The Mechanics and Control of Undulatory Robotic Locomotion

Thesis by
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Abstract

In this dissertation, we examine a formulation of problems of undulatory robotic locomotion within the context of mechanical systems with nonholonomic constraints and symmetries. Using tools from geometric mechanics, we study the underlying structure found in general problems of locomotion. In doing so, we decompose locomotion into two basic components: internal shape changes and net changes in position and orientation. This decomposition has a natural mathematical interpretation in which the relationship between shape changes and locomotion can be described using a connection on a trivial principal fiber bundle.

We begin by reviewing the processes of Lagrangian reduction and reconstruction for unconstrained mechanical systems with Lie group symmetries, and present new formulations of this process which are easily adapted to accommodate external constraints. Additionally, important physical quantities such as the mechanical connection and reduced mass-inertia matrix can be trivially determined using this formulation. The presence of symmetries then allows us to reduce the necessary calculations to simple matrix manipulations.

The addition of constraints significantly complicates the reduction process; however, we show that for invariant constraints, a meaningful connection can be synthesized by defining a generalized momentum representing the momentum of the system in directions allowed by the constraints. We then prove that the generalized momentum and its governing equation possess certain invariances which allows for a reduction process similar to that found in the unconstrained case. The form of the reduced equations highlights the synthesized connection and the matrix quantities used to calculate these equations.

The use of connections naturally leads to methods for testing controllability and aids in developing intuition regarding the generation of various locomotive gaits. We present accessibility and controllability tests based on taking derivatives of the connection, and relate these tests to taking Lie brackets of the input vector fields.

The theory is illustrated using several examples, in particular the examples of the snakeboard and Hirose snake robot. We interpret each of these examples in light of the theory developed in this thesis, and examine the generation of locomotive gaits using sinusoidal inputs and their relationship to the controllability tests based on Lie brackets.

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Chapter 1

Introduction

The methods by which both living creatures and robotic systems move through their environments are at once extremely complicated and highly commonplace. While it is quite easy to conjure up images of various modes of *locomotion*, such as legged walking, serpentine slithering, or wheeled rolling, there has been little work done in exploring the underlying structure which is common to many types of locomotion. Instead, researchers have focused on studying particular systems, or morphologies, in an attempt to derive strong results about these specific examples. The emphasis of this dissertation is not to replace this type of analysis by a general scheme, but instead to enhance it by providing a firm foundation upon which to analyze problems of locomotion within a unified framework. This framework can be greatly simplified by understanding and utilizing the extra structure inherent in these types of problems.

In the present day use of robots, most are either fixed in one place with an end effector that moves within a bounded workspace, or are based on some type of wheeled platform, which has its own type of environmental restrictions. However, nature has shown us that there are *many* more forms of locomotion than are presently used by roboticists. While some uses of robotic locomotion were studied in the early days of robotics, a recent trend has been towards incorporating these alternative modes of movement into our repertoire, especially as we move into more diverse and challenging environments. For instance, the study of robotic legged locomotion, which has been ongoing for at least the past thirty years, continues to develop, particularly in pursuit of the holy grail of legged motion— dynamic, bipedal walking and running (refer to [49, 82, 91] for excellent historical reviews on these developments). Legs provide a means for moving through untamed and unexplored environments which are not easily accessible with wheeled vehicles. Alternatively, researchers using snake robots have been able to expand our capabilities by providing machines that could explore fallen or severely damaged buildings (e.g., after an earthquake), narrow and winding nuclear waste storage facilities, or even the internal organs of human beings [22, 36].

Certainly, the study of new and interesting forms of locomotion continues to show great potential. As new fabrication techniques are invented, the number of potential robotic applications increases. For example, ultrasonic and piezoelectric motors have already found industrial applications, having been employed in driving the motion used to focus cameras [15], or using inch-worm-like steps to generate linear motion. They also are beginning to see usage as locomotive devices for very small scale mobile robots [3, 29, 39]. Continued advances in our ability to machine smaller and smaller mechanical parts has forced us to seek out new methods of moving these miniature robotic systems. The advances in microelectromechanical systems (MEMS) technology have been astonishing, and yet it is not possible with current technologies to build a wheeled micro-robot. While this achievement may yet occur, we should also expect to see alternative forms of locomotion being developed, such as walking, swimming [29], or even flying micro-robots.

With these thoughts of the future in mind, we turn now to review some of the previous work done in the area of locomotion. In particular, we are interested in locomotive systems which have potential applications in terms of robotic implementations. The desire here is not necessarily to give a comprehensive review of all of the locomotion literature, but instead to highlight those developments which have been important factors in motivating the current research.

1.1 A Survey of Locomotion

In engaging in the study of locomotion, a natural first step is to try to make inferences from living creatures found in nature. Following along these anthropomorphic lines, one of the original studies of non-traditional means for robotic locomotion was the study of snakes. Hirose [33, 36] first examined snake robots from a biologically inspired point of view. In doing so, he performed a series of experiments designed to investigate the existence of generic patterns of motion in snakes. By videotaping their locomotive patterns, he was able to gain a great deal of insight into this problem, and developed what he termed a *serpenoid* curve— a curve representing the path that a snake would trace out as it slithers forward. The serpenoid curve is characterized by a sinusoidal variation in curvature along its length, and is very suggestive of the serpentine path followed by real snakes. Hirose was also able to show theoretically that a snake that assumed this shape could generate a net forward force by applying torques along the length of its body. Using these results, he successfully built a snake-like robot capable of propelling itself forward using only internal torques (that is, without directly driving the wheels). The robot (see Figure 1.1), called the Active Cord Mechanism Model 3 (ACM III), consisted of a long chain of serially connected segments, each of which sat upon an actively controlled, rotating wheel base (the wheels are designed to act like the belly of a snake in preventing lateral slipping). With all of these developments, however, the control of the position and guidance of the snake robot remained a heuristically derived procedure, without the ability to give precise feedback control for this form of locomotion. The next generation of snake-like robots built by Hirose (called Koryu) were modified to allow each segment, or bay, to move vertically with respect to the neighboring segments, as well as to exert a rotational torque on its nearest neighbor [35]. Furthermore, the wheels in the Koryu robot were no longer allowed to move freely, but instead were controlled to move in unison. Thus, these robots were able to climb stairs and even cross over gaps in the floor (consider, for example, the need to cross a partially collapsed bridge).

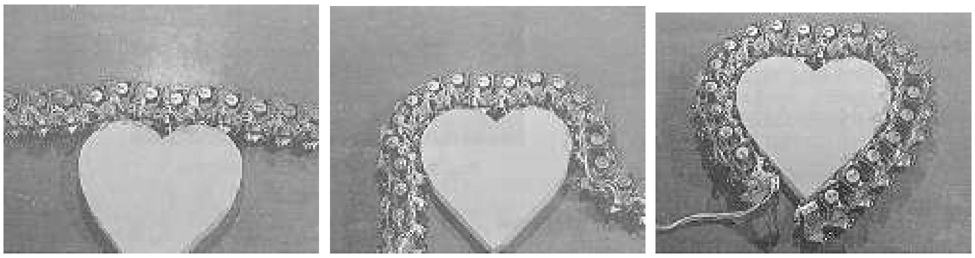

Figure 1.1 The Active Cord Mechanism (ACM III) [33] grasping an object

In [22], Chirikjian and Burdick coined the term *hyper-redundant* to describe robots which have a very large number of independent degrees of freedom. Naturally, snake robots fall into this category, and the optimization algorithms developed by Chirikjian and Burdick have been used quite successfully in a variety of applications, including stable grasping [22] (done by enveloping an object) and local sensor-based path planning/obstacle avoidance [23]. Possible implementations which are currently being developed include satellite grasping and retrieval, hazardous site inspection, including nuclear waste facilities and damaged buildings, and medical applications, such as laparascopic and endoscopic surgical procedures [90]. The initial theoretical motivations for Chirikjian and Burdick were to develop cost-efficient routines for solving the inverse kinematics problem, i.e., determining joint configurations given a specified end-point, or tool, configuration and possibly some type of energy or obstacle constraints. As such, they were really concerned with the *kinematics* of hyper-redundant robots. In investigating locomotion, our concern will be primarily with understanding the *dynamics* of this type of robotic motion. Certain types of snake locomotion, such as sidewinding [17] and inchworm-like motions [43], have previously been investigated, but the problem of dynamic snake locomotion reminiscent of Hirose's snakes has yet to be fully solved.

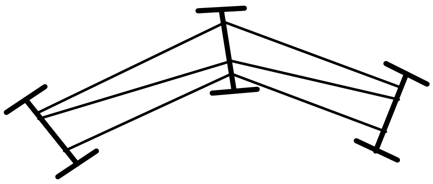

Figure 1.2 Variable Geometry Truss (VGT) assembly (two bays shown)

One particular research effort that has recently begun to address these issues is found in the work of Tsakiris and Krishnaprasad [48]. Using the same Variable Geometry Truss (VGT) mechanism (Figure 1.2) employed by Chirikjian and Burdick (which allows each bay to move in the plane with three degrees of freedom relative to its neighbor), they develop models that employ no-slip wheel constraints and can be used to generate locomotion patterns. They term these models "G-snakes," in reference to the notion that each segment must move within a constrained subset of a Lie group, G. In fact, they show that gaits (i.e., specified input patterns) can be explicitly integrated by quadratures to give the trajectories of the overall G-snake motion.

The framework used in establishing the governing equations for G-snakes in [48] is very similar to that introduced here, and some of the ideas developed by Tsakiris and Krishnaprasad will be used in our treatment of models based on Hirose's original snake robots. The intent here is not simply to rehash Hirose's ideas in a new geometric setting. Instead, the aim is to formulate the problem in terms of some of the intrinsic properties found in general modes of locomotion, and in the process gain an understanding of how to control particular gaits demonstrated by Hirose. In doing so, we hope to demonstrate theoretically how to implement locomotion schemes which to date were found purely heuristically, and to add additional gaits that may not have been realized in previous works.

Interestingly, all of the locomotive robots mentioned above have used *wheeled* approximations to snakes. The class of robots using no-slip wheel constraints (an example of a *nonholonomic*, or non-integrable, constraint) is quite large, and includes almost all mobile robots presently in use. Although there has recently been a growth in the study of "holonomic" (a true misnomer, if there ever was one), or omni-directional mobile robots [81], the bulk of mobile robots in existence use wheels, much like those found on cars or bicycles, to move through their environments. These nonholonomic systems have largely been treated as purely *kinematic* systems, i.e., the dynamics of these mechanical systems are assumed to be constrained in a manner such that only configuration *velocities* need be considered. This assumption is generally quite valid, and has led to some excellent progress in areas such as controllability [11], stabilization [20], and trajectory generation (including the N-trailer problem) [16, 74].

Within the context of locomotion, Kelly and Murray [43] have successfully modeled a large number of locomotive systems using kinematic constraints, with some strong results on controllability and motion generation. They have studied basic inch-worm and sidewinding gaits (using a viscous friction model), as well as some preliminary models of continuous-contact hexapodal walking. They provide results for determining controllability, as well as suggestions for the generation of locomotive walking patterns, or *gaits*. An important part of the structure of the equations they present, which we continue to develop here, is the division of the configuration variables into two classes— *shape* and *position* (see also [79]). In doing so, the aim is to divide locomotion according to its basic structure, by investigating the effect of internal shape or body changes on the generation of motion. Thus, we choose first a set of *position* variables, which describe the position and orientation of the body with respect to some inertial frame. Next, we pick from the remaining configuration variables the *shape* of the system. The cyclic variation of the shape variables induces locomotion. Obviously, it could be argued that in some systems there will be configuration variables which do not directly contribute to locomotion (e.g., the use of arms for posturing in humans). These variables, however, most likely are either used to affect a larger mode of locomotion— for this example, we can think of posturing as a rigid body orientation problem, instead of just a problem of moving in Cartesian coordinates— or they perhaps are extraneous variables in an analysis of locomotion. The goal of our research, then, is to determine useful ways of describing the relationship between shape changes and position changes, using a mathematical construction known as a *connection*.

While there have been great successes studying kinematically constrained systems, there are some systems for which the dynamic effect is *essential* to the motion of the system. These systems include the wobblestone [13, 18, 21] and the snakeboard [57]. For the most part, these *dynamic* nonholonomic systems have not been treated in the literature. Of notable exception is the work of Bloch, McClamroch, and Reyhanoglu [12], where control results were established, with the assumption that the unconstrained directions be fully actuated, and Bloch, Krishnaprasad, Marsden, and Murray [10], which has been a valuable reference and foundation for many of the results presented here. Interestingly, current research in the area of dynamic nonholonomic systems has led to an understanding of how to include within this formulation purely dynamical systems [10, 79] (i.e., systems with symmetries, but no external constraint forces). This is done by treating momentum conservation laws as a type of *internal* nonholonomic constraint. Applications of this approach have primarily been applied to problems in rigid body reorientation, such as the spinning satellite [19, 27, 76, 98] (Figure 1.3) and the falling cat [70]. We see that each of these problems can effectively be thought of as problems of *locomotion*, where the change in position is one of *orientation*. Notice that they quite readily decouple into a set of cyclical internal shape changes (e.g., the wriggling of a cat in mid-air), and a resulting net change in orientation.

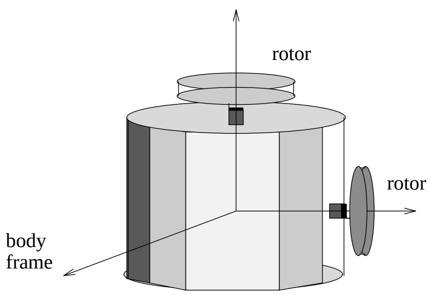

Figure 1.3 Satellite reorientation problem using (two) momentum wheels

Along with these interesting problems of reorientation, there are other locomotion problems that do not necessarily rely on wheel-based constraints. For instance, Shapere and Wilczek [87, 88] have studied the ability of a paramecium to swim through a highly viscous fluid medium using infinitesimal deformations of its external shape. This research relates directly back to the above mentioned structure as they have chosen to decouple the shape deformations from the inertial positioning, and describe locomotion as a net effect of cyclical changes in the internal shape of the body. In the same vein, other researchers have studied the self-propulsion of air bubbles as they propagate through a fluid medium [7, 67]. The means of propulsion here again stems from internal shape deformations leading to a change in position.

Finally, we give a brief review of a distinctly different form of locomotion, namely legged locomotion. This mode of transport will not be given much attention in this dissertation, but it is hoped that the results presented here can be adapted for use with legged robots. The research to date in legged locomotion has largely fallen into two categories: multi-legged (i.e., quadrupeds and hexapods) and bipedal. For the most part, multi-legged research has focused on systems which are *statically stable*. That is, the center of mass of the body is supported over a stable foundation of legs at all times, so that it is not possible for the robot to fall over. On the other hand, research into bipedal locomotion has sought to analyze *dynamically* stable legged locomotion, in which continuous control is necessary to keep from falling down. While some early research into bipeds employed statically (or *quasi-statically*) stable gaits [41, 42], and modern research by Raibert has used dynamic hopping gaits for multi-legged robots [82, 83], the research into legged locomotion has generally been clearly divided into these two regimes. For the most part, this division has also been based on the proposed utility of the legged machines. Multi-legged robots tend to be larger, with greater payloads, and are generally designed for all-terrain transport operations. Bipeds, on the other hand, are more often designed anthropomorphically, for use in exploring unsafe or dangerous environments, or as prosthetic devices. Of notable exception are more recent investigations into "insect"-like robots by Brooks and Beer [6, 28] and Raibert's hopping quadrupeds [83].

One of the earliest successful walking machines of the modern era was the "walking truck" built by Mosher at General Electric in the 1960's [58]. This machine was 11 feet tall, weighed 3000 lbs., and was directly controlled by a human operator. It is significant not only for its technical achievement, but also because it would turn out to be one of the last legged robots to be implemented without the use of computer control. One of the first quadrupeds to use digital control was the Phony Pony built by McGhee and Frank [66], which used basic flip-flops to coordinate the robot's two-DOF legs as a finite-state machine. McGhee was also involved in the building of one of the first truly successful computer-controlled legged machine—the Ohio State University Hexapod [94]. This robot has been used for a wide variety of experiments, including climbing stairs, using sensor feedback, and testing out different gaits.

Other multi-legged robots have included the Adaptive Suspension Vehicle at Ohio State [91], several hexapods built in Russia [78, 97], a computer-controlled walking machine made by Sutherland and capable of carrying a human [84], and several very efficient and cleverly designed quadrupeds built by Hirose et al. [34], just to name a few. Obviously, the study of multi-legged robots has a rich history and even today continues to grow and diversify.

One particular study of note, however, has been the hopping robots built by Raibert. Although his robots do not fall into the same framework as the more traditional, anthropomorphically-designed legged robots, they have had a significant impact on the study of legged locomotion. Interestingly, contributions have been made to both multi-legged and bipedal robots using the same fundamental control and design algorithms. Beginning with a basic "pogo-stick" design— a single-legged, vertical hopping robot— Raibert showed that it was possible to stabilize lateral hopping motions using a simple feedback control law [82]. Adding a second leg, he demonstrated running and even basic gymnastics [37]. With the addition of two more legs, he developed quadruped robots which could demonstrate several different gaits, using the concept of a virtual leg [83] (whereby the stabilization routines for the single-legged hopper could be extended to a four-legged machine).

While Raibert's robots must continuously hop in order to maintain stability, there are also several more traditional walking biped robots that have been built. The bulk of the research into bipeds has taken place in Japan, starting with Kato et al. in the early 1970's [41, 93]. This has been followed by several successful research programs, each with a different control philosophy and mechanical design: Miura and Shimoyama developed a 3D stilt-type biped series called Biper [68]; Miyazaki and Arimoto used a reduced-order model approach to control their walking machines [69]; Furusho and Sano used ankle torques to provide a kick action [30]; and Kajita and Tani restricted trajectories to potential energy conserving orbits [40] (the reader is referred to [30] for a very nice review of the biped literature).

Lastly, we mention the biped research of McGeer [65], which is of a very different variety than that described above. It has no computer control at all, but is instead a passive dynamic walker. Although technically a three-legged machine (it has two outside legs that move in unison and provide lateral stability), McGeer's passive walker demonstrates the feasibility of using the energy added by gravity (walking on a downhill slope) to maintain a stable dynamic biped gait. One of the many lessons learned from McGeer's work is that maintaining a stable walking pattern does not necessarily require large amounts of energy or control. While this is obviously not the solution to the full problem of bipedal locomotion, it certainly provides stimulus for deeper thought about the mechanics and control of biped locomotion.

One of the many important aspects of locomotion that has arisen from the study of legged systems is the notion of *gaits*. As a baseline definition, we will say that a gait is a cyclic pattern of internal shape changes that lead to a particular pattern of locomotion. For this work, we will further refine this definition by specifying a gait to be a class of cyclical shape changes with a characteristic set of frequencies and phasing. In this manner, the quadruped gait in which all legs move with the same frequency and phasing is still called a "pronk," regardless of the magnitude of the motion of the legs. Note, however, that this definition can present some difficulties in classification, most notably in the context of legged locomotion, where walking and running may use the same basic frequencies and relative leg phasing. It is for this reason that we include these two notions of a gait, each of which can play an important role in thinking about and beginning to understanding the patterns of locomotion. This distinction aside, we relate the underlying perspective of this work in relation to gaits— that the analysis of locomotion can be simplified by dividing locomotion into internal cyclical shape changes and net motions of the body resulting from these shape changes. From this point of view, gaits very naturally arise by simply changing the cyclic patterns of inputs to the system.

The study of gaits has a particularly long history, dating back at least as far as 1878 where Muybridge published a series of stop-motion photographs showing that in fact a horse does leave the ground while trotting. He went on to accumulate photos from many other animals; even today this compilation remains as a valuable resource for studying gaits and locomotion [75]. In the biological literature, there has been a continued interest in studying gaits found in many different species. For instance, Hildebrand [32] emphasized the symmetry which is present in many gaits. This can take the form of an obvious symmetry, such as the bounding gait of a quadruped (where pairs of legs move in phase), or more subtle symmetries, as are found in the various gallops of a horse [25] (in which pairs of legs move together, but slightly out of phase). The relative phasing of legs for different gaits was studied extensively by McGhee [66], who also examined the duration of the gait cycle for which the foot was in contact with the ground (which he called the duty factor), and Gambaryan [31], who represented patterns of locomotion using various graphical representations, including successive snapshot drawings of animals and their gaits.

One area of modern research into gaits that has particularly influenced our thinking has been the study of coupled nonlinear oscillators. Significant results have been derived regarding the natural oscillation patterns (interpreted as gaits) that occur when oscillators (mathematical models of central pattern generators (CPG's)) are coupled. Also, changing the parameter of the models can break these patterns and form new ones (representing a change of gait). While these studies ignore the mechanics of the actual problem in favor of studying a more neural-based approach, they are able to represent a wide variety of systems and their gaits. Some of these include snakes and fish, where the muscles are thought to be controlled locally by pairs of coupled oscillators spooled in a long chain, and multi-legged systems, where the phasing of each leg is driven by a single oscillator. For instance, Ashwin and Swift [4] generate results for a system of n-weakly coupled oscillators by looking at the spatio-temporal symmetries that arise. Kopell [47] and Rand et al. [85], on the other hand, focus their attention much more directly towards the modeling of living organisms, particularly using coupled oscillators to generate traveling waves moving along the spine of a snake or fish. We will see that this concept of a traveling wave will be very important in examining the locomotion patterns in the snake robots of Hirose. Finally, excellent literature reviews and research results on symmetries and symmetry-breaking of animal gaits can be found in a series of articles put together by Collins and Stewart [24, 25, 26]. The symmetries in this case are discrete symmetries (e.g., permutations), and so are not within the scope of this work. These papers, however, provide intuitive ideas about possible ways to extend the work in this thesis to include legged locomotion systems.

1.2 Theoretical Statement of Purpose

The mathematical purpose of this dissertation is to present new results in the study of mechanical systems with nonholonomic constraints. This statement of purpose is intended primarily for researchers with some familiarity with the nomenclature of differential geometry, particularly in the context of Lie groups and principal fiber bundles. A review of these ideas is given in Chapter 2. We restrict our attention to a class of systems possessing Lie group symmetries and evolving on a trivial principal fiber bundle. This work is an extension of results from geometric mechanics on the reduction of dynamical systems [60, 61], and in particular relies strongly and builds upon the exposition by Bloch, Krishnaprasad, Marsden, and Murray on nonholonomic mechanical systems with symmetry [10].

When people speak of nonholonomic constraints, the most frequently found usage, particularly for applications, is that of a linear velocity constraint, most often arising from some type of external forcing, e.g., wheel constraints or finger contacts. Usually, these systems are assumed to have unconstrained dynamics (most often the control inputs) that are expressible in terms of velocities only. Therefore, no actual dynamics (second derivatives of the configuration variables) appear in these formulations. As mentioned above, controllability results have been derived for systems which *do* involve full Lagrangian dynamics [12], but with the restriction that the unconstrained degrees of freedom be fully actuated. In this work we investigate a formulation of the dynamics that does not place this requirement on the unconstrained variables.

Along with externally applied nonholonomic constraints, we also examine what we call internal, or intrinsic, nonholonomic constraints that arise when the Lagrangian is invariant with respect to the action of a Lie group. These will be written as linear (or affine) velocity constraints, and often take the form of momentum conservation laws, e.g., conservation of linear and angular momentum for the rigid body. The process of reduction entails using the conservation laws to define a *connection* on a principal fiber bundle. The connection relates the dynamics in the symmetry (group) directions to the dynamics of the reduced space. Reduction then consists of writing the dynamics on a reduced space, and is coupled with a reconstruction process, where the motion in the reduced space is used to reconstruct the full dynamics of the original system. Thus, the system's dynamics may be analyzed on a lower dimensional space (often greatly simplifying the problem), without loss of any information. In addition, the reconstruction of the dynamics affords significant insight into the geometry of the problem and is often an invaluable tool for understanding the dynamics of physical systems. Some modern examples mentioned above include the problems of satellite reorientation and the falling cat.

It is well known that holonomic constraints will allow for the persistence of conserved quantities, and more recent studies have identified certain types of nonholonomic constraints which also allow for reduction to be performed using traditional methods [10, 45]. The nonholonomic constraints for which the reduction process has been previously studied can largely be broken down into two types. First, it is possible that the constraints are such that they preserve a sub-group of symmetries, and so reduction is performed using the invariances which correspond to this sub-group. This category is very closely related to the class of systems with holonomic constraints, and includes the rolling penny [12] and the ball on a rotating plate [10, 16]. Second, in the *principal kinematic* case [10] (these are often called *Chaplygin* constraints), the constraints are such that all of the group symmetries are annihilated. However, this case is characterized by the existence of the correct number of equations of constraint to replace the conservation laws and define a connection. While the additional structure that arises due to group symmetries is no longer of any use, a similar process of reduction and reconstruction can still be performed using the constraints directly [5, 43, 45].

Thus, the potential systems to which the reduction procedure can be applied forms a large spectrum— at one end is the principal kinematic case, and at the other end are unconstrained systems with internal (momentum) constraints. While the results at each end of the spectrum have been worked out, the aim here is to investigate the middle ground in which the constraints break some of the group symmetries, but do not by themselves define a connection. This type of problem is said to have *mixed constraints*, since both internal and external constraints will appear. It has been shown that for the mixed case there no longer exists a conserved quantity. Instead there exists a momentum along the remaining unconstrained directions, called the *generalized momentum* [10]. The flow of the generalized momentum is governed by a *generalized momentum equation*— a differential equation dependent on the interaction between the constraints and the Lie algebra of the symmetry group (first derived in [10]). The present research has evolved from a desire to investigate possible means of adapting previously existing methods of reduction to systems with nonholonomic constraints which break the group symmetries. The theory is formulated in such a manner as to apply for both constrained and unconstrained systems of this form.

The utility of maintaining a structure similar to that found in unconstrained systems is greater than simply to allow us to describe the system's dynamics on a reduced space. There is also an extensive literature devoted to analyzing certain aspects of reduced systems, including control and stabilization [9, 99], stability of equilibria [60, 89] , and the role of geometric phases in generating motion [61, 70, 88]. It is hoped that these results can be extended to aid in investigating the modified form presented below.

While the constraints *do* add an additional degree of complexity to the analysis, they also give rise to something which does not exist in unconstrained systems with symmetries— the ability to increase or control momentum. This can be an extremely important effect in generating locomotion. The role of the generalized momentum equation on locomotion is more clearly defined and discussed in Chapter 5, where various examples are investigated, including the *Snakeboard*(see [57] for a brief introduction) and the Hirose snake. In both cases, the constraints interact with the group action in a nontrivial manner to produce momentum changes which result in locomotion. This interaction of the constraints with natural group symmetries plays an integral role in defining a connection for these systems. The mathematical properties of a connection allow us to establish greatly simplified results for both the dynamics *and* control of locomotion systems. As we will see, the connection plays a very important role in the study of locomotion, where the reduced space is just the internal shape space of the locomotive system. Thus, it allows for the study of locomotion to be decomposed into the analysis of the dynamics on the shape (base) space, and the process of reconstruction in which the motion in the shape space produces desired locomotion of the body.

The layout of this thesis is as follows. In each case, we have tried to highlight the contributions being made to the existing theory. It should be noted that there are some similarities between parts of this work and that of Bloch et al. [10]. This is largely due to the fact that these two works were done in parallel, and often in conjunction, with each other. This thesis is meant to focus on the original work done by this author, though there is obviously overlap with [10] which is unavoidable.

Chapter 2 provides a brief introduction to the mathematical background and notation necessary to work with constrained systems having Lie group symmetries. This includes a discussion of Lie algebras and principal fiber bundles, which will be used to analyze the basic structure of locomotion. We also define the basic equations of motion to be used for both unconstrained and kinematically constrained systems.

The processes of reduction and reconstruction for the unconstrained case are described in Chapter 3, along with a few illustrative examples. The Lagrangian reduction process has been addressed previously in the literature, and to this we add an interpretation of reduction in terms of body coordinates that allows for the inclusion of invariant constraints, and a local formulation of the reduced equations in terms of simple matrix manipulations. In particular, we show that the local forms of the mechanical connection and the locked inertia tensor can easily be found directly from the matrix structure of the reduced Lagrangian (the function induced on the reduced state space by a group-invariant Lagrangian). Finally, we show that the dynamic constraints can be used with the constrained variational principle to perform reduction using a method which we can extend trivially to systems with external constraints.

Chapter 4 begins with a section on the assumptions being made within this dissertation, along with a constructive method for generating a basis for the constrained Lie algebra (the subspace of the Lie algebra that satisfies the constraints). Also in this chapter we introduce the generalized momentum, originally developed in [10]. First, an alternative development of the generalized momentum equation is given, which includes general external forcing. In the case where the external forces of constraint are themselves group invariant, certain invariances of the generalized momentum arise. Proofs are given to show that the generalized momentum and the generalized momentum equation satisfy certain invariance conditions and thus can be reduced to a lower dimensional base space. It is also shown that the symmetries of the generalized momentum equation can be used to rewrite it in a form which is quadratic in the momenta and the base velocities, and give explicit equations for its calculation. We discuss the construction of a nonholonomic connection (developed in [80] and [10]), and show that if the constraints are invariant, then one can always build this connection. Additionally, the steps necessary to perform the processes of reduction and reconstruction are developed.

In Chapter 5 two locomotion examples illustrating the theory are introduced, namely the snakeboard and a model of Hirose's snake robot. The *Snakeboard* [10, 57, 79, 80] (we use italics and capitals to distinguish between the model presented here and the product manufactured and distributed by Snakeboard USA, Inc.) is a commercially available variant of the skateboard in which the wheels are allowed to pivot freely. By coupling a twisting of the torso in phase with turning the wheels, a rider can effectively generate a snake-like locomotion pattern for this type of skateboard *without* having to kick off the ground. This effect is generated by a coupling of the angular momentum generated by twisting one's body with the forces of constraint generated by the wheels' contact with the ground. The snakeboard has been one of the motivating examples in the development of the theory, and provides a good perspective as to how this theory can be used to study general problems of locomotion. The second example is a theoretical model of some of the early snake robots built by Hirose. It consists of a series of connected segments, each of which sits upon an independently controlled wheel base (though again, the wheels themselves are not driven). In this example, we show that a three link robot kinematically fully specifies the motion of the snake, and describe how additional links can be added. We also give some initial results on three gait patterns, one of which is shown to closely follow the theoretical serpenoid curve of Hirose. We also give some discussion to the notion of picking out optimal gaits for these types of robots.

Finally, in Chapter 6 we develop some preliminary results for control of these types of systems. We begin with initial definitions of accessibility and controllability for nonlinear systems, and continue with a discussion of existing results for the kinematic case developed by Kelly and Murray. However, the systems of interest for this presentation are fully *dynamic*, not just kinematic, and so require the development of new controllability tests. We provide an analysis based on Sussman's conditions for small-time local controllability, and give sufficient conditions for establishing nonlinear accessibility and controllability.

Contributions

This dissertation seeks to provide a general theoretical framework within which a diverse set of locomotion problems can be analyzed. The goal is to demonstrate that there are certain structural properties common to many forms of locomotion that can be used to simplify the problem. To this end, we feel that there are four primary contributions to this work:

1. To give a thorough exposition of the process of reduction for locomotion systems (more generally, for nonholonomic mechanical systems with symmetries), and to show that this type of reduction can always be done when the constraints and Lagrangian function are group invariant,

2. To provide explicit details on the reduced equations and to put them in an easily computable form using a reduced mass-inertia matrix with correction (curvature) terms,

3. To demonstrate the theory using two new examples— the snakeboard and the

Hirose snake— and to give some intuition as to how locomotive gaits can be generated, and

4. To provide initial results on local controllability of locomotive systems which make use of the structure that is developed in the reduction process.

Chapter 2

Background

2.1 Symbols and Notation

We begin by establishing some of the notation used throughout this work. The basic notation and methodology is fairly standard within the geometric mechanics literature, and whenever possible we have attempted to use traditional symbols and definitions. Additionally, we have attempted to provide enough details and intuition in order to allow a reader familiar with [73] to be able to use these methods. The following symbols will be used frequently:

Q	: a smooth n-dimensional configuration manifold.
G	: an l-dimensional Lie group.
M	: the m-dimensional base space $= Q/G$.
$\mathfrak{X}(Q)$	: the set of vector fields over Q.
$\Phi_g : Q \to Q$	: the left action of the group G on Q, such that $\Phi_g(q) = g \cdot q$.
$\mathfrak{g}$	: the Lie algebra of G.
ξ_Q	: the infinitesimal generator for $\xi \in \mathfrak{g}$.
$L : TQ \to \mathbb{R}$	: a Lagrangian, which is simply a function on TQ.
$\langle\!\langle\ ,\ \rangle\!\rangle$	: an inner product based on the kinetic energy metric.
$\langle\ ;\ \rangle$	: the natural pairing between covectors and vectors.

We will also use indicial notation throughout this dissertation. This includes an implied summation whereby "up" (superscripted) indices are paired with "down" (subscripted) indices to form a sum. Thus, for appropriately indexed quantities, $\alpha_i v^i = \sum_i \alpha_i v^i$. This will also come up quite often when differentiating by indexed quantities: $\frac{\partial L}{\partial q^i} v^i = \sum_i \frac{\partial L}{\partial q^i} v^i$. Notice that an "up" index in the denominator becomes a "down" index for the purposes of summation. Whenever possible, we will use distinct types of indices to represent different ranges of summation. For instance, $a, b, c, \ldots$ will be used to index group or fiber variables, and so the index runs $1, 2, \ldots, l$. Similarly, $i, j, k, \ldots$ will be used for base variables, in which case the range is $1, \ldots, m$.

In referring to differential geometric objects such as manifolds, tangent bundles, and dual spaces, we follow the notation found in Boothby [14]. Let M be a $\mathcal{C}^\infty$-

manifold of dimension m, and let r be a point in M. The *tangent space* of M at r, denoted T_rM, is the linear vector space which best approximates M at r. The *tangent bundle*, denoted TM, is the disjoint union over M of each of its tangent spaces. We denote by $v_r \in T_rM$ a *tangent vector* at r, and define a (smooth) *vector field* X on M to be a smooth mapping $X : M \to TM : r \mapsto X_r$ which assigns to each point $r \in M$ a tangent vector $X_r \in T_rM$. The set of all vector fields over M is denoted $\mathfrak{X}(M)$.

Let $f : M \to N$ be a smooth mapping between manifolds M and N. Then we write $T_rf : T_rM \to T_{f(r)}N$ to denote the *tangent map* or *differential* of f. This is often seen in other notations as f_* (Boothby), Df, or df. We use the operator T because it allows for easy reference to the base point of the differentiation (i.e., T_r), and because it mimics the standard notation for tangent spaces.

Given a finite-dimensional tangent space, T_rM, we call the space whose elements are linear functions from T_rM to $\mathbb{R}$ the *dual space*, T_r^*M. Note that using similar definitions as above, we can take the disjoint union of T_r^*M over M to define the cotangent bundle and use this to define covector fields. An element $\omega \in T_r^*M$ is called a *dual vector*, or *covector*. As dual vectors are linear functions on T_rM, we will write the *natural pairing* of $\omega \in T_r^*M$ with $v_r \in T_rM$ as $\langle \omega; v_r \rangle = \omega(v_r)$. The tangent map given above, $T_rf : T_rM \to T_{f(r)}N$, uniquely determines a dual linear map (or just *dual*), $T_r^*f : T_{f(r)}^*N \to T_r^*M$, by the relation

$$\langle T_r^* f\omega; v_r \rangle = \langle \omega; T_r f v_r \rangle,$$

for all $v_r \in T_rM, \omega \in T_r^*M$. Note that if A is a matrix representation of a linear map, the dual map for A is just its transpose, i.e., $A^* = A^T$.

2.2 Lie Groups and Associated Structures

As mentioned in Chapter 1, one point of commonality among the various problems of locomotion and reorientation is that the motion of the system evolves on a simple spatial manifold, such as $SE(2)$ or $SO(3)$ (respectively, translation and rotation in the plane and spatial rotation of the rigid body). For example, locomotive systems like snakes, inch-worms, and paramecia can be modeled as simple planar objects, and hence move in $SE(2)$. We can also consider problems of rigid body reorientation on $SO(3)$, e.g., the falling cat or a spinning satellite, or even more complex locomotion such as birds and fish which move in $SE(3)$. These manifolds are all examples of Lie groups, and so we would like to make use of the mathematical structure inherent in working with Lie groups. While examples evolving in $SE(3)$ will not be presented here, the results we derive in this work are equally valid for any Lie group of symmetries, including $SE(3)$.

2.2.1 Lie Groups

Let G be a differentiable ($\mathcal{C}^\infty$) manifold which is at the same time a group. For $g, h \in G$, let hg denote the product of g and h.

Definition 2.1 *[14] The manifold G is said to be a* Lie group *if the product mapping, $hg : G \times G \to G$ and the inverse mapping, $g^{-1} : G \to G$ are both $\mathcal{C}^\infty$ mappings. We denote by e the* identity *element of G, such that $e = gg^{-1}$.*

Example 2.2 *As an example, we look at $SE(2)$, the group of rotations and translations in the plane. A point $g = (x, y, \theta) \in SE(2)$ can be represented using homogeneous coordinates:*

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}.$$

In doing so, the product of two elements in $SE(2)$ is given simply by matrix multiplication. Thus, the element $hg \in SE(2)$, for $h = (a^1, a^2, \alpha)$, is given by

$$hg = (a^1 + x\cos\alpha - y\sin\alpha, a^2 + x\sin\alpha + y\cos\alpha, \theta + \alpha).$$

Similarly, we can write down the inverse mapping as

$$g^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & -x\cos\theta - y\sin\theta \\ -\sin\theta & \cos\theta & x\sin\theta - y\cos\theta \\ 0 & 0 & 1 \end{pmatrix},$$

which is also a smooth operation.

Note that because matrix multiplication in general does not commute, multiplication by g on the right differs from multiplication by g on the left. Lie groups for which this is true are called *non-Abelian*, and naturally come equipped with two maps, $L_g : G \to G : h \mapsto gh$ and $R_g : G \to G : h \mapsto hg$, called respectively *left* and *right translation* (or *action*) of G on G. The terms "left" and "right" apply obviously to matrix groups such as $SE(2)$, as multiplication on the left and multiplication on the right.

Definition 2.3 *The* adjoint action *of G on G is defined to be the inner automorphism $\mathcal{I}_g : G \to G$ given by $\mathcal{I}_g h = L_g(R_{g^{-1}} h)$.*

The adjoint action in some ways measures the non-commutativity of the left and right actions. If H is an Abelian group, then the adjoint action reduces to the identity on H: $\mathcal{I}_h g = g$ for all $h, g \in H$. When considering motion along non-Abelian groups, a choice must be made as to whether to represent translation by the left or right action. The adjoint action forms a means of transforming between these two choices of representations. For the purposes of this dissertation, we will almost exclusively make use of the left action, though the results contained here can quite easily be formulated in terms of right actions.

2.2.2 Algebras

In discussing algebras, we consider objects over the field of real numbers, though definitions can be generalized over a commutative ring with a unit. We use Lang [50] as a reference.

Definition 2.4 *An* algebra *A is a vector space with a product satisfying*

$$a(uv) = (au)v = u(av)$$

for every $a \in \mathbb{R}$ and $u, v \in A$. A vector subspace I of A is called a left ideal *(respectively,* right ideal*) if for every $i \in I$, $ui \in I$ (resp., $iu \in I$) for all $u \in A$. A subspace I is said to be a* two-sided ideal *if it is both a left and right ideal.*

While an ideal is not necessarily a subalgebra, the quotient of an algebra by a two-sided ideal inherits a natural algebra structure from A.

2.2.3 The Lie Algebra of G

Associated with the Lie group, G, is a Lie algebra, $\mathfrak{g}$. We use Varadarajan [96] and Boothby [14] as references for the basic concepts on Lie algebras.

Definition 2.5 *A vector space $\mathfrak{g}$ over $\mathbb{R}$ is said to be a (real)* Lie algebra *if it possesses a* Lie bracket*— that is, a map*

$$(X, Y) \to [X, Y], \qquad X, Y, [X, Y] \in \mathfrak{g}$$

of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ satisfying the following:

1. *Bilinearity over $\mathbb{R}$:*

$$[\alpha^i X_i, \beta^j Y_j] = \alpha^i \beta^j [X_i, Y_j], \quad \text{for } \alpha^i, \beta^j \in \mathbb{R},$$

2. *Skew commutativity:*

$$[X, Y] = -[Y, X],$$

3. *The* Jacobi identity*:*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

Notice that Condition 2 implies that $[X, X] = 0$ for all $X \in \mathfrak{g}$. Let $f_1, \ldots, f_l$ be a basis for $\mathfrak{g}$ (as a vector space). Then the *structure constants* of $\mathfrak{g}$ relative to this basis are uniquely determined by

$$[f_a, f_b] = c^d_{ab} f_d.$$

It is easily shown that the Lie algebra is isomorphic to the tangent space of G at the identity, i.e., that $\mathfrak{g} \simeq T_e G$. The Lie group structure allows us to represent group velocities (i.e., vectors in T_gG) in terms of Lie algebra elements. This is done by pulling back group velocities to the identity using the *lifted* left action, $T_gL_{g^{-1}}$. In a similar manner, we can think of generating a left-invariant vector field on G by pushing a Lie algebra element forward using T_eL_g. Thus, if $\xi \in \mathfrak{g}$,

$$X_\xi(g) = T_eL_g\xi$$

defines a left-invariant vector field on G. We can associate with this vector field a curve in G. Let $\phi_\xi : \mathbb{R} \to G : t \mapsto \exp t\xi$ be the integral curve of X_ξ passing through e at $t = 0$. Thus, $\frac{d}{dt}(\phi_\xi)|_{t=0} = \xi$. The function $\exp : \mathfrak{g} \to G : \xi \mapsto \phi_\xi(1)$ is called the *exponential mapping* of $\mathfrak{g}$ into G.

For non-Abelian groups, the non-commutativity of the left and right actions implies that there are actually *two* natural ways to map $\mathfrak{g}$ to a vector field on G—one using the left action as above, and another using the right action. In the context of robotic manipulation, this arises when writing velocities as screws, in having to make a choice between body and spatial representations [73]. To be more explicit, let $\xi \in \mathfrak{g}$ represent a group velocity, $v_g \in T_gG$, which has been pulled back to the Lie algebra. We use superscripts "b" and "s" to denote body and spatial representations, respectively, which gives

$$\xi^b = T_gL_{g^{-1}}v_g \quad \text{and} \quad \xi^s = T_gR_{g^{-1}}v_g.$$

The relationship between spatial and body velocities can be written in terms of the adjoint action of G on $\mathfrak{g}$, which is determined by taking the tangent map of the adjoint action Ad_g on G. By an abuse of notation, this mapping is also labeled Ad_g.

Definition 2.6 *The* (lifted) adjoint action *of G on $\mathfrak{g}$ is defined to be the map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{Ad}_g\,\xi = T_{g^{-1}}L_g(T_eR_{g^{-1}}\xi)$ for $\xi \in \mathfrak{g}$.*

Thus, $\xi^b = \mathrm{Ad}_{g^{-1}}\,\xi^s$.

The adjoint action will come up frequently in what is to follow as a means of mapping between reduced representations of tangent vectors based on the left and right translations. The distinction between the adjoint action on G and that on $\mathfrak{g}$ will not be made textually, but should be clear from the context. When working with homogeneous coordinates of a rigid body, it will be useful to think of the adjoint action as providing a mapping from objects defined in terms of spatial coordinates to their alternative representation in terms of body coordinates. We will also see that the dual adjoint map Ad_g^* will come up in mapping between the two representations of dual elements to the Lie algebra. Traditionally, reduction methods have been formulated in terms of spatial coordinates, but we will see that the use of body coordinates can be quite valuable in certain examples.

2.2.4 Principal Fiber Bundles

Lie groups naturally arise in the study of locomotion as a means of describing position and orientation. As mentioned above, we choose the remaining variables to describe the internal shape of the system. The *shape space* has a natural mathematical interpretation as a quotient space, $M = Q/G$, and will often be referred to as the *reduced* or *base* space. The entire structure forms a *principal fiber bundle.* To define this type of structure, we first must describe the left (or right) action of G on Q.

Definition 2.7 *[62] A (*left*)* action *of a Lie group G on Q is a smooth mapping $\Phi : G \times Q \to Q$ such that:*

1. *$\Phi(e, q) = q$ for all $q \in Q$, and*
2. *$\Phi(g, \Phi(h, q)) = \Phi(L_g h, q)$ for all $g, h \in G$ and $q \in Q$.*

We will normally only be interested in the action as a mapping from Q into Q, and so will write the action as $\Phi_g : Q \to Q$, where $\Phi_g(q) = \Phi(g, q)$. As a shorthand, $\Phi_g(q)$ will often be written as $g \cdot q$, or just gq.

Definition 2.8 *An action is said to be* free *if it has no fixed points, i.e., if the relation $\Phi_g(q) = q$ implies $g = e$ for each $q \in Q$.*

Given the action of G on Q, along with the natural quotient space structure, we define a principal fiber bundle in the following manner.

Definition 2.9 *A* principal fiber bundle *over M with group G consists of a manifold Q and a free left action of G on Q satisfying the following:*

1. *M is the quotient space of Q defined by the G-induced equivalence relation, $M = Q/G$, where the* canonical projection *$\pi : Q \to M = Q/G$ is differentiable, and*
2. *Q is locally* trivial. *That is, every point $q \in Q$ has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $G \times U$. Thus, there exists a diffeomorphism $\psi : \pi^{-1}(U) \to G \times U$ given by $\psi(q) = (\varphi(q), \pi(q))$, for which $\varphi : \pi^{-1}(U) \to G$ satisfies $\varphi(\Phi_g q) = L_g \varphi(q)$ for all $g \in G$ and $q \in U$.*

A principal fiber bundle is commonly denoted by $Q(M, G)$, where Q is called the total space, *M the* base space, *and G the* structure group *or* fiber space.

The geometric structure found in problems of locomotion, however, is most often of a form that can be written *globally* as the product of the structure group and base space, i.e., as $Q = G \times M$. For this reason we restrict our attention to systems of this form, which are said to evolve on *trivial* principal fiber bundles.

Definition 2.10 *A* trivial principal fiber bundle *is a manifold $Q = G \times M$ such that G acts freely on Q on the left by trivially extending L_g to act on Q. Letting Φ_g denote the action of G on Q, this implies that $\Phi_h(g, r) = (L_h g, r)$, for $h \in G$ and $(g, r) \in G \times M = Q$.*

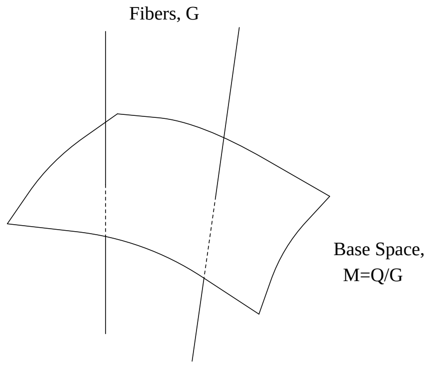

Figure 2.1 Trivial fiber bundle picture

In the trivial bundle case, we can easily picture Q as a base manifold M with *fiber*, G, attached at each point $r \in M$ (see Figure 2.1). This bundle comes naturally equipped with *two* canonical projections; namely, on the first and second factors, given by the maps $\pi_1 : Q \to G : (g, r) \mapsto g$ and $\pi_2 : Q \to M : (g, r) \mapsto r$.

Definition 2.11 *The* lifted action *is the map* $T\Phi_g : TQ \to TQ : (q, v) \mapsto (\Phi_g(q), T_q\Phi_g(v))$ *for all* $g \in G$ *and* $q \in Q$.

In the literature, the action of G on T_qQ is very often denoted $g\dot{q} = T_q\Phi_g(\dot{q})$, for $\dot{q} \in T_qQ$. Similarly, there will be times when we wish to express only the effect of the action on TG, which we denote as $h\dot{g} = T_gL_h\dot{g}$, for $\dot{g} \in T_gG$. Notice that the action on TG and the action on TQ are related by $T_q\Phi_h(\dot{g}, 0) = (T_gL_h\dot{g}, 0)$. We should also mention that the lifted action defined here is the same as the action denoted Φ_g^T by Abraham and Marsden in [1], when considered pointwise in G. Also defined in [1] is a very important quantity that provides an infinitesimal description of the action of Φ_g on Q.

Definition 2.12 *[1] If* $\xi \in T_eG$*, then* $\Phi^\xi : \mathbb{R} \times Q \to Q : (t, q) \mapsto \Phi(\exp t\xi, q)$ *is an* $\mathbb{R}$*-action on* Q*, that is,* Φ^ξ *is a flow on* Q*. The corresponding vector field on* Q *given by*

$$\xi_Q(q) = \frac{d}{dt}\Phi(\exp t\xi, q)\big|_{t=0}$$

is called the infinitesimal generator *of the action corresponding to* ξ.

A very important relationship to realize when working with trivial principal fiber bundles is that the infinitesimal generator is naturally the Lie algebra element pushed forward via the *right* action on G. To see this, let $\xi \in \mathfrak{g}$ and $q = (g, r) \in Q$. Then, $\Phi^\xi : \mathbb{R} \times Q \to Q : (t, (g, r)) \mapsto ((\exp t\xi)g, r)$, or $\Phi^\xi(t) = (R_g \exp t\xi, r)$. This implies that $\xi_Q(q) = (T_eR_g\xi, 0) \in T_qQ$.

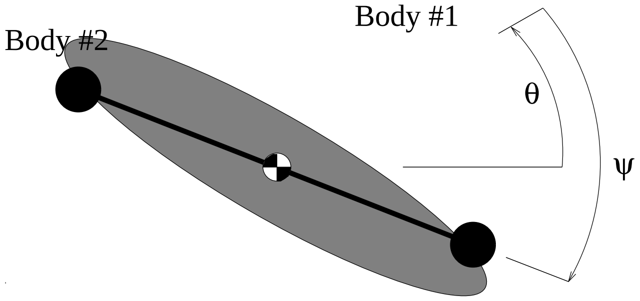

Figure 2.2 Elroy's beanie

2.2.5 Symmetries and Invariances

When we speak of a mechanical system with Lie group symmetries, we really mean that certain quantities, particularly the Lagrangian, remain invariant under the action of the Lie group, G. From a differential geometric viewpoint, this will manifest itself as invariance under the pull-back of the lifted action (see Marsden, Montgomery, and Ratiu [61] for more details). In the context of reduction, the primary assumption will be that the Lagrangian is invariant. This is a sufficient condition for establishing the existence of a conservation law for holonomically constrained systems. With the introduction of nonholonomic constraints, however, we see that the conservation laws *may* be broken. This necessitates our defining invariance for vector fields and one-forms, as these objects will be used to represent the nonholonomic constraints.

Definition 2.13 *A Lagrangian function, $L : TQ \to \mathbb{R}$, a vector field, $X \in \mathfrak{X}(Q)$, and a one-form, $\omega \in \mathfrak{X}^*(Q)$, are said to be G-*invariant *if they are invariant with respect to the lifted action, $T\Phi_g$, i.e., if, respectively, $L(\Phi_g(q), T_q\Phi_g v_q) = L(q, v_q)$, $T_q\Phi_g X(q) = X(\Phi_g(q))$, and $T_q^*\Phi_g\omega(\Phi_g(q)) = \omega(q)$, for all $g \in G, v_q \in T_qQ$.*

Invariance for vector fields and one-forms is perhaps most easily seen in terms of the following commutative diagrams.

$$\begin{tikzcd} Q \arrow[r, "X"] \arrow[d, "\Phi_g"'] & T_qQ \arrow[d, "T_q\Phi_g"] \\ Q \arrow[r, "X"'] & T_{gq}Q \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "\omega"] \arrow[d, "\Phi_g"'] & T_q^*Q \\ Q \arrow[r, "\omega"'] & T_{gq}^*Q \arrow[u, "T_{gq}^*\Phi_{g^{-1}}"'] \end{tikzcd}$$

Thus, for invariance of a vector field X and a one-form ω, respectively, the above diagrams must commute.

Example 2.14 Elroy's Beanie

To illustrate the ideas presented above, we examine a system consisting of two planar rigid bodies attached at their centers of mass, shown in Figure 2.2. This is perhaps the simplest example of a dynamical system with non-Abelian Lie group symmetries in which the configuration space is a fiber bundle over a nontrivial shape space. We will allow the rigid bodies to move freely in the plane (though still pinned together), and assume the existence of control torques between the two bodies (for the center of mass fixed in the plane, this problem is often referred to as Elroy's beanie).

Let $(x, y, \theta) \in SE(2)$ *be the position and orientation of the center of mass of body #1, and let* $\psi \in \mathbb{S}$ *be the relative angle between body #1 and body #2. Also, denote by* m *the total mass of the system, and* J *and* J_ψ *the inertias of body #1 and body #2, respectively. Thus, the configuration space is the fiber bundle* $Q = SE(2) \times \mathbb{S} = G \times M$*, with fiber coordinates* $g = (x, y, \theta)$*, and base coordinate* $r = \psi$*. The Lagrangian has no potential energy term, so*

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_\psi(\dot{\theta} + \dot{\psi})^2.$$

Notice that the Lagrangian is actually independent of the configuration variables. In such cases where the Lagrangian is solely a function of velocities, the position variables $g = (x, y, \theta)$ *are called* cyclic *variables (and here also* $r = \psi$ *is cyclic). In the case of cyclic variables, reduction is always possible using a Lie group with addition as its product (e.g., for* $g = (x, y, \theta)$*, we* could *use the group* $\mathbb{R}^2 \times \mathbb{S}$ *with addition). In this particular example, however, we can use a slightly more structured Lie group, namely* $SE(2)$*, the group of translations and rotations in the plane.*

First, we show that the Lagrangian is G*-invariant. The group action is that of* $G = SE(2)$ *on* Q*, similar to Example 2.2 above:*

$$\Phi_g(q) = \begin{pmatrix} a^1 + x\cos\alpha - y\sin\alpha \\ a^2 + x\sin\alpha + y\cos\alpha \\ \alpha + \theta \\ \psi \end{pmatrix} \tag{2.1}$$

and the lifted action is

$$T_q\Phi_g(\dot{q}) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \dot{x}\cos\alpha - \dot{y}\sin\alpha \\ \dot{x}\sin\alpha + \dot{y}\cos\alpha \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}, \tag{2.2}$$

where $g = (a^1, a^2, \alpha) \in SE(2)$*. Notice that since we are working with matrix groups, we can easily write the dual map* $T_q^*\Phi_g$ *as*

$$T_q^*\Phi_g = (T_q\Phi_g)^T = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A straightforward calculation shows that the Lagrangian is invariant:

$$\begin{aligned}
L(\Phi_g q, T_q\Phi_g \dot{q}) &= \frac{1}{2}m((\dot{x}\cos\alpha - \dot{y}\sin\alpha)^2 + (\dot{x}\sin\alpha - \dot{y}\cos\alpha)^2) \\
&\quad + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_\psi(\dot{\theta}+\dot{\psi})^2 \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_\psi(\dot{\theta}+\dot{\psi})^2 \\
&= L(q,\dot{q}).
\end{aligned}$$

Finally, we compute the adjoint action of G on $\mathfrak{g}$ by

$$\begin{aligned}
\mathrm{Ad}_g &= T_e(R_{g^{-1}}L_g) = T_g R_{g^{-1}} T_e L_g \\
&= \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

In order to represent the Lie algebra, let $f_1,\dots,f_l$ denote a basis for $\mathfrak{g}$, and $f^1,\dots,f^l$ be its dual (a basis for $\mathfrak{g}^*$). Then we can write an element of $\mathfrak{g}$ as $\xi = \xi^i f_i$, and compute the infinitesimal generator for ξ. For this example, we use the standard basis for $se(2)$ which arises by identifying the Lie algebra with T_eG, where G has coordinates (x,y,θ). The exponential mapping, which takes elements in $\mathfrak{g}$ to elements in G, is given by

$$\exp(t\xi) = \left(\frac{\xi^1}{\xi^3}\sin\xi^3 t + \frac{\xi^2}{\xi^3}(\cos\xi^3 t - 1), \frac{\xi^2}{\xi^3}\sin\xi^3 t + \frac{\xi^1}{\xi^3}(1-\cos\xi^3 t), \xi^3 t\right) \in G$$

(notice that for $\xi^3 = 0$, l'Hospital's rule implies that $\exp(t\xi) = (\xi^1 t, \xi^2 t, 0)$). Then we find that

$$\begin{aligned}
\xi_Q(q) &= \frac{\mathrm{d}}{\mathrm{ds}}[\Phi(\exp s\xi, q)]\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{ds}}[(x+\frac{\xi^2}{\xi^3})\cos\xi^3 s + (\frac{\xi^1}{\xi^3} - y)\sin\xi^3 s - \frac{\xi^2}{\xi^3}, \\
&\qquad (y - \frac{\xi^1}{\xi^3})\cos\xi^3 s + (\frac{\xi^2}{\xi^3} + x)\sin\xi^3 s + \frac{\xi^1}{\xi^3}, \theta + \xi^3 s, \psi]\Big|_{s=0} \\
&= (\xi^1 - \xi^3 y, \xi^2 + \xi^3 x, \xi^3, 0) = (T_e R_g \xi, 0),
\end{aligned}$$

where $\xi_Q \in \mathfrak{X}(Q)$.

2.3 The Euler-Lagrange Equations

We begin this section by looking at how to define a general Lagrangian system. We do this using the *integral Lagrange-d'Alembert principle*, which includes general forcing functions. For much of the discussion to follow, however, we will gain certain advantages by restricting ourselves to study only mechanical Lagrangian systems.

This should not seem too restrictive in the context of locomotion systems, which are most often described as mechanical (Lagrangian) systems.

Assume the existence of a Lagrangian function, $L(q, v)$, and a forcing function, $\tau(q, v)$, both on TQ. In order to specify the dynamics of the Lagrangian system associated with L, we use a variational principle to seek extrema of the function L integrated over possible paths. Thus, we will take variations along a curve $c : [a, b] \in \mathbb{R} \to Q$, with fixed endpoints, i.e., such that $\delta c(a) = \delta c(b) = 0$.

Proposition 2.15 *[62] A curve, $c(t)$, is said to satisfy the* ***integral Lagrange-d'Alembert principle*** *if*

$$\delta \int_a^b L(c(t), \dot{c}(t))dt + \int_a^b \tau(c(t), \dot{c}(t))\delta c\, dt = 0,$$

for any given variation δc that vanishes at the endpoints.

Applying the variation and simplifying the result using integration by parts allows us to express this principle in a differential form which will be more useful to us. This is the *Euler-Lagrange equations* with external forces, given in local coordinates $(q^i, \dot{q}^i)$ on TQ by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \tau_i.$$

This formulation of the equations of motion holds true for general Lagrangians. For the purposes of studying locomotion, though, it will often be useful to consider a subset of these general systems, namely *mechanical* systems [1]. A mechanical system is characterized by a decomposition of the Lagrangian into two terms: kinetic energy, T, and potential energy, V, such that $L = T - V$. The kinetic energy can be defined in terms of a metric function, $\mathbb{G}(q) : T_qQ \times T_qQ \to \mathbb{R}$ by

$$T = \mathbb{G}(q)(v_q, w_q) = \langle\langle v_q, w_q \rangle\rangle,$$

for $v_q, w_q \in T_qQ$ (hence forward we will drop the q-dependence of $\mathbb{G}$). The potential energy is any function on Q, and very often is used to represent conservative forces.

For a mechanical system, we can make the equations of motion slightly more explicit:

$$\ddot{q}^j = \mathbb{G}^{ij}\left(\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^i \partial \dot{q}^k}\dot{q}^k + \tau_i\right),$$

where $[\mathbb{G}^{ij}]$ is the inverse of the metric tensor, $\mathbb{G}_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$.

2.4 Noether's Theorem

The use of momentum conservation laws is central to our study of locomotion and reduction. The concept of a conservation law is a well-known physical principle that has a very nice mathematical interpretation in the context of Lie group symmetries. For unconstrained systems, the group invariance of the Lagrangian function, L, implies that configuration variables used to describe the unforced dynamics of the system are in some sense redundant. That is, they are more than sufficient to fully describe the trajectories of the system. Therefore, there is a conserved quantity— a constant of the motion— that allows us to factor out this redundancy in the process called reduction. The content of the conservation laws for Lie group symmetries is given by *Noether's theorem.*

Theorem 2.16 (Noether) *Given L, a G-invariant Lagrangian, the* **momentum mapping** $J : TQ \to \mathfrak{g}^*$*, given by*

$$\langle \mathrm{J}(v_q); \xi \rangle = \langle\!\langle v_q, \xi_Q \rangle\!\rangle, \ \forall\ \xi \in \mathfrak{g},$$

is a constant of the motion defined by the Euler-Lagrange equations.

Thus, given an initial velocity for the system, v_0, there is a related value of the momentum map, $\mu = \mathrm{J}(v_0) \in \mathfrak{g}^*$. If $c : [0, T] \to Q$ is the solution to the Euler-Lagrange equations with $c(0) = v_0$, and $c'(t)$ is the tangent to this curve at $c(t)$, then Noether's theorem implies that $J(c'(t)) = \mu$ for all $t \in [0, T]$. For an l-dimensional Lie group, this effectively implies that the conservation laws define l *internal* affine constraints on the system. And, since μ is fixed by the choice of initial condition, we see that the allowable dynamics of the system must exist within a well-defined affine subspace of TQ for all $t \in [0, T]$. For $\mu = 0$, this becomes a linear subspace (i.e., it contains the zero tangent vector), and further simplifications will occur. The process of reduction, in which the internally constrained dynamics are factored out, is taken up in Chapter 3 below.

2.5 Distributions and Frobenius' Theorem

As mentioned above, the conservation laws in systems with symmetries are affected in a nontrivial manner by the presence of nonholonomic constraints. Thus, we will need to develop some of the language of distributions used when working with constraints.

Definition 2.17 *Let Q be a differentiable manifold. A* distribution*, D, on Q is a subbundle of TQ. The dimension of D at $q \in Q$, denoted* $\dim D_q$*, is called the* rank *of D at q. If $m = \dim D(q)$ is constant in a neighborhood U of $q \in Q$, then we can write D_x in terms of m linearly independent $\mathcal{C}^\infty$-vector fields $X_1, \ldots, X_m$ for each $x \in U$, called a* local basis *of D.*

In anticipation of the controllability results in Sections 6.2 and 6.3, we also introduce Frobenius' theorem.

Definition 2.18 *A distribution D is said to be* involutive *if $[X, Y] \in D$ for each $X, Y \in D$. If N is a connected $\mathcal{C}^\infty$ submanifold of Q such that $T_qN \subset D_q$ for each*

$q \in N$, then we shall say that N is an integral manifold *of D. A distribution is said to be* integrable *if for each point $q \in Q$ there exists a local integral manifold $N \ni q$ such that $TN = D\big|_N$.*

Theorem 2.19 (Frobenius) *A distribution D on a manifold Q is integrable if and only if it is involutive.*

Frobenius' theorem states that integrability and involutivity are (locally) equivalent notions. The content of this theorem is very interesting when applied to driftless systems. It says that if a given set of control vector fields, $X_1, \ldots, X_m$, and all of its iterated Lie brackets form an integrable distribution, then the motion of the control system is locally restricted to a local integral manifold. Thus, on a manifold Q, it is possible to move from one point to any another point in a local neighborhood only if the integrable distribution spans TQ around that point.

2.6 Equations of Motion with Nonholonomic Constraints

In the presence of constraints, we must slightly modify the equations of motion in order to incorporate external constraint forces.

Definition 2.20 *A* constraint distribution *$\mathcal{D}$ is a distribution on Q composed of the allowable directions of motion at each point $q \in Q$, written $\mathcal{D}_q$. A curve $c : [a, b] \to Q$ is said to* satisfy the constraints *if $\dot{c}(t) \in \mathcal{D}_{c(t)}$, $\forall t \in [a, b]$.*

We will throughout this work require dim $\mathcal{D}$ to be constant over Q. Making this restriction allows us to write $\mathcal{D}$ as the kernel of a set of one-forms over Q. An interesting, open research question is the effect of allowing more general constraints, but it is unclear at present how to use the Lagrangian formulation in this setting. One possibility that is currently under investigation is the use of alternative formulations for the equations of motion, including the Gibbs-Appell equations [54].

We saw above that $\mathcal{D}$ can be expressed in terms of a local basis of vector fields. In order to formulate the equations of motion, however, it will prove easier to use an alternative method for representing $\mathcal{D}$, in terms of linear functionals on the velocities, or one-forms. Given k linear constraints, we can write them as a vector-valued set of k equations:

$$\omega^i_j(q)\dot{q}^j = 0, \qquad \text{for} \quad i = 1, \ldots, k, \tag{2.3}$$

where $\omega^1, \ldots, \omega^k$ have a natural interpretation as one-forms over Q. Let $dq^1, \ldots, dq^n$ be a basis for T^*Q, then the constraints can be written as $\omega^i = \omega^i_j dq^j$, and

$$\mathcal{D} = \{v \in TQ |\, \langle \omega^a; v\rangle \omega^a_j v^j = 0, \text{for } a = 1, \ldots, k\}.$$

We denote by $(L, \mathcal{D})$ the constrained system with Lagrangian L and constraint distribution $\mathcal{D}$. It now makes sense to describe the dynamics of $(L, \mathcal{D})$ by writing its equations of motions. To do so, we employ the following constrained variational principle.

Definition 2.21 *A curve, $c : [a,b] \to Q$, is said to satisfy the* Lagrange-d'Alembert equations *of motion for a constrained system if the following constrained variational principle holds:*

$$\delta \int_a^b L(c(t), \dot{c}(t))dt = 0,$$

where variations are taken over all possible curves, but with the restriction that $\delta c \in \mathcal{D}$ and that the resultant curve satisfy $\dot{c}(t) \in \mathcal{D}_{c(t)}$.

For this principle, we have chosen the method of taking the variations before applying the constraints. The alternative is to impose the constraints before taking the variation, thereby restricting the possible variations. This is known as the vakanomic principle, and the problems associated with this method are discussed by Lewis and Murray in [55].

Define coordinates for TQ as $(q^i, \dot{q}^i)$, for $i = 1, \ldots, n$. We can rewrite the equations of motion more explicitly by using the method of Lagrange multipliers. A good summary of how this can be done using the principle of virtual work is given in [53].

Definition 2.22 *A curve $c : [a,b] \to Q$ is said to satisfy the* nonholonomic constrained variational principle *if $\dot{c}(t) \in \mathcal{D}_{c(t)}$ for all $t \in [a,b]$ and*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(c(t), \dot{c}(t))\right) - \frac{\partial L}{\partial q^i}(c(t), \dot{c}(t)) = \lambda_a \omega_i^a + \tau_i. \tag{2.4}$$

This defines a set of n second order and k first order differential equations. For mechanical systems, the Lagrange multipliers, $\lambda_1, \ldots, \lambda_k$, can be solved for algebraically and eliminated in order to yield the equations of motion with constraints.

In order to establish results on momentum and reduction for systems with constraints, we would like to be able to eliminate the Lagrange multipliers without going through this process. One way to do this is to recognize $\omega^1, \ldots, \omega^k$ as one-forms, and recall that all vectors in the constraint distribution (i.e., all allowable velocities) will lie in the null space of these one-forms. Noting this, we rewrite Eq. 2.4 in a more natural geometric setting as a set of n one-forms:

$$\beta_L(q, \dot{q}, t) = \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t))\right) - \frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) + \lambda_a \omega_i^a - \tau_i\right\} dq^i, \tag{2.5}$$

where the λ_a's are at this point undetermined. The λ_a's naturally enter as scaling factors for the constraint one-forms. An interesting calculation is to show that the terms involving L which arise from the unconstrained Euler-Lagrange equations fit naturally in a geometric context as one-forms, since they transform through coordinate changes as such. The same is true of the forcing function τ, which is more traditionally thought of as a one-form. β_L provides an alternative, coordinate version for the equations of motion, written simply as:

Definition 2.23 *A curve, $c : [a, b] \to Q$, is said to satisfy the Lagrange-d'Alembert equations of motion for a constrained system if*

$$\beta_L(c, \dot{c}, t) = 0$$

and $\dot{c}(t) \in \mathcal{D}_{c(t)}\ \forall t \in [a, b]$.

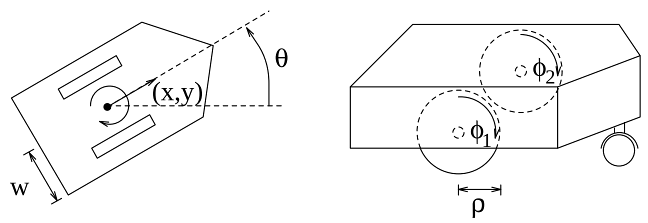

Figure 2.3 Two-wheeled planar mobile robot.

Example 2.24 Two-wheeled mobile robot

Consider the two-wheeled planar mobile robot presented by Kelly and Murray in [43] and shown in Figure 2.3. The robot's position, $(x, y, \theta) \in SE(2)$, is measured via a frame located at the center of the wheel base. The position of the wheels is measured relative to vertical and is denoted (ϕ_1, ϕ_2). Each wheel is controlled independently and is assumed to roll without slipping. The configuration space is then $Q = G \times M = SE(2) \times (\mathbb{S} \times \mathbb{S})$. Let m denote the mass of the robot, J its inertia about the center of mass, and J_w the inertia of each wheel about its pivot point. Then the Lagrangian is just

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_w(\dot{\phi}_1^2 + \dot{\phi}_2^2).$$

The group action is that of $G = SE(2)$ on Q, similar to Example 2.14 above:

$$\Phi_g(q) = \begin{pmatrix} a^1 + x\cos\alpha - y\sin\alpha \\ a^2 + x\sin\alpha + y\cos\alpha \\ \alpha + \theta \\ \phi_1 \\ \phi_2 \end{pmatrix} \qquad (2.6)$$

and the lifted action is

$$T_q\Phi_g(\dot{q}) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}\cos\alpha - \dot{y}\sin\alpha \\ \dot{x}\sin\alpha + \dot{y}\cos\alpha \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}, \tag{2.7}$$

where $g = (a^1, a^2, \alpha) \in SE(2)$. A straightforward calculation like the one above for Elroy's beanie (Example 2.14) shows that the Lagrangian is invariant.

The constraints defining the no-slip condition can be written as in Eq. 2.3:

$$\begin{aligned} \dot{x}\cos\theta + \dot{y}\sin\theta - \frac{\rho}{2}(\dot{\phi}_1 + \dot{\phi}_2) &= 0 \\ -\dot{x}\sin\theta + \dot{y}\cos\theta &= 0 \\ \dot{\theta} - \frac{\rho}{2w}(\dot{\phi}_1 - \dot{\phi}_2) &= 0, \end{aligned} \tag{2.8}$$

and similar calculations readily show that the constraints are also G-invariant. Using Lagrange multipliers, we can write the dynamical equations as

$$\begin{aligned} m\ddot{x} + \lambda_1\cos\theta - \lambda_2\sin\theta &= 0 \\ m\ddot{y} + \lambda_1\sin\theta + \lambda_2\cos\theta &= 0 \\ J\ddot{\theta} + \lambda_3 &= 0 \\ J_w\ddot{\phi}_1 - \frac{\rho}{2}\lambda_1 - \frac{\rho}{2w}\lambda_3 &= \tau_1 \\ J_w\ddot{\phi}_2 - \frac{\rho}{2}\lambda_1 + \frac{\rho}{2w}\lambda_3 &= \tau_2, \end{aligned}$$

along with the constraint equations given by Eqs. 2.8. Differentiating Eqs. 2.8, solving for the unknown multipliers, and eliminating the group variables yields the reduced base equations:

$$(J_w + \frac{J\rho^2}{4w} + \frac{m\rho^2}{4})\ddot{\phi}_1 + (-\frac{J\rho^2}{4w} + \frac{m\rho^2}{4})\ddot{\phi}_2 = \tau_1 \tag{2.9}$$

$$(-\frac{J\rho^2}{4w} + \frac{m\rho^2}{4})\ddot{\phi}_1 + (J_w + \frac{J\rho^2}{4w} + \frac{m\rho^2}{4})\ddot{\phi}_2 = \tau_2. \tag{2.10}$$

For the mobile robot, this process of eliminating the Lagrange multipliers is straightforward, but non-trivial. In more complicated examples, the resulting equations for the base space can become rather unwieldy. This provides additional motivation for us to re-examine the elimination of these extra variables and unknown multipliers using techniques based on the geometry of the problem.

Notice that Eqs. 2.8–2.10 fully specify the motion of the two-wheeled mobile robot. This example illustrates what are called Chaplygin, or principal kinematic, constraints, in which there are the same number of constraints as the dimension of the Lie group, and for which it is possible to invert the group velocities in terms of the base variables. The division of variables given by these equations is precisely the division that we would like to have when describing locomotion. The shape variables are separated out in a manner that clearly highlights their independence from the group variables (Eqs. 2.9 and 2.10), and by which it is easy to show that they are controllable given the proper input torques. At the same time, Eq. 2.8 can be inverted in the following manner, which makes explicit the dependence of the group variables on the shape variables:

$$T_g L_{g^{-1}} \dot{g} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = -\begin{pmatrix} \frac{\rho}{2} & \frac{\rho}{2} \\ 0 & 0 \\ \frac{\rho}{2w} & -\frac{\rho}{2w} \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

Notice that the changes in position are described by first order differential equations as defined by the kinematic constraints. Written in this form, we see that the constraint equations explicitly define the role of shape changes in generating net motion along the fiber— that is, in generating locomotion. When we write the equations in the form of Eq. 2.11, we are implicitly defining a *connection* for this trivial principal fiber bundle. The definition and usage of a connection in the setting of general locomotion systems is the subject of Chapters 3 and 4.

Chapter 3

Lagrangian Reduction in the Absence of Constraints

The process of Lagrangian reduction in the unconstrained case consists of splitting the dynamics of the system according to its symmetries and then reducing the system to a lower dimensional space in which the symmetries have been modded out. In this chapter, we discuss methods for performing this splitting on a trivial principal fiber bundle. For comparison, we briefly review in Section 3.3 the traditional reduction procedure for unconstrained systems (for a more thorough presentation, the reader is referred to [60, 61, 63]), and then in Section 3.4 develop an alternative procedure which makes use of the body coordinate representation. This alternate form can be quite useful in the presence of body (left-invariant) forces, and, as we show in the sequel, generalizes quite easily to the addition of invariant nonholonomic constraints arising in problems of locomotion.

3.1 Connections on Principal Fiber Bundles

We seek a splitting of the dynamics that is compatible with the Lie group symmetries, and which naturally highlights the structure of the shape space for particular locomotion systems. Thus, we must initially define two subspaces: one which contains the group directions and one which encodes the pertinent information regarding the internal shape of the system (and, as we will see, may include information regarding the constraints). The process of reduction, then, consists of modding out by the group and reducing to a lower dimensional shape space. For a general principal fiber bundle, $Q(M,G)$, there is a natural way of defining the set of vectors tangent to the group orbits.

Definition 3.1 *The* vertical subbundle *(also called the* fiber distribution*) is the subbundle of TQ defined by*

$$VQ = \bigcup_{q \in Q} V_q Q = \bigcup_{q \in Q} \{ v_q \in T_q Q \mid v_q \in \ker T_q \pi_2 \}.$$

Vectors in VQ are said to be vertical.

In other words, VQ is the disjoint union over Q of each subspace of T_qQ which is tangent to the fiber (and hence each $v_q \in V_qQ$ is in the kernel of the projection to the base space). For a trivial bundle these are all vectors of the form $(v_g, 0)$, for $v_g \in T_{\pi_1(q)}G$.

For a general principal fiber bundle, there is no canonical way to define a complementary space to VQ. Thus, depending on the problem we may have different criteria for choosing a *horizontal subbundle*. For example, in a trivial principal fiber bundle, we can naturally define a horizontal subbundle by choosing vectors tangent to the base manifold, i.e., vectors of the form $v_q = (0, v_r)$ with nonzero components only in TM. Alternatively, for systems which possess a metric, e.g., mechanical systems, horizontal vectors are chosen to be orthogonal (with respect to the metric) to vertical vectors. In the presence of constraints, however, these are not necessarily the most appropriate choices to make. In the Chaplygin or principal kinematic case, the horizontal distribution consists simply of those vectors which satisfy the constraints. For systems with mixed nonholonomic constraints (that is, external kinematic and internal momentum constraints), however, the choice is not so clear. In Section 4.5 an alternative choice of horizontal will be discussed for systems with mixed nonholonomic constraints.

For the purposes of locomotion, the vertical distribution really describes the *net* velocities of the rigid body, moving as a whole. Thus, a purely vertical motion would represent a change in inertial position, with no accompanying change in internal shape. For the mobile robot example above, the vertical subbundle consists of all vectors whose last two components are zero. This corresponds to pure rotation and translation in the plane, without any spinning of the wheels. Suppose that we made the choice of horizontal according to the natural division into base and fiber spaces given by a trivial fiber bundle, i.e., using the splitting $TQ = TG \times TM$. Then, for the mobile robot horizontal vectors would be those involving only a pure rotation of the wheels. As a constrained system, however, this decoupling of the variables does not make good physical sense, since the robot cannot move in the plane without moving the wheels, and vice versa. In fact, it is exactly this notion which leads us to choose a different horizontal subbundle, or *connection*, which more appropriately reflects the interaction between the wheel (shape) motion, and the movement along the vertical (fiber) distribution. The interaction between shape and position changes is implicitly defined by the constraints, and so we will build a connection based on the information encoded by the external constraints.

Mathematically, the connection defines the relationship between the tangent bundle on the base space and a G-invariant (horizontal) subbundle of TQ. It provides the means to "lift" vectors from TM (shape velocities) to the appropriate vectors in TQ (shape plus position velocities) during reconstruction. Although the definition of a connection is not really standard within the literature, we follow a very common definition given by Kobayashi and Nomizu [44].

Definition 3.2 *A* connection *is an assignment of a* horizontal subbundle*, $H_qQ \subset T_qQ$, for each point $q \in Q$ such that*

1. $T_qQ = V_qQ \oplus H_qQ$,

2. *$T_q\Phi_g H_q Q = H_{g\cdot q}Q$, for every $q \in Q$ and $g \in G$, and*

3. *H_qQ depends smoothly on q.*

The direct sum of condition (1) implies that T_qQ can everywhere be divided into the vertical subspace given by V_qQ and a *horizontal subspace*, given by H_qQ. Every vector $v \in T_qQ$ can then be uniquely written in terms of its horizontal and vertical decomposition: $v_q = h_q + w_q$, where $h_q \in H_qQ$ and $w_q \in V_qQ$. Define in this manner the maps **ver** and **hor** to be the projections onto the vertical and horizontal subspaces, respectively. Using the decomposition $v = h + w$ (assumed to hold pointwise over Q), this implies simply that $h = \mathbf{hor}\, v$ and $w = \mathbf{ver}\, v$. Another benefit of working with connections is that the horizontal subspace defined by the connection is everywhere isomorphic to the tangent space of the base space: $H_qQ \simeq T_{\pi_2(q)}M$. The *horizontal lift* maps vectors in $T_{\pi_2(q)}M$ to their corresponding lifted vectors in $H_q \subset T_qQ$ under this identification.

We have defined the connection as a G-invariant horizontal distribution that is complementary to VQ. An alternate definition that is often used (and encodes the same information) is given by the *connection one-form*.

Definition 3.3 *A* principal connection one-form, $\mathcal{A}$, *is a Lie algebra-valued one-form on Q satisfying the following properties:*

1. *$\mathcal{A}(q) \cdot \xi_Q(q) = \xi$, $\xi \in \mathfrak{g}$, and*

2. *$\mathcal{A}(\Phi_g q) \cdot T_q\Phi_g \dot{q} = \mathrm{Ad}_g\, \mathcal{A}(q) \cdot \dot{q}$.*

Condition 1 implies that $\mathcal{A}$ takes vectors in T_qQ to the Lie algebra elements associated with their vertical components. Thus, it extracts from a vector the components in the group direction, relative to some choice of horizontal. This is possible because for each vertical vector $w_q \in V_qQ$, there is associated a unique Lie algebra element, $\xi \in \mathfrak{g}$, such that ξ generates w_q. Recalling the definition of an infinitesimal generator, this means that

$$w_q = \xi_Q(q) = \frac{d}{ds}\left(\Phi(\exp(s\xi), q)\right)|_{s=0}, \tag{3.1}$$

where $\exp : \mathfrak{g} \to G$ is the exponential mapping of a Lie algebra element to the corresponding element in its Lie group (see Boothby [14] for more details). The connection one-form, then, takes the vector, $v_q \in T_qQ$, and returns the Lie algebra element associated with the vertical component of v_q, namely $\xi = \mathcal{A}(v_q)$, where $\xi_Q(q) = \mathbf{ver}\, v_q$. Note that the connection one-form is defined so that $\xi \in \mathfrak{g}$ implies $\mathcal{A}(\xi_Q) = \xi$. Conversely, if $h_q \in H_qQ$, then $\mathcal{A}(h_q) = 0$, i.e., the connection evaluates to zero on horizontal vectors.

An interesting simplification occurs when working with a connection on a trivial principal fiber bundle. In this setting, one can always write the connection in a simple local form which is very illuminating (used in [10]). Let $q = (g, r) \in G \times M = Q$. Then we see that the information encoded in a connection can be distilled down to a map $A(r) : T_rM \to \mathfrak{g}$, as described in the following proposition.

Proposition 3.4 *Let $\mathcal{A}$ be a principal connection one-form on $Q(M,G)$. Then $\mathcal{A}$ can be written in a local trivialization as*

$$\mathcal{A}\cdot\dot{q} = \mathrm{Ad}_g(g^{-1}\dot{g} + \mathrm{A}(r)\dot{r}),$$

where $g^{-1}\dot{g}$ is understood to imply the lifted action of g^{-1} on $\dot{g}\in T_gG$ given by $T_gL_{g^{-1}}$. We call A *the* ***local form*** *of $\mathcal{A}$.*

Proof: Recalling the above conditions for a connection one-form, Condition 1 implies that $\mathcal{A}(g,r)\cdot(\dot{g},0) = \dot{g}g^{-1}$. Thus

$$\mathcal{A}(q)\cdot(\dot{g},\dot{r}) = \dot{g}g^{-1} + B(g,r)\dot{r}.$$

Furthermore, Condition 2 implies that

$$\mathrm{Ad}_h\,\dot{g}g^{-1} + \mathrm{Ad}_h\,B(g,r)\dot{r} = h\dot{g}g^{-1}h^{-1} + B(hg,r)\dot{r},$$

or

$$\mathrm{Ad}_h\,B(g,r) = B(hg,r).$$

Setting $h=g^{-1}$ gives the desired result, with $B(g,r) = \mathrm{Ad}_g\,B(e,r) = \mathrm{Ad}_g\,A(r)$. ■

Notice that the heart of the above formula is the local form of the connection, A. While A is itself independent of g, it is mapped through the adjoint action in order to write the result in spatial coordinates. For this reason, we remark that the body coordinate representation can be important for writing this one-form, since the information encoded by the connection can be fully decoupled from the fiber variables. If we denote the body form of the connection by $\mathcal{A}^b$, then

$$\mathcal{A}^b = \mathrm{Ad}_{g^{-1}}\mathcal{A} = g^{-1}\,\mathrm{d}g + \mathrm{A}(r)\,\mathrm{d}r.$$

3.2 The Momentum Map and Mechanical Connection One-Form

There are three primary elements involved in the reduction procedure for unconstrained systems with symmetries: the momentum map, the locked inertia tensor, and the connection one-form. These quantities are all standard to the reduction literature. The reader will notice, however, that in this presentation we also define body coordinate representations of the momentum map and the connection one-form, which will be used in the derivations below. We recall here for convenience the definition of the momentum map given above in Noether's theorem.

Definition 3.5 *For a mechanical system, the* momentum map *is defined to be the map,* $\mathrm{J}: TQ\to\mathfrak{g}^*$ *which satisfies the following:*

$$\langle \mathrm{J}(v_q);\xi\rangle = \langle\langle v_q,\xi_Q\rangle\rangle,$$

for all $\xi \in \mathfrak{g}$ *and* $v_q \in T_qQ$.

We can also define the momentum map in body coordinates:

Definition 3.6 *The* body momentum map *is defined to be the map,* $\mathrm{J}^b : TQ \to \mathfrak{g}^*$ *which satisfies the following:*

$$\begin{aligned}\langle \mathrm{J}^b(v_q); \xi\rangle &= \langle\!\langle v_q, (\mathrm{Ad}_{g^{-1}}\,\xi)_Q(gq)\rangle\!\rangle \\ &= \langle\!\langle v_q, T_{gq}\Phi_{g^{-1}}\xi_Q(gq)\rangle\!\rangle,\end{aligned}$$

for all $\xi \in \mathfrak{g}$ *and* $v_q \in T_qQ$.

Body coordinates are not normally used for performing reduction, but in the presence of constraints, it appears to be a useful reference frame to use. Obviously the two are related, via the transformation:

$$\mathrm{J}^b = \mathrm{Ad}_g^*\,\mathrm{J}$$

Noether's theorem states that for unconstrained systems the momentum map in spatial coordinates is conserved along trajectories. This will not be true in general for the body momentum map, except in the case that G is Abelian, for which $\mathrm{Ad}_g^* = \mathrm{id}$ (the identity mapping). We remark, however, that in the special case that the spatial momentum is zero, i.e., when $\mu = \mathrm{J}(\dot{q}) = 0$, the body momentum defined by $p = \mathrm{J}^b = \mathrm{Ad}_g^*\,\mu$ is also zero *and* is also constant (shown below in Section 3.4.3). Thus, for unconstrained systems with zero momentum, the body and spatial representations of the equations are almost identical. The choice of which form to use in this case is purely based on convenience of representation.

Mechanical systems naturally possess a metric on TQ, implicitly defined by the kinetic energy. The existence of such a metric allows one to define the *mechanical connection* for systems with symmetries. The mechanical connection one-form is related to the momentum map via the *locked inertia tensor*, defined as follows:

Definition 3.7 *The* locked inertia tensor *is the map,* $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ *which satisfies*

$$\langle \mathbb{I}(q)\xi; \eta\rangle = \langle\!\langle \xi_Q, \eta_Q\rangle\!\rangle$$

for all $\xi, \eta \in \mathfrak{g}$.

The connection one-form— in this case, a *mechanical connection*— is given by

$$\mathcal{A} = \mathbb{I}^{-1}(q)\,\mathrm{J},$$

and satisfies all of the properties of the connection one-form given above. Again, we will define a dual quantity taken in body coordinates, namely

$$\mathcal{A}^b = \mathbb{I}^{-1}(q)\,\mathrm{J}^b\,.$$

Notice that the locked inertia tensor used here is the same in both body and spatial coordinates. This is due to the fact that the locked inertia tensor is Ad_g-invariant, in the sense that

$$\mathrm{Ad}_g^* \, \mathbb{I}(\Phi_g q) \, \mathrm{Ad}_g = \mathbb{I}(q).$$

The one-forms $\mathcal{A}$ and $\mathcal{A}^b$ are also related in a manner similar to the momentum maps, via the adjoint mapping:

$$\mathcal{A}^b = \mathrm{Ad}_{g^{-1}} \mathcal{A}.$$

It should be noted, however, that the Lie algebra valued one-form $\mathcal{A}^b$ is *not* a connection one-form, since when evaluated on infinitesimal generators it does not return the Lie algebra element that generates it, i.e., $\mathcal{A}^b \cdot \xi_Q = Ad_{g^{-1}}\xi \neq \xi$. It will arise, though, when building a connection for systems with constraints.

Example 3.8 Elroy's Beanie *(cont.)*

We return to the example of Elroy's beanie in order to illustrate the concepts that arise when building a connection. Recall from above that the Lagrangian is G-invariant. This suggests the existence of a conserved quantity, and so we will formulate the momentum map and mechanical connection for this system. We would also like to investigate the distinction between body and spatial coordinates, and so recall the adjoint action given above:

$$\mathrm{Ad}_g = \begin{pmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{pmatrix}.$$

Also shown above, the infinitesimal generator for $\xi = \xi^i f_i \in \mathfrak{g}$ is

$$\xi_Q(q) = (\xi^1 - \xi^3 y, \xi^2 + \xi^3 x, \xi^3, 0).$$

Let $\mathrm{J} = \mathrm{J}_i f^i \in \mathfrak{g}^*$, and then compute the momentum map as

$$\begin{aligned} \langle \mathrm{J}(v_q); \xi \rangle = \mathrm{J}_i \xi^i &= \langle\!\langle v_q, \xi_Q(q) \rangle\!\rangle \\ &= (v_x, v_y, v_\theta, v_\psi) \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & J + J_\psi & J_\psi \\ 0 & 0 & J_\psi & J_\psi \end{pmatrix} \begin{pmatrix} \xi^1 - y\xi^3 \\ \xi^2 + x\xi^3 \\ \xi^3 \\ 0 \end{pmatrix} \\ &= \xi^1(mv_x) + \xi^2(mv_y) + \xi^3(mxv_y - myv_x + (J + J_\psi)v_\theta + J_\psi v_\psi), \end{aligned}$$

where $v_q = (v_x, v_y, v_\theta, v_\psi)$. We find, then, that

$$\mathrm{J}(v_q) = mv_x f^1 + mv_y f^2 + (mxv_y - myv_x + (J + J_\psi)v_\theta + J_\psi v_\psi) f^3.$$

The body momentum map is given by

$$\mathrm{J}^b(v_q) = \mathrm{Ad}_g^* \, \mathrm{J}(v_q) = T_e L_g^* (mv_x, mv_y, (J + J_\psi)v_\theta + J_\psi v_\psi).$$

Next, to determine the mechanical connection for this system, we calculate an expression for the locked inertia tensor:

$$\mathbb{I} = \begin{pmatrix} m & 0 & -my \\ 0 & m & mx \\ -my & mx & J + J_\psi + mx^2 + my^2 \end{pmatrix}$$

Then the mechanical connection is easily computed to be

$$\begin{aligned} \mathcal{A}(v_q) &= \mathbb{I}^{-1}(q)\,\mathrm{J}(v_q) \\ &= (v_x + y(v_\theta + \frac{J_\psi}{J+J_\psi}v_\psi),\, v_y - x(v_\theta + \frac{J_\psi}{J+J_\psi}v_\psi),\, v_\theta + \frac{J_\psi}{J+J_\psi}v_\psi), \end{aligned}$$

while in the body representation this becomes:

$$\begin{aligned} \mathcal{A}^b(v_q) &= \mathrm{Ad}_{g^{-1}}\,\mathcal{A} \\ &= (v_x\cos\theta + v_y\sin\theta, -v_x\sin\theta + v_y\cos\theta, v_\theta + \frac{J_\psi}{J+J_\psi}v_\psi) \\ &= T_eL_{g^{-1}}v_g + (0, 0, \frac{J_\psi}{J+J_\psi}v_\psi), \end{aligned}$$

where $v_g = (v_x, v_y, v_\theta)$ and $T_eL_{g^{-1}}v_g \in \mathfrak{g}$. Thus, the local form of the connection is just

$$A = \begin{pmatrix} 0 \\ 0 \\ \frac{J_\psi}{J+J_\psi} \end{pmatrix}.$$

Note that while this is a very simple expression, it is enough information to completely encode the connection for this problem. In order to give some interpretation of how this information is encoded, let us look at the connection evaluated on trajectories with initial momentum equal to zero. For this case, the fiber equations become

$$g^{-1}\dot{g} = -A(r)\dot{r} = -\begin{pmatrix} 0 \\ 0 \\ \frac{J_\psi}{J+J_\psi} \end{pmatrix}\dot{\psi}.$$

Thus, the internal constraints given by the connection imply that a rotation of the rotor (given by a nonzero value for $\dot{\psi}$) yields an opposite motion in the body representation of the θ direction (scaled by $\frac{J_\psi}{J+J_\psi}$). This implies that the velocity of the rotor (body #2) and the central body (body #1) are directly coupled via $\dot{\theta} = \frac{J_\psi}{J+J_\psi}\dot{\psi}$. In more general locomotion problems, the connection will be used to encode more complex interactions with the shape variables and with the momentum of the system.

3.3 Reduction Using the Routhian

The reduction procedure for an Abelian group can be traced back to Routh (1860), but the full method for the non-Abelian case was not developed until much more recently [10, 60, 63]. The purpose of reduction will be to drop the dynamical equations to the quotient space, here just the base space, $M = Q/G$. By doing this, the total dynamics for the system will be encoded in a set of differential equations evolving solely on the base space. Let us examine how this will be done.

First, we express the Lagrangian in terms of kinetic and potential parts:

$$L(q, v_q) = \frac{1}{2}\langle\langle v_q, v_q\rangle\rangle - V(q). \tag{3.2}$$

Next, recall that we can decompose any vector into horizontal and vertical components: $v = \mathbf{hor}\, v + \mathbf{ver}\, v = \mathbf{hor}\, v + (\mathcal{A}(v))_Q$, where $(\mathcal{A}(v))_Q$ is vertical by definition. Using the definition for J, we can rewrite this as $v_q = \mathbf{hor}\, v_q + (\mathbb{I}^{-1}\mathrm{J}(v_q))_Q$. Substituting into Eq. 3.2 and expanding gives

$$L(q, v_q) = \frac{1}{2}\langle\langle \mathbf{hor}\, v_q, \mathbf{hor}\, v_q\rangle\rangle + \frac{1}{2}\langle\langle(\mathbb{I}^{-1}\mathrm{J}(v_q))_Q, (\mathbb{I}^{-1}\mathrm{J}(v_q))_Q\rangle\rangle - V(q).$$

Noether's equation for the unconstrained case implies that $\mathrm{J} = \mu = \text{const}$ along trajectories, so we will restrict orbits to those which lie in the level set given by $\mathrm{J} = \mu \in \mathfrak{g}^*$. If $\mu = 0$, then we see that the Lagrangian can be written solely in terms of horizontal vectors, since $\mathcal{A} = \mathbb{I}^{-1}\mathrm{J} = 0$ when evaluated along trajectories. As such, the variational principle will drop directly through the quotient, to give dynamical equations on M.

In the case where $\mu \neq 0$, we must perform a momentum shift in order to allow us to drop the variational principle down to the base space. As such, we expect some correction factors to enter into the variational principle.

Again, we restrict to a level set, $\mathrm{J} = \mu$, where now μ can be any element in $\mathfrak{g}^*$. Define the Routhian, R^μ, to be

$$\mathrm{R}^\mu = L(q, v_q) - \langle\mu; \mathcal{A}(v_q)\rangle.$$

The momentum map associated with the Routhian is identically zero:

$$\begin{aligned}
\langle\frac{\partial \mathrm{R}^\mu}{\partial v}; \xi_Q\rangle &= \langle\frac{\partial L}{\partial v} - \frac{\partial\langle\mu; \mathcal{A}(v)\rangle}{\partial v}; \xi_Q\rangle \\
&= \langle\frac{\partial L}{\partial v}; \xi_Q\rangle - \langle\langle\mu; \mathcal{A}(\cdot)\rangle; \xi_Q\rangle \\
&= \langle\mathbb{F}L(v); \xi_Q\rangle - \langle\mu; \mathcal{A}(\xi_Q)\rangle \\
&= \langle\mathrm{J}(v); \xi\rangle - \langle\mu; \xi\rangle \\
&= \langle\mu - \mu; \xi\rangle = 0,
\end{aligned}$$

for all $\xi \in \mathfrak{g}$. Furthermore, by performing the same substitution done above for the general Lagrangian, with $v_q = \mathbf{hor}\, v_q + \mathcal{A}(v_q)_Q$, and restricting to a level set where

$\mu = \text{const}$, we find that

$$\mathrm{R}^\mu = \frac{1}{2}\langle\langle \mathbf{hor}\, v, \mathbf{hor}\, v \rangle\rangle - V^\mu,$$

where $V^\mu = V(q) - \frac{1}{2}\langle \mu; \mathbb{I}^{-1}\mu \rangle$. A straightforward calculation shows that

$$\frac{d}{dt}\left(\frac{\partial \mathrm{R}^\mu}{\partial \dot{q}^i}\right) - \frac{\partial \mathrm{R}^\mu}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \left(\frac{\partial \mathcal{A}^k_j}{\partial q^i} - \frac{\partial \mathcal{A}^k_i}{\partial q^j}\right)\mu_k \dot{q}^j.$$

Since $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$, the reduced variational principle can be written as

$$\frac{d}{dt}\left(\frac{\partial \mathrm{R}^\mu}{\partial \dot{q}^i}\right) - \frac{\partial \mathrm{R}^\mu}{\partial q^i} = \beta^k_{ij}\mu_k \dot{q}^j, \tag{3.3}$$

where

$$\beta^k_{ij} = \frac{\partial \mathcal{A}^k_j}{\partial q^i} - \frac{\partial \mathcal{A}^k_i}{\partial q^j}.$$

The β^k_{ij}'s are directly related to the components of the curvature (exterior derivative) of the connection one-form, and appear naturally in the reduced equations for the base space as correction terms.

Example 3.9 The Rigid Body

Let us recall the classical example of the rigid body, free to rotate in space ($SO(3)$). This example will serve to illustrate various concepts to follow, particularly in contrast to constrained systems. This is merely provided as an illustrative example, and so the full details are not provided here. For these details, the reader is referred to [60, 64].

Let $R \in SO(3)$ denote the configuration of the rigid body with respect to some reference frame. As alluded to above, for motion on a non-Abelian Lie group, we can naturally write the velocity in terms of the Lie algebra in two different ways. The spatial angular velocity *of the body, $\omega \in so(3)$, can be written as a skew symmetric matrix,*

$$\hat{\omega} = \dot{R}R^{-1}.$$

The spatial angular velocity is the instantaneous angular velocity of the body as viewed from an inertial reference frame. We can also write this velocity as an element of $\mathbb{R}^3$ using the relationship $\hat{\omega}v = \omega \times v$, for $v \in \mathbb{R}^3$. The spatial angular velocity corresponds to the velocity of the rigid body measured in the coordinates of an external frame.

The body angular velocity, *$\hat{\Omega} = R^{-1}\dot{R}$, is related to the spatial velocity by the adjoint action, which for $\Omega, \omega \in \mathbb{R}^3$ can be written as*

$$\Omega = \mathrm{Ad}_{R^{-1}}\, \omega = R^{-1}\omega.$$

The body angular velocity basically corresponds to the angular velocity of a reference frame attached to the rigid body with respect to some inertial reference frame, but written in coordinates of the body frame. Using the locked inertia tensor $\mathbb{I}$*, we can write the Lagrangian for the rigid body as* $L = \frac{1}{2}\langle \mathbb{I}R^{-1}\omega; R^{-1}\omega\rangle = \langle \mathbb{I}\Omega; \Omega\rangle$*. Similar to above, we define the* body *angular momentum,* $\Pi = \mathbb{I}\Omega$*, and the* spatial *angular momentum,* $\pi = \mathrm{Ad}^*_R \Pi = R\Pi$*. From these, we have two means of expressing the equations of motion for the rigid body.*

The first of these describes the well known conservation law for a spinning body in the absence of external forces:

$$\dot{\pi} = 0.$$

The second form is simply the Euler equations written in terms of the body angular momentum:

$$\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi. \tag{3.4}$$

For unforced systems, the first set of equations will obviously be most useful, and will define the reduction and reconstruction process. Namely, reduction will consist of restricting to the level set, $\pi = \mathrm{const}$*. Reconstruction, then, will require solving for the motion of the rigid body using the first order ODE's given by*

$$\dot{R}(t) = \pi R(t).$$

The reason for highlighting the alternative formulation in terms of the body momentum is that it will be the form which is most convenient to use in the presence of invariant constraints. Also, we see that the body form of the momentum equation is independent of the group variables and so provides an alternative means of studying the dynamics in the presence of body forces, e.g., a satellite with thrusters instead of internal rotors. Thus, both forms will play an important role in reduction, depending on the type of problem being solved.

3.4 Reduction Using the Constrained Lagrangian

3.4.1 The Reduced Lagrangian

In order to do reduction using body coordinates, we begin by reducing to the space induced by modding out the group, i.e., the space $TQ/G = \mathfrak{g} \times TM$. As coordinates on this space, we will use $(\xi, r, \dot{r})$, where $\xi = \xi^b = g^{-1}\dot{g}$. Given the invariance of the Lagrangian, it is easy to see that we can immediately rewrite it in terms of these partially reduced coordinates.

Definition 3.10 *The* reduced Lagrangian *is the function* $l : TQ/G \to \mathbb{R}$ *induced by a* G*-invariant Lagrangian function, given by*

$$l(\xi, r, \dot{r}) = L(g^{-1}g, r, \dot{r}, g^{-1}\dot{g}).$$

In the case of mechanical systems, where $L(v_q) = T(v_q) - V(q) = \frac{1}{2}\langle\langle v_q, v_q\rangle\rangle - V(q)$, Murray [72] describes a splitting of l that enables one to write down the local forms of the locked inertia tensor and connection directly by looking at the reduced mass-inertia matrix. We use body coordinates to write down the reduced Lagrangian, although we will see that this can also be done easily using spatial coordinates. In doing so, we present a splitting of the bundle structure that diagonalizes the reduced mass-inertia matrix. The aim here is to present several options, with the thought that each one may be an important representation, dependent on the problem at hand.

Proposition 3.11 *Given L to be G-invariant, the reduced Lagrangian can be written as*

$$l(r, \dot{r}, \xi) = \frac{1}{2}(\xi^T, \dot{r}^T)\begin{pmatrix} I & IA \\ A^T I & m(r) \end{pmatrix}\begin{pmatrix} \xi \\ \dot{r} \end{pmatrix} - V(r). \tag{3.5}$$

We will call this decomposition of the reduced Lagrangian the (left) invariant decomposition *of l.*

Proof: We begin by writing L using a matrix representation of the metric on Q:

$$L(g, r, \dot{g}, \dot{r}) = \frac{1}{2}(\dot{g}^T, \dot{r}^T)\begin{pmatrix} \mathbb{G}_{11} & \mathbb{G}_{12} \\ \mathbb{G}_{21} & \mathbb{G}_{22} \end{pmatrix}\begin{pmatrix} \dot{g} \\ \dot{r} \end{pmatrix} - V(g, r),$$

with $\mathbb{G}_{12} = \mathbb{G}_{21}^T$.

Recalling that $\mathrm{J} = \mathbb{I}\mathcal{A}$ and $\mathcal{A}(\dot{g}, \dot{r}) = \xi^s + \mathrm{Ad}_g\, A(r)\dot{r}$, we have

$$\dot{g}^T \mathbb{G}_{11}\dot{g} = \langle\langle(\dot{g}, 0), (\dot{g}, 0)\rangle\rangle = \langle \mathrm{J}(\dot{g}, 0); \xi^s\rangle = \langle \mathbb{I}\mathcal{A}(\dot{g}, 0); \xi^s\rangle = \langle \mathbb{I}\xi^s; \xi^s\rangle.$$

Similarly for $\mathbb{G}_{21}$,

$$\dot{r}^T \mathbb{G}_{21}\dot{g} = \langle\langle(0, \dot{r}), (\dot{g}, 0)\rangle\rangle = \langle \mathrm{J}(0, \dot{r}); \xi^s\rangle = \langle \mathbb{I}\mathcal{A}(0, \dot{r}); \xi^s\rangle = \langle \mathbb{I}\,\mathrm{Ad}_g\, A(r)\dot{r}; \xi^s\rangle.$$

Then, setting

$$m(r) = \mathbb{G}_{22}(e, r), \text{ and } V(r) = V(e, r),$$

we have

$$l^s(r, \dot{r}, \xi^s) = \frac{1}{2}((\xi^s)^T, \dot{r}^T)\begin{pmatrix} \mathbb{I} & \mathbb{I}\,\mathrm{Ad}_g\, A \\ A^T\,\mathrm{Ad}_g^*\,\mathbb{I} & m \end{pmatrix}\begin{pmatrix} \xi^s \\ \dot{r} \end{pmatrix} - V(r),$$

since $\mathrm{Ad}_g^* = \mathrm{Ad}_g^T$ for matrix manipulations.

We can then convert to a body representation using the adjoint relation $\xi = \xi^b = \mathrm{Ad}_{g^{-1}}\,\xi^s$ and the local form of the locked inertia tensor, $I(r) = \mathbb{I}(e, r) = \mathrm{Ad}_g^*\,\mathbb{I}\,\mathrm{Ad}_g$:

$$l = l^b(r, \dot{r}, \xi) = \frac{1}{2}(\xi^T, \dot{r}^T)\begin{pmatrix} I & IA \\ A^T I & m \end{pmatrix}\begin{pmatrix} \xi \\ \dot{r} \end{pmatrix} - V(r).$$

Finally, we remark that by defining $\Omega = \xi + A\dot{r}$, the invariant decomposition of l takes on a block diagonal form:

$$l^{\Omega}(r, \dot{r}, \Omega) = \frac{1}{2}(\Omega^T, \dot{r}^T) \begin{pmatrix} I & 0 \\ 0 & m - A^T I A \end{pmatrix} \begin{pmatrix} \Omega \\ \dot{r} \end{pmatrix} - V(r).$$

■

The term $m - A^T I A$ will appear again below as the reduced mass-inertia matrix that naturally arises when writing Lagrange's equations on the base space.

3.4.2 The Reduced Nonholonomic Variational Principle

In formulating the reduced equations for a system with nonholonomic constraints, we will find it convenient to employ a reduced version of the equations found using the nonholonomic variational principle. Since the equations defining the momenta can also be considered to be a nonholonomic constraint, we find that this variational principle can then be used to perform reduction for unconstrained mechanical systems with symmetries.

Given a set of constraint one-forms, $\omega^1, \ldots, \omega^k$, we can split these one-forms using the trivial bundle structure, as

$$\omega^a = \omega^a_b dg^b + \omega^a_i dr^i, \quad b = 1, \ldots, l, \ i = 1, \ldots, m.$$

Similarly, we can divide the forcing function into $\tau = \tau_b dg^b + \tau_i dr^i$.

Recall the nonholonomic constrained variational principle defined above,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \omega^a_i + \tau_i.$$

We can use the invariance of the Lagrangian to write these equations on the partially reduced space of $\mathfrak{g} \times TM$.

Proposition 3.12 *The* **reduced nonholonomic constrained variational principle** *on* $\mathfrak{g} \times TM$ *is given by*

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi^a}\right) - \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi^a} = \lambda_c \omega^c_b g^b_a + \tau_b g^b_a \tag{3.6}$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{r}^i}\right) - \frac{\partial l}{\partial r^i} = \lambda_c \omega^c_i + \tau_i, \tag{3.7}$$

where g^b_a *denotes the coordinate version of the lifted left action* $T_e L_g$ *on* $T_e G = \mathfrak{g}$, *and* ad^*_ξ *is the dual of the* **adjoint action of** $\mathfrak{g}$ **on** $\mathfrak{g}$ *such that* $\mathrm{ad}_\xi \eta = [\xi, \eta]$ *and* $\mathrm{ad}^*_\xi p = \langle p; [\xi, \cdot] \rangle$.

Proof: We show this in coordinates, using the following calculations:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}^a}\right) &= \frac{d}{dt}\left(\frac{\partial l}{\partial \xi^b}\frac{\partial \xi^b}{\partial \dot{g}^a}\right) \\
&= (g^{-1})^b_a \frac{d}{dt}\frac{\partial l}{\partial \xi^b} + \frac{d}{dt}(g^{-1})^b_a \frac{\partial l}{\partial \xi^b} \\
&= (g^{-1})^b_a \frac{d}{dt}\frac{\partial l}{\partial \xi^b} + \frac{\partial (g^{-1})^b_a}{\partial g^c}\dot{g}^c \frac{\partial l}{\partial \xi^b} \\
&= (g^{-1})^b_a \frac{d}{dt}\frac{\partial l}{\partial \xi^b} + \frac{\partial (g^{-1})^b_a}{\partial g^c} g^c_d \xi^d \frac{\partial l}{\partial \xi^b},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial g^a} &= \frac{\partial l}{\partial \xi^b}\frac{\partial \xi^b}{\partial g^a} \\
&= \frac{\partial\left((g^{-1})^b_c \dot{g}^c\right)}{\partial g^a}\frac{\partial l}{\partial \xi^b} \\
&= \frac{\partial (g^{-1})^b_c}{\partial g^a} g^c_d \xi^d \frac{\partial l}{\partial \xi^b}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
[\xi, \eta]^b &= (g^{-1})^b_a [g^c_d \xi^d, g^f_e \eta^e]^a \\
&= (g^{-1})^b_a \left(\frac{\partial g^a_e \eta^e}{\partial g^c} g^c_d \xi^d - \frac{\partial g^a_d \xi^d}{\partial g^c} g^c_e \eta^e\right) \\
&= (g^{-1})^b_a \left(\frac{\partial g^a_e}{\partial g^c} g^c_d - \frac{\partial g^a_d}{\partial g^c} g^c_e\right) \xi^d \eta^e.
\end{aligned}$$

where $(g^{-1})^b_a$ represents the lifted action of G on T_gG such that $\xi^b = (g^{-1})^b_a \dot{g}^a \in \mathfrak{g}$.

We see from these calculations that the Euler-Lagrange equations in the group directions can be rewritten as

$$(g^{-1})^b_a \frac{d}{dt}\frac{\partial l}{\partial \xi^b} + \frac{\partial (g^{-1})^b_a}{\partial g^c} g^c_d \xi^d \frac{\partial l}{\partial \xi^b} - \frac{\partial (g^{-1})^b_c}{\partial g^a} g^c_d \xi^d \frac{\partial l}{\partial \xi^b} = \lambda_\alpha \omega^\alpha_a + \tau_a. \tag{3.8}$$

If we multiply through by g^a_e and use the identity

$$0 = \frac{\partial g^a_e (g^{-1})^b_a}{\partial g^c} = g^a_e \frac{\partial (g^{-1})^b_a}{\partial g^c} + \frac{\partial g^a_e}{\partial g^c}(g^{-1})^b_a \implies g^a_e \frac{\partial (g^{-1})^b_a}{\partial g^c} = -\frac{\partial g^a_e}{\partial g^c}(g^{-1})^b_a,$$

then Eq. 3.8 becomes

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^e} - (g^{-1})^b_a \frac{\partial g^a_e}{\partial g^c} g^c_d \xi^d \frac{\partial l}{\partial \xi^b} + (g^{-1})^b_a \frac{\partial g^a_d}{\partial g^c} g^c_e \xi^d \frac{\partial l}{\partial \xi^b} = \lambda_\alpha \omega^\alpha_a g^a_e + \tau_a g^a_e.$$

Finally, we recognize the second and third terms on the left-hand side of this equation as the dual of the adjoint action acting on $\frac{\partial l}{\partial \xi}$, since

$$\left(\mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi}\right)_e = \left(\frac{\partial l}{\partial \xi}[\xi, \cdot]\right)_e = \frac{\partial l}{\partial \xi^b}\xi^d \left[(g^{-1})_a^b \left(\frac{\partial g_e^a}{\partial g^c} g_d^c - \frac{\partial g_d^a}{\partial g^c} g_e^c\right)\right].$$

Using this, the result follows directly, since the equations on the base space remain unchanged when using the reduced Lagrangian. (As a brief aside, note that these lengthy expressions give the structure constants for the Lie algebra, as

$$C_{de}^b = (g^{-1})_a^b \left(\frac{\partial g_e^a}{\partial g^c} g_d^c - \frac{\partial g_d^a}{\partial g^c} g_e^c\right),$$

where $[\xi, \eta]^b = C_{de}^b \xi^d \eta^e$.) ■

Given the constraints as a principal connection on $Q(M, G)$ (for example, as the mechanical connection or derived from Chaplygin constraints), we can eliminate the Lagrange multipliers in order to write the base equations explicitly. We will use the body coordinate form of the connection:

$$\omega^c = (g^{-1})_a^c dg^a + \mathbb{A}_i^c dr^i.$$

Notice here that we have written the local form of the connection as $\mathbb{A}$ to represent a connection based on external constraints. In the sequel it will be important to distinguish between the local form of the mechanical connection and the local form of the connection that arises due to external constraints, and so we will use the symbols A and $\mathbb{A}$, respectively.

Also, we assume that the forces are written as one-forms, which are also G-invariant. Given this, we can divide τ as

$$\tau = \tau_c^e (g^{-1})_b^c dg^b + \tau_i dr^i, \tag{3.9}$$

where τ_c^e is just τ_b pulled back to the group identity.

Proposition 3.13 *Given a system $(L, \mathcal{D})$ with a connection on Q and G-invariant forces, τ, the reduced equations on TM can be written as*

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{r}^i}\right) - \frac{\partial l}{\partial r^i} = \left[\frac{d}{dt}\left(\frac{\partial l}{\partial \xi^b}\right) - \mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi^b}\right] \mathbb{A}_i^b + \tau_i - \tau_b^e \mathbb{A}_i^b. \tag{3.10}$$

Proof: This proof follows almost directly, given the structure of Eqs. 3.6 and 3.7 above. First, we can use the form of $\omega_a^b = (g^{-1})_a^b dg^a$ to solve for λ_b:

$$\lambda_b = \frac{d}{dt}\left(\frac{\partial l}{\partial \xi^b}\right) - \mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi^b} - \tau_b^e.$$

Then, substituting into Eq. 3.7 for λ_b and $\omega_i^b = \mathbb{A}_i^b$ gives the desired result. ■

3.4.3 The Constrained Lagrangian Approach

In a similar fashion to the Routhian approach, we can develop a reduction method using the body representation of velocities. The advantage of this method for our purposes will be that it extends to constrained systems in a very straightforward manner and is a more natural context for including left-invariant, or body, forces. In contrast to the above where we use $\mathrm{J}_i = \frac{\partial L}{\partial \dot{q}^i}$, we will use here a body momentum map based on the reduced Lagrangian,

$$p = \frac{\partial l}{\partial \xi},$$

and write the constraints defined by the connection as

$$\xi = -A(r)\dot{r} + I^{-1}p. \tag{3.11}$$

Obviously the major drawback about using this representation for unconstrained systems is that the momentum is no longer constant, but now is governed by a momentum equation:

$$\dot{p} = \mathrm{ad}^*_\xi \, p + \tau^e, \tag{3.12}$$

derived using Eqs. 3.6 and 3.9 with $p = \frac{\partial l}{\partial \xi}$. This equation demonstrates that for unconstrained systems without forcing, if the momentum in spatial or body coordinates is initially zero, then it remains fixed in *both* representations. The systems of interest here, however, are those in which the constraints partially (or fully) break the symmetries, and so we do not expect the momenta to be conserved, even if it is initially zero.

Eqs. 3.11 and 3.12 fully define the motion along the fiber. Additionally, ξ in Eq. 3.12 can be substituted for using Eq. 3.11 in order to rewrite Eq. 3.12 in terms of base and momentum variables *only*. This leads us to define the following:

Definition 3.14 *The* extended base space *is the momentum space* $\mathfrak{g}^*$ *appended to the tangent bundle to the base space,* TM.

While reduction for unconstrained systems implies reducing the dynamics directly to the base space, we will see that for systems with nonholonomic constraints the best that can be done is to reduce the dynamics to the extended base space.

To do this, we make use of what we will call the (*reduced*) *constrained Lagrangian.* This is just the reduced Lagrangian with the constraints substituted in:

$$l_c(r, \dot{r}, p) = l(r, \dot{r}, \xi)\Big|_{\xi = -A\dot{r} + I^{-1}p}.$$

Performing straightforward calculations on l_c:

$$\begin{aligned}
\frac{\partial l_c}{\partial \dot r^i} &= \frac{\partial l}{\partial \dot r^i} + \frac{\partial l}{\partial \xi^a}\frac{\partial \xi^a}{\partial \dot r^i} \\
&= \frac{\partial l}{\partial \dot r^i} - \frac{\partial l}{\partial \xi^a} A_i^a, \\
\frac{d}{dt}\left(\frac{\partial l_c}{\partial \dot r^i}\right) &= \frac{d}{dt}\left(\frac{\partial l}{\partial \dot r^i}\right) - \frac{d}{dt}\left(\frac{\partial l}{\partial \xi^a}\right) A_i^a - \frac{\partial l}{\partial \xi^a}\dot A_i^a, \text{ and} \\
\frac{\partial l_c}{\partial r^i} &= \frac{\partial l}{\partial r^i} + \frac{\partial l}{\partial \xi^a}\frac{\partial \xi^a}{\partial r^i} \\
&= \frac{\partial l}{\partial r^i} + \frac{\partial l}{\partial \xi^a}\left(-\frac{\partial A_j^a \dot r^j}{\partial r^i} + \frac{\partial I^{-1}p}{\partial r^i}\right),
\end{aligned}$$

lead us to the following equations for the base dynamics:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial l_c}{\partial \dot r^i}\right) - \frac{\partial l_c}{\partial r^i} &= \frac{d}{dt}\left(\frac{\partial l}{\partial \dot r^i}\right) - \frac{\partial l}{\partial r^i} - \frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) A_i \\
&\quad - \frac{\partial l}{\partial \xi}\left(\dot A_i - \frac{\partial A_j \dot r^j}{\partial r^i} + \frac{\partial I^{-1}p}{\partial r^i}\right) + \tau_i - \tau_b^e A_i^b \\
&= -\operatorname{ad}_\xi^* \frac{\partial l}{\partial \xi} A(\delta r^i) - \frac{\partial l}{\partial \xi}\left(dA(\dot r, \delta r^i) + \frac{\partial I^{-1}p}{\partial r^i}\right) + \tau_i - \tau_b^e A_i^b,
\end{aligned} \tag{3.13}$$

where

$$dA(\dot r, \delta r^i) = \frac{\partial A_i}{\partial r^j}\dot r^j - \frac{\partial (A_j \dot r^j)}{\partial r^i}$$

represents the local curvature form corresponding to the mechanical connection. This formula is really nothing more than a restatement of the reduced nonholonomic constrained variational principle given above in Eq. 3.10. The utility of presenting the equations in this manner comes when we more closely examine the structure of the constrained Lagrangian. This structure allows us to define the base dynamics in an easily computable form which is familiar to most engineers.

First, a straightforward calculation shows that

$$\begin{aligned}
l_c(r, \dot r, p) &= \frac{1}{2}\langle\langle \mathbf{hor}\, \dot q, \mathbf{hor}\, \dot q\rangle\rangle + \frac{1}{2}\langle p; I^{-1}p\rangle - V(r) \\
&= \frac{1}{2}\dot r^T \tilde M \dot r + \frac{1}{2}\langle p; I^{-1}p\rangle - V(r),
\end{aligned}$$

where $\tilde M(r) = m - A^T I A$ is the *reduced mass-inertia matrix*. Using this information, we can write down the base dynamics as

$$\tilde M \ddot r + \dot r^T \tilde C \dot r + \frac{\partial V}{\partial r} + \tilde N = B(r)\tau, \tag{3.14}$$

where

$$\begin{aligned}\tilde{C}_{ijk}(r)\dot{r}^j\dot{r}^k &= \frac{1}{2}\left(\frac{\partial \tilde{M}_{ij}}{\partial r^k} + \frac{\partial \tilde{M}_{ik}}{\partial r^j} - \frac{\partial \tilde{M}_{kj}}{\partial r^i}\right)\dot{r}^j\dot{r}^k,\\ B_i\tau &= \tau_i - \tau_a^e A_i^a, \quad \text{and}\\ \tilde{N} &= \mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi} A(\cdot) + \frac{\partial l}{\partial \xi}\left(dA(\dot{r},\cdot) + \frac{\partial I^{-1}p}{\partial r}(\cdot)\right).\end{aligned}$$

It is interesting to note that the formula for the Routhian in terms of body coordinates is just

$$R^p = \frac{1}{2}\langle\langle \mathbf{hor}\,\dot{q}, \mathbf{hor}\,\dot{q}\rangle\rangle - \frac{1}{2}\langle p; I^{-1}p\rangle,$$

so that $l_c - R^p = \langle p; I^{-1}p\rangle$, as expected. Obviously, though, formulating the reduction in terms of either the Routhian or the constrained Lagrangian must yield the same reduced dynamics, but with different motivating intuition.

Chapter 4

The Generalized Momentum

Much of the progress on locomotion presented here has depended on the development of the theory of reduction and reconstruction for nonholonomically constrained systems with symmetries by Bloch, Krishnaprasad, Marsden, and Murray [10]. In particular, their use of connections has motivated our thinking concerning how theoretically to address the concept of cyclical shape changes that generate locomotion. Also, the development of the generalized momentum equation has allowed us to capture an important aspect of locomotion— the generation of forward velocities described in terms of generalized momenta.

4.1 Assumptions

Recall the constrained dynamical system $(L, \mathcal{D})$ developed in Section 2.6 above. This information alone allows us to construct a generalized momentum equation (see Section 4.3 below), which loosely corresponds to describing momenta along the allowable directions defined by $\mathcal{D}$ ($\dim \mathcal{D} = n - k$). However, in order to use the additional structure provided by the existence of Lie group symmetries, it will be important to assume that $\mathcal{D}$, as well as L, is G-invariant. This invariance can be written as $T_q\Phi_g\mathcal{D}_q = \mathcal{D}_{g \cdot q}$, but we will require a slightly stronger type of invariance.

Assumption 1 *The constraint distribution $\mathcal{D}$ can be expressed in terms of a local basis, $X_1, \ldots, X_{n-k}$, that is G-invariant. That is, the relationship $T_q\Phi_g X_i(q) = X_i(\Phi_g q)$ is satisfied for $i = 1, \ldots, n-k$.*

Alternatively, we will find it convenient to express $\mathcal{D}$ as the kernel of a set of left-invariant differential one-forms, $\omega^1, \ldots, \omega^k$. Recall that invariance for a one-form will imply that $\omega^i(hq) = T^*_{hq}\Phi_{h^{-1}}\omega^i(q)$. The following proposition shows that for systems with a metric, the equivalence between these two representations is quite natural.

Proposition 4.1 *Given a G-invariant basis, $X_1, \ldots, X_{n-k}$, for $\mathcal{D}$, there exist G-invariant one-forms, $\omega^1, \ldots, \omega^k$ such that*

$$\mathcal{D} = \{v \in TQ \mid \langle \omega^i; v \rangle = 0,\ i = 1, \ldots, k\}.$$

Proof: Let $X_1(e,r),\ldots,X_{n-k}(e,r)$ denote the vector fields $X_1,\ldots,X_{n-k}$ evaluated at the identity of G. Next, choose k vector fields, $X_{n-k+1}(e,r),\ldots,X_n(e,r)$ orthogonal to $X_1(e,r),\ldots,X_{n-k}(e,r)$, and push them forward using the action of G on Q. The invariance of the metric assures that these vector fields remain orthogonal to $X_1,\ldots,X_{n-k}$ over all of Q. Finally, define k one-forms by

$$\omega^i = \mathbb{F}L(X_{n-k+i}), \text{ for } i = 1,\ldots,k.$$

These one-forms are left-invariant, since

$$\begin{aligned} T^*_{hq}\Phi_{h^{-1}}\omega^i(q) &= T^*_{hq}\Phi_{h^{-1}}\mathbb{F}L(X_{n-k+i}(q)) \\ &= \mathbb{F}L(T_q\Phi_g X_{n-k+i}(q)) \\ &= \mathbb{F}L(X_{n-k+i}(hq)) \\ &= \omega^i(hq), \end{aligned}$$

and their kernel is the constraint distribution, $\mathcal{D}$. ■

We should mention here that Assumption 1 is a very natural assumption to make when dealing with problems of locomotion. In all of the locomotion problems studied to date, the constraints have been group invariant. Consider for example, the mobile robot with wheel constraints. If we designate a frame that we attach to the body of the robot, then the constraint forces generated by the wheel will act in the same directions, *relative to the body frame*, no matter where we place the robot in the plane. Similarly with other forms of locomotion, we find that the shape of the system determines how the constraints act in relation to the body frame, and that the resulting, or net, motion of the body due to the constraints is the same regardless of the initial position of the system.

We are interested in the invariance of the constraints because it makes it easier to integrate them into the natural symmetries of the system implied by the invariance of the Lagrangian. Thus, it is natural to restrict our attention to those constraints that act in the group direction.

Definition 4.2 *The intersection of the constraints with the fiber distribution, $\mathcal{S} = \mathcal{D} \cap VQ$, is called the* constrained fiber distribution.

Recall from Section 3.1 that for a given fiber vector $v_q \in \mathcal{S}_q$, there corresponds a unique Lie algebra element, ξ, which generates v_q, i.e., $v_q = \xi_Q(q)$. Let X be a vector field in $\mathcal{S}$. Then at each point $q \in Q$, we can determine a Lie algebra element for which X_q is the infinitesimal generator. We do this for all points $q \in Q$ to define the map $\xi^q : Q \to \mathfrak{g}$ that generates X, such that $X(q) = (\xi^q(q))_Q(q)$ for all $q \in Q$. The development of the generalized momentum equation (to be used for reduction and reconstruction) will depend on these Lie algebra elements, and hence will depend on $\mathcal{S}$ being non-empty. For unconstrained systems, there are no restrictions placed on the Lie algebra and so we can choose fixed elements in $\mathfrak{g}$ to form a basis for $\mathcal{S} = VQ$. This is not true, however, for general systems with nonholonomic constraints, since the Lie algebra element necessary to generate S_q will vary over Q. This is a subtle point which is crucial to understanding why the generalized momentum varies with time.

A few additional assumptions are made here regarding the rank of $\mathcal{S}$. These are necessary for the analysis, but do not appear to be restrictive in practical examples.

Assumption 2 *In addition to Assumption 1 (that the constraint distribution be of constant dimension $(n-k)$ over Q), it is assumed that the dimension of the constrained fiber distribution, $\mathcal{S}$, is constant, with $s = \dim \mathcal{S} = n - m - k$.*

Assumption 3 *The constraint distribution contains all of the allowable base directions, i.e., $\mathcal{D} + VQ = TQ$.*

Finally, we add a condition that will be invoked only at certain points in order to make additional simplifications to the analysis.

Condition 4 *The constraint forces do not act in the base directions. Alternatively, this condition may be expressed as $TM + \mathcal{S} = \mathcal{D}$.*

Remarks:

1. Let us briefly comment on the effect of each of these assumptions. Assumption 2 implies that the allowable degrees of freedom along the group orbit does not vary over Q. This should not be confused with the existence of a Lie algebra element which generates the motion along this direction. As will be shown later, the presence of nonholonomic constraints will tend to violate the existence of any such element in the Lie algebra.

2. Assumption 3 guarantees that there are no constraints acting only between the base variables. This allows us to build a meaningful connection relating the unconstrained base (shape) velocities to fiber velocities. Condition 4 will be invoked as a much more restrictive condition than Assumption 3 as it provides further structure to be used in the process of reduction. For a trivial bundle, this condition can be understood as saying that the constraints do not generate any forces in the base direction, and so is more restrictive than Assumption 3. It implies that the reaction forces used in the method of Lagrange multipliers will not appear in the equations for the base variables.

Finally we note that although we represent the problem here on a *trivial* principal fiber bundle, the general case (for which $Q(M,G)$ is a principal fiber bundle) can always be *locally* represented as the trivial product bundle, $Q = G \times M$. This is called a *local trivialization* of $Q(M,G)$, and loosely corresponds to a choice of gauge in gauge field theory. Various problems in locomotion (e.g., the falling cat and the paramecium), have been originally formulated in the language of gauge fields, and so it is important to recognize this parallel in the nomenclature from the physics literature.

4.2 The Constrained Lie Algebra

The three assumptions above lead us to conclude that the constrained fiber distribution, $\mathcal{S}$, is a subspace of TQ consisting of vertical, left-invariant vector fields. As such, they are pointwise isomorphic to a subspace of the Lie algebra. Similar to Bloch et al. [10], we denote by $\mathfrak{g}^q$ the subspace of $\mathfrak{g}$ which generates $\mathcal{S}_q$, and by $\mathfrak{g}^{\mathcal{S}}$ the fiber bundle over Q with fibers $\mathfrak{g}^q$. We call $\mathfrak{g}^{\mathcal{S}}$ the *constrained Lie algebra*. Note that $\mathfrak{g}^{\mathcal{S}} \simeq \mathcal{S}/G$. This bundle will play an important role in the process of reduction. As in [10], we can construct a basis for $\mathfrak{g}^{\mathcal{S}}$ in the following manner. First, pick a basis, $f_1(r), \ldots, f_s(r)$, for $\mathfrak{g}^q$ at the group identity, i.e., at $q = (e, r)$. The infinitesimal generators for these basis elements satisfy the constraints at $g = e$:

$$\langle \omega^i(e,r); (f_\alpha(r))_Q(e,r)\rangle = 0, \;\; i = 1, \ldots, k, \;\; \alpha = 1, \ldots, s \;(s = n - k - m).$$

We can then extend this to a basis for $\mathfrak{g}|_{g=e}$ by choosing k elements orthogonal to $f_1, \ldots, f_s$ relative to the local form of the locked inertia tensor, I. This basis is independent of the group variables. We can establish a basis for $\mathfrak{g}$ by pushing these basis elements forward using the adjoint action:

$$f_a(g,r) = \mathrm{Ad}_g\, f_a(r), \;\; a = 1, \ldots, l. \tag{4.1}$$

We show in the following proposition that the first s elements of this basis for $\mathfrak{g}$ are aligned to form a basis of $\mathfrak{g}^{\mathcal{S}}$.

Proposition 4.3 *The basis for $\mathfrak{g}^{\mathcal{S}}$ given in Eq. 4.1 is such that the infinitesimal generators satisfy the constraints and are left-invariant, i.e.,*

$$\langle \omega^i(q); (f_\alpha(q))_Q\rangle = 0 \quad \text{and} \quad T_q\Phi_h(f_\alpha(q))_Q = (f_\alpha(hq))_Q,$$

for $i = 1, \ldots, k$ and $\alpha = 1, \ldots, s$.

Proof: For a fixed Lie algebra element, $\xi \in \mathfrak{g}$, there is a well known Lie algebra identity given by

$$(\mathrm{Ad}_h\, \xi)_Q(q) = T_{h^{-1}q}\Phi_h \xi_Q(h^{-1}q).$$

On the surface, it is not clear that this law will hold in general for Lie algebra elements which vary over Q. We give this result in a technical lemma.

Lemma 4.4 *Given a left-invariant vector field, ξ^q_Q, on Q, the following relationship must hold:*

$$(\mathrm{Ad}_{g^{-1}}\, \xi^q(g\cdot q))_Q = T_{g\cdot q}\Phi_{g^{-1}}(\xi^q(g\cdot q))_Q(g\cdot q),$$

where $\xi^q \in \mathfrak{g}^q$ is the curve in the constrained Lie algebra whose infinitesimal generator is the section of $\mathcal{S}$ denoted by ξ^q_Q (note that the configuration dependence of ξ^q is written functionally as $\xi^q(q)$).

Proof: First, we need to establish a relationship between the adjoint operator and the exponential mapping (c.f., [1], pp. 256–7 and 269).

Sublemma 4.5 *For every $g \in G$ and $\xi^q \in \mathfrak{g}^q$,*

$$\exp(t \operatorname{Ad}_g \xi^q) = g(\exp t\xi^q)g^{-1}.$$

Proof: As above, let $\mathcal{I}_g$ denote the adjoint action of G on G given by $\mathcal{I}_g : G \to G : h \mapsto ghg^{-1}$q. Then, given $t\xi^q \in \mathfrak{g}^q$, for $t \in \mathbb{R}$, define the mapping $\phi : \mathbb{R} \to G : s \mapsto \mathcal{I}_g \exp(st\xi^q)$. This map defines a one-parameter subgroup of G, which can also be represented as $\phi(s) = \exp(s\eta)$, with $\eta = \frac{d}{ds}\phi(s)\big|_{s=0} = t \cdot T_e\mathcal{I}_g \cdot \xi^q$. Because of this, we can write the following relationship when $\phi(s)$ is evaluated at $s = 1$:

$$g(\exp t\xi^q)g^{-1} = \mathcal{I}_g \exp(t\xi^q) = \phi(1) = \exp(\eta) = \exp(t \cdot T_e\mathcal{I}_g \cdot \xi^q) = \exp(tAd_g\xi^q).$$

▼

We now complete the proof of Lemma 4.4. For $q \in Q$,

$$\begin{aligned}
(\operatorname{Ad}_{g^{-1}} \xi^q(g \cdot q))_Q &= \frac{d}{dt}\Phi(\exp t \operatorname{Ad}_{g^{-1}} \xi^q(g \cdot q), q)\Big|_{t=0} \\
&= \frac{d}{dt}\Phi(g^{-1}(\exp t\xi^q(g \cdot q))g, q)\Big|_{t=0} \\
&= \frac{d}{dt}\Phi_{g^{-1}} \circ \Phi(\exp t\xi^q(g \cdot q), g \cdot q)\Big|_{t=0} \\
&= T_{g\cdot q}\Phi_{g^{-1}} \frac{d}{dt}\Phi(\exp t\xi^q(g \cdot q), g \cdot q)\Big|_{t=0} \\
&= T_{g\cdot q}\Phi_{g^{-1}}(\xi^q(g \cdot q))_Q.
\end{aligned}$$

▼

Lemma 4.4 can then be used to show invariance:

$$\begin{aligned}
(f_\alpha(hq))_Q = f_\alpha(hg, r)_Q &= (\operatorname{Ad}_{hg} f_\alpha(r))_Q \\
&= (\operatorname{Ad}_h f_\alpha(g, r))_Q \\
&= T_q\Phi_h(f_\alpha(g, r))_Q.
\end{aligned}$$

Having shown the G-invariance of the vector fields $(f_\alpha)_Q$, we must still show that they lie in the constraint distribution defined by the ω^i's. We do so by showing that $\langle\omega^i(q); f_\alpha(q)\rangle = 0$ for $q \in Q, i = 1, \ldots, k$, and $\alpha = 1, \ldots, s$.

$$\begin{aligned}
\langle\omega^i(q); (f_\alpha(q))_Q\rangle &= \langle\omega^i(g, r); (\operatorname{Ad}_g f_\alpha(r))_Q\rangle \\
&= \langle\omega^i(g, r); T_{(e,r)}\Phi_g(f_\alpha(r))_Q(e, r)\rangle \\
&= \langle T_e^*\Phi_g\omega^i(g, r); (f_\alpha(r))_Q\rangle \\
&= \langle\omega^i(e, r); f_\alpha(e, r)\rangle \\
&= 0,
\end{aligned}$$

for $i = 1, \ldots, k$, $\alpha = 1, \ldots, s$, and $q \in Q$. Finally, realize that because the first s basis elements were chosen tangent to the group orbit, they will remain tangent to the group orbit when pushed forward through the group action. In other words, $f_\alpha(g, r) \in \mathcal{D} \cap VQ$, for $\alpha = 1, \ldots, s$. ■

This result implies that on a product bundle there is a natural way to construct an invariant basis for the distribution, if it is defined by one-forms which are themselves invariant.

Remark: In the case that the constraints do not act in the base directions (i.e., Condition 4 is satisfied), we can directly generate a basis for $\mathfrak{g}^{\mathcal{S}}$. Using the fact that we are working with a trivial bundle, we can pull back the constraints to the identity by simply setting the group variables to the identity, $g = e$. If Condition 4 is satisfied, then we can identify the invariant one-forms with elements of the dual of the Lie algebra, $\mathfrak{g}^*$. Next, pick a basis for the constrained Lie algebra at the identity, $\mathfrak{g}^q$, where $q = (e, r)$, such that $\langle \omega^i(e, r); f_\alpha(r) \rangle = 0$. As above, denote this basis by $f_1(r), \ldots, f_s(r)$. Having done this, we continue as above by extending the basis to $\mathfrak{g}$ and pushing these basis elements forward using the adjoint mapping by setting $f_a(g, r) = \mathrm{Ad}_g\, f_a(r)$, $a = 1, \ldots, l$.

4.3 An Alternative Derivation of the Generalized Momentum Equation

In this section we present an alternative derivation (to [10]) of the generalized momentum equation, which allows for the inclusion of general forcing functions. One thing to note is that Assumption 1 (that the constraint distribution $\mathcal{D}$ be G-invariant) can be relaxed for this proposition. However, the work in the following sections will use this assumption in order to establish new invariance results concerning the generalized momentum.

Proposition 4.6 *Let $(L, \mathcal{D})$ be a constrained system on $Q(M, G)$ and assume L to be G-invariant. For all curves $c : [a, b] \to Q$ satisfying the nonholonomic constrained variational principle, the following* ***generalized momentum equation*** *holds for all elements, $\xi^q \in \mathfrak{g}^q$:*

$$\frac{d}{dt}p = \frac{\partial L}{\partial \dot{q}^i}\left(\frac{d}{dt}[\xi^q(c(t))]\right)_Q^i + \tau_i\left(\xi^q(c(t))\right)_Q^i \tag{4.2}$$

where

$$p = \frac{\partial L}{\partial \dot{q}^i}\left(\xi^q(c(t))_Q^i\right)$$

is the ***generalized momentum.***

Proof: We begin by restricting the velocities to lie in the constrained fiber distribution, $\mathcal{S} = \mathcal{D} \cap VQ$. Recall from Chapter 2 this implies that each $v_q \in \mathcal{S}_q$ is infinitesimally generated by some Lie algebra element. The map which takes a Lie algebra element to its infinitesimal generator at q (a vector in T_qQ) is an isomorphism when restricted to fibers (e.g., $\mathcal{S} \subset VQ$). As above, given $X \in \mathcal{S}$ define the map $\xi^q : Q \to \mathfrak{g}^{\mathcal{S}}$ by requiring that at each point q the infinitesimal generator of ξ^q is X_q. Thus, for all $q \in Q$ we have $X_q = (\xi^q(q))_Q(q)$. We will assume from now on that this identification is always made, so that we can write a vector field in $\mathcal{S}$ simply as ξ^q_Q with the implication that it is generated by the map ξ^q described above.

Note that since ξ^q is a Lie algebra-valued function it will make sense to take the time-derivative of ξ^q along solution curves. The reader should be aware of this when terms of the form $\frac{d}{dt}\xi^q$ are encountered. The time derivative of ξ^q can be interpreted as an element in the tangent space of the Lie algebra, $T_{\xi^q}\mathfrak{g}$, which itself has a canonical identification with the Lie algebra, i.e., $T\mathfrak{g} \simeq \mathfrak{g} \times \mathfrak{g}$. The identification of $T\mathfrak{g}$ with $\mathfrak{g} \times \mathfrak{g}$ is a canonical isomorphism for all vector spaces. With this identification we can associate the quantity $\frac{d}{dt}\xi^q$ with a vector field on Q by taking its infinitesimal generator.

Choose a section of $\mathcal{S}$. Taking the derivative of the generalized momentum, p, and applying the chain rule yields

$$\frac{d}{dt}p = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(\xi^q_Q)^i\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)(\xi^q_Q)^i + \frac{\partial L}{\partial \dot{q}^i}(T\xi^q_Q \cdot \dot{q} + (\frac{d}{dt}\xi^q)^i_Q). \tag{4.3}$$

Next, recall that the invariance of the Lagrangian implies $L(\Phi_g q, T_q\Phi_g v_q) = L(q, v_q)$. In particular, if we let $g = \exp(s\xi^q)$, then invariance implies that

$$L(\Phi_{\exp(s\xi^q)}q, T_q\Phi_{\exp(s\xi^q)}v_q) = L(q, v_q).$$

By differentiating this expression and evaluating it at $s = 0$ we get

$$\frac{\partial L}{\partial q^i}(\xi^q)^i_Q(c(t)) + \frac{\partial L}{\partial \dot{q}^i}(T\xi^q_Q \cdot \dot{c}(t))^i = 0. \tag{4.4}$$

Finally, recall the formula for β_L given in Section 2.6 (Eq. 2.5) to see that along solution curves to the system $(L, \mathcal{D})$, the natural pairing of β_L with ξ^q_Q is:

$$\begin{aligned} 0 = \langle \beta_L ; \xi^q_Q \rangle &= (\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} + \lambda_a \omega^a_i - \tau_i)(\xi^q_Q)^i \\ &= (\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} - \tau_i)(\xi^q_Q)^i. \end{aligned} \tag{4.5}$$

Substituting Eq. 4.4 into Eq. 4.5 yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \cdot (\xi^q)^i_Q + \frac{\partial L}{\partial \dot{q}^i}(T\xi^q_Q \cdot \dot{q}) = \tau_i(\xi^q_Q)^i. \tag{4.6}$$

Then, subtracting Eq. 4.6 from Eq. 4.3 gives the generalized momentum equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(\xi_Q^q)^i\right) = \frac{\partial L}{\partial \dot{q}^i}(\frac{d}{dt}\xi^q)_Q^i(t) + \tau_i(\xi^q)_Q^i. \tag{4.7}$$

■

Remarks:

1) Note that the generalized momentum defined in this proposition is similar to the momentum defined in the unconstrained case as $p = \frac{\partial l}{\partial \xi}$. In fact, we will see below that for invariant constraints the generalized momentum has a natural interpretation as the unconstrained momentum projected onto the unconstrained directions. In the limiting case where there are no constraints, then, the momentum defined by $p = \frac{\partial l}{\partial \xi}$ is returned. For this reason, we will make a slight abuse of notation and denote the generalized momentum also by p, with the realization that it is equally valid for unconstrained systems.

2) An important implication of Proposition 4.6 is that the time derivative of the momentum is no longer zero (even if p is initially zero), but is instead governed by a differential equation reflecting the time evolution of the Lie algebra elements that define the constrained fiber distribution. For systems in which the constraints are invariant, there are additional properties described in the next section that can be derived. Recall, however, that the proof of the current proposition does not require this assumption.

3) Equation 4.7 can also be applied to unconstrained systems. This could prove to be useful in the case that the external torques, τ, are applied in the fiber directions. In such cases, Eq. 4.7 implies that the momentum associated with the unconstrained system would no longer be conserved. Instead, it is governed by a differential equation which depends on the interaction between the forcing function and the section chosen to define the momentum. The reduction scheme presented in the sections to follow, then, may be important for analyzing the effect of forcing in the group directions for unconstrained systems. Of course, if the forcing is applied only to the base space, then more standard techniques for Lagrangian reduction apply (see [63] for more details).

4) Although the generalized momentum law is written here in coordinates, it can also be expressed in a more intrinsic manner, which is independent of the choice of a particular coordinate system for Q. Having made the identification, $T\mathfrak{g} \simeq \mathfrak{g} \times \mathfrak{g}$, Eq. 4.7 can be written as

$$\frac{d}{dt}\langle \mathbb{F}L(c(t)); \xi_Q^q\rangle = \langle \mathbb{F}L(c(t)); \left(\frac{d}{dt}\xi^q\right)_Q\rangle + \langle \tau; (\xi^q)_Q\rangle,$$

where $\mathbb{F}L$ is the Legendre transformation, defined to be the fiber derivative of the Lagrangian (for more information on this, see [1]).

Corollary 4.7 *Given the above basis for the constrained Lie algebra, $f_1(g,r), \ldots, f_s(g,r) \in \mathfrak{g}^{\mathcal{S}}$, the generalized momentum equation along trajecto-*

ries becomes

$$\frac{d}{dt}(p_\alpha) = \frac{\partial L}{\partial \dot{q}^i}(\frac{d}{dt} f_\alpha)^i_Q(t) + \tau_i (f_\alpha)^i_Q,$$

where $p_\alpha = \frac{\partial L}{\partial \dot{q}^i}(f_\alpha)^i_Q$.

4.4 Invariance of the Generalized Momentum

Obviously, the generalized momentum defined above is no longer a constant. The question arises as to whether there are any other properties of the generalized momentum which will prove useful to our analysis. The first property which we would expect is for the momentum to be G-invariant. Assuming the constraint distribution to be G-invariant, this property is shown in the following proposition.

Proposition 4.8 *Given a constrained system* $(L, \mathcal{D})$ *for which there exist* G*-invariant sections,* $X_1, \ldots, X_s \subset \mathcal{S}$*, the generalized momentum given by* $p_\alpha(v_q) = \langle \mathbb{F}L(v_q); X_\alpha(v_q)\rangle$ *is* G*-invariant, i.e.,* $p_\alpha(T_q\Phi_g v_q) = p_\alpha(v_q)$*, for* $\alpha = 1, \ldots, s$.

Proof: The proof of this proposition is a direct result of the invariance properties of the Legendre transformation and the section chosen for the constrained fiber distribution. To see that the Legendre transformation is indeed invariant, first recall the invariance equation for the Lagrangian, $L = L \circ T\Phi_g$. Taking the fiber derivative of this expression yields

$$\mathbb{F}L(v) \cdot w = \mathbb{F}L\left(T\Phi_g(v)\right) \cdot T\Phi_g(w) \;\forall\, v, w \in TQ,$$

which can alternately be expressed as

$$T^*_{g\cdot q}\Phi_{g^{-1}}\mathbb{F}L(v_q) = \mathbb{F}L(T_q\Phi_g v_q). \tag{4.8}$$

In other words, the following diagram commutes:

$$\begin{tikzcd} TQ \arrow[r, "\mathbb{F}L"] \arrow[d, "T\Phi_g"'] & T^*Q \arrow[d, "T^*\Phi_g"] \\ TQ \arrow[r, "\mathbb{F}L"'] & T^*Q \end{tikzcd}$$

Recall that the invariance assumption for the constraint distribution $\mathcal{D}$ is $T_q\Phi_g X_\alpha(q) = X_\alpha(\Phi_g q)$, for $\alpha = 1, \ldots, s$. Invariance of the generalized momentum is then a trivial statement:

$$\begin{aligned} p_\alpha(T_q\Phi_g v_q) &= \langle \mathbb{F}L(T_q\Phi_g v_q); X_\alpha(\Phi_g q)\rangle \\ &= \langle T^*_{g\cdot q}\Phi_{g^{-1}}\mathbb{F}L(v_q); T_q\Phi_g X_\alpha(q)\rangle \\ &= \langle \mathbb{F}L(v_q); X_\alpha(q)\rangle \\ &= p_\alpha(v_q), \end{aligned}$$

for $\alpha = 1, \ldots, s$. ■

4.5 The Nonholonomic Connection

Given that the trajectories for the dynamical system are constrained to lie in the subspace, $\mathcal{D}$, the natural question is to ask if it is possible to use the information encoded in $\mathcal{D}$, along with the metric, to define a meaningful connection on $Q(M, G)$. As mentioned earlier, given a trivial principal fiber bundle with a metric, there are natural ways to build a connection. For example, the metric can be used to define a mechanical connection as in the unconstrained case. This however, does not make use of the constraints, and so the information encoded in the connection will have no relevance to the actual dynamics of the system. Recall also that in the principal kinematic case, the constraints are sufficient by themselves to define a connection. In general for the mixed case, though, we will need to use the constraints to build part of the connection, and then complete the construction by using the metric. To do so, we begin by defining a particular choice of horizontal.

Proposition 4.9 *The horizontal subbundle,*

$$H_q Q = \{v_q \in T_q Q \mid v_q \in \mathcal{D} \text{ and } \langle\!\langle v_q, w_q \rangle\!\rangle = 0,\ \forall\ w_q \in \mathcal{S}\},$$

defines a connection on $Q(M, G)$.

Proof: We verify this directly by showing that each of the defining conditions for a connection is satisfied (Definition 3.2). First, we must show that $T_q Q = V_q Q \oplus H_q Q$. Let U_q be the subspace of $T_q Q$ such that $T_q Q = \mathcal{D}_q \oplus U_q$. Since vectors in $H_q Q$ are defined to be orthogonal to $\mathcal{S}_q$, we also have the splitting $\mathcal{D}_q = H_q Q \oplus \mathcal{S}_q$. This implies that $T_q Q = V_q Q \oplus H_q Q$, since $V_q Q = \mathcal{S}_q \oplus U_q$.

Next, we must show that $T_q \Phi_g H_q Q = H_{g\cdot q} Q$. First, notice that Assumption 1 requires that $w_q \in \mathcal{D}$ implies $T_{h\cdot q}\Phi_{h^{-1}} w_{h\cdot q} = w_q$. We use this to show that for all v_q in $H_q Q$, $T_q \Phi_h v_q \in H_{h\cdot q} Q$:

$$\begin{aligned}
\langle\!\langle w_{h\cdot q}, T_q \Phi_h v_q \rangle\!\rangle &= \langle \mathbb{F}L(w_{h\cdot q}); T_q \Phi_h v_q \rangle \\
&= \langle T_q^* \Phi_h \mathbb{F}L(w_{h\cdot q}); v_q \rangle \\
&= \langle \mathbb{F}L(T_{h\cdot q}\Phi_{h^{-1}} w_{h\cdot q}); v_q \rangle \\
&= \langle \mathbb{F}L(w_q); v_q \rangle = 0.
\end{aligned}$$

Finally, we remark that condition 3 of the definition is satisfied, since we have assumed the constraint distribution to be smooth and of constant rank over Q. ■

Using the constraints and the induced metric on $\mathfrak{g}^q$, we can synthesize a connection one-form for the mixed constraint problem. First, we define a version of the momentum map for systems with nonholonomic constraints.

Definition 4.10 *The* nonholonomic momentum map *is the map,* $\mathrm{J}^{\mathrm{nhc}} : TQ \to (\mathfrak{g}^{\mathcal{S}})^*$*, defined by*

$$\langle \mathrm{J}^{\mathrm{nhc}}; \xi^q \rangle = \langle\!\langle v_q, \xi^q_Q \rangle\!\rangle,$$

where $\xi^q \in \mathfrak{g}^q$*.*

Similar to above, define the locked inertia tensor for the constrained Lie algebra as

$$\langle \mathbb{I}^c \eta^q; \xi^q \rangle = \langle\!\langle \eta^q_Q, \xi^q_Q \rangle\!\rangle,$$

where $\xi^q, \eta^q \in \mathfrak{g}^q$. Then define $\mathcal{A}^{\mathrm{sym}} : TQ \to \mathfrak{g}^{\mathcal{S}}$ to be

$$\mathcal{A}^{\mathrm{sym}} = (\mathbb{I}^c)^{-1}\, \mathrm{J}^{\mathrm{nhc}}.$$

The one-form $\mathcal{A}^{\mathrm{sym}}$ possesses the special property that the mapping $\mathcal{A}^{\mathrm{sym}}_Q : T_qQ \to T_qQ$ is the identity on all vectors in $\mathcal{S}$.

Next, recall the basis for $\mathfrak{g}^{\mathcal{S}}$ given in Section 4.2 as $f_1(g,r), \ldots, f_s(g,r)$. We can extend this basis using the locked inertia tensor to a basis for $\mathfrak{g}$ by defining k additional basis elements, $f_{s+1}, \ldots, f_{s+k}$. If we denote the complementary subspace to $\mathcal{D}$ by U (i.e., $TQ = \mathcal{D} \oplus U$), then we can use the left-invariant constraint one-forms to define a one-form, $\mathcal{A}^{\mathrm{kin}}$, such that $\mathcal{A}^{\mathrm{kin}}_Q$ is the identity on U. Given this, then we define the *nonholonomic connection one-form* to be the one-form on Q given by

$$\mathcal{A}^{\mathrm{nhc}} = \mathcal{A}^{\mathrm{kin}} + \mathcal{A}^{\mathrm{sym}}.$$

That this is actually a connection one-form follows from the invariance of the constraints and the requirement that $\mathcal{A}^{\mathrm{kin}}$ and $\mathcal{A}^{\mathrm{sym}}$ act appropriately on their respective subspaces. In the next section we give a constructive method for generating the connection which helps to clarify this procedure.

One thing to note about the nonholonomic momentum map is that along trajectories its components evaluate to the generalized momenta. Let $f^1, \ldots, f^{s+k}$ denote the dual basis to $f_1(g,r), \ldots, f_{s+k}$ (recall that $s + k = l$). Then, if we write the nonholonomic momentum map as

$$\mathrm{J}^{\mathrm{nhc}} = \mathrm{J}^{\mathrm{nhc}}_\alpha f^\alpha(g,r),$$

we see that

$$\mathrm{J}^{\mathrm{nhc}}_\alpha = \langle \mathrm{J}^{\mathrm{nhc}}; f_\alpha(g,r) \rangle = p_\alpha,$$

where p_α is the generalized momentum defined above. Thus, along trajectories we can use the connection to define the following constraint equations:

$$\mathcal{A}^{\mathrm{nhc}}(\dot{q}) = g^{-1}\dot{g} + \mathbb{A}\dot{r} = (I^c)^{-1}p, \tag{4.9}$$

where $(I^c)^{-1}p \in \mathfrak{g}^{\mathcal{S}}$ is naturally embedded in $\mathfrak{g}$ at each $q \in Q$.

4.6 Synthesizing the Connection

Before deriving further relationships regarding the generalized momentum, we briefly consider a practical, alternative method for constructing the nonholonomic connection. As necessary, we will make the additional assumption that there exists a metric on TQ. This will be used to provide the additional structure needed to identify the dual of the constrained Lie algebra, $(\mathfrak{g}^{\mathcal{S}})^*$, with a subspace of $\mathfrak{g}^*$. For mechanical systems there is a natural metric associated with the kinetic energy. This metric is used to define the generalized momentum, which is equivalent to the momentum of the system in the directions that satisfy the constraints. Being able to define the momentum in this manner allows us to retain some of the physical intuition we would normally have in unconstrained systems and is an important part of the synthesized connection.

Denote the coordinate representation of the metric, $\langle\!\langle , \rangle\!\rangle$ on TQ by $\mathbb{G}_{ij}$, so that $\langle\!\langle v, w \rangle\!\rangle = \mathbb{G}_{ij} v^i w^j$. For mechanical systems the metric is $\mathbb{G}_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$. As above, let the *synthesized* connection be defined by those vectors in $\mathcal{D}$ which are orthogonal to all vectors in $\mathcal{S}$, with respect to the given metric.

More concretely, recall the dual notation for the constraint distribution which is described by the kernel of a set of one-forms, $\omega^1, \ldots \omega^k$. Also, recall that for $\mathcal{S}$ there is assumed to exist a G-invariant basis of vector fields, $X_1, \ldots, X_s$ (related to these one-forms via Proposition 4.1). Vectors in TQ orthogonal to these basis vectors can alternatively be expressed as the kernel of a different set of one-forms with the aid of the isomorphism known as the *flat* operation. The flat operator, $^\flat : TQ \to T^*Q$, is defined by the relationship $\langle u^\flat ; w \rangle = \langle\!\langle u, w \rangle\!\rangle$, for all $w \in TQ$. In coordinates this is given by $(u^\flat)_i = \mathbb{G}_{ij} u^j$, where $u = u^j \frac{\partial}{\partial q^j}$. Construct from this map $s = l - k$ one-forms, $\omega^{k+1}, \ldots, \omega^{k+s}$, called the *synthesized one-forms*:

$$\omega^{k+\alpha} = (X_\alpha)^\flat, \;\; \alpha = 1, \ldots, s$$

(recall that $l = \dim G$). A quick calculation shows that these synthesized one-forms satisfy an important property:

$$\begin{aligned} p_\alpha(v_q) &= \langle \mathbb{F}L(v_q); X_\alpha \rangle \\ &= \langle X_\alpha^\flat; v_q \rangle \\ &= \langle \omega^{k+\alpha}(q); v_q \rangle, \;\; \alpha = 1, \ldots, s. \end{aligned} \tag{4.10}$$

The importance of this statement is two-fold. First, notice that there are now l independent one-forms on Q, given by $\omega^1, \ldots, \omega^l$. The kernel of these one-forms define l constraints— the same number as the dimension of the Lie group G. This is suggestive of the Chaplygin case in which constraints are used to build an Ehresmann connection over a fiber bundle. Additionally, Eq. 4.10 relates these one-forms to the generalized momenta, which we have just shown to be invariant with respect to the group action. The invariance of the generalized momenta thus implies the G-

invariance of the synthesized one-forms, since

$$\begin{aligned} p_\alpha(v_q) &= p_\alpha(T_q\Phi_g v_q) \\ &\Updownarrow \\ \langle \omega^{k+\alpha}(q); v_q\rangle &= \langle \omega^{k+\alpha}(\Phi_g q); T_q\Phi_g v_q\rangle \\ &= \langle T_q^*\Phi_g \omega^{k+\alpha}(\Phi_g q); v_q\rangle, \ \alpha = 1, \ldots, s. \end{aligned}$$

The horizontal subspace defined by $H_qQ = \{v_q \in T_qQ \mid \langle \omega^a(q); v_q\rangle = 0, \ a = 1, \ldots, l\}$ has already been shown in the previous section to define a connection on $Q(M, G)$. Recall that there are two important assumptions underlying this statement. First, Assumption 1 implies that the invariant distribution can be written using a basis of G-invariant one-forms, $\omega^1, \ldots, \omega^k$. And second, Assumption 3 (that $\mathcal{D} + VQ = TQ$) ensures that the horizontal subspace H_qQ is isomorphic to the tangent space of the base, $T_{\pi(q)}M$. As was mentioned earlier, these assumptions are mostly of a technical nature, and do not seem to be restrictive in light of the examples studied to date.

We can treat the synthesized one-forms as a set of affine constraints and write the entire set of constraints (nonholonomic and synthesized) as a linear operator acting on TQ:

$$\omega(q)\dot{q} = \eta, \tag{4.11}$$

where $\eta = (0, \ldots, 0, p_1, \ldots, p_s)$. The fact that the constraints define a connection implies that in a local trivialization we can rewrite Eq. 4.11 in the following manner,

$$g^{-1}\dot{g} + \mathbb{A}(r)\dot{r} = \gamma(r, p). \tag{4.12}$$

This equation deserves some comment. First, realize that it is actually a set of l equations describing motion along the fiber (the g variables). In the first term, g^{-1} action of $T_gL_{g^{-1}}$ on G. In the second term of Eq. 4.12, the base-dependent matrix $\mathbb{A} : TM \to VQ$ is the *local form of the nonholonomic connection*, and describes the vertical components which would result from horizontally lifting a base vector $\dot{r}$ to TQ. Finally, γ is a vector-valued function of r and the generalized momenta (p is written in vector form as $p = (p_1, \ldots, p_s)$), which relates the dynamic component of the motion (via the momentum) to the fiber velocities. If we compare Eq. 4.12 to the form specified by the connection one-form in Eq. 4.9, then we see that it must be that $\gamma = \tilde{I}^{-1}p$, so that

$$\xi = g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}p. \tag{4.13}$$

Notice that in the above equation we have written $\tilde{I}^{-1}$ instead of $(I^c)^{-1}$. We do this to distinguish the constrained locked inertia tensor, $(I^c)^{-1} : (\mathfrak{g}^{\mathcal{S}})^* \to \mathfrak{g}^{\mathcal{S}}$, from its natural embedding in the Lie algebra as $\tilde{I}^{-1} : (\mathfrak{g}^{\mathcal{S}})^* \to \mathfrak{g}$. This is a subtle, yet important, point for which we will briefly digress in order to clarify. As above, let $f_1, \ldots, f_s$ denote a basis for $\mathfrak{g}^{\mathcal{S}}$. Then, if we let I_{ab} denote the coordinates of the local form of the locked inertia tensor, then

$$(I^c)_{\alpha\beta} = I_{ab} f^a_\alpha f^b_\beta.$$

If we let $B^{\alpha\beta} = ((I^c)^{-1})^{\alpha\beta}$ represent the inverse of I^c (an admittedly cumbersome notation), then

$$(\tilde{I}^{-1})^{a\alpha} = B^{\alpha\beta} f^a_\beta.$$

Using this, it is easy to derive an identity that will come up later:

$$B = \tilde{I}^{-1} I \tilde{I}^{-1}.$$

4.7 Invariance of the Generalized Momentum Equation

The invariance of the generalized momentum will be used to establish a connection on the bundle $Q(M,G)$ (recall that the horizontal subbundle required by condition (2) for a connection must be G-invariant). In carrying out the reduction process for constrained systems, it will also be useful to show that the generalized momentum equation given in Proposition 4.6 satisfies a similar invariance condition. Invariance of the momentum equation will allow us to decouple the dynamics of the generalized momentum from the fiber velocities. As in Section 3.4.3, it will then be possible to reduce to an extended base space formed by appending the generalized momentum terms to the base space. These issues will be discussed further in Section 4.8. That the generalized momentum would decouple from the fiber dynamics makes good sense for mechanical systems in which the intrinsic dynamics are invariant with respect to some global position and orientation. Further studies are needed to examine the implications of invariance of the momentum equation for other types of invariances which do not have such straightforward physical interpretations as position and orientation.

Proposition 4.11 *Given a constrained mechanical system, $(L, \mathcal{D})$, let $P^c_\alpha(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}^i}(\frac{d}{dt}(\xi^q_\alpha(q))^i_Q) = \frac{d}{dt}p_\alpha$, $\alpha = 1,\ldots,s$, where ξ^q_α is a Lie algebra-valued function over Q that generates the G-invariant vector field $(\xi^q_\alpha)_Q \in \mathcal{S}$. As a function on TQ, P^c_α is G-invariant, or, for all $g \in G$ and $\alpha = 1,\ldots,s$,*

$$P^c_\alpha(\Phi_g q, T_q\Phi_g \dot{q}) = P^c_\alpha(q,\dot{q}).$$

Proof: The time derivative of a Lie algebra-valued function ξ^q can alternatively be written as

$$\frac{d}{dt}\xi^q(q) = T_q\xi^q(q)\cdot\dot{q}.$$

G-invariance implies invariance with respect to each fixed element $g \in G$ (that is, the element g is constant), so that

$$\frac{d}{dt}\left(\xi^q(g\cdot q)\right) = T_{gq}\xi^q(g\cdot q)\cdot\frac{d}{dt}(g\cdot q).$$

This yields a transformation law for the infinitesimal generator of $\frac{d}{dt}\xi^q$:

$$\begin{aligned}((\frac{d}{dt}\xi^q)_Q)(\Phi_g(q), T_q\Phi_g\dot{q}) &= (T_{gq}\xi^q(g\cdot q)\cdot T_q\Phi_g\dot{q})_Q(g\cdot q)\\ &= (T_{gq}\xi^q(g\cdot q)\cdot\frac{d}{dt}(\Phi_g\cdot q))_Q(g\cdot q)\\ &= (\frac{d}{dt}\xi^q(g\cdot q))_Q(\Phi_g(q)).\end{aligned}$$

Next recall the general transformation property of Lie algebra elements shown in Lemma 4.4:

$$(\mathrm{Ad}_{g^{-1}}\,\xi^q(g\cdot q))_Q = T_{g\cdot q}\Phi_{g^{-1}}(\xi^q(g\cdot q))_Q(g\cdot q).$$

Finally, by making the canonical vector space identification, $T_\xi\mathfrak{g} \simeq \mathfrak{g}\times\mathfrak{g}$, the term $\frac{d}{dt}\xi^q(g\cdot q)$ can be written as an element of $\mathfrak{g}$ and we find that

$$\begin{aligned}P^c_\alpha(\Phi_g q, T_q\Phi_g\dot{q}) &= \langle \mathbb{F}L(g\cdot q, T_q\Phi_g\dot{q}); \left(\frac{d}{dt}\xi^q_\alpha\right)_Q(\Phi_g(q), T_q\Phi_g\dot{q})\rangle\\ &= \langle T^*_{g\cdot q}\Phi_{g^{-1}}\mathbb{F}L(g\cdot q, T_q\Phi_g\dot{q}); \left(\frac{d}{dt}\xi^q_\alpha\right)_Q(\Phi_g(q), T_q\Phi_g\dot{q})\rangle\\ &= \langle \mathbb{F}L(q,\dot{q}); T_{g\cdot q}\Phi_{g^{-1}}\left(\frac{d}{dt}\xi^q_\alpha(g\cdot q)\right)_Q(\Phi_g(q))\rangle\\ &= \langle \mathbb{F}L(q,\dot{q}); \left(Ad_{g^{-1}}\frac{d}{dt}\xi^q_\alpha(g\cdot q)\right)_Q(q)\rangle\\ &= \langle \mathbb{F}L(q,\dot{q}); \frac{d}{dt}\left(Ad_{g^{-1}}\xi^q_\alpha(g\cdot q)\right)_Q(q)\rangle\\ &= \langle \mathbb{F}L(q,\dot{q}); \frac{d}{dt}\left(\xi^q_\alpha(q)\right)_Q(q)\rangle\\ &= P^c_\alpha(q,\dot{q}),\end{aligned}$$

and the result is achieved. ■

This proposition implies that we will always be able to write the generalized momentum equation in a reduced form that is independent of the group variables. Bloch et al. [10] show this implicitly when they reduce the generalized momentum equation to

$$\frac{d}{dt}p_\alpha = \langle\frac{\partial l}{\partial\xi};[\xi,f_\alpha]+\frac{\partial f_\alpha}{\partial r^i}\dot{r}^i\rangle + \tau^e f_\alpha,\ \alpha = 1,\ldots,s, \tag{4.14}$$

where $\xi = g^{-1}\dot{g}$ and l is the reduced Lagrangian (we have included G-invariant forcing functions by adding the term $\tau^e f_\alpha$).

The following proposition shows the explicit construction of Eq. 4.14 using similar methods to the ones given in the proof of the generalized momentum equation (Proposition 4.6). It also establishes that the generalized momentum equation can be written in terms of a base-dependent quadratic form of the extended base variables, $(p, \dot{r})$.

Proposition 4.12 *Given the invariance of the Lagrangian and the constraints, the generalized momentum equation is independent of the group variables, and can be written as*

$$\dot{p} = \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T \sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T \sigma_{pp}(r)p + \tilde{\tau}(r), \tag{4.15}$$

which, when unforced, is a quadratic function of p and $\dot{r}$.

Proof: Given the setup of the problem, it is a straightforward calculation to show that the equations of motion governing flow along the fiber can be written

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = \mathrm{ad}^*_\xi \left(\frac{\partial l}{\partial \xi}\right) + \lambda_i \omega^i(e, r) + \tau^e, \tag{4.16}$$

where the $\omega^i(e, r)$'s are the constraints evaluated at $g = e$ as above. Recall the basis for the constrained fiber distribution, $f_1, \ldots, f_s$. Noting that any element of this distribution is in the kernel of the ω^i's, and interpreting Eq. 4.16 as elements of $\mathfrak{g}^*$ acting on $\mathfrak{g}$, we have

$$\begin{aligned}
\frac{d}{dt}(p_\alpha) &= \frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) f_\alpha + \frac{\partial l}{\partial \xi}\frac{d}{dt}(f_\alpha) \\
&= \mathrm{ad}^*_\xi \left(\frac{\partial l}{\partial \xi}\right) f_\alpha + \frac{\partial l}{\partial \xi}\frac{d}{dt}(f_\alpha) + \tau^e f_\alpha \\
&= \langle \frac{\partial l}{\partial \xi}; [\xi, f_\alpha] + \frac{d}{dt}(f_\alpha)\rangle + \tau^e f_\alpha,
\end{aligned}$$

where we have used the definition of the generalized momentum, $p_\alpha = \frac{\partial l}{\partial \xi} \cdot f_\alpha$.

Next, we define structure constants for the constrained Lie algebra. Let $\xi^q, \eta^q \in \mathfrak{g}^q$, and define the structure constants, $\bar{c}^k_{ij}$ by

$$[\xi^q, \eta^q] = \bar{c}^k_{ij}\xi^i \eta^j f_k.$$

As above, we write the local forms of the mechanical connection and the locked inertia tensor as A and I, respectively. Then, using the equation for the body form of the connection:

$$g^{-1}\dot{g} = \xi = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}p,$$

we find that the reduced momentum equation can be rewritten solely in terms of these local forms and the structure constants. First, we note that the following simplification follows directly from the structure of the reduced Lagrangian given in Eq. 3.5:

$$\begin{aligned}\frac{\partial l}{\partial \xi} &= \dot{r}^T \mathrm{A}^T I + \xi^T I \\ &= \left(\dot{r}^T (\mathrm{A} - \mathbb{A})^T + \tilde{I}^{-1} p\right) I.\end{aligned}$$

Then,

$$\begin{aligned}\frac{d}{dt} p_\alpha &= \left\langle \frac{\partial l}{\partial \xi}; [\xi, f_\alpha] + \frac{\partial f_\alpha}{\partial r^i} \dot{r}^i \right\rangle + \tau^e f_\alpha \\ &= \left\langle I_{ab} \left((\mathrm{A}_i^a - \mathbb{A}_i^a) \dot{r}^i + (\tilde{I}^{-1})^{a\gamma} p_\gamma \right); \bar{c}_{cd}^b \left(-\mathbb{A}_j^c \dot{r}^j + (\tilde{I}^{-1})^{c\beta} p_\beta \right) f_\alpha^d + \frac{\partial f_\alpha^b}{\partial r^j} \dot{r}^j \right\rangle + \tilde{\tau} \\ &= I_{ab} (\tilde{I}^{-1})^{a\gamma} \bar{c}_{cd}^b (\tilde{I}^{-1})^{c\beta} f_\alpha^d p_\beta p_\gamma \\ &\quad + I_{ab} \left((\mathrm{A}_j^a - \mathbb{A}_j^a) \bar{c}_{cd}^b (\tilde{I}^{-1})^{c\gamma} f_\alpha^d + (\tilde{I}^{-1})^{a\gamma} (-\bar{c}_{cd}^b \mathbb{A}_j^c f_\alpha^d + \frac{\partial f_\alpha^b}{\partial r^j}) \right) \dot{r}^j p_\gamma \\ &\quad + I_{ab} (\mathrm{A}_i^a - \mathbb{A}_i^a) \left(-\bar{c}_{cd}^b \mathbb{A}_j^c f_\alpha^d + \frac{\partial f_\alpha^b}{\partial r^j} \right) \dot{r}^i \dot{r}^j + \tilde{\tau} \\ &= (\frac{1}{2} p^T \sigma_{pp} p + p^T \sigma_{p\dot{r}} \dot{r} + \dot{r}^T \sigma_{\dot{r}\dot{r}} \dot{r})_\alpha + \tilde{\tau},\end{aligned}$$

as desired. ∎

It is interesting to note how the terms of Eqs. 4.14 and 4.15 simplify in particular examples. Consider first the example of the snakeboard (explicit details are contained in Chapter 5), where the generalized momentum equation does not include terms quadratic in the momentum. These terms would normally arise due to the Lie bracket, $[\xi, f_\alpha]$, in Eq. 4.14. However, since the constrained Lie algebra is one-dimensional (and hence Abelian), we find that $[\xi, f_\alpha] \equiv 0$. On the other hand, for the example of the rigid body, the generalized momentum equation is given by Eq. 3.4:

$$\dot{\Pi} = \Pi \times \mathbb{I}^{-1} \Pi,$$

and we see that *only* terms quadratic in the momentum are present. This is related to the aforementioned fact that for unconstrained systems one can always choose a basis for $\mathfrak{g}^S$ which is fixed and which generates all of VQ. In this situation, $\frac{\partial f_\alpha}{\partial r^i} \equiv 0$.

4.8 Reduction

For an unconstrained system with symmetries, the reduction process involves restating the equations of motion in terms involving only base variables. In other words, the symmetries effectively place constraints on the system which allow for the dynamics given by Lagrange's equations to be dropped to the quotient space, $M = Q/G$. Analysis can then be performed on a lower dimensional space, without losing any of the information of the system. The additional dynamics necessary to describe the motion of the system on the total configuration space are then recovered during the reconstruction procedure.

For systems with constraints, however, it is not necessarily possible to reduce down to a system involving *only* the base variables. The process presented here makes use of the generalized momentum developed in [10] and the synthesized connection defined above to perform reduction on general systems with constraints and symmetries. In doing so, the dynamics are reduced to an *extended base space* formed by appending the generalized momentum terms onto the original base space. For systems in which the constraint distribution is invariant, it will be possible to reduce to the extended base space. As such we can write the dynamics in terms of the extended base variables *only*. This section serves two purposes: first, we demonstrate that this type of reduction can *always* be done for systems in which the constraint distribution is G-invariant, and second, we give explicit matrix calculations which can be used to determine the reduced dynamics in a manner similar to the unconstrained case above. The details given below for reduction are primarily intended to demonstrate the result with some intuition of why it is correct. For additional remarks and a more detailed analysis of reduction in the case of general affine constraints, the reader is referred to [10].

Invariance of the Lagrangian implies that we can continue to define a reduced Lagrangian on $\mathfrak{g} \times TM$ by

$$l(\xi, r, \dot{r}) = L(g^{-1}g, r, g^{-1}\dot{g}, \dot{r}),$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$. Next, recall the affine constraint given above in Eq. 4.12,

$$\xi = g^{-1}\dot{g} = \tilde{I}^{-1}p - \mathbb{A}(r)\dot{r}. \tag{4.17}$$

(Note that we continue to denote the nonholonomic connection by $\mathbb{A}$ and the mechanical connection by A.) With the connection and the momentum equation, we have the same data as was used in the reduction for unconstrained systems above. Following this, we again define the *constrained Lagrangian*, l_c, to be the reduced Lagrangian with the constraints substituted in from Eq. 4.17:

$$l_c(r, \dot{r}, p) = l(\xi, r, \dot{r})\Big|_{\xi = \tilde{I}^{-1}p - \mathbb{A}(r)\dot{r}}. \tag{4.18}$$

The constrained Lagrangian has a very simple intrinsic interpretation which is almost identical to the one found in the unconstrained case.

Proposition 4.13 *The constrained Lagrangian can be written as*

$$\begin{aligned} l_c &= \frac{1}{2}\langle\langle \mathbf{hor}\,\dot q, \mathbf{hor}\,\dot q\rangle\rangle + \frac{1}{2}\langle p; (I^c)^{-1}p\rangle - V(r) \\ &= \frac{1}{2}\dot r^T \tilde M \dot r + \frac{1}{2}\langle p; (I^c)^{-1}p\rangle - V(r), \end{aligned} \tag{4.19}$$

where

$$\tilde M(r) = m - \mathbb{A}^T I \mathbb{A}$$

for all but principal kinematic constraints, in which case the reduced mass-inertia matrix becomes

$$\tilde M(r) = m - A^T I A + (A - \mathbb{A})^T I (A - \mathbb{A}).$$

Proof: We begin by commenting on an interesting point— the reduced mass-inertia matrix, $\tilde M$, for the principal kinematic case actually involves more terms than for the case of mixed constraints. This is shown below, and is due to the fact that the symmetries are completely annihilated in the principal kinematic case. Thus, they do not help to simplify the reduced mass-inertia matrix.

First, we prove a lemma regarding the relationship between the local forms of the mechanical connection and the synthesized connection for systems in which the constraints by themselves do not define a connection.

Lemma 4.14 *In the case of mixed constraints (and trivially for the unconstrained case), the following relationships hold for all $\dot r \in T_r M$:*

$$\langle I(A - \mathbb{A})\dot r; \mathbb{A}\dot r\rangle = 0 \quad \text{and} \quad \langle I(A - \mathbb{A})\dot r; \tilde I^{-1} p\rangle = 0,$$

where I is the local form of the locked inertia tensor.

Proof: The content of these statements is that the space defined by $A - \mathbb{A}$ lies in a space orthogonal (with respect to the locked inertia tensor) to the constrained Lie algebra. To show this, recall the basis for $\mathfrak{g}$ developed above. Denote by $f_1, \ldots, f_s$ the basis for $\mathfrak{g}^q$, and by $f_{s+1}, \ldots, f_l$ the orthogonal completion of this basis to $\mathfrak{g}$, such that

$$\langle I f_\alpha; f_z\rangle = 0, \; \alpha = 1, \ldots, s, \; z = s+1, \ldots, l.$$

Thus, we can write the mechanical connection, $A(r) : T_r M \to \mathfrak{g}$, in terms of this basis as

$$A(r)\dot r = A_i^\alpha \dot r^i f_\alpha + A_i^z \dot r^i f_z,$$

where we have divided A according to this orthogonal splitting of $\mathfrak{g}$. We prove the lemma by showing that $\mathbb{A}_i^\alpha = A_i^\alpha$ and $\tilde I^{-1} p \in \mathfrak{g}^q$ (the latter statement can be assumed by definition). In coordinates, we write the formula for the generalized momentum as

$$p_\beta = \langle \frac{\partial l}{\partial \xi}; f_\beta \rangle = I_{ab}(A_i^a \dot{r}^i + \xi^a) f_\beta^b.$$

If we break up ξ^a as $\xi^a = \xi^\alpha f_\alpha^a + \xi^z f_z^a$, then we can use the orthogonality of the basis for $\mathfrak{g}$ to show that

$$\begin{aligned} p_\beta &= I_{ab}\xi^\alpha f_\alpha^a f_\beta^b + I_{ab} A_i^a \dot{r}^i f_\beta^b \\ &= (I^c)_{\alpha\beta}\xi^\alpha + I_{ab} A_i^a \dot{r}^i f_\beta^b, \end{aligned}$$

where, as above, $(I^c)_{\alpha\beta} = I_{ab} f_\alpha^a f_\beta^b$ is the constrained locked inertia tensor. For convenience, we will denote the inverse of I^c by $B^{\alpha\beta} = ((I^c)_{\alpha\beta})^{-1}$. Then we can solve for ξ^α as

$$\begin{aligned} \xi^\alpha &= B^{\alpha\beta} p_\beta - B^{\alpha\beta} I_{ab} A_i^a \dot{r}^i f_\beta^b \\ &= B^{\alpha\beta} p_\beta - B^{\alpha\beta} I_{ab}(A_i^\alpha \dot{r}^i f_\alpha^a + A_i^z \dot{r}^i f_z^a) f_\beta^b \\ &= B^{\alpha\beta} p_\beta - B^{\alpha\beta}(I_{ab} f_\alpha^a f_\beta^b) A_i^\alpha \dot{r}^i \\ &= B^{\alpha\beta} p_\beta - A_i^\alpha \dot{r}^i. \end{aligned} \tag{4.20}$$

Eq. 4.20 explicitly shows the construction of the local form of the connection as the restriction of the mechanical connection to $\mathfrak{g}^q$. Thus,

$$(\mathbb{A}\dot{r})^a = A_i^\alpha \dot{r}^i f_\alpha^a \quad \text{and} \quad \left(\tilde{I}^{-1} p\right)^a = B^{\alpha\beta} p_\beta f_\alpha^a.$$

Then we can easily see that $(A - \mathbb{A})\dot{r} = A_i^z \dot{r}^i f_z$, $z = s+1, \ldots, l$, which is orthogonal to both $\mathbb{A}\dot{r}$ and $\tilde{I}^{-1}p$ with respect to the local form of the locked inertia tensor, I. ▼

Using this lemma, we can now prove the statement regarding form of the constrained Lagrangian. The inner product of horizontal vectors is given by

$$\begin{aligned} \langle\!\langle \mathbf{hor}\, \dot{q}, \mathbf{hor}\, \dot{q} \rangle\!\rangle &= \begin{pmatrix} -\mathbb{A}\dot{r} \\ \dot{r} \end{pmatrix}^T \begin{pmatrix} I & IA \\ A^T I & m \end{pmatrix} \begin{pmatrix} -\mathbb{A}\dot{r} \\ \dot{r} \end{pmatrix} \\ &= (\mathbb{A}\dot{r})^T I (\mathbb{A}\dot{r}) - 2(\mathbb{A}\dot{r})^T I A \dot{r} + \dot{r}^T m \dot{r} \\ &= \dot{r}^T m \dot{r} - (A\dot{r})^T I (A\dot{r}) + \dot{r}^T (A - \mathbb{A})^T I (A - \mathbb{A})\dot{r} \\ &= \dot{r}^T \tilde{M}(r) \dot{r}. \end{aligned}$$

We compare this with the constrained Lagrangian:

$$\begin{aligned}
l_c(r,\dot r,p) &= l(r,\dot r,\tilde I^{-1}p - \mathbb{A}(r)\dot r) - V(r)\\
&= \frac{1}{2}\dot r^T m\dot r + (A\dot r)^T I(-\mathbb{A}\dot r + \tilde I^{-1}p)\\
&\quad + \frac{1}{2}(-\mathbb{A}\dot r + \tilde I^{-1}p)^T I(-\mathbb{A}\dot r + \tilde I^{-1}p) - V(r)\\
&= \frac{1}{2}\left(\dot r^T m\dot r - (A\dot r)^T I(A\dot r) + \dot r^T(A-\mathbb{A})^T I(A-\mathbb{A})\dot r\right)\\
&\quad + (A\dot r)^T I\tilde I^{-1}p - (\mathbb{A}\dot r)^T I\tilde I^{-1}p + \frac{1}{2}(\tilde I^{-1}p)^T I\tilde I^{-1}p - V(r)\\
&= \frac{1}{2}\langle \mathbf{hor}\,\dot q; \mathbf{hor}\,\dot q\rangle + \frac{1}{2}(\tilde I^{-1}p)^T I\tilde I^{-1}p - V(r)\\
&= \frac{1}{2}\dot r^T\tilde M(r)\dot r + \frac{1}{2}p^T(I^c)^{-1}p - V(r).
\end{aligned}$$

Notice that the term

$$(A\dot r)^T I\tilde I^{-1}p - (\mathbb{A}\dot r)^T I\tilde I^{-1}p = \dot r^T(A-\mathbb{A})^T I\tilde I^{-1}p$$

in the above equation disappears in the principal kinematic case (since $p \equiv 0$) and is equal to zero otherwise (by Lemma 4.14). Finally, we note that in the mixed/unconstrained cases, $\dot r(A-\mathbb{A})^T I\mathbb{A}\dot r = 0$, so that the reduced mass-inertia matrix becomes

$$\tilde M(r) = m - \mathbb{A}^T I\mathbb{A}.$$

■

Using Eq. 3.10 for the reduced equations in the presence of constraints, we can again write down the base equations in terms of the constrained Lagrangian (c.f., Eq. 3.13), but this time using the nonholonomic connection to define the constraints:

$$\frac{d}{dt}\left(\frac{\partial l_c}{\partial \dot r^i}\right) - \frac{\partial l_c}{\partial r^i} = -\operatorname{ad}^*_\xi\frac{\partial l}{\partial \xi}\mathbb{A}(\delta r^i) - \frac{\partial l}{\partial \xi}\left(d\mathbb{A}(\dot r,\delta r^i) + \frac{\partial \tilde I^{-1}p}{\partial r^i}\right) + \tau_i - \tau^e_b\mathbb{A}^b_i.$$

In order to write down the equations in matrix form, we substitute for l_c using Eq. 4.19. This gives the following form:

$$\tilde M\ddot r + \dot r^T\tilde C\dot r + \frac{\partial V}{\partial r} + \tilde N = B(r)\tau, \tag{4.21}$$

where

$$\tilde N = \operatorname{ad}^*_\xi\frac{\partial l}{\partial \xi}\mathbb{A}(\cdot) + \frac{\partial l}{\partial \xi}\left(d\mathbb{A}(\dot r,\cdot) + \frac{\partial \tilde I^{-1}p}{\partial r}(\cdot)\right) - \frac{1}{2}\langle p; \frac{\partial (I^c)^{-1}}{\partial r}p\rangle,$$

and $\tilde C$ and B are as defined in the unconstrained case with the mechanical connection replaced by the nonholonomic connection, $A \to \mathbb{A}$. Notice that a very interesting thing has happened in the case of mixed constraints— the mechanical connection is no longer present in the reduced equations. This is understandable since the nonholonomic connection $\mathbb{A}$ encodes the information carried by the mechanical connection, and at the same time incorporates the external constraints.

Thus, for a system with symmetries and nonholonomic constraints in which the constraint distribution is group invariant, it is possible to reduce completely to an extended base space. The equations on this extended base space take the form of second order equations on M and first order equations describing the time evolution of p.

Remarks: The process of reduction presented here consists of three steps, each with a meaningful component in relation to understanding locomotion:

1. Establish a *connection* using external constraints and internal symmetries— this describes the effect of shape changes and momenta on net *position* changes,
2. Develop the *generalized momentum equation*— this governs the flow of the *generalized momenta* and describes how the *velocity* of the system in the direction of the constraints changes with variations in shape, and
3. Solve for the *reduced* base equations, which can be written in terms of the base and momenta variables only— these will describe the *internal shape changes* which the locomotive system must undergo in order to move. Control of these variables is very often assumed, and this set of equations may be useful in justifying this assumption.

4.9 Reconstruction

Having reduced the dynamics to the base space, we would next like to reconstruct the full dynamics of the constrained system. The reconstruction process basically consists of taking the same steps described in the previous section, but in reverse. Thus, given input forces on the base space, we first solve for the extended base variables, including the generalized momenta. Then, using the connection, we can determine the motion along the fiber by integrating the constraint equation (Eq. 4.17):

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}p.$$

In some cases, for example kinematic constraints with a solvable Lie algebra [48], this integration can be done explicitly via quadratures. In general, though, this will need to be done using numerical integration.

The reduction procedure can be very important in a controls context, where it is assumed that the base space is fully actuated (particularly in the context of locomotion, where this is the internal shape space). The goal is to control the dynamics of the remaining variables. In this context, it may often be assumed that the base variables are directly controlled, and so the process of reduction is not necessary. However, it will still be important to reduce the generalized momentum equation in order to decouple the momentum dynamics from the fiber variables.

Thus, given the trajectories that the base dynamics are prescribed to follow, we solve for the fiber variables by integrating up through the generalized momentum equations. In a control theoretic context, though, we may be able to reach various conclusions about the motions of the system without having to solve for explicit trajectories. These issues are taken up in Chapter 6.

Chapter 5

Applications of the Theory

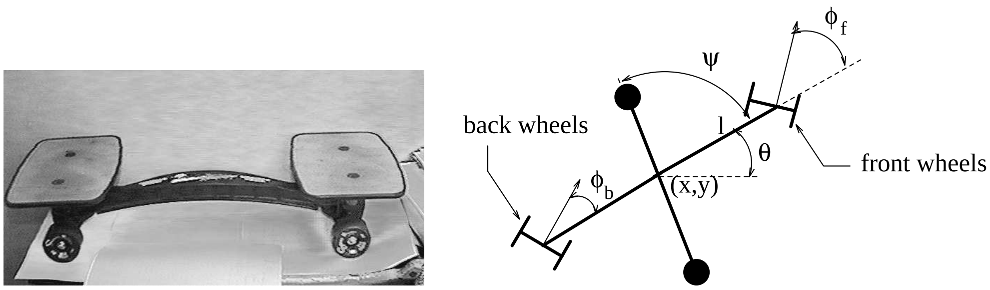

Figure 5.1 The *Snakeboard*, along with a simplified model

5.1 The Snakeboard

Now let us turn to a formulation of the snakeboard problem in terms of the relationships derived above. We briefly recall the description of the snakeboard as given in [57].

The *Snakeboard* (we use italics and capitals to denote the commercially available product licensed by Snakeboard USA, Inc.) is a wheeled vehicle similar to a skateboard, the popular toy/mode of transportation in use by teenagers everywhere. The primary difference between these two products it that the *Snakeboard* has two sets of independently rotating wheel trucks connected by a rigid cross-brace. The unique and attractive feature of the *Snakeboard* is the ability to generate forward motion— even to move up a hill— without kicking off the ground. This is done by performing cyclical twisting motions of the torso, synchronized with a pivoting of the feet. Through a coupling of angular momentum changes and ground contact forces, a snake-like pattern of forward motion is generated.

For our purposes, we use a simplified model of the *Snakeboard*, which we call the snakeboard (Figure 5.1), and in which the human torso is replaced by a mechanical rotor. The configuration manifold for the snakeboard is $Q = SE(2) \times \mathbb{S} \times \mathbb{S} \times \mathbb{S}$.

$SE(2)$ is the group of rigid motions in the plane, and describes the position of the board with respect to some inertial reference frame. $\mathbb{S}$ denotes the group of rotations on $\mathbb{R}^2$. As coordinates for Q we shall use $(x, y, \theta, \psi, \phi_b, \phi_f)$ where (x, y, θ) describes the position of the board with respect to a reference frame, ψ is the angle of the rotor with respect to the board, and ϕ_b and ϕ_f are, respectively, the angles of the back and front wheels with respect to the board.

Notice that the snakeboard can be considered an extension of Elroy's beanie (Example 2.14), with the addition of a pair of wheel constraints. The configuration space easily splits into a trivial principal fiber bundle structure, with $q = (g, r)$ given by $g = (x, y, \theta) \in G = SE(2)$ and $r = (\psi, \phi_b, \phi_f) \in M = \mathbb{S} \times \mathbb{S} \times \mathbb{S}$. The group action for $h = (a^1, a^2, \alpha) \in G$ is given by the map:

$$\begin{aligned}\Phi_h(x, y, \theta, \psi, \phi_b, \phi_f) = (&x\cos\alpha - y\sin\alpha + a^1, x\sin\alpha + y\cos\alpha + a^2,\\ &\theta + \alpha, \psi, \phi_b, \phi_f),\end{aligned}$$

and the lifted action takes the form:

$$T_q\Phi_h(\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}, \dot{\phi}_b, \dot{\phi}_f) = (\dot{x}\cos\alpha - \dot{y}\sin\alpha, \dot{x}\sin\alpha + \dot{y}\cos\alpha, \theta, \dot{\psi}, \dot{\phi}_b, \dot{\phi}_f).$$

Notice that Φ_h and $T_q\Phi_h$ act only on the group variables. The projection π_1 is the canonical projection on the first three coordinates, and π_2 is the projection on the last three coordinates.

The snakeboard is a dynamic system and so we must include the effects of masses and inertias of the rotor, body, and wheels. The parameters we will use for this problem are:

m	: the mass of the board,
J	: the inertia of the board,
J_r	: the inertia of the rotor,
J_w	: the inertia of the wheels (assumed to be the same), and
l	: the length from the board's center of mass to the wheels.

For the snakeboard, the unconstrained Lagrangian is comprised only of kinetic energy terms:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_r(\dot{\psi} + \dot{\theta})^2 + \frac{1}{2}J_w((\dot{\phi}_b + \dot{\theta})^2 + (\dot{\phi}_f + \dot{\theta})^2).$$

It is assumed that the control torques will be applied to the rotor and the front and rear wheel axes. Hence, there is no forcing in the fiber directions. The wheels of the snakeboard are assumed to roll without lateral sliding. As such, the back wheels provide a nonholonomic constraint of the form

$$-\sin(\phi_b + \theta)\dot{x} + \cos(\phi_b + \theta)\dot{y} - l\cos(\phi_b)\dot{\theta} = 0. \tag{5.1}$$

Similarly at the front wheels the constraint is

$$-\sin(\phi_f + \theta)\dot{x} + \cos(\phi_f + \theta)\dot{y} + l\cos(\phi_f)\dot{\theta} = 0. \tag{5.2}$$

For our formulation, we will write the constraints as the kernel of a set of one-forms on Q. To be specific, all velocities must lie in $\ker\{\omega^1, \omega^2\}$, where

$$\omega^1 = -\sin(\phi_b + \theta)dx + \cos(\phi_b + \theta)dy - l\cos(\phi_b)d\theta \tag{5.3}$$

$$\omega^2 = -\sin(\phi_f + \theta)dx + \cos(\phi_f + \theta)dy + l\cos(\phi_f)d\theta \tag{5.4}$$

A quick set of calculations shows that both the Lagrangian and the constraint one-forms are invariant with respect to the lifted group action. Thus, we can write down the constraint distribution, $\mathcal{D}_q = \{v_q \in T_qQ \mid v_q \in \ker\{\omega^1, \omega^2\}\}$, in terms of left-invariant vector fields:

$$\mathcal{D}_q = \operatorname{span}\{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi_b}, \frac{\partial}{\partial \phi_f}\},$$

where

$$\begin{aligned} a &= -l[\cos\phi_b \cos(\phi_f + \theta) + \cos\phi_f \cos(\phi_b + \theta)] \\ b &= -l[\cos\phi_b \sin(\phi_f + \theta) + \cos\phi_f \sin(\phi_b + \theta)] \\ c &= \sin(\phi_b - \phi_f). \end{aligned}$$

We remark that there is an isolated singularity for this distribution at $\phi_b = \phi_f = \pm\frac{\pi}{2}$. This corresponds to the wheels turned perpendicular to the length of the board. It is easy to picture the singularity as allowing two independent motions— pure rotation about the center of mass at the same time as pure translation perpendicular to the board. In the case of the commercial *Snakeboard*, physical limitations do not allow the wheels to reach these angles (they are restricted to approximately $\pm\frac{\pi}{4}$rad of rotation). For this reason, we restrict our analysis to regions in the configuration space that do not include the points $\phi_b = \phi_f = \pm\frac{\pi}{2}$.

5.1.1 The Generalized Momentum Equation

The snakeboard has a non-trivial interaction between the constraints and the symmetries that admits a generalized momentum. Thus, it provides a fairly simple example of a system with mixed constraints (the reader is also referred to a similar example of the Roller Racer discussed in [95]). It is exactly the evolution of the momentum described in the generalized momentum equation that leads to the generation of net forward velocities for the snakeboard.

The initial step in deriving the conservation law will be to choose a left-invariant section of $\mathcal{S}$ (note that for the snakeboard $\mathcal{S}$ is one-dimensional). This can easily be done based on the form of $\mathcal{D}_q$ given above, but we choose here to illustrate the method discussed in Section 4.2. As described in that section, the fact that the constraints given by Eqs. 5.3 and 5.4 are left-invariant and satisfy Condition 4 leads to a natural way of choosing the section. First, use the lifted action to pull back these one-forms to $\mathfrak{g}^*$. This is easily done by evaluating ω^1 and ω^2 at the identity, which yields

$$\begin{aligned}
\omega_e^1 &= -\sin(\phi_b)dx + \cos(\phi_b)dy - l\cos(\phi_b)d\theta \\
\omega_e^2 &= -\sin(\phi_f)dx + \cos(\phi_f)dy + l\cos(\phi_f)d\theta.
\end{aligned}$$

Note that we have identified elements of the cotangent bundle at the group identity with elements of $\mathfrak{g}^*$. The kernel of these dual elements defines a subspace of $\mathfrak{g}$ given by

$$f_1(r) = a^e\frac{\partial}{\partial x} + b^e\frac{\partial}{\partial y} + c^e\frac{\partial}{\partial \theta}$$

where

$$\begin{aligned}
a^e &= -2l\cos\phi_b\cos(\phi_f) \\
b^e &= -l\sin(\phi_b + \phi_f) \\
c^e &= \sin(\phi_b - \phi_f).
\end{aligned}$$

From this we are able to define a basis for $\mathfrak{g}^{\mathcal{S}}$ using the adjoint mapping:

$$\begin{aligned}
f_1(g, r) &= \mathrm{Ad}_g f_1(r) \\
&= (a + yc)\frac{\partial}{\partial x} + (b - xc)\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta},
\end{aligned}$$

where a, b, c are as above. The infinitesimal generator of $f_1(g, r)$ on Q,

$$(f_1(g, r))_Q = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}$$

gives a basis for the constrained fiber distribution,

$$\mathcal{S}_q = \mathrm{span}\{f_1(g, r)_Q\}.$$

Next, we calculate the nonholonomic momentum. Let f^1 denote the dual basis element for $(\mathfrak{g}^{\mathcal{S}})^*$ (dual to $f_1(g, r)$), and let

$$\mathrm{J}^{\mathrm{nhc}}(v_q) = p(v_q)f^1.$$

Then the nonholonomic momentum map is defined by the relation

$$\begin{aligned}
\langle \mathrm{J}^{\mathrm{nhc}}(v_q); \xi f_1(g, r)\rangle &= \langle \mathbb{F}L(v_q); \xi(f_1(g, r))_Q(q)\rangle \\
&= \Big\langle (mv_x, mv_y, \hat{J}v_\theta + J_r v_\psi + J_w(v_b + v_f), J_r v_\psi, J_w v_b, J_w v_f); \\
&\qquad \xi(a, b, c, 0, 0, 0)\Big\rangle \qquad (5.5) \\
&= \xi(mav_x + mbv_y + \hat{J}cv_\theta + J_r cv_\psi + J_w c(v_b + v_f)),
\end{aligned}$$

where $\hat{J} = J + J_r + 2J_w$ is the sum of the moments of inertia. Thus, along trajectories

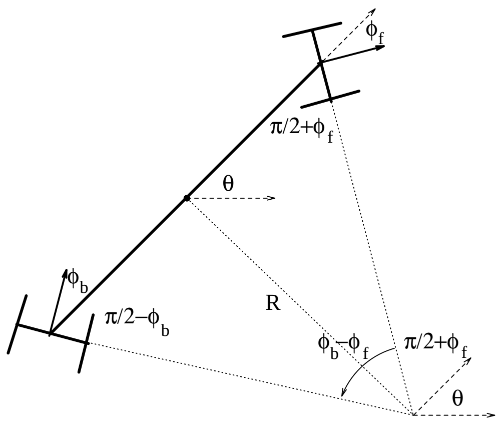

Figure 5.2 Defining the instantaneous center of rotation

(where $v_q = \dot{q}$),

$$p = ma\dot{x} + mb\dot{y} + \hat{J}c\dot{\theta} + J_r c\dot{\psi} + J_w c(\dot{\phi}_b + \dot{\phi}_f).$$

This choice of momentum corresponds to choosing the momentum of the snakeboard along the constrained fiber distribution, or instantaneously around the center of rotation defined by the wheel constraints. This is shown in Figure 5.2, although the actual momentum chosen differs by a scaling factor of $\sin(\phi_b - \phi_f)$.

Before continuing with the derivation of the generalized momentum equation, we first would like to make two simplifying assumptions which will greatly reduce the complexity of the derivations to follow. First, we make the assumption that the wheels are controlled to move out of phase with each other. In other words, let $\phi = \phi_b = -\phi_f$. Second, along the lines of Bloch et al., we assume that $\hat{J} = J + J_r + 2J_w = ml^2$. The first assumption is motivated by research experience suggesting that this type of phasing can give all of the basic locomotive gaits. The second assumption is used mainly to simplify the equations.

Given this, the Lagrangian takes the form:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}J_r\dot{\psi}(\dot{\psi} + 2\dot{\theta}) + J_w\dot{\phi}^2.$$

We can also then rewrite the nonholonomic momentum map as

$$\mathrm{J}^{\mathrm{nhc}}(q, \dot{q}) = (-2ml\cos^2\phi\cos\theta\dot{x} - 2ml\cos^2\phi\sin\theta\dot{y} + ml^2\sin 2\phi\dot{\theta} + J_r\sin 2\phi\dot{\psi})f^1.$$

Next, we use the locked inertia tensor in order to define the connection. Using the formula,

$$\langle I^c\eta^q; \xi^q\rangle = \langle\langle \eta^q_Q, \xi^q_Q \rangle\rangle,$$

with $\xi^q, \eta^q \in \mathfrak{g}^{\mathcal{S}}$, we have

$$I^c = 4ml^2 \cos^2 \phi.$$

Then, the group symmetry part of the nonholonomic connection one-form is defined to be

$$\mathcal{A}^{\text{sym}} = (I^c)^{-1}\, \mathrm{J}^{\text{nhc}} = (-\frac{1}{2l}(\cos\theta\, \mathrm{d}x - \sin\theta\, \mathrm{d}y) + \frac{1}{2}\tan\phi\, \mathrm{d}\theta + \frac{J_r}{2ml^2}\tan\phi\, \mathrm{d}\psi) f_1.$$

This defines a horizontal subspace for the constraint distribution, $\mathcal{D}$, as a bundle over M. Further, we can use the relationship,

$$\mathcal{A}^{\text{sym}} = (I^c)^{-1} p,$$

along with the constraint one-forms, ω^1 and ω^2, to synthesize a connection on the full state space, TQ. To do so, write these one-forms in matrix form as constraint equations evaluated along trajectories, $(q, \dot{q})$:

$$W(r) g^{-1}\dot{g} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{J_r}{2ml^2}\tan\phi & 0 \end{pmatrix} \dot{r} = \begin{pmatrix} 0 \\ 0 \\ \frac{p}{4ml^2\cos^2\phi} \end{pmatrix}, \tag{5.6}$$

where

$$W(r) = \begin{pmatrix} \sin\phi & \cos\phi & l\cos\phi \\ -\sin\phi & \cos\phi & -l\cos\phi \\ -\frac{1}{2l} & 0 & \frac{1}{2}\tan\phi \end{pmatrix}.$$

Notice that $\det[W(r)] = 1$ and that multiplying by $W(r)^{-1}$ yields the local form for the body one-form, $\mathcal{A}^b$, on TQ given by

$$\mathcal{A}^b \dot{q} = g^{-1}\dot{g} + \mathbb{A}\dot{r},$$

with

$$\mathbb{A} = \begin{pmatrix} -\frac{J_r}{2ml}\sin 2\phi & 0 \\ 0 & 0 \\ \frac{J_r}{2ml^2}\sin^2\phi & 0 \end{pmatrix}. \tag{5.7}$$

Furthermore, if the momentum is given by p, then the allowable trajectories for this system must satisfy

$$\mathcal{A}^b(q) \cdot \dot{q} = \Gamma(r) p = \begin{pmatrix} -\frac{1}{2l} \\ 0 \\ \frac{1}{2ml^2}\tan\phi \end{pmatrix} p. \tag{5.8}$$

Eq. 5.8 fully specifies the trajectories along the fiber in terms of base and momentum variables. Next, we examine the flow of the momentum p governed by the generalized momentum equation, Eq. 4.2. For the snakeboard, this equation is

$$\begin{aligned}\dot{p} &= \frac{\partial L}{\partial \dot{q}}\left(\frac{d}{dt}(f_1)\right)_Q \\ &= m\dot{a}\dot{x} + m\dot{b}\dot{y} + \dot{c}(ml^2\dot{\theta} + J_r\dot{\psi}) \\ &= 2ml(\cos\theta\sin 2\phi\,\dot{\phi} + \sin\theta\cos^2\phi\,\dot{\theta})\dot{x} + 2ml(\sin\theta\sin 2\phi\,\dot{\phi} - \cos\theta\cos^2\phi\,\dot{\theta})\dot{y} \\ &\quad + 2ml^2\cos 2\phi\,\dot{\phi}\dot{\theta} + 2J_r\cos 2\phi\,\dot{\phi}\dot{\psi}.\end{aligned}$$

However, as we saw in Section 4.7, it is possible to rewrite this equation in terms of the base and momentum variables *only*, i.e., to eliminate the dependence of the fiber coordinates. Substituting for $(\dot{x}, \dot{y}, \dot{\theta})$ from Eq. 5.8 gives the generalized momentum equation in a reduced form:

$$\dot{p} = 2J_r\cos^2\phi\,\dot{\phi}\dot{\psi} - \tan\phi\,\dot{\phi}p.$$

The dynamic equations governing the base variables can then be determined using Eq. 4.21 as described in Section 4.8. For convenience, we recall the general form of this equation here:

$$\tilde{M}\ddot{r} + \dot{r}^T\tilde{C}\dot{r} + \frac{\partial V}{\partial r} + \tilde{N} = B(r)\tau.$$

The local form of the nonholonomic connection is given in Eq. 5.7 above, and we remark that we can write down the local form of the locked inertia tensor directly from the reduced Lagrangian,

$$l = (\xi^T, \dot{r}^T)\begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & ml^2 & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & 2J_w \end{pmatrix}\begin{pmatrix} \xi \\ \dot{r} \end{pmatrix},$$

as

$$I = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & ml^2 \end{pmatrix}.$$

Similarly, we see that the mechanical connection is just

$$A = I^{-1}IA = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{ml^2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ J_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{J_r}{ml^2} & 0 \end{pmatrix}.$$

Thus, the constrained Lagrangian is just

$$
\begin{aligned}
l_c &= \frac{1}{2}\dot{r}^T \tilde{M}\dot{r} + \frac{1}{2}\langle p; (I^c)^{-1}p\rangle - V(r) \\
&= \frac{1}{2}(\dot{\psi}, \dot{\phi})\Bigg(\begin{pmatrix} J_r & 0 \\ 0 & 2J_w \end{pmatrix} \\
&\qquad - \begin{pmatrix} -\frac{J_r}{ml}\sin\phi\cos\phi & 0 & \frac{J_r}{ml^2}\sin^2\phi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{J_r}{l}\sin\phi\cos\phi & 0 \\ 0 & 0 \\ J_r\sin^2\phi & 0 \end{pmatrix} \Bigg) \begin{pmatrix} \dot{\psi} \\ \dot{\phi} \end{pmatrix} \\
&\qquad + \frac{1}{2}\frac{p^2}{4ml^2\cos^2\phi} \\
&= \left(J_r - \frac{J_r^2}{ml^2}\sin^2\phi \right)\dot{\psi}^2 + 2J_w\dot{\phi}^2 + \frac{p^2}{8ml^2\cos^2\phi}.
\end{aligned}
$$

Having calculated the reduced mass-inertia matrix, $\tilde{M}$, we then calculate

$$
\begin{aligned}
\tilde{C}_{ijk}(r)\dot{r}^j\dot{r}^k &= \frac{1}{2}\left(\frac{\partial \tilde{M}_{ij}}{\partial r^k} + \frac{\partial \tilde{M}_{ik}}{\partial r^j} - \frac{\partial \tilde{M}_{kj}}{\partial r^i} \right)\dot{r}^j\dot{r}^k \\
&= \begin{pmatrix} -\frac{J_r^2}{ml^2}\sin 2\phi\,\dot{\psi}\dot{\phi} \\ \frac{J_r^2}{ml^2}\sin\phi\cos\phi\,\dot{\psi}^2 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{N} &= \mathrm{ad}^*_{\xi}\frac{\partial l}{\partial \xi}\mathbb{A}(\cdot) + \frac{\partial l}{\partial \xi}\left(d\mathbb{A}(\dot{r},\cdot) + \frac{\partial \tilde{I}^{-1}p}{\partial r}(\cdot) \right) - \frac{1}{2}\langle p; \frac{\partial (I^c)^{-1}}{\partial r}p\rangle \\
&= \begin{pmatrix} \frac{J_r^2}{ml^2}\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} + \frac{J_r}{2ml^2}\dot{\phi}p \\ -\frac{J_r^2}{ml^2}\sin\phi\cos\phi\,\dot{\psi}^2 \end{pmatrix}.
\end{aligned}
$$

Plugging this into Eq. 4.21 gives the base equations:

$$
(1 - \frac{J_r}{ml^2}\sin^2\phi)\ddot{\psi} = \frac{J_r}{2ml^2}\sin 2\phi\,\dot{\phi}\dot{\psi} - \frac{1}{2ml^2}\dot{\phi}p + \frac{1}{J_r}\tau_{\psi} \tag{5.9}
$$

$$
\ddot{\phi} = \frac{1}{2J_w}\tau_{\phi}. \tag{5.10}
$$

One additional remark— the snakeboard satisfies Condition 4, and so we notice that the base equations do not involve the Lagrange multipliers. For problems that satisfy this condition, it can sometimes be easier to write down the base equations directly by substituting the constraint equations from Eq. 5.6 into the full set of dynamical equations. This will not always be the case, as there are some situations in which it is easier to work with the variables on the reduced space. One example of this is the ball rolling on a rotating plate [10], in which reduction allows one to bypass finding an amenable parameterization for $SO(3)$, and instead work with just the Lie algebra, $\mathfrak{so}(3)$, and the reduced space of the plane ($\mathbb{R}^2$).

To demonstrate this procedure, we will perform the calculations for the snakeboard. First, we write down the Euler-Lagrange equations for the base variables:

$$\begin{aligned} J_r(\ddot{\psi} + \ddot{\theta}) &= \tau_\psi \\ 2J_w\ddot{\phi} &= \tau_\phi \end{aligned}$$

Notice that the Lagrange multipliers do not enter into these equations. Then we can substitute for $\dot{\theta}$ using the fiber equations, Eq. 5.6:

$$\dot{\theta} = -\frac{J_r}{ml^2}\sin^2\phi\,\dot{\psi} + \frac{1}{2ml^2}\tan\phi\,p.$$

Differentiating with respect to time gives:

$$\begin{aligned} \ddot{\theta} &= -\frac{J_r}{ml^2}(\sin 2\phi\,\dot{\phi}\dot{\psi} + \sin^2\phi\,\ddot{\psi}) + \frac{1}{2ml^2}(\sec^2\phi\,\dot{\phi}p + \tan\phi\,\dot{p}) \\ &= -\frac{J_r}{ml^2}(\sin 2\phi\,\dot{\phi}\dot{\psi} + \sin^2\phi\,\ddot{\psi}) + \frac{1}{2ml^2}(\sec^2\phi\,\dot{\phi}p) \\ &\qquad + \frac{1}{2ml^2}\tan\phi(2J_r\cos^2\phi\,\dot{\phi}\dot{\psi} - \tan\phi\,\dot{\phi}p) \\ &= -\frac{J_r}{ml^2}\sin^2\phi\,\ddot{\psi} - \frac{J_r}{2ml^2}\sin 2\phi\,\dot{\phi}\dot{\psi} + \frac{1}{2ml^2}\dot{\phi}p, \end{aligned}$$

where we have used the generalized momentum equation,

$$\dot{p} = 2J_r\cos^2\phi\,\dot{\phi}\dot{\psi} - \tan\phi\,\dot{\phi}p,$$

to substitute for $\dot{p}$. Thus, we can write the dynamics on the base space as above:

$$\begin{aligned} (1 - \frac{J_r}{ml^2}\sin^2\phi)\ddot{\psi} &= \frac{J_r}{2ml^2}\sin 2\phi\,\dot{\phi}\dot{\psi} - \frac{1}{2ml^2}\dot{\phi}p + \frac{1}{J_r}\tau_\psi \\ \ddot{\phi} &= \frac{1}{2J_w}\tau_\phi. \end{aligned}$$

Notice the natural appearance of the reduced mass-inertia matrix in these equations:

$$\tilde{M}(r) = \begin{pmatrix} (J_r - \frac{J_r^2}{ml^2}\sin^2\phi) & 0 \\ 0 & 2J_w \end{pmatrix}.$$

5.1.2 Reconstruction

To recapitulate the ideas presented so far for the snakeboard, we have used the generalized momentum equation and the tools associated with it in order to simplify the dynamics to the following form:

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}p, \tag{5.11}$$

$$\dot{p} = \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T\sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T\sigma_{pp}(r)p + \tilde{\tau}(r), \quad \text{and} \tag{5.12}$$

$$\tilde{M}\ddot{r} + \dot{r}^T\tilde{C}\dot{r} + \frac{\partial V}{\partial r} + \tilde{N} = B(r)\tau. \tag{5.13}$$

Thus we have reduced the equations from 6 second order equations with 3 first order constraint equations (equivalently, 15 first order equations) to 3 second order equations and 4 first order equations (10 first order equations). A few words of what this implies for both reduction and reconstruction are in order. Suppose that we are given a base integral curve (a curve, $c_b(t)$ on M) which we would like to follow, along with some initial point $v_q \in TQ$ such that $\pi_2(v_q) = c_b(0)$. The initial velocity v_q defines a starting value for the momentum, $p(0)$, and Eq. 5.12 defines the time evolution of the generalized momentum, $p(t)$. Assuming that this is solved for and that we have complete actuation of the base variables, then we can determine explicitly the control torques necessary to follow the specified base curve. This can *always* be done, provided that the constraints plus the generalized momentum equation can be used to build a connection. This will be true as long as there are no constraints acting purely to constrain the base variables, which has been true in all of the examples studied to date. The process of reconstruction is then completed by lifting the base curve up through the Lie algebra via the connection (Eq. 5.11) in order to solve for the dynamics on the fiber.

Most importantly, however, we have brought out some of the intuitively illuminating intrinsic structure of the problem's dynamics. This was done by writing the connection in a manner that directly relates shape changes to momentum and position changes. The next step will be to examine controllability results which use the specialized structure of the connection. This topic is taken up in Chapter 6.

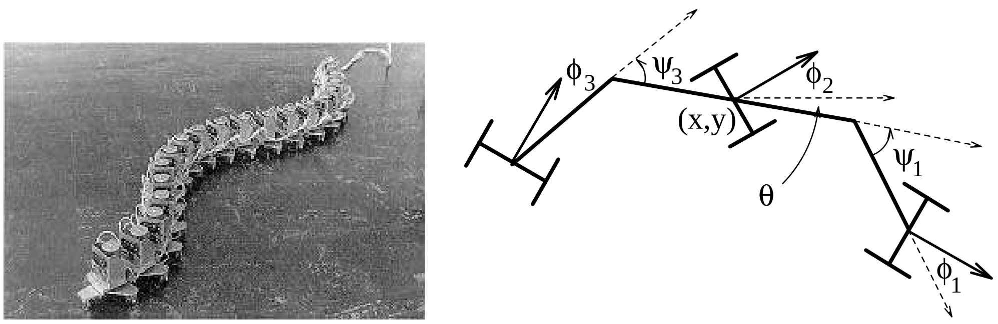

Figure 5.3 The kinematic snake

5.2 The Kinematic Snake of Hirose

In this section, we present an example of a principal kinematic system in which there are sufficient constraints to define a connection. The system presented here is based on the ACM III snake robot built by Hirose [35] (a model is shown in Figure 5.3), where certain assumptions are made regarding the actuation of the individual segments. The basic principles relating the ACM III to a real snake are based on the assumption that the body of a real snake has a small coefficient of friction along the length of its belly and a high coefficient of friction transverse to its length. In this approximation, the snake is prevented by friction from slipping laterally, while at the same time able to move each point on its body forward without impedance. Hirose used this differential friction as the basis for choosing wheeled segments to guide the snake in its motion. Discretization of the snake's backbone curve allows us to model the snake as a finite number of such wheeled segments. The reader is referred to [33] for more detail.

We present here an analysis of Hirose's snake from the standpoint of the material presented above on nonholonomic constraints and symmetries. Obviously, there may be different perspectives from which to view this system. For instance, Hirose has studied the generation of locomotion for these systems by using force balances on each pair of segments [33, 35]. The advantage in the current presentation will be that we seek to divide out the dynamics according to the pieces that are important to locomotion, and to put the equations in a form amenable to treatment by existing control and stabilization theory.

Any model of the Hirose snake must be able to describe a many-segmented body which locomotes using only internal torques. We begin by examining the three segment model shown in Figure 5.3, since it will define a Chaplygin system. In this model, we assume that there are control inputs at each of the two segment joints and each of the three wheel pivot joints. As in the snakeboard, the wheels themselves are assumed to rotate freely, i.e., they are unactuated. The reason for initially looking at this particular model is that the three wheel constraints define a connection for this problem. Thus, away from singularities, the positioning and motion of the snake in the plane is fully determined by the shape variables shown in Figure 5.3.

By recognizing that the three wheel constraints define a connection, we are naturally led to a method for handling additional body segments. The technique will consist of using the first three segments to define the motion in $SE(2)$, and then using the wheel constraints of the additional bays as the governing equations for these segments. Thus, we develop a system that has a "following" behavior, in which the lead segments define the path to be traced, and the additional segments are constrained to follow this lead. In a real snake, the additional segments serve a useful purpose in providing greater stability for the snake, and can be used to perform more complicated maneuvers, such as crossing over gaps in the floor or pushing off objects to move along a slippery surface.

For the three segment snake drawn in Figure 5.3, we label the center point of the middle segment (and hence the center of mass when the snake is fully extended) by

$(x, y, \theta) \in SE(2)$, the wheel angles of segments 1, 2, and 3 by $(\phi_1, \phi_2, \phi_3) \in \mathbb{S}\times\mathbb{S}\times\mathbb{S}$, respectively, and the relative orientation of segment 1 with respect to segment 2 and segment 2 with respect to segment 3 by $(\psi_1, \psi_3) \in \mathbb{S}\times\mathbb{S}$, respectively. We will treat the Hirose snake as a purely kinematic system, and so we do not consider the dynamics that arise due to masses and inertias of the wheels and body segments.

Each no-slip wheel constraint takes the following general form:

$$(-\sin\tilde{\phi}_i, \cos\tilde{\phi}_i) \cdot \begin{pmatrix} \dot{\bar{x}}_i \\ \dot{\bar{y}}_i \end{pmatrix} = 0,$$

where $\tilde{\phi}_i$ is the absolute angle (measured with respect to horizontal) of the i^{th} wheel, and $(\bar{x}_i, \bar{y}_i)$ is the Cartesian positioning of the center of rotation for the i^{th} wheel. Using this notation, we find that

$$\begin{aligned} \bar{x}_1 &= x + l\cos\theta + l\cos(\theta - \psi_1), & \bar{y}_1 &= y + l\sin\theta + l\sin(\theta - \psi_1), \\ \bar{x}_2 &= x, & \bar{y}_2 &= y, \\ \bar{x}_3 &= x - l\cos\theta - l\cos(\theta + \psi_3), & \bar{y}_3 &= y - l\sin\theta - l\sin(\theta + \psi_3), \end{aligned}$$

and

$$\begin{aligned} \tilde{\phi}_1 &= \phi_1 - \psi_1 + \theta, \\ \tilde{\phi}_2 &= \phi_2 + \theta, \\ \tilde{\phi}_3 &= \phi_3 + \psi_3 + \theta. \end{aligned}$$

Thus, the constraint equations can be written as

$$\begin{aligned} -\sin\tilde{\phi}_1\,\dot{x} + \cos\tilde{\phi}_1\,\dot{y} - l(\cos\phi_1 + \cos(\phi_1 - \psi_1))\dot{\theta} &= -l\cos\phi_1\,\dot{\psi}_1, \\ -\sin\tilde{\phi}_2\,\dot{x} + \cos\tilde{\phi}_2\,\dot{y} &= 0, \\ -\sin\tilde{\phi}_3\,\dot{x} + \cos\tilde{\phi}_3\,\dot{y} - l(\cos\phi_3 + \cos(\phi_3 + \psi_3))\dot{\theta} &= l\cos\phi_3\,\dot{\psi}_3, \end{aligned} \tag{5.14}$$

which can be written in a form similar to the snakeboard, but without a generalized momentum term:

$$W(r)T_gL_{g^{-1}}\dot{g} = \Gamma(r)\dot{r}.$$

This supplies three constraints on the three dimensional Lie group, $G = SE(2)$. Straightforward calculations similar to the two-wheeled mobile robot above show that the constraints are G-invariant. Therefore, the kernel of these constraints defines a connection on the trivial principal fiber bundle $Q = G \times M = SE(2) \times \mathbb{S}\times\mathbb{S}\times\mathbb{S}\times\mathbb{S}\times\mathbb{S}$. We can invert the constraint equations directly to write the local form of the connection one-form as

$$\xi = -\mathbb{A}(r)\dot{r}.$$

This gives the following:

$$\xi^1 = -\frac{l^2 \cos\phi_2}{\det W}\left[\dot\psi_1 \cos\phi_1 \left(\cos(\phi_3 - \psi_3) + \cos\phi_3\right) + \dot\psi_3 \cos\phi_3 \left(\cos(\phi_1 + \psi_1) + \cos\phi_1\right)\right] \tag{5.15}$$

$$\begin{aligned}\xi^2 &= -\frac{l^2 \sin\phi_2}{\det W}\left[\dot\psi_1 \cos\phi_1 \left(\cos(\phi_3 - \psi_3) + \cos\phi_3\right) + \dot\psi_3 \cos\phi_3 \left(\cos(\phi_1 + \psi_1) + \cos\phi_1\right)\right] \qquad (5.16)\\ &= \xi^1 \tan\phi_2\end{aligned}$$

$$\xi^3 = \frac{l}{\det W}\left[\dot\psi_1 \cos\phi_1 \sin(\phi_3 + \psi_3 - \phi_2) - \dot\psi_3 \cos\phi_3 \sin(\phi_1 - \psi_1 - \phi_2)\right], \tag{5.17}$$

where

$$\begin{aligned}\det W = l[&\sin(\phi_1 - \psi_1 - \phi_2)(\cos(\phi_3 + \psi_3) + \cos\phi_3)\\ &+ \sin(\phi_3 + \psi_3 - \phi_2)(\cos(\phi_1 - \psi_1) + \cos\phi_1)].\end{aligned}$$

Next, we examine a possible method for extending the snake to an arbitrary number of segments. Suppose that we add a fourth segment. Let ϕ_4 and ψ_4 denote the angles of the wheels and the body segment, respectively. Then, following the above notation,

$$\begin{aligned}\tilde\phi_4 &= \phi_4 + \psi_3 + \psi_4 + \theta,\\ \bar x_4 &= x - l\cos\theta - 2l\cos(\theta + \psi_3) - l\cos(\theta + \psi_3 + \psi_4),\\ \bar y_4 &= y - l\sin\theta - 2l\sin(\theta + \psi_3) - l\sin(\theta + \psi_3 + \psi_4),\end{aligned}$$

and the constraint becomes

$$\begin{aligned}-\sin\tilde\phi_4\, \dot x + \cos\tilde\phi_4\, \dot y - l(\cos(\phi_4 + \psi_3 + \psi_4) + 2\cos(\phi_4 + \psi_4) + \cos\phi_4)\dot\theta&\\ - l(2\cos(\phi_4 + \psi_4) + \cos\phi_4)\dot\psi_3 = l\cos\phi_4\, \dot\psi_4.&\end{aligned} \tag{5.18}$$

Observe that we have added two additional degrees of freedom— one wheel angle, ϕ_4, and one inter-segment angle, ψ_4— and added one kinematic constraint given by Eq. 5.18. As with the first three body segments, we will control both ϕ_4 and ψ_4, but now are forced to satisfy the constraint as well. This is easily done, however, by choosing to control the wheel angle, while inverting Eq. 5.18 to establish a governing equation for ψ_4. Doing this, we find that

$$\begin{aligned}\dot\psi_4 = \frac{1}{\cos\phi_4}\Big(&\frac{1}{l}\big(- \sin\tilde\phi_4^e\, \xi^1 + \cos\tilde\phi_4^e\, \xi^2\big) - (\cos\tilde\phi_4^e + 2\cos(\phi_4 + \psi_4) + \cos\phi_4)\xi^3\\ &- (2\cos(\phi_4 + \psi_4) + \cos\phi_4)\dot\psi_3\Big),\end{aligned}$$

where $\tilde\phi_4^e = \phi_4 + \psi_3 + \psi_4$, i.e., $\tilde\phi_4\big|_{g=e} = \tilde\phi_4\big|_{\theta=0}$. Notice that the process of solving for $\dot\psi_4$ yields a term with $\cos\phi_4$ in the denominator, which is nonzero for all values of ϕ_4 that we will consider here (we continue the assumption, as in the snakeboard, that the wheels cannot pivot to an angle of $\pm\frac{\pi}{2}$).

Repeating this process, we can add as many additional segments as we desire, with the guarantee that each of the following segments properly satisfy all of the constraints. To illustrate, we will do this for a fifth constraint, of the form

$$\begin{aligned} l(2\cos(\phi_5+\psi_5)+&\cos\phi_5)\dot{\psi}_4 + l\cos\phi_5\,\dot{\psi}_5 \\ &= -\sin\tilde{\phi}_5^e\,\xi^1 + \cos\tilde{\phi}_5^e\,\xi^2 \\ &\quad - l(\cos\tilde{\phi}_5^e + 2\cos(\phi_5+\psi_4+\psi_5) + 2\cos(\phi_5+\psi_5) + \cos\phi_5)\xi^3 \\ &\quad - l(2\cos(\phi_5+\psi_4+\psi_5) + 2\cos(\phi_5+\psi_5) + \cos\phi_5)\dot{\psi}_3, \end{aligned} \tag{5.19}$$

where again, $\tilde{\phi}_5^e = \tilde{\phi}_5\big|_{g=e} = \phi_5 + \psi_3 + \psi_4 + \psi_5$. For a five link snake, we can thus invert Eqs. 5.18 and 5.19 to determine governing equations for ψ_4 and ψ_5.

One point to notice is that as we continue to add constraints, it will always be possible to arrange the equations in the following form:

$$B(\phi_4,\ldots,\phi_k)\begin{pmatrix}\dot{\phi}_4\\ \vdots\\ \dot{\phi}_k\end{pmatrix} = C(\phi_4,\ldots,\phi_k,\psi_3,\ldots,\psi_k)\begin{pmatrix}\xi\\ \dot{\psi}_3\end{pmatrix},$$

where k is the total number of body segments for the snake. Also, the matrix B is lower triangular, with determinant

$$\det(B) = \prod_{j=4}^{k} l\cos\phi_j,$$

and so is always invertible.

To this point, we have not touched on a very important aspect of this problem, namely the generation of actual locomotion patterns for the snake. This topic is the subject of the next chapter on controllability and gaits.

Chapter 6

Controllability and Gaits

6.1 Background and Formulation

For systems of the form we are discussing, the Lagrangian and the constraints are left-invariant, i.e.,

$$L(\Phi_h q, T_q \Phi_h \dot{q}) = L(q, \dot{q}) \quad \text{and} \quad \omega^i(hq) = T^*_{hq} \Phi_{h^{-1}} \omega(q).$$

In such cases, it was shown in Chapter 4 (Eq. 4.15) that the equations of motion can be transformed into the following form (we assume forcing only in the base directions):

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}(r)p, \tag{6.1}$$

$$\dot{p} = \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T \sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T \sigma_{pp}(r)p, \tag{6.2}$$

$$\ddot{r} = u. \tag{6.3}$$

For the terms, $\sigma_{\dot{r}\dot{r}}$, $\sigma_{p\dot{r}}$, and σ_{pp} of the generalized momentum equation, we note that the proof of invariance given in Section 4.7 implies that they are strictly functions of the base variables, r. Therefore, the generalized momentum equation can be written as a function of the base and momentum variables *only*. In the case that $\sigma_{\dot{r}\dot{r}} \equiv 0$, the system can be written in standard form for a nonlinear control system with affine inputs (where the inputs appear linearly with an affine drift term):

$$\dot{z} = f(z) + h_i(z)v^i, \tag{6.4}$$

where $z = (g, p, r) \in N = G \times \left(\mathfrak{g}^{\mathcal{S}}\right)^* \times M$. While this can be done for the principal kinematic and unconstrained cases, it will not be true in general. In order to write the system in the standard control form of Eq. 6.4, we must first dynamically extend the control inputs by redefining them as higher derivatives of the input variables. Equivalently, this can be thought of as specifying the accelerations to be the control inputs (this is done implicitly in Eq. 6.3). This makes sense for analyzing a fully dynamic, mechanical system, where the inputs enter as control torques acting at the level of accelerations. Let $u = \dot{v}$, and define the manifold N for our problem as $N = G \times \left(\mathfrak{g}^{\mathcal{S}}\right)^* \times TM$. Then, using $z = (g, p, r, \dot{r}) \in N$, we see that Eqs. 6.1–6.3 can be written in the standard control form of Eq. 6.4 with

$$f(z) = \begin{pmatrix} g(-\mathbb{A}\dot{r} + \tilde{I}^{-1}p) \\ \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}\dot{r} + p^T\sigma_{p\dot{r}}\dot{r} + \frac{1}{2}p^T\sigma_{pp}p \\ \dot{r} \\ 0 \end{pmatrix} \quad \text{and} \quad h_i(z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_i \end{pmatrix}, \tag{6.5}$$

where e_i is the m-vector ($m = \dim M$) with a "1" in the i^{th} row and "0" otherwise.

Example 6.1 Snakeboard *(cont.)*

Returning to the snakeboard example of Chapter 5, recall the equations for the base dynamics given by Eqs. 5.9 and 5.10:

$$\begin{aligned}(1 - \frac{J_r}{ml^2}\sin^2\phi)\ddot{\psi} &= \frac{J_r}{2ml^2}\sin 2\phi\,\dot{\phi}\dot{\psi} - \frac{1}{2ml^2}\dot{\phi}p + \frac{1}{J_r}\tau_\psi \\ \ddot{\phi} &= \frac{1}{2J_w}\tau_\phi,\end{aligned}$$

One of the results proved in [57] is that the base dynamics are controllable. This is easily seen from the equations above— we can directly invert for any desired base dynamics, given the flow of the momentum as a state. Therefore, we can feedback transform these equations into control form as

$$\ddot{\psi} = u_\psi \quad \text{and} \quad \ddot{\phi} = u_\phi.$$

Letting $z = (x, y, \theta, p, \psi, \phi, \dot{\psi}, \dot{\phi}) \in N$, we can then write the snakeboard equations in the form of Eqs. 6.5:

$$f = \begin{pmatrix} \frac{\cos\theta\left(-p+\dot{\psi}J_r\sin 2\phi\right)}{2ml} \\ \frac{\sin\theta\left(-p+\dot{\psi}J_r\sin 2\phi\right)}{2ml} \\ \frac{-2\dot{\psi}J_r\sin^2\phi+p\tan\phi}{2ml^2} \\ 2\dot{\phi}\dot{\psi}J_r\cos^2\phi - \dot{\phi}p\tan\phi \\ \dot{\psi} \\ \dot{\phi} \\ 0 \\ 0 \end{pmatrix}, \quad h_\psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad h_\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

6.1.1 Free Lie Algebras

In order to investigate controllability properties for these types of nonlinear control systems with drift, we will use the strongest conditions of which we are currently aware, given by Sussman in [92]. To use this theory, we will need to develop a rigorous notion of the degree of a Lie bracket. This is given in terms of free Lie algebras. We refer to Serre [86] for the relevant notation. The notation defined in this section can be a bit cumbersome, and so the reader not interested in these details is encouraged to glance briefly at Proposition 6.5 before moving on to the next section. It will certainly suffice to have an intuitive idea of the degree of a Lie bracket in order to understand the sections that follow.

Definition 6.2 *A* magma *is a set M with a product map from $M \times M$ into M and denoted $(m_1, m_2) \mapsto m_1 \cdot m_2$.*

Given another set $\mathbf{X}$ (assumed here to be finite), we can construct a magma in the following manner. Let $\mathbf{X}_1 = \mathbf{X}$, and inductively define $\mathbf{X}_n = \coprod_{p+q=n} \mathbf{X}_p \times \mathbf{X}_q$, where $\coprod$ represents the disjoint union of these product operations.

Definition 6.3 *The* free magma *on $\mathbf{X}$ is the set*

$$M_{\mathbf{X}} = \coprod_{n=1}^{\infty} \mathbf{X}_n$$

with magma map taking $M_{\mathbf{X}} \times M_{\mathbf{X}}$ to $M_{\mathbf{X}}$ given by the canonical inclusion of $\mathbf{X}_p \times \mathbf{X}_q \to \mathbf{X}_{p+q} \subset M_{\mathbf{X}}$ resulting from the definition of $\mathbf{X}_n$ above. The length *of an element $w \in M_{\mathbf{X}}$ is the unique integer n such that $w \in \mathbf{X}_n$.*

Next we define the *free algebra* $A_{\mathbf{X}}$ to be the $\mathbb{R}$-algebra generated by $M_{\mathbf{X}}$. An element $\alpha \in A_{\mathbf{X}}$ is then a finite sum $\alpha = \sum_{m \in M_{\mathbf{X}}} c_m m$, with $c_m \in \mathbb{R}$. The product in $A_{\mathbf{X}}$ derives naturally from the magma map on $M_{\mathbf{X}}$. Let $\mathcal{I}$ denote the two-sided ideal of $A_{\mathbf{X}}$ generated by elements of the form $a \cdot a$ and $a \cdot (b \cdot c) + c \cdot (a \cdot b) + b \cdot (c \cdot a)$ (recall the definition of a two-sided ideal given in Chapter 2). Note that this is quite suggestive of the defining identities of a Lie algebra.

Definition 6.4 *The* free Lie algebra *$L_{\mathbf{X}}$ is the quotient algebra given by $A_{\mathbf{X}}/\mathcal{I}$. We write the inherited product on $L_{\mathbf{X}}$ as $[\cdot, \cdot]$.*

If we denote by $\mathrm{Br}(\mathbf{X})$ the subset of $L_{\mathbf{X}}$ containing purely products of elements in $\mathbf{X}$, then we see that $\mathrm{Br}(\mathbf{X})$ generates $L_{\mathbf{X}}$ as a vector space over $\mathbb{R}$. In order to write down a linearly independent generating set for $L_{\mathbf{X}}$ we could use a Philip Hall basis [74, 86], but instead we will rely on a result of Lewis [53]:

Proposition 6.5 *Every element of the free Lie algebra $L_{\mathbf{X}}$ can be written as a linear combination of repeated brackets of the form*

$$[X_k, [X_{k-1}, [\ldots, [X_2, X_1] \ldots]]],$$

where $X_i \in \mathbf{X}$, $i = 1, \ldots, k$.

It should be clear now that we can use this free Lie algebra to define a notion of degree for a Lie bracket. Let the set $\mathbf{X} = (X_0, \ldots, X_m)$ be a finite sequence of indeterminates. Then we inherit a sense of length from the magma $M_{\mathbf{X}}$.

Definition 6.6 *Let the* degree of B relative to X_a*, denoted* $\delta^a(B)$*, be the integer number of times that* X_a *appears in the bracket* B*. The* degree *of* $B \in \mathrm{Br}(\mathbf{X})$ *is then given by*

$$\delta(B) = \sum_{a=0}^{m} \delta^a(B). \tag{6.6}$$

To illustrate this, suppose that $m = 2$. Then the degrees for

$$Y = [X_0, [X_1, X_2], [X_0, X_1]\,] \quad \text{and} \quad Z = [X_1, X_2]$$

are $\delta^0(Y) = 2$, $\delta^1(Y) = 2$, $\delta^2(Y) = 1$ and $\delta^0(Z) = 0$, $\delta^1(Z) = 1$, $\delta^2(Z) = 1$, respectively.

Finally, in order to make use of the controllability results given by Sussman in [92], we must also define the *θ-degree* of a bracket.

Definition 6.7 *For each* $\theta \in [1, \infty)$*, the* θ-degree *of* $Y \in \mathrm{Br}(\mathbf{X})$*, denoted* $\delta_\theta(Y)$*, is given by*

$$\delta_\theta(Y) = \frac{1}{\theta}\delta^0(Y) + \sum_{i=1}^{m} \delta^i(Y). \tag{6.7}$$

The interesting point to notice about the θ-degree of a bracket is that it holds in the limit as $\theta \to \infty$. Thus, in contrast to the regular notion of the degree of a bracket, we see for the example brackets above that

$$\delta(Y) = 5 > \delta_\infty(Y) = 3, \text{ but } \quad \delta(Z) = 2 = \delta_\infty(Z).$$

Thus, depending on the number of times the drift vector field, X_0, appears, the two measures of brackets can be quite different. It will always be true, however, that $\delta(Y) \geq \delta_\theta(Y)$, for $Y \in \mathrm{Br}(\mathbf{X})$.

Remark: To recap, then, we have used the magma $M_{\mathbf{X}}$ to capture the notion of iterated brackets. The free Lie algebra $L_{\mathbf{X}}$ was then defined to be the $\mathbb{R}$-algebra formed from $M_{\mathbf{X}}$, with certain elements being modded out (as determined by the defining relations for a Lie algebra). Then we can use indeterminate elements $X_0, \ldots, X_m$ to give a precise description of the degree of a bracket.

Lastly, we need a means of evaluating elements in $L_{\mathbf{X}}$ in terms of vector fields for a given control problem. Let $\mathbf{g} = (g_0, g_1, \ldots, g_m)$ be a set of vector fields on a manifold N, so that each g_i is an element of the set of all partial differential operators on $\mathcal{C}^\infty(N)$. Then if we write an element $B \in A_{\mathbf{X}}$ as $B = a^I X_I$, where $I = (i_1, \ldots, i_k)$, we can define the *evaluation map* by substituting in the vector fields for the indeterminates, i.e., by "plugging in the g_i's for the X_i's":

$$\mathrm{Ev}_z(\mathbf{g}) : A_{\mathbf{X}} \to D(N) : a^I X_I \mapsto a^I g_I,$$

where $g_I = g_{i_1} g_{i_2} \ldots g_{i_k}$ and $z \in N$. The evaluation map can be restricted to $L_{\mathbf{X}}$ (and hence $\mathrm{Br}(\mathbf{X})$) to give a surjective homomorphism from $L_{\mathbf{X}}$ onto L_{g}, where L_{g} is the Lie algebra of vector fields generated by $\mathbf{g}$.

6.1.2 Accessibility and Controllability

In order to discuss control theoretic issues regarding a particular system, we must start by precisely defining the types of control goals we seek. In nonlinear control theory, there are two commonly used notions of control— *accessibility* and *controllability*. Putting aside technical definitions for a moment, we would like our control goal to be something like the following: a system will be said to be controllable if, given any initial point q_i and final point q_f, there exists an admissible control law u which drives the system from q_i to q_f. For general nonlinear systems, the notion of *small-time local controllability*, in which controllability is shown for local neighborhoods of q_i, will be the closest we can come to our goal of controllability. Note that it is still very much a local condition, and, while it is a much stronger condition than accessibility, it is also much more difficult to satisfy. Here we give definitions for these terms and present an example of how they differ.

Let $\mathcal{R}^V(z_0, T)$ denote the set of reachable points in N from z_0 at time $T > 0$, using admissible controls, $u(t)$, and such that the trajectories remain in the neighborhood V of z_0 for all $t \leq T$. Furthermore, let

$$\mathcal{R}_T^V(z_0) = \cup_{t \leq T} \mathcal{R}^V(z_0, t)$$

be the set of all reachable points from z_0 within time T. These two definitions lead us naturally to define the following:

Definition 6.8 *[77] The system given by Eq. 6.4 is* locally accessible *if for all $z \in N$, $R_T^V(z)$ contains a non-empty open set of N for all neighborhoods V of z and all $T > 0$.*

Definition 6.9 *[92] The system given by Eq. 6.4 is called* small-time locally controllable *(STLC) if for any neighborhood V, time $T > 0$ and $z \in N$, z is an interior point of $\mathcal{R}_T^V(z)$ for all $T > 0$.*

For driftless systems, local accessibility and local controllability are equivalent. Notice, however, that the general types of systems in which we are interested will *require* the presence of a drift vector field, since this is how the momenta enter into the dynamic equations (notice the $\tilde{I}^{-1}p$ term in Eq. 6.3). To give a motivating example of how these definitions differ, consider the problem of controlling an airplane in flight. The airplane can in a coarse sense be thought of as a system that is locally accessible, since it can basically reach an open set of points relative to its forward trajectory. It is, however, obviously not STLC, since the open neighborhood that it can reach after flying for some *small* time T does not contain the point at which it started. Notice that here we emphasize that this only holds for small time, or in a local neighborhood. If our requirement for a system to be controllable were only that it be able to move between two points, then the airplane would satisfy this condition, since it could perform a circle in order to return to the starting point. It is unclear as to what sense of controllability will be most important for the purposes of locomotion. To date, however, there are no strong theoretical results concerning questions of global nonlinear controllability, and so we must be satisfied with investigating small-time local controllability.

6.1.3 The Lie Algebra Rank Condition

For general systems of the form:

$$\dot{z} = f(z) + h_i(z)u^i, \quad z \in N,$$

a standard method for determining accessibility is to compute the *accessibility distribution.* To do so, we define a sequence of distributions. Let

$$\Delta_0 = \operatorname{span}\{f, h_1, \ldots, h_m\}$$

(the span taken over $\mathcal{C}^\infty$ functions on N), and iteratively define

$$\Delta_k = \Delta_{k-1} + \operatorname{span}\{[X,Y] \mid X, Y \in \Delta_{k-1}\}.$$

This is a nondecreasing sequence of distributions on N, and so terminates at some k_{f}, under certain regularity conditions. We will call $\Delta_{k_{\mathrm{f}}}$ the *accessibility distribution*, and denote it by C:

$$C = \Delta_{k_{\mathrm{f}}} = \Delta_\infty.$$

A standard result from nonlinear control theory (based on Frobenius' Theorem), known as the Lie algebra rank condition (LARC), equates accessibility with the condition $C = TN$.

Theorem 6.10 *(LARC) If $\dim C(z) = \dim T_z N$ for all $z \in N$, then the system given by Eq. 6.4 is locally accessible.*

As a means of illustrating the calculations necessary to compute the accessibility distribution, we include the following example from Nijmeijer and van der Schaft [77] (Example 3.14).

Example 6.11 *Let $N = \mathbb{R}^2$ with coordinates (z_1, z_2). Consider the system*

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ z_1^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

We can write C as the span of the vector fields

$$f = z_1^2 \frac{\partial}{\partial z_2} \qquad h = \frac{\partial}{\partial z_1}$$
$$[h, f] = 2z_1 \frac{\partial}{\partial z_2} \qquad [h, [h, f]] = 2\frac{\partial}{\partial z_2}.$$

Thus, $\dim C(z) = \dim \mathbb{R}^2 = 2$ *for all* $(z_1, z_2) \in N$ *and so Theorem 6.10 implies that this system is locally accessible. Notice, however, that this system possesses a drift term so that* $\dot{z}_2 \geq 0$ *everywhere. Given a starting point,* (z_1^0, z_2^0)*, the reachable sets will consist only of points with* $z_2 \geq z_2^0$ *and hence will not contain* (z_1^0, z_2^0) *in their interior. Therefore, this system is* not *STLC.*

6.1.4 The Principal Kinematic Case

Kelly and Murray [43] have derived controllability results for the *principal kinematic* case. The kinematic case implies a driftless system where accessibility and controllability are equivalent. However, the conditions they give for controllability will be useful in the present context for checking accessibility and controllability in systems where momentum terms drive the system. In the kinematic case, however, momenta arising from symmetries are annihilated by the nonholonomic constraints. Therefore, $p \equiv 0$, and the equations of motion reduce to

$$\begin{aligned} g^{-1}\dot{g} &= -\mathbb{A}(r)\dot{r} \\ \dot{r} &= u. \end{aligned} \tag{6.8}$$

Given specified control inputs, the local form of the connection, $\mathbb{A}(r)$, thus determines the motion in the full configuration space.

Notice also that for the purposes of establishing controllability, Eq. 6.8 can be viewed as a special case of the general result for nonholonomic mechanical systems given by Bloch, Reyhanoglu, and McClamroch (Theorem 5 in [12]). This result, however, does not make use of the special structure found in these types of problems. By using this structure, Kelly and Murray were able to derive straightforward computational conditions for controllability and suggest methods for generating desired trajectories. In their paper, they establish two important results that will be useful later. First, they observe that by taking the appropriate derivatives, the controllability analysis can be performed on the Lie algebra, i.e., at $g = e$. Furthermore, they show that the controllability of a kinematic system can be determined solely from the local form of the connection, $\mathbb{A}$, its curvature, and higher covariant derivatives. The reader unfamiliar with exterior derivatives of differential forms is referred to [2].

Definition 6.12 *Given a connection form $\mathcal{A}$ on Q, the* curvature form *is the 2-form $D\mathcal{A}$ determined by evaluating the exterior derivative of $\mathcal{A}$ on horizontal vectors. In other words,*

$$D\mathcal{A}(X, Y) = d\mathcal{A}(\mathbf{hor}\, X, \mathbf{hor}\, Y),$$

for $X, Y \in \mathfrak{X}(Q)$.

Noting that the connection form evaluates to zero on horizontal vectors, we can rewrite the curvature form as

$$D\mathcal{A}(X,Y) = d\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)],$$

where $[\mathcal{A}(X), \mathcal{A}(Y)]$ is the Lie bracket on $\mathfrak{g}$.

The result is then developed using the local curvature form given by

$$D\mathbb{A}(X,Y) = d\mathbb{A}(X,Y) + [\mathbb{A}(X), \mathbb{A}(Y)],$$

where now $X, Y \in \mathfrak{X}(M)$ are base vector fields.

If we rewrite Eq. 6.8 as

$$\dot{q} = X_i^h u^i,$$

with

$$X_i^h = \begin{pmatrix} -g\mathbb{A}(e_i) \\ e_i \end{pmatrix}$$

(recall that e_i is the vector in $T_r M$ with a 1 in the i^{th} row), then it is shown in [43] that each of the brackets in the accessibility distribution C can be expressed in terms of derivatives of the connection. For example, the first order brackets between control vector fields can be expressed in terms of the curvature:

$$[X_i^h, X_j^h] = \begin{pmatrix} -gD\mathbb{A}(e_i, e_j) \\ 0 \end{pmatrix},$$

and the next higher order bracket in similar fashion:

$$[\,[X_i^h, X_j^h], X_k^h] = \begin{pmatrix} -g(L_{e_k} D\mathbb{A}(e_i, e_j) - [\mathbb{A}(e_k), D\mathbb{A}(e_i, e_j)]) \\ 0 \end{pmatrix}.$$

Noting this, they construct a series of subspaces of $\mathfrak{g}$ given by repeatedly taking higher derivatives of the connection:

$$\begin{aligned}
\mathfrak{h}_1 &= \operatorname{span}\{\mathbb{A}(X) : X \in T_r M\} \\
\mathfrak{h}_2 &= \operatorname{span}\{D\mathbb{A}(X,Y) : X, Y \in T_r M\} \\
\mathfrak{h}_3 &= \operatorname{span}\{L_Z D\mathbb{A}(X,Y) - [\mathbb{A}(Z), D\mathbb{A}(X,Y)], \\
&\qquad\qquad [D\mathbb{A}(X,Y), D\mathbb{A}(W,Z)] : W, X, Y, Z \in T_r M\} \\
&\ \ \vdots \\
\mathfrak{h}_k &= \operatorname{span}\{L_X \xi - [\mathbb{A}(Z), \xi], [\eta, \xi] : X \in T_r M, \xi \in \mathfrak{h}_{k-1}, \eta \in \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{k-1}\}.
\end{aligned} \tag{6.9}$$

Notice that in the above equations, the connection has been placed in the appropriate mathematical context as a Lie algebra valued one-form on M. Thus, derivatives of $\mathbb{A}$ will take their values in $\mathfrak{g}$ when evaluated along the appropriate vector fields on M. We point out that the curvature of the connection is defined with respect to the structure equations as

$$D\mathbb{A}(X,Y) = d\mathbb{A}(X,Y) + [\mathbb{A}(X), \mathbb{A}(Y)],$$

where $[\mathbb{A}(X), \mathbb{A}(Y)]$ is the Lie bracket on $\mathfrak{g}$ and d represents exterior differentiation.

Next recall that for driftless systems local controllability and local accessibility are equivalent, so that the results given below in terms of the accessibility distribution will apply to controllability for systems with purely kinematic constraints. Kelly and Murray define two types of local controllability, adapted for problems of locomotion. *Fiber controllability* implies that we can use control inputs to move to any position in the fiber, but without regards to the intermediate or final conditions of the controlled variables. On the other hand, *total controllability* is a slightly stronger condition, basically equivalent to STLC, which includes the ability to fully specify the motion of the controlled variables.

Proposition 6.13 *[43] The system given by Eqs. 6.8 is locally fiber controllable at $q \in Q$ if and only if*

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots,$$

and is locally totally controllable if and only if

$$\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots.$$

The subspaces, $\mathfrak{h}_k \subset \mathfrak{g}$, will be used below to give sufficient conditions for local accessibility (and later controllability) of the general mixed case given by Eqs. 6.5.

In order to illustrate the above definitions (and to make clearer the distinction between fiber and total controllability), we include the example of the two-wheeled mobile robot, presented by Kelly and Murray in [43].

Example 6.14 Two-wheeled Mobile Robot *(cont.)*

Recall the two-wheeled planar mobile robot described in Chapter 2. The configuration space is $Q = G \times M = SE(2) \times (\mathbb{S} \times \mathbb{S})$, with coordinates $q = (x, y, \theta, \phi_1, \phi_2)$. The constraints defining the no-slip condition can be written as in Eq. 6.8 so as to highlight their Lie group structure:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = - \begin{pmatrix} \frac{\rho}{2} & \frac{\rho}{2} \\ 0 & 0 \\ \frac{\rho}{2w} & -\frac{\rho}{2w} \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

From this, it is clear that the local form of the connection is given by

$$\mathbb{A}(r) = \begin{pmatrix} \frac{\rho}{2} & \frac{\rho}{2} \\ 0 & 0 \\ \frac{\rho}{2w} & -\frac{\rho}{2w} \end{pmatrix}. \tag{6.10}$$

Also, we note that it is easy to show that the base directions are controllable using Eqs. 2.9 and 2.10.

The connection is used to define $\mathfrak{h}_1$ for the controllability calculations, and so

$$\mathfrak{h}_1 = \operatorname{span}\{(1, 0, \frac{1}{w})^T, (1, 0, -\frac{1}{w})^T\}.$$

In order to compute the curvature, $D\mathbb{A}$, we use the formula $D\mathbb{A} = d\mathbb{A} + [\mathbb{A}, \mathbb{A}]$, for which we will need the structure constants of the Lie algebra. A straightforward calculation shows that for $\xi, \eta \in \mathfrak{g}$,

$$[\xi, \eta] = \begin{pmatrix} \xi^2\eta^3 - \xi^3\eta^2 \\ \xi^1\eta^3 - \xi^3\eta^1 \\ 0 \end{pmatrix},$$

using the standard basis for $se(2)$. If we write $\mathbb{A}$ using differential forms as

$$\mathbb{A} = \begin{pmatrix} \frac{\rho}{2}d\phi_1 + \frac{\rho}{2}d\phi_2 \\ 0 \\ \frac{\rho}{2w}d\phi_1 - \frac{\rho}{2w}d\phi_2 \end{pmatrix},$$

then it is easy to see that $d\mathbb{A} = 0$. Calculating the bracket, we get

$$D\mathbb{A} = [\mathbb{A}, \mathbb{A}] = -\begin{pmatrix} 0 \\ \frac{\rho^2}{2w}d\phi_1 \wedge d\phi_2 \\ 0 \end{pmatrix}.$$

Clearly, the Lie algebra element

$$(0, 1, 0)^T \in \mathfrak{h}_2$$

is in the span of $D\mathbb{A}$, when applied to the appropriate tangent vectors. Thus, the two-wheeled mobile robot is fiber controllable, since $\mathfrak{h}_1 + \mathfrak{h}_2 = \mathfrak{g}$. However, Kelly and Murray show in [43] that the higher order derivatives of $\mathbb{A}(r)$ will never lead to terms with nonzero elements in the third slot (i.e., terms like $(*, *, 1)$), and so the mobile robot in this example is not *totally controllable (since $\mathfrak{h}_2 + \mathfrak{h}_3 + \cdots \neq \mathfrak{g}$). This surprising result is related to the geometric relationship between the two wheels, and the paths which they must follow.*

6.1.5 Unconstrained Systems with Symmetries

In the same manner as for the principal kinematic case above, Montgomery [71] showed that similar tests can be used to show controllability for an unconstrained dynamical system with Lie group symmetries. His result applies to the case where the spatial momentum μ is zero (and hence the body momentum $p = \mathrm{Ad}_g^* \mu = 0$), so that all motion is horizontal. For this situation, we see that, since the momentum is zero and constant (recall Noether's theorem for unconstrained systems), the equations reduce to those of the principal kinematic case (Eq. 6.8),

$$\begin{aligned} g^{-1}\dot{g} &= -A(r)\dot{r} \\ \dot{r} &= u, \end{aligned}$$

where A is again the mechanical connection. Using the same construction above, his result states that if

$$\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots,$$

then any two configurations q_0 and q_1 can be connected by a horizontal path, i.e., one which satisfies the $\mu = 0$ constraint. In other words, even though we have a fully dynamical system, it is possible to give simple controllability conditions based on the connection. Notice, however, that for $\mu \not\equiv 0$ this presents a drift term which implies that controllability and accessibility are no longer equivalent, and so Chow's theorem (LARC) implies only accessibility. One of our goals in the following sections is to derive tests for general systems with symmetries and constraints in order to establish basic controllability results.

6.2 Local Accessibility

We begin this section by examining a few of the lower order brackets in the accessibility distribution, C, which play an important role in the accessibility and controllability analyses to follow. Notice that we have chosen the control vector fields in such a manner that they are mutually orthogonal, and such that

$$[h_i, h_j] = 0, \quad \forall\, i, j \in \{1, \ldots, m\}.$$

The remaining first order brackets (those in Δ_1) will be of the form of a control vector field bracketed with the drift vector field. A quick calculation shows that

$$\alpha_i := [h_i, f] = \begin{pmatrix} -\mathbb{A}_i(r) \\ (\sigma_{\dot{r}\dot{r}})_{ij}\dot{r}^j + (\sigma_{p\dot{r}})_i^j p_j \\ e_i \\ 0 \end{pmatrix}.$$

At this point we direct the reader's attention to the similarity between this set of vector fields and those for the kinematic case. If one disregards the variables which are eliminated in the kinematic case, i.e., the momentum and acceleration variables, then the two sets of equations are identical. A loose mathematical interpretation of this similarity is that the bracket operation pairing the drift and torque controls (given by $\alpha_i = [h_i, f]$) yields a vector field that is "equivalent" to having integrated the input control torques, converting them to something approximating velocity controls. Hence they take on a form reminiscent of the kinematic case, where the control inputs are velocities. This, of course, is just a naive way of describing the similarities between the brackets α_i and the inputs in the principal kinematic case.

Moving to the second order brackets, an interesting thing happens when we bracket h_i with α_j:

$$\beta_{ij} := [h_i, \alpha_j] = [h_i, [h_j, f]\,] = \begin{pmatrix} 0 \\ (\sigma_{\dot r \dot r})_{ij} \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the $\sigma_{\dot r \dot r}$ term, which is a cross-coupling term for the base variables, directly affects the momentum variables via the β_{ij} brackets. Viewing this coupling as a map, $\sigma_{\dot r \dot r} : TM \times TM \to \mathbb{R}^p$, then $\sigma_{\dot r \dot r}$ being surjective implies that all of the momentum directions can be generated via this second order bracket. This mapping will be quite useful for a variety of reasons, as detailed below.

Proposition 6.15 *Assume that $\sigma_{\dot r \dot r}$ is onto and that*

$$\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \cdots ,$$

where the $\mathfrak{h}_k$'s are defined as above (Eqs. 6.9) using the local form of the connection given in Eqs. 6.5. Then the system given by Eqs. 6.5 is locally accessible.

Proof: To show accessibility, we need to show that the distribution Δ_∞ spans TN at each point z. The assumption on $\sigma_{\dot r \dot r}$ implies that the bracket given by $[h_i, [h_j, f]\,]$ will span the momentum directions, so it remains only to show that Δ_∞ contains the fiber and base directions. To do this, we begin with Δ_1. It will contain vectors of the form,

$$h_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_i \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} -\mathbb{A}_i(r) \\ (\sigma_{\dot r \dot r})_{ij}\dot r^j + (\sigma_{p \dot r})_i^j p_j \\ e_i \\ 0 \end{pmatrix}.$$

Thus, the base directions (velocity and acceleration vectors on M) will be contained in Δ_1. Next, we examine Δ_2. It will contain the vectors, α_i, and also vectors of the form,

$$\beta_{ij} = \begin{pmatrix} 0 \\ (\sigma_{\dot{r}\dot{r}})_{ij} \\ 0 \\ 0 \end{pmatrix}.$$

Thus, for each $z \in N$ we can cancel off the terms in α_i which act in the momentum direction, and so define a new set of vector fields to operate on:

$$\tilde{\alpha}_i = \begin{pmatrix} -\mathbb{A}_i(r) \\ 0 \\ e_i \\ 0 \end{pmatrix}.$$

Using these vector fields, we define

$$\tilde{\Delta}_2 := \operatorname{span}\{f, h_i, \beta_{ij}, \tilde{\alpha}_i\}.$$

and the subsequent distributions, $\tilde{\Delta}_k$, similar to before. Then $\tilde{\Delta}_\infty \subset \Delta_\infty$. As in the kinematic case, higher order bracketing of $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ will lead to higher order derivatives of the connection, $\mathbb{A}(r)$. By the assumption that $\mathfrak{h} = \mathfrak{g}_2 + \mathfrak{g}_3 + \ldots$, we have it that $\tilde{\Delta}_\infty = T_z N$, for each $z \in N$. The result follows since $\Delta_\infty \supset \tilde{\Delta}_\infty = T_z N$. ■

The criterion given in Proposition 6.15 will be used in the following sections as a basis for checking local controllability and for demonstrating accessibility and controllability properties for the snakeboard example.

6.3 Local Controllability

Unfortunately, for nonlinear systems with drift we have seen above that local accessibility may be quite different from local controllability. In order to provide a result for controllability, we will need to show that certain of the higher order brackets either vanish or can be written as a linear combination of lower order brackets. This result is due to Sussman [92], and is the strongest statement of local controllability for nonlinear control systems with drift of which we are currently aware. For further details on this construction, please refer to [12, 92].

Let $h_0 := f$ so that $\Delta_0 = \operatorname{span}\{h_0 = f, h_1, \ldots, h_m\}$. As above, we will represent brackets of these vector fields using a finite sequence of indeterminates, $\{X_0, \ldots, X_m\}$. We have already defined the degree of $X \in \mathrm{Br}(\mathbf{X})$ and the θ-degree of X to be the sum and scaled sum of the δ^i's, respectively.

We also need to define a symmetrizer operation, $\beta(X)$, as

$$\beta(X) = \sum_{\pi \in S_m} \tilde{\pi}(X),$$

where S_m is the group of permutations on $\{1, \ldots, m\}$, and $\tilde{\pi}$ is the operator on $L(\mathbf{X})$ that fixes X_0 and permutes $X_1, \ldots, X_m$, sending X_i to $X_{\pi(i)}$. Thus, $\beta(X)$ is the sum of all permutations of the bracket X that leaves X_0 fixed. This is a technical definition used by Sussman, but which we will not actually need to compute in practice.

Then, we have the following theorem due to Sussman:

Theorem 6.16 *[92] Given the system of Eq. 6.4, with $h_0(z_0) = f(z_0) = 0$ at an equilibrium point $z_0 \in N$, assume that $\mathbf{g} = (h_0, \ldots, h_m)$ satisfies the LARC at z_0. Further, assume that there is a $\theta \in [1, \infty)$ such that whenever $X \in \mathrm{Br}(\mathbf{X})$ is a bracket for which $\delta^0(X)$ is odd and $\delta^1(X), \ldots, \delta^m(X)$ are all even, then there exist brackets $Y_1, \ldots, Y_k$ such that*

$$\mathrm{Ev}_{z_0}(\mathbf{g})(\beta(X)) = \alpha^i \mathrm{Ev}_{z_0}(\mathbf{g})(Y_i),$$

for some $\alpha^1, \ldots, \alpha^k \in \mathbb{R}$, and

$$\delta_\theta(Y_i) < \delta_\theta(X), \ i = 1, \ldots, m.$$

Then the system defined by Eq. 6.4 is STLC from z_0.

For ease of use, we restate this in a slightly weaker corollary that will help to simplify the necessary calculations.

Corollary 6.17 *Given the system of Eq. 6.4, with $h_0(z_0) = f(z_0) = 0$ at an equilibrium point $z_0 \in N$, assume that $\mathbf{g} = (h_0, \ldots, h_m)$ satisfies the LARC at z_0. Further, assume that whenever $X \in \mathrm{Br}(\mathbf{X})$ is a bracket for which $\delta^0(X)$ is odd and $\delta^1(X), \ldots, \delta^m(X)$ are all even, then there exist brackets $Y_1, \ldots, Y_k$ such that*

$$\mathrm{Ev}_{z_0}(\mathbf{g})(X) = \alpha^i \mathrm{Ev}_{z_0}(\mathbf{g})(Y_i),$$

for some $\alpha^1, \ldots, \alpha^k \in \mathbb{R}$, and

$$\delta(Y_i) < \delta(X), \ i = 1, \ldots, m.$$

Then the system defined by Eq. 6.4 is STLC from z_0.

Proof: The two things to show here are that $\delta(Y_i) < \delta(X)$ implies $\delta_\theta(Y_i) < \delta_\theta(X)$ and that we can remove the symmetrization operation, β. The first statement is obvious, since $\delta(X) = \delta_\theta(X)|_{\theta=1}$. To prove the statement regarding the symmetrizer, β, notice that if X is a bracket for which $\delta^0(X)$ is odd and $\delta^1(X), \ldots, \delta^m(X)$ are all even, then

$$\beta(X) = \sum_{\pi \in S_m} \tilde{\pi}(X)$$

will also have $\delta^0(X)$ odd and each $\delta^1(X), \ldots, \delta^m(X)$ even, since it is just a permutation of the order of the elements. Thus, it is covered by the assumption that any bracket having these relative degrees must be expressible by a linear combination of lower order brackets. ■

With this corollary in mind, we define a "bad" bracket to be those brackets for which the drift term appears an odd number of times and for which the control vector fields each appear an even number of times (including zero times). The sufficient conditions for small-time local controllability, then, can be simply restated as requiring that all "bad" brackets be expressible in terms of brackets of lower degree.

Proposition 6.18 *Assume that the system defined by Eqs. 6.5 is locally accessible, that the map $\sigma_{\dot{r}\dot{r}}$ is onto, and that $(\sigma_{\dot{r}\dot{r}})_{ii} \equiv 0$ for $i = 1, \ldots, m$ (no summation over i). Then this system is small-time locally controllable (STLC) from all equilibrium points, $z_0 \in N$.*

Proof: In order to show controllability, we begin by demonstrating that all "bad" brackets as defined by Sussman will either be zero or be expressible in terms of lower order "good" brackets (in fact, of order 3). This, along with the assumption that the LARC is satisfied (using the results from the kinematic case), will give the result via Corollary 6.17.

First, we restrict our attention to the point $z_0 = (0, 0, 0, 0) \in G \times \mathbb{R}^p \times M \times T_r M$. It is easy to show that the result will hold for all equilibrium points, $z \in N$ (of the form $z = (g, 0, r, 0)$), by translating Eqs. 6.5 appropriately. Also notice that $f(z_0) = 0$, satisfying the first requirement of Theorem 6.16.

Next, recall the definition of the degree of a bracket and notice two important facts that must be true of any bad bracket X: 1) $\delta(X)$ must be odd, and 2) $\delta^0(X) \neq \sum_{i=1}^m \delta^i(X)$. These are both made true by virtue of there being exactly one odd term in the summation of Eq. 6.7. The first condition implies that all even order brackets are necessarily "good" brackets, while the second condition implies that for bad brackets the quantity

$$\gamma(X) := \delta^0(X) - \sum_{i=1}^m \delta^i(X)$$

is always odd, and thus never zero.

More specific to the system of Eqs. 6.5, let $\mathcal{O}(k)$ denote a function in $(z, \dot{z})$ which is a homogeneous polynomial of order k in $(\dot{r}, p)$. Thus, $f(z) = (\mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(1), 0)$. A straightforward set of calculations shows that for any bracket involving the drift vector field, f, bracketing by f will increase the order of these functions by 1, and that bracketing by any of the h_i's will decrease the order of the bracket by 1. Thus, we will find that for any bad bracket, X, (which by definition must contain at least one X_0 in the bracket), it will evaluate to a vector field with the form $\mathrm{Ev}_{z_0}(\mathbf{g})(X) = (\mathcal{O}(\gamma(X)), \mathcal{O}(\gamma(X)+1), \mathcal{O}(\gamma(X)), 0)$[1], or will be identically zero, e.g., any bracket involving $[h_1, h_2] \equiv 0$. Viewed this way, it is easy to see that all bad brackets for which $\gamma(X) \neq -1$ must have $\mathrm{Ev}_{z_0}(\mathbf{g})(X) = 0$.

[1] We have allowed $\gamma(X)$ to be negative, and so define $\mathcal{O}(k) = 0$ for all $k < 0$.

Thus, the only bad brackets that we need worry about are those with $\gamma(X) = -1$, for which $\mathrm{Ev}_{z_0}(\mathbf{g})(X) = (0, \mathcal{O}(0), 0, 0)$. These are brackets which lie in the momentum direction. But we have already assumed that the map $\sigma_{\dot{r}\dot{r}}$ is onto, which means that these directions are captured by a bracket of degree 3:

$$\beta_{ij} = \mathrm{Ev}_{z_0}(\mathbf{g})([X_i, [X_j, X_0]\,]) = \begin{pmatrix} 0 \\ -(\sigma_{\dot{r}\dot{r}})_{ij} \\ 0 \\ 0 \end{pmatrix}.$$

Unfortunately, brackets of the form $[X_i, [X_i, X_0]\,]$ $(i = j)$ are also bad brackets, which explains the necessity of assuming that $(\sigma_{\dot{r}\dot{r}})_{ii} \equiv 0$. Given this, however, we see that any bad bracket which is not zero at z_0 can be rewritten in terms of brackets of the form $[h_i, [h_j, f]\,]$, where $i \neq j$. ■

6.4 Locomotive Gaits

Let us briefly consider an important aspect of locomotion that is intricately related to the study of control for these types of systems. A very common observation of locomotion is that it is most often generated by *cyclical* shape changes [24, 32]. The motion takes on a characteristic form, called a *gait*.

Definition 6.19 *A locomotive* gait *is a specified cyclic pattern of internal shape changes (inputs) which couple to produce a net motion.*

One very interesting phenomenon that arises in the study of locomotion is the presence of a very limited set of basic motion patterns. For each species, there usually exist at most a handful of gaits, often tailored for specific needs or environments. For instance, a human will walk or run, depending on the desired speed, but may also hop or skip (though these two gaits do not seem to serve any evolutionary function). On the other hand, snakes will generally move in a serpentine fashion, but can adapt to other environments. For instance, on a slippery surface, a snake may push off the walls of its environment and move in a concertina (inch-worm) gait. Also, snakes in the desert are known to use a sidewinding gait in order to minimize the amount of time that body surfaces spend in contact with the hot sand, and maximize the time that surfaces are off the ground and hence cooled by the air. What is interesting about all of this is that there is a small set of gaits that are used, and almost universally these gaits are based on a single frequency of oscillation. In studying locomotion, and in particular when examining related control issues, it will be important to ask the question of how our models and control laws reflect these naturally occurring patterns of motion.

6.5 Examples

We provide here a brief discussion of the gaits that have been found for our two examples, the snakeboard and the Hirose snake. Obviously, the analysis of gaits is intricately related to issues of controllability for locomotion systems.

6.5.1 The Snakeboard

We return to the snakeboard example to investigate controllability and gait patterns. Obviously, the bracket of the control inputs, $[h_\psi, h_\phi]$, is identically zero. The only other first order brackets are those mixing the drift vector field with the control inputs:

$$\begin{aligned}\alpha_\psi = [h_\psi, f] = \Big(&\frac{J_r \sin 2\phi}{2ml}\cos\theta, \frac{J_r \sin 2\phi}{2ml}\sin\theta, \frac{-J_r \sin^2\phi}{ml^2}, \\ &2\dot{\phi} J_r \cos^2\phi, 1, 0, 0, 0\Big)^T\end{aligned}$$

and

$$\alpha_\phi = [h_\phi, f] = \Big(0, \quad 0, \quad 0, \quad 2\dot{\psi} J_r \cos^2\phi - p\tan\phi, \quad 0, \quad 1, \quad 0, \quad 0\Big)^T.$$

Notice that these vector fields have "1's" in the appropriate velocity directions. As mentioned above, this loosely corresponds to *integrating* the control torques to velocity controls. Notice that this will also encode the information given by the local form of the connection, $\mathbb{A}(r)$, since the connection relates input velocities to fiber velocities.

The vector fields above imply control of the base (assumed to be controllable). In order to show accessibility and controllability (STLC), the first criteria to be satisfied are the conditions on $\sigma_{\dot{r}\dot{r}}$, given by the following third order brackets. First, we need the diagonal elements of $\sigma_{\dot{r}\dot{r}}$ to be zero. This is seen to be true via a direct calculation:

$$\beta_{\phi\phi} = \beta_{\psi\psi} = 0.$$

Then we look at off diagonal terms to show that $\sigma_{\dot{r}\dot{r}}$ is onto (and hence that the momentum direction is contained in the accessibility distribution). To see this, we simply write down the necessary bracket:

$$\beta_{\phi\psi} = [h_\psi, \alpha_\phi] = \Big(0, \quad 0, \quad 0, \quad 2J_r\cos^2\phi, \quad 0, \quad 0, \quad 0, \quad 0\Big)^T,$$

which is nonzero for all $\phi \neq \frac{\pi}{2}$.

Finally, to demonstrate that the snakeboard is controllable, we need show that $\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \ldots$, using the connection, $\mathbb{A}(r)$. We begin by computing $[\alpha_\phi, \alpha_\psi]$, which gives us the curvature of the connection, $D\mathbb{A}$. This yields terms of the form:

$$\Big(\tfrac{J_r}{ml}\cos 2\phi, \quad 0, \quad -\tfrac{J_r}{ml^2}\sin 2\phi\Big)^T \in \mathfrak{h}_2.$$

Then, $[\alpha_\phi, [\alpha_\phi, \alpha_\psi]]$ yields

$$\Big(-\tfrac{2J_r}{ml}\sin 2\phi, \quad 0, \quad -\tfrac{2J_r}{ml^2}\cos 2\phi\Big)^T \in \mathfrak{h}_3,$$

and $[\alpha_\phi,[\alpha_\psi,[\alpha_\psi,[\alpha_\psi,\alpha_\phi]]]]$ gives

$$\left(0,\quad \frac{2{J_r}^2}{m^2l^3}\cos 2\phi,\quad 0\right)^T \in \mathfrak{h}_5.$$

Thus, $\mathfrak{g}=\mathfrak{h}_2+\mathfrak{h}_3+\mathfrak{h}_5$, and the conditions for Proposition 6.18 are satisfied.

The reader should note that while the condition that $(\sigma_{\dot r\dot r})_{ii}\equiv 0$ in Proposition 6.18 may appear slightly artificial, it is *required* for satisfying Sussman's criterion for controllability. This is due to the fact that there are no other brackets of lesser degree (or for that matter, lesser θ-degree) that are nonzero and with elements in the momentum directions. In fact, research by Lewis and Murray [56] suggests that similar conditions may be needed for general mechanical systems. They study accessibility and controllability for unconstrained mechanical systems (not necessarily with symmetries), and report similar conditions on third order brackets of the type $[h_i,[f,h_i]]$. In their case, these brackets are allowed to be nonzero if they are contained in the control input vector field; however, it is not difficult to show that for our purposes these brackets must be identically zero.

Finally, having shown that the snakeboard is controllable, we return to the question of how these calculations relate to the gait patterns demonstrated by the snakeboard. A major part of this issue, then, is asking the question, "what role do the connection and its derivatives *really* play in describing the actual motion of the system?" In particular, "what is the relationship between the connection and its derivatives and locomotive gaits?" Although the results at present are only qualitative, they certainly suggest that we are on the right track. Along with this, they provide some hints as to what directions to follow in future research.

Extensive simulations of the snakeboard gaits can be found in [57], some of which are included here to provide a new perspective on how these results fit into the present context. To date, there have been three basic gait patterns studied for the snakeboard: the "drive" (or "serpentine") gait, the "rotate" gait, and the "parallel parking" gait. In each of these, we assume complete control of the base variables, ϕ and ψ, and specify their trajectories as sinusoidal inputs of the form:

$$\phi = a_\phi\sin(\omega_\phi t) \quad\text{and}\quad \psi = a_\psi\sin(\omega_\psi t).$$

A gait will be referenced by an integer ratio of the form $\omega_\phi:\omega_\psi$, corresponding to the ratio between ω_ϕ and ω_ψ. For instance, a 3:2 gait (the parallel parking gait) would correspond to $\omega_\phi=3$ and $\omega_\psi=2$. For the simulations, the following parameters were used:

m	: 6 kg
J	: 0.06 kg·m^2
J_r	: 0.167 kg·m^2
J_w	: 0.00167 kg·m^2
l	: 0.3 m

These values roughly reflect the physical parameters used to build a working prototype snakeboard (shown in Figure 6.1).

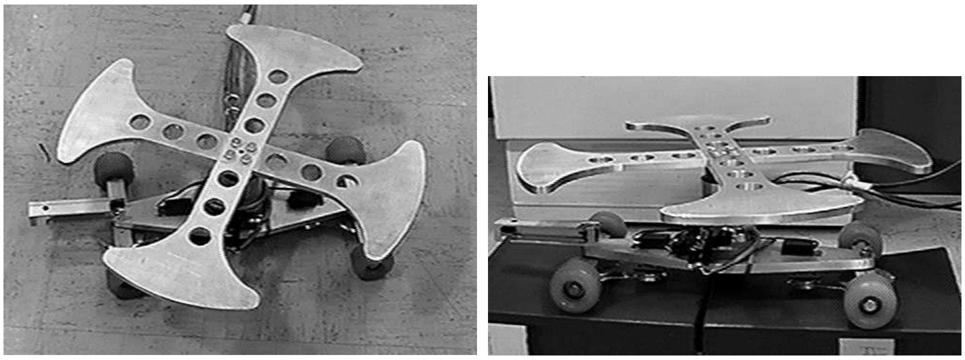

Figure 6.1 A working demo version based on the snakeboard model

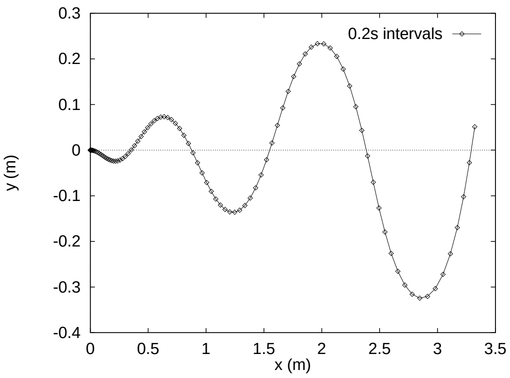

Figure 6.2 Position of the center of mass for the 1:1 (drive) gait

The "drive" gait

The drive gait is characterized by a 1:1 frequency ratio, and demonstrates a forward, serpentine motion resembling that of a snake. A simulation of this gait is shown in Figure 6.2, using the parameters: $a_\phi = 0.7$ rad, $a_\psi = -1$ rad, and $\omega_\phi = \omega_\psi = 1$ rad/sec. We remark that the scaling of the axes given in this figure and those to follow is chosen so as to maximize the visibility and spread of the data presented in these figures, and so this must be taken into account when interpreting the results in terms of physical quantities. Notice that in Figure 6.2 the amplitude of the motion in the transverse or y-direction steadily increases. This is due to the fact that momentum is continually being built up by this gait. Human riders use feedback to control this effect, and are visibly seen modifying their input patterns once a desired speed is reached.

In relationship to the Lie bracket calculations, we notice that the 1:1 frequency

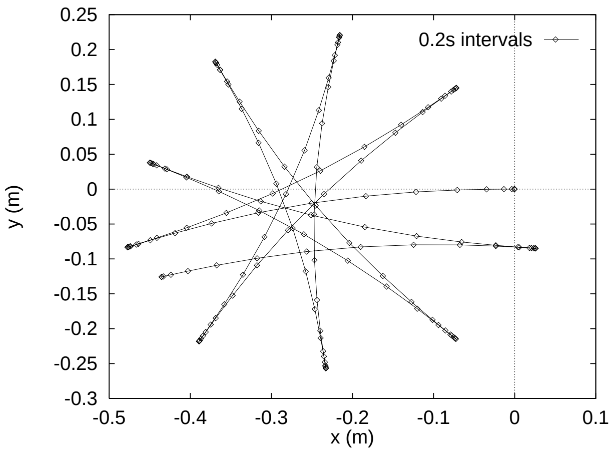

Figure 6.3 Position of the center of mass for the 2:1 (rotate) gait

ratio has a direct correspondence to the 1:1 bracket, $[\alpha_\phi, \alpha_\psi]$. In fact, evaluated at $\phi = 0$ (the center of the wheels' rotation), the bracket gives a Lie algebra element of

$$\left(\tfrac{J_r}{ml}, \quad 0, \quad 0\right)^T.$$

This is written in the body frame of the board, and so corresponds to forward motion, along the length of the board.

The "rotate" gait
The rotate gait uses a 2:1 frequency ratio, and generates a rotational motion (in θ) that leaves the (x, y) position relatively unchanged in the mean. The input parameters for the simulation shown in Figure 6.3 were $a_\phi = 0.7$ rad, $a_\psi = 1$ rad, $\omega_\phi = 2$ rad/sec, and $\omega_\psi = 1$ rad/sec. The snakeboard moves steadily around a central point, while undergoing large rotations– moving π radians, or one half rotation, in approximately four cycles.

Again, we return to examine the correspondence of this motion with the Lie bracket. We see that the necessary bracket direction, the θ-direction, is given by a 2:1 Lie bracket. Namely, $[\alpha_\phi, [\alpha_\phi, \alpha_\psi]]\big|_{\phi=0}$ produces the element

$$\left(0, \quad 0, \quad -\tfrac{2J_r}{ml^2}\right)^T.$$

The "parking" gait
The final gait studied is the parallel parking gait, so called because its motion resembles that of a car performing a parallel parking maneuver (see Figure 6.4). It is based on a 3:2 frequency ratio and generates a net lateral motion, transverse to the length of the board. The parameters used in the simulation were $a_\phi = 0.7$ rad,

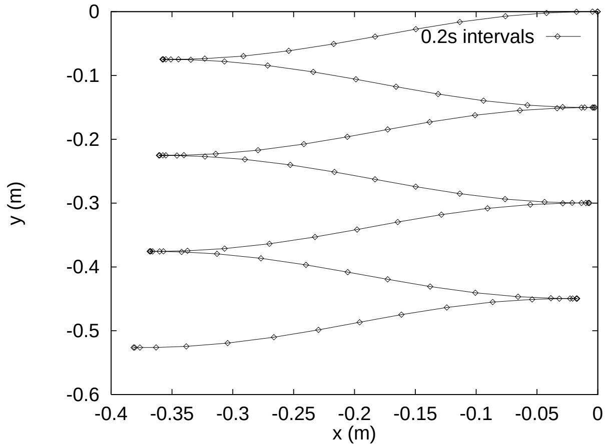

Figure 6.4 Position of the center of mass for the 3:2 (parking) gait

$a_\psi = 1$ rad, $\omega_\phi = 3$ rad/sec, and $\omega_\psi = 2$ rad/sec.

The 3:2 bracket, $[\alpha_\phi, [\alpha_\psi, [\alpha_\psi, [\alpha_\psi, \alpha_\phi]]]]$, in which α_ψ appears 3 times and α_ϕ appears twice, gives

$$\left(0, \quad \tfrac{2J_r{}^2}{m^2 l^3}, \quad 0\right)^T.$$

Other permutations of the fifth order, 3:2 bracket give Lie algebra elements that are either in the same direction or are identically zero. The nonzero entry in the second position of the Lie algebra element above corresponds directly to the direction transverse to the board, namely the y-direction when the board is at $\theta = 0$.

6.5.2 The Kinematic Snake

With the kinematic snake of Hirose, there is obviously a principal gait pattern in which we are most interested— the gait found in common snakes as they slide along the ground. This gait was described by Hirose as being closest to a "serpenoid" curve, and we show below that the serpentine gait generated by our theoretical model is strikingly similar to the pattern of the serpenoid curve. We also present two other gaits not normally seen in nature, but which arise for the particular model we are using. These could be implemented in a snake robot based on Hirose's ACM III.

At this point, we do not have useful results regarding controllability of the kinematic snake, but this is the goal of work in progress. One of the factors hindering this effort is the presence of singularities in this type of model. These will occur any time the axes of the three wheels of the kinematic snake intersect at a point, which occurs frequently for the gaits we are examining. These singularities force us to choose carefully the form of the inputs. Further work is obviously necessary, perhaps with extensions to include in the model slipping of the wheels and friction forces acting internally and externally.

The gaits presented below are only a selection of the more interesting gaits that have been explored. As with the snakeboard, they are all based on integrally related frequencies of the shape inputs. The ratios we give will relate the frequency of bending of the inter-segment angle, ψ_i, versus the frequency of the rotation of the wheels, measured in ϕ_i. Thus, a 2:1 gait represents the segments bending at twice the speed as the turning of the wheels. Unlike the snakeboard, however, the relative phasing of each of these angles will play a critical role in generating locomotion, as well as in avoiding the kinematic singularities (while phasing is important for the snakeboard, it is not nearly as crucial as it is for the kinematic snake).

The "serpentine" gait

We begin the analysis by examining the serpentine gait, which arises when using a 1:1 frequency ratio. For this, we use sinusoids of the form:

$$\phi_i = a_i^\phi \sin(\omega_i^\phi t + p_i^\phi),$$

where similar values for ψ_i will be superscripted with a ψ. The serpentine gait is demonstrated using common values for the amplitude and frequencies, so that

$$\begin{aligned} a_1^\phi = a_3^\phi = 0.2 = -a_2^\phi, &\qquad a_1^\psi = a_3^\psi = 0.6, \\ \omega_1^\phi = \omega_2^\phi = \omega_3^\phi = 1, &\qquad \omega_1^\psi = \omega_3^\psi = 1, \end{aligned}$$

and with the length from the wheel base to inter-segment pivot point set to 0.1m (hence a full segment would measure 0.2m).

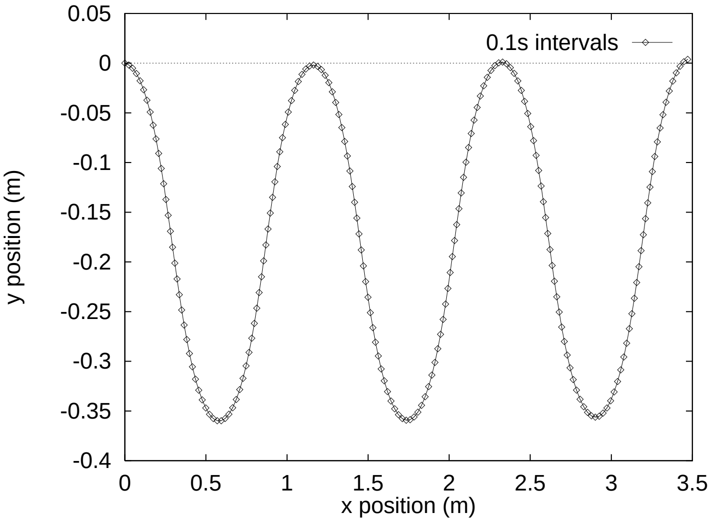

Figure 6.5 A plot of (x, y) for the kinematic snake in serpentine mode

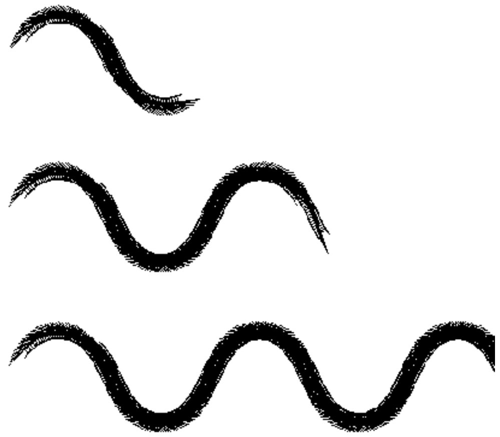

Figure 6.6 A trace of the kinematic snake in serpentine mode

For the phasing, we send a traveling wave down the length of the snake (done by using an increasing value for the phase of the wheels), while forcing the inter-segment angle to move 90° out of phase with their corresponding wheels. Thus, for the simulation shown in Figure 6.5, the phases are given by

$$\begin{array}{lll} p_1^\phi = -\frac{\pi}{16}, & p_2^\phi = 0, & p_3^\phi = \frac{\pi}{16}, \\ p_1^\psi = \frac{7\pi}{16}, & & p_3^\psi = \frac{9\pi}{16}. \end{array}$$

Notice that each of the wheel angles differs by $\frac{\pi}{16}$, while the joint angles are $\frac{\pi}{2}$ out of phase of their respective wheel angles. To give an idea of how this resembles the motion of a snake, we include a trace of the serpentine motion in Figure 6.6.

By varying the magnitudes of the wheel angles (or the inter-segment joint angles), slightly different patterns of locomotion are found to occur. Figure 6.7 shows the resultant gaits for three different values of $a_1^\psi = a_3^\psi = a^\psi$. Each of these simulated gaits is run for the same length of time, which indicates that certain parameter values will result in a greater distance being traveled. This information can be very useful in designing an actual snake robot by helping to optimize the parameters chosen for locomotion. We present in Figure 6.8 two parameter sweeps (one on ϕ and on ψ) which show obvious peaks indicating possible optimal parameter choices, given the phasing between segments ($\frac{\pi}{16}$rad) and segment length (0.2m).

One point of interest is to examine how this motion compares with the serpenoid curve proposed by Hirose [33]. We generate this curve using the following

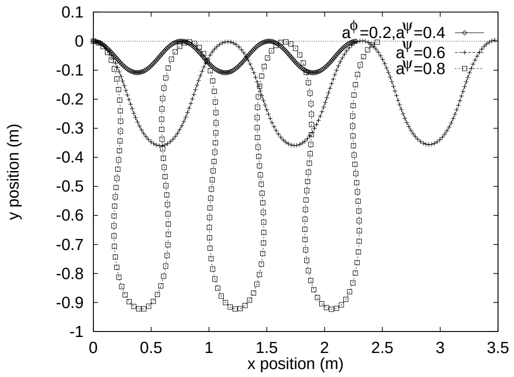

Figure 6.7 Three different shapes for the serpentine gait

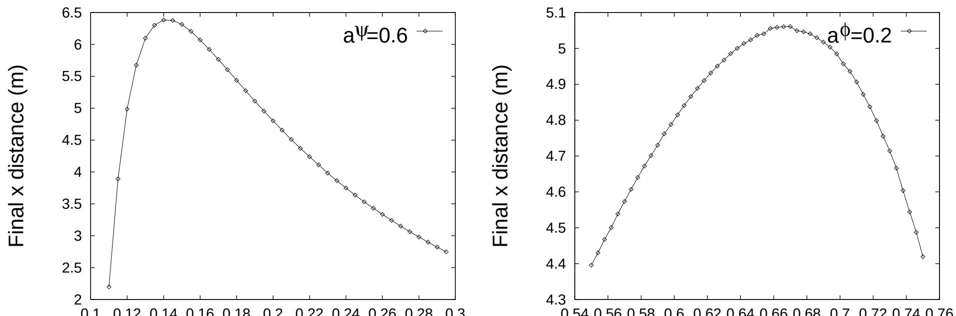

Figure 6.8 Joint angle parameter sweeps versus "forward" distance traveled

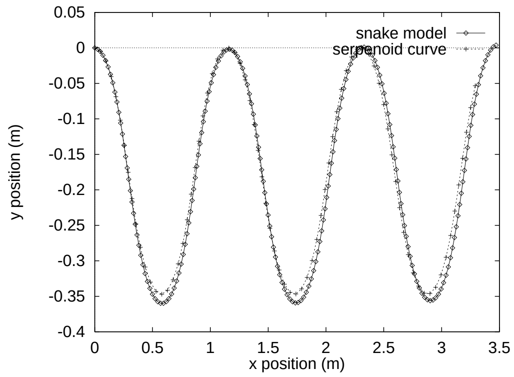

Figure 6.9 A comparison of the kinematic snake model versus the serpenoid curve

parameterization:

$$\dot{x} = \cos\left(\alpha \sin(\beta s)\right) \tag{6.11}$$

$$\dot{y} = -\sin\left(\alpha \sin(\beta s)\right). \tag{6.12}$$

We have observed that the parameters α and β can be chosen such that the serpenoid curve defined by Eqs. 6.11 and 6.12 can be made to match arbitrarily closely any of the serpentine patterns generated using the 1:1 gait. An example of this is given in Figure 6.9, with $\alpha = 0.85$ and $\beta = \frac{23\pi}{16} \simeq 4.52$rad.

As a final note on the serpentine gait, we mention that it seems to work well when additional segments are added using the methods described in Chapter 5 above. At this point we measure this only qualitatively, and refer to Figure 6.10 in which the trace of a 5-segment snake robot is given. Notice that the additional two segments seem to follow the leading three segments quite well. Further analysis of these extensions is still required to explore the possibility of adding segments when following other gaits and patterns of motion.

The "rotate" gait

Finally, there are two other types of gaits, both producing a net rotation, though by very different types of motion. The first gait uses a more "natural" type of gait for a snake, characterized by forward and backward motions similar to the snakeboard rotate gait (and reminiscent of how humans turn a car in tight situations, e.g., a "three-point turn"). The other gait does not seem practical for a real snake, but might be employed by a mobile robot. The frequency ratios for these two gaits are the inverses of each other. For the more "natural" gait, shown in Figure 6.11, the frequency ratio is 2:1, with the parameters used being the same as above, except that $\omega_1^{\psi} = \omega_2^{\psi} = 2$ and all joint magnitudes, $a_1^{\phi}, \ldots, a_2^{\psi}$ being set to 0.4 and 0.5, respectively.

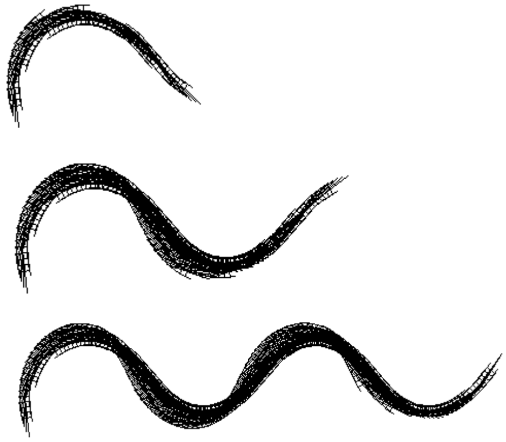

Figure 6.10 Traces of the 5-link kinematic snake

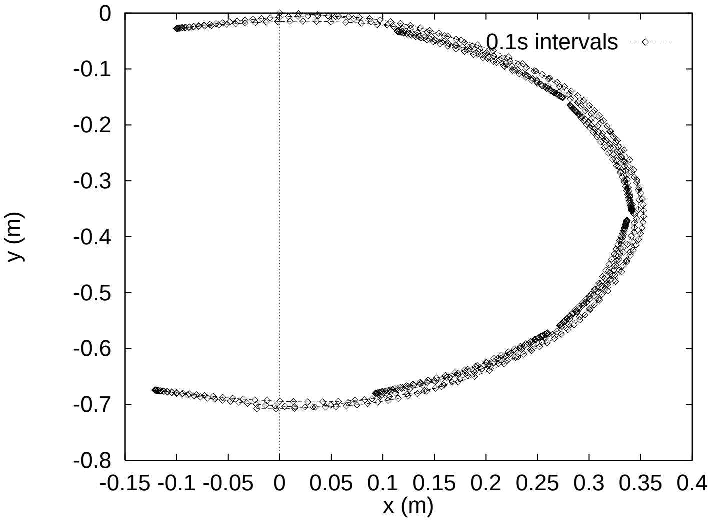

Figure 6.11 A central body segment trace of the 2:1 gait

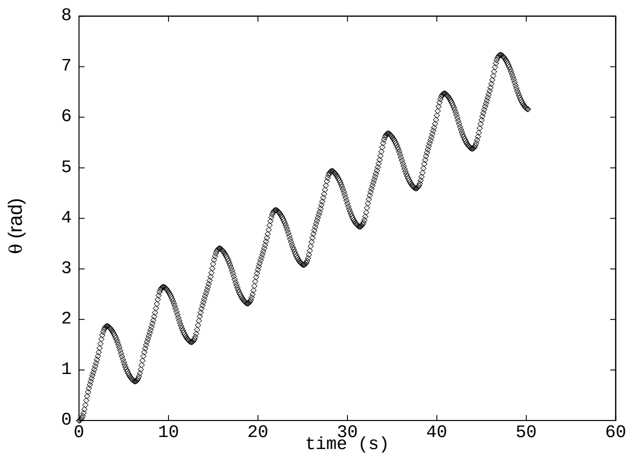

Figure 6.12 A trace of the angle θ for the 1:2 gait

Finally, the 1:2 rotate gait, with the parameters set so that $a^{\phi} = \frac{\pi}{4}$ and $a^{\psi} = \frac{3\pi}{8}$, is shown in Figure 6.12. Notice that we have plotted the variation of θ, the angle of the central body segment, with respect to time. This is because the actual (x, y) position of this segment moves only very insignificantly during the motion of this gait.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

The primary goal of this work has been to develop and explore new results in the theory of nonholonomic mechanical systems with symmetries, emphasizing in particular how these results can be used as a unifying framework for analyzing locomotion. Obviously, the theory presented here is not restricted to just locomotion systems. In fact, it can be applied to general principal fiber bundles with G-invariant Lagrangian functions [10]. From an engineering perspective, however, we have found the theory to be quite revealing when restricted to mechanical systems on trivial principal fiber bundles, which covers a large spectrum of locomotion systems. The intent of this thesis has been to present these ideas with enough mathematical rigor to justify the results, but at a level that is approachable by engineers interested in studying problems of locomotion.

In Chapter 2 we have provided a brief introduction to the tools needed to understand the processes involved in Lagrangian reduction and, in particular, for the setting of constrained systems. The important symbols and notation are first highlighted, followed by a development of the theory of Lie groups, including the concepts of Lie algebras, principal fiber bundles, and symmetries for functions, vector fields, and one-forms. The second half of Chapter 2 was devoted to developing the equations of motion for dynamical systems with constraints, and theorems that are standard to the literature, including Noether's theorem for unconstrained systems and Frobenius' theorems for distributions. The theory in Chapter 2 is illustrated through the use of two examples: Elroy's beanie and the two-wheeled mobile robot. These examples were chosen to give some intuition into unconstrained systems and systems with principal kinematic constraints, respectively.

Having established the notation and basic theoretical foundations, we returned in Chapter 3 to discuss the process of Lagrangian reduction in the absence of constraints. Basic to this theory is the use of a connection on a principal fiber bundle. Along with this, we described the momentum map and connection one-form. In Chapter 3, we have attempted to give some motivation as to why the connection is a very important concept that can be used to help understand the generation of net locomotion from basic shape changes. We also presented two methods for Lagrangian reduction in unconstrained systems: the more traditional Routhian method and an alternative formulation in terms of the constrained Lagrangian. The utility of the constrained Lagrangian method lies in the fact that it is quite straightforward for unconstrained systems, and generalizes easily to the case in which nonholonomic constraints are present. It also allows us to write down the reduced base equations using simple matrix operations.

In Chapter 4, we have given a complete exposition of the reduction and reconstruction procedure for mechanical systems with symmetries and constraints. This chapter starts with explicit statements of the main assumptions used in developing the generalized momentum, and includes a constructive method for generating a basis for the constrained Lie algebra. The remainder of Chapter 4 was centered around the generalized momentum developed by Bloch et al. [10]. The generalized momentum describes the momentum of the system along the constrained symmetry directions. An alternative derivation using the nonholonomic variational principle was used to develop the generalized momentum equation, which includes forcing functions in the directions of symmetry. The generalized momentum (and its governing equation) is very important for locomotion systems since the momenta (and hence, velocities) along the unconstrained directions effectively describe the net motion of the system. Thus, the generalized momentum equation plays an important role in determining how the internal shape changes can be used to build a net velocity for a locomotive body. We also showed that the momentum and momentum equation can always be chosen to be invariant, given that the constraints are themselves G-invariant. These facts were then used to synthesize a nonholonomic connection for this system (along the lines of [10]). Using this connection, it was demonstrated how to reduce the equations of motion to an extended base space, using basic matrix manipulations. Using these formulas, the reduced dynamics on the base space can easily be computed using standard symbolic manipulation packages such as Mathematica or Maple. Finally, we briefly discussed the process of reconstruction, using the connection and generalized momentum equation.

The theory presented herein has been applied to many different examples in locomotion and beyond. In Chapter 5, we have presented two new examples which illustrate various facets of the theory. The treatment of the snakeboard is the more comprehensive of these two examples, both in the analysis performed and in the ways in which it illustrates the theory. The model we use for the snakeboard clearly falls in the category of "mixed" constraints, so that there is a non-trivial generalized momentum term. By reducing to the base space, it was easily shown that the base dynamics are controllable. This fact was used later in Chapter 6 to discuss issues of controllability for the snakeboard. The kinematic snake of Hirose, on the other hand, demonstrates the theory as applied to the principal kinematic or Chaplygin case. Thus, there is no generalized momentum equation, but the same principles of building a connection on a trivial principal fiber bundle apply. We have used the decomposition into base and fiber variables to gain an insight into how to extend our model to include many more segments in the body of the snake.

Finally, in Chapter 6 we have established initial results concerning accessibility and controllability tests for nonholonomically constrained systems with symmetries.

After reviewing some concepts on free Lie algebras, we presented a summary of the relevant results from the principal kinematic and unconstrained cases (both of which can be considered as driftless problems). These results were then used as a basis for establishing sufficiency tests for accessibility and controllability in the general case in which the generalized momentum enters as a drift term. We concluded Chapter 6 with a discussion of gaits and a summary of the many gaits found in the examples presented in Chapter 5.

7.2 Future Work

We have presented in this thesis a comprehensive theory that can be used as a basis for further study of locomotion problems. This theory, however, has opened up as many questions for future research as it has answered regarding our basic understanding of locomotion. This section is divided into a series of subsections, each of which is devoted to a separate area of possible future research.

7.2.1 Averaging Theory for Lie Groups

One of the more obvious goals in studying locomotion is to begin to develop methods for generating and tracking trajectories. Until basic steps toward this goal are taken, the feasibility of actual robotic implementations of alternative modes of locomotion is severely limited. In Section 6.4, we have presented a first cut at this goal, by providing basic gaits that generate certain directions of motion. However, due to the nonlinearities of the problem, the concept of using superposition to blend two gaits into other hybrid gaits is not possible.

In this section, we discuss in some detail results on averaging theory for Lie groups developed by Leonard and Krishnaprasad [51]. The structure that they use most closely resembles the structure found in the principal kinematic case, and so these would be the first types of problems to examine in attempting to extend their theory. As a basic synopsis of what they have done, the theory allows one to give explicit results for (locally) equating Lie bracket directions with the motion that is generated by small-amplitude periodic controls. They also develop a constructive control algorithm designed to exploit the averaging results [52]. While the results are only valid for drift-free systems in which the local form of the connection is constant, it is hoped that future research will reveal how their techniques can be applied to control systems on fiber bundles and to systems with drift.

First, we write the control system as

$$g^{-1}\dot{g} = \epsilon U(t), \qquad U(t) = f_i u^i(t), \tag{7.1}$$

where $f_1, \ldots, f_l$ form a basis for $\mathfrak{g}$, $u^i(t) \in \mathbb{R}$ are the control inputs, and $\epsilon > 0$ is a scaling parameter which reflects the use of small-amplitude inputs, such that ϵu^i is small. Although we have defined Eq. 7.1 using a full set of inputs, in general it is assumed that some of the u^i are identically zero. In fact, from the perspective of control theory, the really interesting cases occur when there are fewer controls than states.

For the purposes of averaging, certain terms will arise repeatedly. We use periodic inputs of common period T, and so let

$$\tilde{u}^i(t) = \int_0^t u^i(\tau)d\tau$$

and

$$u^i_{av} = \frac{1}{T}\tilde{u}^i(T).$$

Also, let $\tilde{U} = \tilde{u}^i f_i$. They make the additional assumption that $u^i_{av} = 0$ for each i, which implies that each $\tilde{u}^i$ also has common period T.

Along with demonstrating that the Lie brackets specify the directions of motion, Leonard and Krishnaprasad also show that the magnitude of that motion is proportional to certain areas bounded by the inputs. With this in mind, define the quantities

$$Area^{ij}(T) = \frac{1}{2}\int_0^T \left(\tilde{u}^i(\tau)\dot{\tilde{u}}^j(\tau) - \tilde{u}^j(\tau)\dot{\tilde{u}}^i(\tau)\right) d\tau$$

and

$$m^{ijk}(T) = \frac{1}{3}\int_0^T \left(\tilde{u}^i(\tau)\dot{\tilde{u}}^j(\tau) - \tilde{u}^j(\tau)\dot{\tilde{u}}^i(\tau)\right) \tilde{u}^k(\tau)d\tau$$

to be, respectively, the areas and the moments defined by the simply connected, closed curves $\tilde{u}^i$, $\tilde{u}^j$, and $\tilde{u}^k$. As might be expected, $Area^{ij}$ will give the contribution given by the second order brackets, while m^{ijk} will do the same for third order brackets.

In representing the solution to Eq. 7.1, they use the *single exponential* representation given by Magnus [59]. Under certain conditions (discussed in [51]), the solution to Eq. 7.1 with $g(0) = e$ (the group identity) can be written as

$$g(t) = e^{Z(t)},$$

where

$$Z(t) = \epsilon\int_0^t U(\tau)d\tau + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau + \frac{\epsilon^3}{12}\int_0^t [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]\,]d\tau + \ldots$$

In a local region, we will denote a norm on $\mathfrak{g}$ by $\|\cdot\|$ and from this construct a metric on G, denoted $\hat{d}(\cdot,\cdot)$. Using these definitions, we are ready to state their main theorem.

Theorem 7.1 *(Area-Moment Rule) In the appropriate local neighborhoods, let* Z_0

be an initial condition such that $g(0) = e^{Z_0}$ *and* $\| Z_0 \| = O(\epsilon^2)$, *and define*

$$Z^{(3)}(t, \epsilon) = \epsilon\tilde{U} + \frac{\epsilon^2 t}{2T} Area^{ij}[f_i, f_j] - \frac{\epsilon^3 t}{2T} m^{ijk}[[f_i, f_j], f_k] + Z_0^{(3)},$$

$$g^{(3)}(t) = e^{Z^{(3)}(t)},$$

where $\| Z_0 - Z_0^{(3)} \| = O(\epsilon^3)$. *Then*

$$\hat{d}(g(t), g^{(3)}(t)) = O(\epsilon^3), \ \forall\, t \in [0, b/\epsilon],$$

where b *is chosen such that convergence requirements from [59] are satisfied.*

Thus, we have an approximation to the fiber trajectory, $g^{(3)}(t) = \exp Z^{(3)}(t)$, which has first order brackets scaled by $Area^{ij}$ and ϵ^2, and second order brackets (degree 3 brackets) scaled by m^{ijk} and ϵ^3. The result implies that this approximation will be within $O(\epsilon^3)$ of the actual trajectory, $g(t)$, for a time of order $\frac{1}{\epsilon}$.

As was mentioned above, a natural extension to this framework would be to derive results in the case that the connection is a function of the base variables, i.e., $A = A(r)$. In this case, Eq. 7.1 would become

$$g^{-1}\dot{g} = \epsilon U(t), \qquad U(t) = f_i(r)u^i(t).$$

Other future work includes extending these results to the unconstrained case with nonzero momentum (where some type of momentum shift might be possible) and to some specific classes of systems with drift, for example, systems in which the drift has a zero average over one period. We note here, however, that for unconstrained systems in which the reduced Lagrangian is a function only of ξ and $\dot{r}$ (i.e., it is independent of the shape variables, r), then the above theorem on averaging applies directly.

7.2.2 Optimal Control and Optimality of Gaits

In discussing the control of robotic systems, a natural question that arises is whether there are some controls which are "optimal." How we define optimal is very much left up to us, but most often optimal controls are chosen so as to minimize some cost function, very often a norm of the control inputs. For the purposes of studying locomotion, issues of optimal control can play a significant role in two ways.

First, since we are really interested in having a locomotive system move from point A to point B, we would certainly like to be able to say that the robot is using controls that optimize (minimize) the amount of energy required to move along this path. This will be particularly true for fully autonomous systems which may have only a finite power reserve, so that energy expenditures for moving the robot play a critical role in the duration of the experiment (and hence to some extent the utility of the robot).

Additionally, the study of optimal control may provide us with the answers to more fundamental (and philosophical) questions regarding the optimality of gaits chosen by animals in nature. For instance, does the serpentine motion of the snake somehow optimize its use of energy in generating locomotion? A similar question could also then be asked for animals which use various gaits for different operating regimes, e.g., horses changing gaits at different speeds. While there are obviously many other factors entering into the evolutionary choice of particular gaits that would hinder us from rigorously showing that a specific gait is chosen for the reason of optimality, it would at least provide us with some evidence to support these claims.

There is extensive literature on optimal control, including work done specifically for reduced systems, mostly using the Pontryagin maximum principle combined with Poisson reduction. We highlight here one result of a slightly different nature which applies directly to nonholonomic mechanical systems with symmetries. In [46], Koon and Marsden describe a method that uses Lagrangian reduction directly to establish necessary conditions for the existence of optimal controls for these types of systems. Given a cost function on the shape velocities, $C(\dot{r})$, they use Lagrange multipliers to relax the constraints and in the process define a new Lagrangian, $\mathcal{L}$, for the optimal control problem:

$$\mathcal{L} = C(\dot{r}) + \langle \lambda(t); \xi + \mathbb{A}(r)\dot{r} - \tilde{I}^{-1}p\rangle + \langle \dot{p} - \dot{r}^T\sigma_{\dot{r}\dot{r}}\dot{r} - \dot{r}^T\sigma_{\dot{r}p} - p^T\sigma_{pp}p; \kappa(t)\rangle,$$

where $\lambda(t) \in \mathfrak{g}^*$ and $\kappa(t) \in \mathfrak{g}^{\mathcal{S}}$. Then the main result is given by the following theorem.

Theorem 7.2 *If $q(t) = (g(t), r(t))$ is a (regular) optimal trajectory for the above stated optimal control problem, then there exist $\lambda \in \mathfrak{g}^*$ and $\kappa \in \mathfrak{g}^{\mathcal{S}}$ such that the reduced curve, $(\xi(t), r(t), \dot{r}(t)) \in TQ/G$ satisfies the reduced Euler-Lagrange equations:*

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} &= 0 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi} - \mathrm{ad}^*_{\xi}\frac{\partial \mathcal{L}}{\partial \xi} &= 0 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}} - \frac{\partial \mathcal{L}}{\partial p} &= 0, \end{aligned}$$

along with the fiber and momentum equations given by

$$\begin{aligned} \xi &= -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}p \\ \dot{p} &= \dot{r}^T\sigma_{\dot{r}\dot{r}}\dot{r} + \dot{r}^T\sigma_{\dot{r}p} + p^T\sigma_{pp}p. \end{aligned}$$

With this general formulation (for mechanical systems with symmetries), they also provide detailed coordinate calculations in bundle coordinates. Finally, they apply the theory to examples, including the snakeboard, to give necessary conditions for optimal controls using a cost function of $C(\dot{\psi}, \dot{\phi}) = \frac{1}{2}(\dot{\psi}^2 + \dot{\phi}^2)$. The equations they obtain are fairly simple in form, but further work is still necessary to interpret these results in terms of basic gait patterns and locomotion.

7.2.3 Other Control Issues

The accessibility and controllability tests presented in Chapter 6 are just the beginning of the research needed to fully address the question of controllability for locomotion systems. As such, they offer easily computable tests that work well for some initial examples, but which may not be strong enough in more general applications. Along with a desire to extend these results, there are obviously many other related control issues that should be considered.

For example, Bloch and Crouch implicitly show in [8] that Brockett's sufficiency test for exponential stabilizability does not hold for mechanical systems with nonholonomic constraints. This implies that we cannot use smooth state feedback to exponentially stabilize a locomotion system about an equilibrium point. In order to achieve this, we would need to explore other types of control laws, such as discontinuous or time-varying feedback. It would be interesting to see how these types of controllers might be synthesized using the knowledge of gait patterns and connections.

7.2.4 Extensions to Other Systems

In this dissertation, we have discussed three particular examples of locomotion systems, as well as several problems in mechanics to which this theory applies. A very important topic to consider when discussing future work is the possibility of extending the theory to further examples. To this extent, we consider a few areas of interest.

The class of undulatory locomotion systems is quite large, with many more forms of locomotion than have been discussed here. One subset of undulatory systems that has not been directly addressed consists of examples such as swimming and flying, where the motion is not generated by pushing off the ground (or some rigid surface), but instead comes from a complex interaction of the body with the ambient environment (i.e., water, air, etc.). In some limiting cases, models have been developed which make use of the same structure of connections and trivial principal fiber bundles, including the paramecia studied by Shapere and Wilczek [87, 88] and the viscous models of snakes and inchworms developed by Kelly and Murray [43]. However, these models do not include the full fluid dynamics which occur for swimming in moderate to high Reynolds number. It is not clear, even, if the theory of reduction and connections can be used in these cases, though there is some reason to believe that the presence of symmetries (common to basically *all* locomotion systems) can be incorporated into the analysis. Having an understanding of these other locomotion regimes can be quite useful from a robotics perspective. The topic of underwater research has been widely discussed recently, and it would be very interesting to explore fish-like methods of propulsion, as this appears to be a very efficient form of locomotion in a fluid environment. Similarly, the construction of micro-robots will most certainly require a better understanding of the mechanics of swimming (as motivation, we include in Figure 7.1 a picture of a micro-robot built by Fukuda et al. [29] which uses a propulsion mechanism similar to a water-bug).

Also suggested by the title, there are certain problems of locomotion which ob-

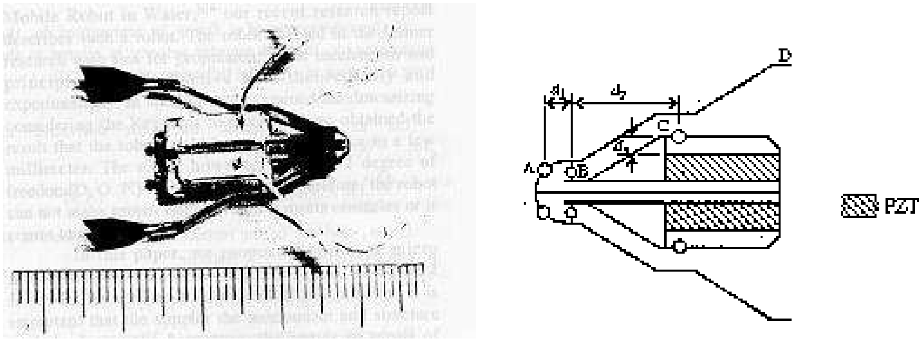

Figure 7.1 Underwater micro-mobile robot and schematic drawing

viously lie outside the theory presented in this dissertation. Namely, it is fairly clear how this theory can be used for general problems of *undulatory* locomotion, such as snakes, inchworms, and wheeled robots, but it is not nearly as clear how these ideas can be extended to legged robots. The reason for this is that the kinematic constraints for legged systems (which manifest themselves as no-slip constraints between the legs and the ground) are not continuous. While the fundamental concept of internal shape changes leading to locomotion may still be important to consider, it is no longer possible to model the interaction in a straightforward and continuous manner. In order to make use of the results from nonholonomic mechanical systems with symmetries, one possibility may be to recognize the symmetries which occur in the interchangeability of the legs. This would involve a reduction by a *discrete* group, e.g., the group of permutations on the configuration variables. Using reduction for a legged system could then allow us to study the dynamics of a smaller set of legs, and make inferences about how they drive the locomotion patterns of the total system. This would be very much like Raibert's use of a "virtual" leg in analyzing quadrupeds [82]. Use of discrete symmetries could then lead to analysis of different gait patterns being generated by different symmetries, similar to the discussions by Collins and Stewart [24, 26] of coupled nonlinear oscillators.

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