(\sigma t)\rho(0) \\ &= \exp(\sigma t)\rho(v) \\ &= \exp(\sigma t)\|v\|_2. \end{aligned}$$

Reconstructing the full solution $x(t) = \rho(t)y(t)$ where $y(t) = v/\|v\|_2$ yields $x(t) = \exp(\sigma t)v$ which is, of course, the correct answer. Finally, Lemma 2.16 merely reaffirms the well known fact that uniform asymptotic stability and exponential stability are equivalent for linear systems.

2.2 Homogeneous Approximations of Vector Fields

This section discusses nilpotent homogeneous approximations of sets of vector fields. The vector fields are the input vector fields of the controllable driftless system,

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m. \tag{2.8}$$

The entire analysis is local so we assume that vector fields are defined on $\mathbb{R}^n$. Furthermore, the vector fields are taken to be analytic. A brief review of Appendix A may be helpful at this point to familiarize the reader with some terminology and definitions. We are interested in obtaining an approximation, in the sense described below, of the set of vector fields $\{X_1, \ldots, X_m\}$. The Lie bracket of vector fields is $[\cdot, \cdot]$.

Let $\mathcal{L}(X_1, \ldots, X_m)$ be the Lie algebra generated by the set $\{X_1, \ldots, X_m\}$. Every element of $\mathcal{L}$ is a linear combination of repeated Lie brackets of the form,

$$[X_{\pi_k}, [X_{\pi_{k-1}}, [\ldots [X_{\pi_2}, X_{\pi_1}] \ldots]]],$$

where X_{π_i} is in the set $X_1, \ldots, X_m$ and $k = 0, 1, 2 \ldots$ [36].

For any algebra $\mathcal{A}$, a countable family of subspaces $\mathcal{F}_j$ is a *filtration* of $\mathcal{A}$ if

$$\{0\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots, \quad \mathcal{A} = \bigcup_{j \geq 0} \mathcal{F}_j, \quad \mathcal{F}_i \cdot \mathcal{F}_j \subset \mathcal{F}_{i+j}.$$

The following definition specifies a special filtration of the Lie algebra of a finite set of generating vector fields.

Definition 2.18 The *control filtration*, $\mathcal{F}^X$, of $\mathcal{L}(X_1, \ldots, X_m)$ is a sequence of subspaces defined as,

$$\begin{aligned}
\mathcal{F}_0^X &= \{0\}, \\
\mathcal{F}_1^X &= \text{span}\{X_1, \ldots, X_m\}, \\
\mathcal{F}_2^X &= \text{span}\{X_1, \ldots, X_m, [X_1, X_2], \ldots, [X_1, X_2], \ldots, [X_{m-1}, X_m]\}, \\
&\vdots \\
\mathcal{F}_k^X &= \text{span}\{\text{all products of i-tuples from } \{X_1, \ldots, X_m\}, \text{for } i \leq k\}, \\
&\vdots
\end{aligned} \tag{2.9}$$

and $\mathcal{F}^X = \{\mathcal{F}_j^X\}_{j \geq 0}$.

From the characterization of elements of $\mathcal{L}$ and the definition of the filtration it is easy to see that

$$\mathcal{F}_i^X \cdot \mathcal{F}_j^X = [\mathcal{F}_i^X, \mathcal{F}_j^X] \subset \mathcal{F}_{i+j}^X$$
$$\mathcal{L} = \bigcup_{i \geq 0} \mathcal{F}_i^X,$$

so that $\mathcal{F}^X$ is indeed a filtration.

The set of vector fields is approximated about a specific point, $x_0 \in \mathbb{R}^n$, which is a desired equilibrium point here. Now let $F_i(x_0)$ be the subspace of $\mathbb{R}^n$ (more precisely the tangent space, $T_{x_0}\mathbb{R}^n$, of $\mathbb{R}^n$ at x_0) spanned by $Z(x_0)$ where $Z \in \mathcal{F}_i^X$. This yields an increasing sequence of vector subspaces,

$$\{0\} = F_0(x_0) \subset F_1(x_0) \subset \cdots \subset F_i(x_0) \subset \cdots \subset \mathbb{R}^n.$$

This sequence must be stationary after some integer since it is assumed that the Lie algebra has full rank at x_0. In other words, since the system (2.8) is controllable $\dim F_k(x_0) = n$ for all k greater that some minimal integer N. Now we count the growth in the dimension of the subspaces and set $n_1 = \dim F_1(x_0), n_2 = \dim F_2(x_0), \ldots, n_N = n = \dim F_N(x_0)$. The following dilation is defined,

Definition 2.19 The *dilation adapted to the filtration* (at the point x_0) is the map,

$$\Delta_\lambda^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n),$$

where the scalings satisfy $r_i = 1$ for $1 \leq i \leq n_1$, $r_i = 2$ for $n_1 + 1 \leq i \leq n_2$, etc.

Henceforth, in order to simplify the notation in the expressions to follow it is assumed that $x_0 = 0$. This is achieved with a translation of the origin of the coordinate system.

Definition 2.20 The *local coordinates adapted to the filtration* $\mathcal{F}^X$ (denoted by y) are related to the original coordinates (denoted by x) by the local analytic diffeomorphism derived from composing flows of vector fields from the filtration,

$$x = \Phi(y) = \psi_{X_{\pi_1}}^{y_1} \circ \psi_{X_{\pi_2}}^{y_2} \circ \cdots \circ \psi_{X_{\pi_n}}^{y_n}(0), \tag{2.10}$$

where $\psi_X^t(x_0) = \psi_X(t, 0, x_0)$ denotes the flow of the vector field X and,

i) $X_{\pi_i} \in \mathcal{F}_j^X \quad \text{for} \quad n_{j-1} + 1 \leq i \leq n_j$,

ii) $\dim\{X_{\pi_1}, \ldots, X_{\pi_n}\} = n$.

A vector field written in a local coordinate system will explicitly show the dependence, i.e., $X(x)$ is written in x-coordinates while $X(y)$ is the same vector field written in y-coordinates. The importance of the local coordinates adapted to $\mathcal{F}^X$ is explained by the following theorem,

Theorem 2.21 (Theorem 2.1, [18]) *Let $\mathcal{L}$ be a Lie algebra of vector fields on $\mathbb{R}^n$ and $\mathcal{F} = \{\mathcal{F}_j\}_{j\geq 0}$ an increasing filtration of $\mathcal{L}$ at zero with Δ_λ the dilation adapted to $\mathcal{F}$ and y the local coordinates adapted to $\mathcal{F}$. Then if $X \in \mathcal{F}_l$,*

$$X(y) = X^l(y) + X^{l-1}(y) + X^{l-2}(y) + \cdots ,$$

where $X^j(y)$ is a vector field homogeneous of degree j with respect to Δ^r_λ.

In other words, if $X(y) \in \mathcal{F}_l$ is expanded in terms of vector fields which are homogeneous with respect to Δ_λ, $X(y) = \sum_{j=r_n}^{-\infty} X^j(y)$, then $X^{r_n}(y) = \cdots = X^{l+1}(y) = 0$ and the "leading order" vector field, $X^l(y)$, is degree l with respect to Δ_λ. This leading order vector field is termed the *$\mathcal{F}$-approximation of $X \in \mathcal{F}_l$ in the $\mathcal{F}$-adapted coordinates.* An important property of the $\mathcal{F}$-approximation is given by the following proposition,

Proposition 2.22 (Corollary 2.2.1, [18]) *Let $\mathcal{F} = \{\mathcal{F}_j^X\}$ be the control filtration of $\mathcal{L}(X_1, \ldots, X_m)$ and $\{\mathcal{F}_j^Y\}_{j\geq 0}$ be the equivalently defined filtration of $\mathcal{L}(Y_1, \ldots, Y_m)$ where Y_i is the $\mathcal{F}$-approximation of $X_i, i = 1, \ldots, m$. Furthermore, let F_l^X and F_l^Y be the corresponding increasing sequence of vector subspaces of $\mathbb{R}^n$. Then,*

$$F_l^X(0) = F_l^Y(0), \quad l = 0, 1, \ldots .$$

Remark 2.23 Some readers may find it irksome that the approximation process relies on a special local coordinate system. In other words, the approximation described above does not seem to have a coordinate free representation. This, however, is not the case. Bellaïche et al. [1] have defined the notion of *local order* which they use to give the approximation a more intrinsic meaning. Their approximation coincides with the $\mathcal{F}$-approximation when the vector fields are written in local coordinates adapted to $\mathcal{F}$. In addition, the $\mathcal{F}$-approximation of the generating set are homogeneous degree one vector fields and generate a nilpotent Lie algebra themselves [18, Proposition 2.3]. Nilpotency is a coordinate free property.

When implementing a feedback law the equations must be written is some coordinate system. Coordinates adapted to $\mathcal{F}$ are chosen in this thesis since the homogeneous nature of the $\mathcal{F}$-approximation are exploited. A simple example may help to clarify some of these points.

Example 2.24 Consider the two vector fields on $\mathbb{R}^3$ given by,

$$\begin{aligned} X_1(x) &= \partial/\partial x_1 + \partial/\partial x_2 \\ X_2(x) &= (a + x_1)\partial/\partial x_2 + (ab + ax_2 + bx_1 + x_1x_2)\partial/\partial x_3. \end{aligned}$$

An homogeneous approximation of $\{X_1, X_2\}$ around $x = 0$ is desired for various values of a and b. For $a \neq 0$ the dimension of F_1 is 2. The Lie bracket of X_1 and X_2 at $x = 0$ evaluates to be $[X_1, X_2](0) = -\partial/\partial x_2 - (a+b)\partial/\partial x_3$ so $\dim F_2 = 3$.

The dilation scaling powers are $r_1 = r_2 = 1$ and $r_3 = 2$. The coordinates adapted to $\mathcal{F}$ may be computed using the formula in equation (2.10), however the linear transformation $x = A_1 y$ with

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 0 & ab & 1 \end{pmatrix},$$

suffices in placing X_1 and X_2 into suitable coordinates since,

$$\begin{aligned} X_1(y) &= \partial/\partial y_1 \\ X_2(y) &= (1 + y_1/a)\partial/\partial y_2 + (ay_1 + a^2 y_2 + y_1^2 + aY_1 y_2)\partial/\partial y_3. \end{aligned}$$

Thus the $\mathcal{F}$-approximation of these vector fields is,

$$\begin{aligned} Y_1(y) &= \partial/\partial y_1 \\ Y_2(y) &= \partial/\partial y_2 + (ay_1 + a^2 y_2)\partial/\partial y_3. \end{aligned}$$

These vector fields are homogeneous degree one (since $X_1, X_2 \in \mathcal{F}_1^X$) with respect to the dilation $\Delta_\lambda y = (\lambda y_1, \lambda y_2, \lambda^2 y_3)$. The terms which are truncated from $X_1(y)$ and $X_2(y)$ are higher order with respect to this dilation.

When $a = 0$ and $X_2(0) = 0$, more brackets are required since the dimensions of $\mathcal{F}_1$ and $\mathcal{F}_2$ drop to 1 and 2 respectively. In particular $[X_1,[X_1,X_2]](0) = 2\partial/\partial x_3$ suffices since the set $\{X_1(0), [X_1,X_2](0), [X_1,[X_1,X_2]](0)\}$ is linearly independent. In this case $n_1 = 1$, $n_2 = 2$ and $n_3 = 3$ so the new dilation scaling powers are $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. The $\mathcal{F}$-adapted local coordinates may be used to calculate the linear mapping $x = A_2 y$ where,

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -b & 2 \end{pmatrix},$$

which places X_1 and X_2 into the form,

$$\begin{aligned} X_1(y) &= \partial/\partial y_1 \\ X_2(y) &= y_1\partial/\partial y_2 + \frac{1}{2}y_1(y_1 - y_2)\partial/\partial y_3. \end{aligned}$$

The $\mathcal{F}$-approximation are the vector fields,

$$\begin{aligned} Y_1(y) &= \partial/\partial y_1 \\ Y_2(y) &= y_1\partial/\partial y_2 + \frac{1}{2}y_1^2\partial/\partial y_3. \end{aligned}$$

Both of these vector fields are homogeneous degree one with respect to the new dilation. To conclude this example, the filtration, dilation and $\mathcal{F}$-approximation may change from point to point, however the approximation is always defined at a particular point if the Lie algebra has full rank there.

2.3 Lyapunov Functions for Homogeneous Degree Zero Vector Fields

This section reviews converse Lyapunov stability theory for homogeneous systems and gives an extension for degree zero periodic vector fields. These results are important since the feedbacks derived in this dissertation exponentially stabilize an approximation of the driftless system and the higher order (with respect to a dilation) terms neglected in the approximation process are shown to not locally change the stability of the system. The main theorem by Rosier in [39] states that given an autonomous continuous homogeneous (with respect to some dilation Δ_λ) vector field $\dot{x} = f(x)$ with asymptotically stable equilibrium point $x = 0$, there exists a Δ_λ-homogeneous Lyapunov function smooth on $\mathbb{R}^n \setminus \{0\}$ and differentiable as many times as desired at the origin. Rosier defines the new homogeneous Lyapunov function as

$$\overline{V}(x) = \begin{cases} \int_0^\infty \frac{1}{h^{k+1}}(f \circ V)(h^{r_1}x_1, \ldots, h^{r_n}x_n)dh & \text{if} \quad x \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if} \quad x = 0, \end{cases} \tag{2.11}$$

where $V(x)$ is a smooth Lyapunov function whose existence is guaranteed by the converse theorems in Kurzweil [22] and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$f = \begin{cases} 0 & \text{on} \quad (-\infty, 1], \\ 1 & \text{on} \quad [2, \infty), \end{cases}$$

with $f' \geq 0$. The integer $k > 0$ controls the degree of differentiability of $\overline{V}(x)$ at the origin.

Rosier's converse theorem extends to the class of continuous, time-periodic, homogeneous degree *zero* systems, $\dot{x} = X(t, x)$, with asymptotically stable equilibrium point $x = 0$. This fact is stated as a proposition. In coordinates X is written as $X(t, x) = \sum_{i=1}^n a_i(t, x)\partial/\partial x_i$.

Theorem 2.25 (extension of [39]) *Suppose the differential equation $\dot{x} = X(t, x)$ satisfies the following properties,*

i) X is continuous in t and x,

ii) $X(t, 0) = 0 \ \forall t$,

iii) $X(t + T, x) = X(t, x) \ \forall x$,

iv) X is homogeneous degree zero *(in x) with respect to the dilation $\Delta_\lambda = (\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n)$,*

v) the solution $x(t) = 0$ is asymptotically stable.

Let p be a positive integer and k a real number larger than $p \cdot \max r_i$. Then there exists a function $\overline{V} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that,

a) $\overline{V}(t, x)$ is smooth for $x \in \mathbb{R}^n \setminus \{0\}$, and C^p at $x = 0$,

b) $\overline{V}(t, 0) = 0, \ \overline{V}(t, x) > 0 \ x \neq 0$

c) $\overline{V}$ is degree k with respect to Δ_λ i.e. $\overline{V}(t, \Delta_\lambda x) = \lambda^k \overline{V}(t, x)$, *d)* $\overline{V}(t+T,x) = \overline{V}(t,x)\ \forall x$ *and smooth with respect to* t,

e) $\frac{d\overline{V}}{dt}(t,x) = \frac{\partial \overline{V}}{\partial t}(t,x) + \nabla\overline{V}(t,x)\cdot X(t,x) < 0\ \forall x \neq 0$.

Proof: The Lyapunov function $\overline{V}$ is constructed from a smooth Lyapunov function $\tilde{V}(t,x)$ which has the property that $d\tilde{V}/dt < 0$ for all $x \neq 0$ and $\tilde{V}(t+T,x) = \tilde{(}t,x)\ \forall x$ [22]. The construction of $\overline{V}$ is given by equation (2.11) with $V(x)$ replaced by $\tilde{V}(t,x)$. The proof of properties (a) to (c) are identical to the proofs in [39]. The periodicity of $\overline{V}$ with respect to t is easily verified from the definition. The j^{th} partial of $\overline{V}$ with respect to t for $x \neq 0$ may be computed explicitly from the definition by differentiating under the integral sign. The important fact to note is that the integrand is a sum of products between $f^{(i)}, i = 1,\ldots,k$ and $\partial^i\tilde{V}/\partial x^i, i = 1,\ldots,k$ where $f^{(i)}$ is the i^{th} derivative of f. Since the derivatives of f have compact support then the integral is well defined. Smoothness follows since every term in the integrand is smooth. Finally the derivative of $\overline{V}$ along solutions of X is,

$$\begin{aligned}
\frac{d\overline{V}}{dt}(x,t) &= \frac{\partial\overline{V}}{\partial t}(x,t) + \sum_{i=1}^{n} f_i(x,t)\frac{\partial\overline{V}}{\partial x_i}(x,t) \\
&= \int_0^\infty \frac{1}{h^{k+1}} f'(\tilde{V}(t,\Delta_h x))\frac{\partial\tilde{V}}{\partial t}(t,\Delta_h x)dh \\
&\quad + \sum_{i=1}^{n}\int_0^\infty \frac{h^{r_i}}{h^{k+1}} f'(V(t,\Delta_h x))a_i(t,x)\frac{\partial\tilde{V}}{\partial x_i}(t,\Delta_h x)dh \qquad (2.12) \\
&= \int_0^\infty \frac{1}{h^{k+1}} f'(V(t,\Delta_h x)) \cdot \\
&\quad \left[\frac{\partial\tilde{V}}{\partial t} + \sum_{i=1}^{n}\left(a_i\frac{\partial\tilde{V}}{\partial x_i}\right)\right](t,\Delta_h x)dh.
\end{aligned}$$

The integrand is nonpositive since $\tilde{V}(t,x)$ is a Lyapunov function for $\dot{x} = X(t,x)$. Thus, the time derivative of $\overline{V}(t,x)$ is negative for all t,$x \neq 0$. ■

It is tempting to believe that this converse theorem holds for any continuous, time-periodic, homogeneous degree $\tau \geq 0$ vector field instead of the $\tau = 0$ case which is studied here. However, as demonstrated above this construction is guaranteed to yield a *Lyapunov* function for $\dot{x} = X(t,x)$ only when it is homogeneous degree zero. To see this, suppose $X(t,x)$ is degree $\tau > 0$. The total time derivative of $\overline{V}(t,x)$ along nonzero trajectories of $\dot{x} = X(t,x)$ is,

$$\frac{d\overline{V}}{dt}(t,x) = \int_0^\infty \frac{1}{h^{k+1}} f'(\tilde{V}(t,\Delta_h x))\left[\frac{\partial\tilde{V}}{\partial t} + \frac{1}{h^\tau}\sum_{i=1}^{n}\left(f_i\frac{\partial\tilde{V}}{\partial x_i}\right)\right](t,\Delta_h x)dh. \qquad (2.13)$$

The sign definiteness of $\partial\tilde{V}/\partial t$ and $\sum_i f_i\partial\tilde{V}/\partial x_i$ as separate entities is not known and when $\tau \neq 0$ the terms in the integrand of (2.13) are weighted by different amounts due to the presence of the $1/h^\tau$ factor. Thus, even though

$\overline{V}(x,t)$ is positive definite for any $\tau \geq 0$, the sign definiteness of its total time derivative is not known in the $\tau \neq 0$ case. An example illustrating the failure of this construction in the case $\tau \neq 0$ is reviewed below.

Example 2.26 The scalar system

$$\dot{x} = (-\alpha + \cos t)x^3 \quad \alpha > 1, \tag{2.14}$$

is globally uniformly asymptotically stable. The vector field defined by equation (2.14) is also homogeneous degree -2 with respect to the standard dilation. Any autonomous positive definite function on $\mathbb{R}$ is a Lyapunov function for this system, however to demonstrate the precise failure of the algorithm a time-periodic Lyapunov function is required. This Lyapunov function is constructed in the standard manner: integrate a positive definite function along solutions of (2.14) with the initial condition and starting time as parameters. These calculations are carried out explicitly below. The general solution of (2.14), denoted $\psi(t, t_0, x_0)$, is given by the formula

$$\psi(t, t_0, x_0) = \frac{x_0}{\sqrt{(2\alpha(t-t_0) - 2\sin t + 2\sin t_0)x_0^2 + 1}}.$$

Define the following function

$$\begin{aligned} V(t,x) &= \int_t^\infty |\psi(\tau, t, x)|^4 d\tau && (2.15)\\ &= \int_t^\infty \frac{x^4}{\left((2\alpha(\tau - t) - 2\sin\tau + 2\sin t)x^2 + 1\right)^2} d\tau && (2.16)\\ &= \int_0^\infty \frac{x^4}{\left((2\alpha s - 2\sin(s+t) + 2\sin t)x^2 + 1\right)^2} ds. && (2.17)\end{aligned}$$

The 4^{th} power is used in the integrand to ensure convergence. It is easily verified that this function is positive definite and 2π-periodic with total derivative $dV/dt = -x^4$. Hence equation (2.17) defines a Lyapunov function for the system in equation (2.14). The partial derivatives of V are required for the analysis to follow and are given by,

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) &= \int_0^\infty \frac{-2x^6(-2\cos(s+t) + 2\cos t)}{\left((2\alpha s - 2\sin(s+t) + 2\sin t)x^2 + 1\right)^3} ds, \\ \frac{\partial V}{\partial x}(t,x) &= \int_0^\infty \left[\frac{4x^3}{\left((2\alpha s - 2\sin(s+t) + 2\sin t)x^2 + 1\right)^2} \right. \\ &\qquad \left. - \frac{4x^5(2\alpha s - 2\sin(s+t) + 2\sin t)}{\left((2\alpha s - 2\sin(s+t) + 2\sin t)x^2 + 1\right)^3} \right] ds. \end{aligned} \tag{2.18}$$

V is clearly *not* homogeneous. In order to "homogenize" V, Rosier's algorithm is applied. This requires picking a smooth function $f : \mathbb{R} \to \mathbb{R}$ so that the new positive definite homogeneous function, denoted $\overline{V}(t,x)$, may be defined by the expression given in equation (2.11). The candidate

$$f(t) = \begin{cases} 0 & t \in (-\infty, 1] \\ t-1 & t \in [1,2] \\ 1 & t \in [2,\infty) \end{cases} \tag{2.19}$$

is not smooth at the points $t = 1, 2$. However it is possible to smooth f in a neighborhood of these points so that function given by (2.11) with the "smoothed" f approximates arbitrarily closely (2.11) defined with (2.19). Thus for computations we may use (2.19) instead of a smoothed version. The newly constructed function $\overline{V}$ is positive definite and homogeneous. The partial derivatives are

$$\begin{aligned} \frac{\partial \overline{V}}{\partial t}(t,x) &= \int_0^\infty \frac{1}{h^{k+1}} f'(V(t,hx)) \cdot \frac{\partial V}{\partial t}(t,hx)dh \\ \frac{\partial \overline{V}}{\partial x}(t,x) &= \int_0^\infty \frac{1}{h^{k}} f'(V(t,hx)) \cdot \frac{\partial V}{\partial x}(t,hx)dh. \end{aligned} \tag{2.20}$$

As in the computation of $\overline{V}$, the partials in these equations may be approximated as closely as desired since f is modified on a set of arbitrarily small measure and is smooth as required there. Since V is positive definite and nondecreasing for every fixed t the set $\{x \in \mathbb{R} : V(t,x) = 1\}$ consists of two points x_a and x_b for every t. Furthermore, V is symmetric so $|x_a| = |x_b|$. Define $l(t)$ as the magnitude of the points which solve $V(t,x) = 1$ for every $t \in [0, 2\pi)$. Similarly, define $u(t)$ as the magnitude of points which solve $V(t,x) = 2$ for $t \in [0, 2\pi)$. The expressions for the partial derivatives of $\overline{V}$ reduce to

$$\begin{aligned} \frac{\partial \overline{V}}{\partial t}(t,x) &= \int_{\frac{l(t)}{|x|}}^{\frac{u(t)}{|x|}} \frac{1}{h^{k+1}} \frac{\partial V}{\partial t}(t,hx)dh \\ &= |x|^k \int_{l(t)}^{u(t)} \frac{1}{s^{k+1}} \frac{\partial V}{\partial t}(t,s)ds \\ &= |x|^k Q_1(t) \end{aligned} \tag{2.21}$$

$$\begin{aligned} \frac{\partial \overline{V}}{\partial x}(t,x) &= \int_{\frac{l(t)}{|x|}}^{\frac{u(t)}{|x|}} \frac{1}{h^{k}} \frac{\partial V}{\partial t}(t,hx)dh \\ &= \operatorname{sgn}(x)|x|^{k-1} \int_{l(t)}^{u(t)} \frac{1}{s^{k}} \frac{\partial V}{\partial t}(t,s)ds \\ &= \operatorname{sgn}(x)|x|^{k-1} Q_2(t) \end{aligned} \tag{2.22}$$

where we have used the fact that $\partial V/\partial t(t,-x) = \partial V/\partial t(t,x)$ and $\partial V/\partial x(t,-x) = -\partial V/\partial x(t,x)$. The integrals in (2.21) and (2.22) are merely 2π-periodic functions of time, denoted by Q_1 and Q_2. Since $\partial \overline{V}/\partial x(t,x)\dot{x}$ is order $k+2$ then

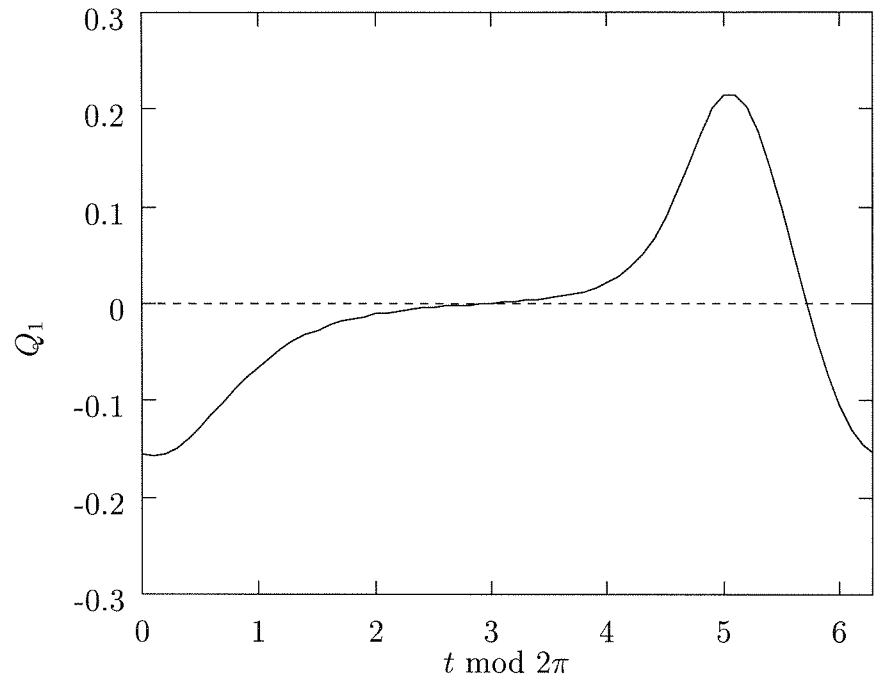

Figure 2.1: 2π-periodic coefficient in $\frac{\partial \overline{V}}{\partial t}$.

for some neighborhood of the origin the term $\partial \overline{V}/\partial t(t,x) = |x|^k Q_1(t)$ dominates the total derivative $d\overline{V}/dt$. Thus, the sign of Q_1 determines the sign of the derivative of $\overline{V}$ along trajectories of (2.14). $Q_1(t)$ is computed numerically from the expressions in equations (2.21) and (2.18). The results are shown in Figure 2.1. $\alpha = 1.1$ in this example. Note that Q_1 changes sign so that $d\overline{V}/dt$ is not sign definite at the origin. Thus, even though $\overline{V}$ is a positive definite homogeneous function it is not a Lyapunov function of the system (2.14).

The most important case for the analysis in this thesis is the converse Lyapunov theorem for degree zero systems.

An important theorem concerning the stability of perturbed degree zero vector fields wraps up this section.

Proposition 2.27 *Let $x = 0$ be an asymptotically stable equilibrium point of the T-periodic continuous homogeneous degree zero vector field $\dot{x} = X(t,x)$. Consider the perturbed system*

$$\dot{x} = X(t,x) + R(t,x). \tag{2.23}$$

Assume each component of $R(t,x)$ may be uniformly bounded by,

$$|R_i(t,x)| \leq m\rho^{r_i+1}(x) \quad i = 1, \ldots, n, \quad x \in U,$$

where U is an open neighborhood of the origin and $\rho(\cdot)$ is a homogeneous norm compatible with the dilation that leaves the unperturbed equation invariant. Then $x = 0$ remains a locally exponentially stable equilibrium of the perturbed equation (2.23).

Proof: By Theorem 2.25, there exists a positive definite, decrescent, T-periodic in t, continuous homogeneous degree $l > 0$ function which is smooth on $\mathbb{R}^n \setminus \{0\}$ with a negative definite derivative along trajectories of the unperturbed equation,

$$\frac{d\overline{V}}{dt}(x,t) = \frac{\partial \overline{V}}{\partial t}(x,t) + \sum_{i=1}^{n} f_i(x,t)\frac{\partial \overline{V}}{\partial x_i} < 0 \quad \forall t, x \neq 0.$$

Note that dV/dt is homogeneous degree l. Setting

$$M = \min_{t\in[0,T),\rho(x)=1} -\frac{dV}{dt}(x,t) > 0,$$

we obtain,

$$\frac{d\overline{V}}{dt} \leq -M\rho^l(x) \quad \forall t, x \neq 0.$$

Evaluating $\overline{V}$ along trajectories of the perturbed equation (2.23),

$$\begin{aligned}\left.\frac{d\overline{V}}{dt}\right|_{(2.23)} &\leq -M\rho^l(x) + \sum_{i=1}^{n} |R_i(x,t)| \left|\frac{\partial \overline{V}}{\partial x_i}(x,t)\right| \\ &\leq -M\rho^l(x) + n\overline{m}m\rho^{l+1}(x) \quad \forall t,\ \forall x,\end{aligned}$$

where the bound $|\partial\overline{V}/\partial x_i| \leq \overline{m}\rho^{l-r_i}(x)$ is derived from the fact that homogeneous degree p functions, continuous on $\mathbb{R}^n \setminus \{0\}$, may be majorized by the homogeneous norm raised to the power p. $\overline{V}$ has the bounds $\alpha\rho^l(x) \leq \overline{V}(t,x) \leq \beta\rho^l(x)$ for some $\alpha, \beta > 0$. Choose $c > 0$ such that

$$-Mc + nm\overline{m}c^{1+1/l} < 0,$$

and define,

$$U = \{x|\rho^l(x) < c\}.$$

Thus for $x \in U,\ t \in [0,T)$, $\overline{V}$ is decreasing along trajectories of (2.23). Start with $x_0 \in \{x|\rho^l(x) < \frac{\alpha}{\beta}c\} \subset U$. The solution $\psi(t,t_0,x_0)$ will remain in U on some interval $t \in [t_0, s]$ with $s > 0$. During this interval,

$$\begin{aligned}\rho^l(\psi(t,t_0,x_0)) &\leq \frac{1}{\alpha}\overline{V}(t,\psi(t,t_0,x_0)) \\ &\leq \frac{1}{\alpha}\overline{V}(t_0,x_0) \\ &\leq \frac{\beta}{\alpha}\rho^l(x_0) \\ &\leq c.\end{aligned}$$

However this implies that the solution remains in U for all $t > t_0$. Since the function $-Mz + n\overline{m}mz^{1+1/l}$ is majorized by $(-M + n\overline{m}mc^{1/l})z$ for $z \in [0, c]$ then the following bound holds for all $t \geq t_0$ and for all $x \in U$,

$$
\begin{aligned}
\left.\frac{d\overline{V}}{dt}\right|_{(2.23)} &\leq -L\rho^l(x) \quad \text{where} L = M - n\overline{m}mc^{1/l} \\
&\leq -\frac{L}{\beta}\overline{V} \\
\implies \overline{V}(\psi(\tau, x, t), \tau) &\leq \overline{V}(\psi(t, x, t), t)e^{-\frac{L}{\beta}(\tau - t)} \\
\implies \rho(\psi(\tau, x, t)) &\leq \left(\frac{\beta}{\alpha}\right)^{1/l} \rho(x)e^{-\frac{L}{l\beta}(\tau - t)} \quad x \in S^1.
\end{aligned}
$$

■

Chapter 3

Analysis Results for Homogeneous Systems

This chapter presents analysis results which are useful for establishing some properties of the closed-loop systems derived in Chapter 4. The feedbacks in this thesis are not Lipschitz functions. Hence, existence but not uniqueness of the system solutions is guaranteed. However, conditions on the feedbacks are given which are sufficient to ensure uniqueness.

An averaging result for time-periodic homogeneous degree zero differential equations is proven. The motivation for this theorem comes from the synthesis approach which uses perturbation arguments to derive exponential stabilizers for driftless systems. For example, a small parameter is introduced into the feedbacks which allows the designer to approximate the system solutions. A set of differential equations of lower dimension is obtained with the parameter as a scale factor multiplying the vector field. This new set of equations is not Lipschitz but is still homogeneous. Since the equations exhibit explicit time dependence, they may be averaged to obtain a "simpler" system. The averaging theorem is applied to conclude asymptotic stability of the original system given asymptotic stability of the averaged system for sufficiently small parameter values.

The synthesis approaches in this thesis rely on Lyapunov analysis rather than approximated solutions of the closed-loop equations so the averaging result is not applied in latter chapters. However it is included because it is a general result for degree zero systems.

3.1 Uniqueness of Solutions

Uniqueness of solutions of ordinary differential equations is an important property for a mathematical model of any physical process. Uniqueness of solutions gives a precise mathematical interpretation of the physical concept of determinism. The models of the driftless systems considered in this thesis are analytic so the only possible way for nonunique solutions to arise occurs when the control designer specifies feedback functions which do not have sufficient regularity to guarantee uniqueness in the closed-loop model. Of course, the physical system implemented with these feedbacks will exhibit deterministic behavior. Thus, the problem is with the mathematical model and its capacity to predict the future behavior of the physical system.

Virtually the only way to analyze the performance of nonlinear control systems is through extensive simulation. The simulations require a mathematical model of the physical process. A numerical simulation of a model with nonunique solutions exhibits discontinuous dependence with respect to initial conditions on any finite time interval. Thus, the numerical simulation may not give a good indication of physical system response.

Homogeneous degree zero systems are of primary interest to us. An example of a degree zero closed-loop driftless system with nonunique solutions is given below but first the *maximal* and the *minimal* solutions of a scalar differential equation are recalled.

Definition 3.1 Consider the scaler differential equation $\dot{y} = f(t, y)$ where f is continuous in $|t - a| \leq T, |y - c| \leq K$. Then there exists a *maximal* and a *minimal* solution $y_M(t)$ and $y_m(t)$ such that $y_m(t) \leq y(t) \leq y_M(t)$ for any other solution $y(t)$ such that $y(a) = y_M(a) = y_m(a)$ [2].

Example 3.2 Consider the three state driftless system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1. \end{aligned} \tag{3.1}$$

This system is the prototype driftless control system which will be used in numerous examples throughout the thesis. An asymptotically stabilizing feedback is,

$$\begin{aligned} u_1 &= -x_1 + \sqrt[s]{x_3} \cos t \\ u_2 &= -x_2 + \sqrt{|x_3|} \sin t, \end{aligned} \tag{3.2}$$

where $\sqrt[s]{\cdot}$ is the "signed" square root,

$$\sqrt[s]{t} = \begin{cases} \sqrt{t} & t > 0 \\ 0 & t = 0 \\ -\sqrt{|t|} & t < 0 \end{cases} .$$

The closed-loop system is homogeneous degree zero with respect to the dilation with the scalings $r = (1, 1, 2)$. The system may be rigorously shown to be ρ-exponential stable. The feedbacks are continuous but not Lipschitz and it is shown below that there are solutions which are not unique. First consider the following differential equation,

$$\dot{y} = \alpha_1(t) + \alpha_2(t) \sqrt[s]{y}, \tag{3.3}$$

where $\alpha_i, i = 1, 2$ are continuous functions. There are two cases to consider,

i) Assume $\alpha_1(0) \geq 0$ and $\alpha_2(0) > 0$. Then there exists some $c > 0$ and $T > 0$ such that $\alpha_2(t) \geq c$ for $t \in I = [0, T]$. The function $c\sqrt[3]{y}$ is a lower bound for the right hand side of equation (3.3) for $t \in I$ and $y \geq 0$. Thus the maximal solution of (3.3) with $y_M(0) = 0$ is an upper bound for all solutions of $\dot{x} = c\sqrt[3]{x}$ with $x(0) = 0$ for $t \in I$ [2, Chapter 6]. Since $x(t) = \frac{1}{4}(ct)^2$ is one solution then $y_M(t) \geq \frac{1}{4}(ct)^2$.

ii) Now assume $\alpha_1(0) \leq 0$ with the same assumptions and bounds for α_2. The function $c\sqrt[3]{y}$ is an upper bound for the right hand side of equation (3.3). Thus the minimal solution of (3.3) with $y_m(0)$ is a lower bound for all solutions of $\dot{x} = c\sqrt[3]{x}$ with $x(0) = 0$ so $y_m(t) \leq -\frac{1}{4}(ct)^2$ for $t \in I$.

To illustrate nonuniqueness of solutions of the original system (3.1) with feedback (3.2) consider the initial conditions $x_1(0) = 0, x_2(0) > 0$ and $x_3(0) = 0$. Let $(x_1(0), x_2(t), x_3(t))$ be a solution with these initial conditions. Regardless of the behavior of $x_1(t)$ and $x_2(t)$, the equation for x_3, i.e., $\dot{x}_3 = x_2(-x_1 + \sqrt[3]{x_3}\cos t)$, may be viewed as the system (3.3) and falls into either case i) or ii). Hence, there must exist at least one solution of the system that satisfies $x_3(t) \geq \frac{1}{4}(ct)^2$ or $x_3(t) \leq -\frac{1}{4}(ct)^2$ for some time interval and some c depending on $x_2(0)$. However, another solution with the same initial condition is $(x_1(t) \equiv 0, x_2(t) = x_2(0)\exp(-t), x_3(t) \equiv 0)$ (this may be verified by direct substitution into the closed-loop equations).

Numerical simulations demonstrating the nonunique behavior are shown in Figure 3.2. The difference in initial conditions in these two simulations is $2e^{-20}$. The solutions *do not* approach one another even as the minute difference in initial conditions is further decreased. Thus, the solutions do not exhibit continuous dependence on the initial conditions.

This situation is to be avoided and we would like to specify conditions on the vector field which guarantees uniqueness. A homogeneous vector field is completely specified by the values assumed on the set $\{x : \rho(x) = 1\}$ so any smoothness imposed on the vector field here is automatically extended to $\mathbb{R}^n \setminus \{0\}$ via the dilation. In order to avoid the uniqueness problems demonstrated above we may assume the vector field to be locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, i.e., for every $x \in \mathbb{R}^n \setminus \{0\}$ there exists a neighborhood of x and some $0 < L < \infty$ such that the vector field satisfies $\|X(t, y) - X(t, z)\| \leq L|y - z|$ for all y and z in this neighborhood. This does not imply that the vector field is Lipschitz in any neighborhood of the origin. This is stated in the following lemma for degree zero vector fields.

Lemma 3.3 *Let $X(t,x)$ be a continuous homogeneous degree zero vector field, Lipschitz on $\mathbb{R}^n \setminus \{0\}$, with the dilation scalings $r_1 = 1 \leq \cdots \leq r_n$. The vector field is not Lipschitz in any neighborhood of the origin if $r_i > 1$ for some i.*

Proof: A vector field is Lipschitz if each component is Lipschitz. Denote the first component as $a(t, x)$. The function a is continuous and homogeneous degree

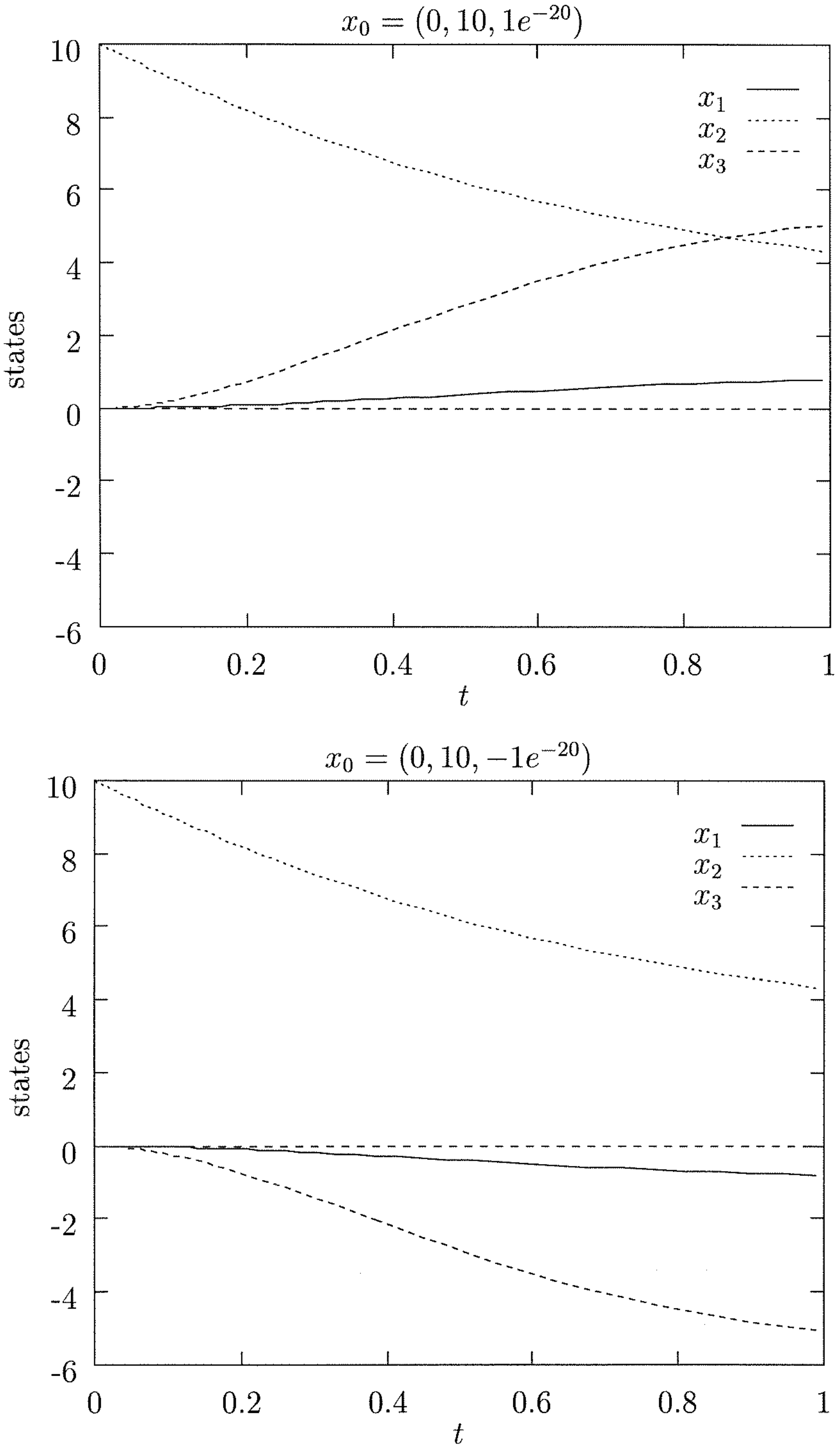

Figure 3.1: Nonunique solutions.

one. Choose $x_0 \in \mathbb{R}^n \setminus \{0\}$ and suppose a has Lipschitz constant L in some neighborhood of $x = (x_1, \ldots, x_n)$. Define the upper right Dini derivative [30],

$$D_i^+ a(x) = \limsup_{h \to 0^+} \frac{a(x_1, \ldots, x_{i-1}, h + x_i, x_{i+1}, \ldots, x_n) - a(x)}{h}.$$

Assume that $D_i^+ a(x) = c \neq 0$. Note that $|c| \leq L$ by virtue of the Lipschitz bound. Furthermore,

$$\begin{aligned} &D_i^+ a(\Delta_\lambda x) \\ &\quad = \limsup_{h \to 0^+} \frac{a(\lambda^{r_1} x_1, \ldots, \lambda^{r_{i-1}} x_{i-1}, h + \lambda^{r_i} x_i, \lambda^{r_{i+1}} x_{i+1}, \ldots, \lambda^{r_n} x_n) - a(\Delta_\lambda x)}{h} \\ &\quad = \limsup_{h \to 0^+} \lambda^{1-r_i} \frac{a(x_1, \ldots, x_{i-1}, h/\lambda^{r_i} + x_i, x_{i+1}, \ldots, x_n) - a(x)}{h/\lambda^{r_i}} \\ &\quad = \lambda^{1-r_i} c. \end{aligned}$$

If $r_i > 1$ then $\lim_{\lambda \to 0} |D_i^+ a(\Delta_\lambda x_0)| \to \infty$. Hence, a cannot be Lipschitz in any neighborhood of zero. ■

Hence even with the assumption that the vector field is Lipschitz on $\mathbb{R}^n \setminus \{0\}$ it is not necessarily Lipschitz at zero. It is still possible to conclude uniqueness of solutions in this case though. This is proven in the next lemma.

Lemma 3.4 *Suppose $X(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is an homogeneous vector field in x of order 0 with respect to a given dilation Δ_λ, uniformly bounded with respect to t and $x = 0$ an isolated equilibrium point. Furthermore suppose that X is locally Lipschitz everywhere except $x = 0$, where it is continuous. Then the flow of X is unique.*

Proof: The point $x = 0$ is the only point where uniqueness may fail since X is not necessarily Lipschitz there. However no solution through a point $p \neq 0$ can reach the origin in finite time because this implies that $\rho(\psi(t, t_0, p) \to 0$ in finite time. This is not possible since the equation describing the evolution of ρ is $\dot{\rho} = Q(t, y)\rho$, where Q is a continuous function of y and uniformly bounded in t. The point y evolves on a compact set so the there always exists a bound

$$M \doteq \sup_{(t,y)} |Q(t, y)|.$$

The following inequalities on ρ hold as a result of the bound on Q,

$$c_1 e^{-M(t-t_0)} \leq \rho(x(t - t_0)) \leq c_2 e^{M(t-t_0)},$$

where the c_i's are positive constants. Similarly a solution cannot leave the origin in finite time. If this were possible then the time reversed vector field (which has the same bounds on $\rho(x(t - t_0))$ as its forward time counter part) has a solution which reaches the origin in finite time. This contradicts the above result. Thus solutions cannot leave or reach the origin in finite time. ■

3.2 Averaging Results

In this section we present an averaging result which will be useful for analyzing the closed-loop equations. First we introduce the class of systems of interest. Consider the differential equation

$$\dot{x} = \epsilon X(t, x, \epsilon), \tag{3.4}$$

where X is a continuous map from $\mathbb{R}^n \times \mathbb{R}^n \times [0, \bar{e})$ into $\mathbb{R}^n$, T-periodic with respect to t and $X(t, 0, \epsilon) = 0$ for all t in $(-\infty, \infty)$. Time is rescaled so that the period is always 2π. We further restrict our attention to a class of homogeneous degree zero vector fields (with respect to the dilation $\Delta_\lambda = (\lambda^{r_1}x_1, \lambda^{r_2}x_2, \ldots, \lambda^{r_n}x_n)$). A solution of (3.4) through the point x_0 at time t_0 is denoted $\psi(t, t_0, x_0)$.

In the averaging theorem we will infer stability (instability) of the zero solution of equation (3.4) from stability (instability) of the zero solution of the *averaged* system,

$$\dot{x} = \epsilon X_0(x), \tag{3.5}$$

where

$$X_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x, 0)dt. \tag{3.6}$$

The vector field in (3.4) is 2π-periodic in t so the average in (3.6) is equivalent to

$$X_0(x) = \frac{1}{2\pi} \int_0^{2\pi} X(t, x, 0)dt.$$

Note that X_0 is homogeneous of order zero with respect to the dilation Δ_λ. Before the averaging result is stated we prove a lemma.

Define the one-parameter family of diffeomorphisms on the extended phase space of equation (3.4) which leave it invariant,

$$\begin{aligned} \Psi_\lambda : S^1 \times \mathbb{R}^n &\to S^1 \times \mathbb{R}^n \\ (t, x) &\mapsto (t, \Delta_\lambda(x)) \qquad \lambda > 0. \end{aligned}$$

We also define three nested homogeneous balls in the extend phase space

$$B_{c_i} = \{(t, x) \in S^1 \times \mathbb{R}^n | \rho(x) \leq c_i\} \quad i = 1, 2, 3,$$

where $c_1 > c_2 > c_3 > 0$ and ρ is a homogeneous norm compatible with Δ_λ.

Lemma 3.5 (Scaling Lemma). *For time periodic homogeneous order zero vector fields (3.4) and given the B_{c_i}'s defined above, suppose we know the following facts,*

1. $(t_0, x_0) \in B_{c_2}$ *implies* $(t, \psi(t, t_0, x_0)) \in B_{c_1}$ *for all* $t > t_0$,

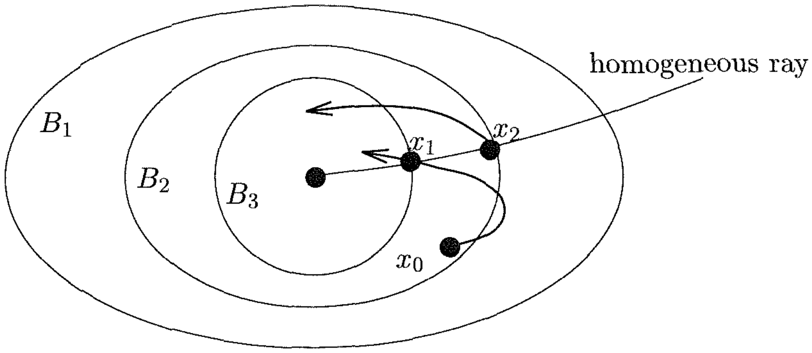

Figure 3.2: Homogeneous balls used in proof: use the dilation to map x_2 to x_1 thus extending the trajectory starting at x_0.

2. *there exists a* $\overline{T} > 0$ *such that* $(t_0, x_0) \in B_{c_2}$ *implies* $(t, \psi(t, t_0, x_0)) \in B_{c_3}$ *for all* $t > \overline{T}$,
3. *the trajectories of the system (3.4) are unique.*

Then the zero solution of (3.4) is asymptotically stable.

Proof: We first prove stability. Start the system (3.4) with initial conditions in B_{c_2}. Then $(t, \psi(t, t_0, x_0)) \in B_{c_1}$ for all $t > t_0$. In other words, $\rho(x_0) < c_2$ implies $\rho(\psi(t, t_0, x_0)) < c_1$ for all $t > t_0$. This condition may be extend to any neighborhood using the mapping Δ_λ. Suppose the bound $\rho(\psi(t, t_0, x_0)) < e_1$ for all $t > t_0$ is desired, then restrict $\rho(x_0) < e_1 \frac{c_2}{c_1}$. This is demonstrated below,

$$\begin{aligned}
\rho(\psi(t, t_0, x_0)) &= \rho\left(\psi(t, t_0, \Delta_{e_1/c_1}(\Delta_{c_1/e_1}(x_0)))\right) \\
&= \rho\left(\Delta_{e_1/c_1}\left(\psi(t, t_0, (\Delta_{c_1/e_1}(x_0)))\right)\right) \\
&= \frac{e_1}{c_1}\rho(\psi(t, t_0, \Delta_{\frac{c_1}{e_1}}(x_0))) \\
&\leq e_1 \quad \text{since } \rho(\Delta_{c_1/e_1}(x_0)) = \frac{c_1}{e_1}\rho(x_0) < \frac{c_1}{e_1}\frac{c_2}{c_1}e_1 = c_2.
\end{aligned} \tag{3.7}$$

This is stability of the zero solution. See Figure 3.2 for a picture.

To demonstrate asymptotic stability we proceed in a similar manner. Define the annulus,

$$A_2 = B_{c_2} \setminus B_{c_3}.$$

We know that solutions with initial conditions in A_2 enter B_{c_3} in finite time $\overline{T}$ and remain there. Since the differential equation is invariant under Ψ then we may map the annulus A_2 to another annulus that sits inside A_2 and shares a common boundary. This way solutions starting in A_2 are extended into the new annulus (because of the invariance) and can only remain in the new annulus for a finite time. Since the system trajectories are unique then the extended trajectory must be the continuation of the initial trajectory.

Define the sequence of nested annuli,

$$A_i = \Psi_{c_3/c_2}(A_{i-1}), \qquad i = 3, 4, \ldots.$$

Note that the outer boundary of A_i, denoted $\partial^o A_i$, is the inner boundary of A_{i-1}, denoted $\partial^i A_{i-1}$, because,

$$\begin{aligned}
\partial^o A_i &= \Psi_{c_3/c_2}(\partial^0 A_{i-1}) \\
&= \Psi_{(c_3/c_2)^{i-2}}(\partial^0 A_2) \\
&= \Psi_{(c_3/c_2)^{i-3}}\left(\Psi_{c_3/c_2}(\partial^o A_2)\right) \\
&= \Psi_{(c_3/c_2)^{i-3}}\left(\partial^i A_2\right) \\
&= \partial^i A_{i-1}.
\end{aligned}$$

The properties of solutions with initial conditions in A_2 are shared by the other annuli. Hence, an initial condition in A_i must enter A_{i+1} in time $\overline{T}$. Extending this to the larger annulus defined by

$$\Sigma_N = \bigcup_{i=2,\ldots,N} A_i, \qquad N > 2$$

implies that solutions with initial conditions here will enter the set A_{N+1} in a time no less that $N\overline{T}$ and can never reenter Σ_N. Thus we pick $\Delta = c_2$ and for $\epsilon > 0$ choose $\tilde{t} = m\overline{T}$ where m satisfies $\left(\frac{c_3}{c_2}\right)^m < \epsilon$. This is equivalent to asymptotic stability. ∎

Remark 3.6 This lemma actually demonstrates *exponential* stability of the zero solution because the time taken to leave any given annulus is independent of the "size" of the annulus (this is a result of the vector field having degree zero with respect to the dilation Δ_λ). At time $t > m\overline{T}$ any solution may be bounded by a homogeneous ball with size proportional to $\left(\frac{c_3}{c_2}\right)^m$. Hence, this bound plus stability of the solutions may be recast as an exponential stability result with respect to the homogeneous norm ρ.

Theorem 3.7 *Assume that the solutions of the equation (3.4) are unique. Suppose $y = 0$ is an asymptotically stable fixed point of the associated averaged system $\dot{y} = \epsilon X_0(y)$. Then for $\epsilon > 0$ sufficiently small, the solution $x = 0$ is exponentially stable for the full equations (3.4).*

This result is already well known for C^1 vector fields where $x = 0$ is a hyperbolic fixed point. Proving the theorem when the vector field is differentiable is straightforward since the standard averaging change of coordinates places the vector field into a form where the time-varying part is bounded with an arbitrarily small Lipschitz constant (by making ϵ sufficiently small). Hence, if $x = 0$ is a hyperbolic fixed point of X_0 then the stability of the full system is determined by the stability of X_0 for ϵ sufficiently small. Unfortunately this proof does not extend to our case since the averaging change of coordinates tends to "mix" the new coordinates so that the transformed vector field is no longer homogeneous. However we may get a *total stability* result in the new coordinates which will imply certain strong behavior of the solutions of the original homogeneous system. The idea of the proof uses the fact that in the new coordinates we may choose ϵ small enough so that we may make a ball about the origin attractive and invariant. Mapping this ball back to the original coordinates implies the same for solutions of equation (3.4). Now we may use the homogeneity of the vector field to extend the solutions to an arbitrarily small attractive neighborhood of the origin. The details are now presented.

Proof: We first recall the usual averaging results. The reader is referred to Hale [15] (Lemma V3.1, Lemma V3.2 and Lemma 5 of the appendix). For any compact set Ω in $\mathbb{R}^n$ there exists an ϵ_0 and a function $u(t, x, \epsilon)$ such that the averaging transformation,

$$x = y + \epsilon u(t, y, \epsilon) \quad (t, y, \epsilon) \in \mathbb{R} \times \Omega \times [0, \epsilon_0), \tag{3.8}$$

applied to (3.4) yields the equation

$$\dot{y} = \epsilon f_0(y) + \epsilon F(t, y, \epsilon), \tag{3.9}$$

where X_0 is the averaged vector field as defined above. $F(t, y, \epsilon)$ is continuous for $(t, y, \epsilon) \in \mathbb{R} \times \Omega \times [0, \epsilon_0)$ and $F(t, y, 0) = 0$. The function u possesses the following properties on $\mathbb{R} \times \Omega \times [0, \epsilon_0)$:

1. $u(t, x, \epsilon)$ is periodic with period 2π (same period as the vector field),
2. has continuous derivatives with respect to t and derivatives of an arbitrary specified order with respect to x.
3. ϵu and $\epsilon \frac{\partial u}{\partial y}$ approach 0 as $\epsilon \to 0$ uniformly in $t \in \mathbb{R}^n$ and $y \in \Omega$.

The solution $y = 0$ of (3.5) is asymptotically stable so there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ with the following properties [22],

1. V is as smooth,
2. $V(0) = 0, V(y) > 0$ for all $y \neq 0$, and V is radially unbounded,
3. $\nabla V \cdot f_0(y) < 0$ for all $x \neq 0$.

Consider the compact sets defined by

$$D_\alpha = \{y \in \mathbb{R}^n | V(y) \leq \alpha\} \quad \alpha > 0.$$

The boundaries of these sets are denoted ∂D_α. Given D_α, define constants

$$\overline{\sigma}_{D_\alpha} = \max_{y \in \partial D_\alpha} \rho(y) \qquad \underline{\sigma}_{D_\alpha} = \min_{y \in \partial D_\alpha} \rho(y).$$

Choose $c_1 > 0$ such that $D_{c_1} \subset \Omega$. Now find c_2, and corresponding D_{c_2}, such that $\overline{\sigma}_{D_{c_2}} < \underline{\sigma}_{D_{c_1}}/2$. This may always be done because V is positive definite and continuous. Evaluating V along solutions of the transformed vector field (3.9) yields

$$\frac{dV}{dt} = \epsilon\nabla V \cdot X_0(x) + \epsilon\nabla V \cdot F(t, y, \epsilon).$$

On the compact set $D_{c_1} \setminus D_{c_2}$ calculate

$$\beta = \min_{y\in D_{c_1}\setminus D_{c_2}} -\nabla V \cdot f_0(y),$$

which is clearly greater than zero. We also define $M(\epsilon)$ as

$$M(\epsilon) = \max_{y\in D_{c_1},\ t\in S^1} |\nabla V \cdot F(t, y, \epsilon)|.$$

$M(\epsilon)$ is continuous because F is a continuous function of ϵ and $M(0) = 0$ since $F(\cdot, \cdot, 0) = 0$. The averaging transformation will not, in general, respect the dilation scaling. Hence the vector field $F(t, x, \epsilon)$ will not be homogeneous. For example we may be forced to bound F with homogeneous functions of lower order than X_0 and hence asymptotic stability cannot be concluded with this Lyapunov analysis. On the annulus $D_{c_1} \setminus D_{c_2}$ the time derivative of V is bounded by

$$\frac{dV}{dt} \le \epsilon(-\beta + M(\epsilon)).$$

Now choose $\tilde{\epsilon} \in (0, \epsilon_0)$ such that $M(\tilde{\epsilon}) \le \frac{\beta}{2}$. The choice of $\tilde{\epsilon}$ renders D_{c_1} and D_{c_2} invariant. Trajectories through points in $D_{c_1} \setminus D_{c_2}$ will reach D_{c_2} in a finite time no greater than

$$\overline{T} = \frac{2(c_1 - c_2)}{\tilde{\epsilon}\beta},$$

because $\dot{V} < -\epsilon\beta/2$ on $D_{c_1} \setminus D_{c_2}$. Choosing any $\epsilon \in (0, \tilde{\epsilon})$ does not change the invariance or attractive nature of the sets. The only modification in this case is $\overline{T}$. The functional relationship of $\overline{T}$ is exactly the one given above with $\tilde{\epsilon}$ replaced by the new ϵ. In the y-coordinates we can't say anything more about the stability of the zero solution. However, we may map the D_i's back to the extended phase space of (3.4) with the diffeomorphism (3.8). This will result in a warped version of $S^1 \times D_i$'s. We would like to bound these warped sets with homogeneous balls and apply the scaling lemma to conclude asymptotic stability. This is worked out in detail below.

Recall the map $x = y + \epsilon u(t, y, \epsilon)$ is at least a C^1 diffeomorphism for $(t, y, \epsilon) \in S^1 \times \Omega \times\ [0, \epsilon_0)$. As $\epsilon \to 0$ this map approaches the identity. Since u is 2π periodic in t it is useful to define the following diffeomorphism between $S^1 \times \Omega$ and the extended phase space of the vector field in (3.4), $S^1 \times \mathbb{R}^n$,

$$\varphi_\epsilon(t, y) = (t, y + \epsilon u(t, y, \epsilon)).$$

Define the compact sets in $S^1 \times \mathbb{R}^n$,

$$E_{c_1} = \varphi_\epsilon(t, D_{c_1}) \qquad E_{c_2} = \varphi_\epsilon(t, D_{c_2}).$$

For fixed t, $E_{c_1} \to (t, D_{c_i})$ as $\epsilon \to 0$. The boundaries of E_{c_i} are denoted ∂E_{c_i}. As for the sets D_α, we define the quantities,

$$\overline{\sigma}_{E_{c_i}} = \max_{(t,x)\in\partial E_{c_i}} \rho(x) \qquad \underline{\sigma}_{E_{c_i}} = \min_{(t,x)\in\partial E_{c_i}} \rho(x).$$

Note that

$$\overline{\sigma}_{E_{c_i}} \to \overline{\sigma}_{D_{c_i}} \qquad \underline{\sigma}_{E_{c_i}} \to \underline{\sigma}_{D_{c_i}}, \tag{3.10}$$

as $\epsilon \to 0$ since $\partial E_{c_i} \to (t, \partial D_{c_i})$ for each $t \in S^1$. It is possible for $\overline{\sigma}_{E_{c_2}} > \underline{\sigma}_{E_{c_1}}$ for the choice of $\tilde{\epsilon}$ made above (at different times of course). The relations in (3.10) imply ϵ may be further decreased to ensure $\overline{\sigma}_{E_{c_2}} < \underline{\sigma}_{E_{c_1}}$ since $\overline{\sigma}_{E_{c_2}} \to \overline{\sigma}_{D_{c_2}}$ as $\epsilon \to 0$ (recall $\overline{\sigma}_{D_{c_2}} < \underline{\sigma}_{D_{c_1}}/2$ from the choice of c_1 and c_2). Hence $\partial E_{c_2} \cap \partial E_{c_1} = \emptyset$. Now we may define homogeneous balls that are proper subsets of one another. Define the homogeneous balls in $S^1 \times \mathbb{R}^n$,

$$\begin{aligned}
B_{\underline{\sigma}_{E_{c_1}}} &= \{(t,x) \in S^1 \times \mathbb{R}^n \,|\, \rho(x) \leq \underline{\sigma}_{E_{c_1}}\} \\
B_{\overline{\sigma}_{E_{c_1}}} &= \{(t,x) \in S^1 \times \mathbb{R}^n \,|\, \rho(x) \leq \overline{\sigma}_{E_{c_1}}\} \\
B_{\overline{\sigma}_{E_{c_2}}} &= \{(t,x) \in S^1 \times \mathbb{R}^n \,|\, \rho(x) \leq \overline{\sigma}_{E_{c_2}}\}.
\end{aligned}$$

The previous choice of ϵ leads to the following inclusions,

$$E_{c_2} \subset B_{\overline{\sigma}_{E_{c_2}}} \subset B_{\underline{\sigma}_{E_{c_1}}} \subset E_{c_1} \subset B_{\overline{\sigma}_{E_{c_1}}}.$$

Now we will say a few words about solutions with initial conditions in these sets. E_{c_1} is invariant under (3.4) because D_{c_1} is invariant under (3.9) and the diffeomorphism (3.8) takes D_{c_1} into E_{c_1}. Furthermore, solutions of (3.4) with initial conditions in E_{c_1} will reach the set E_{c_2} in no less than time $\overline{T}$ and remain thereafter because these corresponding facts hold for D_{c_1} and D_{c_2} and the (3.9) maps D_{c_2} to E_{c_2}. Hence, solutions through points in $B_{\underline{\sigma}_{E_{c_1}}}$ are constrained to remain in $B_{\overline{\sigma}_{E_{c_1}}}$ and furthermore must enter $B_{\overline{\sigma}_{E_{c_2}}}$ in finite time, $\overline{T}$, and remain there. Now apply Lemma 3.5 with $B_{c_1} = B_{\overline{\sigma}_{E_{c_1}}}$, $B_{c_2} = B_{\underline{\sigma}_{E_{c_1}}}$ and $B_{c_3} = B_{\overline{\sigma}_{E_{c_2}}}$ to conclude asymptotic stability of the zero solution. ∎

Remark 3.8 The same arguments may be used to show that trajectories moving from B_3 to the outer boundary of B_2 imply the origin is unstable. Furthermore this theorem is only ***sufficient*** to guarantee ρ-exponential stability of the original system. For example, the homogeneous degree zero (with respect to $r = (1,1,2)$) system

$$\begin{aligned}
\dot{x}_1 &= \epsilon(-x_1 + \frac{x_3}{\rho(x)} \cos t) \\
\dot{x}_2 &= \epsilon(-x_2 \pm \frac{x_3^2}{\rho^3(x)} \sin t) \\
\dot{x}_3 &= \epsilon x_2(-x_1 + \frac{x_3}{\rho(x)} \cos t) \\
\rho(x) &= (x_1^4 + x_2^4 + x_3^2)^{1/4},
\end{aligned}$$

is ρ-exponentially stable for the choice of "+" in $\dot{x}_2$ and unstable for the "−" case even though the averaged system for both cases is *stable* (but not asymptotically stable),

$$\begin{aligned}\dot{x}_1 &= -\epsilon x_1\\ \dot{x}_2 &= -\epsilon x_2\\ \dot{x}_3 &= -\epsilon x_1 x_2.\end{aligned}$$

This is in contrast to the results for C^2 systems where local exponential stability of the averaged system is necessary and sufficient for local exponential stability of the original system for ϵ sufficiently small.

The following example illustrates the application of the averaging theorem.

Example 3.9 Consider the following ordinary differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{3}x_1 + \frac{x_2}{\rho(x)}\cos^2 t \\ -\frac{1}{2}\frac{x_1^2 x_2}{\rho(x)} + x_1^2\cos t + x_2 \sin t \end{pmatrix} \quad \epsilon > 0, \tag{3.11}$$

where $\rho(x) = (x_1^4 + x_2^2)^{1/4}$. This system is homogeneous with respect the dilation $\Delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2)$, smooth on $\mathbb{R}^n \setminus \{0\}$, 2π-periodic with respect to t and not Lipschitz in any neighborhood of the origin. Uniqueness of solutions follows from Lemma 3.4. The averaged system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{3}x_1 + \frac{1}{2}\frac{x_2}{\rho} \\ -\frac{1}{2}\frac{x_1^2 x_2}{\rho} \end{pmatrix}. \tag{3.12}$$

A positive definite function and its derivative along solutions of the averaged system are,

$$\begin{aligned} V &= x_1^4 + x_2^2 \\ \frac{dV}{dt} &= \epsilon\left(-\frac{4}{3}x_1^4 + 2\frac{x_1^3 x_2}{\rho} - \frac{x_1^2 x_2^2}{\rho^2}\right). \end{aligned}$$

Since both functions are homogeneous with respect to Δ_λ, each function is uniquely determined by its values on the homogeneous sphere S^1_Δ. A plot of dV/dt on S^1_Δ (parametrized by the angle θ from the positive x_1 axis in a counter-clockwise direction) is shown in Figure 3.3 (ϵ is taken to be 1 in this plot since it only scales the value of dV/dt). dV/dt is negative semidefinite so asymptotic stability cannot be concluded without further analysis. However we show below that the system is asymptotically (and hence ρ-exponentially) stable by invoking LaSalle's theorem. The set in $\mathbb{R}^n \setminus \{0\}$ at which $dV/dt = 0$ are the points where $dV/dt = 0$ on S^1_Δ scaled with the dilation for all $\lambda > 0$. This set is invariant if the vector field is tangent to this set. A necessary condition for this occurrence is that the inner product between a tangent vector to this set and a normal vector to the vector field at this point be zero. A tangent vector to the set where $dV/dt = 0$ is just the Euler homogeneous vector field evaluated at the correct point: $X_E = x_1\partial/\partial x_1 + 2x_2\partial/\partial x_2$. On the other hand,

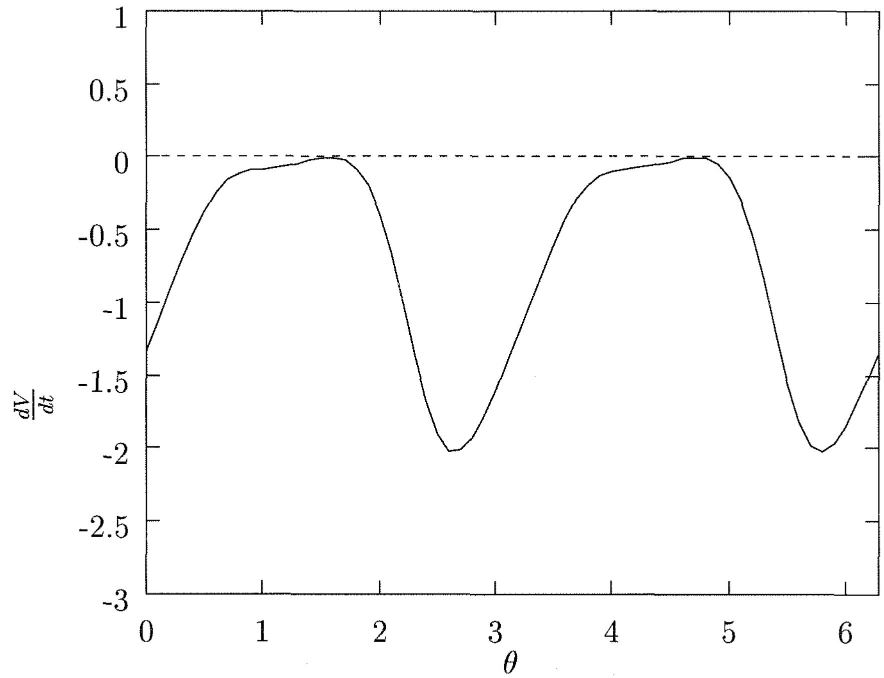

Figure 3.3: Lyapunov function derivative on S^1_Δ.

a normal vector to the vector field at the point where $dV/dt = 0$ is merely $D_x V$. Since $\langle X_E, D_x V \rangle(x) = 4V(x)$ then the vector field is always transverse to the set where $dV/dt = 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Hence the system is ρ-exponentially stable. Thus we conclude that the original system is ρ-exponentially stable for ϵ sufficiently small. Figure 3.4 compares the solutions of the original and averaged system for $\epsilon = 0.1$. The simulation in Figure 3.5 verifies that the system (3.11) is unstable for $\epsilon = 1$.

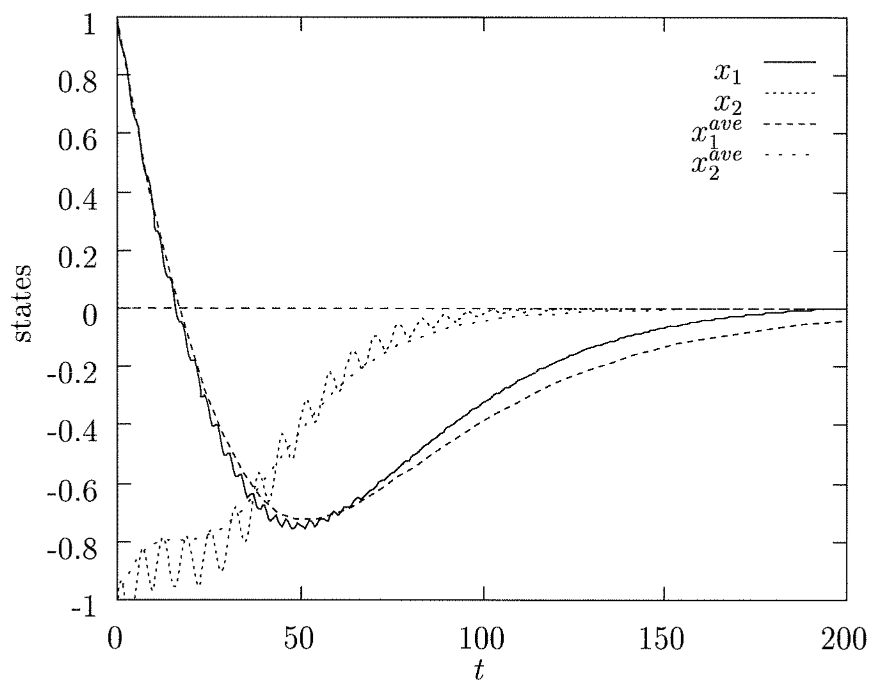

Figure 3.4: Simulation of averaged system (3.12) and original system (3.11) for $\epsilon = 0.1$.

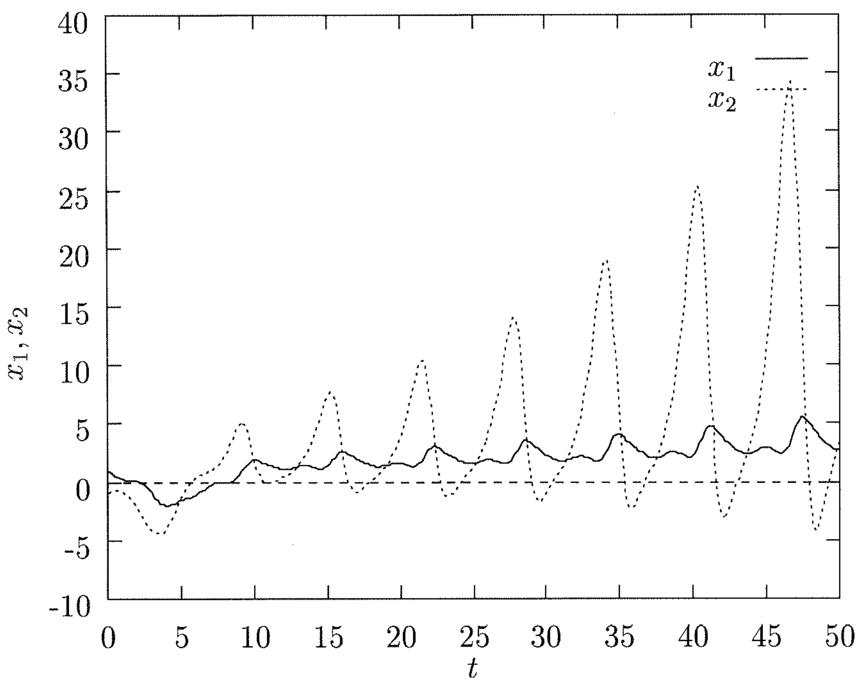

Figure 3.5: System (3.11) is unstable for $\epsilon = 1$.

Chapter 4

Applications to Driftless Control Systems

The objective of this chapter is to apply the material introduced in the previous chapters to produce algorithms that yield ρ-exponential stabilizers for the driftless control system,

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m. \tag{4.1}$$

However, before discussing the algorithms, Section 4.1 shows that the rate of convergence of an asymptotically stabilizing Lipschitz feedback cannot be bounded by a decaying exponential envelope. The algorithms rely on a local homogeneous approximation of the input vector fields. Section 4.2 applies the approximation procedure reviewed in Section 2.2 to driftless systems. The first algorithm discussed in Section 4.3 is an extension of a result by Pomet [37]. The second algorithm gives sufficient conditions for a smooth stabilizer to be modified to a ρ-exponential stabilizer. Both algorithms yield functions which are homogeneous degree one with respect to the dilation specified during the approximation process. The dilation must have some scaling power greater than 1 since a least one level of Lie brackets are required for controllability. The feedbacks are not Lipschitz at the origin in this case. However, given the fact that Lipschitz feedback cannot exponentially stabilize a driftless system it is remarkable that the non-Lipschtiz nature of the ρ-exponential stabilizers is a result of requiring the functions to be degree one with respect to the dilation. Choosing the feedbacks to be homogeneous is natural since it preserves the homogeneous structure of the approximation.

The chapter ends on a practical aspect of ρ-exponential stabilizer design. Many driftless systems are based on kinematic models of mechanical systems. The control inputs are velocities for these models. The velocities of mechanical systems cannot be exactly specified since the control action is realized be the application of forces. Section 4.4.2 proves that ρ-exponential stabilizers may be extended to systems with actuator dynamics modeled by integrators. The extended controllers command forces and still ρ-exponentially stabilize the system. This section also demonsrates that filtering the state measurements do not destabilize the system if the filter bandwidth is sufficiently high.

4.1 Limitations of Lipschitz Feedback

Before discussing various synthesis methods a result on the regularity of exponentially stabilizing feedback functions is proven. In particular it is shown that feedbacks which are Lipschitz in the state cannot exponentially stabilize, in the usual sense, a controllable driftless system to a point. The following theorem is the main result of this section.

Theorem 4.1 *Suppose the input vector fields of the driftless system (4.1) are* C^1 *and the feedbacks* $u_i(t,x), i = 1, \ldots, m$, *which are measurable in* t *and Lipschitz with respect to* x, *asymptotically stabilize the point* $x = 0$. *Then there does* not *exist an* $\beta > 0$ *and* $\alpha > 0$ *such that,*

$$\|\psi(t, t_0, x_0)\|_2 \leq \beta \|x_0\|_2 e^{-\alpha(t-t_0)}.$$

This theorem states that in order to achieve exponential stability the feedback must necessarily be non-Lipschitz.

Before proving Theorem 4.1, some results from nonsmooth analysis will be reviewed [6].

Definition 4.2 The *generalized Jacobian* at $x \in \mathbb{R}^n$ of a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ is defined as the set:

$$\partial F(x) \doteq \text{co} \, \{\lim DF(x_i) | x_i \to x, x_i \notin \Omega_F\},$$

where Ω_F is the set of measure zero where the standard Jacobian of F, DF, is not defined.

In general, ∂F is a set valued map when F is Lipschitz but not C^1. Set valued maps are also called *multifunctions.* Some useful properties of ∂F are:

i) ∂F is upper semicontinuous and

ii) $\partial F(x)$ is a convex compact subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$.

Additional properties are given in [6].

When $X(t,x)$ is measurable in t and Lipschitz in x denote the flow of the corresponding differential equation $\dot{x} = X(t,x)$ as $\psi(t,\tau,x)$. If X is not C^1 in x then there is no notion of the classical linearization about any solution. However using the definition of generalized Jacobian a natural extension of the linearization is called a *differential inclusion.*

Definition 4.3 The *linearization* of X about the trajectory $\psi(t,\tau,x)$ is represented by the differential inclusion

$$\dot{y}(s) \in \partial_x X(s, \psi(s,\tau,x)) y(s), \quad s \in [\tau, t].$$

The right-hand side, $\partial_x X(s, \psi(s, \tau, x))$, is a set valued map which depends on the parameter s and as a consequence there is a *set* of "solutions" of the differential inclusion associated with any given trajectory $\psi(t, \tau, x)$ of the original system. The solutions of the differential inclusion are defined in the following manner. A *measurable selection* of $\partial_x X(s, \psi(s, \tau, x))$ is a measurable function $\gamma : [\tau, t] \to \mathbb{R}^n$ such that $\gamma(s) \in \partial_x X(s, \psi(s, \tau, x))$. The existence of such functions is guaranteed by the hypothesis on X. Define $\phi(t, \tau)$ as the set of all linear matrix solutions to the system,

$$\dot{Y}(s) = \gamma(s)Y(s), \quad Y(\tau) = I,$$

for some measurable selection γ. The *plenary hull* of $\phi(t, \tau)$, denoted $R(t, \tau)$, is the set

$$R(t, \tau) = \{M | \langle v, Mw \rangle \leq \max[\langle v, Nw \rangle | N \in \phi(t, \tau)] \; \forall v, w \in \mathbb{R}^n\}. \quad (4.2)$$

The utility of the preceding definitions becomes apparent in the following relationship between the generalized Jacobian of the flow and the plenary hull,

Theorem 4.4 ([6],Theorem 7.4.1) *The map $F(x) \doteq \psi(t, \tau, x)$ is Lipschitz for all t, τ and satisfies $\partial F(x) \subset R(t, \tau)$.*

The idea behind the proof of Theorem 4.1 is as follows. If the flow ψ satisfies an exponential stability criterion then there exist elements in ∂F that cannot be in R since R has a special form for driftless systems with Lipschitz feedback. This contradicts the statement of Theorem 4.4. Theorem 4.1 is proven using two propositions.

Proposition 4.5 *For the system (4.1) with Lipschitz feedback, there exists a set of coordinates such that all elements of $R(t, \tau)$ about the solution $x(t) = 0$ have the form,*

$$\begin{bmatrix} \star & \star \\ \mathbf{0}_{(n-r)\times r} & I_{n-r} \end{bmatrix}, \quad (4.3)$$

where $\mathbf{0}_{(n-r)\times r}$ is an $(n - r) \times r$ matrix of zeros, and I_{n-r} is an $n - r$ identity matrix.

Proof: Define the matrix $B \in \mathbb{R}^{n \times m}$,

$$B = \begin{bmatrix} X_1(0) & X_2(0) & \dots & X_m(0) \end{bmatrix},$$

and $r = \text{rank}(B) \leq m$. The closed-loop system is denoted $\dot{x} = X(t, x)$. We first show that the generalized Jacobian of $f(t, x)$ with respect to x has the form,

$$\partial X(t, 0) = B \partial U(t, 0),$$

where $\partial U(t, 0)$ is the generalized Jacobian (with respect to x) of the control map

$$U(t, x) = (u_1(t, x), \dots, u_m(t, x)).$$

Define the C^1 map,

$$\begin{aligned} G : \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{R}^n \\ (z, v) &\mapsto X_1(z)v_1 + \cdots + X_m(z)v_m, \end{aligned}$$

and the Lipschitz map with parameter t,

$$\begin{aligned} E_t : \mathbb{R}^n &\to \mathbb{R}^n \times \mathbb{R}^m \\ y &\mapsto (y, U(t, y)). \end{aligned}$$

$E_t(0) = 0$ since $x = 0$ is an equilibrium point of the closed-loop system. Applying the generalized Jacobian chain rule to,

$$X(t, x) = G \circ E_t(x),$$

results in $\partial X(t, 0) = DG(E_t(0))\partial E_t(0)$ [6, Theorem 2.6.6]. G is continuously differentiable so $DG(E_t(0)) = (\mathbf{0}_{n\times n}|B)$. Using the definition of generalized Jacobian,

$$\begin{aligned} \partial E_t(0) &= \text{co }\{\lim DE_t(x_i) \,|\, x_i \to 0, x_i \notin \Omega_U\} \\ &= \text{co }\left\{\lim \begin{bmatrix} I_{n\times n} \\ D_x U(t, x_i) \end{bmatrix} \,|\, x_i \to 0, x_i \notin \Omega_U \right\} \\ &= \begin{bmatrix} I_{n\times n} \\ \text{co }\{\lim D_x U(t, x_i) \,|\, x_i \to 0, x_i \notin \Omega_U\} \end{bmatrix} \\ &= \begin{bmatrix} I_{n\times n} \\ \partial U(t, 0) \end{bmatrix}. \end{aligned}$$

The set Ω_U is a set of measure zero where the derivative of U with respect to x is not defined. Hence,

$$\begin{aligned} \partial X(t, 0) &= [\mathbf{0}_{n\times n} \,|\, B] \begin{bmatrix} I_{n\times n} \\ \partial U(t, 0) \end{bmatrix} \\ &= B \cdot \partial U(0, t). \end{aligned}$$

Thus, the differential inclusion of equation (4.1) about the solution $x(t) = 0$ is the system,

$$\dot{y}(s) \in B \cdot \partial U(s, 0) y(s).$$

B is a rank r matrix and may be expressed in suitable coordinates as,

$$B = \begin{bmatrix} \star \\ \mathbf{0}_{(n-r)\times m} \end{bmatrix}.$$

Thus, any measurable selection of $B\partial U(0, t)$ is a function of the form,

$$\gamma(t) = \begin{bmatrix} \star \\ \mathbf{0}_{(n-r)\times n} \end{bmatrix},$$

and all so elements of $\phi(t,\tau)$ must fix the last $n-r$ coordinates directions,

$$\begin{bmatrix} \star & \star \\ \mathbf{0}_{(n-r)\times r} & I_{n-r} \end{bmatrix}.$$

Let M represent an element of the plenary hull of $\phi(t,\tau)$. Partition M as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $M_{11} \in \mathbb{R}^{r\times r}$, $M_{12} \in \mathbb{R}^{r\times(n-r)}$, $M_{21} \in \mathbb{R}^{(n-r)\times r}$, $M_{22} \in \mathbb{R}^{r\times r}$. Choosing the vectors v and w in definition (4.2) as

$$v = \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix}, \tilde{v} \in \mathbb{R}^{n-r} \qquad w = \begin{pmatrix} \tilde{w} \\ 0 \end{pmatrix}, \tilde{w} \in \mathbb{R}^{r},$$

implies

$$\langle \tilde{v}, M_{21}\tilde{w}\rangle \leq 0 \Rightarrow M_{12} = \mathbf{0}_{(n-r)\times r}.$$

Similarly,

$$\langle \tilde{v}, M_{22}\tilde{w}\rangle \leq \langle \tilde{v}, \tilde{w}\rangle \quad \forall \tilde{v} \in \mathbb{R}^{n-r}, \tilde{w} \in \mathbb{R}^{n-r} \Rightarrow M_{22} = I_{n-r}.$$

Thus any element of $R(t,\tau)$ must have the form

$$\begin{bmatrix} \star & \star \\ \mathbf{0} & I_{n-r} \end{bmatrix}.$$

■

The last proposition required for the proof of Theorem 4.1 is established next. Assuming that the closed-loop system (4.1) with Lipschitz feedback is exponentially stable then there exists the following bound on the solutions,

$$\|\psi(t,\tau,x)\| \leq \beta\|x\|e^{-\alpha(t-\tau)},$$

where $\alpha > 0$ and $\beta > 0$. The difference $t-\tau$ may be chosen large enough so that the constant $\beta e^{\alpha(t-\tau)} \leq 1/2$. The map $F(x) \doteq \psi(t,\tau,x)$ then satisfies,

$$\|F(x)\| \leq \frac{1}{2}\|x\|. \tag{4.4}$$

This bound leads to the last proposition.

Proposition 4.6 *Suppose a Lipschitz map $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the bound (4.4). Then for any $v \in \mathbb{R}^n$ there exists $Z \in \partial F(0)$ such that $\|Zv\| \leq 1/2\|v\|$.*

The proof of this proposition uses a mean value theorem for set valued maps,

Theorem 4.7 ([6],Proposition 2.6.5) *Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map then,*

$$F(y) - F(x) \in co\ \partial F([x, y])(y - x),$$

where the set $co\ \partial F([x, y])$ is the convex hull of all points in $\partial F(z)$ with z on the straight line segment joining x and y.

Proof of Proposition 4.6: We first show that given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{co}\ \partial F([y, x]) \subset \partial F(0) + \epsilon \boldsymbol{B}, \forall |x| < \delta, |y| < \delta,$$

where $\boldsymbol{B}$ is the unit ball of $n \times n$ matrices. From the upper semicontinuity of the generalized Jacobian, given $\epsilon > 0$ there exists $\delta > 0$ such that,

$$\partial F(x) \subset \partial F(0) + \epsilon \boldsymbol{B}, \forall |x| < \delta.$$

Pick x and y with norm less than δ and choose arbitrary elements $X \in \partial F(x), Y \in \partial F(y)$. Combining the following relationships,

$$\begin{aligned} tX &\in t(\partial F(0) + \epsilon \boldsymbol{B}) \\ (1 - t)Y &\in (1 - t)(\partial F(0) + \epsilon \boldsymbol{B}), \end{aligned}$$

yields with $t \in [0, 1]$,

$$tX + (1 - t)Y \in \text{co}\ \{\partial F(0) + \epsilon \boldsymbol{B}\}.$$

However, the set $\{\partial F(0) + \epsilon \boldsymbol{B}\}$ is convex since $\partial F(0)$ is convex. Thus the convex combination of any matrices in $\partial F(x)$ and $\partial F(y)$ is also in the set $\partial F(0) + \epsilon \boldsymbol{B}$. Since

$$\text{co}\ \partial F([y, x]) = \text{co} \left[\bigcup_{z \in [x,y]} \partial F(z) \right],$$

then

$$\text{co}\ \partial F([x, y]) \subset \partial F(0) + \epsilon \boldsymbol{B}, \quad \forall |x| < \delta, |y| < \delta.$$

An arbitrary vector $v \in \mathbb{R}^n$ may be scaled by $\lambda > 0$ so that $\tilde{v} \doteq \lambda v$ has norm $|\tilde{v}| < \delta$. Since

$$F(\tilde{v}) \in \text{co}\ \partial F([0, \tilde{v}])\tilde{v},$$

then there exists $Z \in \text{co}\partial F([0, \tilde{v}])$ such that $F(\tilde{v}) = Z\tilde{v}$. However, the map Z is ϵ-close to $\partial F(0)$ and the bound (4.4) implies that $|Z\tilde{v}| \leq 1/2|\tilde{v}|$ or, what is the same, $|Zv| \leq 1/2|v|$. By shrinking ϵ to zero and scaling the point v to be in the corresponding δ-ball, we obtain a sequence of matrices $\{Z_i\}$ which contract v by at least a factor of $1/2$ and satisfy

$$Z_j \in \text{co}\ \partial F(0) + \epsilon \boldsymbol{B}, \quad \forall \epsilon > 0, j > N(\epsilon),$$

for some integer $N(\epsilon)$. The sequence $\{Z_i\}$ is bounded so there exists a convergent subsequence $\{Z_{\pi_i}\}$ which must converge to a member of $\partial F(0)$ since $\partial F(0)$ is compact,

$$\lim_{i \to \infty} Z_{\pi_i} = Z \in \partial F(0).$$

Finally,

$$\begin{aligned}|Zv| &\le |Z_{\pi_i}v| + |(Z - Z_{\pi_i})v| \\ &\le \frac{1}{2}|v| + \epsilon, \quad i > N(\epsilon) \\ \implies |Zv| &\le \frac{1}{2}|v|.\end{aligned}$$

■

Theorem 4.1 is proven with Propositions 4.5 and 4.6.

Proof of Theorem 4.1: Proposition 4.5 implies that every element of $R(t,\tau)$ must fix $n-r$ directions for all t,τ. However, *assuming* exponential stability of the flow implies, for sufficiently large $t-\tau$, that there exist a matrix $Z \in \partial F(0)$ which contracts an arbitrary vector $v \in \mathbb{R}^n$ (Proposition 4.6). This contradicts the statement of Theorem 4.4 when v is chosen as a vector fixed by $R(t,\tau)$ since the matrix $Z \in \partial F(0)$ which contracts v cannot be in $R(t,\tau)$. ■

4.2 Homogeneous Approximations of Driftless Control Systems

Instead of working with the original set of input vector fields of the control system (4.1), an approximation that makes sense in terms of stabilization about a desired point x_0 is desired. The Jacobian linearization of this system about any point is not useful in any control theoretic context since the linearized system is *not* controllable. For example, the linearization of equation (4.1) about the point x_0 is

$$\dot{\xi} = X_1(x_0)u_1 + \cdots + X_m(x_0)u_m, \tag{4.5}$$

where $x = \xi + x_0$. Since the number of inputs m is less than the state dimension n then,

$$\mathrm{rank}[X_1(x_0) \cdots X_m(x_0)] \le m,$$

and so the linearized system (4.5) is not controllable. However if the Lie algebra of the set of analytic input vector fields has rank n at x_0 then the results reviewed in Section 2.2 show that there exists a homogeneous degree one approximate system written in the (new) coordinates as

$$\dot{y} = X_1^1(y)u_1 + \cdots + X_m^1(y)u_m. \tag{4.6}$$

Furthermore $\dim \mathcal{F}^{\tilde{X}}(x_0) = \dim \mathcal{F}^X(x_0)$ so that controllability of (4.1) is transferred to the approximating driftless system (4.6). The natural dilation associated with the system depends only on the dimension of the span of the subspaces of the filtration. Thus, the scaling powers in the dilation depend only on the point about which the approximation is made and not on any particular coordinate representation (although the homogeneous structure is evident only in the coordinates adapted to the filtration).

The use of homogeneous feedback is strongly motivated by the existence of a controllable homogeneous approximating system (4.6). If homogeneous degree one control functions $u_i(t, y)$ can be found such that $y = 0$ is a uniformly asymptotically stable equilibrium point of the closed-loop system then $y = 0$ is exponentially stable with respect to the homogeneous norm ρ since the closed-loop vector field is degree zero (Property 2.16). Thus, the stability type is not the familiar exponential stability definition but rather ρ-exponential stability. As pointed out in Section 2.1.3, ρ-exponential stability can be locally recast into the bound,

$$\|\psi(t, t_0, x_0)\|_2 \leq M\|x_0\|_2^{1/\sigma} e^{-\alpha(t-t_0)} \quad \text{for some } M > 0, \alpha > 0, \sigma > 1,$$

where ψ represents the flow of the system. Thus each state is bounded by a decaying exponential envelope but the dependence on the initial condition is allowed to be more general than that in the usual definition of exponential stability. The higher order perturbing terms, present when one considers the full set of equations in y-coordinates, do not locally change the stability type of the origin. In other words the original control system with feedback,

$$\dot{y} = (X_1^1(y) + X_1^0(y) + \cdots)u_1(t, y) + \cdots + (X_m^1(y) + X_m^0(y) + \cdots)u_m(t, y),$$

is still locally ρ-exponentially stable. This is a consequence of the converse Lyapunov theorem for homogeneous vector fields (Proposition 2.27) and is proven in Proposition 4.10. The standing assumption in the remainder of the thesis is that the system (4.1) has been transformed to the adapted coordinates and that a degree one homogeneous approximation has been computed. This approximation will be used exclusively in the sequel. An example of the approximation process is given in Appendix B.

The quest for (locally) ρ-exponentially stabilizing feedback has been reduced to the search for time-periodic asymptotically stabilizing degree one functions for the approximate system. The dilation associated with the input vector field approximations and feedbacks will always have $r_n > 1$ since at least one level of Lie brackets is required to achieve controllability of the system. Thus the degree one feedbacks are not Lipschitz at the origin even though they may be locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. The proof of this fact is essentially the same as the proof of Lemma 3.4. The non-Lipschitz feedbacks seem more reasonable in light of the facts established in Section 4.1. Since the closed-loop system is not Lipschitz it becomes apparent why a broader notion of exponential stability, namely ρ-exponential stability, is required for the systems.

The coordinates adapted to the filtration are found by composing the flows of n nonlinear differential equations. The calculations may be performed by hand for any given system. For large systems this can be an arduous task and so automated computation with a computer is desirable. An algorithm which performs this task is given in Appendix B.

Finally, certain systems may be transformed exactly into a nilpotent homogeneous form. In other words, there exists a diffeomorphism of the state and *input* such that the new system representation is its own nilpotent homogeneous approximation. In this situation no approximation is involved and the model is valid up to the boundary where the diffeomorphisms are no longer defined. In [31], necessary and sufficient conditions are given for the transformation of two-input driftless systems into "chained" (or equivalently "power") form.

4.3 Synthesis Methods

This section presents two methods for synthesizing ρ-exponential stabilizers. In other words, given the driftless system (4.1), under what conditions can feedbacks be constructed so that the closed-loop system is locally ρ-exponentially stable? The algorithms do not cover every analytic driftless system, however many practical physical examples satisfy the condition in the theorems. The previous section outlined the objective of the algorithms: generate uniformly asymptotically stabilizing homogeneous degree one feedbacks for the homogeneous approximate driftless system (4.6). Lyapunov analysis is useful for proving *asymptotic* stability while the requirement that the closed-loop system is degree zero enforces ρ-exponential stability.

Driftless systems fail Brockett's necessary condition so no continuous time-invariant feedback can stabilize the system to a point. However Coron's Theorem A.10 in Appendix A proves that time-periodic continuous feedback is sufficient to stabilize the system to a point. The proof of Theorem A.10 is not constructive in any practical sense so algorithms are still required. The prototype three-dimensional system (3.1) is used to illustrate how time periodicity overcomes the topological obstruction of Brockett's condition. The system is repeated here for convenience,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2. \tag{4.7}$$

The x_1 and x_2 variables may be directly manipulated since the control inputs are equal to the time derivatives of these variables. It is not obvious how to manipulate x_3 by changing the inputs. The Lie bracket of X_1 and X_2 is $[X_1, X_2] = -\partial/\partial x_3$. The motion in Lie bracket vector field is modeled by the infinitesimal loop where,

$$\left.\begin{aligned} u_1(t) &= 1 \\ u_2(t) &= 0 \end{aligned}\right\} t \in [0, \epsilon)$$
$$\left.\begin{aligned} u_1(t) &= 0 \\ u_2(t) &= 1 \end{aligned}\right\} t \in [\epsilon, 2\epsilon)$$
$$\left.\begin{aligned} u_1(t) &= -1 \\ u_2(t) &= 0 \end{aligned}\right\} t \in [2\epsilon, 3\epsilon)$$

$$\left.\begin{aligned} u_1(t) &= 0 \\ u_2(t) &= -1 \end{aligned}\right\} t \in [3\epsilon, 4\epsilon).$$

In the limit as $\epsilon \to 0$ the initial point has moved an infinitesimal amount in the $[X_1, X_2]$ direction. An asymptotically stabilizing feedback is derived below based on this property. A continuous finite time analog of this input sequence is $u_1(t) = a \cos t$ and $u_2(t) = b \sin t$ where a and b are coefficients which are chosen later. After one period, the initial point $(x_1(0), x_2(0), x_3(0))$ has moved to $(x_1(0), x_2(0), x_3(0) - ab\pi)$. Thus, x_1 and x_2 have returned to their initial states and x_3 has moved on amount proportional to $-ab$. Now choose $a = x_3$ and $b = x_3^2$ so that $u_1 = x_3 \cos t$ and $u_2 = x_3^2 \sin t$. This choice will "push" $x_3(t)$ to zero given any initial condition $x_3(0)$. The x_1 and x_2 variables are also required to go to zero for stabilization. One way to enforce this is to add a term to each of the control functions which has a stabilizing effect. One way to accomplish this is to modify the feedbacks to

$$\begin{aligned} u_1 &= -x_1 + x_3 \cos t \\ u_2 &= -x_2 + x_3^2 \sin t. \end{aligned} \tag{4.8}$$

Although these feedbacks where derived heuristically, the system (4.7) with feedback (4.8) is locally asymptotically stable. Rigorous proof of this fact uses center manifold analysis of the closed-loop system. This is sketched below.

The time-periodic terms in the feedback in equations (4.8) may be replaced by the variables of an appended harmonic oscillator. The closed-loop system becomes,

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 z_1 \\ \dot{x}_2 &= -x_2 + x_3^2 z_2 \\ \dot{x}_3 &= x_2(-x_1 + x_3 z_1) \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1. \end{aligned}$$

The center manifold variables are (x_3, z_1, z_2). Representing x_1 and x_2 as a graph over the center manifold variables yields,

$$\begin{aligned} x_1 &= \frac{1}{2} x_3 z_1 + \frac{1}{2} x_3 z_2 + \mathcal{O}(3) \\ x_2 &= -\frac{1}{2} x_3^2 z_1 + \frac{1}{2} x_3^2 z_2 + \mathcal{O}(4). \end{aligned}$$

Substituting these expressions into the $\dot{x}_3$ equation yields,

$$\dot{x}_3 = -\frac{1}{4} (z_1 - z_2)^2 x_3^3 + \mathcal{O}(6). \tag{4.9}$$

Since $z_1 = \cos(t + \theta_0)$ and $z_2 = \sin(t + \theta_0)$ for some θ_0, then the center manifold system (4.9) is locally uniformly asymptotically stable and so $x_1 \to 0$ and $x_2 \to 0$ too. Note that the rate of convergence cannot be bounded by an exponential envelope. This is due to the fact that the feedback is Lipschitz.

Now suppose the frequency of the oscillator variables is set to zero (so the closed-loop system is time-invariant now) and the variables are frozen at $z_1(t) = \cos\theta_0$ and $z_2(t) = \sin\theta_0$. In this case the following two manifolds of equilibrium points passing through $(x_1, x_2, x_3) = 0$ appears,

$$\begin{array}{rl} (\gamma_1 \cos\theta_0, 0, \gamma_1) & \gamma_1 \in \mathbb{R} \\ (\gamma_2 \cos\theta_0, \gamma_2^2 \sin\theta_0, \gamma_2) & \gamma_2 \in \mathbb{R}\,. \end{array}$$

Obviously the origin cannot be asymptotically stable. Thus the time-periodic components in the feedback provide a time-periodic sign change similar to the Lie bracket calculation. Destroying the periodic sign change apparently leads to the formation of equilibrium points arbitrarily close to the origin as shown above. Brockett's condition is not applicable when the system is time-periodic since the closed-loop system may always be interpreted as an autonomous system with a series of oscillator appended to the states. The oscillator states are not required to converge to the origin and so the origin is not an asymptotically stable equilibrium point.

The first algorithm is an extension of the algorithm in [37]. The time periodicity is explicitly introduced into the system by a function which, in the absence of any other feedback, renders every solution of the system time-periodic. It is then a matter of perturbing these trajectories so the state converges to the origin. The details of this procedure are discussed in the next section. The second algorithm assumes the existence of a smooth time-periodic asymptotically stabilizing feedback and sets forth sufficient conditions under which the smooth feedback can be converted into a ρ-exponential stabilizer.

4.3.1 Extension of Pomet's algorithm.

An algorithm for the construction of local ρ-exponentially stabilizing feedbacks is described in this section. It is based on an extension of Pomet's algorithm [37]. Using the approximation in Section 4.2, which was based on the analysis from Section 2.2, the following truncated driftless control system is associated with the original control system:

$$\dot{x} = \sum_{i=1}^{m} X_i^1(x) u_i. \tag{4.10}$$

The X_i^1 are analytic vector fields, homogeneous degree 1 with respect to the dilation, Δ_λ, defined in the approximation process. The algorithm in [37] may be modified to provide stabilizers for (4.10) when the input vector fields of equation (4.10) satisfy the following condition,

$$\begin{aligned} \operatorname{rank} \{ & X_1^1, X_2^1, \ldots, X_m^1, \\ & [X_1^1, X_2^1], \ldots, [X_1^1, X_m^1], \ldots, \\ & \operatorname{ad}^j_{X_1^1} X_2^1, \ldots, \operatorname{ad}^j_{X_1^1} X_m^1, \ldots \} (x_0) = n. \end{aligned} \tag{4.11}$$

The point x_0 is the desired equilibrium point. The superscript "1" will be dropped for the remainder of this section but it is understood that the input vector fields are degree one.

A heuristic overview of how the algorithm works is presented before embarking on the construction of the feedbacks and proofs. Supposing the input vector fields satisfy (4.11), a 2π-periodic function of time, $\alpha(t,x)$, is chosen so that all nonzero solutions of $\alpha(t,x)X_1(x)$ are 2π-periodic and $x = 0$ is an equilibrium point. In order to define a positive definite function on the phase space, each closed periodic "loop" is assigned a positive number. This is accomplished by defining a positive definite function on a Poincaré map associated with the flow of αX_1. In other words, the flow is sampled at $t_0 \in [0, 2\pi)$ and then a positive definite function is applied to the value of the flow at this time. This resulting number is denoted $V(t,x)$. The feedback u_1 is defined to be the open loop part, α, *minus* the Lie derivative of $V(x,t)$ with respect to the vector field X_1. The remaining inputs $u_i, i = 1, \ldots, m$, are defined to be the negative of the Lie derivative of $V(x,t)$ with respect to X_i. This choice of feedbacks guarantee that $x = 0$ is *stable*. Under some extra conditions the feedback can be shown to be uniformly asymptotically stabilizing.

The extension of Pomet's algorithm to ρ-exponentially stabilize systems of the form (4.10) is now developed. The following modification of Proposition 1 in [37] is made (as in [37], the vector field X_1 plays a particular role),

Proposition 4.8 *Let $\alpha : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a time-periodic, smooth on $\mathbb{R}\times\mathbb{R}^n\setminus\{0\}$, homogeneous degree one function with respect to Δ_λ. Assume α also satisfies the following conditions,*

$$\begin{aligned} \alpha(t+2\pi, x) &= \alpha(t,x) \quad \forall t, x \\ \alpha(-t, x) &= -\alpha(t,x) \quad \forall t, x \\ \alpha(t, 0) &= 0 \quad \forall t. \end{aligned} \tag{4.12}$$

Let $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a function defined as,

$$V(t,x) = \varrho(\psi(0,t,x)),$$

where $\varrho : \mathbb{R}^n \to \mathbb{R}$ is any positive definite homogeneous degree 2 function that is smooth on $\mathbb{R}^n \setminus \{0\}$. Here $\psi(t, t_0, x_0)$ represents the flow of the vector field $\alpha(t,x)X_1(x)$ evaluated at time t and passing through x_0 at time t_0. The function V has the following properties,

1. *V is smooth on $\mathbb{R}\times\mathbb{R}^n \setminus \{0\}$,*
2. *V is homogeneous degree 2 with respect to Δ_λ,*
3. *V is 2π-periodic with respect to t: $V(t+2\pi, x) = V(t,x)$,*
4. *$V(t,x) = 0 \iff x = 0$,*
5. *$\frac{\partial}{\partial x}V(t,x) \neq 0 \quad \forall x \neq 0$ (the gradient at 0 may not be defined),*
6. *$V(t,x)$ is a proper map $\forall t \in [0, 2\pi)$.*

Proof: The product of the scalar degree one *function* $\alpha(t,x)$ with the degree one *vector field* $X_1(x)$ defines a degree *zero* vector field $(\alpha X_1)(t,x)$, by the convention established in Definition 2.1. This new vector field is smooth on $\mathbb{R}^n \setminus \{0\}$ and its flow is complete. Completeness follows from the dilation scaling property enjoyed by solutions of degree zero vector fields and the exponential upper bound on the growth of solutions, i.e., the bound established in the proof of Lemma 3.4. Hence, $\psi(t,t_0,x)$ is a homeomorphism $\forall t, t_0, x$ and a smooth diffeomorphism $\forall t, t_0$ and $x \neq 0$. Item (1) is obvious since V is the composition of functions which are smooth on $\mathbb{R}^n \setminus \{0\}$. Also note that the flow satisfies $\Delta_\lambda \psi(t,t_0,x_0) = \psi(t,t_0,\Delta_\lambda x_0)$ since the vector field is degree zero with respect to Δ_λ. Item (2) follows from $V(t,\Delta_\lambda x) = \varrho(\psi(0,t,\Delta_\lambda x)) = \varrho(\Delta_\lambda \psi(0,t,x)) = \lambda^2 V(t,x)$. The periodicity of ψ with respect to t and t_0 must first be established before proving Item (3). The first fact to show is that $\psi(-t,0,x) = \psi(t,0,x)$. This is accomplished by showing that $\psi(-t,t_0,x)$ also satisfies the equation $\dot{x} = \alpha X_1$. Let $s = -t$ then,

$$\begin{aligned}
\frac{d\psi}{ds}(s,t_0,x) &= -\frac{d\psi}{dt}(s,t_0,x) \\
&= -\alpha(s,\psi(s,t_0,x))X_1(\psi(s,t_0,x)) \\
\implies \frac{d\psi}{dt}(-t,t_0,x) &= \alpha(t,\psi(-t,t_0,x))X_1(\psi(-t,t_0,x)).
\end{aligned}$$

When $t_0 = 0$ then $\psi(-t,0,x) = \psi(t,0,x)$ since initial conditions match. In particular $\psi(\pi,-\pi,x) = x$. The differential equation is periodic so time translated solutions must also satisfy the equation: $\psi(t+n2\pi, t_0+n2\pi, x) = \psi(t,t_0,x)$ for all (t,t_0,x). These facts show that for any t the following is true,

$$\begin{aligned}
\psi(t+2\pi,t,x) &= \psi(t+2\pi,\pi,\psi(\pi,t_0,x)) \\
&= \psi(t+2\pi,\pi,\psi(\pi,-\pi,\psi(-\pi,t,x))) \\
&= \psi(t+2\pi,\pi,\psi(-\pi,t,x)) \\
&= \psi(t,-\pi,\psi(-\pi,t,x)) \\
&= x.
\end{aligned}$$

Generalizing to an arbitrary 2π time shift,

$$\begin{aligned}
\psi(t+n2\pi,t,x) &= \psi(t+n2\pi,t+(n-1)2\pi,\psi(t+(n-1)2\pi,t,x)) \\
&= \psi(t+(n-1)2\pi,t+(n-2)2\pi,\psi(t+(n-2)2\pi,t,x)) \\
&\vdots \\
&= \psi(t+2\pi,t,x) \\
&= x.
\end{aligned}$$

Lastly, the starting time is arbitrary since,

$$\begin{aligned}
\psi(t+n2\pi,t_0,x) &= \psi(t+n2\pi,\psi(t,t_0,x)) \\
&= \psi(t,t_0,x).
\end{aligned}$$

Thus the flow is 2π-periodic with respect to its first argument. The flow is also 2π-periodic with respect to its second argument,

$$\begin{aligned}\psi(t,t_0,x) &= \psi(t+n2\pi,t_0+n2\pi,x)\\ &= \psi(t,t_0+n2\pi,x).\end{aligned}$$

Item (3) is easily shown now since

$$V(t+n2\pi,x) = \varrho(\psi(0,t+n2\pi,x)) = \varrho(\psi(0,t,x)) = V(t,x).$$

Item (4) follows from the fact ϱ is positive definite and the origin and any nonzero x cannot lie on the same trajectory. Item (5) may be written for $x \neq 0$,

$$\frac{\partial}{\partial x}V(t,x) = \nabla\varrho(0,t,x)\cdot D_x\psi(0,t,x).$$

$\nabla\varrho(y) \neq 0$ for $y \neq 0$ from Property 2.7 and $D_x\psi$ is full rank for nonzero x. Lastly, $V(t,x)$ is proper for any $t \in [0,2\pi)$ since it satisfies the bounds $c_1\rho^2(x) < V(t,x) < c_2\rho^2(x)$. ■

The following choice of inputs u_i render (4.10) stable,

$$\begin{aligned}u_1(t,x) &= \alpha(t,x) - L_{X_1}V(x,t)\\ u_2(t,x) &= -L_{X_2}V(x,t)\\ &\vdots\\ u_m(t,x) &= -L_{X_m}V(x,t).\end{aligned} \tag{4.13}$$

Note that these control functions are smooth functions of t and $x \in \mathbb{R}^n \setminus \{0\}$. Under additional assumptions $x = 0$ is exponentially stable with respect the homogeneous norm.

Theorem 4.9 *Suppose the approximate system satisfies (4.11) and an α satisfying Proposition 4.8 is chosen. If the following conditions are satisfied,*

$$\left.\begin{array}{r}L_{X_1}V(t,x) = \ldots = L_{X_m}V(t,x) = 0\\ \alpha(t,x) = \frac{\partial\alpha}{\partial t}(t,x) = \frac{\partial^2\alpha}{\partial t^2}(t,x) = \ldots = 0\end{array}\right\} \Rightarrow x = 0, \tag{4.14}$$

then $x = 0$ is a globally δ-exponentially stable equilibrium point of (4.10) with respect to the dilation when the feedback (4.13) is applied.

The proof of the theorem is very similar to the one given by Pomet however an outline is given for the sake of completeness. Most of the modifications of his proof are in establishing that certain functions and flows have the properties specified in his paper. The majority of this extra work was shown in Proposition 4.8. The conditions of the theorem guarantee uniform asymptotic convergence of the trajectories to the origin. The fact that the closed-loop system is degree zero with respect to $\Delta_\lambda$implies that the system is ρ-exponentially stable.

Proof: The closed-loop system, that is system (4.10) with the feedback (4.13), is degree zero with respect to Δ_λ. This is evident from the fact that the Lie derivatives of the degree two function V with respect to the degree one vector fields X_i is a degree one function. This implies that the control functions are degree one since $u_1 = \alpha - L_{X_1}V$ and $u_j = -L_{X_j}V$, $j = 2, \ldots, m$. Scaling a degree one vector field by a degree function yields a degree zero vector field so the closed-loop system must be degree zero. Thus ρ-exponential stability is equivalent to uniform asymptotic stability by Lemma 2.16.

The proof that feedback (4.13) is uniformly asymptotically stabilizing is shown below. First note that the derivative of V along solutions of the system $\dot{x} = \alpha X_1$ is zero since the value of V on a trajectory of this system is constant,

$$\begin{aligned} V(t, \psi(t, t_0, x)) &= \varrho(\psi(0, t, \psi(t, t_0, x))) \\ &= \varrho(\psi(0, t_0, x)), \end{aligned}$$

where ψ is the flow of $\dot{x} = \alpha X_1$. The derivative of V along trajectories of the *closed-loop* system is

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial t} + \nabla V \cdot \left(\alpha X_1 - (L_{X_1}V)X_1 - \sum_{i=2}^{m}(L_{X_i}V)X_i \right) \\ &= -\sum_{i=1}^{m}(L_{X_i}V)^2. \end{aligned}$$

The time derivative is negative semidefinite and since V is a proper function with respect to x then all solutions are bounded. For notational simplicity the superscript "1" will be omitted from the vector fields X_i^1 in the remainder of the proof.

LaSalle's theorem is used to show asymptotic stability. LaSalle's theorem is applicable in this case because the system is time-periodic. It is sufficient to show, for asymptotic stability, that no nontrivial trajectories of the closed-loop system are contained in the set where $\dot{V} = 0$. The time is identified with the circle S^1 since the system is time-periodic. The set where the time derivative of V is zero is,

$$A = \{(t, x) \mid \dot{V}(t, x) = 0\} = \{(t, x) \in S^1 \times \mathbb{R}^n \mid L_{X_i}V(t, x) = 0,\ i = 1, \ldots, m\}.$$

The closed-loop system restricts to the vector field αX_1 on the set A. The time derivative of the functions $L_{X_i}V$ with respect to solutions passing through points in A is,

$$\begin{aligned} \frac{d}{dt}(L_{X_i}V)(t, x) &= \frac{\partial}{\partial t}(L_{X_i}V)(t, x) + (\nabla L_{X_i}V \alpha X_1)(t, x) \\ &= \frac{\partial}{\partial t}(L_{X_i}V) + L_{\mathrm{ad}_{X_1}X_i}V + L_{X_i}L_{\alpha X_1}V \\ &= L_{X_i}(\frac{\partial V}{\partial t} + L_{\alpha X_1}V)(t, x) + L_{X_i}L_{\alpha X_1}V \\ &= L_{X_i}L_{\alpha X_1}V \quad \forall (t, x) \in A. \end{aligned}$$

Induction may be used to show that,

$$\frac{d^j}{dt^j}(L_{X_i}V) = L_{\mathrm{ad}^j_{\alpha X_1}X_i}V \quad \forall (t,x) \in A, j \geq 0.$$

Since the functions $L_{X_i}V$ are zero on trajectories which stay in A then,

$$L_{\mathrm{ad}^j_{\alpha X_1}X_i}V(t,x) = 0 \quad \forall (t,x) \in A, j \geq 0. \tag{4.15}$$

Now assume that $\alpha(t,x) \neq 0$ at some point $(\tilde{t}, \tilde{x} \neq 0) \in A$. The Lie bracket identity,

$$[fX, gY] = fg[X,Y] + f \cdot L_X g \cdot Y - g \cdot L_Y f \cdot X,$$

where f and g are functions may used to show that,

$$\mathrm{rank}\left\{ \mathrm{ad}^j_{X_1}X_i \quad \begin{matrix} k = 1, \ldots, m \\ j \geq 0 \end{matrix} \right\} = \mathrm{rank}\left\{ \mathrm{ad}^j_{\alpha X_1}X_i \quad \begin{matrix} k = 1, \ldots, m \\ j \geq 0 \end{matrix} \right\}, \tag{4.16}$$

when $\alpha(\tilde{t}, \tilde{x}) \neq 0$. However the condition in equation (4.11) shows that the rank of the set in equation (4.16) must be equal to n. The only way for the expressions in equation (4.15) to be satisfied is for $\nabla V(\tilde{t}, \tilde{x}) = 0$ when $\alpha \neq 0$. However from Item 5 in Proposition 4.8, $V(t,x) \neq 0$ for all $x \neq 0$. Thus the rank of equation (4.16) must be less than n which implies that $\alpha(t,x)$ (and all of its time derivatives) must be zero on this trajectory in A. The hypothesis of the theorem may be used to conclude that $x = 0$ is the only solution in A which is consistent with the analysis given above. Thus the origin is uniformly asymptotically stable. The fact that the closed-loop system is degree zero with respect to Δ_λ implies that the origin is ρ-exponentially stable where ρ is any homogeneous norm compatible with the dilation. ∎

In practice it may be difficult to verify the conditions in the theorem to conclude asymptotic stability. It is useful to choose α such that $\alpha(t,x) = 0 \Leftrightarrow x = 0$. For example, $\alpha(x,t) = \rho(x)\sin t$, where ρ is any smooth homogeneous norm, satisfies the hypothesis of the theorem. The feedback is smooth on $\mathbb{R}^n \setminus \{0\}$ and so the solutions of the closed-loop system are unique by Lemma 3.4.

The proposition below demonstrates that the feedbacks locally ρ-exponentially stabilize the full system. In other words, the terms neglected in the truncated system do not locally change the stability of the equilibrium point.

Proposition 4.10 *Suppose the conditions of Theorem 4.9 hold. Then the feedback (4.13) locally δ-exponentially stabilizes the original system (4.1).*

Proof: Consider the feedback (4.13) applied to the system in equations (4.1) written in the special local coordinates, $\dot{x} = \sum_{i=1}^n X_i^1(x)u_i(x,t) + R(x,t)$, where $R(x,t) = \sum_{i=1}^n \left(\sum_{j=1}^\infty X_i^{1-j}(x)\right) u_i(x,t)$. The m vector fields $\sum_{j=1}^\infty X_i^{1-j}(x), i = 1, \ldots, m$, are analytic and the k^{th} component is a sum of homogeneous polynomials of degree greater than or equal to r_k so that the absolute value of k^{th}

component is bounded by $c_i \rho^{r_k}(x)$ in a sufficiently small neighborhood of the origin. Since the u_i are homogeneous degree one functions then the absolute value of the k^{th} component of $R(x,t)$ may be bounded by a scalar times ρ^{r_k+1} in a neighborhood of the origin. The local stability result follows from application of Proposition 2.27. ∎

Certain driftless control systems may be transformed to exactly a nilpotent homogeneous form. Examples are the "chained form" or "power form" systems [32, 44]. In this case Theorem 4.9 provides a globally δ-exponentially stabilizing feedback since there are no "higher order" perturbing terms.

Finally the algorithm may be summarized as,

1. Compute the local coordinate change which places the input vector fields in form
$$\dot{x} = \sum_{i=1}^{m} \left(X_i^1(x) + X_i^0(x) + X_i^{-1}(x) + \cdots \right) u_i.$$
2. If the relation
$$\begin{aligned} \operatorname{rank} \{ & X_1^1, X_2^1, \ldots, X_m^1, \\ & [X_1^1, X_2^1], \ldots, [X_1^1, X_m^1], \ldots, \\ & \operatorname{ad}^j_{X_1^1} X_2^1, \ldots, \operatorname{ad}^j_{X_1^1} X_m^1, \ldots \} (0) = n \end{aligned}$$
is satisfied then continue with the procedure.
3. Construct homogeneous degree one feedbacks, using the approximate control system,
$$\dot{x} = X_1^1(x) u_1 + \cdots + X_m^1(x) u_m,$$
according to Proposition 4.8 and equation (4.13).
4. These feedback applied to the original system are still locally δ-exponentially stabilizing by Proposition 4.10.

The following example applies this algorithm to the prototype three-dimensional example given by equations (4.7).

Example 4.11 The input vector fields in the system given by equation (4.7) are degree one with respect the dilation $\Delta_\lambda = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$. The set of input vector fields are their own nilpotent homogeneous approximation with respect to this dilation. The X_1^1 vector field is chosen to be $X_1^1 = \partial/\partial x_2$ and X_2^1 is $X_2^1 = \partial/\partial x_1 + x_2 \partial/\partial x_3$. A smooth homogeneous norm which is compatible with this dilation is,
$$\rho(x) = (x_1^4 + x_2^4 + x_3^2)^{\frac{1}{4}}.$$

The open loop input is defined as $\alpha(t,x) = \rho(x)\sin t$. The conditions of Theorem 4.9 are satisfied with this choice. Let $\psi(t,t_0,x_0)$ denote the flow of the vector field,

$$\begin{aligned}\dot{x} &= \alpha(t,x)X_1^1(x) \\ &= \rho(x)\sin t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.\end{aligned}$$

One choice for the positive definite degree 2 function ϱ is

$$\varrho(p) = \frac{1}{2}\left(p_1^2 + p_2^2 + \frac{p_3^2}{\rho^2(p)}\right).$$

Hence, the Lyapunov function V is defined as $V(t,x) = \varrho(\psi(0,t,x))$. This function cannot be computed explicitly so numerical computation is required. The feedbacks are defined as

$$\begin{aligned}u_1 &= \alpha(t,x) - L_{X_1^1}V(t,x) \\ u_2 &= -L_{X_2^1}V(t,x).\end{aligned} \tag{4.17}$$

The Lie derivatives of V with respect to the input vector fields must also be calculated numerically. A numerical simulation of the system with the feedback (4.17) is shown in Figure 4.1. The exponential decay of the states is evident from the log plot in Figure 4.2

4.3.2 Modification of smooth controllers into ρ-exponential stabilizers

This section discusses a very useful method to modify many uniformly asymptotically stabilizing feedbacks into exponential stabilizers. Our primary motivation in this section is in providing feedbacks which are easy to implement. Smooth asymptotically stabilizing controllers are often written in terms of elementary functions and operations and are straightforward to implement but suffer from slow convergence rates. We now pose the question: when can a uniformly asymptotically stabilizing controller be modified into an exponential stabilizer? If the modifications can be performed in real-time then the method would show promise as a way of implementing exponential stabilizers with slightly more computation than required by the smooth stabilizers.

We assume that the input vector fields are already in "homogeneous" coordinates. In other words, the controller asymptotically stabilizes the homogeneous approximation discussed in Section 4.2. The dilation associated with the approximation is denoted Δ_λ. Recall the Euler vector field, $X_E(x)$, corresponding to this dilation is represented by the equations $\dot{x}_i = r_i x_i$, $i = 1,\ldots,n$. The following proposition specifies the condition under which an asymptotic stabilizer can be modified into an exponential stabilizer. The closed-loop system is denoted $\dot{x} = X(t,x)$ with the feedback functions $u_i(t,x), i = 1,\ldots,m$. Most

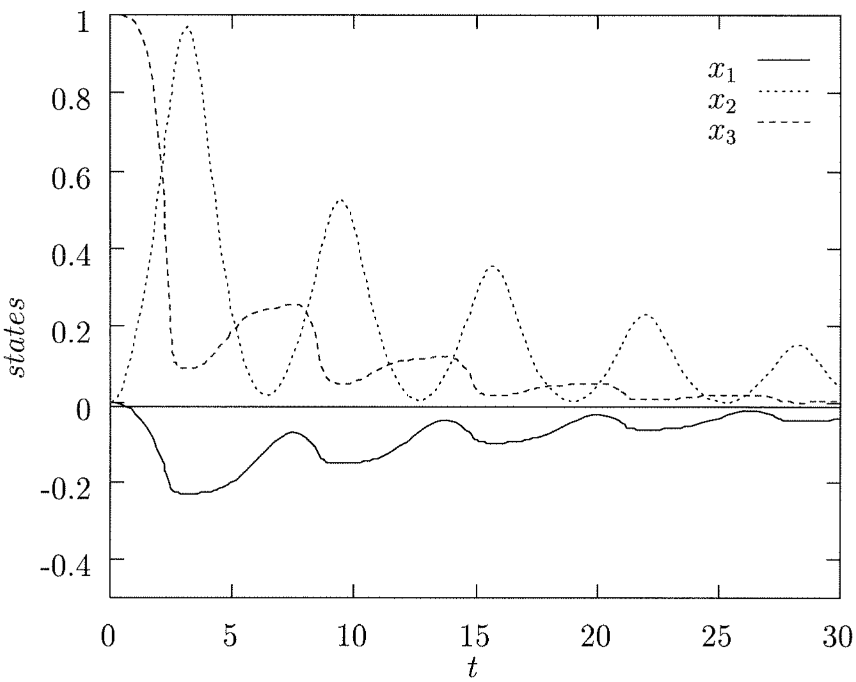

Figure 4.1: Extension of Pomet's algorithm.

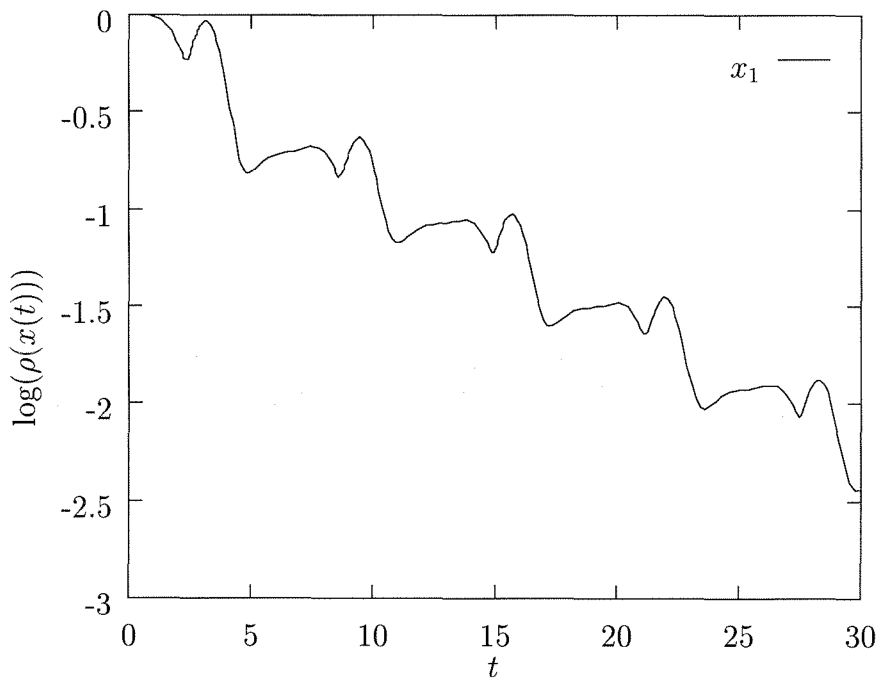

Figure 4.2: Log of the homogeneous norm of the state.

smooth stabilizing controllers are time-periodic so we restrict ourselves to this case.

Theorem 4.12 *Suppose there exists a T-periodic Lyapunov function, $V(t,x)$, for the T-periodic smooth vector field $\dot{x} = X(t,x)$ such that for some constant $C > 0$ the family of level sets parametrized by t,*

$$G_t^C = \{x | V(t,x) = C\},$$

are transversal to the Euler vector field for all $t \in [0,T)$. Under this hypothesis, the original feedbacks may be modified to the following T-periodic ρ-exponentially stabilizing feedbacks,

$$\tilde{u}_i(t,x) = \tilde{\rho}(t,x) u_i(t, \gamma_t(x)) \quad i = 1, \ldots, m.$$

$\tilde{\rho} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ is a uniquely defined homogeneous degree one function such that,

$$\tilde{\rho}(t,x)|_{x \in G_t^C} = 1.$$

The map $\gamma_t : \mathbb{R}^n \setminus \{0\} \to G_t^C$ returns the point on the set G_t^C which lies on the same homogeneous ray as x, i.e. $\gamma_t(x) = \delta_\lambda x = \overline{x} \in G_t^C$ for some scaling $\lambda > 0$.

Remark 4.13 In many cases the stabilizing feedback is derived from Lyapunov analysis and so the closed-loop system has a function which may be tested for the properties given in the proposition.

Proof: We first show that $\tilde{\rho}$ and γ_t are well defined quantities. We assume that the Lyapunov function is smooth in all of its arguments and that the original feedback functions, u_i, are smooth. We define the value of the function $g : \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ to be the $\lambda \in \mathbb{R}^+$ which solves,

$$F(\lambda, t, x) \doteq V(t, \delta_\lambda x) - C = 0.$$

In other words, $g(t,x) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ returns the dilation scaling factor required to map the point $x \neq 0$ to the point $\overline{x} \in G_t$ on the same homogeneous ray at time t. $\overline{x}$ is unique since the transversality condition implies that the projection $\pi|_{G_t^C} : G_T^C \to S_\Delta^{n-1}$ is a local diffeomorphism. Furthermore, since G_t^C is compact and connected [46, Theorem 3.7] there is only one point in the preimage of $(\pi|_{G_t^C})^{-1}(y), y \in S_\Delta^{n-1}$. Hence the projection is a global diffeomorphism between G_t^C and S_Δ^{n-1} for each fixed t. The map from x to $\overline{x}$ is $(\pi|_{G_t^C})^{-1} \circ \pi$ and $g(t,x) = \rho(\overline{x})/\rho(x)$. The smoothness of g is determined with the implicit function theorem as shown below. Suppose that (λ, t, x) satisfies (4.3.2), then we compute,

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x) &= \left(-\frac{1}{\partial F / \partial \lambda} \frac{\partial F}{\partial t} \right)(t, \Delta_\lambda x) \\ &= \left(-\frac{1}{\partial F / \partial \lambda} \frac{\partial V}{\partial t} \right)(t, \Delta_\lambda x). \end{aligned}$$

The quantity $\partial F/\partial\lambda(t,\Delta_\lambda x)$ is nonzero since,

$$\begin{aligned}
\frac{\partial F}{\partial \lambda}(t,\Delta_\lambda x) &= \frac{\partial V}{\partial \lambda}(t,\Delta_\lambda x)\\
&= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,\Delta_\lambda x) r_i \lambda^{r_i-1} x_i\\
&= \frac{1}{\lambda}\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,\Delta_\lambda x) r_i \lambda^{r_i} x_i\\
&= \frac{1}{\lambda} L_{X_E} V(t,\Delta_\lambda x).
\end{aligned}$$

This last condition is precisely the transversality condition on the set G_t^C. Thus, $\partial F/\partial\lambda(t,\Delta_\lambda x) \neq 0$ and the implicit function theorem states that

$$\frac{\partial g}{\partial t}(t,x) = \left(-\frac{\lambda}{L_{X_E}V}\frac{\partial V}{\partial t}\right)(t,\Delta_\lambda x).$$

Similarly,

$$\frac{\partial g}{\partial x_i}(t,x) = \left(-\frac{\lambda^{r_i+1}}{L_{X_E}V}\frac{\partial V}{\partial x_i}\right)(t,\Delta_\lambda x).$$

Note that $\lambda = g(t,x)$ in these computations. We show that g is *degree* -1. Suppose $g(t,x)=\lambda$, then $g(t,\delta_\sigma x)$ is the λ_0 that solves $V(t,\delta_{\lambda_0}\delta_\sigma x) - C = 0$. Since $\delta_{\lambda_0}\delta_\sigma x = \delta_{\lambda_0\sigma}x$ then $\lambda = \lambda_0\sigma$ so $g(t,\delta_\sigma x) = \lambda/\sigma = g(t,x)/\sigma$.

The function $\gamma : \mathbb{R}\times\mathbb{R}^n\setminus\{0\}\to G_t$ is,

$$\gamma(t,x) \doteq \delta_{g(t,x)}x.$$

Note that $\gamma(t,\Delta_\lambda x) = \gamma(t,x), \forall\lambda>0$. $\tilde\rho : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^+$ is defined as,

$$\tilde\rho(t,x) \doteq \begin{cases} \frac{1}{g(t,x)} & x\neq 0\\ 0 & x=0 \end{cases}.$$

Furthermore, for any $\overline{x}\in G_t$, $\tilde\rho(t,\overline{x}) = 1$ since $\gamma(t,\overline{x}) = \overline{x}$. The definitions may be used to show that $\gamma(t,\cdot)$ is smooth on $\mathbb{R}^n\setminus\{0\}$and $\tilde\rho(t,\cdot)$ is continuous on $\mathbb{R}^n$ and smooth $\mathbb{R}^n\setminus\{0\}$. Furthermore, $\tilde\rho$ is homogeneous degree 1. T-periodicity of $\tilde\rho$ and γ is evident from the fact that V is T-periodic.

The modified feedbacks are defined as,

$$\tilde u_i(t,x) \doteq \tilde\rho(t,x)u_i(t,\gamma(t,x)).$$

These functions are degree one since,

$$\begin{aligned}
\tilde u_i(t,\Delta_\lambda x) &= \tilde\rho(t,\Delta_\lambda x)u_i(t,\gamma(t,\Delta_\lambda x))\\
&= \lambda\tilde\rho(t,x)u_i(t,\gamma(t,x))\\
&= \lambda\tilde u_i(t,x).
\end{aligned}$$

These functions agree with the original feedbacks on G_t^C i.e. for $\overline{x} \in G_t^C$, $\tilde{u}_i(t, \overline{x}) = u_i(t, \overline{x})$. We assume that the input vector fields are already in homogeneous form and that the u_i are uniformly asymptotically stabilizing. We now show that the closed-loop system, denoted $\dot{x} = \tilde{X}(t, x)$, with the modified feedback is exponentially stable. The closed-loop systems is degree zero since the feedback is degree one and the input vector fields are degree one. Hence, all we need to show is uniform asymptotic stability with the modified feedbacks. This is accomplished with the following degree k positive definite function,

$$\begin{aligned} \tilde{V} : \mathbb{R} \times \mathbb{R}^n &\to \mathbb{R}^+ \\ (t, x) &\to \tilde{\rho}^k(t, x), \end{aligned}$$

where k is any positive integer. The time derivative of $\tilde{V}$ for $x \neq 0$ is,

$$\begin{aligned} \frac{d\tilde{V}}{dt}(t, x) &= \left(\frac{d}{dt} \frac{1}{g} \right)(t, x) \\ &= -\frac{k}{g^{k+1}(t, x)} \left(\frac{\partial g}{\partial t}(t, x) + D_x g(t, x)(\tilde{X}) \right)(t, x) \\ &= -\frac{k}{g^{k+1}(t, x)} \left(-\frac{g(t, x)}{L_{X_E} V(t, \overline{x})} \frac{\partial V}{\partial t}(t, \overline{x}) \right. \\ &\quad \left. -\frac{1}{L_{X_E} V(t, \overline{x})} \sum_{i=1}^{n} g^{r_i+1}(t, x) \frac{\partial V}{\partial x_i}(t, \overline{x}) \tilde{X}_i(t, x) \right) \quad \overline{x} = \delta_{g(t,x)} x \in G_t \\ &= \frac{k}{g^k(t, x) L_{X_E} V(t, \overline{x})} \left(\frac{\partial V}{\partial t}(t, \overline{x}) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, \overline{x}) \tilde{X}_i(t, \delta_{g(t,x)} x) \right) \\ &= \frac{k}{g^k(t, x) L_{X_E} V(t, \overline{x})} \left(\frac{\partial V}{\partial t}(t, \overline{x}) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, \overline{x}) X_i(t, \overline{x}) \right) \\ &= \frac{k \tilde{\rho}^k(t, x)}{L_{X_E} V(t, \overline{x})} \frac{dV}{dt}(t, \overline{x}). \\ &= \left(\frac{k}{L_{X_E} V} \frac{dV}{dt} \right)(t, \overline{x}) \cdot \tilde{V}(t, x). \end{aligned}$$

The only remaining fact to show is that $L_{X_E} V(t, \overline{x}) > 0$. $L_{X_E} V(t, \overline{x})$ is constant sign from transversity so initially assume that this quantity is negative. For ϵ sufficiently small the points in the sets $G_t^{C+\epsilon}$ and $G_t^{C-\epsilon}$ also satisfy $L_{X_E} V < 0$. As shown above, these sets are diffeomorphic to spheres (for t fixed) and so separate $\mathbb{R}^n$ into an exterior and interior domain. Fix an arbitrary $t_0 \in [0, T)$. The trajectory of X_E pierces each set only once and since $L_{X_E} V < 0$ then we conclude that $G_{t_0}^{C+\epsilon}$ sits inside the interior domain of $G_{t_0}^C$ which sits inside the interior domain of $G_{t_0}^{C-\epsilon}$. This holds for all t since t_0 is arbitrary. If we start the system $\dot{x} = X(t, x)$ with an initial condition (τ, x) in the set $G_\tau^{C-\epsilon}$ then at some time later the trajectory enters the ball radius of $\min_{t \in [0,t), x \in G_t^{C+\epsilon}} \|x\|$ by asymptotic stability. Thus at some $\tau' > \tau$ the trajectory crosses $G_{\tau'}^{C+\epsilon}$ but

$V(\tau', x(\tau')) = C + \epsilon > V(\tau, x(\tau)) = C - \epsilon$ which contradicts the fact that $\dot{V} < 0$. Hence, $L_{X_E} V(t, \overline{x}) > 0$ and the system with modified feedbacks is asymptotically stabile. ρ-exponential stability follows from the fact that the closed-loop system is degree zero. ■

The new feedback is as smooth on $\mathbb{R}^n \setminus \{0\}$ as the original feedback restricted to the level set of the Lyapunov function in the proof of Theorem 4.12. The original feedback is assumed to be at least Lipschitz and so solutions of the closed-loop system with the modified feedback are unique by Lemma 3.4.

The following example demonstrates the algorithm on the prototype driftless system (4.7).

Example 4.14 This example uses the three-dimensional two input driftless system (4.7) to illustrate the algorithm. A smooth asymptotically stabilizing feedback for the system are the functions

$$\begin{aligned} u_1(t, x) &= -x_1 + x_3 \cos t, \\ u_2(t, x) &= -x_2 + x_3^2 \sin t. \end{aligned}$$

Asymptotic stability of the closed-loop system can be shown using the following Lyapunov function,

$$V(t, x) = \left(x_1 - \frac{x_3}{2}(\cos t + \sin t)\right)^2 + \left(x_2 - \frac{x_3^2}{2}(\sin t - \cos t)\right)^2 + x_3^2.$$

Thus we need to check the transversality condition with a level of the Lyapunov function. V may be approximated by the quadratic form $\tilde{V} = \langle x, Bx \rangle$ for C sufficiently small, where

$$B = \begin{pmatrix} 1 & 0 & -\frac{1}{2}\alpha \\ 0 & 1 & 0 \\ -\frac{1}{2}\alpha & 0 & 1 + \frac{1}{4}\alpha^2 \end{pmatrix},$$

and $\alpha = \cos t + \sin t \in [-\sqrt{2}, \sqrt{2}]$. The inner product between the level sets of $\tilde{V}$ and the Euler vector field is

$$\begin{aligned} L_{X_E} \tilde{V} &= \langle x, \mathrm{diag}[r_i] Bx \rangle \\ &= \langle x, \tilde{B} x \rangle, \end{aligned}$$

where $\tilde{B}$ is the symmetric matrix

$$\tilde{B} = \begin{pmatrix} 1 & 0 & -\frac{3}{4}\alpha \\ 0 & 1 & 0 \\ -\frac{3}{4}\alpha & 0 & 2 + \frac{1}{2}\alpha^2 \end{pmatrix}.$$

Since $\tilde{B}$ is positive definite for all $\alpha \in [-\sqrt{2}, \sqrt{2}]$ the Euler vector field is transverse to any level set of $\tilde{V}$ and hence any level set of V for C sufficiently small.

Experimentation reveals that value of $C = 1$ works well. The modification of the feedbacks is carried out as specified in the proof. What makes this method attractive from an implementation point of view is the fact that the function $g(t,x)$ is easily computed by searching over a single *scalar* parameter λ such that $V(t, \Delta_\lambda x) = C$. In addition $V(t, \Delta_\lambda x)$ is a monotone increasing function of λ in a neighborhood of the λ which satisfies this expression. This search may be performed efficiently in real-time. Once the value of λ has been computed which satisfies $V(t, \Delta_\lambda x) = 1$ then we set $\tilde{\rho}(t,x) = 1/\lambda$ and $\overline{x} = \gamma(t,x) = \Delta_\lambda x$. The modified feedbacks are

$$\begin{aligned}
\tilde{u}_1(t,x) &= \frac{1}{\lambda}\left(-\overline{x}_1 + \overline{x}_3 \cos t\right), \\
&= \frac{1}{\lambda}\left(-\lambda x_1 + \lambda^2 x_3 \cos t\right), \\
&= -x_1 + \lambda x_3 \cos t, \\
\tilde{u}_2(t,x) &= \frac{1}{\lambda}\left(-\overline{x}_2 + \overline{x}_3^2 \sin t\right), \\
&= \frac{1}{\lambda}\left(-\lambda x_2 + \lambda^4 x_3^2 \sin t\right), \\
&= -x_2 + \lambda^3 x_3^2 \sin t.
\end{aligned}$$

Simulations comparing the performance of these feedbacks with the original smooth feedbacks are shown in Figure 4.3. The ρ-exponential stabilizer returns the system to a small neighborhood of the origin much faster than the smooth controller from which it was derived. The energy in the control signals

$$E_T(u) = \int_0^T (u_1^2(t) + u_2^2(t))dt,$$

is shown in Figure 4.4. Note that the ρ-exponential stabilizer requires bounded energy to return the system to the origin. Center manifold analysis may be used to show that the rate of decay of the closed-loop system with the smooth controller is bounded by a constant times $1/\sqrt{t}$ for large t. Hence, the smooth control law consumes an unbounded amount of energy to return the system to the origin. Another important fact is that the smooth and modified control laws match on the set where $V(t,x) = C$ so the maximum controller effort commanded by the exponential stabilizer does not exceed that of the smooth control law for all initial conditions satisfying $V(t, x(0)) \leq C$.

Note that the system is robust to any higher order perturbing terms which were neglected in the approximation process. This is consequence of Proposition 4.10.

4.4 Practical Considerations

The previous sections presented several methods for obtaining local ρ-exponentially stabilizing feedbacks. This section presents several results of practical significance for degree zero systems.

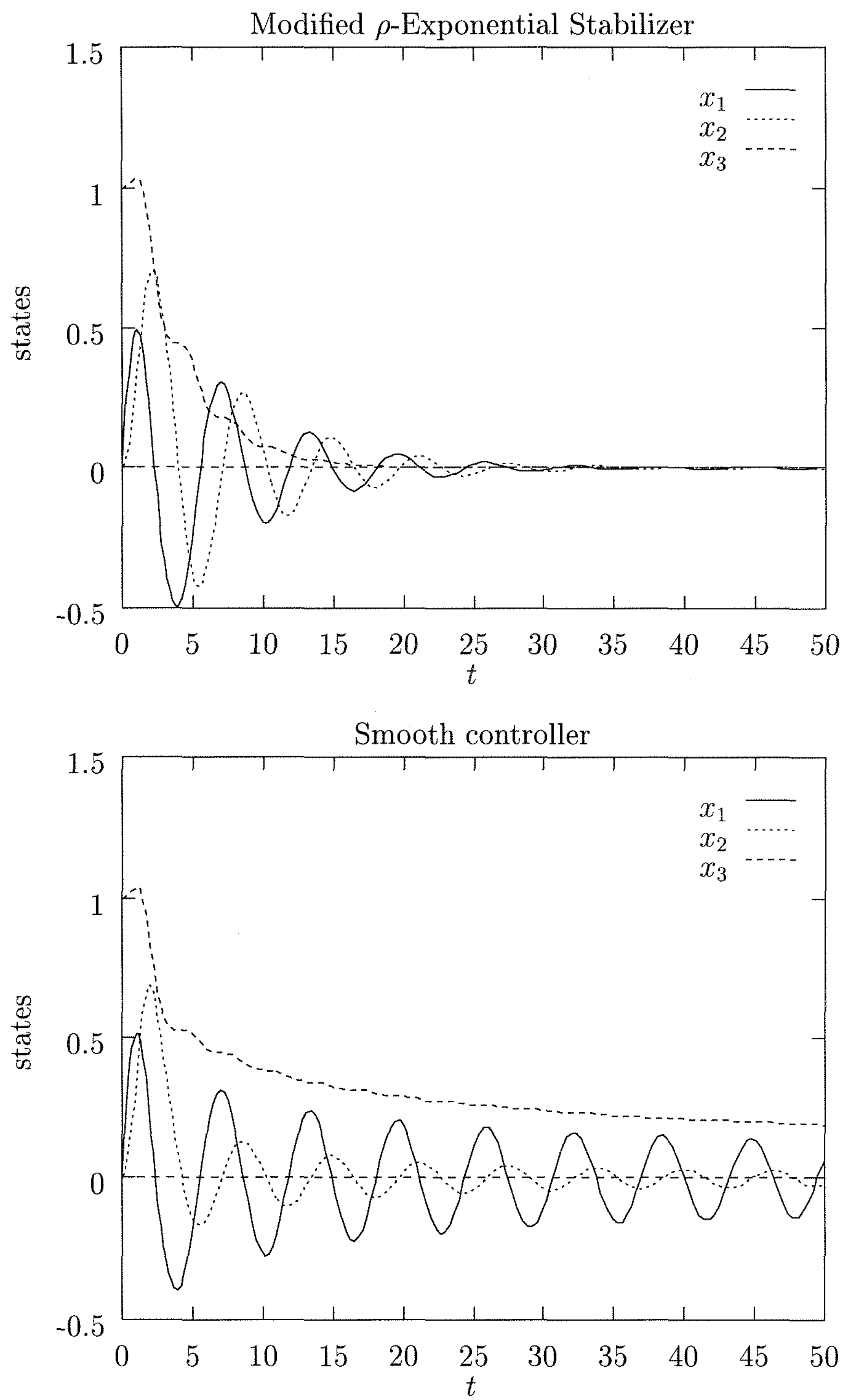

Figure 4.3: Comparison of modified feedback and smooth feedback.

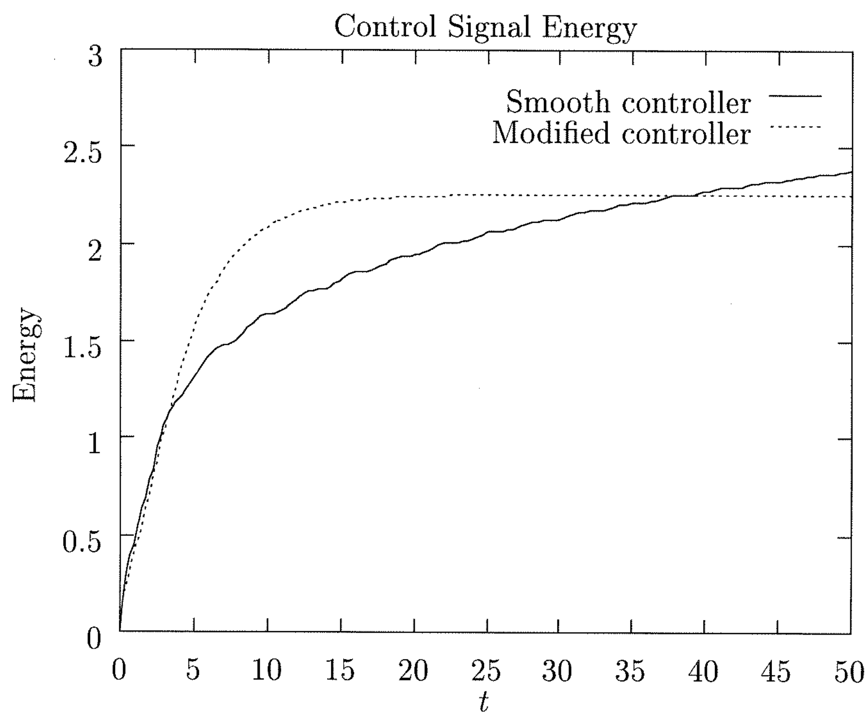

Figure 4.4: Comparison of energy in control signal.

Since a linearization of the closed-loop system does not exist we must be certain that standard control practices, such as filtering the measurements, do not destabilize the system. Lowpass filters are often used to smooth sensor measurements to avoid aliasing during digital sampling. We show below that the inclusion of lowpass filters in the loop do not change the ρ-exponentially property of solutions provided the filter bandwidth is sufficiently high. This result is reminiscent of the stability theory for singularly perturbed systems. However, since the linearization is not defined, the usual singular perturbation results are not applicable.

Many driftless control systems represent kinematic models in which the control inputs are velocities. A simple model for including actuator dynamics is to extend the kinematic model to a system with a set of integrators preceeding each input. The inputs into the integrators represent the control commands in this case. This section demonstrates how a ρ-exponentially stabilizing controller can be converted to a controller which stabilizes the system with integrators.

Another concern is the increased sensitivity to noise around the equilibrium point due to the non-Lipschitz nature of the feedback. The benefit of non-Lipschitz feedback is an increased rate of convergence. However the non-Lipschitz feedbacks can present some additional complications. In particular, if the output of the controller is specified directly by non-Lipschitz functions then any disturbance in the signals processed through these functions can lead to large *control rates.* This is mitigated by filtering the output of the non-Lipschitz functions. The framework established for studying actuator dynamics may be applied to this problem as well except that the additional integrators now become states of the controller. Several examples illustrate the applications of these results.

4.4.1 Filtering of measurements

Every system with a digital controller must include some form of measurement filtering to avoid aliasing. In linear systems, or nonlinear systems with well defined linearization, the "dynamics" often dictate the filter bandwidth: the cutoff frequency of the filter is chosen to be higher that the frequency band where active control is desired. Driftless systems have no intrinsic time scale associated with them because turning off the control inputs freezes the state. Lowpass filtering of the state measurements is still required though to prevent aliasing. This section proves that including a simple lowpass filter of the form $\frac{1}{s/\omega_c+1}$ in the loop does not destabilize a uniformly asmptotically stable degree zero system provided the cutoff frequency is sufficiently high. This fact is not immediately obvious especially for some of the planar systems studied by Kawski [19]. For example, Kawski has shown that the system,

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_2 - x_1^3, \end{aligned} \tag{4.18}$$

may be asymptotically stabilized with the feedback,

$$u = -kx_1 + kx_2^{\frac{1}{3}},$$

for k sufficiently large. This system is homogeneous degree zero with respect to the dilation $\Delta_\lambda x = (\lambda x_1, \lambda^3 x_2)$. The interesting point is that the linearization of equation (4.19) has an uncontrollable unstable mode. Hence, in this example the non-Lipschitz feedback succeeds in stabilizing the system where a C^1 feedback cannot. It is not unreasonable to think that placing a lowpass filter in the loop may destroy the asymptotic stability of the closed-loop system because of the phase lag of the filtered signal. However, this is not the case as shown in the following specialized singular perturbation result for degree zero systems.

Proposition 4.15 *Suppose the system*

$$\dot{x} = X(t,x) \quad x \in \mathbb{R}^n, \tag{4.19}$$

is continuous,time-periodic and degree zero with respect to the dilation Δ_λ. Let $x = 0$ be an asymptotically (and hence ρ-exponentially) stable equilibrium point. Then for $k > 0$ sufficiently large, the filtered *system,*

$$\begin{aligned} \dot{x} &= X(t,y) \\ \dot{y} &= -kI_n(y-x) \end{aligned} \quad (x,y) \in \mathbb{R}^{2n}, \tag{4.20}$$

is also ρ-exponentially stable with respect to the new dilation $\tilde{\Delta}_\lambda(x,y) = (\Delta_\lambda x, \Delta_\lambda y)$.

Proof: The filtered system (4.20) is degree zero with respect to $\tilde{\Delta}_\lambda$ so asymptotic stability need only be shown. Suppose a homogeneous norm associated with Δ_λ is $\rho(x) = (x_1^{c/r_1} + \cdots + x_n^{c/r_n})^{\frac{1}{c}}$ where c is evenly divisible by r_i, $i = 1, \ldots, n$. A smooth homogeneous norm for the filtered system is,

$$\tilde{\rho}(x,y) = \left(x_1^{c/r_1} + \cdots + x_n^{c/r_n} + y_1^{c/r_1} + \cdots + y_n^{c/r_n}\right)^{\frac{1}{c}}.$$

The compatible projection is $\tilde{\pi} : \mathbb{R}^{2n} \setminus \{0\} \to S_{\tilde{\Delta}}$ where,

$$\tilde{\pi}(x,y) = \left(\frac{x_1}{\tilde{\rho}^{r_1}(x,y)}, \ldots, \frac{x_n}{\tilde{\rho}^{r_n}(x,y)}, \frac{y_1}{\tilde{\rho}^{r_1}(x,y)}, \ldots, \frac{y_n}{\tilde{\rho}^{r_n}(x,y)}\right),$$

and the sphere is $S_{\tilde{\Delta}} = \{(x,y) \in \mathbb{R}^{2n} | \tilde{\rho}(x,y) = 1\}$.

Since $x = 0$ is a ρ-exponentially stable equilibrium point of the original system there exists a Lyapunov function $V(t,x)$, which is smooth on $\mathbb{R}^n \setminus \{0\}$ and degree c with respect Δ_λ, such that

$$\begin{aligned} \frac{dV}{dt}(t,x)|_{(4.19)} &= \frac{\partial V}{\partial t}(t,x) + (\nabla V \cdot X)(t,x) \\ &\le -b_1 \rho^c(x), \end{aligned}$$

for some $b_1 > 0$. Note that the $\partial V/\partial x_i$ is degree $c - r_i > 0$, for $i = 1, \ldots, n$ and so must be continuous. Now consider the new positive definite function $\tilde{V}(t,x,y)$ for the filtered system,

$$\tilde{V}(t,x,y) = V(t,x) + \sum_{i=1}^{n} \frac{r_i}{c}(y_i - x_i)^{\frac{c}{r_i}}.$$

The derivative of $\tilde{V}$ along solutions of the filtered system is

$$\begin{aligned}\frac{d\tilde{V}}{dt}(t,x,y)|_{(4.20)} = \frac{\partial V}{\partial t}(t,x,y) + \nabla V(t,x) \cdot X(t,y) - \sum_{i=1}^{n}(y_i - x_i)^{\frac{c}{r_i}-1} X_i(t,y) \\ - k\sum_{i=1}^{n}(y_i - x_i)^{\frac{c}{r_i}},\end{aligned}$$

where X_i denotes the i^{th} component of the vector field X. The time derivative $d\tilde{V}/dt$ is homogeneous degree c with respect to $\tilde{\Delta}_\lambda$ and is continuous in all of its arguments. The objective is to show that $d\tilde{V}/dt$ is negative definite for $k > 0$ sufficiently large. If $(\overline{x}, \overline{y})$ denotes a point on $S_{\tilde{\Delta}}$ then the time derivative of $\tilde{V}$ along solutions of equation (4.20) is,

$$\begin{aligned}\frac{d\tilde{V}}{dt}(t,x,y) = \tilde{\rho}^c(x,y)\Bigg[\frac{\partial V}{\partial t}(t,\overline{x}) + \nabla V(t,\overline{x}) \cdot X(t,\overline{y}) - \sum_{i=1}^{n}(\overline{y}_i - \overline{x}_i)^{\frac{c}{r_i}-1} X_i(t,\overline{y}) \\ -k\sum_{i=1}^{n}(\overline{y}_i - \overline{x}_i)^{\frac{c}{r_i}}\Bigg].\end{aligned}$$

The terms in the square brackets are continuous on $S_{\tilde{\Delta}}$ since they are the restriction continuous functions. Define the set $G = \{(\overline{x}, \overline{y}) \in S_{\tilde{\Delta}} |\, \overline{x} = \overline{y}\}$. For all $(\overline{x}, \overline{y})$ in G the following bound holds,

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,\overline{x}) + \nabla V(t,\overline{x}) \cdot X(t,\overline{y}) - \sum_{i=1}^{n}(\overline{y}_i - \overline{x}_i)^{\frac{c}{r_i}-1} X_i(t,\overline{y}) \\ &= \frac{\partial V}{\partial t}(t,\overline{x}) + \nabla V(t,\overline{x}) \cdot X(t,\overline{x}) \\ &\le \quad -b_1\rho(\overline{x}) \\ &= -b_1\frac{\rho(x)}{\tilde{\rho}(x,x)} \\ &= -b_1\left(\frac{1}{2}\right)^{\frac{1}{c}}.\end{aligned}$$

By continuity there must exist an open neighborhood $U \subset S_{\tilde{\Delta}}$ of G such that

$$\frac{\partial V}{\partial t}(t,\overline{x}) + \nabla V(t,\overline{x}) \cdot X(t,\overline{y}) - \sum_{i=1}^{n}(\overline{y}_i - \overline{x}_i)^{\frac{c}{r_i}-1} X_i(t,\overline{y}) < -b_1\left(\frac{1}{2}\right)^{\frac{1}{c}+1}.$$

Now define the constants,

$$M = \max_{(\overline{x},\overline{y})\in S_{\tilde{\Delta}}\setminus U} \frac{\partial V}{\partial t}(t,\overline{x}) + \nabla V(t,\overline{x})\cdot X(t,\overline{y}) - \sum_{i=1}^{n}(\overline{y}_i - \overline{x}_i)^{\frac{c}{r_i}-1}X_i(t,\overline{y})$$

$$m = \min_{(\overline{x},\overline{y})\in S_{\tilde{\Delta}}\setminus U} \sum_{i=1}^{n}(\overline{y}_i - \overline{x}_i)^{\frac{c}{r_i}}.$$

Note that m must be greater than zero.

Back to evaluating the time derivative of $\tilde{V}$,

$$\frac{d\tilde{V}}{dt}(t,x,y) \leq \begin{cases} -b_1\left(\frac{1}{2}\right)^{\frac{1}{c}+1}\tilde{\rho}(x,y) & (\pi(x),\pi(y))\in U \\ (M-mk)\tilde{\rho}(x,y) & (\pi(x),\pi(y))\in S_{\tilde{\Delta}}\setminus U \end{cases}.$$

Choosing $k>0$ such that $M-mk<0$ makes the time derivative of $\tilde{V}$ negative definite. Thus the filtered system (4.20) is ρ-exponentially stable with respect to $\tilde{\Delta}_\lambda$. ■

Remark 4.16 The proposition may be applied "recursively" to show that lowpass filters with faster rolloff also preserve stability for sufficiently high bandwidth.

Lowpass filtering of the state measurements may also attenuate the effect of noise on the size of the ball of convergence versus the bound on the noise. If there is some persistently acting noise disturbance, the state will not converge to zero but will be confined to some ball around the origin. If most of the noise spectrum is above the cutoff frequency of the lowpass filter then the ball size will shrink. However, introducing a filter reduces the rate of convergence of the states since the filtered state lags in phase behind the actual state. The quantitative aspects of the trade-off between choosing a filter cutoff frequency to maximize the attenuation of sensor noise versus closed-loop rate of convergence is not explored here except in the following example.

Example 4.17 The system in equations (4.7) will be used to illustrate the reduction in the size of the ball of convergence when noise is present in the measurement and the measurements are filtered. As noted in previous examples the system in equation (4.7) is homogeneous degree one with respect to the dilation $\delta_\lambda(z) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$. A (globally) ρ-exponentially stabilizing controller is given by

$$\begin{aligned} u_1(t,x) &= -x_1 + \tfrac{x_3}{\rho(x)}\cos t \\ u_2(t,x) &= -x_2 + \tfrac{x_3^2}{\rho^3(x)}\sin t, \end{aligned} \tag{4.21}$$

where ρ is defined as $\rho(x) = (x_1^4 + x_2^4 + x_3^2)^{(1/4)}$. A rigorous proof that this controller ρ-exponentially stabilizes the system may be found in Appendix C.

This controller is chosen since it is easy to manipulate. The closed-loop system with lowpass filter is model as,

$$
\begin{aligned}
\dot{x}_1 &= -y_1 + \frac{y_3}{\rho(y)} \cos t \\
\dot{x}_2 &= -y_2 + \frac{y_3^2}{\rho^3(y)} \sin t \\
\dot{x}_3 &= x_2(-y_1 + \frac{y_3}{\rho(y)} \cos t) \\
\dot{y}_1 &= -k(y_1 - x_1) \\
\dot{y}_2 &= -k(y_2 - x_2) \\
\dot{y}_3 &= -k(y_3 - x_3).
\end{aligned}
$$

The filtered system is ρ-exponentially stable for k sufficiently large. The system is most sensitive to noise introduced in the x_3 variable. The argument for this is simple. Suppose noise is introduced into the x_3 variable and the maximum amplitude of the states is measured. Now reduce the noise until the amplitude of x_3 is some factor γ of its original value. Since the system is homogeneous, the homogeneous ball which bounds the trajectories of the system will have the x_1 and x_2 amplitudes scaled by $\sqrt{\gamma}$.

A sinusoid of constant amplitude, $n(t) = d \sin \omega t$, is added to x_3 to model noise. Figure 4.5 shows a numerical simulation of the system with and without the lowpass filter. The parameters for the simulation are $k = 3$, $d = 0.2$ and $\omega = 10$. The size of the ball that the trajectories are confined to is decreased with the addition of the filter. Figure 4.6 points out another issue of importance: the control *rate* is very high even in the system with the filter implemented. The reason the rate is large is due to the fact that the control functions given by equations (4.21) are not Lipschitz at the origin. Thus a smooth signal passed through these functions in a neighborhood of the origin can have an arbitrarily large time derivative. This situation is highly undesirable since control rate limits often exist in practice. The next section on dynamic extension shows how the controller output can be smoothed and still preserve ρ-exponential stability.

4.4.2 Torque inputs and dynamic extension

Traditionally, stabilization of driftless systems has concentrated on the use of kinematic models of the system for control design. That is, the velocity of the system is assumed to be a direct input which can be manipulated. Based on these kinematic models, a number of researcher have developed control strategies which result in asymptotic or exponential stabilization of the system around an equilibrium point. In the exponential case, structural limitations require that the control laws be nondifferentiable at the equilibrium point. This raises questions about the applicability of such controls to physical systems in which the torques, and not the velocities, are the control inputs to the system.

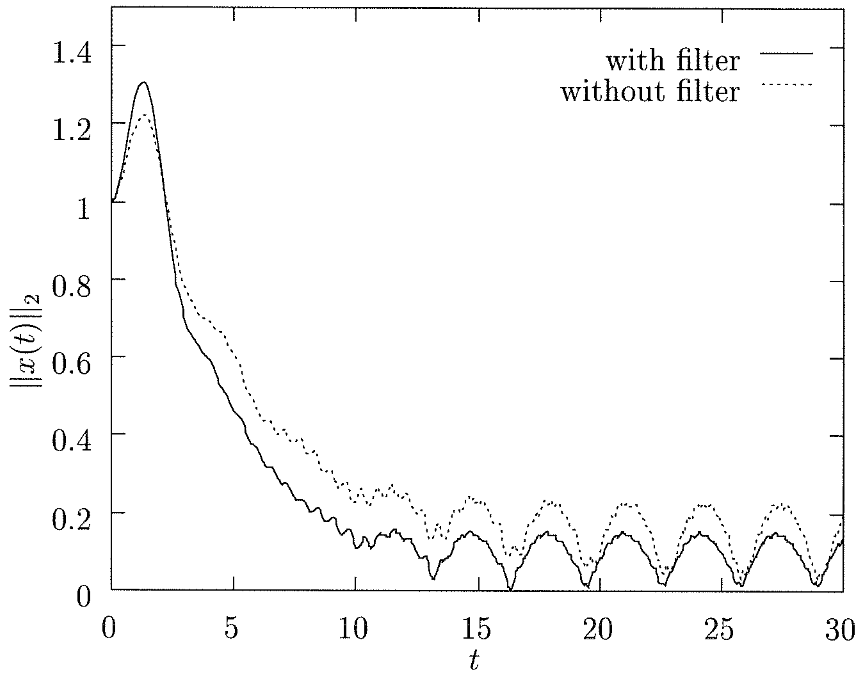

Figure 4.5: Norm of states for the systems.

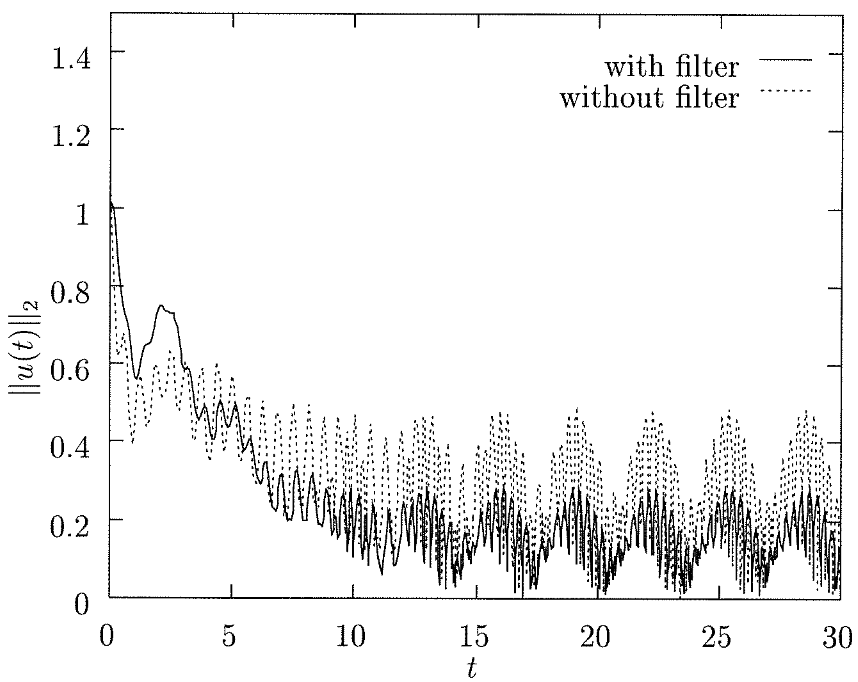

Figure 4.6: Norm of control command.

This section develops some tools for synthesizing control laws for mobile robots and other driftless systems that are controlled by input torques. The main result gives a set of conditions under which a kinematic controller (i.e., one which assumes the velocities are the inputs) can be converted to a dynamic controller (one which uses the torques as the inputs).

We concentrate on the class of control systems of the form

$$\begin{aligned} \dot{x} &= X_1(x)u_1 + \cdots + X_m(x)u_m \quad && x \in \mathbb{R}^n \\ \dot{u} &= v && u, v \in \mathbb{R}^m. \end{aligned} \tag{4.22}$$

The system

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m \tag{4.23}$$

describes the "kinematic portion" of the system and, for mobile robots, is derived from the Pfaffian constraints which describe the condition that the wheels roll but not slide. We model the dynamic portion of the system via a simple set of integrators. For many, but not all, systems, more complicated dynamic behavior can be converted to this form using a state-feedback control law. We call equation (4.22) the *dynamic* system and equation (4.23) the *kinematic* system. The blanket hypothesis for the systems in this section are:

Assumption 4.18

1. the vector fields X_i are degree one with respect to a given dilation Δ_λ,
2. the controls $u_i = \alpha_i, i = 1, \ldots, m$ are uniformly asymptotically stabilizing feedbacks (for the kinematic system) which are degree one in x with respect to Δ_λ, smooth and time-periodic in t and smooth on $x \in \mathbb{R}^n \setminus \{0\}$,
3. $\operatorname{rank}[X_1(0) \cdots X_m(0)] = m$.

For smooth controllers, extending kinematic controllers to dynamic controllers is straightforward and has been explored, for example, by Walsh and Bushnell [45]. However, due to the nondifferentiable nature of exponential stabilizers we consider here, the usual control Lyapunov approach does not directly apply and must be modified to verify that the extended controller is well-defined and continuous. The use of continuous functions is important in applications since discontinuous control inputs usually are smoothed by the control electronics and/or the system dynamics and hence cannot be applied in practice, possibly resulting in loss of exponential rate of convergence. The main result of this section is stated in the proposition below.

Proposition 4.19 *Let $u = \alpha(t, x)$ be a feedback satisfying the conditions of Assumption 4.18. Then the feedback*

$$v_i = L_{\alpha X}\alpha_i + \frac{\partial \alpha_i}{\partial t} + k(\alpha_i - u_i), \quad i = 1, \ldots, m \tag{4.24}$$

globally exponentially stabilizes the dynamic system (4.22) for $k > 0$ sufficiently large.

The notation αX is used to denote the vector field $\sum_i \alpha_i X_i$. Controller (4.24) is continuous for all (t, x, u) and smooth for all $x \neq 0$. Furthermore, the control law is homogeneous of degree one with respect to the extended dilation,

$$\tilde{\Delta}_\lambda(x, u) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n, \lambda u_1, \ldots, \lambda u_m). \tag{4.25}$$

Thus the closed-loop system remains degree zero with this feedback.

Proof: The closed-loop kinematic system is time-periodic, degree zero and asymptotically stable. This implies that there exists a time-periodic homogeneous Lyapunov function $V(t, x)$ such that $V(t, x) > 0$ for all $x \neq 0$ and all t which is strictly decreasing when $u = \alpha(t, x)$. This requires the extension of Rosier's converse Lyapunov theorem to time-periodic homogeneous degree zero systems developed in Section 2.3. The Lyapunov function may be chosen to be degree two with respect to Δ_λ. Thus the following bounds exist:

$$\begin{array}{c} c_1\rho^2(x) \leq V(t,x) \leq c_2\rho^2(x) \\ \frac{dV}{dt}\big|_{\dot{x}=\alpha X}(t,x) \leq -c_3\rho^2(x), \end{array} \tag{4.26}$$

for some $c_i > 0$ and where ρ is a homogeneous norm with respect to Δ_λ.

For the dynamic system with feedback (4.24) we use the following function,

$$W(t, x, u) = V(t, x) + \frac{1}{2}\sum_{i=1}^{m}(\alpha_i(t, x) - u_i)^2. \tag{4.27}$$

This function is positive definite on the extended phase space (x, u) and so is a candidate for a Lyapunov function. W is also degree two with respect to the extended dilation $\tilde{\Delta}$ defined in (4.25). Continuous partials of W with respect to x do not necessarily exist when $x = 0$, however when $x \neq 0$ the derivative of (4.27) along the trajectories of the system (4.22) with feedback (4.24) is,

$$\dot{W} = \dot{V} + \sum_{i=1}^{m}\left(\sum_{l=1}^{m}\left(\sum_{j=1}^{n}\frac{\partial \alpha_i}{\partial x_j}X_l^{(j)}\right)u_l + \frac{\partial \alpha_i}{\partial t} - v_i\right)(\alpha_i - u_i), \quad x \neq 0$$

where $X_l^{(j)}$ represents the j^{th} component of the l^{th} input vector field. Substituting the expression for v_i and writing

$$L_{X\alpha}\alpha_i = \sum_{l=1}^{m}\left(\sum_{j=1}^{n}\frac{\partial \alpha_i}{\partial x_j}X_l^{(j)}\right)\alpha_l,$$

the time derivative of W when $x \neq 0$ becomes,

$$\dot{W} = \dot{V} + \sum_{i=1}^{m}\left(\sum_{l=1}^{m}\left(\sum_{j=1}^{n}\frac{\partial \alpha_i}{\partial x_j}X_l^{(j)}\right)(u_l - \alpha_l) - k(\alpha_i - u_i)\right)(\alpha_i - u_i)$$

$$=\dot{V} + (\alpha - u)^T(-kI_m + Q(t,x))(\alpha - u).$$

I_m denotes the $m \times m$ identity matrix and $Q(t,x)$ is an $m \times m$ matrix with ij^{th} component given by,

$$[Q]_{ij} = -\frac{1}{2}(L_{X_i}\alpha_j + L_{X_j}\alpha_i). \tag{4.28}$$

$L_{X_i}\alpha_j$ is a degree zero function and so is not necessarily defined at $x = 0$.

A useful observation is that $\dot{V}$ is a continuous function of x,

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{l=1}^{m} u_l L_{X_l} V, \tag{4.29}$$

since $\partial V/\partial t$ is degree two and the $L_{X_l}V, l = 1, \dots, m$, are degree one functions.

The condition in Assumption 4.18 that $\operatorname{rank}[X_1(0) \cdots X_m(0)] = m$ guarantees that no non-trivial trajectory of the closed-loop system is contained in the set $Z = \{(x,u) : x = 0, u \neq 0\}$. This is shown by considering the set of vectors $[I_n \; 0_{n\times m}]^T$ which are orthogonal to the set Z. The dot product of these vectors with the closed-loop vector field is

$$\begin{aligned} \begin{bmatrix} I_n \\ 0_{m\times n} \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} (0,u) &= \sum_{i=1}^{m} X_i(0) u_i \\ &= 0 \Leftrightarrow u = 0. \end{aligned}$$

Thus, if a trajectory passes through the set Z at time t^* then $\frac{dW}{dt}(t^*)$ may not be defined however $\frac{dW}{dt}(t^* \pm \epsilon)$ *is* defined for all $\epsilon > 0$ sufficiently small. Thus the upper right Dini derivative of $W(t)$,

$$D^+W(t^*) \doteq \limsup_{\epsilon \to 0^+} \frac{W(t^* + \epsilon) - W(t^*)}{\epsilon},$$

is equal to the right-hand derivative of $dW/dt(t^*)$ since dW/dt is continuous at $t^* + \epsilon$ for $\epsilon > 0$ sufficiently small:

$$D^+W(t^*) = \lim_{\epsilon \to 0^+} \frac{dW}{dt}(t^* + \epsilon).$$

Substituting the original expression for $\dot{W}$ when $x \neq 0$ into the expression for D^+W yields and recalling that $\dot{V}$ is continuous in all arguments,

$$\begin{aligned} D^+W(t^*) &= \lim_{\epsilon \to 0^+} \left[\frac{dV}{dt} + (\alpha - u)^T(-kI_m + Q)(\alpha - u) \right]_{t = t^* + \epsilon} \\ &= \frac{dV}{dt}(t^*, x(t^*)) - k\|\alpha(t^*, x(t^*)) - u(t^*)\|^2 \\ &\quad + \lim_{\epsilon \to 0^+} \left[(\alpha - u)^T Q (\alpha - u) \right]_{t^* + \epsilon} \\ &\leq \frac{dV}{dt}(t^*, x(t^*)) + (-k + q)\|\alpha(t^*, x(t^*)) - u(t^*)\|^2 \end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm and

$$q = \sup_{t\in[0,2\pi),x\neq 0} \|Q(t,x)\|_F. \tag{4.30}$$

$\|\cdot\|_F$ denotes the Frobenius norm. q is well defined since Q is degree zero and assumes all of its values when restricted to the homogeneous sphere $\{x : \rho(x) = 1\}$. The above bound is also valid for $\dot{W}$ when $x \neq 0$ so,

$$D^+W(t) \leq \frac{dV}{dt} + (-k+q)\|\alpha - u\|^2 \quad \forall t, x, u.$$

Substituting the expression for $\dot{V}$ from (4.29) yields,

$$D^+W \leq \frac{\partial V}{\partial t} + \sum_{k=1}^{m} \alpha_k L_{X_k} V + \sum_{k=1}^{m} (u_k - \alpha_k) L_{X_k} V + (-k+q)\|\alpha - u\|^2 \ \forall t, x, u. \tag{4.31}$$

The first two terms on the right side of the inequality are the time derivative of V along trajectories of the system when $u = \alpha(t,x)$ and may be bounded by $-c_3\rho^2(x)$ from equation (4.26). The third term to the right of the inequality may be bounded by $c_4\rho(x)\|u - \alpha\|$ for some $c_4 > 0$. Substituting these bounds into equation (4.31) yields,

$$\begin{aligned} D^+W &\leq -c_3\rho^2(x) + c_4\rho(x)\|u-\alpha\| + (-k+q)\|\alpha - u\|^2 \\ &= \begin{pmatrix} \rho(x) & \|u - \alpha(t,x)\| \end{pmatrix} \begin{pmatrix} -c_3 & \frac{1}{2}c_4 \\ \frac{1}{2}c_4 & -k+q \end{pmatrix} \begin{pmatrix} \rho(x) \\ \|u-\alpha(t,x)\| \end{pmatrix}. \end{aligned}$$

This bound is negative definite when $k > k^* \doteq q + \frac{1}{4}\frac{c_2^2}{c_1}$. Furthermore the bound is degree two with respect to the dilation $\tilde{\Delta}_\lambda$ so,

$$D^+W \leq -\tilde{k}W,$$

for some $\tilde{k} > 0$ whenever $k > k^*$. The differential inequality from [23, Theorem 1.4.1] implies,

$$W(t) \leq W(0)e^{-\tilde{k}t}.$$

Hence, the system is asymptotically stable. Exponential stability follows from the fact that the closed-loop system is degree zero with respect to the extended dilation $\tilde{\Delta}_\lambda$ defined in equation (4.25). This completes the proof. ■

The states u also approach $\alpha(t,x)$ exponentially since the time derivative of $\|u-\alpha\|^2$ may be written as,

$$\begin{aligned} \frac{d}{dt}\|u-\alpha\|^2 &= (u-\alpha)^T(-kI_m + Q)(u-\alpha) \\ &\leq (-k+q)\|u-\alpha\|^2, \end{aligned}$$

where Q and q are defined in equations (4.28) and (4.30).

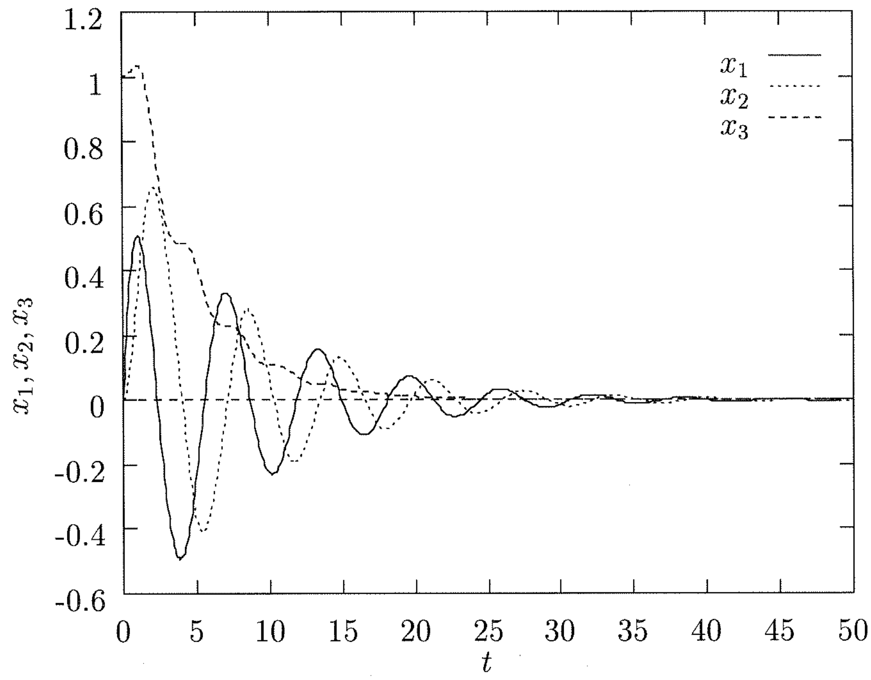

Figure 4.7: Kinematic state response.

In many situations one is forced to rely on a local homogeneous approximation of the kinematic system and the closed system is only locally exponentially stable. In this case, the construction in the proof of the proposition can still be used but gives only a local exponentially stabilizing controller for the dynamic system. The region of convergence may be smaller for the dynamic system than for the original kinematic system since we require that while u is converging to $\alpha(x,t)$, the state must remain within the region of attraction of the original controller. The region of attraction can be enlarged by increasing the rate of convergence of u to α (up to the limits of the actuators).

The form of the control law shows that it can be regarded as a combined control law consisting of a feedforward portion, which drives the system along the desired trajectory when $u = \alpha(x,t)$, and a feedback portion, which stabilizes the the (extended) state space equation $u = \alpha(x,t)$. The following example illustrates the procedure.

Example 4.20 We illustrate the dynamic extension procedure with system (4.7) and the feedback in equation (4.21) This feedback is extended to the system with integrators,

$$\begin{aligned} \dot{x}_1 &= u_1 & \dot{u}_1 &= v_1 \\ \dot{x}_2 &= u_2 & \dot{u}_2 &= v_2 \\ \dot{x}_3 &= x_2 u_1, \end{aligned} \tag{4.32}$$

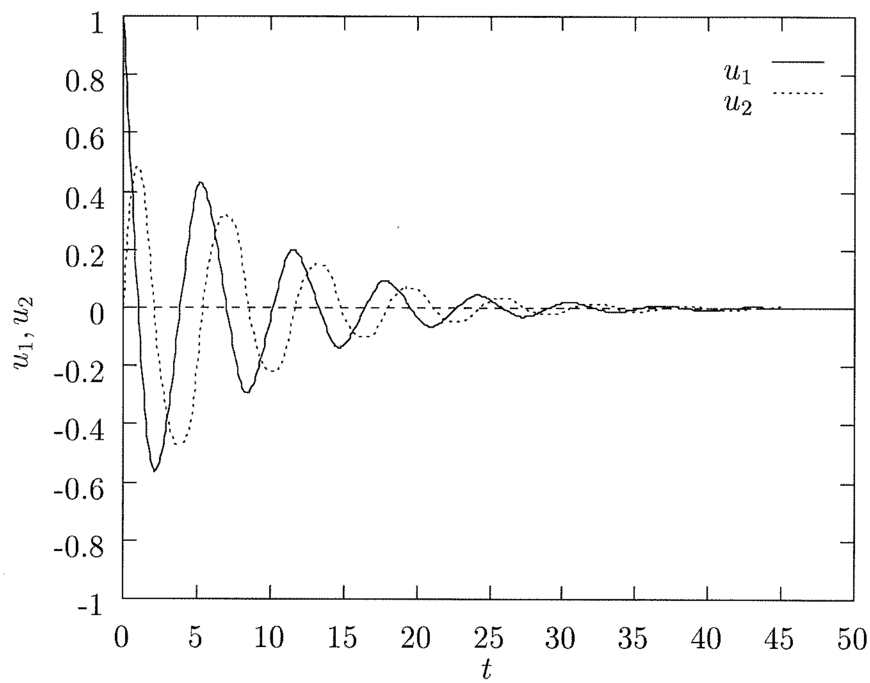

Figure 4.8: Extended state response.

where the new feedback functions v_i are computed to be

$$v_i = L_{X_\alpha}\alpha_i + \frac{\partial \alpha_i}{\partial t} + k(\alpha_i - u_i). \tag{4.33}$$

The terms $L_{X_\alpha}\alpha_i$ are

$$\begin{aligned} L_{X_\alpha}\alpha_1 =& \alpha_1(x,t)\left(-1 - \frac{x_1^3 x_3}{\rho^5}\cos t\right) \\ &+ \alpha_2(x,t)\left(-\frac{x_2^3 x_3}{\rho^5}\cos t\right) \\ &+ x_2\alpha_1(x,t)\left(\frac{\cos t}{\rho} - \frac{1}{2}\frac{x_3^2}{\rho^5}\cos t\right) \\ L_{X_\alpha}\alpha_2 =& \alpha_1(x,t)\left(-3\frac{x_1^3 x_3^2}{\rho^7}\sin t\right) \\ &+ \alpha_2(x,t)\left(-1 - 3\frac{x_2^3 x_3^2}{\rho^7}\sin t\right) \\ &+ x_2\alpha_1(x,t)\left(2\frac{x_3}{\rho^3}\sin t - \frac{3}{2}\frac{x_3^3}{\rho^7}\sin t\right). \end{aligned}$$

The remaining terms in (4.33) are easily computed from the definitions of α_i. Note that the new system (4.32) is invariant with respect to the extended dilation

$$\tilde{\Delta}_\lambda(x,u) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda u_1, \lambda u_2).$$

Hence, uniform asymptotic stability is equivalent to exponential stability with respect to a homogeneous norm compatible with $\tilde{\Delta}_\lambda$. Simulations of the extended system and control inputs are shown in Figures 4.7, 4.8, and 4.9 with a value of $k = 5$.

An experimental version of the system, with optional trailers attached to the robot, is described in Chapter 5. The wheels are driven by stepper motors and hence the torque controller is embedded in the dynamics of the motors. However, the results presented show that there are no discontinuities in the time trajectories of the velocity inputs, and hence controlling the torques (via a set of integrators) is feasible.

The driftless system extended with integrators in equation (4.22) was used above to demonstrate how controllers can be derived for systems in which the integrators represent simple inertial or actuator dynamics. In this case the controller outputs are the "v" variables in equation (4.22). The simulations in Example 4.17 point out the disadvantages of having the control output specified directly from non-Lipschitz functions: noisy measurements can saturate the control output rate. This saturation is evident in Figure 4.6 and would also plague the extended system when the control output is the v variables. To ameliorate this condition, the controller output must be filtered to remove the high frequencies. However instead of passing the control output through lowpass filters the system setup in equation (4.22) which we have already explored may be

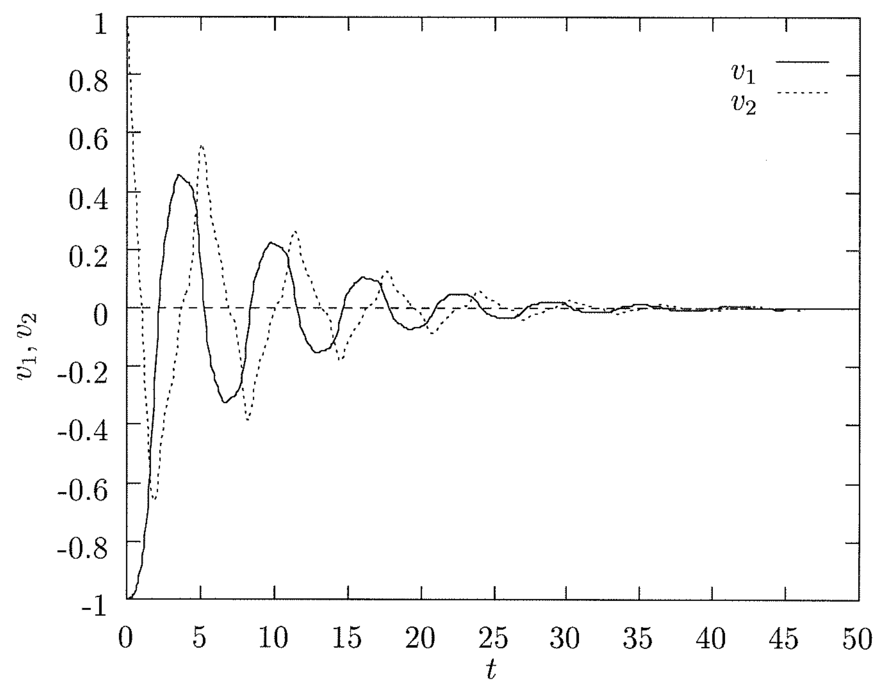

Figure 4.9: Controller output.

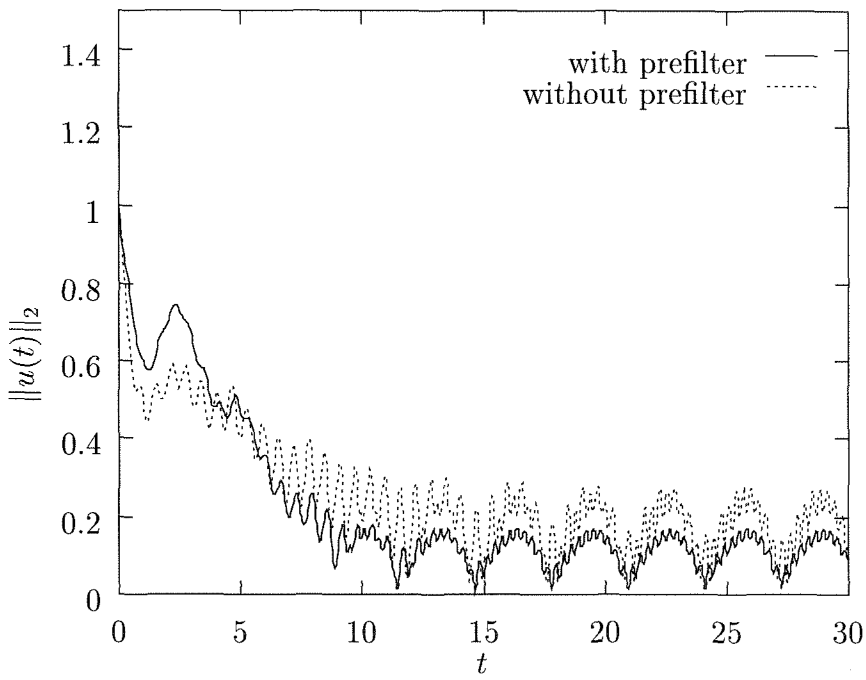

Figure 4.10: Norm of control output.

used. In this case the states u do not represent actuator dynamics but are states of the controller itself. The control output u is guaranteed to be continuously differential (assuming the noise added to the state measurements is continuous) since it satisfies a differential equation with continuous right hand side. Thus the controller is dynamic now and ρ-exponential stability is maintained. The example below illustrates the smoothing of the control output when noise is present.

Example 4.21 The numerical simulations use the extended model in equation (4.32) with the feedback (4.33) derived from equations (4.21). Figure 4.10 is a plot of the norm of the control output where the extended system is interpreted as a dynamic controller now. The measurement noise was modeled as the sinusoid $0.2 \sin 10t$ added to the x_3 variable as in Example 4.17. The controller parameters and gains where chosen to be those in Example 4.20. The figure also contains the results of a simulation in which a prefilter for the measurements is included. The prefilter parameters are the same as those in Example 4.17. Both graphs of the dynamic controller output show considerable smoothing compared to their counterparts in Figure 4.6. The prefiltering reduces the size of the ball that bounds the states and the dynamic extension smooths the control output to avoid actuator saturation.

Chapter 5

Experimental Validation

5.1 Description of Experiment

This chapter presents experimental results on the use of time-varying feedback controllers for stabilizing mechanical systems with nonholonomic constraints. In particular, the system to be controlled is a two-wheeled mobile robot towing a trailer. The experiments demonstrate point stabilization using the methods developed in the previous chapters. Many of the techniques and experimental results described here are also applicable to more practical problems such as parallel parking and backing into a loading dock. A picture of the experimental apparatus is shown in Figure 5.1.

The fundamental assumption in modeling the kinematics and dynamics of a mobile robot is that the wheels of the robot roll without slipping. This means that each wheel (or pair of wheels connected by an axle) is free to roll in the direction that it is pointing and spin around the vertical axis. This is clearly an idealization and one of the questions which we hope to answer is to what extent this model is accurate enough for use in control design. This problem naturally leads to driftless control systems since the state represents by the car and trailer configuration and the inputs are the forward and angular velocities of the front wheels. Even for the simple kinematic wheel, the number of states is three and the number of inputs is two. No inertial effects are involved, i.e. the standing assumption is that the motors of the physical system provide the required forces and torques to effect the velocities specified by the controller. The driftless models for many different configurations of car and trailer may be found in Sørdalen [38]. These experiments use two specific models described later.

For most of the controllers which are implemented, the kinematic equations are converted into a special normal form, called "chained form" [32]. A system

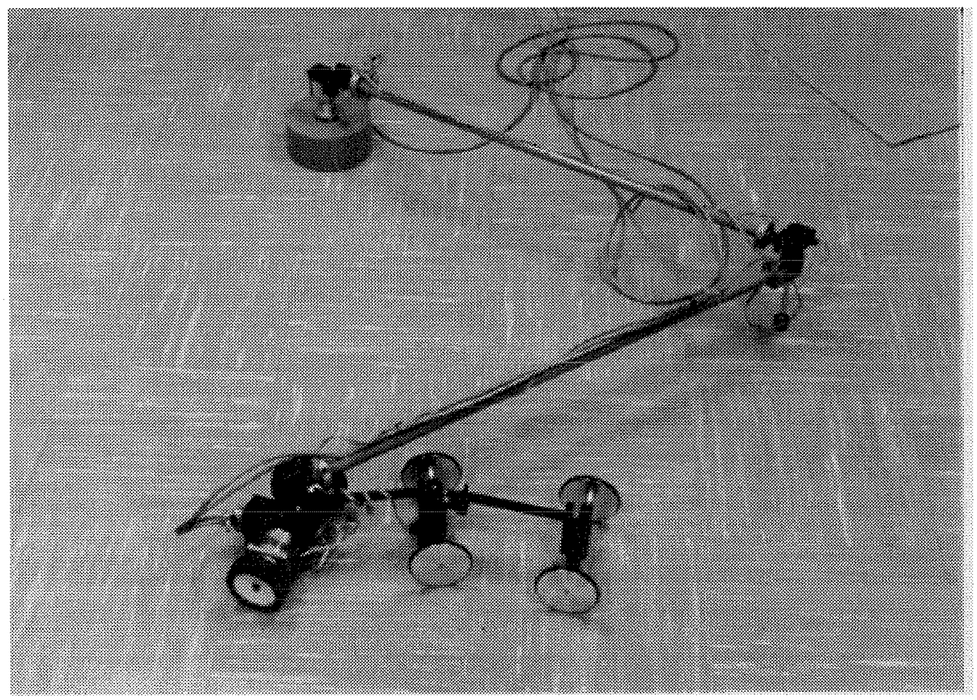

Figure 5.1: The nonholomobile mobile robot.

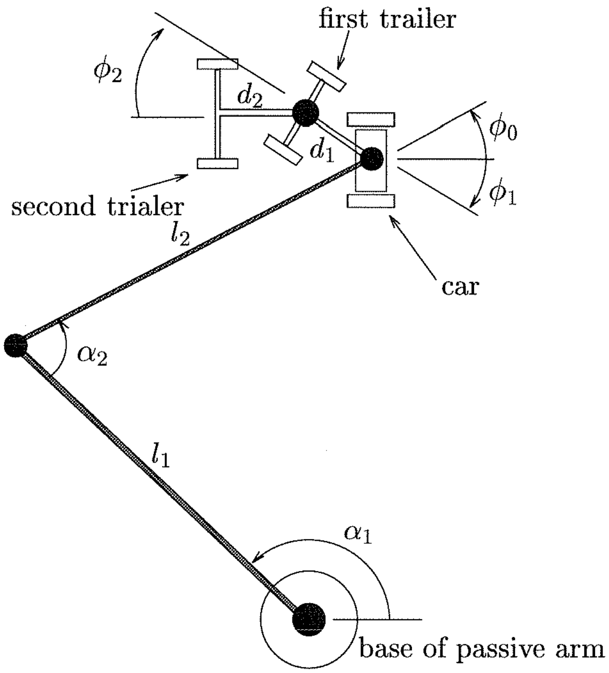

Figure 5.2: Experimental apparatus.

in chained form is written as

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
\dot{x}_4 &= x_3 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1.
\end{aligned} \tag{5.1}$$

Necessary and sufficient conditions for feedback transforming a system into chained form are given in [31].

The object of the experiments is to stabilize the system about a given position and orientation using feedback. The car is a two-wheeled device with each wheel driven separately by a stepper motor. The position and orientation of the system are sensed using a passive two link manipulator with the base fixed to the floor and the distal end attached to the car. Optical encoders at the manipulator joints and on the car return angle information. Refer to Figure 5.2 for the locations of the encoders and kinematics of the arm.

Once coordinate frames for the car and manipulator are chosen, the forward kinematics of the manipulator is computed to locate the position and orientation of the car. The orientation of the trailers is provided by encoders mounted

Parameter	Length(cm)
l_1	88.9
l_2	84.6
d_1	19.0
d_2	19.0
car wheelbase	10.0
wheel radii	4.0

Table 5.1: Kinematic parameters.

on the car and first trailer. The orientations of the car and trailers may be referenced with respect to a fixed horizontal or given relative to the preceding car or trailer. The map from one convention to the other is a simple kinematic change of coordinates and so is not presented here. Similarly, it may be desirable to reference the position of the system with respect to the rear trailer instead of the car. Again, since the transformation is straightforward it is not included. When discussing a particular kinematic model of the system it is assumed that any preliminary computations have been performed so that the position and orientation information provided by the encoders is compatible with the model.

The important kinematic parameters of the aggregate system are listed in Table 1. The link lengths of the manipulator are denoted l_1 and l_2. The trailer lengths are denoted d_1 and d_2.

The optical encoders are quadrature encoders providing 2000 counts per revolution or an accuracy of 0.18 degrees. They provide about 1 mm of resolution when the manipulator is fully extended. This was judged satisfactory for the kind of positioning experiment performed here. Each encoder signal is decoded with a quadrature decoder. These decoders keep a running pulse count of the encoder output. The real time software checks the buffer of the individual decoders to determine the angle that the encoder has turned with respect to its initial reset position. The decoders reside on a prototype card attached to an IBM PC.

The car is powered by two 4-phase permanent magnet stepper motors. The motors are configured so that a single step is 0.9 degrees. The motors can handle a maximum step rate of approximately 500 steps per second and still provide sufficient torque to accelerate the vehicle. Saturation of the motors occurs at about 600 steps per second. A parallel port chip enables/disables the motors and specifies the direction of rotation. The step rate is set by the output of a programmable interval timer. The step rates of the motors can be varied from more than 400 steps/sec to less than 1 step/sec in increments of less than 1 step/sec. This resolution was deemed sufficient for this experiment. When the stepper motors are used in this configuration they are controlled in an open-loop manner. For example, the control laws compute desired velocities based on the position and orientation of the system. The velocities are then converted into the equivalent "steps per second." The implicit assumption with this method is that the motors can apply the torque required to overcome inertial effects to maintain the proper speed. There is no direct way to verify that the desired velocity is actually achieved. However, since the control laws are continuous the input to the motors is naturally ramped. An alternative is to use DC servo motors but this requires more hardware. The experimental results demonstrate that the stepper motors perform quite well.

Real-time control was implemented in software using the Sparrow real-time control kernel [34]. This package controls servo loop execution, provides a simplified interface to sensor and actuator hardware, and allows data capture and dumping. Using the Sparrow software, a 200 Hz servo loop was used to implement a 5th order digital Butterworth filter with 10 Hz cut-off frequency for smoothing all sensor inputs. The sample rate for the feedback control law was 20 Hz. This was implemented by computing the control action every 10th iteration of the servo loop. Data was captured at the 20 Hz sample rate.

The kinematic models are presented below. The car with no trailer is represented by the following set of equations:

$$\begin{aligned} \dot{x} &= \cos\theta_0 v \\ \dot{y} &= \sin\theta_0 v \\ \dot{\theta}_0 &= \omega. \end{aligned} \tag{5.2}$$

The scalar v is the forward velocity of the car and ω is its angular velocity. These are inputs determined by the control law. The Cartesian position of the car is denoted (x, y). The car with a single trailer represents a 4-dimensional nonholonomic system with the model,

$$\begin{aligned} \dot{x} &= \cos\theta_1 v \\ \dot{y} &= \sin\theta_1 v \\ \dot{\theta}_0 &= w \\ \dot{\theta}_1 &= \frac{1}{d_1}\tan(\theta_0 - \theta_1)v. \end{aligned} \tag{5.3}$$

With this particular model x and y are the position of the *trailer.* The forward velocity of the trailer is denoted v and ω is the angular velocity of the car. The forward velocity of the car is computed as $v_{car} = \cos(\theta_0 - \theta_1)v$. The control law computes v and ω and then the car velocity, v_{car}, is determined using the previous expression. Finally, the software determines the appropriate step rate for each motor. Figure 5.3 shows the coordinate system used for each model.

5.2 Control Laws

We now discuss application of these ideas to the stabilization of the car-trailer system. Consider the situation in which the input vector fields of the nonholonomic system are homogeneous of degree one with respect to some dilation. A

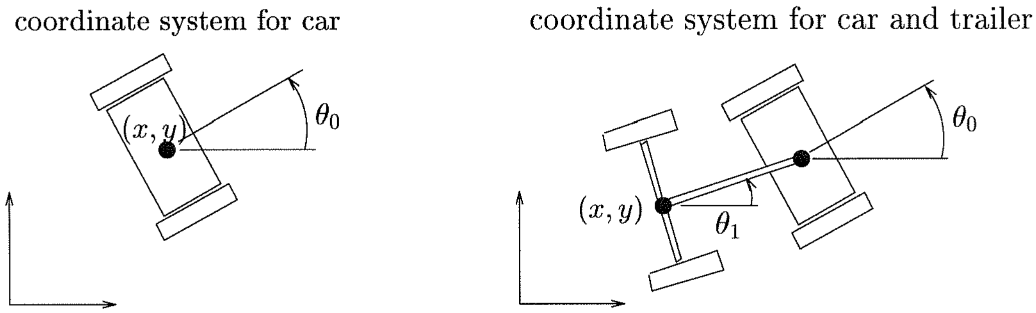

Figure 5.3: Coordinate systems.

feedback that is a homogeneous function of degree one makes the closed-loop vector field homogeneous of order zero (using the convention described above). If this feedback is uniformly stabilizing in time then each state may be bounded by a decaying exponential envelope. For a car and trailer system the so-called chained form coordinates of the input vector fields are homogeneous of degree one with respect to a dilation with powers assigned to a particular state corresponding to the number of Lie brackets of the input vector fields required to span that state direction.

The stabilizing feedbacks for the systems in power form are motivated from the discussions in [28] and [29]. The actual feedbacks are derived from optimizing the rate of convergence as observed in numerical simulations. There does not yet exist a computational method for generating Lyapunov functions that may be used for analysis of asymptotically stable homogeneous vector fields. Converse theorems do exist, however they are not useful for specific examples since knowledge of the flow is assumed in constructing the Lyapunov function.

Recall the kinematic model of the car and no trailers. A transformation that converts equation (5.2) into a set of "almost" homogeneous vector fields is given by

$$\begin{aligned} z_1 &= \theta_0 \\ z_2 &= x\cos\theta_0 + y\sin\theta_0 \\ z_3 &= x\sin\theta_0 - y\cos\theta_0. \end{aligned} \tag{5.4}$$

This particular change of coordinates has the advantage of being a global diffeomorphism. One can confirm that the vector fields in these coordinates have the form

$$\begin{aligned} \dot{z}_1 &= u_1 \\ \dot{z}_2 &= u_2 - z_3 u_2 \\ \dot{z}_3 &= z_1 u_2, \end{aligned} \tag{5.5}$$

where $u_1 = \omega$ and $u_2 = v$. This system is nilpotent but not homogeneous because of the z_3u_2 present in the first equation. This term actually improves the convergence properties of the system with the feedbacks given below. One may verify this by using center manifold analysis on the system with the smooth feedback. Hence, we essentially ignore this term when designing the feedbacks. The dilation that corresponds to these vector fields is

$$\Delta_\lambda(z_1, z_2, z_3) = (\lambda z_1, \lambda z_2, \lambda^2 z_3) \quad \lambda > 0, \tag{5.6}$$

and the homogeneous norm

$$\rho(z) = (z_1^4 + z_2^4 + z_3^2)^{\frac{1}{4}}. \tag{5.7}$$

A control law motivated by [29] is

$$\begin{aligned} u_1 &= -c_{11} z_1 + c_{12} \frac{z_3}{\rho(z)} \cos \Omega t \\ u_2 &= -c_{21} z_2 + c_{22} \frac{z_3^2}{\rho^3(z)} \sin \Omega t, \end{aligned} \tag{5.8}$$

where the c_{ij} are positive real parameters which may be adjusted to modify the system response. Ω is the frequency of the time periodic component of the control. A proof that this control law is asymptotically stabilizing is given in Appendix C. These are homogeneous functions of order 1 with respect to (5.7), are smooth on $\mathbb{R}^n \backslash \{0\}$ and continuous at the origin. If the closed-loop system is asymptotically stable then it is actually exponentially stable with respect to the homogeneous norm (5.7).

If one is interested in globally smooth feedback there are a number of results available. We compare our homogeneous feedback to two smooth controllers derived by significantly different methods. The first smooth controller is just a smooth version of (5.8),

$$\begin{aligned} u_1 &= -c_{11} z_1 + c_{12} z_3 \cos \Omega t \\ u_2 &= -c_{21} z_2 + c_{22} z_3 \sin \Omega t, \end{aligned} \tag{5.9}$$

where the c_{ij} are parameters. More details on the properties of this feedback may be found in [28, 44]. The control law is written for the system in chained form so the preliminary coordinate transformation (5.4) is required. This smooth feedback is contrasted to a controller derived from Pomet's method [37],

$$\begin{aligned} v &= -c_v(x \cos \theta_0 + (y(1 + \frac{1}{\Omega^2} \sin^2 \Omega t) - \frac{1}{\Omega} \theta_0 \sin \Omega t) \sin \theta_0) \\ w &= y \cos \Omega t - c_w(\theta_0 - \frac{1}{\Omega} y \sin \Omega t), \end{aligned} \tag{5.10}$$

where c_v and c_w are positive parameters. Note that the control law is given in the original coordinates. Pomet's method may be used to generate a feedback for the system written in chained form, however one could argue that an intrinsic advantage to this method is the fact that special coordinates are *not* required.

We adopt this interpretation and so derive the feedback based on (5.2). This feedback was generated from choosing

$$\begin{aligned}
\alpha(t,x) &= y\cos\Omega t \\
V(t,x) &= \frac{1}{2}\left(x^2 + y^2 + (\theta_0 - \frac{1}{\Omega}y\sin\Omega t)^2\right).
\end{aligned}$$

Refer to [37] for the notation.

The system with one trailer is now discussed. Recall the 4-dimensional set of kinematic equations describing the system (5.3). The diffeomorphism and input transformation that places the model into chained form is

$$\begin{aligned}
z_1 &= x \\
z_2 &= \frac{1}{d_1}\sec^3\theta_1\tan(\theta_0 - \theta_1) \\
z_3 &= \tan\theta_1 \\
z_4 &= y,
\end{aligned} \tag{5.11}$$

and the inputs are computed from

$$\begin{aligned}
u_1 &= \cos\theta_1 v \\
u_2 &= \sec^3\theta_1\tan(\theta_0-\theta_1)(\frac{3}{d_1^2}\tan\theta_1\tan(\theta_0-\theta_1) - \frac{1}{d_1^2}\sec(\theta_0-\theta_1))v + \\
&\frac{1}{d_1}\sec^3\theta_1\sec^2(\theta_0-\theta_1)\omega.
\end{aligned} \tag{5.12}$$

The expression of the vector fields in these coordinates is

$$\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= z_2 u_1 \\
\dot{z}_4 &= z_3 u_1.
\end{aligned} \tag{5.13}$$

This system is homogeneous of degree 1 with respect to the dilation

$$\Delta_\lambda(z) = (\lambda z_1, \lambda z_2, \lambda^2 z_3, \lambda^3 z_4). \tag{5.14}$$

A particular choice of homogeneous norm is

$$\rho(z) = (z_1^{12} + z_2^{12} + z_3^6 + z_4^4)^{\frac{1}{12}}. \tag{5.15}$$

The feedback that is implemented has the form

$$\begin{aligned}
u_1 &= -c_{11}z_1 + c_{12}\left(\frac{z_3^2}{\rho^3} + \frac{z_4^2}{\rho^5}\right)(\cos\Omega t - \sin\Omega t), \\
u_2 &= -c_{21}z_2 + c_{22}\frac{z_3}{\rho}\cos 2\Omega t + c_{23}\frac{z_4}{\rho^2}\cos 3\Omega t,
\end{aligned} \tag{5.16}$$

where the c_{ij} are positive parameters. This feedback is homogeneous of degree 1 and so the closed-loop vector field is homogeneous of degree 0 with respect to (5.14). Numerical simulations of these models will be compared to actual data in the next section. A stabilizing feedback will necessarily stabilize at an exponential rate.

The smooth controller for the 4-dimensional system that is implemented is from [28, 44]. The system is written in chained form and the feedback takes the form,

$$\begin{aligned} u_1 &= -c_{11}z_1 + c_{12}\left(z_3^2 + z_3\right)(\cos\Omega t - \sin\Omega t), \\ u_2 &= -c_{21}^1 z_2 + c_{22}z_3\cos 2\Omega t + c_{23}z_4\cos 3\Omega t, \end{aligned} \tag{5.17}$$

5.3 Experimental Results

The experimental results are presented in this section. The first part compares open loop trajectories generated by a nonholonomic path planning algorithm to numerical simulations of the equations. These results motivate the need for feedback. The physical parameters in Table 5.1 were measured with a metal tape measure and so the accuracy of these measurements is limited to several millimeters. This will lead to errors in the computation of the position of the system. The most compelling reason to employ feedback is to make the system insensitive to such errors and so approximate measurement of the system position should be adequate if the feedback is "good." It is difficult to perform a detailed robustness analysis on these systems but the fact that the closed-loop systems perform quite well is testimony to some degree of robustness possessed by the feedback.

The results with feedback are presented following the open-loop experiments. Some thought must be given to the interpretation of the results if a comparison between several types of controllers is made on the same system. The rate at which the system approaches its equilibrium position from different initial positions is a reasonable criterion to assess the controller performance. In any application the control effort is a real limitation on the achievable performance. This limitation is embodied in the fact that the stepper motors saturate at about 500 steps/sec. Therefore it is reasonable to choose, as a means of comparison between different controllers, a fixed neighborhood of the equilibrium point where it is desired that each control law stabilize the system with initial conditions in this neighborhood, but at the same time not saturate the motors. The individual control laws may be "tuned" to take full advantage of the actuator in this neighborhood. We compare the controllers in this manner. Outside the neighborhood, where the motors saturate, saturation functions may be used to increase the domain of attraction [43]. However, since we are interested in the long term behavior of the system, we need only consider initial conditions inside the neighborhood where the saturation function have no affect.

5.3.1 Open loop inputs

We now present some experimental results using open-loop inputs to the car (no trailers). The velocity inputs are computed by representing the velocities as the sum of harmonic components with unknown amplitudes. The system is converted to chained form and integrated with the desired initial conditions. The final position is enforced resulting in a set of polynomials with the amplitudes of the harmonic functions as the indeterminates. The actual system trajectories are shown is Figure 5.4. The numerical simulation demonstrating that the open loop inputs steer the mathematical model to the origin is also shown in Figure 5.4.

The inputs where chosen to returns the car to the origin with zero attitude. The initial conditions where chosen so as to match those of a feedback experiment presented in the next subsection. The initial conditions for computing the velocity inputs and the numerical simulation are

$$x = -0.5945m \quad y = 0.3299m \quad \theta_0 = 0.8262rad.$$

The initial conditions of the experimental apparatus are

$$x = -0.5923m \quad y = 0.3296m \quad \theta_0 = 0.8294rad.$$

The responses are qualitatively very similar however disturbances and modeling error contribute to the large discrepancy between the actual and desired final position of the car (20 cm in the y position and 9 degrees in orientation).

The careful designer could probably do better than this at the expense of more detailed models for the system. However, the objective of this experiment is not to perform such an analysis of open-loop control schemes but rather motivate the use of feedback.

5.3.2 Stabilization of the car

Experimental results with feedback are now presented for the car. Figure 5.5 compares the exponentially stabilizing homogeneous controller (5.8) and the smooth asymptotic controller (5.9), both of which use the coordinate change (5.4). The figure compares the time response of the the system with both controllers to a set of initial conditions very close to those used with the open-loop experiment. Figure 5.6 contains a step rate comparison of both controllers. The car uses two motors and the step rate input into one motor is plotted for both experiments. Note that the peak step rate amplitude of the smooth asymptotic controller is higher than the peak amplitude of the exponential homogeneous control law. Figure 5.6 also contains a log plot of the y-variable. The exponential convergence of the homogeneous controller is evident.

Figure 5.7 presents experimental results with the Pomet feedback (5.10). The controller exhibits large effort during the initial transient period of the system response. Figure 5.8 shows numerical simulations of the homogeneous control law and Pomet's smooth control law with the initial conditions of the

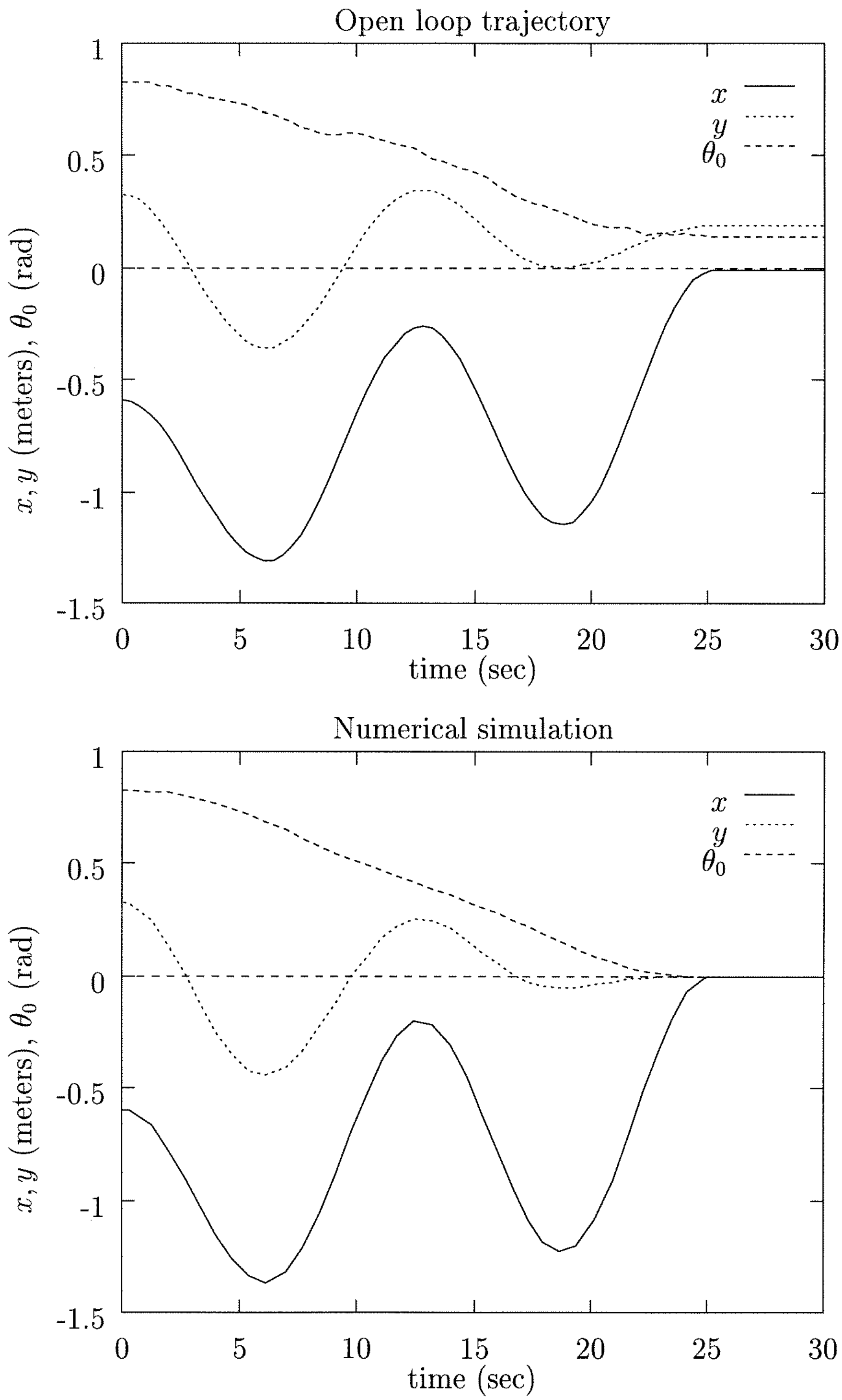

Figure 5.4: Open-loop control.

	Homogeneous (5.8)	Smooth (5.9)	Pomet (5.10)
c_{11}	0.3	0.3	·
c_{12}	0.4	0.4	·
c_{21}	1.0	1.0	·
c_{22}	3.0	5.0	·
c_v	·	·	1.0
c_ω	·	·	1.0
Ω	2.0	2.0	2.0

Table 5.2: Control law parameters for the car with no trailers.

simulation set to the initial data of the experiments in Figures 5.5 and 5.7. The simulations are very close to the actual response. Simulations for all of the other cases (smooth controller and the controllers for the car and trailer) are not shown since the results are qualitatively similar to the experimental data. The simulations are used to adjust the parameters of the controllers, the final tuning being performed on the actual system after the simulations yield the desired response. Note that the smooth controllers are asymptotically stabilizing the system but the rate is very slow. The control parameters used in these experiments are found in Table 5.2.

5.3.3 Stabilization of the car and one trailer

The stabilization results for the car and one trailer are discussed below. Particular attention should be paid to the behavior of the y-variable. Figure 5.9 compares closed-loop behavior of the exponentially stabilizing homogeneous control law (5.16) and the smooth asymptotic control law (5.17) with the same initial conditions. No specific initial condition was chosen to make one controller perform "better" than another. The step-rates generated by each control law are shown in Figure 5.10. The peak step rate for both controllers is approximately 300 steps/sec. The $\log(|y|)$ plot is useful for assessing the convergence rate of the system. This will be discussed in more detail in the next section. The parameters used in the experiments with one trailer are given in Table 5.3.

5.3.4 Discussion

The first aspect of the experimental results to note is the rate at which y approaches zero. For the controllers which rely on chained form, the y variable is identified with the "slowest" state. Thus the rate at which this state decays is of practical interest. It is useful to plot $\log(|y|)$ to study this behavior. The fact that y in the homogeneous controllers' response may be bounded above by a straight line (see the log plots in Figures 5.6 and 5.10) indicates that y is approaching zero at an exponential rate. The average rate of convergence

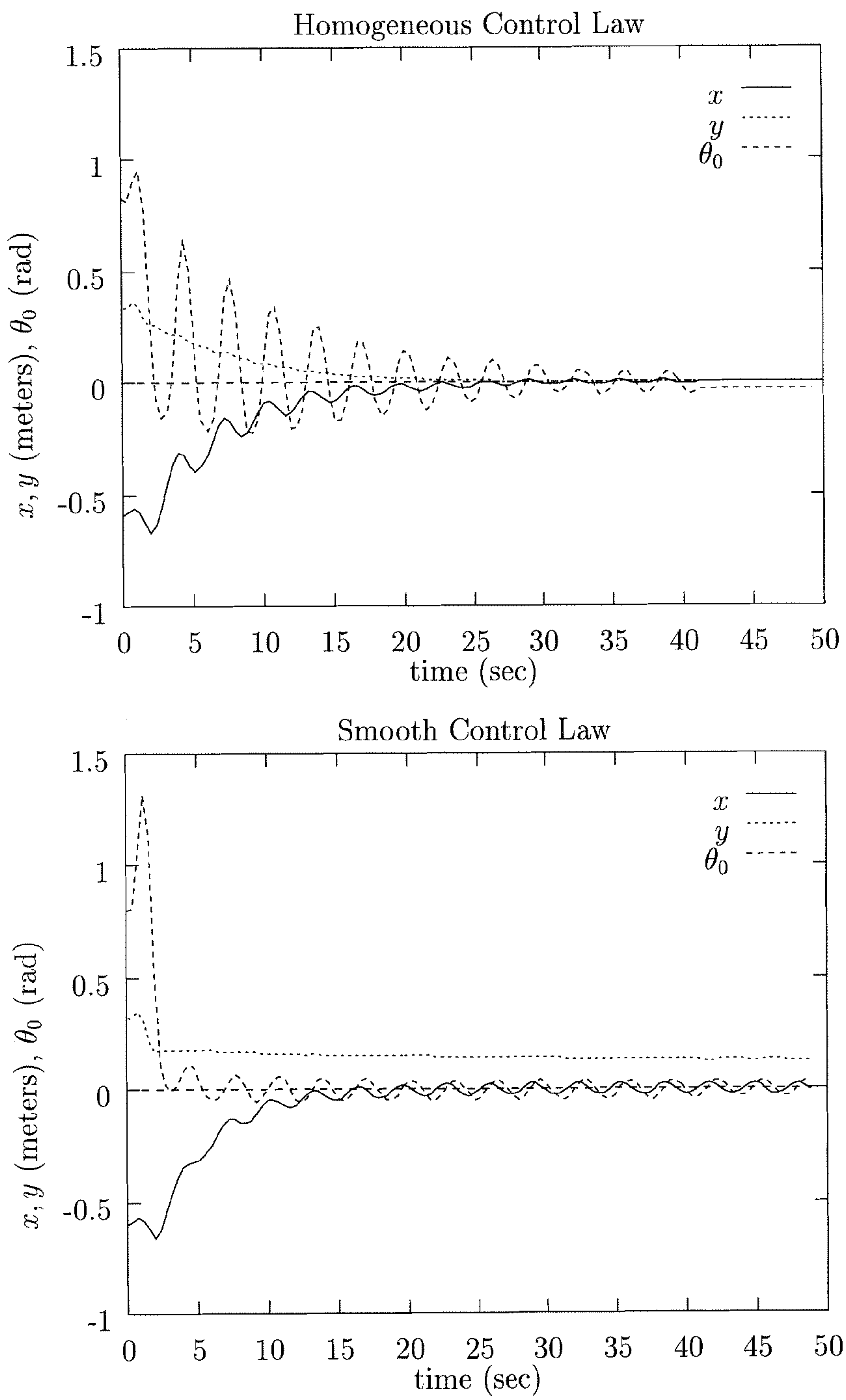

Figure 5.5: Experimental comparison of homogeneous and smooth feedbacks for the car with no trailers.

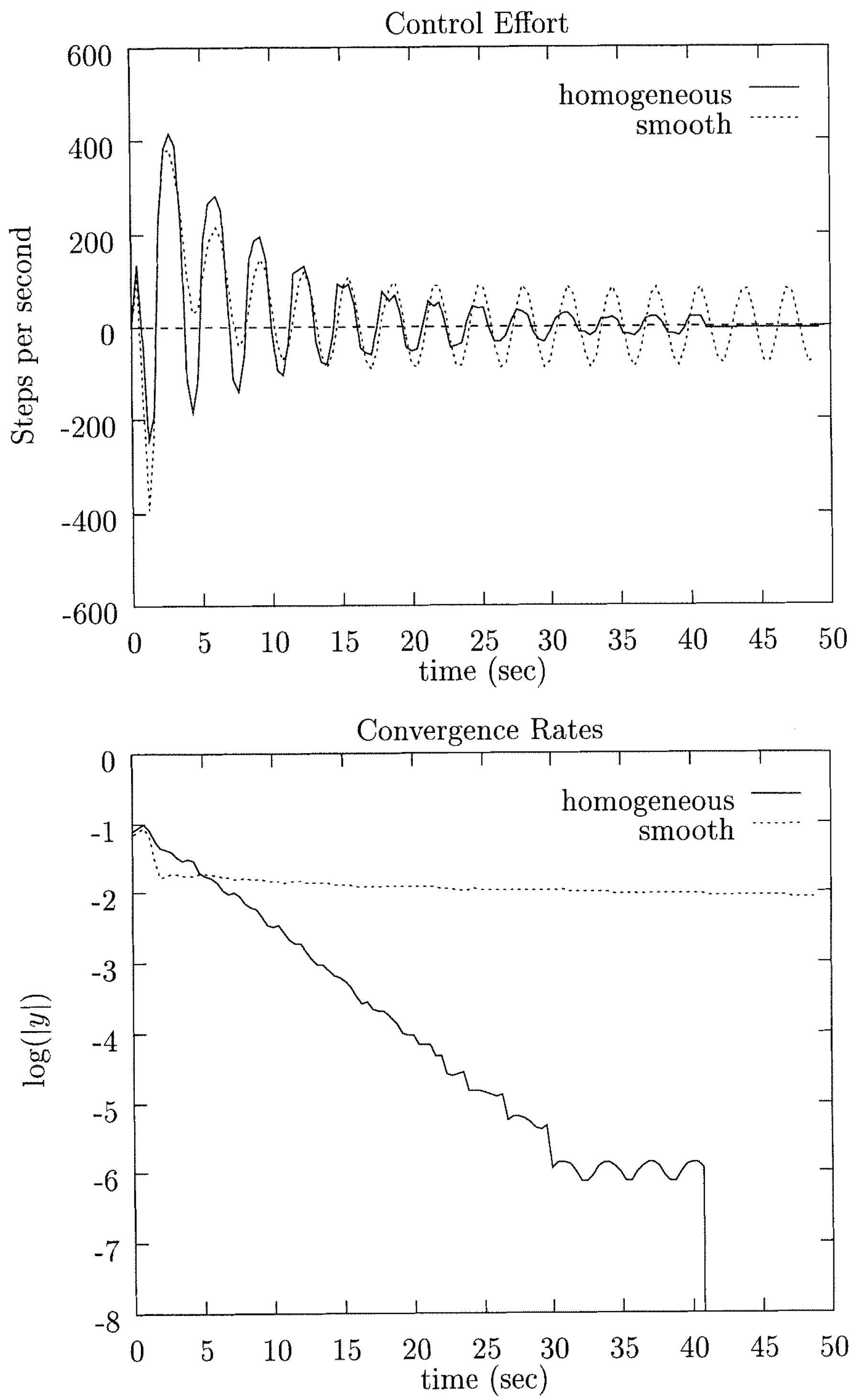

Figure 5.6: Experimental controller effort and convergence rates for the car with no trailers.

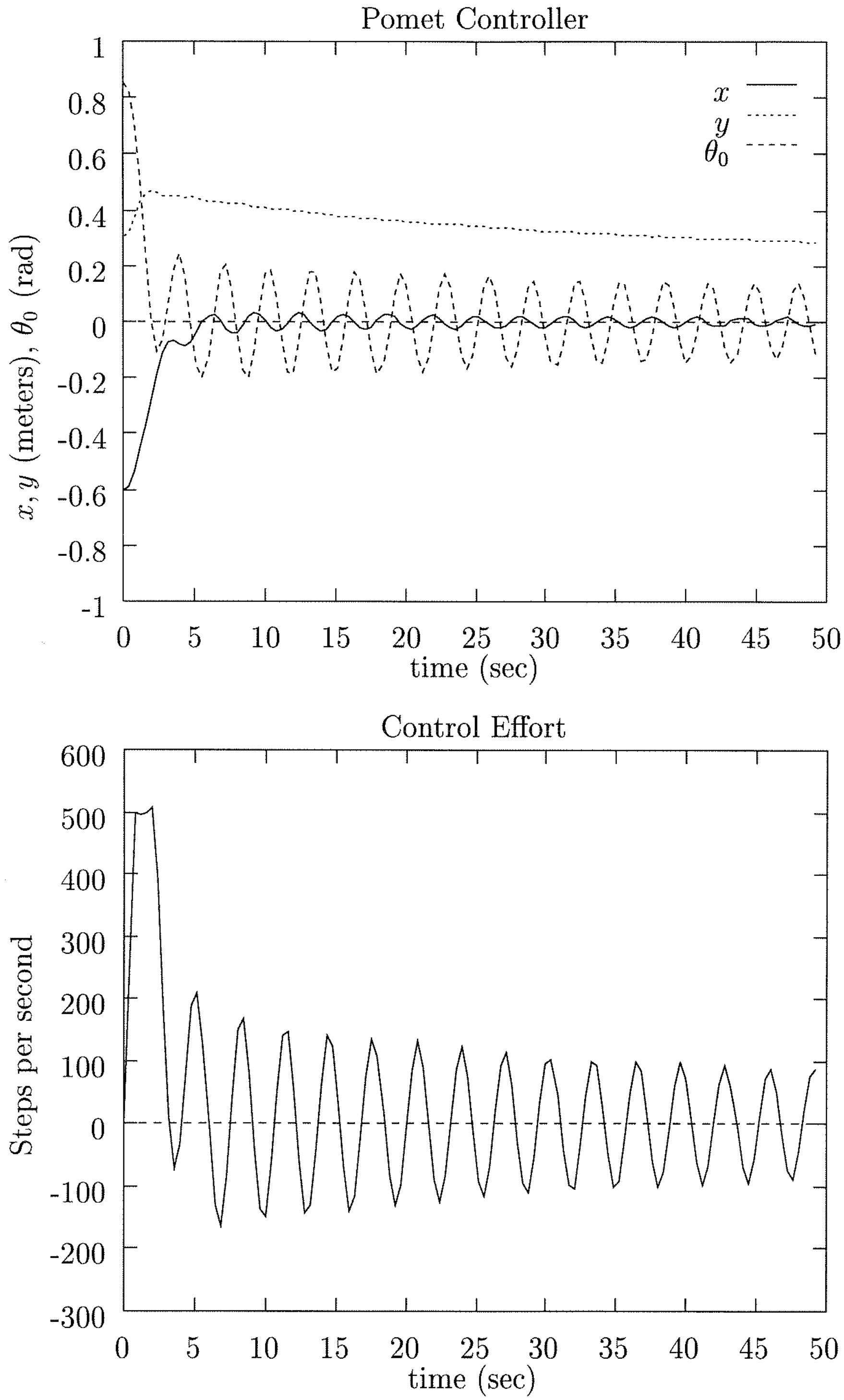

Figure 5.7: Pomet control law response and actuator effort for the experimental system with no trailers.

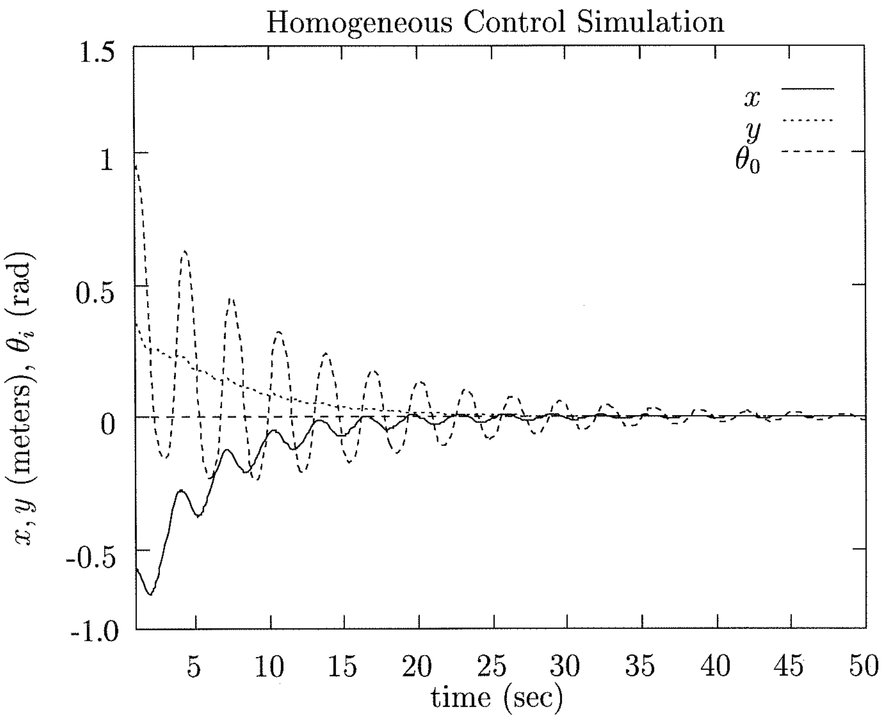

Figure 5.8: Numerical simulation of the homogeneous and smooth Pomet controllers for the car with no trailers.

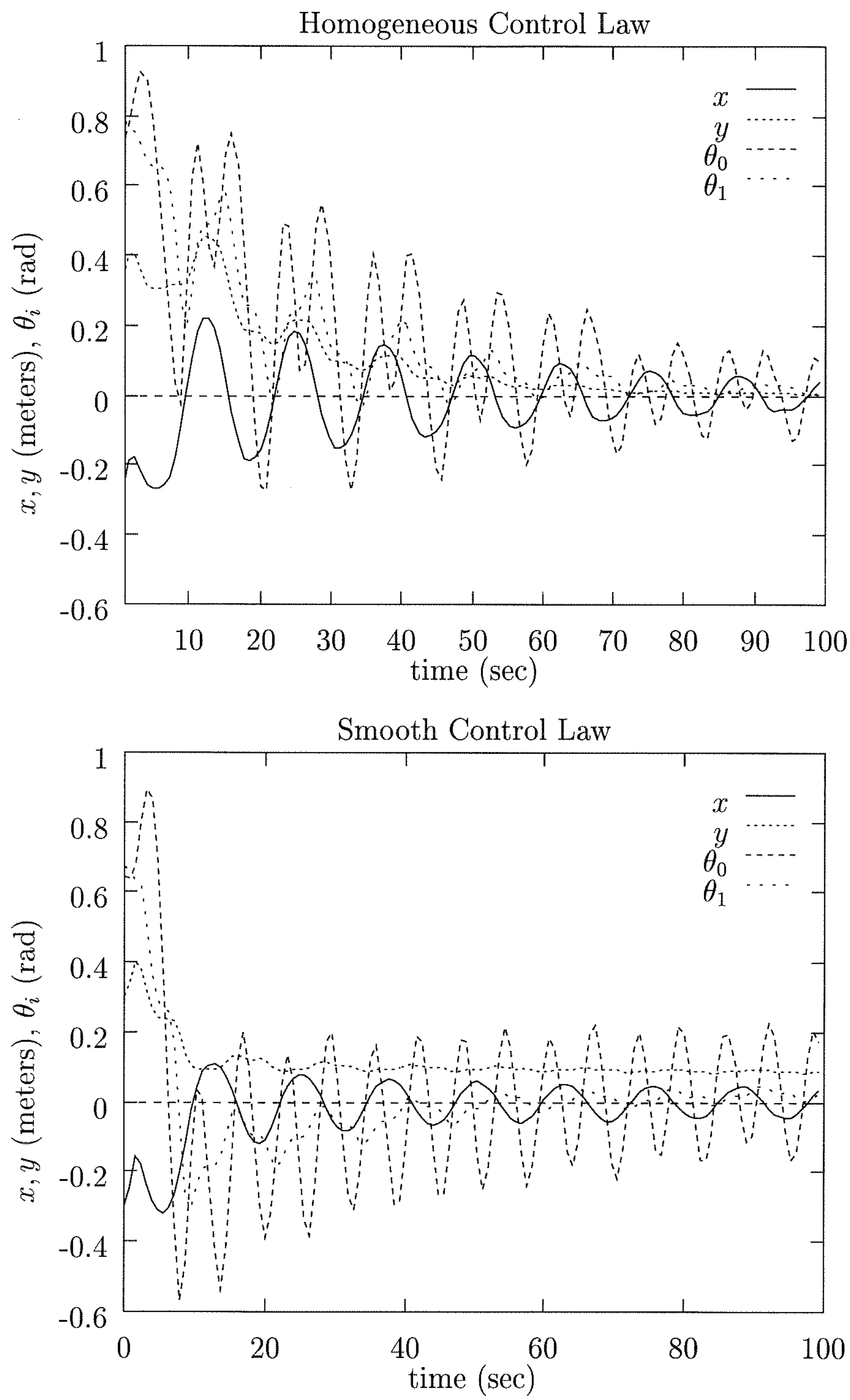

Figure 5.9: Experimental comparison of homogeneous and smooth feedbacks for the car with one trailer.

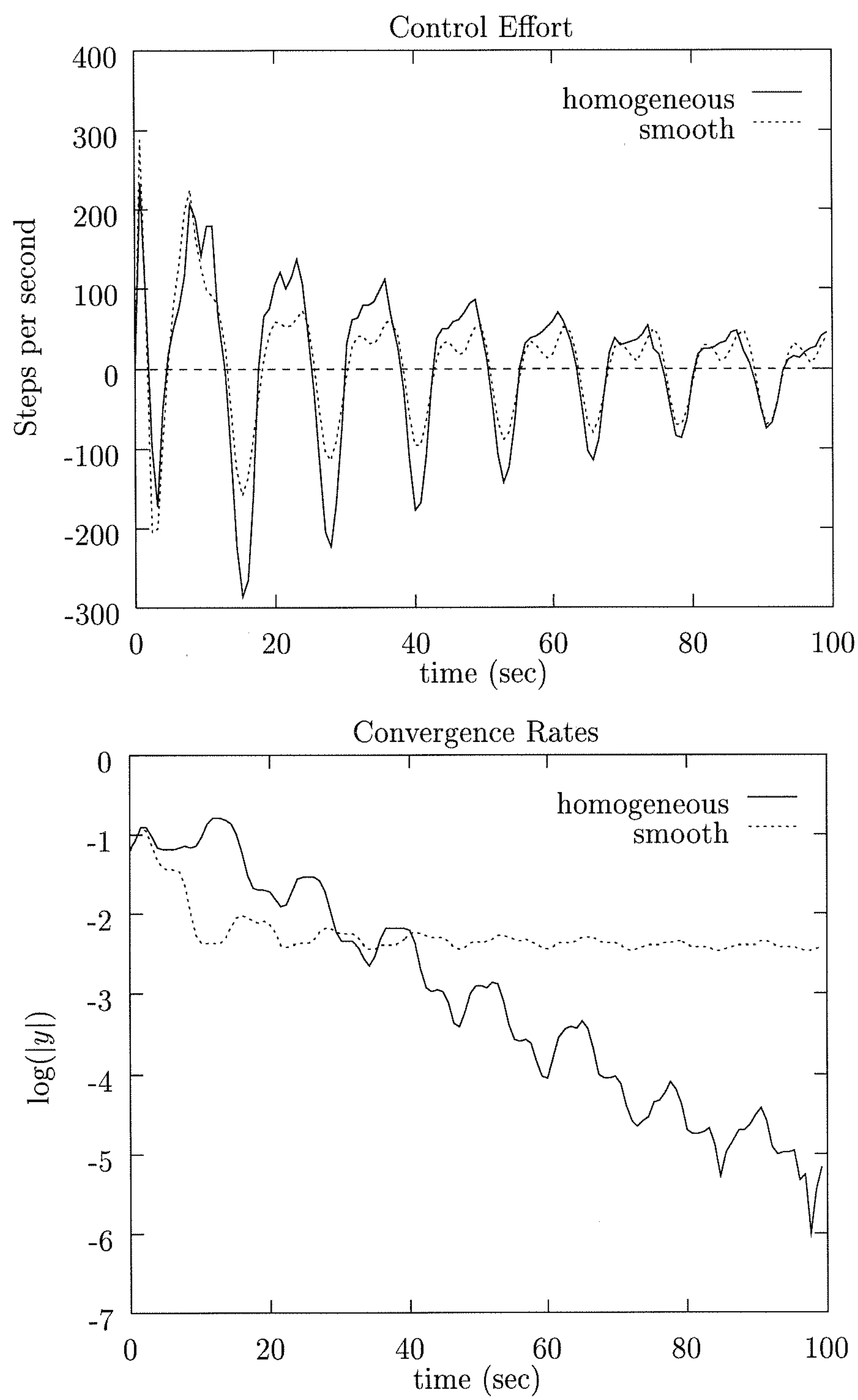

Figure 5.10: Experimental controller effort and convergence rates for the car with one trailer.

	Homogeneous (5.16)	Smooth (5.17)
$c_{u_1}^1$	0.5	0.5
$c_{u_1}^2$	0.6	0.6
$c_{u_2}^1$	0.5	0.5
$c_{u_2}^2$	0.5	0.5
$c_{u_2}^3$	0.5	0.5
Ω	0.5	0.5

Table 5.3: Control law parameters for the car and one trailer.

is equal to the average slope on the plots. The smooth controller in chained form decays at an algebraic rate. This is also evident from the log plots. The Pomet controller is written in the original physical coordinates so there is no distinguished "slow" state. However center manifold analysis may be used to show that the rate of decay of y determines the rate of convergence for the entire system. The discrete nature of the motors places a lower bound on how close the system can come to the origin. This may cause *hunting.* However this is a shortcoming of the hardware, not a limitation of the controller, and may be dealt with by ad hoc means (such as switching the controller off in some small neighborhood of the equilibrium point).

A few words should be said concerning the choice of fundamental period of the control laws and the digital filtering. First, the period, Ω, was chosen in order to maximize the rate of convergence but at the same time not saturate the motors. Analysis of the systems in chained form clearly demonstrates that shorter periods result in faster convergence times but at the expense of increased motor speed. Second, the bandwidth of the digital filter was chosen to be high enough to guarantee asymptotic stability. This is basically the special perturbation result proven in Proposition 4.15. The bandwidth was determined experimentally by balancing the tradeoff between measurement smoothing and convergence rate.

We now discuss control design related aspects for the individual problems. The controllers used in these experiments do not differentiate between length scales. For example, the (x, y) position of the car may be expressed in cm, m or even km. Hence as long as the actuators don't saturate, the region of convergence in terms of the linear variables is rather arbitrary. The response of the system depends critically on the length scale chosen though. For the homogeneous systems this is embodied by the *shape* of the corresponding homogeneous ball: homogeneous balls when the lengths are measured in kilometers and the angles in radians look much different than the balls with the lengths measured in meters. The length scale must be chosen so that the system response is satisfactory. The definition of "satisfactory" depends on the particular application
.

The three-dimensional system (car and no trailers) uses a length scale of 1 meter and angle scale of 1 radian. However the length scale for the system with the car and one trailer is the length of the trailer itself, i.e. one "unit" of length is 19 cm. A length scale of one meter leads to undesirable behavior because, for example, the homogeneous ball with $y = 1$ mm on its boundary also has $x = 10$ cm on its boundary! The finite precision of the actuators and sensors will invariably cause hunting in a neighborhood of the origin. This neighborhood is actually a homogeneous ball, for homogeneous closed-loop vector fields, and if the length scale is not chosen carefully can lead to large excursions of x with respect to small changes in y. This type of behavior is characteristic of any homogeneous vector field. Our selection of the trailer length as the length scale mitigates this undesirable behavior for the homogeneous feedback.

The hunting behavior is demonstrated for the car and trailer in Figure 5.11 when the characteristic length is taken as 1 m. The second figure shows that the hunting occurs in a homogeneous ball with $\rho \approx 0.2$. The arguments for picking a good length scale to eliminate hunting are not as compelling for smooth feedbacks since all of the analysis may be performed with any of the usual p-norms.

Lastly, we discuss a very important concept that is germane to *any* control systems design requiring a diffeomorphism to place the model into a desired coordinate representation. The singular values of the linearization of the diffeomorphism (5.11), at various points in the phase space, indicates the amount of "stretching" performed on the variables by the transformation. A controller that depends on an ill-conditioned transformation may exhibit extreme sensitivity to small changes in certain state variables. This is exemplified in Figure 5.12 where a poor length scale was chosen for the transformation. Numerical simulations of the system imply closed-loop stability but the actual response does not look stable. Plotting the chained form variables shows that the z_2 variable is dominant and is quite noisy. This results in very poor performance of the system. The length scale chosen for this experiment is 5 meters and the controller is the homogeneous controller which uses transformation (5.11). However, this behavior is caused by the transformation and is observed with any controller implementation. The trailer length is actually $d_1 = 0.19/5 \approx 0.038$ as far as the diffeomorphism is concerned. The condition number of the diffeomorphism evaluated at the origin is 52.7. This is due primarily to a singular value with magnitude 37.2. The amplification of the physical data occurs in the $\theta_0 - \theta_1$ "input" direction to the z_2 "output" direction. This is illustrated by performing a singular value decomposition on the linearization of the transformation at the origin,

$$\begin{pmatrix} z_1 \\ z_2 \\ z_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{d_1} & & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ \theta_0 \\ \theta_1 \end{pmatrix}.$$

Thus, when $\theta_0 - \theta_1$ crosses zero the same occurs to the z_2 variable except it is amplified by an order magnitude.

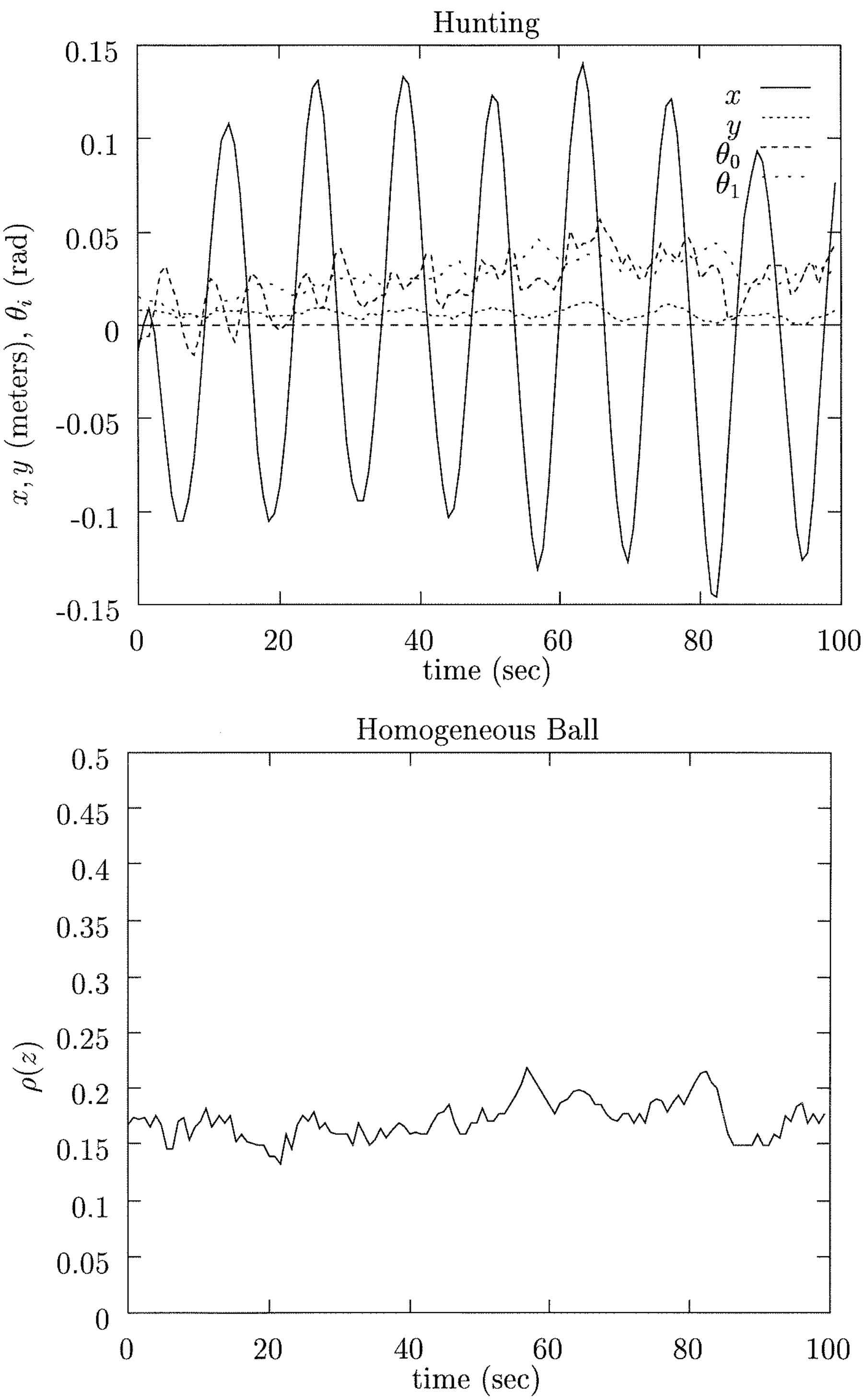

Figure 5.11: Hunting inside a homogeneous ball.

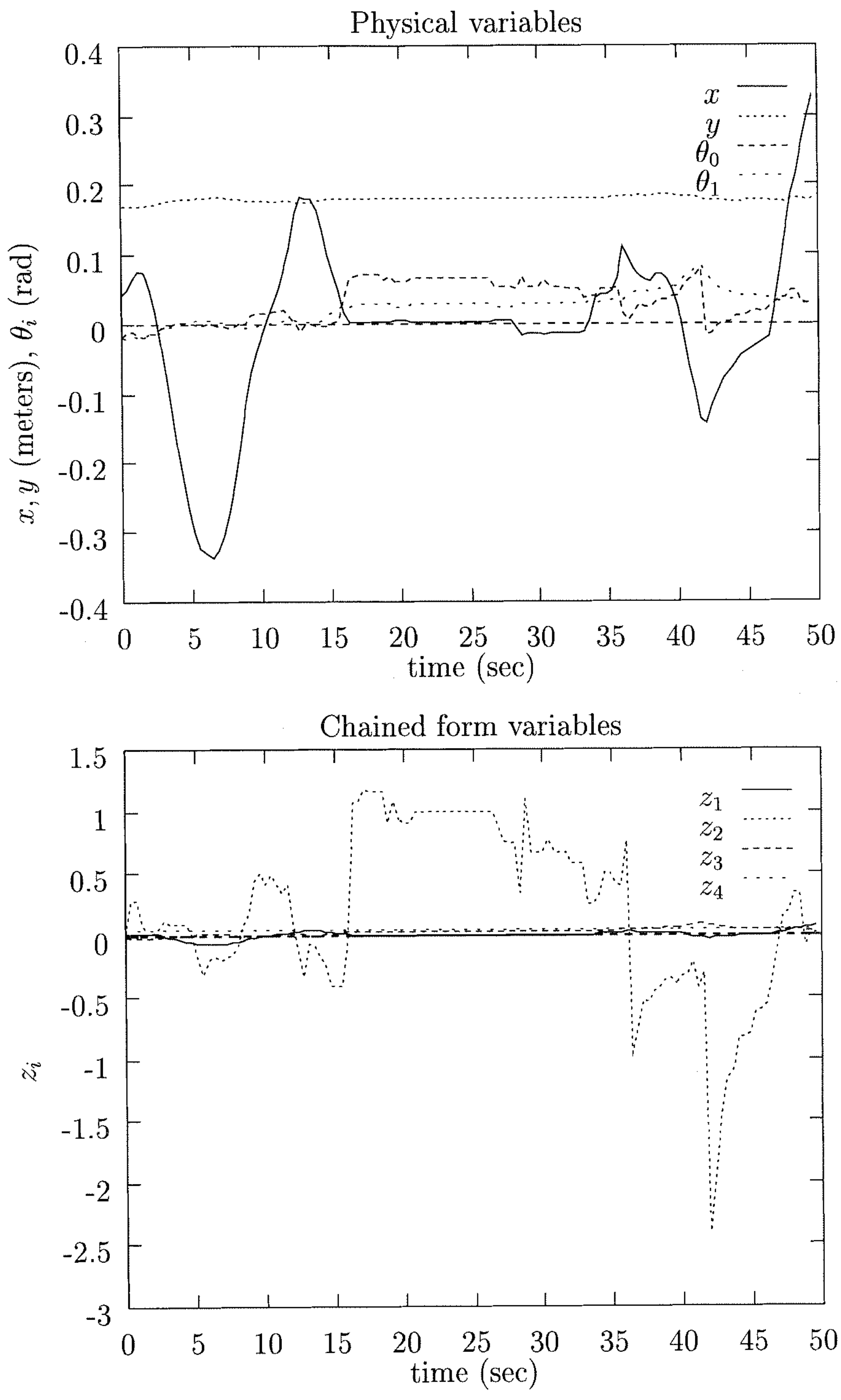

Figure 5.12: Affect of ill-conditioned diffeomorphism on system performance.

We overcome the ill-conditioning by scaling the linear measurements with respect to the trailer length. Even for wheeled systems judicious choice of length scale may not solve the ill-conditioning problem. For example, consider the situation in which the ratio of two kinematic parameters is large: a length scale cannot be chosen to normalize both parameters to one. Finally, an important point to note is that the 3-dimensional system has no characteristic length associated with the kinematic model and the transformation specified by equations (5.4) has condition number 1 at all points in the phase space for any desired length scale.

The issue of transformation conditioning has not been addressed in the nonlinear systems literature but, as illustrated here, has a large impact on the performance. Control practitioners are well aware of the potential dangers of model inversion for linear systems. Our transformation may be interpreted as a kinematic inversion as opposed to the dynamic inversion often used in linear synthesis. One should expect the same problems to arise in the nonlinear setting as well.

Chapter 6

Conclusion

This thesis has presented an approach to obtain explicit ρ-exponentially stabilizing control laws for a large class of driftless systems. The feedbacks rely on a fundamental approximation of the system and they preserve the structure of this approximation. This leads to a slightly modified notion of exponential stability. Although the synthesis methods in this thesis do not cover all controllable analytic driftless systems, many application areas satisfy the conditions required by the methods. The feedbacks are necessarily non-Lipschitz for exponential stabilization and this property naturally arises from the synthesis procedure. By requiring the feedbacks to be smooth everywhere except the desired equilibrium point the closed-loop solutions are guaranteed to be unique.

A specialized singular perturbation result for degree zero systems proves that lowpass filters in the loop do not change the ρ-exponential stability of the system. This fact is not obvious since the system linearization is not defined. Of course, lowpass filtering of measurements is always used in applications where the controller is implemented digitally. Another aspect of practical significance is the fact that the control variables are often velocities in the driftless models. The extension of kinematic controllers to controllers which stabilize the driftless system plus a set of integrators is given. This framework is also used to show how dynamic controllers, which include the integrators as states, may be used to smooth the control rate commanded by the controller. In this case the control action is continuously differentiable.

This entire paradigm was put to the experimental test with the nonholomobile. The experiments demonstrated the superior performance of the ρ-exponential stabilizers as opposed to traditional smooth feedbacks. The experiments also revealed the importance of diffeomorphism condition number for nonlinear control systems.

6.1 Future Directions

Even though the synthesis methods in this dissertation do not cover the most general class of driftless systems there are other issues which deserve just as much attention. Several research areas, pertaining to driftless systems and to nonlinear systems in general, are given below.

Quantitative Analysis. Many of the results proven here are of a perturbative nature. In other words, a given property manifests itself for "ϵ sufficiently small" or "k sufficiently large." This is certainly true of the averaging theorem, lowpass filtering and dynamic extension results. These results are qualitative in the sense that they do not actually exhibit an ϵ or k for which the results hold but merely imply the existence of such numbers. Results of this nature are usually the first ones to be proven in analysis because they are the easiest to formulate and solve. This does not diminish their importance in *systems theory* but somewhat limits their usefulness in practical applications. A useful set of design tools would assign values to ϵ and k and show the tradeoff between domain of attraction, convergence rate, and the effects of noise on control effort and control rate. Another useful tool would explore methods to optimize the convergence rate of the closed-loop system. Many of these issues can be partially solved with the use of a Lyapunov function. However, there is currently no way to choose or construct the Lyapunov function which gives the least conservative estimates of the quantities of interest.

Robustness. The converse Lyapunov theorems used in this thesis were used to show that terms neglected in the *approximation* of the model do not affect the stability of the system. Perturbations of the model itself were not considered. When the model is written in the coordinates adapted to its filtration it is a simple matter to characterize the perturbations which do not affect the stability. These results are still of a quantitative nature though. A more useful result is the characterization of the perturbations in the original coordinates which do not change the stability of the system. For example, the kinematic wheel given by the model,

$$\begin{aligned}\dot{x} &= \cos\theta v \\ \dot{y} &= \sin\theta v \\ \dot{\theta} &= \omega,\end{aligned}$$

is locally asymptotically stabilized by the feedback,

$$\begin{aligned}v &= -z_2 + \frac{z_3^2}{\rho^3(z)}\sin t \\ \omega &= -z_1 + \frac{z_3}{\rho(z)}\cos t \\ \rho(z) &= (z_1^4 + z_2^4 + z_3^2)^{\frac{1}{4}},\end{aligned}$$

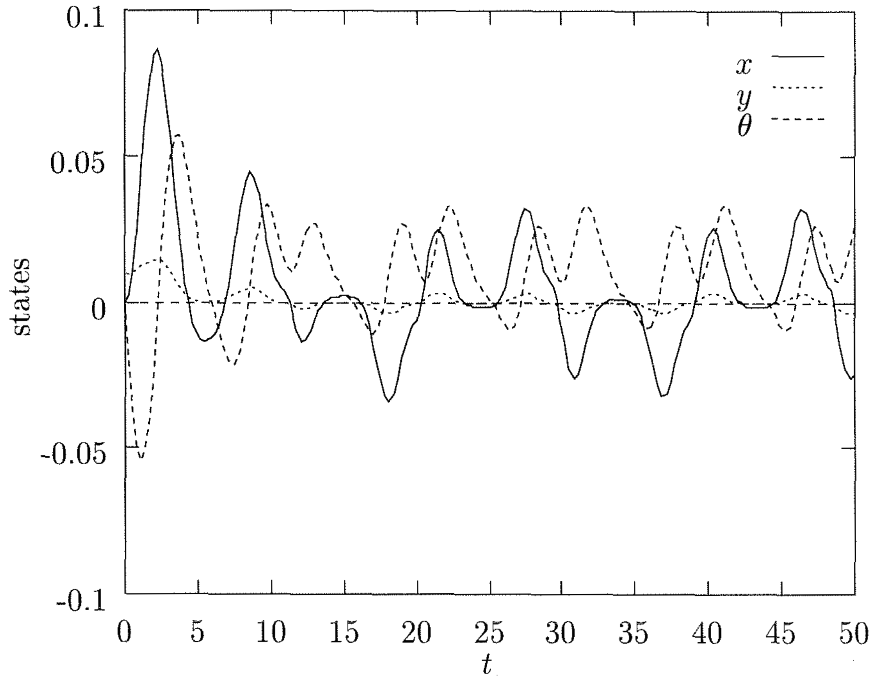

Figure 6.1: Unstable equilibrium.

where

$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta. \end{aligned}$$

The convergence rate is exponential in the sense of equation (2.2) for the (x, y, θ) variables. A small perturbation of the $\dot{y}$ equation to $\dot{y} = (\epsilon + \sin\theta)v$ destabilizes the origin as shown in Figure 6.1 ($\epsilon = 0.1$ and the initial conditions are $(0, 0.01, 0)$). This perturbation is actually degree two compared to the degree one approximations of the vector fields in the coordinates adapted the filtration. Unlike linear input-output models, the states of many nonlinear models represent physical quantities. In this case, the class of physically meaningful perturbations must be considered. In the kinematic wheel example, the perturbation used in the $\dot{y}$ equation is not justified from a physical basis and so we expect our controller to perform well even though an arbitrarily small perturbation in the model can destabilize the desired equilibrium point.

Systems with Drift Vector Fields. Control systems with drift vector fields have the form,

$$\dot{x} = X_0(t, x) + \sum_{i=1}^{M} X_i(t, x)u_i.$$

Homogeneous approximations and feedbacks have been applied to these systems in certain low dimensional cases by Hermes [17, 18] and Kawski [20, 19]. A system with drift vector field is more difficult to analyze when the linearization is not controllable. In fact no necessary and sufficient conditions have been found for STLC [41]. Thus systems with drift are a more challenging class to control. Egeland in [11] has recently proposed a model for an underwater vehicle which has homogeneous strucure. The drift vector field is due to the intertial effects of the vehicle and does not fit within the framework of equations (4.22). The model also fails Brockett's condition for continuous time-invariant stabilization. However, an asymptotically stabilizing controller was derived by further developing ideas from the methods used by Sørdalen [38] who stabilized driftless systems in chained form. These references demonstrate that homogeneous approximations and feedbacks have an important and fundamental role to play in systems with drift vector fields.

Appendix A

Review of Controllability and Stabilization Results for Driftless Control Systems

This appendix reviews the basic local controllability properties of nonlinear affine control systems. A necessary condition for continuous stabilization of driftless systems and related results are also covered. An excellent reference for the controllability results reviewed in this appendix is Nijmeijer and van der Schaft's book [36].

A.1 Controllability

Controllability for nonlinear systems is developed for general affine systems of the form

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} X_i(x)u_i, \quad u \in U \subset \mathbb{R}^m,\ x \in V \subset \mathbb{R}^n \tag{A.1}$$

where V is an open subset of $\mathbb{R}^n$, the X_j are smooth vector fields defined on V, and the u_i are real valued functions of time. The following assumption is made concerning the type of control input that is admissible:

Assumption A.1 An admissible input u satisfies the following two conditions,

i) the input space U is such that the set of associated vector fields of the system (A.1)

$$\mathcal{F} = \{X_0 + \sum_{i=1}^{m} X_i u_i | (u_1, \ldots, u_m) \in U\}$$

contains the vector fields $X_0, X_1, \ldots, X_m$ and,

ii) an admissible control is a piecewise constant function which is piecewise continuous from the right.

See Sontag [41] for an explanation as to why the set of control inputs may be restricted to those satisfying the assumption instead of a more general class of functions. The trajectory of the system through the point x with admissible input u is denoted $\psi(t, 0, x; u)$. The definition of controllability is

Definition A.2 The nonlinear control system (A.1) is called *controllable* if for any two points $x_1, x_2 \in V$ there exists an admissible control of finite duration $u : [0, T] \to U$ such that $\psi(T, 0, x_1, u) = x_2$.

An object of fundamental importance for controllability is the so called *accessibility algebra* of the system (A.1).

Definition A.3 The *accessibility algebra* $\mathcal{C}$ of the system (A.1) is the smallest subalgebra of the Lie algebra of smooth vector fields on V that contains $X_0, X_1, \ldots, X_m$.

The *accessibility distribution* C is the distribution

$$C(x) = \text{span}\{X(x)|X \text{ a vector field in } \mathcal{C}\}, x \in V. \tag{A.2}$$

Let $R^W(x, T)$ denote the set of reachable points from x at time $T > 0$ following trajectories which remain in the neighborhood W of x for all $t < T$. Furthermore, define

$$R_T^W(x) = \bigcup_{\tau \leq T} R^W(x, T).$$

The following theorem states that an open neighborhood of points may be reach from x if the accessibility distribution has rank n at x,

Theorem A.4 *Assume that*

$$\dim C(x) = n,$$

for the system (A.1). Then for any neighborhood W of x and $T > 0$ the set $R_T^W(x)$ contains a non-empty open set of V.

This theorem does not imply the system (A.1) is controllable about a given point. However, for driftless systems, i.e. $X_0 = 0$, a full rank accessibility distribution does imply local controllability,

Proposition A.5 *Suppose $X_0 = 0$ in (A.1) then if $\dim C(x) = n$ then R_T^W contains a neighborhood of x for all neighborhoods W of x and $T > 0$.*

Finally, another stronger definition of controllability is the notion of *small time local controllability*, abbreviated STLC. The definition is reviewed below since several references in the introduction assume the STLC property. Another important property of driftless systems on $\mathbb{R}^n$ is that controllability and STLC are equivalent.

Definition A.6 The system (A.1) is *STLC* (at zero) if for any t_1, $\alpha > 0$ the set of all points which can be reached at time t_1 via solutions, initiating from zero, by using measurable controls $t \to u(t) = (u_1(t), \ldots, u_m(t))$ satisfying $|u_i(t)| \leq \alpha$, contains a neighborhood of zero.

A.2 Stabilization

The following necessary condition for continuous autonomous stabilization was first brought to the attention of the controls community by Brockett [4]. The class of systems are ordinary differential equations which are continuous in the state and parameter u (to be thought of as a control variable),

$$\dot{x} = f(x, u). \tag{A.3}$$

Theorem A.7 (Brockett) *Assume that (A.3) admits a continuous stabilizing feedback $u(x)$. Then for each $\epsilon > 0$ there is a positive number δ such that, for each y with $\|y\| < \delta$, the equation*

$$y = f(x, u)$$

is solvable on the set $\|x\| < \epsilon, \|u - u(0)\| < \epsilon$.

Thus the image of $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ must cover a neighborhood of the origin. The driftless system

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m, \tag{A.4}$$

where the rank the input vector fields $[X_1|\cdots|X_m]$ is less than n (the state dimension) always fails this condition. Hence there does not exist a continuous autonomous feedback which renders any point asymptotically stable. However Coron [7] showed that controllable driftless systems may be asymptotically stabilized with a smooth time-periodic feedback,

Theorem A.8 (Coron) *Consider the driftless system given by equation (A.4) with smooth input vector fields. Assume that for all x in $\mathbb{R}^n \setminus \{0\}$ there are vector fields $Y_1, \ldots, Y_r$ in the Lie algebra generated by $\{X_1, \ldots, X_m\}$ such that $\{Y_1(x), \ldots, Y_r(x)\}$ span $\mathbb{R}^n$. Then $x = 0$ can be globally asymptotically stabilized by means of a smooth time-periodic feedback law $u_i = u_i(t, x)$.*

Thus even though Brockett's condition precludes an autonomous continuous stabilizing feedback, Coron's result demonstrates that an explicitly time varying continuous feedback can asymptotically stabilize the driftless system.

The limitations of smooth feedback were discussed in Chapter 4. It is of interest to know whether faster rates of convergence can be achieved for driftless systems. Another existence result by Coron states that controllable driftless systems may be stabilized to the origin in finite time with continuous time-periodic feedback. The definitions and results below are taken from [8].

Definition A.9 The continuous system satisfying the small time local controllability condition given in Definition A.6 is *locally asymptotically stabilizable by means of a T-periodic feedback* law if there exists $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ such that

$$\begin{gathered} u \in C^\infty(\mathbb{R}^n \setminus \{0\} \times \mathbb{R}; \mathbb{R}^m) \cap C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m), \\ u(0,t) = 0 \;\; \forall t \in \mathbb{R}, \\ u(x, t+T) = u(x,t) \;\; \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \end{gathered}$$

and

i) $\exists \delta > 0$ such that for $|x_0| < \delta, t_0 \leq t_1$ there exists one and only one solution on $[t_o, t_1]$ of $\dot{x} = f(x, u(t,x)), x(t_0) = x_0$,

ii) $0 \in \mathbb{R}^n$ is a locally asymptotically stable point of $\dot{x} = f(x, u(t,x))$.

If such a u exists the the system is said to be T-LAS. If, moreover, for all small enough x_0,

$$\dot{x} = f(x, u(t,x)) \text{ and } x(0) = x_0 \quad \Longrightarrow \quad x(T) = 0,$$

then the system is termed T-Locally Stabilizable (T-LS).

As mention above, for driftless systems the STLC condition is equivalent to controllability. The result of primary interest from Coron's paper is

Theorem A.10 (Coron) *Assume $f(x,u) = \sum_{i=1} X_i(x) u_i$. Then $\dot{x} = f(x,u)$ is T-LS for all positive T.*

The feedbacks presented in this thesis do not stabilize the equilibrium point in finite time however they are continuous and smooth on $\mathbb{R}^n \setminus \{0\}$ as is required by exponential rates of stabilization.

Appendix B

Computation of Coordinates Adapted to a Filtration

This appendix introduces a simple algorithm for computing coordinates adapted to a filtration. We are interested in approximating a set $\{X_1, \ldots, X_m\}$ of *input* vector fields which generate a full-rank Lie algebra by a set of controllable nilpotent vector fields. The coordinate change is polynomial with order equal to the degree of nonholonomy of the vector fields. The algorithm has the advantage of being simple in concept and easily implementable with a symbolic manipulation package. Even though the algorithm is developed in the context of driftless control systems, it may be made more broadly applicable with some minor modifications. More motivation may be found in [18].

B.1 Background and Motivation

For notation and basic definitions the reader is referred to Chapter 2. Related computational results may be found in [42]. This algorithm is developed for approximating the vector fields of the following nonholonomic control system,

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m, \quad x \in \mathbb{R}^n, \tag{B.1}$$

where $X_i : \mathbb{R}^n \to \mathbb{R}^n$ is an analytic, nonzero vector field in a neighborhood of, without loss of generality, the origin. We assume $n > m$. The coordinates adapted to the filtration described in Chapter 2 are generated by the following local diffeomorphism x- and z-coordinates,

$$x = \Phi(z) = \psi^{z_1}_{X_{\pi_1}} \circ \psi^{z_2}_{X_{\pi_2}} \circ \cdots \circ \psi^{z_n}_{X_{\pi_n}}(0), \tag{B.2}$$

where ψ^t_X denotes the flow of X for time t and the vector fields X_{π_i} are selected from the filtration according to equation (2.9). The usual notation for flows is

$\psi(t, t_0, x)$. The new notation is used to simplify the writing of the composed flows and to denote the vector field with which flow is associated. The construction of this transformation does require solving differential equations which can be an arduous task. The physical response of the system with the stabilizing feedback will depend on which vector fields from the filtration are chosen to compute the transformation. However this is not explored in this appendix. Finally, we must distinguish between "degree with respect to a dilation" and "degree in a Taylor series expansion" (which is actually degree with respect to the standard dilation). It should be clear which definition is intended from the context.

B.2 Results

This section describes the algorithm and its application for the approximation problem discussed above. The diffeomorphism (B.2) can certainly be computed by hand for low dimensional problems. However, this is undesirable if a number of transformations are desired by picking different vector fields from the filtration. For higher dimensional systems the computations become pedantic and a symbolic manipulator "solution" is desired.

The vector fields of the filtration are analytic so the transformation (B.2) is analytic in a neighborhood of the origin. It is intuitive that a sufficiently high degree Taylor approximation of (B.2) should suffice to place the $X_i \in F_1$ into proper form. Recall that $X_i = X_i^1 + X_i^0 + X_i^{-1} + \cdots$, where the superscript denotes the degree with respect to the dilation, in the coordinates adapted to the filtration. Thus, computing (B.2) does not eliminate the "higher" order terms in the new coordinates. Given this fact it is less compelling to compute (B.2) exactly. We should be satisfied with an approximation of (B.2) as long as $X_i = X_i^1 + \tilde{X}_i^0 + \tilde{X}_i^{-1} + \ldots$ in the approximated coordinates (ie. the "degree one" part of X_i remains unchanged).

The main obstruction to implementing (B.2) on a computer is the fact that symbolic manipulators are unable to integrate general nonlinear vector fields symbolically. The idea behind the algorithm is to perform preliminary coordinate changes so that the vector fields are trivial to integrate. The X_{π_i} may be "straightened out" since they are nonzero in a neighborhood of the origin. Integrating the straightened out system is trivial. Unfortunately, this coordinate change requires knowledge of the flow $\psi^t_{X_{\pi_i}}$. However, we may perform a sequence of transformations on each X_{π_i} that successively removes higher order terms up to some prespecified order. The flow of $\psi^t_{X_i}$ may be approximated accurately by a simple symbolic integration in these coordinates.

We now show that if the degree of nonholonomy of the set $\{X_1, \ldots, X_m\}$ is p then it is sufficient to compute (B.2) to order p to get the correct degree one (with respect to the dilation) approximation, X_i^1, of X_i. Suppose we have the full diffeomorphism $x = \Phi(z)$ given by (B.2). Then $X_i(x)$ given in the coordinates adapted to the filtration is

$$\begin{aligned} X_i(z) &= \left(\Phi^{-1}\right)_* X_i(\Phi(z)) \\ &= X_i^1(z) + X_i^0(z) + \dots . \end{aligned} \tag{B.3}$$

The degree of nonholonomy p implies that the variable z_n has a weight p in the dilation. Variables z_1 through z_m have weight 1 since each $X_i \in F_1, i-1, \dots, m$. Thus, the highest order Taylor series terms in X_i^1 are monomials of $z_1, z_2, \dots, z_m$ with degree $p-1$ appearing in the expression for $\dot{z}_n$. Hence terms of degree $p-1$ must be preserved by the approximation of Φ. To compute $X_i^1(z)$ through degree $p-1$ we compute each of the terms in (B.3) through degree $p-1$. This implies that we must at least retain terms through degree $p-1$ in $X_i(x)$ and $\Phi(z)$. Now suppose

$$\Phi_*(z) = A + R(z), \tag{B.4}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular and $R(0) = 0$. The Jacobian of the inverse map may be computed as,

$$\begin{aligned} \left(\Phi^{-1}\right)_*(z) &= \left(\Phi_*(z)\right)^{-1} \\ &= (A + R(z))^{-1} \\ &= A^{-1}(I + R(z)A^{-1})^{-1} \\ &= A^{-1}\left(I - R(z)A^{-1} + \left(R(z)A^{-1}\right)^2 + \dots\right) \\ &= A^{-1}\left(I - R(z)A^{-1} + \left(R(z)A^{-1}\right)^2 + \dots + \left(R(z)A^{-1}\right)^{p-1}\right) + \mathcal{O}(p). \end{aligned} \tag{B.5}$$

In practice, $R(z)$ is truncated at order p so that the computation of $\left(\Phi^{-1}\right)_*$ is correct to order p. Thus an approximation of $\Phi(z)$ through degree p terms is sufficient to calculate $X_i^1(z)$ in the new coordinates.

Computing each of the diffeomorphisms, $\psi_{X_{\pi_i}}^{z_i}$, in equation (B.2) through order p will ensure that Φ is approximated through order p. Now we demonstrate how to approximate the flow $\psi_{X_{\pi_i}}^t$ for $i = 1, \dots, n$. Suppose that an initial linear change of coordinates has been performed so that $X_{\pi_i}(0) = e_i$. To remove the linear terms in the vector field X_{π_i} apply a change of coordinates as follows,

$$x = y + h_2(y), \tag{B.6}$$

where h_2 is composed of degree 2 monomials in y_i and is yet undetermined. Changing to y-coordinates yields,

$$\begin{aligned} X_{\pi_i}(y) &= (I + D_y h_2(y))^{-1} X_{\pi_i}(y + h_2(y)) \\ &= (I - D_y h_2(y) + (D_y h_2(y))^2 - \dots) X_{\pi_i}(y + h_2(y)) \\ &= X_{\pi_i}(0) + X_{\pi_i}^{(1)}(y) - D_y h_2(y) X_{\pi_i}(0) + \mathcal{O}(y^2), \end{aligned} \tag{B.7}$$

where $X_{\pi_i}^{(1)}$ denotes Taylor terms of degree one of the vector field and $D_y h_2$ is the Jacobian of h_2. To eliminate the linear terms we choose h_2 such that the following is satisfied,

$$D_y h_2(y) X_{\pi_i}(0) = X_{\pi_i}^{(1)}(y). \tag{B.8}$$

One way to select h_2 is to set

$$h_2(y) = \int X_{\pi_i}^{(1)}(y) dy_i. \tag{B.9}$$

This choice guarantees that relation (B.8) is always true. The kernel of this operation is all degree 2 monomials which do not contain y_i. Even though these terms allow extra degrees of freedom in the coordinate transformation, we show below that the choice given by equation (B.9) is the most desirable for our algorithm. Once h_2 is determined we can formally change the vector field into the y-coordinates retaining all terms of degree p or smaller. It should be obvious that one can proceed in an analogous manner to eliminate the degree k terms, $X_{\pi_i}^{(k)}$, by specifying a transformation $x = y + h_{k+1}(y)$ where

$$h_{k+1}(y) = \int X_{\pi_i}^{(k)}(y) dy_i \tag{B.10}$$

consists of degree $k+1$ monomials in y. Thus, successive near-identity transformations are used place X_{π_i} into the following form

$$X_{\pi_i} = e_i + \mathcal{O}(p+1). \tag{B.11}$$

This process is analogous to the normal form computations for fixed points of vector fields with the important distinction that the operation

$$D_y h_{k+1} X_{\pi_i}(0)$$

is always a surjection from vector valued monomials of degree $k+1$ to vector valued monomials of degree k. Thus we are able to remove terms of arbitrary degree unlike the normal form case around a degenerate fixed point. Furthermore, we are not concerned if this process converges for infinite sequences of transformations since the computations are terminated at some fixed order. Approximate integration of the vector field (B.11) can now be accomplished.

Now we approximate the solution of equation (B.11). Suppose the initial condition is $x_i(0) = (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$. Then the trajectory of equation (B.11) through this initial condition is

$$\psi_{X_{\pi_i}}^{\tau}(x(0)) = (t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) + \mathcal{O}(\tilde{t}^{p+1}), \tag{B.12}$$

where $\tilde{t} = (t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)$. This is easily shown by expanding the solution of (B.11) in a multiple power series in the time parameter τ *and* the initial condition $x(0) = (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$. The existence of this power series is guaranteed by the analyticity of the vector field.

Now we have enough background to summarize the algorithm:

i) Begin with a preliminary linear change of coordinates $y = A^{-1}x$ where

$$\begin{aligned} A &= D_z\Phi(0) \\ &= (X_{\pi_1}(0), X_{\pi_2}(0), \ldots, X_{\pi_n}(0)) . \end{aligned}$$

A is nonsingular by assumption. We abuse notation by writing the vector fields in these new coordinates as $X_{\pi_i}(x)$ except now $X_{\pi_i}(0) = e_i$.

ii) Now we successively straighten out each vector field in equation (B.2) and approximate its flow starting with X_{π_n}. Perform the sequence of transformations described above to "straighten out" X_{π_n} through order p to yield $X_{\pi_n} = e_n + \mathcal{O}(p+1)$. Approximate the solution from the initial condition $x(0) = 0$,

$$\psi^{z_n}_{X_{\pi_n}}(x(0)) = (0, \ldots, 0, z_n) + O(z_n^{p+1}),$$

where $z = (z_1, z_2, \ldots, z_n)$. However at this stage only z_n is present.

iii) Compose these intermediate transformations into one transformation and truncate any terms with degree $p + 1$ or greater. Denote this polynomial change of coordinates as $x = \psi_n(y)$. The construction is such that x represents the original coordinates and y represents the new coordinate system.

iv) Now transform the remaining vector fields X_{π_i}, $i = 1, \ldots, n-1$, with ψ_n:

$$\left(\psi_n^{-1}\right)_* X_{\pi_i}(\psi_n(y)).$$

Abusing notation once more, denote the vector fields in these coordinates as $X_{\pi_i}(x)$, $i = 1, \ldots, n-1$. Note that $\left(\psi_n^{-1}\right)_*$ may be approximated through order p using the computation from equation (B.5) (in this case $A = I$ so some symbolic operations are saved).

v) Now straighten out $X_{\pi_{n-1}}(x)$ through order p so that in the new coordinates $X_{\pi_{n-1}} = e_{n-1} + \mathcal{O}(p+1)$. Denote the order p transformation which accomplishes this ψ_{n-1}. Before the approximate solution of $X_{\pi_{n-1}}$ is calculated, the initial condition $x(0) = (0, \ldots, 0, z_n)$ must be transformed into the new coordinate representation. In other words the following equation must be solved for $y(0)$,

$$(0, \ldots, 0, z_n) = \psi_{n-1}(y_1(0), \ldots, y_n(0)). \tag{B.13}$$

If we make the specific choice of the h_i given by equation (B.10) when constructing ψ_{n-1} then every term in ψ_{n-1} of degree greater than one has a y_{n-1} factor. Thus the unique solution of (B.13) in a neighborhood of the origin is easily verified to be $y_i(0) = x_i(0)$, $i = 1, \ldots, n$ i.e. $y_i(0) = 0$ for $i = 1, \ldots, n-1$ and $y(n) = z_n$. Had we decided to exercise the extra degrees of freedom in computing ψ_{n-1} then the inversion of (B.13) would have been much more involved.

vi) The flow of $X_{\pi_{n-1}}$ from the initial condition $x(0) = (0, \ldots, 0, z_n)$ is approximated as

$$\psi^{z_{n-1}}_{X_{\pi_{n-1}}}(x(0)) = (0, \ldots, 0, z_{n-1}, z_n) + \mathcal{O}(z^{p+1}).$$

Transform X_{π_i}, $i = 1, \ldots, n-2$, into the new coordinates using ψ_{n-1} and proceed to straighten out $X_{\pi_{n-2}}$. Once again, special choice of the h_i's ensures that the initial condition in the new coordinates is the same as the initial condition in the old coordinates.

vii) Proceeding in this manner we finally straighten out X_{π_1} with ψ_1 and the flow is given by

$$\begin{aligned}\psi^{z_1}_{X_{\pi_1}}(0, z_2, z_3, \ldots, z_n) &= (z_1, z_2, \ldots, z_n) + \mathcal{O}(z^{p+1}) \\ &= F(z) + \mathcal{O}(z^{p+1}).\end{aligned} \tag{B.14}$$

viii) The local diffeomorphisms ψ_i, $i = 1, \ldots, n$ are polynomial of order p. They relate the new z-coordinates to the *original* x-coordinates by following transformation,

$$x = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_1(F(z)). \tag{B.15}$$

By virtue of our choices for h_i, $F(z) = (z_1, z_2, \ldots, z_n)$ although in the more general case F would be a local diffeomorphism itself. $\Psi(z)$ denotes the order p truncation of (B.15) and has the property that,

$$\Phi(z) = \Psi(z) + \mathcal{O}(z^{p+1}) \tag{B.16}$$

by the construction above.

The original vector fields X_i, $i = 1, \ldots, m$, may be expressed in the z-coordinates to yield the decomposition $X_i = X_i^1 + X_i^0 + X_i^{-1} + \ldots$.

This algorithm is rather involved for hand computations. However, its simplicity and the fact that the flows of the vector fields are trivial in the correct coordinate system makes the algorithm amenable to symbolic programming.

B.3 Improvements to the Algorithm

An improvement qualifies as any modification that reduces the number of symbolic manipulations yet still produces a correct homogeneous degree one approximation of X_i. There are several potential improvements which may accelerate the execution time of the program. The first improvement is to note that Ψ need only be computed through order $p-1$ (i.e., all terms of degree p or higher are neglected) because the degree p terms remaining in the transformed vector fields will either be components of a degree 1 vector field or higher *with respect to* the dilation. For example, if X_{π_1} is chosen to be $X_1 \in F_1$ then $X_1^1 = e_1$ if Ψ is computed through order p as opposed to $X_1^1 = e_1 + r_{p-1}$, where r_{p-1} are degree $p-1$ monomials, in the situation when Ψ is computed through order $p-1$.

A second improvement may be realized by computing the ψ_i to various orders of accuracy depending on i. For example, because we are interested in the approximation of vector fields $X \in F_1$, the z_n variable, which has weight p with respect to the dilation, appears *nowhere* in X^1 in the coordinates adapted to the filtration. Thus the flow $\psi^{z_n}_{X_{\pi_n}}(0)$ need not be calculated through order p,as it is above, to achieve the same approximation of X^1_i. This implies that the order of ψ_n may be lowered from p. Similarly, z_1 has weight 1 in the dilation so that we would always want to compute the flow of X_{π_1} to order p (or order $p-1$ given the argument in the first paragraph of this section) to preserve terms like z_1^{p-1} which may appear in the $\dot{z}_n$ component of $X_{\pi_i}(z)$.

The results of a Mathematica program which implements the algorithm on an example are shown below.

Example B.1 Consider the four-dimensional two input driftless control system with input vector fields

$$X_1(x) = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + (x_1+x_2)\frac{\partial}{\partial x_3} + (x_1 + \frac{1}{2}x_1^2 + x_2)\frac{\partial}{\partial x_4} \tag{B.17}$$

$$X_2(x) = x_1\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + (x_1+x_3)\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}. \tag{B.18}$$

The set $\{X_1, X_2\}$ is controllable since

$$\operatorname{rank}[X_1, X_2, \operatorname{ad}^2_{X_1}X_2, \operatorname{ad}^3_{X_1}X_2](0) = 4.$$

We wish to approximate the set (B.18) by homogeneous degree one vector fields. Choosing the X_{π_i} vector fields in the algorithm as $X_{\pi_1} = X_1, X_{\pi_2} = X_2, X_{\pi_3} = \operatorname{ad}^2_{X_1}X_2, X_{\pi_4} = \operatorname{ad}^3_{X_1}X_2$ results in the following approximate system

$$X_1^1(y) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{1}{2}y_1^2y_2 \end{pmatrix} \quad X_2^1(y) = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}y_1^2 + y_1y_2 \\ \frac{1}{2}y_1^2y_2 \end{pmatrix}.$$

These vector fields are degree one with respect to the dilation with scaling powers $r = (1,1,3,4)$. The (local) diffeomorphism between the x- and y-coordinates is

$$x = T \circ f(y),$$

where T is

$$T(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix},$$

and $f(y)$ is

$z_1 = y1$

$$z_2 = -2y_1^2 + \frac{1}{6}y_1^3 + y2 - 3y_1y_2 - \frac{3}{2}y_1^2y_2 - 3y_1y_3 - \frac{3}{2}y_1^2y_3 - \frac{5}{2}y_3^2 - \frac{5}{2}y_2y_3^2$$
$$- 3y_1y_4 - \frac{3}{2}y_1^2y_4 - 5y_2y_4 - \frac{5}{2}y_2^2y_4 - 5y_3y_4 - 5y_2y_3y_4 + \frac{5}{2}y_4^2 + \frac{5}{2}y_2y_4^2$$
$$z_3 = \frac{3}{2}y_1^2 - \frac{1}{6}y_1^3 + 3y_1y_2 + \frac{3}{2}y_1^2y_2 + y_3 + 3y_1y_3 + \frac{3}{2}y_1^2y_3 + 2y_3^2 + 2y_2y_3^2 + 3y_1y_4 +$$
$$\frac{3}{2}y_1^2y_4 + 4y_2y_4 + 2y_2^2y_4 + 4y_3y_4 + 4y_2y_3y_4 - 2y_4^2 - 2y_2y_4^2$$
$$z_4 = \frac{1}{2}y_1^2 + y_1y_2 + \frac{1}{2}y_1^2y_2 + y_1y_3 + \frac{1}{2}y_1^2y_3 + \frac{1}{2}y_3^2 + \frac{1}{2}y_2y_3^2 + y_4 + y_1y_4 + \frac{1}{2}y_1^2y_4$$
$$+ y_2y_4 + \frac{1}{2}y_2^2y_4 + y_3y_4 + y_2y_3y_4 - \frac{1}{2}y_4^2 - \frac{1}{2}y_2y_4^2.$$

Appendix C

Stability Proof for Three-Dimensional System

There is often a trade-off between the complexity of a controller derived through an algorithmic process versus the simplicity of a controller derived through more hueristic means. The extension of Pomet's algorithm and the modification of smooth controllers to ρ-exponential stabilizers yield controllers for which asymptotic stability is automatic since construction of an appropriate Lyapunov function is part of the process. In the first case, the controller is determined numerically in all but some special situations. In the second case, the controller is a scaled version of an explicit smooth control law except that the scaling must be determined numerically. Hence, controllers derived by these methods cannot, in general, be written down explicitly. However a simple explicit control law is often desirable in real-time applications.

This appendix contains a proof of asymptotic stability for the prototype three-dimensional two input driftless system (3.1) with the following feedback,

$$\begin{aligned} u_1 &= -x_1 + \frac{x_3}{\rho(x)}\cos t \\ u_2 &= -x_2 + \frac{x_3^2}{\rho^3(x)}\sin t \\ \rho(x) &= (x_1^4 + x_2^4 + x_3^2)^{1/4}. \end{aligned} \tag{C.1}$$

The system (3.1) is its own nilpotent homogeneous degree one approximation with respect to the dilation

$$\Delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3). \tag{C.2}$$

The closed-loop system is degree zero since the feedback functions (C.1) are degree one with respect to the dilation.

The feedback (C.1) is a hueristic modification of the smooth feedback,

$$\begin{aligned} u_1 &= -\,x_1 + x_3 \cos t \\ u_2 &= -\,x_2 + x_3^2 \sin t. \end{aligned} \tag{C.3}$$

Center manifold analysis proves that this feedback is locally uniformly asymptotically stabilizing (see the introduction to Section 4.3). The smooth feedback functions may be made degree one by rescaling each term by a power of $\rho(x)$ in order to make that term degree one. This results in the feedback given by equations (C.1). There is no guarantee that the new feedback is stabilizing. However, we show below that the feedback (C.1) is asymptotically stabilizing.

The function,

$$V(t,x) = (x_1 - \frac{x_3}{2}(\cos t + \sin t))^2 + (x_2 - \frac{x_3^2}{2}(\sin t - \cos t) + x_3^2, \tag{C.4}$$

is a Lyapunov function for system (3.1) with feedback (C.3). A numerical calculation will reveal that the time derivative of (C.4) along the closed-loop system with (C.1) is indefinite. However, the "whole" function V is not required. As in the proof of Theorem 4.12, if we can identify a level set of V which may be defined as the level set of a homogeneous function and the time derivative of V, evaluated at all points on the level set, is negative then the system is asymptotically stable. Denote the family of level sets of $V(t,x)$ (parametrized by t) for some constant C as,

$$G_t^C = \{x | V(t,x) = C > 0\}.$$

Suppose there exists a C such that the level set G_t^C, for each fixed t, is transverse to the Euler vector field corresponding to the dilation (C.2), $X_E = x_1\partial/\partial x_1 + x_2\partial/\partial x_2 + 2x_3\partial/\partial x_3$. This conditions guarantees that the level sets may be defined as the level sets of a *homogeneous* function (homogeneous with respect to (C.2)). Furthermore, if the time derivative of V evaluated at all (t,x) with $x \in G_t^C$ along the closed-loop system with (C.1) is negative, then closed-loop system is asymptotically stable. The arguments to show this are essentially the proof of Theorem 4.12.

The proof relies on brute force numerical computations. Figure C.1 is a plot of

$$\min_{x \in G_t^{0.85}} L_{X_E} V,$$

versus $t \in [0, 2\pi)$. The function is always positive so the level sets $G_t^{0.85}$ are transverse to the Euler vector field. Figure C.2 plots the function,

$$\max_{x \in G_t^{0.85}} \frac{\partial V}{\partial t} + L_X V,$$

versus t where X denotes the closed-loop system with feedback (C.1). The time derivative of V is negative on the set $G_t^{0.85}$ since the function is negative. Thus the closed-loop system is ρ-exponentially stable with respect to the dilation (C.2).

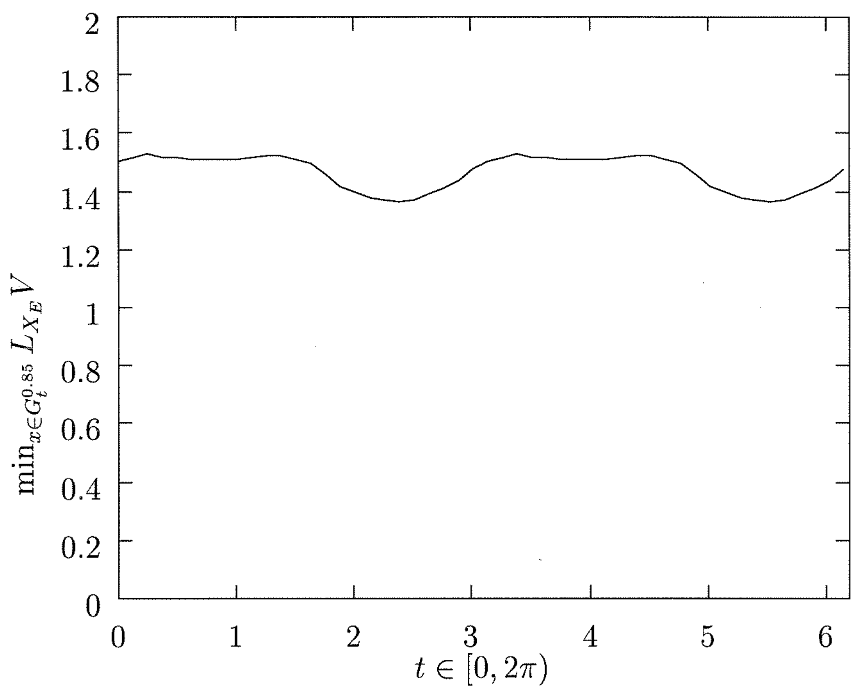

Figure C.1: Level sets of $V(t,x) = 0.85$ are transverse to X_E.

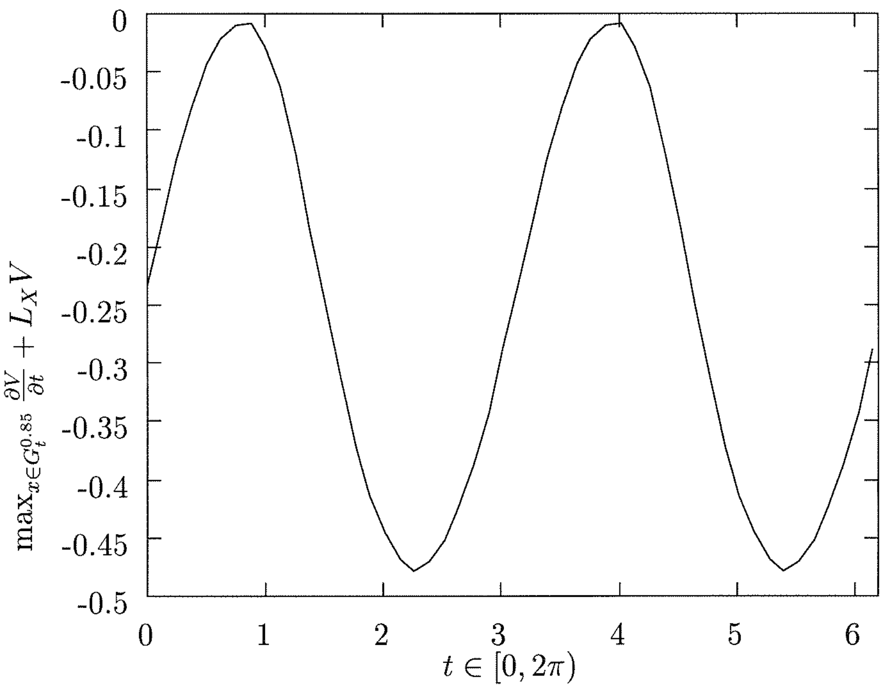

Figure C.2: Closed-loop system is asymptotically stable.

Bibliography

[1] A. Bellaïche, J.P.Laumond, and J.J. Risler. Nilpotent infinitesimal approximation to a control Lie algebra. In *IFAC Symposium on Nonlinear Control Systems Design (NOLCOS)*, pages 174–181, 1992.

[2] G. Birkoff and G-C. Rota. *Ordinary Differential Equations.* Wiley, 1989.

[3] A. M. Bloch, M. Reyhanoglu, and N.H. McClamroch. Control and stabilization of nonholonomic dynamic systems. *IEEE Transactions on Automatic Control*, 37(11):1746–1757, 1992.

[4] R. W. Brockett. Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman, and H. J. Sussman, editors, *Differential Geometric Control Theory*, pages 181–191. Birkhauser, 1983.

[5] R. W. Brockett. On the rectification of vibratory motion. *Sensors and Actuators*, 20(1–2):91–96, 1989.

[6] F.H. Clarke. *Optimization and Nonsmooth Analysis.* SIAM, 1990.

[7] J-M. Coron. Global asymptotic stabilization for controllable systems without drift. *Mathematics of Control, Signals, and Systems*, 5:295–312, 1991.

[8] J-M. Coron. Links between local controllability and local continuous stabilization. In *IFAC Symposium on Nonlinear Control Systems Design (NOLCOS)*, pages 477–482, 1992.

[9] C. Canudas de Wit and O.J. Sørdalen. Examples of piecewise smooth stabilization of driftless NL systems with less inputs than states. In *IFAC Symposium on Nonlinear Control Systems Design (NOLCOS)*, pages 26–30, June 1992.

[10] C. Canudas de Wit and O.J. Sørdalen. Exponential stabilization of mobile robots with nonholonomic constraints. *IEEE Transactions on Automatic Control*, 37(11):1791–1797, 1992.

[11] O. Egeland and E. Berglund. Control of an underwater vehicle with nonholonomic acceleration constraints. In *IFAC Symposium on Robot Control (SYROCO)*, pages 959–964, 1994.

[12] C. Fernandes, L. Gurvits, and Z. Li. Foundations of nonholonomic motion planning. Technical Report 577, Courant Institute of Mathematical Sciences, New York University, 1991.

[13] R. W. Goodman. *Nilpotent Lie Groups: Structure and Applications to Analysis.* Springer-Verlag, 1976. Lecture Notes in Mathematics, No. 562.

[14] W. Hahn. *Stability of Motion.* Springer-Verlag, 1967.

[15] J. Hale. *Ordinary Differential Equations.* Wiley-Interscience, 1969.

[16] H. Hermes. Homogeneous coordinates and continuous asymptotically stabilizing feedback controls. In Saber Elaydi, editor, *Lecture Notes in Pure and Applied Mathematics*, volume 127, pages 249–260. Marcel Dekker, 1989.

[17] H. Hermes. Asymptotically stabilizing feedback controls and the nonlinear regulator problem. *SIAM Journal of Control and Optimization*, 29(1):185–196, 1991.

[18] H. Hermes. Nilpotent and high-order approximations of vector field systems. *SIAM Review*, 33(2):238–264, 1991.

[19] M. Kawski. Stabilization of nonlinear systems in the plane. *Systems and Control Letters*, 12:169–175, 1989.

[20] M. Kawski. Homogeneous stabilizing feedback laws. *Control-Theory and Advanced Technology*, 6(4):497–516, 1990.

[21] P. S. Krishnaprasad and R. Yang. Geometric phases, anholonomy, and optimal movement. In *IEEE International Conference on Robotics and Automation*, pages 2185–2189, 1991.

[22] J. Kurzweil. On the inversion of Ljapunov's second theorem on stability of motion. *American Mathematical Society Translations*, 24:19–77, 1963.

[23] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities*, volume I. Academic Press, 1969.

[24] J-P. Laumond. Feasible trajectories for mobile robots with kinematic and environment constraints. In *Int. Conf. on Intelligent Autonomous Systems*, pages 346–354, Amsterdam, 1987.

[25] N. E. Leonard and P. S. Krishnaprasad. Averaging for attitude control and motion planning. In *IEEE Control and Decision Conference*, pages 3098–3104, 1993.

[26] Z. Li and J. Canny. Motion of two rigid bodies with rolling constraint. *IEEE Transactions on Robotics and Automation*, 6(1):62–71, 1990.

[27] Z. Li, P. Hsu, and S. Sastry. Grasping and coordinated manipulation by a multifingered robot hand. *International Journal of Robotics Research*, 8(4):33–50, 1989.

[28] R. T. M'Closkey and R. M. Murray. Convergence rates for nonholonmic systems in power form. In *American Control Conference*, pages 2967–2972, 1993.

[29] R. T. M'Closkey and R. M. Murray. Exponential covergence of nonholonomic systems: Some analysis tools. In *IEEE Control and Decision Conference*, pages 943–948, 1993.

[30] E. D. McShane. *Integration*. Princeton University Press, 1944.

[31] R. M. Murray. Nilpotent bases for a class of non-integrable distributions with applications to trajectory generation for nonholonomic systems. *Mathematics of Control, Signals, and Systems*, 1994. (in press).

[32] R. M. Murray and S. S. Sastry. Nonholonomic motion planning: Steering using sinusoids. *IEEE Transactions on Automatic Control*, 38(5):700–716, 1993.

[33] R. M. Murray, G. Walsh, and S. S. Sastry. Stabilization and tracking for nonholonomic systems using time-varying state feedback. In *IFAC Symposium on Nonlinear Control Systems Design (NOLCOS)*, pages 182–187, Bordeaux, France, 1992.

[34] R. M. Murray and E. Wemhoff. Sparrow refernece manual v 2.0. Technical report, California Institute of Technology, 1994.

[35] R.M. Murray, Z. Li, and S.S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, London, 1994.

[36] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer-Verlag, 1990.

[37] J-B. Pomet. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems and Control Letters*, 18(2):147–158, 1992.

[38] O. J. Sørdalen. *Feedback Control of Nonholonomic Mobile Robots*. PhD thesis, The Norwegian Institute of Technology, Trondheim, Norway, 1993.

[39] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems and Control Letters*, 19(6):467–473, 1992.

[40] C. Samson. Velocity and torque feedback control of a nonholonomic cart. In *Proc. in Advanced Robot Control*, volume 162. Springer-Verlag, 1991.

[41] E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer-Verlag, 1990.

[42] G. Stefani. Polynomial approximations to control systems and local controllability. In *IEEE Control and Decision Conference*, pages 33–38, 1985.

[43] A. R. Teel. Semi-global stabilization of minimum phase nonlinear systems in special normal forms. *Systems and Control Letters*, 19:187–192, 1992.

[44] A. R. Teel, R.M. Murray, and G. Walsh. Nonholonomic control systems: From steering to stabilization with sinusoids. In *IEEE Control and Decision Conference*, pages 1603–1609, 1992. (To appear in Int. J. Control).

[45] G. Walsh and L. Bushnell. Stabilization of multiple input chained form control systems. In *IEEE Control and Decision Conference*, pages 959–964, 1993.

[46] F. W. Wilson. The structure of the level surfaces of a Lyapunov function. *Journal of Differential Equations*, 3:323–329, 1967.