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"LMI approach to mixed performance objective controllers: application to Robust H_2 Synthesis"

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LMI approach to mixed performance objective controllers: application to Robust $\mathcal{H}_2$ Synthesis

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Abstract

The problem of synthesizing a controller for plants subject to arbitrary, finite energy disturbances and white noise disturbances via Linear Matrix Inequalities (LMIs) is presented. This is achieved by considering white noise disturbances as belonging to a constrained set in l_2. In the case of where only white noise disturbances are present, the procedure reduces to standard $\mathcal{H}_2$ synthesis. When arbitrary, finite energy disturbances are also present, the procedure may be used to synthesize general mixed performance objective controllers, and for certain cases, Robust $\mathcal{H}_2$ controllers.

1 Introduction

In the standard robust control paradigm, the signal space which characterizes performance is equivalent to that which captures a system's uncertainty. For example, $\mathcal{H}_\infty$ tools are used when dealing with bounded energy (or power) gain uncertainty (see [12], [10], [18]), while when working with l_∞ disturbances, the uncertainty is assumed to be of finite amplitude gain (see [9]). While it is often the case that the particular characterization of the uncertainty is not critical to the design process, the signal space used to characterize the performance often is. In particular, one of the common complaints among control design engineers which use $\mathcal{H}_\infty$ methods is that the resulting designs tend to be sluggish and overly conservative. As an alternative, $\mathcal{H}_2$ designs are often employed, but they lack the robustness properties of $\mathcal{H}_\infty$ designs (see [5]) which can readily be extended to encompass a system's uncertainty. The attractive feature of $\mathcal{H}_2$ designs is their gain interpretation; they minimize the power output when the disturbances are assumed to be white noise or impulses. This is in contrast to $\mathcal{H}_\infty$ designs, which minimize the energy to energy (or power to power) gain; in many applications, modeling the disturbances as arbitrary signals is a poor modeling choice, and thus $\mathcal{H}_\infty$ designs lead to low performance controllers.

A desirable control design strategy would be one which has the input-output gain interpretation of the $\mathcal{H}_2$ norm, but can readily accommodate $\mathcal{H}_\infty$ bounds on the uncertainty.

In [15], a framework is developed whereby white noise signals are captured in a deterministic setting. The main motivation behind this approach was the reconciling of the worst case setting, natural when considering robustness issues, with the stochastic setting. This framework proved very natural when addressing the so-called *Robust $\mathcal{H}_2$ Analysis* problem, which was solved in [13] and [14].

This approach will be used in this paper to tackle the problem of *Robust $\mathcal{H}_2$ Synthesis* for a restricted class of problems; in particular, rank one synthesis problems with time varying uncertainty. This will be achieved by solving an auxiliary problem, that of synthesizing a controller for plants subject to arbitrary, finite energy disturbances and white noise disturbances.

Other work in this area includes [2], [6], [19] and [17]. Also worth noting is research on the so called *mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$* problem, where only nominal stability and nominal $\mathcal{H}_2$ performance are considered (see [1], [8]).

The paper is organized as follows: we begin with some mathematical preliminaries, followed by a review of the notions introduced in [15] with regards to capturing noise signals as elements of a set. The problem is then posed and solved, followed by a discussion on computational issues. The problem of robust disturbance rejection is then addressed, and it is shown how these types of problems can be re-formulated in the general problem setup previously solved. We conclude with an example which illustrates the tools developed and their numerical properties.

2 Preliminaries

Most of the notation in this paper is standard. We restrict ourselves to discrete time systems. The space of square summable sequences is denoted l_2; when the spatial structure is relevant, it is referred to as l_2^p. The 2-norm of a signal d in l_2 is denoted $\|d\|$. The unit ball of l_2 signals is denoted $\mathbf{Bl}_2$, and consists of all l_2 signals whose norm is less than or equal to one. The discrete time, unit delay operator is denoted λ. The truncation operator P_T is defined as

$$(P_T x)(t) := \begin{cases} x(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \tag{1}$$

Causal, finite dimensional, linear time invariant systems will be denoted FDLTI. A causal linear map G over l_2 is bounded if the restriction of G to l_2 is a bounded operator, with induced l_2 norm denoted $\|G\|$. The transfer function representation of FDLTI system G

is denoted $\hat{G}(\lambda)$. The linear fractional transformation (LFT) between two systems G and K is denoted $G \star K$, and is defined as:

$$G \star K := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{2}$$

where

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

when the inverse of $(I - G_{22}K)$ is well defined. $\mathbf{B}\Delta$ is the set of linear, but otherwise arbitrary, operators whose induced 2-norm is less than or equal to 1. For two subsets of l_2, S_1 and S_2, the *approximation error* of S_1 with S_2 is defined as

$$D(S_1, S_2) := \sup_{s_1 \in S_1} \inf_{s_2 \in S_2} \|s_1 - s_2\| \tag{3}$$

$D(S_1, S_2)$ is a measure of how well S_2 captures the elements of S_1. For a constant matrix A, its maximal element is denoted

$$\|A\|_{\mathrm{M}} := \max_{i,j} |A_{i,j}| \tag{4}$$

3 Deterministic Noise Sets

We begin by reviewing the notions introduced in [15] to capture white noise in sets. Given a signal $n \in l_2^m$, its *autocorrelation function* is defined as

$$R_n(\tau) := \sum_{t=-\infty}^{\infty} n(t)n^{\mathrm{T}}(t + \tau) \tag{5}$$

Note that there is no time averaging in the definition above, as would be used for power signals, since we are dealing with finite energy signals. Given positive integer N and positive number γ, we define the following set of autocorrelation functions:

$$\mathcal{R}_{N,\gamma}^m := \left\{ R(\tau) : \mathbb{Z} \to \mathbb{R}^{m \times m} \;\middle|\; \begin{array}{l} R(-\tau) = R^{\mathrm{T}}(\tau) \\ \|R(0) - I\|_{\mathrm{M}} \le \gamma \\ \|R(\tau)\|_{\mathrm{M}} \le \gamma, \;\; 1 \le \tau \le N \end{array} \right\} \tag{6}$$

and corresponding signal set

$$\mathcal{W}_{N,\gamma}^m := \left\{ n \in l_2^m | R_n(\tau) \in \mathcal{R}_{N,\gamma}^m \right\} \tag{7}$$

In particular, when $\gamma = 0$, we have

$$\mathcal{W}_N^m := \mathcal{W}_{N,0}^m \tag{8}$$

It is shown in [15] how the worst case gain from $\mathcal{W}_N^m$ to l_2 approaches the $\mathcal{H}_2$ norm in the limit as N goes to infinity, the rate of convergence being exponential.

4 Problem Formulation

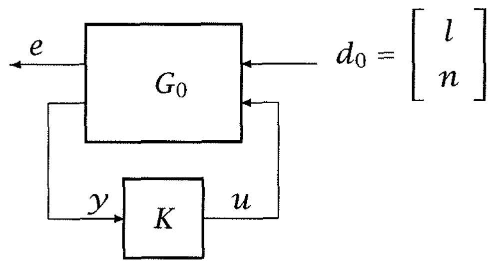

Figure 1: Problem Formulation

The problem formulation is as follows; given positive integer N and FDLTI system G_0

$$G_0 = \begin{bmatrix} G_{11}^l & G_{11}^n & G_{12} \\ G_{21}^l & G_{21}^n & G_{22} \end{bmatrix} \tag{9}$$

find internally stabilizing (see [21]) FDLTI system K such that

$$\sup_{l \in \mathbf{Bl}_2, n \in \mathcal{W}_N^m} \|(G_0 \star K)d_0\|^2 < 1 \tag{10}$$

The performance objective is absorbed into plant G_0; in general, the performance objective will be decreased until a controller no longer exists which satisfies (10). We will later show how the solution to the above problem may be used to solve a variety of robustness problems. The above problem formulation is interesting in its own right, however; it may be the case that some of the disturbance signals may be arbitrary, in this case l, while the others are white noise, such as n. For example, this may be the case when tracking a reference signal l (which may be weighted to restrict tracking over a certain frequency range), in the presence of sensor noise or other random disturbance n.

5 Solution

We will provide a solution to the above problem in a series of steps. The first consists of recalling the solution of controller synthesis when the disturbances are subject to implicit constraints, presented in [3]. The next step consists of parametrizing sets $\mathcal{W}_N^m$ in image form. The final step is to combine these image representations with the solution provided in the first step to solve (10).

5.1 Synthesis with implicit constraints

The following is a review of the main results in [3]. Consider the following subset of l_2,

$$\mathcal{H} = \left\{ d \in l_2 \;\middle|\; \begin{array}{ll} \|H_i d\|^2 \le 1 & 1 \le i \le c \\ \|L_j d\|^2 = \|J_j d\|^2 & 1 \le j \le \bar{c} \end{array} \right\} \tag{11}$$

where the H_i, L_j and J_j are constant matrices. Consider the following constrained feasibility problem: let system G and set $\mathcal{H}$ be given. Find a stabilizing controller K such that

$$\sup_{d \in \mathcal{H}} \|(G \star K)d\|^2 < 1 \tag{12}$$

It is shown in [3] that such a controller K exists if and only if there exist symmetric matrices $\mathbf{S}, \mathbf{T}, \mathbf{W}, \mathbf{Z}$ (of a given spatial structure) such that

$$V\left(R^T \begin{bmatrix} \mathbf{S} & 0 \\ 0 & I \end{bmatrix} R - \begin{bmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{Z} \end{bmatrix}\right) V^T < 0 \tag{13}$$

$$U^T\left(R \begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{W} \end{bmatrix} R^T - \begin{bmatrix} \mathbf{T} & 0 \\ 0 & I \end{bmatrix}\right) U < 0$$

$$\begin{bmatrix} \mathbf{T} & I \\ I & \mathbf{S} \end{bmatrix} > 0$$

$$\begin{bmatrix} \mathbf{Z} & I \\ I & \mathbf{W} \end{bmatrix} > 0$$

$$C^T \mathbf{Z} C < 1$$

where U, V, R and C are constant matrices which depend on the state space representation for G and the constraint set $\mathcal{H}$. A state space representation for K may then be constructed from R, $\mathbf{S}$ and $\mathbf{T}$ (the details may be found in [3] and [11]). The above is a convex feasibility problem, and may be solved using numerical packages such as *LMI Lab* [7].

5.2 Image Representation for $\mathcal{W}_N^m$

The solution provided in the previous section cannot be utilized to directly solve (10); in order to directly specify $\mathcal{W}_N^m$, H_i, L_j and J_j need to be systems, not constant matrices (see [2]). What we would like to do is construct an alternate representation for $\mathcal{W}_N^m$ which is consistent with the solution provided in the previous section.

Let N and m be given. Define

$$\begin{aligned} \hat{U}_k(\lambda) &:= \frac{1+\lambda^k}{\sqrt{8}N} && && (14) \\ \hat{\bar{U}}_k(\lambda) &:= \frac{1-\lambda^k}{\sqrt{8}N} && 1 \le k \le N \\ U &:= [U_1\ \bar{U}_1\ \ldots\ U_N\ \bar{U}_N] \\ V &:= \text{diag}\ [U,\ U,\ \ldots,\ U] \end{aligned}$$

where $V \in \mathcal{RH}_\infty^{m\times(2mN)}$, ie., V consists of m copies of U along the "diagonal". Then it may be verified that

$$\hat{V}\hat{V}^* = \frac{1}{2N} I_m \tag{15}$$

ie., $\hat{V}$ and $\hat{U}$ are co-inner, and $\left\|\hat{V}\right\|_\infty = \left\|\hat{U}\right\|_\infty = \frac{1}{\sqrt{2N}}$.

Define $\tilde{n} \in l_2^{2mN}$ as follows:

$$\begin{aligned} \tilde{n} &:= (\tilde{n}_1,\ \ldots,\ \tilde{n}_m) && (16) \\ \tilde{n}_i &:= (n_{i,1},\ \bar{n}_{i,1},\ \ldots,\ n_{i,N},\ \bar{n}_{i,N}) \end{aligned}$$

and the following set of constraints:

$$\begin{aligned} &\mathbf{C}_1: && \langle n_{i,k}, n_{i,k}\rangle \le 1,\ \langle \bar{n}_{i,k}, \bar{n}_{i,k}\rangle \le 1 && 1 \le i \le m, && 1 \le k \le N \\ &\mathbf{C}_2: && \langle n_{i,k}, n_{j,k}\rangle - \langle \bar{n}_{i,k}, \bar{n}_{j,k}\rangle = 0 && 1 \le i < j \le m, && 1 \le k \le N \\ &\mathbf{C}_3: && \langle n_{i,k}, \bar{n}_{j,k}\rangle - \langle \bar{n}_{i,k}, n_{j,k}\rangle = 0 && 1 \le i < j \le m, && 1 \le k \le N \\ &\mathbf{C}_4: && \langle n_{i,k}, n_{j,k}\rangle + \langle \bar{n}_{i,k}, \bar{n}_{j,k}\rangle = 0 && 1 \le i < j \le m, && k = 1 \end{aligned}$$

The constraint set $\mathcal{N}$ is then defined as:

$$\mathcal{N} := \{\tilde{n} \in l_2^{2mN} \mid \mathbf{C}_1,\ \mathbf{C}_2,\ \mathbf{C}_3,\ \mathbf{C}_4 \text{ are satisfied}\} \tag{17}$$

The image set $\widetilde{\mathcal{W}}_{N,\gamma}^m$ may now be defined:

$$\widetilde{\mathcal{W}}_{N,\gamma}^m := \{n \in l_2^m \mid n = V\tilde{n}, \tilde{n} \in \mathcal{N}, \|n_i\| \ge 1-\gamma \text{ for } 1 \le i \le m\} \tag{18}$$

For $\gamma = 1$, we define

$$\widetilde{\mathcal{W}}_N^m := \widetilde{\mathcal{W}}_{N,1}^m \tag{19}$$

which corresponds to no explicit norm constraint on n_i. The following theorem outlines how $\mathcal{W}_N^m$ and $\widetilde{\mathcal{W}}_{N,\gamma}^m$ are related, and is crucial to the synthesis results which follow:

Theorem 1

1. $\widetilde{\mathcal{W}}^m_{N,0} = \mathcal{W}^m_N$.
2. $\mathcal{W}^m_N \subset \widetilde{\mathcal{W}}^m_{N,\gamma}$ *for* $\gamma \geq 0$.
3. $D(\widetilde{\mathcal{W}}^m_{N,\gamma}, \mathcal{W}^m_N)$ *is upper semi-continuous as a function of* γ *at* $\gamma = 0$.

Before proving Theorem 1, we will need the following two Lemmas:

Lemma 1

$$S := \{n \in l_2^m | 2NV^*n \in \mathcal{N}, \|n_i\|^2 = 1 \text{ for } 1 \leq i \leq m\} = \widetilde{\mathcal{W}}^m_{N,0} \tag{20}$$

Proof of Lemma 1:

It is clear by setting $\tilde{n} = 2NV^*n$ that $S \subset \widetilde{\mathcal{W}}^m_{N,0}$; we will thus show that $\widetilde{\mathcal{W}}^m_{N,0} \subset S$. Let $n \in \widetilde{\mathcal{W}}^m_{N,0}$, with corresponding $\tilde{n}$. For each component, by constraints $\mathbf{C}_1$, $\|\tilde{n}_i\|^2 \leq 2N$, which implies $\|n_i\|^2 \leq 1$. Thus $\|n_i\|^2 = 1$, $\|\tilde{n}_i\|^2 = 2N$.

Since $\hat{U}$ is co-inner, $\exists \hat{U}_\perp \in \mathcal{RH}_\infty^{(2N-1)\times 2N}$ such that $\sqrt{2N}\begin{bmatrix} \hat{U} \\ \hat{U}_\perp \end{bmatrix}$ is unitary. Thus for each i, $\tilde{n}_i$ can be uniquely decomposed as

$$\tilde{n}_i = U^*v_i + U_\perp^* w_i \tag{21}$$

where $v_i \in l_2$, $w_i \in l_2^{2N-1}$. Furthermore, $\|\tilde{n}_i\|^2 = \|U^*v_i\|^2 + \|U_\perp^* w_i\|^2$. From this we may conclude that $v_i = 2Nn_i$, and that $\|v_i\| = \|U^*v_i\|^2 = \|\tilde{n}_i\|^2 = 2N$; thus $w_i = 0$, and $\tilde{n}_i = 2NU^*n_i$. This gives $\tilde{n} = 2NV^*n$, as required.

□

Lemma 2 *Given* $R \in \mathcal{R}^m_{N,\gamma}$, *where* $\gamma < \frac{1}{m(N+1)^2}$, *there exists a signal* $x \in l_2^+$ *such that* $R_x(\tau) = R(\tau)$ *for* $\tau \in [-N, N]$.

The proof of Lemma 2 may be found in the Appendix. We are now in a position to prove Theorem 1:

Proof of Theorem 1:

We begin by showing how n is constrained when $2NV^*n \in \mathcal{N}$. Let $\tilde{n} = 2NV^*n$. Thus

$$\begin{aligned} n_{i,k} &= 2NU_k^* n_i \\ \tilde{n}_{i,k} &= 2N\bar{U}_k^* n_i \qquad \text{for } 1 \leq i \leq m,\ 1 \leq k \leq N \end{aligned} \tag{22}$$

For the above, it can be verified that constraints $\mathbf{C}_1$ to $\mathbf{C}_4$ are equivalent to :

$\mathbf{C}_1$:	$\|n_i\|^2 + \langle n_i, \lambda^k n_i\rangle \leq 1$	$1 \leq i \leq m,$	$1 \leq k \leq N$
	$\|n_i\|^2 - \langle n_i, \lambda^k n_i\rangle \leq 1$	$1 \leq i \leq m,$	$1 \leq k \leq N$
$\mathbf{C}_2$:	$\langle n_i, \lambda^k n_j\rangle + \langle n_i, \lambda^{-k} n_j\rangle = 0$	$1 \leq i < j \leq m,$	$1 \leq k \leq N$
$\mathbf{C}_3$:	$\langle n_i, \lambda^k n_j\rangle - \langle n_i, \lambda^{-k} n_j\rangle = 0$	$1 \leq i < j \leq m,$	$1 \leq k \leq N$
$\mathbf{C}_4$:	$\langle n_i, n_j\rangle = 0$	$1 \leq i < j \leq m$	