$\mathcal{H}_\infty$ optimization with spatial constraints

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Abstract

A generalized $\mathcal{H}_\infty$ synthesis problem where non-euclidian spatial norms on the disturbances and output error are used is posed and solved. The solution takes the form of a linear matrix inequality. Some problems which fall into this class are presented. In particular, solutions are presented to two problems: a variant of $\mathcal{H}_\infty$ synthesis where norm constraints on each component of the disturbance can be imposed, and synthesis for a certain class of robust performance problems.

1 Introduction

One of the strongest cases for the $\mathcal{H}_\infty$ framework is that a system's uncertainty can naturally be incorporated into the design process [22], [5]. The Ricatti based solution in [6] generated the first reliable and computationally attractive algorithms for general $\mathcal{H}_\infty$ synthesis.

Recently, there has been a lot of research activity centered around problems which can be converted to *linear matrix inequalities*, or LMIs (the reader is referred to [1] for a thorough treatment of LMIs and how they apply to control theory). This is mainly due to the widely accepted view that a problem which can be converted to an LMI is as good as an analytical solution; reliable computational methods, such as the interior point methods of Nesterov and Nemirovsky [12], and emerging software packages, such as the LMI Control Toolbox [8], justify this claim. In terms of $\mathcal{H}_\infty$ control theory, LMI solutions to the $\mathcal{H}_\infty$ suboptimal control problem may be found in [7] and [13]. Even though the LMI approach is attractive from the standpoint that no assumptions on the problem data are required (as opposed to the invariant zero assumptions of the Ricatti based approach), arguably the real strength of the LMI formulation is that it can be generalized to encompass other types of $\mathcal{H}_\infty$ related problems, such as the gain scheduling results of Packard [13].

In this paper, the LMI framework is used to tackle a variant of the $\mathcal{H}_\infty$ control problem where non-euclidian spatial norms are used for the disturbances and output error. In the style of Megretski and Treil [11], "S procedure losslessness" types of arguments are used to show that the derived conditions are both necessary and sufficient for a solution to exist.

A direct consequence of this problem formulation is the ability to synthesize suboptimal controllers for problems where norm constraints on each component of the disturbance are imposed, termed *Square* $\mathcal{H}_\infty$ design.

In addition, this formulation allows for robust control synthesis for plants subject to a certain class of structured uncertainty; in particular, when the uncertainty consists of a full block with norm bounded elements. The conditions provided are both necessary and sufficient for a solution to exist. It is shown that many types of robust performance problems can be converted to this form, and solved exactly.

The paper is organized as follows: we begin with some mathematical preliminaries, followed by the problem formulation. Problems which can be cast into this framework are then discussed. We conclude by presenting the solution to the general problem.

2 Preliminaries

Most of the notation in this paper is standard. We restrict ourselves to discrete time systems, although most of the results in this paper extend to continuous time systems. The space of square summable sequences is denoted l_2; when the spatial structure is relevant, it is referred to as l_2^p. The 2-norm of a signal d in l_2 is denoted $\|d\|$. The discrete time, unit delay operator is denoted λ. The induced l_2 norm of a bounded linear operator Φ over l_2 is denoted $\|\Phi\|$. The term *system* will be used to denote causal, finite dimensional, linear, time invariant operators over l_2. A system G is *stable* if it is bounded. The linear fractional transformation (LFT) between two systems G and K is denoted $G \star K$, and is defined as:

$$G \star K := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{1}$$

where

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

when the inverse of $(I - G_{22}K)$ is well defined and causal.

For two subsets of l_2, S_1 and S_2, the *approximation error* of S_1 with S_2 is defined as

$$D(S_1, S_2) := \sup_{s_1 \in S_1} \inf_{s_2 \in S_2} \|s_1 - s_2\| \tag{2}$$

$D(S_1, S_2)$ is a measure of how well S_2 approximates the elements of S_1.

3 Problem Formulation

Consider the feedback interconnection of systems G and K in Figure 1. The closed loop map from d to e is

$$M := G \star K \tag{3}$$

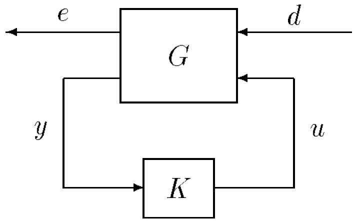

Figure 1: Synthesis Formulation

K will be referred to as a *stabilizing controller* if the closed loop map of Figure 1 is internally stable [23].

Let $\mathcal{H}$ be the following subset of l_2, referred to as the *constraint set*:

$$\mathcal{H} = \left\{ d \in l_2 \left| \begin{array}{ll} \|H_i d\| \leq 1 & 1 \leq i \leq \bar{i} \\ \|L_j d\| = \|J_j d\| & 1 \leq j \leq \bar{j} \end{array} \right. \right\} \tag{4}$$

where the H_i, L_j and J_j are constant matrices of dimension compatible with the spatial structure of d. It is assumed that constraint set $\mathcal{H}$ is bounded.

Define $E(\cdot)$ to be the following function on l_2, referred to as the *cost criterion*:

$$E(e) := \sum_{k=1}^{\bar{k}} \|P_k e\| \tag{5}$$

where the P_k are constant matrices of dimension compatible with the spatial structure of e.

Problem Formulation

Given system G, constraint set $\mathcal{H}$, cost criterion E, and performance specification γ, find a stabilizing controller K such that

$$\sup_{d \in \mathcal{H}} E(Md) < \gamma \tag{6}$$

As will be shown, the solution takes the form of an LMI. Before presenting the solution to the above problem (see Section 6), different types of synthesis problems which can be cast into the above framework will be presented. It is worth noting that the problems which follow only require constraint set $\mathcal{H}$ to be defined in terms of constraint matrices H_i, i.e., no equality constraints are required; these latter types of constraints are used in [2] to solve a certain class of robust synthesis problems where the disturbances are modeled as white noise inputs.

4 Square $\mathcal{H}_\infty$

The reason for the title of this section will become apparent shortly, although it is admittedly an abuse of notation. Consider the standard MIMO $\mathcal{H}_\infty$ synthesis problem, depicted in Figure 1, where $e \in l_2^p$ and $d \in l_2^m$: it is required to find a controller K which minimizes the energy output of the closed loop system M subject to all possible unit energy (by linearity) disturbance inputs.

If we consider the motivation for this problem, it seems reasonable to lump the cost as $\|e\| = \sqrt{\|e_1\|^2 + \cdots + \|e_p\|^2}$; it is required to keep the error e small in some sense, and large deviations are penalized more than smaller ones. It isn't so clear, however, why the disturbance size is lumped together as $\|d\| = \sqrt{\|d_1\|^2 + \cdots + \|d_m\|^2}$; if the d_i are physically motivated, their magnitudes will, in general, be independent. One would expect specifications of the form $\|d_i\| \leq \alpha_i$ for each component (this is, incidentally, one of the arguments for l_1 design (see [4]) versus $\mathcal{H}_\infty$ design, albeit only on the spatial aspect of the norm). Let us assume, without loss of generality, that $\alpha_i = \frac{1}{\sqrt{m}}$, since these constants can be absorbed into generalized plant description G. To capture these constraints into the standard $\mathcal{H}_\infty$ setup, one would have to cover the given allowable disturbance set by the following round constraint:

$$\sum_{i=1}^{m} \|d_i\|^2 \leq 1 \tag{7}$$

which corresponds to the diagram of Figure 2 for $m = 2$.

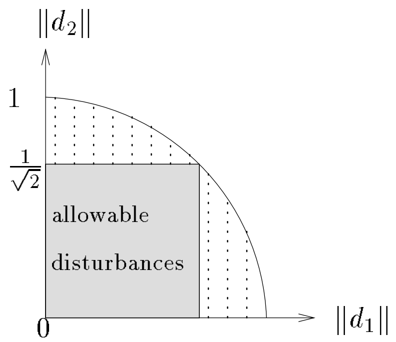

Figure 2: Square vs Round Spatial Constraint

The "square" disturbance set which we want to design for lies inside the round set. Let Φ be a linear operator. Define the following induced norm:

$$\|\Phi\|_{\text{SQ}} := \sup_{\|d_i\| \leq \frac{1}{\sqrt{m}}} \|\Phi d\| \tag{8}$$

referred to as the *square* norm. The following relationship between $\|\Phi\|_{\text{SQ}}$ and $\|\Phi\|$ follows immediately from the above definition:

$$\|\Phi\|_{\text{SQ}} \leq \|\Phi\| \leq \sqrt{m}\|\Phi\|_{\text{SQ}} \tag{9}$$

with the bounds above being tight for any m. Thus the $\mathcal{H}_\infty$ norm of a system may be $\sqrt{m}$ times larger than the square norm, and occurs when only one disturbance has an effect on the output error.

It is straightforward to construct a simple example such that *synthesis* will give this gap. By this, it is meant that doing $\mathcal{H}_\infty$ optimization results in a square norm which is $\sqrt{m}$ times greater than if the optimization were done directly with the square criterion. It should be noted, however, that the simplicity of this example stems from the assumed controller structure. One needs to work somewhat harder to construct an example which fits into the framework of Figure 1.

Consider the following *static* equations:

$$
\begin{aligned}
e_1 &= (1+\epsilon Q)d_1 && (10)\\
e_i &= (1-Q)d_i, \;\; 2 \le i \le m \\
\epsilon &> 0 && (11)
\end{aligned}
$$

where $Q \in \mathbb{R}$ is the design variable. The $\mathcal{H}_\infty$ design problem reduces to

$$
\inf_{Q\in\mathbb{R}} \bar{\sigma} \begin{bmatrix} 1+\epsilon Q & & & \\ & 1-Q & & \\ & & \ddots & \\ & & & 1-Q \end{bmatrix} \tag{12}
$$

This infimum is 1 for all $\epsilon > 0$, and is uniquely achieved by $Q = 0$, the global minimum. The square norm for this design is also 1.

If, however, one chooses $Q = 1$, the resulting $\mathcal{H}_\infty$ and square norms are $1+\epsilon$ and $\frac{1+\epsilon}{\sqrt{m}}$, respectively. Thus by letting ϵ go to zero, by the bounds in (9), one can come arbitrarily close to the optimal square norm, and a gap which approaches $\sqrt{m}$.

Thus if the size of each component of the disturbance is known, one might want to perform the design directly with the square, versus the round, spatial constraint on the disturbance (hence the name Square $\mathcal{H}_\infty$, since the signal norm is still l_2):

Square $\mathcal{H}_\infty$ Synthesis

Given system G and performance specification γ_s, find a stabilizing controller K such that $\|M\|_{SQ} < \gamma_s$.

The Square $\mathcal{H}_\infty$ problem can be cast into the general framework outlined in Section 3 by setting $\gamma = \gamma_s$, $E(e) = \|e\|$, and taking the set $\mathcal{H}$ to be

$$
\mathcal{H} = \{\|d_i\| \le \frac{1}{\sqrt{m}}, \;\; 1 \le i \le m\} \tag{13}
$$

or equivalently, by choosing

$$H_i = \sqrt{m}\mathbf{e_i}^{\mathbf{T}}, \quad 1 \leq i \leq c = m \tag{14}$$

where the $\mathbf{e_i}$ are standard basis vectors in $\mathbb{R}^m$.

Note that in definition of the square norm, $d \in l_2^m$ was partitioned into m scalar valued signals. In general, one could define the square norm for any partition of d. In particular, the H_i can be chosen to allow for some or all of the d_i to be vector valued as opposed to scalar valued; as an extreme case, by choosing $H = I$ (only one constraint), standard $\mathcal{H}_\infty$ synthesis is recovered.

It should be noted that the argument for choosing a square versus a round spatial constraint is based on a worst case design methodology; the round set must cover the square set to account for all possible disturbances. If, however, one wishes to relax the worst case assumption, the round set could be seen as a means to prevent all the components from achieving their maximum energy content (ie, the design is performed with the round set inside the square set). In this context, the square design is the more conservative one. This, however, is simply a scaling argument ($\sqrt{m}$ to be exact); the resulting $\mathcal{H}_\infty$ design is the same whether done inside or outside the "square". Regardless of the interpretation, $\mathcal{H}_\infty$ design may not be the wisest thing to do (as illustrated by the above example, where it is clear that the square design is better than the $\mathcal{H}_\infty$ design, irrespective of the gain interpretation). This usually stems from requiring optimality, or near-optimality. This is often remedied by considering sub-optimal designs, and employing other criteria for choosing a controller (for example, maximum entropy controllers [9]). In this light, Square $\mathcal{H}_\infty$ design should be seen as an additional tool in the $\mathcal{H}_\infty$ methodology; the specific application will determine whether it yields better designs.

5 Robustness Problems

In this section, a certain class of synthesis problems where the plant is subject to structured uncertainty will be solved. The general setup is first introduced, followed by two examples of the types of problems which fall into this class.

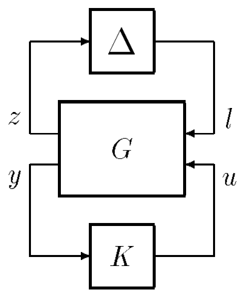

Figure 3: Synthesis for robust stability

Consider the setup of Figure 3; variables z and l are partitioned into $m_{\mathbf{z}}$ and $m_{\mathbf{l}}$ components (not necessarily scalar valued). This partition induces a corresponding one for Δ:

$$l_i = \sum_{j=1}^{m_{\mathbf{z}}} \Delta_{ij} z_j, \quad 1 \leq i \leq m_{\mathbf{l}} \tag{15}$$

For given $m_{\mathbf{z}}$ and $m_{\mathbf{l}}$, the set $\mathbf{F}\Delta$ is defined as follows:

$$\mathbf{F}\Delta := \{\Delta | \ \ \Delta \text{ linear }, \|\Delta_{ij}\| \leq 1\} \tag{16}$$

(these norm bounded sets are typically used to capture modeling errors and uncertainty. In many cases, non-linear modeling errors can be captured as well by "covering" them with norm-bounded linear operators [3]).

K is referred to as a *robustly stabilizing controller* for G and $\mathbf{F}\Delta$ if K is a stabilizing controller for G and

$$\sup_{\Delta \in \mathbf{F}\Delta} \| (I - M\Delta)^{-1} \| < \infty \tag{17}$$

where $M := G \star K$. This condition establishes the stability and well-posedness of the closed loop system for all allowable uncertainty. If the condition above is satisfied, the closed loop system of Figure 3 is said to be *robustly stable*.

The following theorem establishes the equivalence of finding a robustly stabilizing controller K to a problem which fits into the general formulation of Section 3:

Theorem 1

I. *K is a robustly stabilizing controller for G and* $\mathbf{F}\Delta$

if and only if

II. *K is a stabilizing controller for G and*

$$\sup_{\|l_i\| \leq 1} \sum_{j=1}^{m_{\mathbf{z}}} \|z_j\| < 1 \tag{18}$$

where $z = Ml$.

Proof:

II $\Rightarrow$ I: Assume that K is not a robustly stabilizing controller, violating (17). Thus $\forall \epsilon > 0$, $\exists n, \tilde{z} \in l_2$ and $\Delta \in \mathbf{F}\Delta$ such that $\|n\| < \epsilon$, $\sum_{j=1}^{m_{\mathbf{z}}} \|\tilde{z}_j\| = 1$, and $\tilde{z} = (I - M\Delta)^{-1} n$. Setting $l = \Delta\tilde{z}$ results in $z = \tilde{z} - n$, with $\|l_i\| \leq 1$. Since ϵ is arbitrary, (18) is contradicted.

I $\Rightarrow$ II: Since K is a robustly stabilizing controller, it immediately follows that K is a stabilizing controller for G. Furthermore, (17) implies that there exists $r > 1$ such that

$$\sup_{\Delta \in \mathbf{F}\Delta} \| (I - rM\Delta)^{-1} \| < \infty \tag{19}$$

Assume that (18) is not satisfied. Then $\exists l \in l_2$ such that $\sum_{j=1}^{m_{\mathbf{z}}} \|z_j\| = \gamma \geq \frac{1}{r}$. Define

$$\begin{aligned} \Delta_{ij}(\cdot) &:= \tfrac{1}{r\gamma\|z_j\|} l_i \langle z_j, \cdot \rangle, \ \|z_j\| > 0 \\ &:= 0 \qquad\qquad\qquad\quad \|z_j\| = 0 \end{aligned} \tag{20}$$

It follows that $\|\Delta_{ij}\| \leq 1$ and $\Delta z = \frac{1}{r} l$, contradicting (19)

■

Condition (18) is of the form which fits into the general framework presented in Section 3. Note that the Δ constructed when proving the necessity of condition **II** is in general non-causal. In the style of [19], however, it can be shown that the results still hold when Δ is required to be causal.

5.1 Robust Performance

In Figure 3, partition z into $\bar{z}$ and e, and l into $\bar{l}$ and $\bar{d}$. e and $\bar{d}$ are themselves partitioned into $m_{\mathbf{e}}$ and $m_{\mathbf{d}}$ components. This induces the following partition of Δ

$$\Delta = \begin{bmatrix} \Delta^{lz} & \Delta^{le} \\ \Delta^{dz} & \Delta^{de} \end{bmatrix} \tag{21}$$

and results in the diagram of Figure 4:

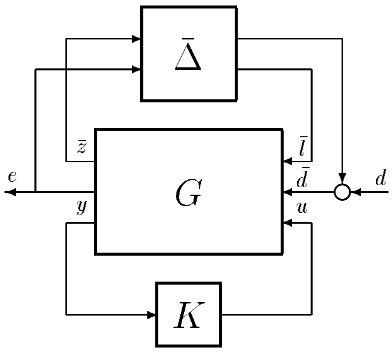

Figure 4: Robust Performance

where $d = \Delta^{de} e$ and $\bar{\Delta} := \begin{bmatrix} \Delta^{lz} & \Delta^{le} \\ \Delta^{dz} & 0 \end{bmatrix}$. $\bar{\mathbf{F}}\Delta$ can be defined analogously. Similar to the construction of Theorem 1, it can be shown that $\sum_{j=1}^{m_{\mathbf{e}}} \|e_j\| \geq \|d_i\|$ for each i if and only if there exists Δ^{de}, $\|\Delta_{ij}^{de}\| \leq 1$, such that $d = \Delta^{de} e$. The following corollary follows immediately:

Corollary 1 I. *K is a robustly stabilizing controller for G and $\bar{\mathbf{F}}\Delta$, and*

$$\sup_{\|d_i\| \leq 1} \sup_{\bar{\Delta} \in \bar{\mathbf{F}}\Delta} \sum_{j=1}^{m_{\mathbf{e}}} \|e_j\| < 1 \tag{22}$$

if and only if

II. *K is a stabilizing controller for G and*

$$\sup_{\|l_i\| \le 1} \sum_{j=1}^{m_z} \|z_j\| < 1 \tag{23}$$

5.1.1 Examples

Consider the setup of Figure 5. Given P, it is required to design K such that disturbances

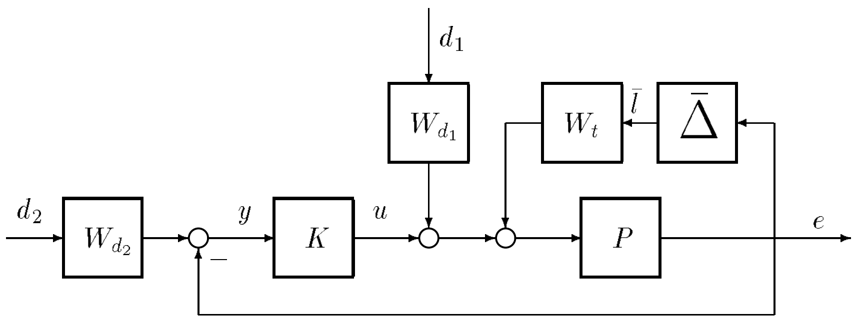

Figure 5: Robust Disturbance Rejection

d_1, along with measurement errors d_2, have a small effect on plant output e. The plant is subject to multiplicative, unstructured uncertainty $\bar{\Delta}$, with associated weight W_t. The exact problem formulation is the following: find a controller K such that the closed loop system is robustly stable and

$$\sup_{\|d_1\| \le 1, \|d_2\| \le 1} \sup_{\|\bar{\Delta}\| \le 1} \|e\| < 1 \tag{24}$$

This can be converted to the setup of Figure 4 by noting that $\bar{l} = \bar{\Delta} e$. Also note that if either of d_1 or d_2 are vector valued signals, they can further be partitioned and bounded separately.

Remark: A similar type of problem is solved in [18], where the uncertainty is taken to be real parametric, and enters the problem linearly in the characteristic polynomial; the solution, in this case, takes the form of an infinite dimensional convex optimization problem.

■

Consider the setup of Figure 6. Given P, it is required to design K such that s tracks d. Equivalently, letting W_y be a weight which captures the range over which tracking is desired, it is required to keep e_1 small. The plant is subject to multiplicative, unstructured uncertainty $\bar{\Delta}$, with associated weight W_t. It may also be required to bound the control effort; this is

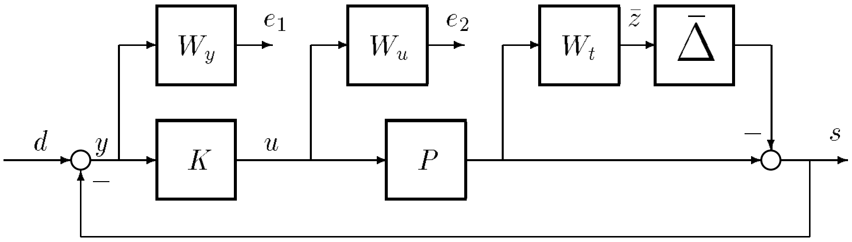

Figure 6: Robust Tracking

done by weighting the controller output by W_u, and requiring e_2 to be small. Formally, it is required to find K such that the closed loop system is robustly stable and

$$\sup_{\|d\|\leq 1} \sup_{\|\bar{\Delta}\|\leq 1} \|e\| < 1 \tag{25}$$

This problem can also be converted to the setup of Figure 4 by defining $\bar{d} := d + \bar{\Delta}\bar{z}$.

■

6 Solution

As will be shown, the solution takes the form of an LMI. There are two basic tools that are used to arrive at this solution. The first is the so called *S-procedure* (see [20]), the process of transforming a problem to one involving multipliers; similar to the results of Megretski and Treil [11], it can be shown that the S-procedure applied to our problem formulation is lossless, or non-conservative. The second tool is the LMI formulation of the discrete time $\mathcal{H}_\infty$ synthesis problem of Gahinet [7] and Packard [13].

We begin by showing that the equality constraints of $\mathcal{H}$ are in some sense continuous. Define the following set

$$\mathcal{H}^\epsilon = \left\{ d \in l_2 \left| \begin{array}{ll} \|H_i d\| \leq 1 + \epsilon & 1 \leq i \leq \bar{i} \\ | \, \|L_j d\| - \|J_j d\| \, | \leq \epsilon & 1 \leq j \leq \bar{j} \end{array} \right. \right\} \tag{26}$$

The following Lemma states that the difference between $\mathcal{H}$ and $\mathcal{H}^\epsilon$ can be made arbitrarily small:

Lemma 1 *$D(\mathcal{H}^\epsilon, \mathcal{H})$ is upper semi-continuous as a function of ϵ at $\epsilon = 0$.*

The proof of the above is rather long and technical, and is thus omitted. The main idea behind the proof may be found in [2].

The following theorem states how the analysis condition with constraint set $\mathcal{H}$ and cost criterion E can be converted to a scaled $\mathcal{H}_\infty$ condition:

Theorem 2 *Given stable system M, constraint set $\mathcal{H}$, and cost criterion E,*

I.

$$\sup_{d\in\mathcal{H}} E(Md) < \gamma \tag{27}$$

if and only if

II. $\exists s_i > 0, t_k > 0, r_j \in \mathbb{R}$ such that

$$\|T^{\frac{-1}{2}} PMS^{\frac{-1}{2}}\|_\infty < 1 \tag{28}$$

where

$$\sum_{i=0}^{\bar{i}} s_i < \gamma, \quad \sum_{k=0}^{\bar{k}} t_k < \gamma \tag{29}$$

$$\begin{aligned} P^{\mathrm{T}} &= [\, P_1^{\mathrm{T}} \; \cdots \; P_{\bar{k}}^{\mathrm{T}} \,] \\ S &= \sum_{i=1}^{\bar{i}} s_i H_i^{\mathrm{T}} H_i + \sum_{j=1}^{\bar{j}} r_j \left(L_j^{\mathrm{T}} L_j - J_j^{\mathrm{T}} J_j \right) > 0 \\ T &= \mathrm{diag}\,(t_1, \cdots, t_{\bar{k}}) > 0 \end{aligned}$$

Proof:

II $\Longrightarrow$ **I**: Assume that **II** is satisfied. Then it can be shown that there exists $\epsilon > 0$ such that $\forall d \in \mathcal{H}$,

$$\sum_{k=1}^{\bar{k}} \frac{1}{t_k} \|P_k Md\|^2 \le \gamma - \epsilon \|d\|^2 \tag{30}$$

By doing a change of variables $\bar{P}_k = \frac{1}{\sqrt{k}} P_k$, it follows that

$$\sum_{k=1}^{\bar{k}} \|\bar{P}_k Md\|^2 \le \gamma - \epsilon \|d\|^2 \tag{31}$$

and by the Cauchy-Schwartz inequality,

$$E(Md) := \sum_{k=1}^{\bar{k}} \sqrt{t_k} \|\bar{P}_k Md\| \le \sqrt{\gamma}\sqrt{\gamma - \epsilon\|d\|^2} \tag{32}$$

This clearly implies that the supremum in (27) is less than or equal to γ. Since M is bounded, however, the $\epsilon \|d\|^2$ term ensures that this supremum is strictly less than γ.

I $\Longrightarrow$ II: Assume that (27) is satisfied, where without loss of generality, it is assumed that $\gamma = 1$ (this constant can be absorbed into M). Define the following bounded set in $l_2^{\bar{k}}$:

$$\mathcal{E} := \left\{ e \in l_2^{\bar{k}} \middle| \|e_k\|^2 \leq 1 \;\; 1 \leq k \leq \bar{k} \right\} \tag{33}$$

It follows that there exists $\beta > 0$ such that

$$\sup_{d \in \mathcal{H}} \sup_{e \in \mathcal{E}} \left(\langle P^{\mathrm{T}} e, Md \rangle + 2\beta \|e\|^2 \right) < 1 \tag{34}$$

Define the following functions on l_2:

$$\begin{array}{lll} \sigma_0(d,e) & := \langle P^{\mathrm{T}} e, Md \rangle + 2\beta \|e\|^2 - 1 & \\ \sigma_k^{\mathcal{E}}(e) & := 1 - \|e_k\|^2 & 1 \leq k \leq \bar{k} \\ \sigma_i^{\mathcal{H}}(d,e) & := \langle P^{\mathrm{T}} e, Md \rangle - \|H_i d\|^2 & 1 \leq i \leq \bar{i} \\ \bar{\sigma}_j^{\mathcal{H}}(d) & := \|J_j d\|^2 - \|L_j d\|^2 & 1 \leq j \leq \bar{j} \end{array} \tag{35}$$

and the following set:

$$\nabla := \left\{ \Lambda(d,e) \mid d, e \in l_2, \; \|e\|^2 \leq \frac{1}{\beta^2}, \; \|d\|^2 \leq \frac{R_{\mathbf{h}}^2}{\beta} \right\} \tag{36}$$

where

$$\Lambda(d,e) := \left(\sigma_0(d,e), \sigma_k^{\mathcal{E}}(d), \sigma_i^{\mathcal{H}}(d,e), \bar{\sigma}_j^{\mathcal{H}}(d) \right) \in R^{1+\bar{k}+\bar{i}+\bar{j}} \tag{37}$$

and $R_{\mathbf{h}}$ is an upper bound for the largest element in $\mathcal{H}$. The following Lemma is essential to the proof:

Lemma 2 *$\bar{\nabla}$, the closure of ∇, is convex and compact.*

The proof of the above lemma is similar to that in [17]. The main idea is the following: if $\Lambda(d_1, e_1)$ and $\Lambda(d_2, e_2)$ are any two elements of ∇, then

$$\Lambda(\sqrt{\alpha} d_1 + \sqrt{1-\alpha} \lambda^{\tau} d_2, \sqrt{\alpha} e_1 + \sqrt{1-\alpha} \lambda^{\tau} e_2)$$

approaches $\alpha \Lambda(d_1, e_1) + (1-\alpha) \Lambda(d_2, e_2)$ for any $\alpha \in [0,1]$ as $\tau \longrightarrow \infty$.

Define

$$\mathbf{X} := \left\{ X \in R^{1+\bar{k}+\bar{i}+\bar{j}} \mid X = \left(x_0, x_k^{\mathcal{E}}, x_i^{\mathcal{H}}, 0_j \right), \;\; x_0, x_k^{\mathcal{E}}, x_i^{\mathcal{H}} \geq 0 \right\} \tag{38}$$

From (34) and (35), it can be shown that $\nabla \bigcap \mathbf{X} = \phi$. By Lemma 1, however, this result can be strengthened to $\bar{\nabla} \bigcap \mathbf{X} = \phi$. By a separating hyperplane argument [10] and the compactness of $\bar{\nabla}$, it follows that there exists X^* in the dual of $\mathbf{X}$ such that

$$\langle \Lambda, X \rangle \leq \langle \Lambda, X^* \rangle \leq 0 \quad \forall \Lambda \in \bar{\nabla}, \; X \in \mathbf{X} \tag{39}$$

Let $X^* =: (x_0^*, t_k^*, s_i^*, r_j^*)$. Since $\bar{\nabla}$ is compact, all the elements of X^* except the r_j^* terms can be assumed to be strictly positive. Defining

$$t_k := \frac{t_k^*}{x_0^*} > 0, \; s_i := \frac{s_i^*}{x_0^*} > 0, \; r_j := \frac{r_j^*}{x_0^*} \tag{40}$$

results in the following inequality for $\|e\|^2 \leq \frac{1}{\beta^2}$, $\|d\|^2 \leq \frac{R_{\mathbf{h}}^2}{\beta}$:

$$\begin{aligned} \langle P^{\mathbf{T}} e, Md \rangle < 1 - 2\beta\|e\|^2 \; &+ \sum_{k=1}^{\bar{k}} t_k \left(\|e_k\|^2 - 1 \right) \\ &+ \sum_{i=1}^{\bar{i}} s_i \left(\|H_i d\|^2 - \langle P^{\mathbf{T}} e, Md \rangle \right) \\ &+ \sum_{j=1}^{\bar{j}} r_j \left(\|L_j d\|^2 - \|J_j d\|^2 \right) \end{aligned} \tag{41}$$

Setting $e = d = 0$ implies that $\sum_{k=1}^{\bar{k}} t_k < 1$. Setting $d = 0$ and $\|e\|^2 = \frac{1}{\beta}$ implies that $t_k > \beta$. Define $\bar{e} := \sqrt{\beta} T^{\frac{1}{2}} e$ and $\bar{d} := \sqrt{\beta} d$. If $\|\bar{e}\| = 1$ and $\|\bar{d}\| \leq R_{\mathbf{h}}$, then $\frac{1}{\beta} \leq \|e\|^2 \leq \frac{1}{\beta^2}$ and $\|d\|^2 \leq \frac{R_{\mathbf{h}}^2}{\beta}$, resulting in the following inequality:

$$\begin{aligned} \langle \bar{e}, T^{\frac{-1}{2}} PM\bar{d} \rangle < 1 \; &+ \sum_{i=1}^{\bar{i}} s_i \left(\|H_i d\|^2 - \langle \bar{e}, T^{\frac{-1}{2}} PM\bar{d} \rangle \right) \\ &+ \sum_{j=1}^{\bar{j}} r_j \left(\|L_j d\|^2 - \|J_j d\|^2 \right) - \beta \end{aligned} \tag{42}$$

The above implies that for all $\bar{d} \in \mathcal{H}$ and $\|\bar{e}\|^2 = 1$, $\langle \bar{e}, T^{\frac{-1}{2}} PM\bar{d} \rangle \leq 1 - \frac{\beta}{1+\sum s_i}$, and consequently,

$$\sup_{\bar{d} \in \mathcal{H}} \|T^{\frac{-1}{2}} PM\bar{d}\| < 1 \tag{43}$$

This completes the first part of the construction, and yields scales T. Scales S can be constructed in an analogous fashion by considering quadratic functions

$$\begin{aligned} \sigma_0(d) &:= \|T^{\frac{-1}{2}} PMd\|^2 + 2\bar{\beta}\|d\|^2 - 1 \\ \sigma_i(d) &:= 1 - \|H_i d\|^2 \\ \bar{\sigma}_j(d) &:= \|J_j d\|^2 - \|L_j d\|^2 \end{aligned} \tag{44}$$

The details are omitted. ∎

The above theorem only address the issue of analysis, since M is a given system. For fixed scales, however, (28) is a standard $\mathcal{H}_\infty$ optimization problem, and results in the following corollary:

Corollary 2 *The constrained synthesis problem of Section 3 is solvable if and only if there exist scales S and T satisfying (29) and a stabilizing controller K for G such that*

$$\|T^{\frac{-1}{2}}(\bar{G} \star K)S^{\frac{-1}{2}}\|_\infty < 1 \tag{45}$$

where $\bar{G} = \begin{bmatrix} P & \\ & I \end{bmatrix} G$.

The next step is to invoke the LMI $\mathcal{H}_\infty$ solution. The specific way in which S and T enter equation (45) is such that the convexity of the $\mathcal{H}_\infty$ solution is preserved, and results in the following:

Theorem 3 *Given minimal state space representation for* $\bar{G} := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$, *there exists a stabilizing controller K and positive definite S and T as in (29) satisfying (45) if and only if there exists positive definite matrices X, Y, $\bar{S}$, and $\bar{T}$ such that*

$$V\left(R^{\mathrm{T}}\begin{bmatrix} X & 0 \\ 0 & \bar{T} \end{bmatrix} R - \begin{bmatrix} X & 0 \\ 0 & S \end{bmatrix}\right) V^{\mathrm{T}} < 0 \tag{46}$$

$$U^{\mathrm{T}}\left(R\begin{bmatrix} Y & 0 \\ 0 & \bar{S} \end{bmatrix} R^{\mathrm{T}} - \begin{bmatrix} Y & 0 \\ 0 & T \end{bmatrix}\right) U < 0 \tag{47}$$

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0, \quad \begin{bmatrix} S & I \\ I & \bar{S} \end{bmatrix} > 0, \quad \begin{bmatrix} T & I \\ I & \bar{T} \end{bmatrix} > 0 \tag{48}$$

where

$$R = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$$

$$\begin{bmatrix} C_2 & D_{21} \\ V_1 & V_2 \end{bmatrix} \text{ full column rank, } C_2 V_1^{\mathrm{T}} + D_{21} V_2^{\mathrm{T}} = 0 \tag{49}$$

$$\begin{bmatrix} B_2 & U_1 \\ D_{12} & U_2 \end{bmatrix} \text{ full row rank, } \qquad U_1^{\mathrm{T}} B_2 + U_2^{\mathrm{T}} D_{12} = 0$$

The proof follows by substituting the problem data into the $\mathcal{H}_\infty$ solution in [13], and some standard results on matrix inequalities. If the system of LMIs above is feasible, a state space description of the controller may be constructed from the state space description of $\bar{G}$ and matrices X and Y, as in [13]. Equivalently, one could take scales S and T which makes the above system of LMIs feasible and solve (45) using the standard Ricatti based approach of [6].

7 Conclusions

A generalized $\mathcal{H}_\infty$ synthesis problem where non-euclidian spatial norms on the disturbances and output error are used has been presented. This added freedom in the problem formulation enables one to synthesize controllers for the Square $\mathcal{H}_\infty$ problem, and for a certain class of uncertain systems.

This framework, however, can be used for many more types of problems. In [2], for example, by adopting the notion of deterministic white noise put forth by Paganini [16], the robust performance problems presented in this paper can be generalized to the case where the exogenous disturbance is a white noise signal, thus providing a solution to the so-called Robust $\mathcal{H}_2$ Synthesis problem for a restricted class of problems. The LPV results in [13] can also be generalized in this direction. Finally, it can be shown that some of the results in Section 5 can be generalized to the case where the uncertainty is linear time invariant, similar to the results in [21], by imposing frequency content constraints on the disturbances.

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