A bound on the number of integrators needed to linearize a two-input control system

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For nonlinear control systems with two inputs we consider the problem of dynamic feedback linearization. For a restricted class of dynamic compensators that correspond to adding chains of integrators to the inputs, we give an upper bound for the order of the compensator that needs to be considered. Moreover, we show by an example that this bound is sharp.

Key words: Nonlinear control, dynamic feedback, linearization, Pfaffian system.

1 Introduction

The problem of dynamic state feedback linearization for nonlinear continuous control systems arose in the late 1980s when it was realized that only few system are exact, or static, feedback linearizable. Although partial results towards a solution of the problem of dynamically feedback linearizing a nonlinear control system have been obtained, see e.g. [4,12], the complete problem is still unresolved. A particularly troubling issue is the question of whether an upper bound exists on the order of the compensator that dynamically feedback linearizes a given system.

A particular class of compensators, which we will call dynamic extensions, is given by adding chains of integrators to the input channels; there has been an interest in finding conditions for linearization under this restricted class of compensators, see e.g. [5]. In this article, we prove that if a control system is linearizable by such a dynamic extension, then it is also linearizable by a (possibly different) dynamic extension whose order is bounded by $2n-2$. Moreover, we provide an example which shows that this bound cannot be improved.

2 Preliminaries

We start this section with a short review of various constructions and fix the notation. Brunovsky showed, in [2], that any controllable linear system $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n, u \in \mathbb{R}^p$ can be converted, via a linear state transformation and a linear feedback, to a canonical form given by p chains of integrators:

$$\begin{array}{ccc} \dot{x}_1^1 = u^1 & \dots & \dot{x}_1^p = u^p \\ \dot{x}_2^1 = x_1^1 & \dots & \dot{x}_2^p = x_1^p \\ \vdots & & \vdots \\ \dot{x}_{k_1}^1 = x_{k_1-1}^1 & & \vdots \\ & & \dot{x}_{k_p}^p = x_{k_p-1}^p \end{array} \tag{1}$$

with $n = k_1 + \cdots + k_p$. For nonlinear systems, Brockett, [1], was one of the first people to work on the problem of exact linearization. Jacubczyk and Respondek, [11], and Hunt, Su and Meyer [9] gave necessary and sufficient conditions for a control affine system,

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)u^i,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^p. \tag{2}$$

to be feedback linearizable. System (2) is called *feedback linerizable* if there exist a feedback $u = \alpha(x) + \beta(x)v, v \in \mathbb{R}^p$ and a state transformation that transforms the system into a controllable linear system, or for that sake, into a system in Brunovsky normal form (1). The solution was consecutively generalized to the case of fully nonlinear systems

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^p, \tag{3}$$

by van der Schaft, [20], and rewritten in the language of differential forms by Gardner and Shadwick, [8].

More generally, one can consider a feedback with internal dynamics of the form

$$\begin{aligned} \dot{z} &= F(x,z,v),\ z \in \mathbb{R}^r,\ v \in \mathbb{R}^p, \\ u &= \alpha(x,z,v), \end{aligned} \tag{4}$$

also called a *compensator of order* r, where r is the number of states that are added to the system. The control system (3) is called *dynamic feedback linearizable*, if there exists a compensator (4) and a state transformation (in the (x,z)-coordinates) that transforms (3), (4) into a controllable linear system.

In this article, we will consider only the restricted class of compensators called *dynamic extensions*. In a dynamic extension, the only added states are derivatives of the original inputs (see also [5,17]),

$$\begin{cases} \dot{u}_0^1 = u_1^1, \dot{u}_1^1 = u_2^1, \ldots, \dot{u}_{r_1-1}^1 = u_{r_1}^1 \\ \vdots \\ \dot{u}_0^p = u_1^p, \dot{u}_1^p = u_2^p, \ldots, \dot{u}_{r_p-1}^p = u_{r_p}^p \end{cases} \tag{5}$$

for certain integers $r_1 \geq 0, \ldots, r_p \geq 0$. This is sometimes thought of as adding a certain number of integrators to each input channel; in this case, we have added r_i derivatives to the i^{th} input channel. For convenience of notation we have set $u_0^i = u^i$. In our notation, u_k^i is the k^{th} derivative of the i^{th} input.

We will call a nonlinear control system (3) *linearizable by dynamic extension* if (3) together with the dynamic extension (5) is feedback linearizable. For this restricted class of compensators, we can establish an upper bound on the number of integrators that need to be considered for a 2-input control system. Our main tool is that of a derived system of a Pfaffian system (a module of smooth differential 1-forms over the smooth functions), see [3,8] for more discussion of Pfaffian systems, exterior derivatives, and derived systems.

If I is Pfaffian system, then the *derived systems* of I are defined inductively by $I^{(0)} = I$,

$$I^{(k+1)} = \operatorname{span}\{\omega \in I^{(k)} \ : \ \mathrm{d}\omega \equiv 0 \bmod I^{(k)}\}.$$

Derived systems of a Pfaffian system are easily calculated in practice; an example of such a calculation is shown later in example 6. The derived systems form a chain of Pfaffian systems called the *derived flag*:

$$I \supset I^{(1)} \supset I^{(2)} \supset \cdots \supset I^{(k)} \supset \cdots .$$

If there is an integer k such that $I^{(k)} = I^{(k+1)}$, then $I^{(l)} = I^{(k)}$, for all $l \geq k$, and we say that the derived flag stabilizes with $I^{(k)}$. A nonlinear control system is strongly accessible if and only if $I^{(l)} = \{0\}$ for some integer l, see also [15].

Consider a control system Σ described by (3). Associate with Σ the Pfaffian system given by

$$I = \mathrm{span}\{\Omega^1, \dots, \Omega^n\},$$

where $\Omega^i = \mathrm{d}x^i - f^i(x,u)\,\mathrm{d}t$. Conditions for linearizing the system using state feedback are easily formulated, we refer to [8,16] for a demonstration of the following result.

Theorem 1 *A control system I is feedback linearizable if and only if*

(i) $I^{(n-1)} = \{0\}$,
(ii) each derived system $I^{(k)}, k \geq 0$ has constant dimension, and, for all $\omega \in I^{(k)}$,

$$\mathrm{d}\omega \equiv 0 \mod I^{(k)},\ \mathrm{d}t. \tag{6}$$

Note that the points at which the dimension of a Pfaffian system is locally constant always forms an open dense set. Therefore, if a given system does not have the same dimension at all points, we restrict to an open set of points where the dimension is constant.

After applying a dynamic extension of the form (5) to a nonlinear control system (3), we obtain the Pfaffian system corresponding to the combined system given by the sum of the two Pfaffian systems:

$$J = I \oplus \mathrm{span}\{\Pi^i_j \ :\ i = 1, \dots, p, j = 1, \dots, r_i\}, \tag{7}$$

where $\Pi^i_j = \mathrm{d}u^i_{j-1} - u^i_j\,\mathrm{d}t$ are the one-forms that are added in the dynamic extension. The following result signifies that if all inputs are differentiated at least once in a dynamic extension, the order of the extension can be immediately reduced.

Proposition 2 *If the system $\dot{x} = f(x,u)$ is linearizable by dynamic extension* (5) *with all indices $r_i \geq 1$, then the system is also linearizable via a dynamic extension of type* (5) *with indices $r_i - 1, i = 1, \dots, p$.*

PROOF. Assume that (3) is linearizable via a compensator of type (5) with all indices $r_i \geq 1$, and denote its Pfaffian system by J. Let $\bar{J}$ be the Pfaffian system corresponding to the system obtained from (3) by adjoining the compensator with indices $r_i - 1$. Then for each integer $k \geq 0$, $\bar{J}^{(k)} = J^{(k+1)}$. It follows from theorem 1 that $\bar{J}$ is also linearizable. $\square$

A repeated application of this proposition will show that if (3) is linearizable via a dynamic extension (5), then one may assume that at least one of the indices $r_i = 0$.

The main result of this paper relies on making a judicious choice for a generating set of 1-forms for the derived systems of J. In an effort to keep expressions compact, we introduce the following convenient notation relative to the dynamic extension (5): by u_k we denote the sequence of the k-th derivative of all inputs whose k^{th} derivative is defined by (5), that is $u_k := \{u_k^i : i = 1, \ldots, p \text{ and } k \leq r_i\}$. For example, $u_0 = u_0^1, \ldots, u_0^p$ is just the sequence of all the original inputs.

Proposition 3 *Let J be the Pfaffian system (7) associated with the control system (3) to which a dynamic extension with indices r_i has been applied. The k-th derived system of J has a set of generators given by the 1-forms $\Pi_j^i, j \leq r_i - k$ and a collection of 1-forms ω^i which are in the span of the original 1-forms Ω^j and whose coefficients are smooth functions of at most $k-1$ derivatives of the inputs:*

$$\omega^i = \sum_{j=1}^{n} W_j^i(x, u_0, \ldots, u_{k-1})\Omega^j. \tag{8}$$

Moreover, if the system (3) is control affine then we may choose

$$\omega^i = \sum_{j=1}^{n} W_j^i(x, u_0, \ldots, u_{k-2})\Omega^j. \tag{9}$$

PROOF. We first prove the statement for the general control system (3), using induction on the order k of the derived systems. For $k = 0$ the statement is true, since the original basis suffices and thus ω^i can be chosen to be equal to Ω^i; no input derivatives appear.

Assume that the statement is true for $k \geq 0$, i.e.

$$J^{(k)} = \operatorname{span}\{\omega^1, \ldots, \omega^m, \Pi_j^i \;:\; i = 1, \ldots, n,\; 1 \leq j \leq r_i - k\}, \tag{10}$$

where each ω^i is of the form (8). We calculate $J^{(k+1)}$. Let $\eta^1, \ldots, \eta^{n-m}$ be a selection of 1-forms from the set $\{\Omega^1, \ldots, \Omega^n\}$ that form a complement to $\{\omega^1, \ldots, \omega^m\}$ in the original Pfaffian system I; any 1-form in I can uniquely be decomposed in terms of the coframe ω^i, η^a. By definition, $J^{(k+1)}$ consists of those one-forms in $J^{(k)}$ whose exterior derivative is equal to zero modulo $J^{(k)}$. Any one-form in $J^{(k)}$ is a linear combination of the generators of $J^{(k)}$ as given in (10). Thus, the one-forms in $J^{(k+1)}$ are defined by functions λ_i, λ_i^j such that

$$d(\sum_{i=1}^{m} \lambda_i \omega^i + \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq r_i - k}} \lambda_i^j \Pi_j^i) \equiv 0 \bmod J^{(k)}.$$

The exterior derivative can be distributed inside the sums, giving:

$$\sum_{i=1}^{m}(\,\mathrm{d}\lambda_i \wedge \omega^i + \lambda_i\,\mathrm{d}\omega^i) + \sum_{\substack{1\le i\le p\\ 1\le j\le r_i-k}} (\,\mathrm{d}\lambda_i^j \wedge \Pi_j^i + \lambda_i^j\,\mathrm{d}\Pi_j^i) \equiv 0 \bmod J^{(k)}.$$

Because the ω^i and Π_j^i are in $J^{(k)}$, they can be assimilated into $J^{(k)}$ on the right-hand side. Thus, the equation will be satisfied if and only if

$$\sum_{i=1}^{m}\lambda_i\,\mathrm{d}\omega^i + \sum_{\substack{1\le i\le p\\ 1\le j\le r_i-k}} \lambda_i^j\,\mathrm{d}\Pi_j^i \equiv 0 \bmod J^{(k)}.$$

Since $\mathrm{d}\Pi_j^i = -\Pi_{j+1}^i \wedge \mathrm{d}t$, the terms $\mathrm{d}\Pi_j^i$ for $j < r_i - k$ can also be assimilated into $J^{(k)}$ on the right-hand side. The equation will be satisfied if and only if $\lambda_i^{r_i-k} = 0, i = 1, \ldots, p$, and

$$\sum_{i=1}^{m}\lambda_i\,\mathrm{d}\omega^i \in J^{(k)}. \tag{11}$$

Now, using the expression (8) for ω^i, and setting $\eta^0 = \mathrm{d}t, \eta^{n-m+j} = \mathrm{d}u^j, j = 1, \ldots, p$,

$$\begin{aligned}
\mathrm{d}\omega^i &= \sum_{j=1}^{n}\left(\mathrm{d}W_j^i \wedge \Omega^j + W_j^i\,\mathrm{d}\Omega^j\right)\\
&= \sum_{j,l}\frac{\partial W_j^i}{\partial x^l}\,\mathrm{d}x^l \wedge \Omega^j + \sum_{\substack{j,l\\ q\le k-1}}\frac{\partial W_j^i}{\partial u_q^l}\,\mathrm{d}u_q^l \wedge \Omega^j\\
&\qquad + \sum_{j,l,q} W_j^i\left(\frac{\partial f^j}{\partial x^l}\,\mathrm{d}x^l + \frac{\partial f^j}{\partial u^q}\,\mathrm{d}u^q\right)\wedge\,\mathrm{d}t\\
&\equiv \sum_{j<l}\phi_{jl}^i(x, u, \ldots, u_k)\eta^j \wedge \eta^l \bmod J^{(k)},
\end{aligned}$$

for certain smooth functions ϕ_{jk}^i. By stacking these individual equations, we obtain the composite:

$$\begin{pmatrix}\mathrm{d}\omega^1\\ \vdots\\ \mathrm{d}\omega^m\end{pmatrix} \equiv \Phi(x, u, \ldots, u_k)^T \begin{pmatrix}\eta^0 \wedge \eta^1\\ \eta^0 \wedge \eta^2\\ \vdots\\ \eta^{n-m+p-1} \wedge \eta^{n-m+p},\end{pmatrix} \bmod J^{(k)}$$

where $\Phi(x, u, \ldots, u_k)$ is a matrix of smooth functions. It follows that $\sum_i \lambda_i\omega^i \in J^{(k+1)}$ if and only if $(\lambda_1, \ldots, \lambda_m)^T \in \ker\Phi$. Since Φ only depends on the state and the first k derivatives of the input, that is, $x, u, \dots, u_k$, there is a basis, say $\Lambda^1, \dots \Lambda^s$ for $\ker \Phi$ that only depends on the variables $x, u, \dots, u_k$. This implies that the 1-forms $\sum_i \Lambda_i^k \omega^i, k = 1, \dots, s$ are forms in $J^{(k+1)}$ that are of the required form (8). Together with the forms $\Pi_i^j, j \leq r_i - (k+1), i = 1, \dots, p$ they generate $J^{(k+1)}$.

Now assume that the system is control affine, as in (2), and furthermore that $0 = r_1 = \cdots = r_q < r_{q+1} \leq \cdots \leq r_p$. Clearly the statement is still true for $k = 0$. We check that the expressions (9) hold for $k = 1$, and then the general statement follows as before by induction. For $k = 1$, we get, modulo J,

$$\begin{pmatrix} \mathrm{d}\Omega^1 \\ \vdots \\ \mathrm{d}\Omega^n \end{pmatrix} \equiv \begin{pmatrix} g_1(x) \; g_2(x) \; \cdots \; g_q(x) \end{pmatrix} \begin{pmatrix} \mathrm{d}u^1 \wedge \mathrm{d}t \\ \mathrm{d}u^2 \wedge \mathrm{d}t \\ \vdots \\ \mathrm{d}u^q \wedge \mathrm{d}t \end{pmatrix}.$$

Hence we can choose generators for $J^{(1)}$ to be $\sum_i \Lambda_i^k \omega^i, k = 1, \dots, s$, where $\left(\Lambda_1^k, \dots, \Lambda_m^k\right)^T \in \ker (g_1 \dots g_q)$ and hence may be taken as functions of x only; $\Lambda_i^k = \Lambda_i^k(x)$. It is clear that in this case the appearance of the input derivatives in the derived flag is delayed by one level. $\square$

3 Two Input Systems

We will now use the previous results to determine an upper bound for the number of integrators needed to linearize a two-input control system. We note that any (control-affine) system with two inputs and three states (or in general, m inputs and $m + 1$ states), is always dynamically feedback linearizable [4].

Theorem 4 *Let Σ denote a control system with $n \geq 4$ states and 2 inputs. If Σ is linearizable via differentiation of the inputs, then Σ is linearizable with a single chain of integrators of length at most*

(i) $2n - 2$, for a general nonlinear control system (5),
(ii) $2n - 3$, for a control affine system (2).

PROOF. Call the two inputs of Σ respectively u and v. According to proposition 2 we may assume that the dynamic extension only differentiates one of the two inputs. Assume that $r \geq 2n - 2$ and denote by Σ_r the control system that is obtained from Σ by differentiating the input v precisely r times. We show that if Σ_r is feedback linearizable, then Σ_{2n-2} is also feedback linearizable.

Let

$$J = I \oplus \operatorname{span}\{\Pi_1, \dots, \Pi_r\},$$

where $I = \operatorname{span}\{\Omega^1, \dots, \Omega^n\}$, $\Omega^i = \mathrm{d}x^i - f^i(x,u)\,\mathrm{d}t$ and $\Pi_k = \mathrm{d}v_{k-1} - v_k\,\mathrm{d}t$. It is immediate that $\Pi_{r-k} \in J^{(k)}$, but $\Pi_{r-k} \notin J^{(k+1)}$.

Since Σ_r is linearizable, its derived flag has the same structure as the derived flag of a 2 input system in Brunovsky normal form. Therefore the derived systems of J may be described as follows, for some basis $\omega^1, \dots, \omega^n$ for I.

$$J^{(k)} = J^{(k+1)} \oplus \begin{cases} \operatorname{span}\{\omega^{n-k}, \Pi_{r-k}\}, & 0 \le k \le n-A \\ \operatorname{span}\{\Pi_{r-k}\}, & n-A+1 \le k \le r \\ \operatorname{span}\{\omega^{A-r+k-2}\}, & r+1 \le k \le r+A, \end{cases}$$

and J^{r+A+1} is trivial. It is convenient to arrange the generators in towers as follows. One obtains generators for $J^{(k)}$ by excluding the forms from the top k rows in both towers.

$$\begin{array}{cc} \omega^n & \Pi_r \\ \omega^{n-1} & \Pi_{r-1} \\ \vdots & \vdots \\ \omega^A & \Pi_{r-n+A} \\ & \vdots \\ & \Pi_1 \\ & \omega^{A-1} \\ & \vdots \\ & \omega^1 \end{array}$$

The worst case scenario is when $A = 1$, so we will consider that case. The cases when $A > 1$ are treated similarly.

According to proposition 3, the generator $\omega^{n-k} \in J^{(k)}$ $(k = 0, 1, \dots, n-1)$ may be chosen to be of the form

$$\sum_{i=1}^{n} W_i(x^1, \dots, x^n, u_0, v_0, \dots, v_{k-1})\Omega^i. \tag{12}$$

Since Σ_r is linearizable each derived system of J satisfies $\mathrm{d}J^{(k)} \equiv 0 \bmod J^{(k)}, \mathrm{d}t$. Thus the following congruences hold, with $\Pi_i = \mathrm{d}v_{i-1} - v_i\,\mathrm{d}t$,

$$\begin{aligned} \mathrm{d}\omega^1 &\equiv 0 \bmod \omega^1, \Pi_1, \ldots, \Pi_{r-n+1}, \mathrm{d}t \\ &\equiv 0 \bmod \omega^1, \mathrm{d}v_0, \ldots, \mathrm{d}v_{r-n}, \mathrm{d}t. \end{aligned}$$

¿From (12) it follows that $\omega^1 \in J^{n-1}$ depends on at most the first $n-2$ derivatives of v, and thus cannot depend on the variables $v_{n-1}, \ldots, v_{r-n}$ and therefore the congruences (3) may be relaxed to (note that $r - n \geq n - 2$ by assumption),

$$\mathrm{d}\omega^1 \equiv 0 \bmod \omega^1, \mathrm{d}v_0, \ldots, \mathrm{d}v_{n-2}. \tag{13}$$

Now consider Σ_{2n-2}, and denote by $\bar{J}$ the corresponding Pfaffian system. Its derived systems are easily calculated using the derived systems of J:

$$\bar{J}^{(k)} = J^{(k+1)} + \begin{cases} \mathrm{span}\{\omega^{n-k}, \Pi_{2n-2-k}\}, & 0 \leq k \leq n-1 \\ \mathrm{span}\{\Pi_{2n-2-k}\} & n \leq k \leq 2n-3, \end{cases}$$

and $\bar{J}^{(2n-2)} = \{0\}$. It follows from congruence (13) that all of the derived systems $\bar{J}^{(k)}$ satisfy the conditions of theorem 4. Hence Σ is linearizable by differentiating the input v at most $2n-2$ times.

Finally, if the control system is control affine, the bound $2n-3$ follows in the same manner as above, but now using (9) instead of (8) $\square$

Remark 5 Any linearizable control system is necessarily *differentially flat* (see [6,7] for a discussion of flatness). Our result implies that if a two-input system is linearizable by dynamic extension, then there exists a set of flat outputs for the system which are functions of the state and at most $n-2$ derivatives of the input. Since the flat outputs can be taken as the ends of the Brunovsky chains, in the worst case, they will be v and h, where $\mathrm{d}h \in \{J^{(n-1)}, \mathrm{d}t\}$. From the proof, h is a function of x, u and at most the first $n-2$ derivatives of v.

The following example will show that the bound of $2n-3$ can not be improved for control affine systems.

Example 6 Let Σ be the control system described by

$$\begin{cases} \dot{x}^1 = x^2(1+v_0), \\ \dot{x}^2 = x^3 + x^1 v_0, \\ \dot{x}^k = x^{k+1}, \; k = 3, \dots, n-2, \\ \dot{x}^{n-1} = u_0, \\ \dot{x}^n = x^{n-1} v_0. \end{cases} \tag{14}$$

A dynamic extension that differentiates v_0 precisely $2n-3$ times linearizes (14), but differentiating less than $2n-3$ does not suffice. Differentiation of u_0 will never linearize (14). We check these statements here for the situation $n = 4$, the general case is similar.

Define $\Omega^1 = \mathrm{d}x^1 - x^2(1+v_0)\,\mathrm{d}t, \Omega^2 = \mathrm{d}x^2 - (x^3 + x^1 v_0)\,\mathrm{d}t, \Omega^3 = \mathrm{d}x^3 - u_0\,\mathrm{d}t, \Omega^4 = \mathrm{d}x^4 - x^3 v_0\,\mathrm{d}t$. To calculate the first derived system of $I = \{\Omega^1, \Omega^2, \Omega^3, \Omega^4\}$, we compute the exterior derivatives of the one-forms Ω^i:

$$\begin{aligned} \mathrm{d}\Omega^1 &\equiv -x^2\,\mathrm{d}v_0 \wedge \mathrm{d}t \mod I \\ \mathrm{d}\Omega^2 &\equiv -x^1\,\mathrm{d}v_0 \wedge \mathrm{d}t \mod I, \\ \mathrm{d}\Omega^3 &= -\,\mathrm{d}u_0 \wedge \mathrm{d}t, \\ \mathrm{d}\Omega^4 &\equiv -x^3\,\mathrm{d}v_0 \wedge \mathrm{d}t \mod I. \end{aligned}$$

Thus it follows that the first derived system is given by $I^{(1)} = \mathrm{span}\{\omega^1 := \Omega^2 - \frac{x^1}{x^2}\Omega^1, \omega^2 := \Omega^4 - \frac{x^3}{x^2}\Omega^1\}$. Note that since the original system is input-affine, a coefficients of the Ω^1 in a basis for $I^{(1)}$ are functions only of the state x. Take $\{\Omega^1, \Omega^3\}$ to be forms complementary to $I^{(1)}$. We now check the exact linearization conditions (6) for $I^{(1)}$.

$$\begin{aligned} \mathrm{d}\omega^1 &\equiv \frac{(x^1)^2 + (x^2)^2 - x^1 x^3}{(x^2)^2}\Omega^1 \wedge \mathrm{d}t - \Omega^3 \wedge \mathrm{d}t \mod \omega^1 \\ \mathrm{d}\omega^2 &\equiv \frac{1}{x^2}\Omega^1 \wedge \Omega^3 - \frac{u_0 x^2 + x^1 x^3 - (x^3)^2}{(x^2)^2}\Omega^1 \wedge \mathrm{d}t - v_0 \Omega^3 \wedge \mathrm{d}t \quad \mod \omega^1 \end{aligned}$$

The appearance of the $\Omega^1 \wedge \Omega^3$ term in the second congruence shows that $\{I^{(1)}, \mathrm{d}t\}$ is not integrable and thus I is not exactly linearizable.

Consider now a dynamic extension of order 5 on v_0. The augmented Pfaffian system is $J = \{\Omega^1, \dots, \Omega^4, \Pi_1, \dots, \Pi_5\}$ where $\Pi_j = \mathrm{d}v_{j-1} - v_j\,\mathrm{d}t$. It may be verified by direct computation of the exterior derivatives that the derived systems take the following form:

$$
\begin{aligned}
J^{(1)} &= \{\Omega^1, \Omega^2, \Omega^4, \Pi_1, \Pi_2, \Pi_3, \Pi_4\} \\
J^{(2)} &= \{\omega^1 := \Omega^1, \omega^2 := \Omega^2 - \frac{1}{v_0}\Omega^4, \Pi_1, \Pi_2, \Pi_3\} \\
J^{(3)} &= \{\omega := \Omega^2 - \frac{1}{v_0}\Omega^4 - \frac{v_1}{v_0(1+v_0)}\Omega^1, \Pi_1, \Pi_2\} \\
J^{(4)} &= \{\Pi_1\}
\end{aligned}
$$

and that $\{J^{(i)}, \mathrm{d}t\}$ is integrable for $i = 1, \ldots, 4$. However, taking the exterior derivative of ω defined as one of the generators for $J^{(3)}$, yields the congruence

$$
\begin{aligned}
\mathrm{d}\omega \equiv\ & f_1 \Pi_1 \wedge \mathrm{d}t + f_2 \Omega^4 \wedge \Pi_1 + f_3 \Omega^1 \wedge \Pi_1 \\
& + f_4 \Omega^1 \wedge \Pi_2 + f_5 \Omega^1 \wedge \mathrm{d}t \mod \omega
\end{aligned}
$$

for some nonzero functions f_i (the form of which is not important here). From this expression, it is evident that $\mathrm{d}\omega \equiv 0 \mod \omega, \Pi_1, \Pi_2, \mathrm{d}t$ but $\mathrm{d}\omega \not\equiv 0 \mod \omega, \Pi_1, \mathrm{d}t$. If only 3 or 4 derivatives are added to the input channel v, the derived flag has the same general form (with one or two fewer Π's at each level) but from the congruences it can be seen that the system will not be feedback linearizable. It can also be checked that no number of derivatives on the other input channel u will linearize the system.

It is easy to modify the above example to a fully nonlinear system for which the bound $2n-3$ is sharp: simply strip off an integrator in the first input channel, which yields $\dot{x}^1 = x^2(1+v_0)$, $\dot{x}^2 = x^3 + x^1 v_0$, $\dot{x}^k = x^{k+1}$, $k = 3, \ldots, n-2$, $\dot{x}^{n-1} = u_0$, $\dot{x}^n = u_0 v_0$.

4 Discussion

The results from the previous section enable us to attack the problem of linearization via dynamic extension algorithmically: consider all possible compensators (5) with order not exceeding $2n-3$ and in each case check for feedback linearization. It is sometimes possible, a priori, to rule out dynamic extensions that differentiate a certain input, by using the necessary condition derived in [17].

The authors suspect that the method described here to obtain an upper bound for 2 input systems can be generalized to systems with $p \geq 2$ inputs.

As said before, the problem of dynamic feedback linearization with general compensators is still unresolved, and the method used in this paper does not readily generalize to this situation. It should be borne in mind that the general case allows for various interesting examples, such as the systems discussed in [17]

$$
\begin{aligned}
\dot{x}^1 &= u^1 \\
\dot{x}^2 &= u^2 \\
\dot{x}^3 &= \arctan(\tfrac{u^2}{u^1})
\end{aligned}
$$

This system is not linearizable via dynamic extension. But if an initial feedback is applied: $u^1 = v^1 \cos v^2, u^2 = v^1 \sin v^2$ then the resulting system turns out to be linearizable by extending the input v^2 twice ($\dot{v}^2 = v_1^2, \dot{v}_1^2 = v_2^2$).

An interesting case is exhibited by a planar model for the ducted fan, see also [13,14] given by

$$
\begin{aligned}
\dot{x} &= v_x, \;\; \dot{y} = v_y, \;\; \dot{\theta} = v_\theta, \\
m\dot{v}_x &= u^1 \cos\theta - u^2 \sin\theta, \\
m\dot{v}_y &= u^1 \sin\theta + u^2 \cos\theta - mg, \\
J\dot{\theta} &= ru^1,
\end{aligned}
$$

where m, J, r are constants. It turns out that (4) is linearizable via a compensator given by $\dot{u}^2 = u_1^2 + 2v_\theta u^1, \;\; \dot{u}_1^2 = u_2^2$, but not by a dynamic extension.

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