Robust Stability Under Mixed Time Varying, Time Invariant and Parametric Uncertainty

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Abstract

Robustness analysis is considered for systems with structured uncertainty involving a combination of linear time-invariant and linear time-varying perturbations, and parametric uncertainty. A necessary and sufficient condition for robust stability in terms of the structured singular value μ is obtained, based on a finite augmentation of the original problem. The augmentation corresponds to considering the system at a fixed number of frequencies. Sufficient conditions based on scaled small-gain are also considered and characterized.

1 Introduction

A substantial amount of research in recent years has been devoted to analysis and synthesis of control systems to achieve robust stability and performance in the presence of *structured* uncertainty. This implies a decentralized nature of the uncertain perturbation, which is a reasonable modeling choice for complex systems, where uncertainty may be introduced at the subsystem level (see Safonov [17] and Doyle [5] for early treatments of this).

In addition to this "spatial" structure, different assumptions can be made on the dynamic properties of the uncertainty: real parametric, linear time invariant (LTI), linear time varying (LTV) or nonlinear perturbations. All these uncertainty classes arise naturally in modeling. Parametric uncertainty appears frequently in first principles models; LTI perturbations are well suited when

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there is frequency domain information about the system, to be "covered" by a suitable ball of LTI uncertainty. Time-variations can be captured by LTV uncertainty; more generally, an arbitrary LTV operator is equivalent to a norm constraint between signals, and therefore provides very crude uncertainty information which may be used to cover a contractive nonlinearity in the system.

In [5], robustness analysis for structured uncertainty was focused on constant, complex matrices, and led to conditions in terms of the structured singular value μ; an evaluation of (complex) μ across frequency captures the case of LTI perturbations [11]. Subsequent work [7, 19] has considered a combination of uncertain parameters and LTI perturbations, and leads to *mixed* (real/complex) μ-analysis. These conditions are exact, although their evaluation is computationally hard (see [4]) and is therefore usually approached by means of bounds. In particular, a class of sufficient conditions for robust stability can be stated in terms of a scaled small-gain theorem, which can be evaluated via convex optimization. The scales are chosen to commute with the spatial structure of the uncertainty, and can either be constant or varying in frequency.

Robustness analysis under LTV uncertainty has seen major progress in recent years, since it was discovered that the above mentioned convex tests apply exactly to this case. It was found in [8] for the l_∞ setting, and later in [18], [9] for the l_2 case, that the constant scales condition is necessary and sufficient for robust stability under structured LTV perturbations. In reference to the frequency-varying scales test, it was shown in [14] (see also [15]) that this condition is necessary and sufficient for robust stability against *arbitrarily slowly varying* (a slightly larger class than LTI) uncertainty.

This paper considers the situation where a mixed structure of LTV, LTI and parametric perturbations affects the system. This is a very natural problem since the same reasons which lead to "decentralized" uncertainty will often produce uncertain models involving a combination of the above mentioned classes.

The inclusion of time-varying uncertainty precludes a straightforward frequency domain μ-analysis as in mixed LTI/parametric problems. For this reason, we propose an augmentation procedure across a number of frequencies, which is related to a lifting technique for μ analysis by Bercovici et. al. [2], and to a power distribution Lemma from Poolla and Tikku [14]. The main result in this paper is to show that a μ test in the augmented structure is necessary and sufficient for robust stability under mixed LTV/LTI/parametric uncertainties. The augmentation also provides an alternative for the formulation of convex upper bounds, and leads to an exact characterization of these tests.

The paper is organized as follows. In Section 2 we review in more detail the background material described above. In Sections 3 we focus on the combined LTV/LTI problem, and obtain the complex μ test for analysis. In Section 4 we analyze the properties of the corresponding convex upper bounds. In Section 5 parametric uncertainty is also introduced in the problem, and an augmented mixed-μ test is derived. Section 6 contains examples which demonstrate the results, and conclusions are given in Section 7. Preliminary versions of these results were presented in [13].

2 Preliminaries

Robust stability and performance analysis under structured uncertainty has been the focus of a substantial research effort in recent years. This section contains a (by no means exhaustive) summary of previous work related to this paper.

A standard setup for robustness analysis is depicted in Figure 1. This picture represents a robust stability problem, where M is the *nominal* system, which is assumed stable (often, M is finite-dimensional LTI), and Δ is a perturbation, which is assumed to have spatial *structure*

$$\Delta = diag\,[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}] \tag{1}$$

In (1) and throughout this paper, the notation $diag[D_1, \ldots, D_n]$ refers to a block diagonal matrix with blocks D_k. The blocks in Δ can in general represent real parameters or dynamic (LTI, LTV, nonlinear) perturbations.

In this paper we consider linear, discrete-time systems, but the results extend with minor

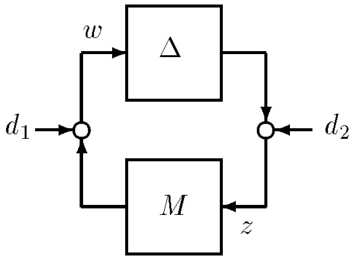

Figure 1: Robust stability

changes to the continuous time case. The l_2 norm is used for signals: l_2^n is the set of $\mathbb{C}^n$ valued square-summable sequences over the positive integers. $\mathcal{L}_C(l_2^n)$ denotes the set of linear, bounded and causal operators in l_2^n. All our perturbation structures will therefore be a subset of the class of structured LTV operators of the form (1) with $\delta_i \in \mathcal{L}_C(l_2^1), \Delta_{L+j} \in \mathcal{L}_C(l_2^{m_j}), 1 \leq i \leq L, 1 \leq j \leq F$. Some of the blocks will additionally be specialized to be LTI operators, or further to be real parameters. The uncertainty is normalized to a ball $\mathbf{B}_{\mathbf{\Delta}} = \{\Delta : \|\Delta\| \leq 1\}$ in the l_2-induced norm.

Assuming M, Δ are causal, the standard notion of stability of the interconnection of Figure 1 is that the map between injected disturbances d_1, d_2 and the signals z, w is causal, and it is a bounded operator when restricted to l_2. If M, Δ are themselves stable, this reduces to testing for the invertibility of $I - \Delta M$. This is captured in the following:

Definition 1 *Assume that $M \in \mathcal{L}(l_2)$. The system of Figure 1 is robustly stable if $I - \Delta M : l_2 \rightarrow l_2$ has a causal, bounded inverse for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. Robust stability is uniform if*

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \left\|(I - \Delta M)^{-1}\right\| < \infty \tag{2}$$

This paper is concerned with tests for robust stability under various assumptions on the uncertain perturbations. An important comment is that these tests can also be used for analyzing robust disturbance rejection: a performance specification in terms of a bound on the induced norm of a transfer function can under fairly general circumstances (see [8, 18]) be converted to a robust stability problem with an additional uncertainty block.

2.1 Constant Matrix Analysis

The effect of the structure of the perturbation in the robustness analysis is apparent when we consider the constant matrix version of the interconnection of Figure 1. Now M , Δ are matrices in $\mathbb{C}^{n\times n}$, Δ still has the spatial structure (1). The invertibility of $I - \Delta M$ is captured by the structured singular value μ [5, 11, 7, 19], defined as follows (where $\bar{\sigma}$ denotes maximum singular value):

Definition 2 *The structured singular value $\mu_\Delta(M)$ of a matrix M with respect to a structure $\mathbf{\Delta}$ is defined as $\mu_\Delta(M) := 0$ if no $\Delta \in \mathbf{\Delta}$ makes $I - \Delta M$ singular, otherwise*

$$\mu_{\mathbf{\Delta}}(M) := \left(min\left\{\bar{\sigma}\left(\Delta\right) : \det[I - \Delta M] = 0\right\}\right)^{-1} \tag{3}$$

Thus $I - \Delta M$ is invertible for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ if and only if $\mu_{\mathbf{\Delta}}(M) < 1$. Two different cases are of interest: complex μ [11] for complex Δ, and mixed μ [19], where some of the blocks in Δ are restricted to be real (see (8) below). In this paper we shall use a common generic notation μ; the distinction will be made explicit whenever is necessary. The following basic property of μ is known as the *main loop theorem* [11]:

Lemma 1 *Given a block structure $\Delta = diag[\Delta_1, \Delta_2]$ and a $\mathbb{C}$-valued matrix, suitably partitioned as $M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, then $\mu_\Delta(M) < 1$ if and only if*

$$\mu_{\Delta_2}(M_{22}) < 1, \qquad \max_{\bar{\sigma}(\Delta_2)\le 1} \mu_{\Delta_1}(M * \Delta_2) < 1$$

where $M * \Delta_2$ is the Linear Fractional Transformation (LFT)

$$M * \Delta_2 := M_{11} + M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21} \tag{4}$$

Since exact computation of the structured singular value is hard, it is usually approximated by upper and lower bounds. For lower bound algorithms refer to [11, 19]. A computationally tractable upper bound is obtained by considering *scaling* matrices which commute with the elements in $\mathbf{\Delta}$. The matrices with that property are of the form

$$X = \mathrm{diag}\left[X_1, \ldots, X_L,\, x_{L+1} I_{m_1}, \ldots, x_{L+F} I_{m_F}\right], \tag{5}$$

Let $\mathbb{X}$ be the set of *positive* matrices of the form (5), define

$$\hat{\mu}_{\mathbf{\Delta}}(M) := \inf_{X \in \mathbb{X}} \bar{\sigma}\left(XMX^{-1}\right) \tag{6}$$

Then $\mu_{\mathbf{\Delta}}(M) \leq \hat{\mu}_{\mathbf{\Delta}}(M)$. Equivalently, the Linear Matrix Inequality (LMI) condition

$$\exists X \in \mathbb{X} : \quad M^* X M - X < 0 \tag{7}$$

is sufficient for $\mu_{\mathbf{\Delta}}(M) < 1$. This convex feasibility condition is attractive for computation (see [3]). It can also be refined for the case of mixed uncertainty structures. If Δ is of the form

$$\Delta = diag\left[\delta_1 I_{r_1}, \ldots, \delta_{L_R} I, \delta_{L_R+1} I \ldots, \delta_{L_R+L_C}, \Delta_{L+1}, \ldots, \Delta_{L+F}\right] \tag{8}$$

where $\delta_1 \ldots \delta_{L_R} \in \mathbb{R}$ (i.e. the first L_R scalar times identity blocks correspond to real parameters), then condition (7) can be tightened (see [7, 19]) to

$$\exists X \in \mathbb{X},\ G \in \mathcal{G} : \quad M^* X M - X + j(M^* G - G^* M) < 0 \tag{9}$$

where the matrices G are of the form $G = diag[G_1, \ldots, G_{L_R}, 0, \ldots, 0]$ with $G_i = G_i^*$.

2.2 Robust Stability Tests

With the notation developed from the constant matrix case, we now summarize known conditions for analysis of the robust stability question in Figure 1, where $M(e^{j\omega})$ is assumed to be always a finite dimensional LTI system.

The first result (see [11]) is that if Δ is LTI uncertainty, robust stability is equivalent to the complex μ-test across frequency

$$\max_{\omega} \mu_{\mathbf{\Delta}}(M(e^{j\omega})) < 1 \tag{10}$$

An analogous condition holds if the uncertainty structure consists of a combination of real parametric and LTI perturbations, the only difference being that complex μ is replaced by mixed μ.

Convex conditions for robust stability analysis follow from the μ-upper bound. The scalings X can be chosen either to be constant, or frequency varying, giving the following two tests:

$$\inf_X \left\| X M(e^{j\omega}) X^{-1} \right\|_\infty < 1 \tag{11}$$

$$\max_\omega \hat{\mu}_{\mathbf{\Delta}}(M(e^{j\omega})) = \inf_{X(\omega)} \left\| X(\omega) M(e^{j\omega}) X(\omega)^{-1} \right\|_\infty < 1 \tag{12}$$

Clearly, (11) implies (12) which in turn implies (10), so both conditions (11-12) are sufficient for the case of LTI uncertainty. Also, for mixed real parametric/LTI perturbations, (12) can be tightened by use of frequency dependent G-scales as in (9).

Recent results have provided an exact characterization of conditions (11) and (12). It was shown independently by Shamma [18] and Megretski [9] that the constant scales test (11) is necessary and sufficient for robust stability under structured LTV perturbations. In the frequency varying case, Poolla and Tikku [14] have recently shown that (12) is necessary and sufficient for robust stability against the class of *arbitrarily slowly varying* structured perturbations. The following definition is from [14].

Definition 3 *An operator $\Delta \in \mathcal{L}_C(l_2)$ has time variation slower than ν if $\|\lambda\Delta - \Delta\lambda\| \leq \nu \|\Delta\|$, where λ is the delay operator. The set of such operators is denoted by $\mathcal{F}(\nu)$.*

The norm $\|\lambda\Delta - \Delta\lambda\|$ is a natural way to capture the rate of time variation of an operator. If this norm is zero, Δ commutes with λ and the operator is time invariant. Therefore small values of ν in Definition 3 correspond to operators which vary "slowly". A value $\nu = 2$ includes all "arbitrarily fast" operators on l_2. The result from [14] is that the (12) holds if and only if the system is robustly stable under perturbations in $\mathcal{F}(\nu)$ for some $\nu > 0$.

It may be argued from these results that μ-analysis should be abandoned in favor of these upper bounds which appear more tractable, especially in the case of (12) which has mild conservatism. There are still good reasons, however, to formulate a problem in terms of μ. In the first place, although the upper bounds have guaranteed polynomial-time computation, the size of the problems can be very large and render the computation impractical. In these cases one often relies heavily on the availability of efficient lower bound algorithms [11, 19, 1] (which have no guarantees but appear to behave well in practice) to compute the analysis. Secondly, if there is parametric uncertainty in the problem, the upper bounds may be substantially conservative (there is no corresponding slowly-varying interpretation). Lower bound algorithms provide a fast method to obtain "bad" parameter values, and can be further employed to assess this conservatism and, if desired, pursue a more refined analysis by branch and bound techniques [10].

2.3 A Power Distribution Lemma

The following Lemma from Poolla and Tikku [14] provides a useful characterization of time varying perturbations, which will be used in this paper.

Lemma 2 *Let $0 \le \omega_1 < \ldots < \omega_r \le \pi$ be distinct frequencies. If the vector valued signals*

$$z = \sum_{k=1}^{r} z_k e^{j\omega_k t}, \quad w = \sum_{k=1}^{r} w_k e^{j\omega_k t} \tag{13}$$

satisfy the power inequality $\sum_{k=1}^{r} \|z_k\|^2 \ge \sum_{k=1}^{r} \|w_k\|^2$, then there exists a linear time-varying, causal operator Δ such that

$$\begin{array}{ll} (i) & \|\Delta\| \le 1 \\ (ii) & \Delta \in \mathcal{F}(\nu),\ \nu = 2\sin(\frac{\omega_r - \omega_1}{2}) \\ (iii) & \Delta z = w + w^{tr},\ w^{tr} \in l_2 \end{array}$$

Heuristically, this lemma says that provided that the total power of z is greater than that of w, a contractive LTV operator can rearrange the power between frequencies, mapping z to w in steady state. The time variation ν required is a function of the amount of "frequency shifting" performed. In contrast, a contractive LTI operator will always decrease the power at every frequency.

3 A μ-Test for Mixed LTV/LTI Analysis

Sections 3 and 4 refer to the system of Figure 1, where we now set

$$\Delta = \begin{bmatrix} \Phi & 0 \\ 0 & \theta \end{bmatrix}, \quad M = \begin{bmatrix} M_{\Phi} & M_{\Phi\theta} \\ M_{\theta\Phi} & M_{\theta} \end{bmatrix}, \tag{14}$$

In (14), M is a finite dimensional LTI discrete time system, which can also be given a state-space representation

$$M(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{15}$$

Equivalently, M is obtained from the LFT $M(\lambda) = \lambda I * S = D + C\lambda(I - \lambda A)^{-1}B$, where λ is the delay operator (or a frequency variable in the unit disk) and S is the constant state-space matrix.

Φ is a causal LTV operator on l_2^m, and θ is a causal LTI operator on l_2^p. Each has a spatial structure analogous to (1), denoted respectively by $\mathbf{\Phi}$ and Θ. $\mathbf{B_\Phi}$ and $\mathbf{B}_\Theta$ denote the unit balls of uncertainty. $\mathbb{X}^\Phi$, $\mathbb{X}^\theta$ will denote the sets of scaling matrices corresponding to each structure. The consideration of real parametric uncertainty is deferred to Section 5.

An important integer parameter determined by the structure is the dimension d of the space of hermitian scaling matrices which commute with the LTV structure $\mathbf{\Phi}$. ($\mathbb{X}^\Phi$ is the positive cone in this space). When $\mathbf{\Phi}$ consists only of full blocks, d is the number of blocks.

The main result in this section is an extension of the exact μ-test (10), for LTI analysis, to the case of mixed LTV/LTI perturbations.

Since LTV structures are usually characterized by tests in terms of $\hat{\mu}$, rather than μ, it is not obvious that a μ-test can capture the mixed LTV/LTI case (and in particular, the LTV case). The main idea to obtain this μ-test is inspired in work by Bercovici et.al.[2], where an *augmentation* or *lifting* in the structure converts the upper bound $\hat{\mu}$ to μ of a larger matrix. The results in [2] apply to constant matrices and are based on operator-theoretic methods, but can also be obtained as a corollary of the more general dynamic results to be presented in this paper, which will be proved by convex analysis methods.

We first consider the augmented matrix

$$\tilde{M}(\omega_1, \ldots, \omega_d) = diag[M(e^{j\omega_1}), \ldots, M(e^{j\omega_d})] \tag{16}$$

which amounts to considering the system M at a fixed number d of frequencies. Next we introduce the following augmented structures in the space of complex matrices:

$$\tilde{\Phi} = \begin{bmatrix} \Phi^{11} & \ldots & \Phi^{1d} \\ \vdots & \ldots & \vdots \\ \Phi^{d1} & \ldots & \Phi^{dd} \end{bmatrix}, \Phi^{ij} \in \mathbf{\Phi}, \;\; i,j = 1 \ldots d \tag{17}$$

$$\tilde{\Delta} = \begin{bmatrix} \Delta^{11} & \ldots & \Delta^{1d} \\ \vdots & \ldots & \vdots \\ \Delta^{d1} & \ldots & \Delta^{dd} \end{bmatrix}, \begin{cases} \Delta^{ii} \in \mathbf{\Delta}, \;\; i = 1 \ldots d \\ \Delta^{ij} = diag[\Phi^{ij}, \; 0], \; i \neq j \end{cases} \tag{18}$$

The augmented structures $\tilde{\mathbf{\Phi}}$, $\tilde{\mathbf{\Delta}}$, are d times larger than the corresponding $\mathbf{\Phi}$, $\mathbf{\Delta}$. For the case of $\tilde{\mathbf{\Phi}}$, it is obtained simply by considering d^2 copies of $\mathbf{\Phi}$, in matrix form. $\tilde{\mathbf{\Delta}}$, which contains $\tilde{\mathbf{\Phi}}$ as a submatrix, is obtained in a similar fashion, the only difference being that the time invariant blocks θ are only "copied" along the diagonal, and the rest of the entries are set to zero. As an illustration, Figure 2 (a) contains the augmented configuration $(\tilde{M}, \tilde{\Delta})$ for the case $d = 2$.

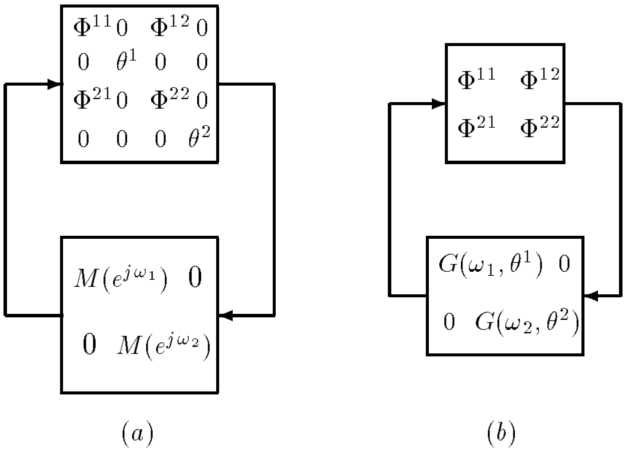

Figure 2: Augmented representations

This structure has a "frequency shifting" interpretation which relates to the remarks made in regard to Lemma 2: the augmentation corresponds to considering a system at a number d of frequencies, and the different treatment of LTI and LTV blocks is due to the fact that only the time-varying perturbations are allowed to "shift energy between frequencies"; this is represented in $\tilde{\Phi}$ by the off-diagonal terms.

It is also convenient to consider the configuration of Figure 2 (b), where the LTI portion is included with the dynamics. Define

$$G(\omega, \theta) := M(e^{j\omega}) * \theta = M_{\Phi} + M_{\Phi\theta}\theta(I - M_{\theta}\theta)^{-1}M_{\theta\Phi} \tag{19}$$

For this LFT to be well defined for $\theta \in \mathbf{B}_{\Theta}$, the following condition must hold:

$$\max_{\omega} \mu_{\Theta}(M_{\theta}(e^{j\omega})) < 1 \tag{20}$$

Under this condition, $G(\omega, \theta)$ is continuous for $\omega \in [-\pi, \pi]$, $\theta \in \mathbf{B}_{\Theta}$. For given $(\omega_k, \theta_k), k = 1 \ldots d$, the matrix $\tilde{G}(\omega_1, \ldots, \omega_d, \theta^1, \ldots, \theta^d) := diag[G(\omega_1, \theta^1) \ldots G(\omega_d, \theta^d)]$ is obtained by LFT between the θ portion of $\tilde{\Delta}$ and the matrix $\tilde{M}(\omega_1, \ldots, \omega_d)$, as shown in Figure 2 ($b$).

We now state the main result:

Theorem 3 *In reference to the system* (14), *the following are equivalent:*

(a) The system is uniformly robustly stable.

(b) Condition (20) holds, and with ω_k varying in $[-\pi, \pi]$, and $\theta^k \in \mathbf{B}_{\Theta}$,

$$\max_{\omega_1,\ldots,\omega_d,\theta^1,\ldots,\theta^d} \mu_{\tilde{\Phi}}(\tilde{G}(\omega_1, \ldots, \omega_d, \theta^1, \ldots, \theta^d)) < 1 \tag{21}$$

(c) With ω_k varying in $[-\pi, \pi]$

$$\max_{\omega_1,\ldots,\omega_d} \mu_{\tilde{\Delta}}(\tilde{M}(\omega_1, \ldots, \omega_d)) < 1 \tag{22}$$

Proof: The equivalence of (b) and (c) is a simple consequence of the main loop theorem (Lemma 1). In fact, for each $\omega_1, \ldots, \omega_d$, Lemma 1 implies that

$$\mu_{\tilde{\Delta}}(\tilde{M}(\omega_1, \ldots, \omega_d)) < 1 \Longleftrightarrow \begin{cases} \mu_{\Theta}(M_{\theta}(e^{j\omega_k})) < 1, \ k = 1 \ldots d \\ \max_{\theta^1,\ldots,\theta^d} \mu_{\tilde{\Phi}}(\tilde{G}(\omega_1, \ldots, \omega_d, \theta^1, \ldots, \theta^d)) < 1 \end{cases} \tag{23}$$

Taking maximum over $\omega_1, \ldots, \omega_d$, the right hand side of (23) gives (20) and (21).

The equivalence of (a) and (b) follows from a more technical convex analysis argument, and is covered in the Appendix.

Theorem 3 provides a necessary and sufficient test for robust stability in terms of a μ-condition (22) which involves a search in d frequency variables. Using a state-space realization as in (15), a state-space test in terms of μ of a constant matrix can be derived, in an analogous way as in the standard LTI problems involving one frequency variable ([11]):

Corollary 4 *The system has uniform robust stability if and only if* $\mu_{\tilde{\Delta}_S}(\tilde{S}) < 1$, *where* $\tilde{S} = diag[S, \ldots, S]$, *and*

$$\tilde{\Delta}_S = \begin{bmatrix} \Delta_S^{11} & \ldots & \Delta_S^{1d} \\ \vdots & \ldots & \vdots \\ \Delta_S^{d1} & \ldots & \Delta_S^{dd} \end{bmatrix} \tag{24}$$

$$\begin{cases} \Delta_S^{ii} = diag[\lambda^i I,\ \Phi^{ii},\ \theta^i],\ \Phi^{ii} \in \mathbf{\Phi},\ i = 1 \ldots d \\ \Delta_S^{ij} = diag[0,\ \Phi^{ij},\ 0],\ \Phi^{ij} \in \mathbf{\Phi},\ i \neq j,\ i,j = 1 \ldots d \end{cases} \tag{25}$$

Proof: $M(\lambda)$ (with λ a complex variable in the closed unit disk $\bar{D}$), is the transfer function of the stable state-space system given by S. The main loop theorem (Lemma 1) implies that

$$\mu_{\tilde{\Delta}_S}(\tilde{S}) < 1 \Longleftrightarrow \max_{\lambda_1, \ldots, \lambda_d \in \bar{D}} \mu_{\tilde{\Delta}}(\tilde{M}(\lambda_1, \ldots, \lambda_d)) < 1 \tag{26}$$

where $\tilde{M}(\lambda_1, \ldots, \lambda_d) := diag[M(\lambda_1), \ldots, M(\lambda_d)]$. The maximum modulus-like property [11] of complex μ implies that the maximum in (26) occurs at the boundary of the disk, and therefore (26) coincides with (c) in Theorem 3.

$\square$

4 Convex Tests for the Mixed LTV/LTI Problem

The previous tests have the theoretical advantage of being exact, but as remarked before, exact μ computation is hard, so practical use of these conditions will involve employing bounds as those in standard software packages such as μ-Tools [1].

In particular, we analyze in this section the upper bounds for this problem (sufficient conditions for robust stability) which lead to convex optimization. For this purpose, the μ conditions obtained in Section 3 can be bounded by use of the constant matrix upper bound $\hat{\mu}$. We focus here on the upper bound $\hat{\mu}_{\tilde{\Delta}}(\tilde{M})$ over the frequencies $\omega_1, \ldots, \omega_d$, which follows from (22). An examination of the augmented structure $\tilde{\Delta}$ shows that the corresponding commuting matrices $\tilde{X}$ are of the form

$$\tilde{X} = diag[X^{\Phi}, X_1^{\theta}, X^{\Phi}, X_2^{\theta}, \ldots, X^{\Phi}, X_d^{\theta}] \tag{27}$$

where $X^{\Phi} \in \mathbb{X}^{\Phi},\ X_k^{\theta} \in \mathbb{X}^{\theta}$.

An alternative is to directly apply scaled small-gain conditions to the original problem. In the case of mixed LTV/LTI analysis as in (14), the natural scaling set is of the "mixed" form

$$X(\omega) = diag[X^{\Phi},\ X^{\theta}(\omega)] \tag{28}$$

where the portion X^{θ} which corresponds to the LTI blocks θ is allowed to vary in frequency, and the portion X^{Φ} corresponding to the LTV blocks is constant. $\mathbb{X}_m$ will denote the set of such scaling functions; without loss of generality they are assumed to be continuous over frequency.

We now show that the two approaches are equivalent.

Proposition 5 *Given the system* (14), *the following are equivalent:*

$$\max_{\omega_1,\ldots,\omega_d} \hat{\mu}_{\tilde{\Delta}}(\tilde{M}(\omega_1, \ldots, \omega_d)) \quad < \quad 1 \tag{29}$$

$$\exists\ X(\omega) \in \mathbb{X}_m : \quad \bar{\sigma}\left(X(\omega) M(e^{j\omega}) X(\omega)^{-1}\right) \quad < \quad 1\ \forall \omega \tag{30}$$

Proof: If an X satisfying (30) is found, then for any choice of frequencies $\omega_1, \ldots, \omega_d$, setting

$$\tilde{X} = diag[X^\Phi, X^\theta(\omega_1), \ldots, X^\Phi, X^\theta(\omega_d)]$$

will result in $\bar{\sigma}\left(\tilde{X}\tilde{M}(\omega_1, \ldots, \omega_d)\tilde{X}^{-1}\right) < 1$, and therefore $\hat{\mu}_{\tilde{\Delta}}(\tilde{M}) < 1$, implying (29).

The converse implication is covered in the Appendix.

As a corollary of Proposition 5 and Theorem 3, condition (30) is *sufficient* for robust stability under mixed LTV/LTI perturbations; this could also be shown via standard small-gain arguments.

From a computational point of view, (30) has the advantage of involving a search over only one frequency variable. A direct approach [1] would be to grid the frequency axis and convert (30) to an LMI condition. Note, however, that the common scale $\mathbb{X}^\Phi$ introduces a coupling in the problem, so one is left with a large LMI condition, with size growing with the number of frequency grid-points.

In comparison, (29) tells us that in fact, d frequency values suffice, and we must only solve a coupled LMI problem of this size. However, since these frequencies are not known a priori, one has to grid a d dimensional space of frequencies. Therefore (29) reduces the size of the coupled LMI at the expense of more gridding. For low values of d, this alternative may be convenient.

We now consider the question of the conservatism of these conditions. The results of [14] reviewed in Section 2 suggest that the conditions become necessary if the LTI perturbations are enlarged to include arbitrarily slowly varying uncertainty. This is the content of the following statement; the proof involves an extension of the techniques of [14], described in the Appendix.

Theorem 6 *The conditions in Proposition 5 are satisfied if and only if there exists $\nu > 0$ such that the system (14) has uniform robust stability for $\Delta = diag[\Phi, \theta]$, Φ arbitrary structured operator, θ structured operator in $\mathcal{F}(\nu)$.*

[1] Another approach would be to parametrize $X^\theta(\omega)$ by basis functions.

5 Combination with Real Parametric Uncertainty

In this section we take a further step in the analysis under combined uncertainty structures; in addition to LTV and LTI blocks, we include real parametric perturbations. We will consider the robustness analysis setup of Figure 3:

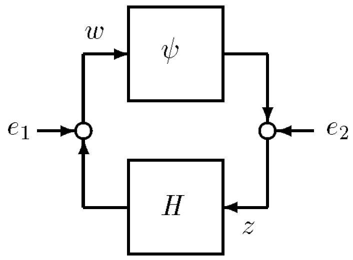

Figure 3: Robust stability problem

$$\Psi = \begin{bmatrix} \Phi & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \delta \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \delta \end{bmatrix} \qquad H = \begin{bmatrix} H_\Phi & H_{\Phi\theta} & H_{\Phi\delta} \\ H_{\theta\Phi} & H_\theta & H_{\theta\delta} \\ H_{\delta\Phi} & H_{\delta\theta} & H_\delta \end{bmatrix} = \begin{bmatrix} H_\Delta & H_{\Delta\delta} \\ H_{\delta\Delta} & H_\delta \end{bmatrix} \tag{31}$$

In (31), $H(e^{j\omega})$ is a stable system. Φ and θ are structured LTV and LTI perturbations as before. Δ is defined as in (14). The additional structured perturbation δ consists of real parametric blocks (e.g. $\delta = diag[\delta_1 I, \ldots, \delta_n I]$, $\delta_i \in \mathbb{R}$). The notation (Ψ, H instead of Δ, M) is chosen to clarify the proofs below.

We wish to obtain a necessary and sufficient condition for robust stability in this class, extending the results in Section 3. Since real parameters are a special case of time invariant uncertainty, at first sight it would appear that Theorem 3 applies directly, yielding a mixed-μ condition on $\tilde{H} = diag[H(e^{j\omega_1}), \ldots, H(e^{j\omega_d})]$, analogous to (22) but with the copies of δ in the augmented structure constrained to be real.

This augmentation would not capture, however, an additional property of real parametric uncertainty: in addition to taking real (rather than complex) values, a real parameter has no dynamics and therefore is constrained to be *constant across frequency*[2]. This suggests a modification of the augmented uncertainty structure for the case of real parameters, where they are forced to be constant across the augmentation. Consider the structure

$$\tilde{\Psi} = \begin{bmatrix} \psi^{11} & \dots & \psi^{1d} \\ \vdots & \dots & \vdots \\ \psi^{d1} & \dots & \psi^{dd} \end{bmatrix}, \begin{cases} \psi^{ii} = diag[\Phi^{ii}, \theta^i, \delta], i = 1 \dots d \\ \psi^{ij} = diag[\Phi^{ij},\ 0,\ 0],\ i \neq j \end{cases} \tag{32}$$

where $\Phi^{ij} \in \mathbf{\Phi}$ and $\theta^i \in \Theta$ as before, but we constrain the copies of δ to be repeated across the augmentation. The $d = 2$ case is depicted in Figure 4 (a).

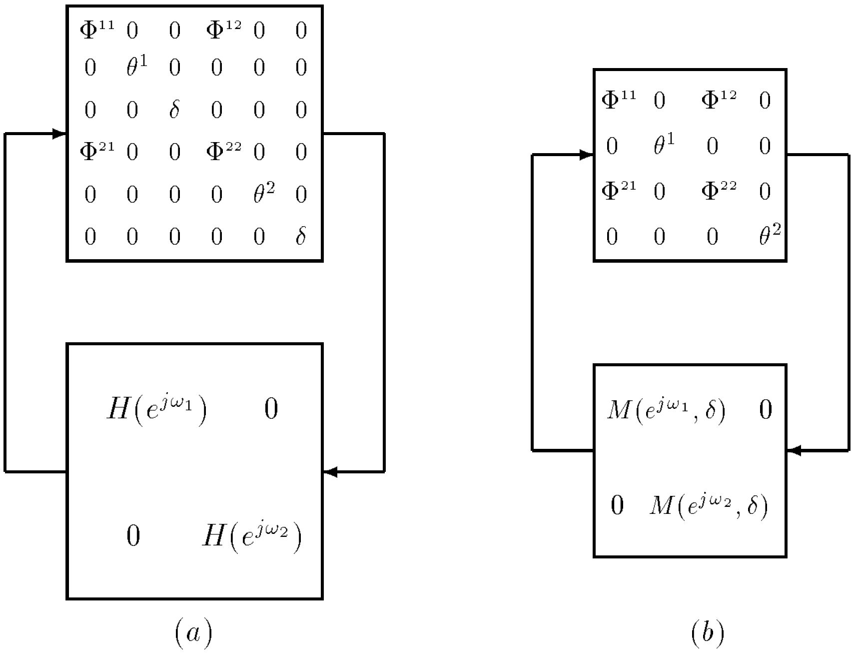

Figure 4: Augmented systems for the real parametric case

It will also be useful to "close the loop" on the real parametric part, which will reduce the

[2]This fact does not come into play in standard mixed μ with LTI/parametric uncertainty, where only one frequency is involved in the destabilization.

problem to the situation of Section 3. Assume that the real μ condition

$$\max_{\omega} \mu_\delta(H_\delta(e^{j\omega})) < 1 \tag{33}$$

holds, then for any fixed $\delta \in \mathbf{B}_\delta$ we can define

$$M(e^{j\omega}, \delta) := H(e^{j\omega}) * \delta = H_\Delta + H_{\Delta\delta}\delta(I - H_\delta\delta)^{-1}H_{\delta\Delta} \tag{34}$$

which is a stable time invariant system for each $\delta \in \mathbf{B}_\delta$.

For fixed δ, and given $\omega_1, \ldots, \omega_d$, the matrix $\tilde{M}(\omega_1, \ldots, \omega_d, \delta) := diag[M(e^{j\omega_1}, \delta) \ldots M(e^{j\omega_d}, \delta)]$ (which corresponds exactly to the augmentation (16)) is obtained by LFT between the δ portion of $\tilde{\Psi}$ and the matrix $\tilde{H}(\omega_1, \ldots, \omega_d)$. The uncertainty structure corresponding to $\tilde{M}$ is $\tilde{\Delta}$ as in (18), which is depicted in Figure 4 (b) for the case $d = 2$.

We now provide the extension of Theorem 3.

Theorem 7 *In reference to the system* (31), *the following are equivalent:*

(a) The system is uniformly robustly stable.

(b) Condition (33) holds, and with ω_k varying in $[-\pi, \pi]$, $\delta \in \mathbf{B}_\delta$

$$\max_{\omega_1, \ldots, \omega_d,\, \delta} \mu_{\tilde{\Delta}}(\tilde{M}(\omega_1, \ldots, \omega_d, \delta)) < 1 \tag{35}$$

(c) With ω_k varying in $[-\pi, \pi]$,

$$\max_{\omega_1, \ldots, \omega_d} \mu_{\tilde{\Psi}}(\tilde{H}(\omega_1, \ldots, \omega_d)) < 1 \tag{36}$$

where μ is the mixed (real/complex) structured singular value with respect to the structure (32).

Proof: The equivalence of (b) and (c) is a direct application of Lemma 1.

Also, (a) implies condition (33), otherwise the real parameters δ would destabilize by themselves. Given (33), we observe that for fixed $\delta \in \mathbf{B}_\delta$,

$$I - \Psi H = \begin{bmatrix} I & -\Delta H_{\Delta\delta}(I - \delta H_\delta)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - \Delta(H * \delta) & 0 \\ -\delta H_{\delta\Delta} & I - \delta H_\delta \end{bmatrix} \tag{37}$$

It follows that the (uniform) robust stability of system (31) is equivalent to the fact that for every δ in $\mathbf{B}_\delta$, the system $(M(e^{j\omega}, \delta), \Delta)$ is (uniformly) robustly stable. Since Δ has the mixed LTV/LTI structure of Theorem 3, this is in turn equivalent to condition (35) in (b).

$\square$

The previous result has reduced the robust stability problem under LTV, LTI and parametric uncertainties to a mixed-μ condition across d-frequencies. We remark the following:

- In analogous manner to Corollary 4, a state-space condition can be derived from (36), which is equivalent to a single mixed μ problem.
- As usual, practical computation of a μ-condition such as (36) must be approached by upper and lower bounds, and possibly branch and bound techniques. Upper bounds will have the form given in (9); analogously to the situation in Section 4, we have the choice of writing these conditions in the original problem or in the augmented problem. In this case, however, these are *not* equivalent: since the augmentation introduces repetition in the real uncertainty, this increases the freedom of the scaling matrices X^δ, G^δ corresponding to these blocks, which provides a way of imposing the condition that δ is constant across frequency. Without augmentation, this condition is not imposed so the bound is weaker.

6 Examples

In this section we illustrate the results of this paper with a series of examples. These have been deliberately constructed so that there is a direct way to answer and interpret the robust stability question, thus providing more insight into the conditions given in the previous sections.

6.1 A system with LTV uncertainty

We consider the interconnection of Figure (5), where G_1, G_2 are LTI SISO systems, and Φ_1, Φ_2 are uncertain perturbations with $\|\Phi_i\| \leq 1$.

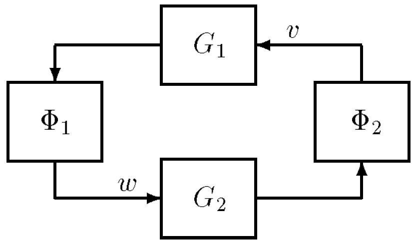

Figure 5: Example of analysis of LTV uncertainty

- If Φ_1, Φ_2 were LTI perturbations, then they would commute with G_1, G_2 and from the small-gain theorem the robust stability condition would be

$$\|G_1 G_2\|_\infty < 1 \tag{38}$$

- From now on we consider the more interesting case when Φ_1, Φ_2 are arbitrary LTV perturbations. It turns out that in this simple configuration, the necessary and sufficient condition for robust stability is

$$\|G_1\|_\infty \|G_2\|_\infty < 1 \tag{39}$$

  This condition is in general stricter than (38), since the two transfer functions $G_1(e^{j\omega})$ and $G_2(e^{j\omega})$ need not achieve their peak gain at the same frequency.

The sufficiency of (39) is clear from small-gain. To explain why it is also necessary, let us particularize in the example (where λ is the delay)

$$G_1 = \frac{0.1}{1 - 0.9\lambda}; \qquad G_2 = \frac{-0.1}{1 + 0.9\lambda} \tag{40}$$

Here both systems have $\mathcal{H}_\infty$ norm equal to one, achieved respectively at frequencies 0, π. Their magnitude frequency response plots are depicted in Figure 6 below. Condition (39) is therefore not satisfied. Let us show that the system can be destabilized by contractive LTV operators. In this case it suffices to consider the time-varying gains $\Phi_1(t) = \Phi_2(t) = (-1)^t$.

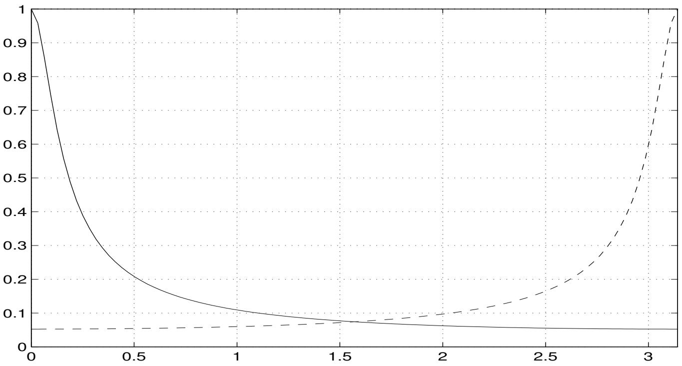

Figure 6: Frequency response of G_1, G_2

In reference to Figure 5, consider a constant signal $v(t) \equiv 1$ at the input of G_1. Since the transfer function G_1 is 1 at $\omega = 0$, the steady state output will be the same signal. The time-varying gain $\Phi_1(t)$ modulates this signal to $w(t) = (-1)^t$, which has all its frequency content at $\omega = \pi$, where G_2 has value 1. This implies the steady state output of G_2 is $w(t)$, which is demodulated back to $v(t)$ by $\Phi_2(t)$. We have a steady state signal in the loop, which violates robust stability. This informal argument can be formalized and extended to arbitrary G_1, G_2, and it illustrates strongly the "frequency-shifting" properties of LTV perturbations.

We should recover the same answer if we do the analysis using the results in Section 3. For this purpose, we first rearrange Figure 5 to an M-Δ setup, where Δ contains in this case only the LTV portion Φ

$$\Delta = \Phi = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix},$$

Since $d = 2$, we must compute the augmentation of Theorem 3 over two frequency variables ω_1, ω_2. Figure 7 contains the resulting function $\mu_{\tilde{\Delta}}(\tilde{M}(\omega_1, \omega_2))$, computed using the software package μ-Tools [1]. We find that the maximum is 1, achieved when the two frequencies take the values 0, π, which is consistent with the previous analysis. Similar results can be obtained using the state-space condition in Corollary 4.

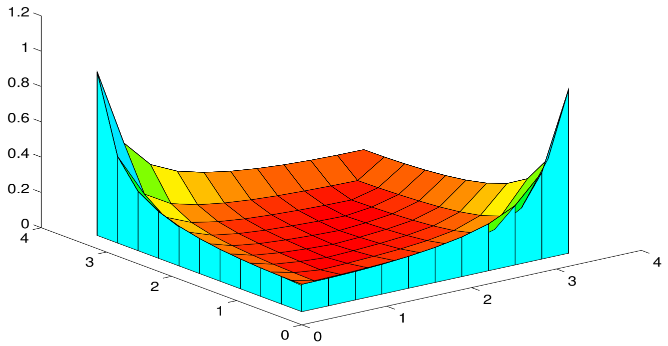

Figure 7: $\mu_{\tilde{\Delta}}(\tilde{M})$ as a function of ω_1, ω_2

6.2 Including LTI uncertainty

We now modify the previous setup to include LTI uncertainty; G_1, G_2 are replaced by

$$G_1 = \frac{0.1}{1-(0.6+0.3\theta_1)\lambda}; \qquad G_2 = \frac{-0.1}{1+(0.6-0.3\theta_2)\lambda}; \qquad \|\theta_i\| \leq 1 \tag{41}$$

which are LFTs $G_i = T_i * \theta_i$ on the LTI perturbations θ_i, as depicted in Figure 8.

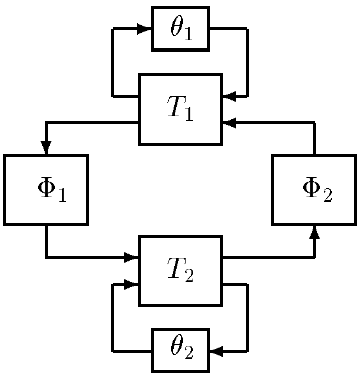

Figure 8: System with LTV and LTI uncertainty

From (39), it is clear that the worst-case θ_i are those which achieve $\max_{\theta_i} \|G_i\|_\infty$. The values in (41) have been chosen so that the worst-case perturbations are $\theta_1 = 1$, $\theta_2 = -1$, which produce G_i as in (40). From the previous analysis the smallest destabilizing perturbation is of norm 1: $\theta_1 = 1$, $\theta_2 = -1$, $\Phi_1 = \Phi_2 = (-1)^t$.

These results are verified when we do the analysis of Theorem 3. After rearranging Figure 8 in the standard setup, and performing the augmentation, we obtain $\max_{\omega_1,\omega_2} \mu_{\tilde{\Delta}}(\tilde{M}(\omega_1,\omega_2)) = 1$ (achieved at $\omega_1 = 0$, $\omega_2 = \pi$).

If the θ_i in (41) correspond to real parameters instead of LTI perturbations, we still obtain the same answer from the robust stability analysis, since the worst-case values obtained above happen to be real. This can also be verified by computing the mixed-μ condition on the augmented system.

6.3 Real parametric vs LTI perturbations

To produce an example where real and LTI perturbations give a different answer, one can use the same structure as in (41), but impose $\theta_1 = \theta_2 = \theta$ (repeated perturbation). In other words,

$$G_1 = \frac{0.1}{1-(0.6+0.3\theta)\lambda}; \qquad G_2 = \frac{-0.1}{1+(0.6-0.3\theta)\lambda} \tag{42}$$

If θ is LTI, the repetition does not alter the results in Section 6.2, since an LTI perturbation can take the values 1 at $\omega = 0$ and -1 at $\omega = \pi$. An example is $\theta(e^{j\omega}) = e^{j\omega}$ (θ is the delay operator λ), which turns the G_i into second order systems with $\|G_1\|_\infty = \|G_2\|_\infty = 1$.

The same answer is obtained from the augmented μ test (22) which gives

$$\max_{\omega_1,\omega_2} \mu_{\tilde{\Delta}}(\tilde{M}(\omega_1,\omega_2)) = 1 \tag{43}$$

achieved at $\omega_1 = 0$, $\omega_2 = \pi$, and a destabilizing perturbation with $\theta^1 = 1$, $\theta^2 = -1$, as expected.

We now change θ for a real parameter $\delta \in [-1,1]$, repeated in G_1, G_2. Since it is constant across frequency, it cannot maximize both $\|G_i\|_\infty$ simultaneously. In fact, straightforward calculations show that

$$\|G_1\|_\infty \|G_2\|_\infty = \frac{1}{16-9\delta^2} \tag{44}$$

which has a maximum of $\frac{1}{7}$ for $\delta \in [-1, 1]$. This implies by (39) that the overall system is stable for $\|\Phi_i\| \leq 1$, $\delta \in [-1, 1]$.

Proceeding by augmentation, (36) gives a value

$$\max_{\omega_1, \omega_2} \mu_{\tilde{\Psi}}(\tilde{H}(\omega_1, \omega_2)) = 0.7906 < 1 \tag{45}$$

In this case we can show directly that the smallest destabilizing perturbation for $\delta \in \mathbb{R}$ and Φ_i LTV, has norm $\frac{1}{0.7906} = 1.2649$. For this purpose, choose $\delta \in [-\gamma, \gamma]$ and $\|\Phi_i\| \leq \gamma$. The full system will be stable as long as

$$\frac{1}{16 - 9\gamma^2} = \max_{\delta \in [-\gamma, \gamma]} (\|G_1\|_\infty \|G_2\|_\infty) < \frac{1}{\gamma^2} \tag{46}$$

Condition (46) is equivalent to $\gamma < \sqrt{8/5} = 1.2649$, as expected.

7 Conclusions

This paper shows that a combination of different classes of uncertain perturbations (LTV,LTI, parametric) can be analyzed with the same mathematical tools as non-mixed problems. A structured singular value condition was obtained, applicable to any combination of these uncertainty classes.

These results allow a totally decoupled approach to uncertainty modeling in complex systems: one can choose the most adequate uncertainty description (LTV, LTI, parametric) at the subsystem level, and obtain an exact condition for robustness analysis of the overall system under the combined uncertainty structure. The price paid in terms of complexity of these conditions is the size of the corresponding augmentation.

From a computational point of view, a number of equivalent conditions have been obtained, and further research is required to determine the most efficient approach for practical problems. In regard to the convex upper bounds which lead to coupled LMI problems across frequency, two alternatives (29) and (30) have been discussed and should be further explored. In relation to lower bound computation, the repeated structure of the augmented systems may be exploited to improve the algorithms.

Finally, scaled small-gain conditions such as (30) can naturally lead to the extension of "D-K iteration" methods for controller synthesis. In the author's opinion, however, synthesis should be based on simpler and more heuristic methods, and these tools with very specialized uncertainty structures are best employed for analysis validating the resulting designs.

Appendix

Convex Analysis Lemmas

The following results from convex analysis will be used in the proofs.

For a set $K \subset \mathbb{R}^d$, $co(K)$ will denote the convex hull of K, i.e. the set of all convex combinations of elements of K. If K is compact, so is $co(K)$.

Lemma 8 (Helly) *Let $\{K_\omega\}_{\omega\in\Omega}$ be family of convex closed sets in $\mathbb{R}^d$, of which at least one is bounded. If $\bigcap_{\omega\in\Omega} K_\omega = \emptyset$ then there exist $d+1$ sets $K_{\omega_i}, i = 1 \ldots d+1$ with empty intersection.*

Lemma 9 *Let K,L be disjoint convex sets in $\mathbb{R}^d$, where K is compact and L is closed. Then there exists a vector $x \in \mathbb{R}^d$, and α, β in $\mathbb{R}$ such that*

$$\langle x, k\rangle \leq \alpha < \beta \leq \langle x, l\rangle \;\; \forall k \in K, l \in L. \tag{47}$$

Lemma 10 *If $K \subset \mathbb{R}^d$, every point in $co(K)$ is a convex combination of $d+1$ points in K; for K compact, every point in the boundary of $co(K)$ is a convex combination of d points in K.*

Proofs: Lemmas 8 and 9 can be found in [16]. The first part of Lemma 10 is a result of Caratheodory (see [16]), and implies that for every $v \in co(K)$, there exists a *simplex* of the form

$$S(v_1, \ldots, v_{d+1}) = \left\{ \sum_{k=1}^{d+1} \alpha_k v_k : \alpha_k \geq 0, \sum_{k=1}^{d+1} \alpha_k = 1 \right\}$$

with vertices $v_k \in K$, which contains v. If the v_k are in a lower dimensional hyperplane, then d points will suffice to generate v. If not, then every point in $S(v_1, \ldots, v_{d+1})$ corresponding to

$\alpha_k > 0\ \forall k$ will be *interior* to $S(v_1, \ldots, v_{d+1}) \subset co(K)$. Therefore for points v in the *boundary* of $co(K)$, one of the α_k's must be 0 and a convex combination of d points will suffice, completing Lemma 10.

Proof of Theorem 3

We now show the equivalence of (a) and (b) in Theorem 3. For simplicity we will write the proof for the case where the structure $\mathbf{\Phi}$ contains only full blocks, i.e. $\mathbf{\Phi} = diag[\Phi_1, \ldots \Phi_d]$, $\Phi_l \in \mathcal{L}(l_2^{m_l})$. The same arguments can be extended to the case of δI blocks in the uncertainty $\mathbf{\Phi}$, in a similar style as in [12]. The following Lemma gives an interpretation of the uncertainty structure $\tilde{\mathbf{\Phi}}$.

Lemma 11 *Assume* (20) *holds. Given* $\omega_1, \ldots, \omega_d$, $\theta^1, \ldots, \theta^d$ *the following are equivalent:*

(i) $\mu_{\tilde{\mathbf{\Phi}}}(\tilde{G}(\omega_1, \ldots, \omega_d, \theta^1, \ldots, \theta^d)) \geq 1$.

(ii) $\exists\ v^1, \ldots, v^d \in \mathbb{C}^m$, *not all zero such that*

$$\sum_{k=1}^{d} \left\| (G(\omega_k, \theta^k) v^k)_l \right\|^2 \geq \sum_{k=1}^{d} \left\| (v^k)_l \right\|^2, \quad l = 1 \ldots d \tag{48}$$

Proof: (i) is equivalent to the existence of $\tilde{\Phi}$, $\bar{\sigma}(\tilde{\Phi}) \leq 1$ and a vector $v = col(v^1, \ldots, v^d) \neq 0$ such that $(I - \tilde{\Phi}\tilde{G}(\omega_1, \ldots, \omega_d, \theta^1, \ldots, \theta^d))v = 0$, or

$$\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix} = \begin{bmatrix} \Phi^{11} & \ldots & \Phi^{1d} \\ \vdots & \ldots & \vdots \\ \Phi^{d1} & \ldots & \Phi^{dd} \end{bmatrix} \begin{bmatrix} G(\omega_1, \theta^1) v^1 \\ \vdots \\ G(\omega_d, \theta^d) v^d \end{bmatrix} \tag{49}$$

Each Φ^{ij} has in turn d subblocks, which will henceforth indexed by the subindex l, and impose a partition on the v^k, $G(\omega_k, \theta^k)v^k$. The equations and variables in (49) can be reordered, to put together the $v_l^1, \ldots, v_l^d$ and $(G(\omega_1, \theta^1)v^1)_l, \ldots, (G(\omega_d, \theta^d)v^d)_l$ for each l, which reduces the structure of $\tilde{\Phi}$ to d full blocks in this new order. Now $\bar{\sigma}(\tilde{\Phi}) \leq 1$ is equivalent to the norm inequalities (48).

□

$(a) \Longrightarrow (b)$

Condition (20) is clearly necessary for robust stability; if it did not hold, the standard results (10) imply that the system could be destabilized by LTI perturbations alone. Therefore $G(\omega, \theta)$ is well defined and continuous.

Assume there exist $\omega_1, \ldots, \omega_d \in [-\pi, \pi], \theta^1, \ldots, \theta^d \in \mathbf{B}_\Theta$ with $\mu_{\tilde{\Phi}}(\tilde{G}(\omega_1, \ldots, \omega_d, \theta^1, \ldots, \theta^d)) \geq 1$. We can use Lemma 11 to obtain $v^1, \ldots, v^d$ satisfying (48).

Fix $\epsilon > 0$; by continuity of G we can perturb the $\theta^1, \ldots, \theta^d$ to have $\bar{\sigma}(\theta^k) < 1$ (strict inequality), and the $\omega_1, \ldots, \omega_d$ to make them *distinct*, satisfying

$$\sum_{k=1}^{d} \left\| (G(\omega_k, \theta^k) v^k)_l \right\|^2 \geq (1-\epsilon)^2 \sum_{k=1}^{d} \left\| (v^k)_l \right\|^2, \quad l = 1 \ldots d \tag{50}$$

We are now in the conditions of an interpolation result given in [6], which states that there exists a causal, stable, rational LTI perturbation $\theta(e^{j\omega}) \in \mathbf{B}_\Theta$ [3], satisfying $\theta(e^{j\omega_k}) = \theta^k$. Now define

$$v(t) = \sum_{k=1}^{d} v^k e^{j\omega_k t}, \quad z(t) = \sum_{k=1}^{d} (G(\omega_k, \theta^k) v^k) e^{j\omega_k t} \tag{51}$$

Then $G(\omega, \theta(e^{j\omega}))v(t) = z(t) + e(t)$, where $e(t)$ is a transient term. For any l, (50) implies that the power of the l-lth component of $z(t)$, is greater than the power of the l-lth component of $v(t)$, times $(1-\epsilon)$. We can invoke Lemma 2 to show the existence of a causal time-varying operator Φ_l which maps $z(t)_l$ to $(1-\epsilon)v(t)_l$, up to a transient term. Constructing $\Phi = diag[\Phi_1 \ldots \Phi_d]$, we obtain

$$(I - \Phi G(\omega, \theta(e^{j\omega})))v(t) = \epsilon v(t) + \bar{e}(t) \tag{52}$$

where $\bar{e}(t) \in l_2$ is transient. This implies that

$$\left\| (I - \Phi(M * \theta))^{-1} \right\| = \left\| (I - \Phi G)^{-1} \right\| \geq \frac{1}{\epsilon} \tag{53}$$

Referring to (14), it can be verified that

$$I - \Delta M = \begin{bmatrix} I & -\Phi M_{\Phi\theta}(I - \theta M_\theta)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - \Phi(M * \theta) & 0 \\ -\theta M_{\theta\Phi} & I - \theta M_\theta \end{bmatrix} \tag{54}$$

[3] [6] constructs a perturbation in the disk algebra $\mathcal{A}(\mathbb{T})$ interpolating a countable number of frequencies; in the case of a finite number of frequencies it can be chosen to be rational

This implies that

$$\left\|(I-\Delta M)^{-1}\right\| \geq \frac{\left\|\begin{bmatrix} I-\Phi(M*\theta) & 0 \\ -\theta M_{\theta\Phi} & I-\theta M_\theta \end{bmatrix}^{-1}\right\|}{\left\|\begin{bmatrix} I & -\Phi M_{\Phi\theta}(I-\theta M_\theta)^{-1} \\ 0 & I \end{bmatrix}\right\|} \tag{55}$$

As $\epsilon \rightarrow 0$, the numerator in (55) tends to infinity from (53), and the denominator is bounded using (20). This implies that $\sup_\Delta \left\|(I-\Delta M)^{-1}\right\| = \infty$, violating uniform robust stability.

$(b) \Longrightarrow (a)$:

For this direction we introduce some notation. For $\omega \in [-\pi, \pi]$, $\theta \in \mathbf{B}_\Theta \subset \mathbb{C}^{p\times p}$, $v \in \mathbb{C}^m$, define

$$\sigma^l_{\omega,\theta}(v) = \|(G(\omega,\theta)v)_l\|^2 - \|v_l\|^2\,,\ l = 1\ldots d \tag{56}$$

$$\Lambda_{\omega,\theta}(v) = (\sigma^1_{\omega,\theta}(v),\ldots,\sigma^d_{\omega,\theta}(v)) \in \mathbb{R}^d \tag{57}$$

$$\nabla = \{\Lambda_{\omega,\theta}(v) :\ \omega \in [-\pi,\pi],\ \theta \in \mathbf{B}_\Theta,\ v \in \mathbb{C}^m,\ \|v\| = 1\} \tag{58}$$

From (20), $\Lambda_{\omega,\theta}(v)$ is continuous in its 3 variables, therefore ∇ is compact in $\mathbb{R}^d$, and so is $co(\nabla)$.

Claim:

$$co(\nabla) \cap (\mathbb{R}^+_0)^d = \emptyset \tag{59}$$

where $\mathbb{R}^+_0$ is the set of nonnegative real numbers.

In fact, if (59) does not hold, then we can find a point y in the boundary of $co(\nabla)$, which falls inside $(\mathbb{R}^+_0)^d$. Invoking Lemma 10, y is a convex combination of d points in ∇. Therefore there exist $\omega_1,\ldots,\omega_d$, $\theta^1,\ldots,\theta^d$, $v^1,\ldots,v^d$ such that

$$y = \sum_{k=1}^d \alpha_k \Lambda_{\omega_k,\theta^k}(v^k) \in (\mathbb{R}^+_0)^d,\ \ \alpha_k \geq 0,\ \ \sum_{k=1}^d \alpha_k = 1. \tag{60}$$

Recalling (56), (57), then (60) implies that (48) holds for the corresponding $G(w_k,\theta^k)$ and $\tilde{v}^k = \sqrt{\alpha_k}v^k$. This violates (b) by Lemma 11, so the claim is proved.

$co(\nabla)$ and $(\mathbb{R}_0^+)^d$ are disjoint closed convex sets in $\mathbb{R}^d$, and $co(\nabla)$ is compact. Therefore, by Lemma 9 there exists $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\alpha, \beta \in \mathbb{R}$ such that

$$\langle x, \Lambda \rangle \leq \alpha < \beta \leq \langle x, y \rangle \quad \forall \Lambda \in \nabla, y \in (\mathbb{R}_0^+)^d \tag{61}$$

Given the special structure of the cone $(\mathbb{R}_0^+)^d$, we can choose $\beta = 0$ and $x_1 > 0, \ldots, x_d > 0$. Using (57), we have

$$x_1 \sigma^1_{\omega,\theta}(v) + \ldots + x_d \sigma^d_{\omega,\theta}(v) \leq \alpha < 0 \qquad \forall \omega \in [-\pi, \pi],\ \theta \in \mathbf{B}_\Theta,\ v \in \mathbb{C}^m,\ \|v\| = 1 \tag{62}$$

Let $X = diag[x_1 I_{m_1} \ldots x_d I_{m_d}]$ which belongs to X^Φ, then (56) and (62) give

$$\|X^{\frac{1}{2}} G(\omega, \theta) v\|^2 - \|X^{\frac{1}{2}} v\|^2 \leq \alpha < 0 \quad \forall \omega,\ \theta,\ v \tag{63}$$

It follows that $\gamma := \max_{\omega,\theta} \bar{\sigma}\left(X^{\frac{1}{2}} G(\omega, \theta) X^{-\frac{1}{2}}\right) < 1$. For any LTI $\theta(e^{j\omega})$ taking values in $\mathbf{B}_\Theta$, and any LTV Φ in $\mathbf{B}_\Phi$, we obtain

$$\left\| X^{\frac{1}{2}} \Phi(M * \theta) X^{-\frac{1}{2}} \right\| \leq \left\| X^{\frac{1}{2}} G(\omega, \theta(e^{j\omega})) X^{-\frac{1}{2}} \right\|_\infty \leq \gamma \tag{64}$$

This implies

$$\left\| \left(I - X^{\frac{1}{2}} \Phi(M * \theta) X^{-\frac{1}{2}} \right)^{-1} \right\| \leq \frac{1}{1 - \gamma} \tag{65}$$

which leads to a uniform bound on $\| \left(I - \Phi(M * \theta)\right)^{-1} \|$. This, together with (55) and assumption (20) gives a uniform bound on $\|(I - \Delta M)^{-1}\|$.

$\square$

7.1 Proof of Proposition 5

It remains to prove that (29) implies (30). (29) implies that

$$\max_{\omega_1, \ldots, \omega_d} \inf_{\tilde{X}} \bar{\sigma}(\tilde{X} \tilde{M}(\omega_1, \ldots, \omega_d) \tilde{X}^{-1}) < 1 \tag{66}$$

For fixed $\omega_1, \ldots, \omega_d$, clearly there exists $\epsilon(\omega_1, \ldots, \omega_d) > 0$ such that

$$\inf_{\frac{1}{\epsilon} I \geq \tilde{X} \geq \epsilon I} \bar{\sigma}(\tilde{X}\tilde{M}(\omega_1, \ldots, \omega_d)\tilde{X}^{-1}) < 1 \tag{67}$$

Since the $\omega_1, \ldots, \omega_d$ vary in a compact set, and $\tilde{M}$ is continuous, it follows that a fixed ϵ can be found satisfying (67) across $\omega_1, \ldots, \omega_d$. Now assume that (30) does not hold. This means that the LMI condition across frequency

$$X(\omega) > 0, \quad M^*(\omega)X(\omega)M(\omega) - X(\omega) < 0 \tag{68}$$

does not have a solution $X \in \mathbb{X}_m$, i.e. it cannot be satisfied with the X^Φ part constant across frequency. As a consequence, the family of sets over ω

$$\mathcal{X}_\omega^\Phi := \left\{ X^\Phi \in \mathbb{X}^\Phi : \exists X = diag[X^\Phi, X^\theta(w)],\ \epsilon I \leq X \leq \frac{1}{\epsilon},\ M(e^{j\omega})^* X M(e^{j\omega}) - X < 0 \right\} \tag{69}$$

have empty intersection for ω ranging in $[-\pi, \pi]$. These sets are convex and compact, and by normalizing the last block of X^Φ to I, they are in a $d-1$ dimensional space. From Lemma 8, there exist d such sets with empty intersection.

Therefore there exist d frequencies $\omega_1, \ldots, \omega_d$ such that $\bar{\sigma}\left(XM(j\omega_k)X^{-1}\right) < 1$ cannot be satisfied with $\epsilon I \leq X \leq \frac{1}{\epsilon}$ and a common X^Φ. This contradicts the fact that ϵ satisfies (67) for all frequencies $\omega_1, \ldots, \omega_d$.

□

7.2 Proof of Theorem 6

The sufficiency of condition (30) for uniform robust stability over $\mathcal{F}(\nu)$ can be proved with very minor modifications to the proof given in [14]; the details are omitted. We now outline the necessity proof, which is essentially a combination of the methods applied in [14] to deal separately with slowly varying perturbations and "arbitrarily fast" perturbations.

Necessity:. For simplicity we assume the structures Θ and Φ consist only of full blocks, $\Phi_l \in \mathcal{L}(l_2^{m_l})$, $\theta_{d+f} \in \mathcal{L}(l_2^{p_f})$. We have $\Delta = diag[\Phi_1, \ldots, \Phi_d, \theta_{d+1}, \ldots, \theta_{d+F}]$.

Assume now that the system does not satisfy (30); we will show uniform robust stability is violated for every $\nu > 0$.

Step 1: We first use Proposition 5 to show the existence of d frequencies $\omega_1, \ldots, \omega_d$ such that

$$\hat{\mu}_{\tilde{\Delta}}(\tilde{M}(\omega_1, \ldots, \omega_d)) \geq 1 \tag{70}$$

At this point, we will perform a *second* finite augmentation of the problem in the style of Section 3. For notational simplicity a matrix form will not be written, but it follows from [2] (or also from the proof in Section 3) that there exists a finite horizon N and vectors $\tilde{v}^1, \ldots, \tilde{v}^N \in \mathbb{C}^{(m+p)\times d}$ which "destabilize" an augmentation of the matrix $\tilde{M}$ in the following sense.

For each $n \in 1 \ldots N$, partition the vector $\tilde{v}^n$ as $(v^k)^n$ in accordance to the blocks of $\tilde{M}$; let $M(e^{j\omega_k})(v^k)^n = (y^k)^n$; the following inequalities hold:

$$\sum_{k,n} \left\| ((y^k)^n)_l \right\|^2 \geq \sum_{k,n} \left\| ((v^k)^n)_l \right\| \quad l = 1 \ldots d \tag{71}$$

$$\sum_{n} \left\| ((y^k)^n)_{d+f} \right\|^2 \geq \sum_{n} \left\| ((v^k)^n)_{d+f} \right\|^2 \ \begin{cases} f = 1 \ldots F \\ k = 1 \ldots d \end{cases} \tag{72}$$

Note that all "energy can be shared" between the repetitions n of this second augmentation. As for the first augmentation in k, only the Φ blocks "transfer energy" as before. Heuristically, this second augmentation will be used with frequencies close to the w_k's. The transfer of energy between these frequencies can be achieved by slowly varying operators.

Step 2: Fix $\nu > 0$ and $\epsilon > 0$. For δ to be chosen below, construct

$$v^k(t) = \sum_{n=0}^{N-1} (v^k)^n e^{j(\omega_k + \frac{n-1}{N}\delta)}, \quad y^k(t) = \sum_{n=0}^{N-1} (y^k)^n e^{j(\omega_k + \frac{n-1}{N}\delta)} \tag{73}$$

By continuity of $M(e^{j\omega})$ around ω_k, $y^k = Mv^k + e^k + w^k$, where w^k is a transient term in l_2, and e^k is a sum of sinusoids such that its power $\|\cdot\|_{\mathcal{P}}$ satisfies $\|e^k\|_{\mathcal{P}} < \epsilon \|v^k\|_{\mathcal{P}}$, for small enough δ.

Also choose δ small enough so that $2\ sin(\frac{\delta}{2}) < \nu$, and so that the intervals $[w_k, w_k + \delta]$ are pairwise disjoint.

From (72) and Lemma 2, for each f, k we can find $\theta^k_{d+f} \in \mathcal{F}(\nu)$ such that $\theta^k_{d+f} y^k_{d+f} = v^k_{d+f}$. Since the Fourier supports of (y^k, v^k) are disjoint for different k, with the same methods as in Lemma 2 it can be shown that there is a single $\theta_{d+f} \in \mathcal{F}(\nu)$, satisfying

$$\theta_{d+f} y^k_{d+f} = v^k_{d+f}, \quad f = 1 \ldots F, \ \ k = 1 \ldots d \tag{74}$$

Now define $v(t) = \sum_{k=1}^d v^k(t)$, $\ y(t) = \sum_{k=1}^d y^k(t)$. We have $y = Mv + e + w$, $w \in l_2$, $\|e\|_{\mathcal{P}} = O(\epsilon)\, \|v\|_{\mathcal{P}}$, and the definition of the v^k, y^k, implies from (71) that

$$\|y_l\|^2 \geq \|v_l\|^2, \ \ l = 1 \ldots d \tag{75}$$

This implies we can construct contractive LTV operators Φ_l, $l = 1 \ldots d$ (with no prescribed rate of variation, since v, y are not confined to a "thin" frequency band) such that $\Phi_l y_l = v_l$. Setting $\Delta = diag[\Phi_l, \ldots, \Phi_d, \theta_{d+1}, \ldots, \theta_{d+F}]$ gives, using (74), $\Delta y = v$. Therefore

$$\|(I - \Delta M)v\|_{\mathcal{P}} = \|\Delta e\|_{\mathcal{P}} = O(\epsilon)\, \|v\|_{\mathcal{P}} \tag{76}$$

Since ϵ is arbitrary, uniform robust l_2-stability is violated within the class $\mathbf{\Delta} = diag[\mathbf{\Phi}, \Theta]$, $\Theta \in \mathcal{F}(\nu)$. Since ν is arbitrary, this completes the proof.

$\square$

Acknowledgements

The author wishes to thank Raffaello D'Andrea for a stimulating discussion, and Geir Dullerud for helpful suggestions. This work was supported by AFOSR.

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