HAMILTONIAN G-SPACES WITH REGULAR MOMENTA *

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Abstract

Let G be a compact connected non-Abelian Lie group and let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space. Call this space a *G-space with regular momenta* if $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, where $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ denotes the regular points of the co-adjoint action of G. Here problems involving a G-space with regular momenta are reduced to problems in an associated lower dimensional Hamiltonian T-space, where $T \subset G$ is a maximal torus. For example two such G-spaces are shown to be equivalent if and only if they have equivalent associated T-spaces. We also give a new construction of a normal form due to Marle (1983), for integrable G-spaces with regular momenta. We show that this construction, which is a kind of non-Abelian generalization of action-angle coordinates, can be reduced to constructing conventional action-angle coordinates in the associated T-space. In particular the normal form applies *globally* if the action-angle coordinates can be constructed globally. We illustrate our results in concrete examples from mechanics, including the rigid body. We also indicate applications to Hamiltonian perturbation theory.

1 Introduction

Before summarizing results in detail, we relate a couple of examples from mechanics. In this way we hope to make the main ideas of the paper transparent and to motivate the more general constructions that follow. We begin with a fairly informal discussion of the spherical pendulum.

The spherical pendulum and the rigid body

The simplest motions of a spherical pendulum are obtained when we exclude external forces including gravity, as the equations of motion then possess a 3D rotational symmetry. Identify configurations of the spherical pendulum with points on the unit sphere $S^2 \subset \mathbb{R}^3$. If we restrict ourselves to initial conditions with non-zero angular momentum $b \in \mathbb{R}^3$, then the pendulum moves on a great circle of S^2, with constant speed. This circle is the intersection of S^2 with the plane Σ_μ passing through the origin and with unit normal $\mu \equiv b/\|b\|$. In fact the motion is equivalent to that of a *planar* pendulum lying in Σ_μ, having the same mass and length, and free of external forces. Thus we imagine the spherical pendulum to be a family of mutually equivalent planar pendula. If we agree to distinguish the planar pendulum corresponding to Σ_μ from the one corresponding to $\Sigma_{-\mu}$, then these pendula can be consistently 'oriented' such

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that any motion of the spherical pendulum corresponds to a counterclockwise (say) planar pendulum motion. This family of planar pendula is parameterized by $\mu \in S^2$, i.e. over the unit momentum sphere. An important observation at this point is that the 3D rotational symmetry of the spherical pendulum drops to a 2D rotational symmetry in each planar pendulum, the latter corresponding to an *Abelian* symmetry group.

It is possible to give an interpretation of these observations in terms of the geometry of a spherical pendulum's phase space P. Remembering that we are disregarding points corresponding to $b = 0$, we may take P to be T^*S^2 with its zero-section removed. The 3D rotational symmetry of the problem corresponds to a Hamiltonian action of $SO(3)$ on P, with respect to which the Hamiltonian is invariant. The family of planar pendula corresponds to a fibration $\pi : P \to S^2$ whose fibres are symplectic submanifolds invariant with respect to the Hamiltonian flow, and which are symplectomorphic to the phase space of a planar pendulum[1]. Each fibre is also invariant with respect to the action of some subgroup of $SO(3)$ isomorphic to $SO(2)$. The Hamiltonian restricted to a fibre corresponds to the usual one for a planar pendulum of the same mass and length, and free of external forces.

The appropriate class of systems for which observations like the above can be generalized is the *G-systems with regular momenta* (defined below). Another example of such a system is the Euler-Poinsot rigid body (a rigid body fixed at one point and free of external torques, including torques due to gravity) as an $SO(3)$-system. After removing points corresponding to zero angular momentum, we shall see that the rigid body becomes an S^2-family of mutually equivalent subsystems, each equivalent to the spherical pendulum system[2]. The usual spherical pendulum Hamiltonian is replaced by a more peculiar one depending on the moments of inertia of the body. As in the previous example, there exists an $SO(2)$ action on the subsystem phase space, with respect to which the subsystem Hamiltonian is invariant. This symmetry is *fixed*, in the sense that the action is independent of the particular moments of inertia. We shall see that the Abelianness of the subsystem symmetry is in fact a completely general phenomenon. When all moments of inertia are equal, one recovers the familiar Hamiltonian for a spherical pendulum free of external torques.

In studying G-systems with regular momenta one is led to first consider the *G-spaces* with regular momenta, which we introduce below. The main purpose of this paper is to study these spaces in detail. Before describing our results let us make a less obvious observation regarding our first example, the spherical pendulum.

One can show that the orbit space $P/SO(3)$, obtained from the phase space P of the spherical pendulum defined above, is isomorphic to $(0, \infty)$, and that the projection $P \to P/SO(3) \cong (0, \infty)$ is a principal $SO(3)$-bundle admitting global sections. Each section induces a diffeomorphism $\phi : SO(3) \times (0, \infty) \to P$. If we choose the section appropriately (to be precise, the section should be *isotropic*), then the symplectic structure pulled back by ϕ to $SO(3) \times (0, \infty)$ is in a certain sense a very natural one. Now in the *planar* pendulum subsystems referred to above, one can construct action-angle coordinates. To be precise, each fibre of π is symplectomorphic to $S^1 \times (0, \infty)$, equipped with the standard symplectic structure $d\theta \wedge dI$. In a sense, $SO(3) \times (0, \infty)$ with its natural symplectic structure is a non-Abelian

[1] With points corresponding to non-positive angular momenta (clockwise rotations and trivial motions) removed.

[2] Strictly speaking (see Sect. 9) we must first replace the symplectic form on T^*S^2 with the *negative* of the standard one, and remove the zero-section.

generalization of $S^1 \times (0,\infty) \cong SO(2) \times (0,\infty)$, and the construction of ϕ can be viewed as a generalization of the construction of action-angle coordinates. We shall see how these observations also generalize to an appropriate class of spaces, namely the *integrable* G-spaces with regular momenta.

We revisit the spherical pendulum and rigid body in more detail towards the end of this paper (Sect. 9).

G-spaces with regular momenta

Let G be a compact and connected Lie group and let $(P', \omega, G, \mathbf{J})$ be a Hamiltonian G-space[3]. Let $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ denote the (open dense) set of regular points of the co-adjoint action of G (not to be confused with the regular *values* of the momentum map). Then as $P \equiv \mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$ is an open G-invariant subset of P', $(P, \omega, G, \mathbf{J})$ is a Hamiltonian G-space for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. We refer to any Hamiltonian G-space $(P, \omega, G, \mathbf{J})$ for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$ as a *G-space with regular momenta*. The classical phase space of the spherical pendulum, as discussed above (with points corresponding to zero angular momentum removed), is an example of such a space. A *G-system* with regular momenta is just a G-space with regular momenta, together with a G-invariant function (Hamiltonian) $H : P \to \mathbb{R}$. Note that whenever G is Abelian, $(P, \omega, G, \mathbf{J})$ is automatically a G-space with regular momenta.

Of course the construction just described of a G-space with regular momenta $(P, \omega, G, \mathbf{J})$ from an arbitrary Hamiltonian G-space $(P', \omega, G, \mathbf{J})$ is only useful if $P \equiv \mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$ is not empty! Indeed it can be shown that all points in a G-space with regular momenta necessarily have isotropy group of dimension not exceeding the rank of G; this immediately rules out many interesting examples[4]. Nevertheless there are non-trivial examples of G-spaces with regular momenta and the remarkably simplified geometry of these spaces makes them attractive candidates for study. It also seems that some insight into the more general case might be gleaned from such a study.

The main simplifying principle that applies to G-spaces with regular momenta is the following:

> *With every Hamiltonian G-space with regular momenta one can associate a Hamiltonian T-space, where T is a maximal torus of G. This space is uniquely defined up to an appropriate equivalence and is lower dimensional when G is non-Abelian. Many problems involving Hamiltonian G-spaces with regular momenta can be reduced to a corresponding problem in its associated T-space.*

The construction that achieves the simplification is a natural fibering $\pi : P \to \mathcal{O}$ of P over an (arbitrary) regular co-adjoint orbit $\mathcal{O}$, whose fibers are symplectic submanifolds (Theorem 4.8). Each fibre is invariant with respect to the action of a maximal torus of G, making it a Hamiltonian T-space. Furthermore, as Hamiltonian T-spaces, the fibres turn out to be mutually equivalent. The fibres are also invariant with respect to the flow of any G-invariant Hamiltonian. Note that although different fibres are invariant with respect to the action of

[3] See Sect. 2 for the definition of these and other terms.

[4] For instance take $P \equiv \mathbb{R}^6$, with the standard symplectic structure, and let $G \equiv SU(3)$ act (faithfully) on P by identifying $\mathbb{R}^6$ with $\mathbb{C}^3$. One finds that for all $x \in P$, $\dim G_x \geq 3 >$rank $G = 2$, and that accordingly $\mathbf{J}(P) \cap \mathfrak{g}^*_{\mathrm{reg}} = \emptyset$. This example is the so-called $1:1:1$ resonance, which occurs in the theory of Hamiltonian normal forms for elliptical equilibria.

different tori, it is possible to 'glue' these actions together to form a *fibre preserving* action on P by a single maximal torus T. Up to conjugacy this action is independent of the particular choice of T. We shall show that this canonical T-action on P is in fact Hamiltonian, making P itself a Hamiltonian T-space in a canonical way. Furthermore we shall discover that the associated momentum map $\mathbf{j} : P \to \mathfrak{t}^*$ is related to the map $\pi : P \to \mathcal{O}$ in a basic way. To avoid confusion in the remainder of this discussion, let it be understood that by the *T-space associated with P* we are referring to a representative fibre of π and not to P itself.

It is worth emphasizing that the discussion so far has made no mention of 'integrability' of any kind. Nor have we made any explicit assumptions about freeness of the G-action (or about regular *values* of the momentum map). All that has been required is that $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$.

An illustration of the principle stated above is that a G-space with regular momenta is in fact completely determined by its associated T-space. More precisely we prove (Theorem 6.1):

> *Two Hamiltonian G-spaces with regular momenta are equivalent $\Leftrightarrow$ their associated T-spaces are equivalent.*

As a corollary one obtains a generalization of Delzant's theorem (Delzant, 1988) on the classification of so-called *completely integrable* Hamiltonian T-spaces to the non-Abelian case of G-spaces with regular momenta (Corollary 6.4).

If we assume that G acts freely, then Marsden-Weinstein symplectic reduction applies to the G-space $(P, \omega, G, \mathbf{J})$. As a second application of the simplifying principle we prove (Lemma 7.3):

> *Symplectic reduction in a G-space with regular momenta gives the same reduced spaces as reduction in its associated Hamiltonian T-space.*

One can therefore view the construction of the associated Hamiltonian T-space as an intermediate step in symplectic reduction in which 'non-Abelian reduction' becomes 'Abelian reduction'.

In the case that G acts freely, a natural definition of integrability is that the reduced spaces are zero-dimensional. It follows immediately that integrability of a G-space with regular momenta is equivalent to integrability of its associated Hamiltonian T-space. As a final illustration of the simplifying principle stated above, we construct a non-Abelian 'normal form' for integrable G-spaces with regular momenta (Theorem 8.8). Indeed we show that these spaces are (at least locally) equivalent to an open subset $G \times U$ of $G \times \mathfrak{t}_0^*$, where $\mathfrak{t}_0^* \subset \mathfrak{t}^*$ is isomorphic to an (open) Weyl chamber in $\mathfrak{g}^*$. The appropriate symplectic structure on $G \times \mathfrak{t}_0^*$ arises by realizing $G \times \mathfrak{t}_0^*$ as the Hamiltonian T-space associated with T^*G, after turning T^*G into a G-space with regular momenta by removing the 'irregular points'. This normal form, which is a kind of non-Abelian generalization of action-angle coordinates, has already been obtained in a more general context by Marle (1983) (but see also Dazord and Delzant (1987)). The novelty here is that putting a G-space with regular momenta into normal form is reduced to constructing conventional action-angle coordinates in the associated Hamiltonian T-space. Furthermore the non-Abelian normal form applies *globally* if in the associated T-space the action-angle coordinates can be constructed globally. Indeed this is the case (after removing a set with open and dense complement) in several examples from mechanics, including the spherical pendulum (as an $SO(3)$-space), the axisymmetric rigid body (as an $SO(3) \times S^1$-space), the $1:1$ resonance (i.e. $\mathbb{R}^2 \cong \mathbb{C}$, as an $SU(2)$-space), the 2D Kepler problem (as an

$SO(3)$-space, using the 'super-symmetry' of the Kepler problem), and the 3D Kepler problem (as an $SO(3) \times S^1$-space, using the usual rotational symmetry of the Kepler problem combined with the S^1 action generated by the Hamiltonian itself). Only the first two examples will be discussed in detail here. We hope all our results will enhance the utility of the normal form, which seems to remain underexploited in applications.

Applications to perturbation theory

This work has been motivated by problems in Hamiltonian dynamics. In particular we have been interested in understanding the following 'symmetry-breaking' problem. Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with regular momenta, and consider a Hamiltonian of the form

$$H_\epsilon = H + \epsilon F \ ,$$

where H is G-invariant, F is fixed but arbitrary, and ϵ is a (small) parameter. If the symmetry group G is sufficiently large then the unperturbed dynamics ($\epsilon = 0$) will be integrable (in an appropriate sense), and there is some chance of extracting information about the dynamics of the perturbed system ($\epsilon \neq 0$), for all sufficiently small ϵ. When G is a torus, action-angle coordinates can be constructed and classical perturbation theory applies. Although conventional 'canonical coordinates' (i.e. action-angle coordinates or their generalizations à la Nekhoroshev (1972)) can be constructed locally if G is non-Abelian, these can prove inadequate for perturbation theory. The problem in the non-Abelian case is that there exist motions of speed $\mathcal{O}(\epsilon)$ that cannot (even locally) be 'pushed to higher order' by $\mathcal{O}(\epsilon)$ coordinate transformations. These *global resonant motions* take trajectories out of locally defined canonical coordinate charts in $\mathcal{O}(1/\epsilon)$ times, which is very short compared with time scales one seeks to establish for other components of the motion. Fasso (1995) has recently given a very good account of this problem. In the non-Abelian case, it makes more sense to use coordinates intrinsic to the non-Abelian geometry of the problem. The non-Abelian normal form $G \times U$ described above is precisely such a system of coordinates. It provides the geometric framework for a 'non-canonical' perturbation theory we shall describe in more detail elsewhere (Perry, 1996). A key advantage of the non-Abelian normal form is that the global resonant motions appear only in the G variables (G being closed and boundaryless), so that even a locally defined coordinate chart proves satisfactory.

We remark that Fasso (1995) has an altogether different solution to this problem using an *atlas* of coordinate charts based on the Nekhoroshev normal form. Fasso's starting point is a symplectically complete fibering[5] of P by tori (in our case the orbits of the canonical T-action). This makes his methods more generally applicable but has the disadvantage of ignoring any non-Abelian symmetry, which may appear very naturally in practice. Also, there seems to be some aspects of the Fasso theory that depend on the particular atlas used.

Paper outline

The paper is organized as follows:

2. Preliminaries
3. On the geometry of the co-adjoint action

[5] See Dazord and Delzant (1987) for a definition.

4. The fibering by invariant Hamiltonian T-spaces
5. The canonical T-action
6. On the classification of G-spaces with regular momenta
7. Symplectic reduction and integrability
8. The non-Abelian generalization of action-angle coordinates
9. Examples

The heart of our results is the existence of a natural equivariant diffeomorphism $\Gamma : \mathfrak{g}^*_{\text{reg}} \to W \times \mathcal{O}$, where W is a Weyl chamber and $\mathcal{O}$ is a regular co-adjoint orbit. We recall the relevant group-theoretical constructions in Sect. 3. Notation and basic definitions are laid out in Sect. 2. Sections 4–8 deal with the theory of G-spaces with regular momenta outlined above. In Sect. 9 we illustrate this theory with examples from mechanics including the Euler-Poinsot rigid body.

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2 Preliminaries

For an introduction to the modern geometric point of view of symmetry in Hamiltonian systems refer to Abraham and Marsden (1978, Chapter 4) or Marsden and Ratiu (1994). For a rapid introduction and recent survey see Marsden (1992). Unless otherwise indicated notation follows Abraham and Marsden (1978).

In Hamiltonian mechanics the 'phase space' is a symplectic manifold (P, ω) and the Hamiltonian H is a smooth function on P determining a vector field X_H on P according to $X_H \lrcorner\, \omega = dH$. By 'smooth' we mean C^∞ or real-analytic. We assume that P is finite-dimensional.

A *symmetry* is a smooth left action on P by some Lie group G, with respect to which the Hamiltonian is invariant. Thus, if the action is denoted $(x, g) \mapsto \Phi_g(x) : P \times G \to P$, then $H \circ \Phi_g = H$ for all $g \in G$. For each $\xi \in \mathfrak{g}$, the *infinitessimal generator* ξ_P is a vector field on P defined by

$$\xi_P(x) \equiv \frac{d}{dt} \Phi_{\exp(t\xi)}(x) \Big|_{t=0} .$$

We assume that the infinitessimal generators are *Hamiltonian* vector fields, and that the corresponding Hamiltonian functions $J_\xi : P \to \mathbb{R}$ (giving $\xi_P = X_{J_\xi}$) are delivered by a smooth map $\mathbf{J} : P \to \mathfrak{g}^*$ according to $J_\xi(x) \equiv \langle \mathbf{J}(x), \xi \rangle$. The map $\mathbf{J} : P \to \mathfrak{g}^*$ is known as a *momentum map*. Note that $\mathfrak{g}$ denotes the Lie algebra of G and $\langle \cdot, \cdot \rangle$ denotes natural pairing between covectors and vectors respectively. When a momentum map exists for an action, that action is called *Hamiltonian*.

Noether's theorem (see e.g. Marsden (1992)) states that $\mathbf{J}(x_t)$ is constant on trajectories $t \mapsto x_t \in P$ of X_H. Thus, for all $\mu \in \mathfrak{g}^*$, the μ-level set $\mathbf{J}^{-1}(\mu)$ is invariant with respect to the Hamiltonian flow for H.

In addition to P, there is a natural action of G on $\mathfrak{g}$ and $\mathfrak{g}^*$. These are the adjoint and co-adjoint actions respectively. In all cases the image of a pair (g, y) under the group action will often be abbreviated $g \cdot y$. Thus $g \cdot x \equiv \Phi_g(x), x \in P$; $g \cdot \xi \equiv \mathrm{Ad}_g \xi, \xi \in \mathfrak{g}$ and $g \cdot \mu \equiv \mathrm{Ad}^*_{g^{-1}} \mu, \mu \in \mathfrak{g}^*$. These all being *left* actions, we have $(g_1 g_2) \cdot y = g_1 \cdot (g_2 \cdot y)$. We write $G \cdot y \equiv \{g \cdot y \mid g \in G\}$, $G_y \equiv \{g \mid g \cdot y = y\}$, $g(S) \equiv \{g \cdot s \mid s \in S\}$ and $G(S) \equiv \cup_{y \in S} G \cdot y$; $G \cdot y$ is the *G-orbit* through y and G_y is the *isotropy group* at y in G. All group quotients G/H are with respect to *left* equivalence: $g_1 \sim g_2$ if $g_1 = g_2 \cdot h$ for some $h \in H$; the equivalence class (left coset) containing g is denoted gH. If a group G acts on manifold X then a point $x \in X$ is called *regular* if $\dim G_x = \min_{x' \in X} \dim G_{x'}$. If x is regular so are all points of $G \cdot x$; $G \cdot x$ is then called a *regular orbit*.

Throughout we assume that the momentum map is *equivariant* with respect to the action of G, i.e. $\mathbf{J}(g \cdot x) = g \cdot \mathbf{J}(x)$. We then refer to $(P, \omega, G, \Phi, \mathbf{J})$ as a *Hamiltonian G-space* and to $(P, \omega, G, \Phi, \mathbf{J}, H)$ as a *Hamiltonian G-system.* Sufficient conditions for the existence of equivariant momentum maps are described in Guillemin and Sternberg (1984b, §25). We will usually write $(P, \omega, G, \mathbf{J})$ (resp. $(P, \omega, G, \mathbf{J}, H)$) instead of $(P, \omega, G, \Phi, \mathbf{J})$ (resp. $(P, \omega, G, \Phi, \mathbf{J}, H)$), the action Φ being understood.

3 On the geometry of the co-adjoint action

Denote by $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$ the regular points of the adjoint action, and by $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ the regular points of the co-adjoint action. The main purpose of this section is to show that there is a natural equivariant diffeomorphism $\Gamma : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O} \times W$ where $\mathcal{O}$ is any regular co-adjoint orbit and $W \subset \mathfrak{g}^*$ is any (open) Weyl chamber in $\mathfrak{g}^*$ (defined below). Furthermore there is a natural isomorphism $i : W \to \mathfrak{t}^*_0$ identifying W with a set $\mathfrak{t}^*_0 \subset \mathfrak{t}^*$; $\mathfrak{t}$ is the Lie algebra of the maximal torus $T \equiv G_\mu$, where μ is any element of W. The isotropy group G_μ is identical for any choice of $\mu \in W$.

Although these results are probably known by experts, it is difficult to find them in a form that is easy to learn without a reasonable knowledge of the 'structure theory' of Lie groups and Lie algebras. Here most of the theory needed is reduced to the following two facts: (i) If G is compact then all maximal Abelian subalgebras of $\mathfrak{g}$ are conjugate under the adjoint action, and (ii) Hermann Weyl's theorem stating that a regular adjoint orbit of G intersects an (open) Weyl chamber in $\mathfrak{g}$ (definition recalled below) in a single point. The interested reader is referred to standard texts (e.g. Varadarajan (1984)) for proofs. Our starting point is the Duflo-Vergne Theorem, which is easily learnt from e.g. Marsden and Ratiu (1994). Experts, after familiarizing themselves with our notation, may want to skip immediately to Proposition 3.15.

Let us briefly motivate our interest in the trivialization. From the diffeomorphism Γ : $\mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O} \times W$, we obtain two projections

$$\begin{array}{rrcl} & \pi_{\mathcal{O}} & : & \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O} \\ \text{and} & \pi_W & : & \mathfrak{g}^*_{\mathrm{reg}} \to W \ . \end{array}$$

In a Hamiltonian G-space $(P, \omega, G, \mathbf{J})$ with regular momenta, the projections lead us to define $\pi \equiv \pi_{\mathcal{O}} \circ \mathbf{J}$ and $\mathbf{j} \equiv i \circ \pi_W \circ \mathbf{J}$, which are well-defined since $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. We will eventually show that the map $\pi : P \to \mathcal{O}$ is a fibering of P by Hamiltonian T-spaces and that $\mathbf{j} : P \to \mathfrak{t}^*$ is the momentum map for a canonical action by T on P, with respect to which the fibres of

π are invariant.

Direct-sum decompositions of $\mathfrak{g}$ and $\mathfrak{g}^*$

3.1 **Theorem** (Duflo and Vergne (1969); see also Weinstein (1983)) *Let G be a finite-dimensional Lie Group (not necessarily compact). Then $\mathfrak{g}^*_{\rm reg}$ is open and dense in $\mathfrak{g}^*$. Furthermore, for all $\mu \in \mathfrak{g}^*_{\rm reg}$, $\mathfrak{g}_\mu$ is Abelian.*

3.2 **Corollary** *If G is compact and connected and $\mu \in \mathfrak{g}^*_{\rm reg}$, then $\mathfrak{g}_\mu$ is a maximal Abelian subalgebra of $\mathfrak{g}$ and G_μ is a maximal torus of G*[6].

In the proof of 3.2 and other results that follow $\rho : \mathfrak{g}^* \to \mathfrak{g}$ is a fixed isomorphism equivariant with respect to the co-adjoint and adjoint actions. Recall that when G is compact such an isomorphism always exists — just take the isomorphism induced by a G-invariant inner product on $\mathfrak{g}$. One obtains such an inner product by averaging an arbitrary inner product over G; averaging makes sense since compact Lie groups admit a bi-invariant probability measure.

Proof of 3.2: By the equivariance of ρ, $\mathfrak{g}_\mu = \mathfrak{g}_{\rho(\mu)}$. By 3.1, $\mathfrak{g}_{\rho(\mu)}$ is therefore Abelian. It must be *maximal* Abelian, for if $[\xi, \mathfrak{g}_{\rho(\mu)}] = 0$ then $[\xi, \rho(\mu)] = 0$, implying $\xi \in \mathfrak{g}_{\rho(\mu)}$. Therefore $\mathfrak{g}_\mu$ is maximal Abelian. Since G is compact and connected, G_μ is also compact and connected (Guillemin and Sternberg, 1984b, Theorems 32.15, 32.16). By a theorem of E. Cartan (see e.g. Kirillov (1976, Theorem 2, §6.3)), there is a unique connected and simply connected Lie group H with Lie algebra $\mathfrak{g}_\mu$, and all connected Lie groups with Lie algebra $\mathfrak{g}_\mu$ must be of the form H/D for some discrete normal subgroup D contained in the center of H. In particular, this applies to G_μ. Since $\mathfrak{g}_\mu$ is Abelian, $H \cong \mathbb{R}^k$ ($k \equiv \dim \mathfrak{g}_\mu$). In particular, H is Abelian. All factor groups of H must also be Abelian, so G_μ is Abelian. Since G_μ is also compact and connected, G_μ is isomorphic to a torus.
□

Henceforth G denotes a compact connected Lie group

3.3 **Lemma** *A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is maximal Abelian $\Leftrightarrow$ $\mathfrak{t} = \mathfrak{g}_\mu$ for some $\mu \in \mathfrak{g}^*_{\rm reg}$.*

Proof: The $\Leftarrow$ part comes from 3.2. To prove $\Rightarrow$ recall (see Varadarajan (1984)) that all maximal Abelian subalgebras are conjugate when G is compact. Thus $\mathfrak{t} = g(\mathfrak{g}_{\mu'})$ for some $g \in G$ and $\mu' \in \mathfrak{g}^*_{\rm reg}$. But $g(\mathfrak{g}_{\mu'}) = \mathfrak{g}_{g\cdot\mu'}$, i.e. $\mathfrak{t} = \mathfrak{g}_\mu$ with $\mu \equiv g \cdot \mu'$.
□

3.4 **Corollary** *If $\mathfrak{t} \subset \mathfrak{g}$ is a maximal Abelian subalgebra then $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$, where $\mathfrak{t}^\perp \equiv [\mathfrak{t}, \mathfrak{g}]$.*

Proof: By 3.3 there exists $\mu \in \mathfrak{g}^*_{\rm reg}$ such that $\mathfrak{t} = \mathfrak{g}_\mu$. Put $\xi \equiv \rho(\mu)$ so that by the G-equivariance of ρ, $\mathfrak{g}_\xi = \mathfrak{g}_\mu = \mathfrak{t}$. With respect to any G-equivariant inner product, ad_ξ is skew-symmetric, so that Ker $\mathrm{ad}_\xi = \mathfrak{t}$ and Im $\mathrm{ad}_\xi = [\xi, \mathfrak{g}]$ are complementary spaces. That is, $\mathfrak{g} = \mathfrak{t} \oplus [\xi, \mathfrak{g}]$. We claim $[\xi, \mathfrak{g}] = [\mathfrak{t}, \mathfrak{g}]$. Since $[\xi, \mathfrak{g}] \subset [\mathfrak{t}, \mathfrak{g}]$, it suffices to show that $[\eta, \mathfrak{g}] \subset [\xi, \mathfrak{g}]$

[6]The proof of the first assertion was communicated to me by Robert Filippini.

for all $\eta \in \mathfrak{t}$. Now $\mathfrak{g} = \mathfrak{t} \oplus [\xi, \mathfrak{g}]$, so $[\eta, \mathfrak{g}] = [\eta, [\xi, \mathfrak{g}]]$, since $[\eta, \mathfrak{t}] = 0$. Thus any element of $[\eta, \mathfrak{g}]$ is of the form $[\eta, [\xi, \beta]]$ for some $\beta \in \mathfrak{g}$. But by the Jacobi identity

$$\begin{aligned} [\eta, [\xi, \beta]] &= -[\beta, [\eta, \xi]] - [\xi, [\beta, \eta]] \\ &= -[\xi, [\beta, \eta]] \ , \end{aligned}$$

since $[\eta, \xi] = 0$. In particular it follows that $[\eta, [\xi, \beta]] \in [\xi, \mathfrak{g}]$, so that any element of $[\eta, \mathfrak{g}]$ is in $[\xi, \mathfrak{g}]$, as required.
□

3.5 **Remark** It is clear from the proof that $\mathfrak{t}^\perp \equiv [\mathfrak{t}, \mathfrak{g}]$ is in fact the orthogonal complement of $\mathfrak{t}$, with respect to any G-invariant inner product on $\mathfrak{g}$.

Write $\underline{\mathfrak{t}} \equiv \mathrm{Ann}\,[\mathfrak{t}, \mathfrak{g}]$ (the annihilator of $[\mathfrak{t}, \mathfrak{g}]$) and $\underline{\mathfrak{t}}^\perp \equiv \mathrm{Ann}\,\mathfrak{t}$, so that assuming $\mathfrak{t}$ is maximal Abelian, we obtain from 3.4 the dual direct sum decomposition

$$\mathfrak{g}^* = \underline{\mathfrak{t}} \oplus \underline{\mathfrak{t}}^\perp \ . \tag{1}$$

We now state alternative characterizations of the subspaces appearing in this decomposition. By definition $\mu \in \underline{\mathfrak{t}}$ if and only if $\langle \mu, [\tau, \xi] \rangle = 0$ for all $\tau \in \mathfrak{t}$ and $\xi \in \mathfrak{g}$, i.e. if and only if $\mathrm{ad}^*_{\mathfrak{t}}\, \mu = 0$. So it follows that

$$\underline{\mathfrak{t}} = \{\mu \in \mathfrak{g}^* \mid \mathfrak{t} \subset \mathfrak{g}_\mu\} \ . \tag{2}$$

Using the fact that $\dim \mathfrak{g}_\mu = \dim \underline{\mathfrak{t}} \Leftrightarrow \mu \in \mathfrak{g}^*_{\mathrm{reg}}$, we also obtain

$$\underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}} = \{\mu \in \mathfrak{g}^* \mid \mathfrak{g}_\mu = \mathfrak{t}\}. \tag{3}$$

Finally, let $\mathrm{T}_\mu(G \cdot \mu)$ denote the tangent space at $\mu \in \mathfrak{g}^*$ to the co-adjoint orbit $G \cdot \mu$. Identifying $\mathrm{T}_\mu(G \cdot \mu)$ with a subspace of $\mathfrak{g}^*$, we have $\mathrm{T}_\mu(G \cdot \mu) \cong \mathrm{ad}^*_{\mathfrak{g}}\, \mu$. We claim that

$$\underline{\mathfrak{t}}^\perp = \mathrm{T}_\mu(G \cdot \mu) \qquad \forall \mu \in \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}} \ . \tag{4}$$

Proof of (4): Let $\mu \in \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$. Then by (3), $\mathfrak{t} = \mathfrak{g}_\mu$. The map $\xi \mapsto \mathrm{ad}^*_\xi \mu : \mathfrak{g} \to \mathfrak{g}^*$ has kernel $\mathfrak{g}_\mu$ and image $\mathrm{T}_\mu(G \cdot \mu)$, implying $\dim \mathfrak{g}_\mu + \dim \mathrm{T}_\mu(G \cdot \mu) = \dim \mathfrak{g}$. It follows that $\dim \mathrm{T}_\mu(G \cdot \mu) = \mathrm{codim}\, \mathfrak{t} = \dim \underline{\mathfrak{t}}^\perp$. To prove (4) it therefore suffices to show that $\mathrm{T}_\mu(G \cdot \mu) \subset \underline{\mathfrak{t}}^\perp$. But elements β of $\mathrm{T}_\mu(G \cdot \mu) = \mathrm{ad}^*_{\mathfrak{g}}\, \mu$ are of the form $\beta = \mathrm{ad}^*_\xi \mu$ for some $\xi \in \mathfrak{g}$, and for all $\tau \in \mathfrak{t}$ we have $\langle \beta, \tau \rangle = \langle \mathrm{ad}^*_\xi\, \mu, \tau \rangle = -\langle \mathrm{ad}^*_\tau\, \mu, \xi \rangle = 0$, since $\mathfrak{t} = \mathfrak{g}_\mu$. Thus $\mathrm{T}_\mu(G \cdot \mu) \subset \mathrm{Ann}\ \mathfrak{t} = \underline{\mathfrak{t}}^\perp$, as required.
□

Weyl chambers

Recall that when $\mathfrak{t} \subset \mathfrak{g}$ is a maximal Abelian subalgebra (G compact) then a connected component of $\mathfrak{t} \cap \mathfrak{g}_{\mathrm{reg}}$ is called a *Weyl chamber in* $\mathfrak{g}$.

3.6 **Definition** Let $\mathfrak{t} \subset \mathfrak{g}$ be any maximal Abelian subalgebra and define (as above) $\underline{\mathfrak{t}} \equiv \mathrm{Ann}\,[\mathfrak{t}, \mathfrak{g}]$. Then any connected component W of $\underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$ is called a *Weyl chamber in* $\mathfrak{g}^*$.

3.7 **Remarks** a. W is *open* in $\underline{\mathfrak{t}}$ (by 3.1 and 3.3).

b. It follows from (3) that through each point $\mu_0 \in \mathfrak{g}^*_{\rm reg}$ there passes a unique Weyl chamber W; it is the connected component of

$$\underline{\mathfrak{g}_{\mu_0}} \cap \mathfrak{g}^*_{\rm reg} \equiv \mathrm{Ann}\,[\mathfrak{g}_{\mu_0}, \mathfrak{g}] \cap \mathfrak{g}^*_{\rm reg} = \{\mu \in \mathfrak{g}^* \mid \mathfrak{g}_\mu = \mathfrak{g}_{\mu_0}\}$$

containing μ_0.

c. Let $\mathfrak{t} \subset \mathfrak{g}$ be maximal Abelian. Then the natural projection $i : \mathfrak{g}^* \to \mathfrak{t}^*$ restricted to $\underline{\mathfrak{t}}$ is an isomorphism, since its kernel is $\underline{\mathfrak{t}}^\perp$ and $\mathfrak{g}^* = \underline{\mathfrak{t}} \oplus \underline{\mathfrak{t}}^\perp$. Let W be a Weyl chamber in $\mathfrak{g}^*$ and put $\mathfrak{t} = \mathfrak{g}_\mu$, where $\mu \in W$ is arbitrary. Then it follows that we may identify $W \subset \underline{\mathfrak{t}}$ with $\mathfrak{t}^*_0 \equiv i(W) \subset \mathfrak{t}^*$.

3.8 **Lemma** *If W is a Weyl chamber in $\mathfrak{g}^*$ then $G_{w_1} = G_{w_2}$ for all $w_1, w_2 \in W$.*

Proof: By (3), $\mathfrak{g}_{w_1} = \mathfrak{g}_{w_2}$. By 3.2, G_{w_1} and G_{w_2} are just tori, so that $\exp_{G_{w_1}} : \mathfrak{g}_{w_1} \to G_{w_1}$ and $\exp_{G_{w_2}} : \mathfrak{g}_{w_2} \to G_{w_2}$ are onto. Therefore $G_{w_1} = G_{w_2}$.
□

3.9 **Lemma** *The intersection of Weyl chambers in $\mathfrak{g}^*$ and (regular) co-adjoint orbits is transversal. More precisely: if $\mu \in \mathfrak{g}^*_{\rm reg}$, W is the Weyl chamber through μ, and $\mathcal{O}$ is the co-adjoint orbit through μ, then $\mathrm{T}_\mu W$ and $\mathrm{T}_\mu\mathcal{O}$ are complementary subspaces of $\mathrm{T}_\mu\mathfrak{g}^* \cong \mathfrak{g}^*$.*

Proof: Put $\mathfrak{t} \equiv \mathfrak{g}_\mu$ so that $\mathfrak{t}$ is maximal Abelian (by 3.2), and that W is an open subset of $\underline{\mathfrak{t}}$ (by 3.7*b* and 3.7a). It follows that $\mathrm{T}_\mu W \cong \underline{\mathfrak{t}}$ and from (4) that $\mathrm{T}_\mu\mathcal{O} \cong \underline{\mathfrak{t}}^\perp$. By (1) we have $\mathfrak{g}^* = \mathrm{T}_\mu W \oplus \mathrm{T}_\mu\mathcal{O}$.
□

3.10 **Lemma** *Let $\rho : \mathfrak{g}^* \to \mathfrak{g}$ be any G-equivariant isomorphism. Then W is a Weyl chamber in $\mathfrak{g}^* \Leftrightarrow \rho(W)$ is a Weyl chamber in $\mathfrak{g}$.*

Proof: Now $\rho(\mathfrak{g}^*_{\rm reg}) = \mathfrak{g}_{\rm reg}$ by the equivariance, so from the definitions it suffices to show that for all maximal Abelian subalgebras $\mathfrak{t}$ we have $\rho(\underline{\mathfrak{t}}) = \mathfrak{t}$. In fact, since $\dim\underline{\mathfrak{t}} = \dim\mathfrak{t}$, it is enough to check that $\rho(\underline{\mathfrak{t}}) \subset \mathfrak{t}$. By (2) and the equivariance of ρ,

$$\rho(\underline{\mathfrak{t}}) = \{\xi \in \mathfrak{g} \mid \mathfrak{t} \subset \mathfrak{g}_\xi\}\ . \tag{5}$$

But $\mathfrak{t} \subset \mathfrak{g}_\xi$ implies $[\tau, \xi] = 0$ for all $\tau \in \mathfrak{t}$, giving $\xi \in \mathfrak{t}$, since $\mathfrak{t}$ is *maximal* Abelian. This shows that the right hand side of (5) is contained in $\mathfrak{t}$ as required.
□

3.11 **Corollary** *Each regular co-adjoint orbit intersects a given Weyl chamber in $\mathfrak{g}^*$ at precisely one point.*

Proof: Follows from 3.10 and Weyl's theorem that regular adjoint orbits intersect a given Weyl chamber in $\mathfrak{g}$ at precisely one point(see e.g. Varadarajan (1984)).
□

The trivialization of $\mathfrak{g}^*$

3.12 **Example** Consider $G = SO(3)$. Make the usual identification $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ by choosing a basis $\{e_1, e_2, e_3\}$ for $\mathfrak{so}(3)$ such that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2 \ .$$

Then the matrix representation of $\mathrm{Ad}^*_{g^{-1}}$ is g itself. We have $\mathfrak{g}^*_{\mathrm{reg}} \cong \mathbb{R}^3 \backslash \{0\}$. The regular orbits are the spheres centered at the origin, of non-zero radius. The Weyl chambers are the open-ended rays emanating from the origin.

Let $\mathcal{O}$ be any orbit (sphere) and W any Weyl chamber (ray). Make the obvious identifications $\mathcal{O} \cong S^2$ and $W \cong (0, \infty)$. Then there is a *diffeomorphism* $\gamma : \mathcal{O} \times W \to \mathfrak{g}^*_{\mathrm{reg}}$ whose inverse is given by

$$\gamma^{-1}(\mu) \equiv (\mu/\|\mu\|, \|\mu\|) \ .$$

□

In the example we obtained an equivariant diffeomorphism $\gamma^{-1} : \mathfrak{g}^*_{\mathrm{reg}} \to W \times \mathcal{O}$, i.e. we succeeded in trivializing the entire (open dense) set of regular points. Indeed this is the case in general as we shall now recall.

Let W be a Weyl chamber in $\mathfrak{g}^*$, and $\mathfrak{t}$ its associated maximal Abelian subalgebra. Let $\mathcal{O}$ be any regular co-adjoint orbit and let $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ be the unique intersection point of W and $\mathcal{O}$; by 3.7*b*, $\mathfrak{t} = \mathfrak{g}_{\mu_0}$ and $\mathfrak{g}_w = \mathfrak{t}$ for all $w \in W$. Conversely one may choose $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ arbitrarily and define W to be the Weyl chamber through μ_0 (as given by 3.7*b*), and $\mathcal{O}$ to be the co-adjoint orbit through μ_0. Recall that for each $\mu \in \mathfrak{g}^*$ we can define a G-equivariant diffeomorphism $\phi_\mu : G/G_\mu \to G \cdot \mu$ by $\phi_\mu(gG_\mu) \equiv g \cdot \mu$. The map ϕ_{μ_0} has inverse $\phi^{-1}_{\mu_0} : \mathcal{O} \to G/G_{\mu_0}$. For any $w \in W$, the domain of ϕ_w is G/G_{μ_0}, because $G_w = G_{\mu_0}$ (by 3.8). Thus the map

$$\begin{gathered} \gamma : \mathcal{O} \times W \to \mathfrak{g}^* \\ \gamma(\mu, w) \equiv \phi_w(\phi^{-1}_{\mu_0}(\mu)) \end{gathered} \tag{6}$$

is well-defined. One verifies that

$$g \cdot \gamma(\mu, w) = \gamma(g \cdot \mu, w) \ . \tag{7}$$

3.13 **Proposition** *γ is a diffeomorphism onto $\mathfrak{g}^*_{\mathrm{reg}}$.*

Proof that γ is a local diffeomorphism: First recall that by 3.9, $\dim \mathcal{O} + \dim W = \dim \mathfrak{g}^*$. Let $\mu \in \mathcal{O} = G \cdot \mu_0$ so that $\mu = g \cdot \mu_0$ for some $g \in G$. Then for any $w \in W$, $\gamma(\mu, w) = \phi_w(gG_{\mu_0}) = \phi_w(gG_w) = g \cdot w$. Since g maps $\underline{\mathfrak{t}}$ isomorphically onto $\underline{g(\mathfrak{t})}$, it follows that γ maps $\{\mu\} \times W$ diffeomorphically onto the Weyl chamber through $g \cdot w$. Conversely, for any fixed w, $\phi_w \circ \phi^{-1}_{\mu_0}$ maps $\mathcal{O} = G \cdot \mu_0$ diffeomorphically onto $G \cdot w$, implying γ maps $\mathcal{O} \times \{w\}$ diffeomorphically onto $G \cdot w$. Fix a point $(\mu, w) \in \mathcal{O} \times W$. The submanifolds $\{\mu\} \times W$ and $\mathcal{O} \times \{w\}$ intersect at (μ, w) transversely in $\mathcal{O} \times W$. We have just shown that the first submanifold is mapped diffeomorphically onto the regular orbit through $g \cdot w$, the second diffeomorphically onto the Weyl chamber through $g \cdot w$. By 3.9 this orbit and chamber also intersect transversally, and so γ is a local diffeomorphism at (μ, w).

□

Proof of surjectivity: Suppose $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$. By 3.11 the orbit $G \cdot \mu$ intersects the chamber W

at some point $w \in W$. It follows that $\mu = g \cdot w$ for some $g \in G$. One verifies $\gamma(g \cdot \mu_0, w) = \mu$, so γ is onto.
□

PROOF OF INJECTIVITY: Suppose $\gamma(\mu_1, w_1) = \gamma(\mu_2, w_2)$. We have $\mu_1 = g_1 \cdot \mu_0$ and $\mu_2 = g_2 \cdot \mu_0$ for some $g_1, g_2 \in G$, so that $\phi_{\mu_0}^{-1}(\mu_1) = g_1 G_{\mu_0}$ and $\phi_{\mu_0}^{-1}(\mu_2) = g_2 G_{\mu_0}$. By supposition $\phi_{w_1}(\phi_{\mu_0}^{-1}(\mu_1)) = \phi_{w_2}(\phi_{\mu_0}^{-1}(\mu_2))$, from which $\phi_{w_1}(g_1 G_{\mu_0}) = \phi_{w_2}(g_2 G_{\mu_0})$ follows. We therefore have $w_1 = (g_1^{-1} g_2) \cdot w_2$. But W intersects each (regular) orbit *once only* (by 3.11), so $g_1^{-1} g_2 = \mathrm{id}_G$, $w_1 = w_2$ and $\mu_1 = \mu_2$.
□

3.14 **Remark** Since W intersects each regular orbit once only and the intersections are transversal, W is a 'slice' at μ_0 for the co-adjoint action in the sense of e.g. Montgomery and Yang (1957). Proposition 3.15 is an example of a 'tubular neighborhood' or 'slice' theorem. General slice theorems state that the neighborhood of an orbit is diffeomorphic to a 'twisted product'; here we have a simple product and the 'neighborhood' is the entire open dense set $\mathfrak{g}^*_{\mathrm{reg}}$. Locally one always has a natural choice of slice, even at irregular points of $\mathfrak{g}^*$. This general construction of (local) slices for the co-adjoint action appears in Guillemin and Sternberg (1984a).

Define $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$ and $\pi_W : \mathfrak{g}^*_{\mathrm{reg}} \to W$ so as to satisfy $\gamma^{-1} = (\pi_{\mathcal{O}}, \pi_W)$, i.e. $\gamma^{-1}(\mu) = (\pi_{\mathcal{O}}(\mu), \pi_W(\mu))$. Then unravelling the definition of γ we deduce that

$$\left.\begin{array}{rcl} \pi_{\mathcal{O}}(g \cdot w) & = & g \cdot \mu_0 \\ \pi_W(g \cdot w) & = & w \end{array}\right\} \quad w \in W, g \in G \ . \tag{8}$$

It is clear from 3.13 that (8) serves also to implicitly define $\pi_{\mathcal{O}}$ and π_W.

The following summarizes many results of this section.

3.15 **Proposition** *Let G be a compact connected Lie group. Each Weyl chamber in $\mathfrak{g}^*$ intersects each co-adjoint orbit at exactly one point and transversally. Let W be a Weyl chamber in $\mathfrak{g}^*$, $\mathfrak{t} \subset \mathfrak{g}$ its associated maximal Abelian subalgebra, and $\mathcal{O}$ a regular co-adjoint orbit. Let $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ be the point of intersection (so that $\mathfrak{t} = \mathfrak{g}_{\mu_0}$) and let $T \equiv G_{\mu_0}$. Then:*

a. *All points in W have isotropy group T, which is a maximal torus of G. Any G-equivariant isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ maps W onto a Weyl chamber in $\mathfrak{g}$ lying in $\mathfrak{t}$. The natural projection $i : \mathfrak{g}^* \to \mathfrak{t}^*$ maps $\underline{\mathfrak{t}} \equiv \mathrm{Ann}\,[\mathfrak{t}, \mathfrak{g}]$ isomorphically onto $\mathfrak{t}^*$ so that $W \subset \underline{\mathfrak{t}}$ may be identified with the open set $\mathfrak{t}^*_0 \equiv i(W) \subset \mathfrak{t}^*$.*

b. *Equation (8) implicitly defines surjective submersions $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$ and $\pi_W : \mathfrak{g}^*_{\mathrm{reg}} \to W$ such that $(\pi_{\mathcal{O}}, \pi_W) : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O} \times W$ is a diffeomorphism. The inverse of this map is γ, the map defined explicitly in (6) above.*

c. *The fibres of $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$ are the Weyl chambers in $\mathfrak{g}^*$. The fibres of $\pi_W : \mathfrak{g}^*_{\mathrm{reg}} \to W$ are the regular co-adjoint orbits.*

Before turning to the main subject matter of this paper we need one last technical detail for later:

Identifications of $\underline{\mathfrak{t}}^\perp$ and $\mathfrak{t}^\perp$ induced by a point in $\mathfrak{t}_0^*$

3.16 **Lemma** *Let W be a Weyl chamber in $\mathfrak{g}^*$ and $\mathfrak{t} \subset \mathfrak{g}$ be the associated maximal Abelian subalgebra (i.e. $\mathfrak{t} = \mathfrak{g}_{\mu_0}$ for any $\mu_0 \in W$). Put $\mathfrak{t}_0^* = i(W)$, where $i : \underline{\mathfrak{t}} \to \mathfrak{t}^*$ is the natural isomorphism (see 3.15a). Then for all $p \in \mathfrak{t}_0^*$ the map*

$$\xi \mapsto \mathrm{ad}_\xi^*(i^{-1}(p)) : \mathfrak{t}^\perp \to \underline{\mathfrak{t}}^\perp$$

is an isomorphism.

Proof: Recall first that $\mathfrak{t}^\perp \equiv [\mathfrak{t}, \mathfrak{g}]$ and $\underline{\mathfrak{t}}^\perp \equiv \mathrm{Ann}\, \mathfrak{t}$. Since $\dim \mathfrak{t}^\perp = \dim \underline{\mathfrak{t}}^\perp$, it suffices to show that $\mathrm{ad}_\xi^*\, i^{-1}(p) = 0 \implies \xi = 0$. Write $\mu \equiv i^{-1}(p) \in W \subset \underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$, so that $\mathfrak{t} = \mathfrak{g}_\mu$ (by (3)). Supposing that $\xi \in \mathfrak{t}^\perp$ satisfies $\mathrm{ad}_\xi^*\, i^{-1}(p) = 0$ we have $\mathrm{ad}_\xi^*\, \mu = 0$, i.e. $\xi \in \mathfrak{g}_\mu = \mathfrak{t}$. But $\mathfrak{t} \cap \mathfrak{t}^\perp = \{0\}$, so $\xi = 0$.
□

3.17 **Definition** For any $p \in \mathfrak{t}_0^*$, let $\lambda_p : \underline{\mathfrak{t}}^\perp \to \mathfrak{t}^\perp$ denote the inverse of the map described in 3.16 above, i.e. the map well-defined by

$$\lambda_p \left(\mathrm{ad}_\xi^*\, i^{-1}(p) \right) = \xi \qquad \forall \xi \in \mathfrak{t}^\perp \ . \tag{9}$$

4 The fibering by Hamiltonian T-spaces

Let $(P, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with regular momenta, i.e. one for which

$$\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}} \ . \tag{10}$$

In what follows H is any G-invariant Hamiltonian, i.e. $(P, \omega, G, \mathbf{J}, H)$ is a Hamiltonian G-system. We continue to restrict attention to the case in which G is compact and connected.

Before discussing the symplectic fibration, let us prove the condition necessarily satisfied by G-spaces with regular momenta, which we stated in the Introduction:

4.1 **Proposition** *Under the assumption (10),* $\dim G_x \leq \mathrm{rank}\ G$*, for all $x \in P$ where* rank G *is the dimension of the maximal tori of G.*

This result comes about because momentum map equivariance already puts a restriction on how un-free the action of G on $\mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$ can be, since $\mathbf{J}$ maps these points to *maximal* co-adjoint orbits. To see this, let us recall the result connecting the isotropy at $x \in P$ to the rank of the momentum map:

4.2 **Lemma** $\mathfrak{g}_x = \mathrm{Ann\ Im\ T}_x\mathbf{J}$.

Proof:

$$\begin{aligned}
\xi \in \mathfrak{g}_x \quad &\Leftrightarrow \quad \xi_P(x) = 0 \\
&\Leftrightarrow \quad \langle X_{J_\xi}(x), v \rangle = 0, \qquad \forall v \in \mathrm{T}_x P \\
&\Leftrightarrow \quad \langle d_x J_\xi, v \rangle = 0, \qquad \forall v \in \mathrm{T}_x P \\
&\Leftrightarrow \quad \langle \mathrm{T}_x \mathbf{J} \cdot v, \xi \rangle = 0, \qquad \forall v \in \mathrm{T}_x P \\
&\Leftrightarrow \quad \xi \in \mathrm{Ann\ Im\ T}_x \mathbf{J} \ .
\end{aligned}$$

□

4.3 **Corollary** *If* $\mathbf{J}$ *is equivariant, then* $\dim \mathfrak{g}_x \leq \dim \mathfrak{g}_{\mathbf{J}(x)}$.

PROOF: Put $\mu \equiv \mathbf{J}(x)$. By equivariance, $\mathrm{T}_x\mathbf{J}$ maps $\mathrm{T}_x(G \cdot x)$ onto $\mathrm{T}_\mu(G \cdot \mu)$. Therefore $\dim \mathrm{Ann}\ \mathrm{Im}\ \mathrm{T}_x\mathbf{J} \leq \mathrm{codim}\ \mathrm{T}_\mu(G \cdot \mu) = \dim \mathfrak{g}_\mu$.
□
Proposition 4.1 now follows from (10) and 4.3.

The symplectic fibering of P

Apply 3.15 to obtain the fibration $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$. Because of (10) we may define a map $\pi : P \to \mathcal{O}$ by $\pi \equiv \pi_{\mathcal{O}} \circ \mathbf{J}$. Note that π is G-equivariant because $\pi_{\mathcal{O}}$ and $\mathbf{J}$ are G-equivariant. The pre-images $\pi^{-1}(\mu)$ are invariant with respect to the flow of X_H by Noether's theorem (Sect. 2). Pick $x \in \pi^{-1}(\mu)$ and $g \in G$. By the equivariance of π, $\pi(g \cdot x) = g \cdot \pi(x) = g \cdot \mu$. If $g \in G_\mu$ then we obtain $\pi(g \cdot x) = \mu$, i.e. $g \cdot x \in \pi^{-1}(\mu)$. Conversely if we require $\pi(g \cdot x) = \mu$, i.e. $g \cdot x \in \pi^{-1}(\mu)$, then we obtain $g \cdot \mu = \mu$, i.e. $g \in G_\mu$. This proves

$$\{g \in G \mid x \in \pi^{-1}(\mu) \implies g \cdot x \in \pi^{-1}(\mu)\} = G_\mu \ . \tag{11}$$

In particular all elements of G_μ map $\pi^{-1}(\mu)$ onto itself.

Choose μ_1 and μ_2 in $\mathcal{O}$ and consider the pre-images $\pi^{-1}(\mu_1)$ and $\pi^{-1}(\mu_2)$. We have $\mu_2 = h \cdot \mu_1$ for some $h \in G$, so $\Phi_h : \pi^{-1}(\mu_1) \to \pi^{-1}(\mu_2)$ is a bijection. The flows of $X_H|\pi^{-1}(\mu_1)$ and $X_H|\pi^{-1}(\mu_2)$ are conjugate because $\Phi_h^* X_H = X_H$. Since $G_{\mu_2} = hG_{\mu_1}h^{-1}$, we have the isomorphism $g \mapsto hgh^{-1} : G_{\mu_1} \to G_{\mu_2}$. Thus the action of G_{μ_1} on $\pi^{-1}(\mu_1)$ and that of G_{μ_2} on $\pi^{-1}(\mu_2)$ are conjugate since

$$\Phi_h(\Phi_g(x)) = \Phi_{hgh^{-1}}(\Phi_h(x)) \ .$$

4.4 **Proposition** *In the terminology of Steenrod (1951),* $\pi : P \to \mathcal{O}$ *admits the structure of a coordinate bundle. The group of the bundle is a subgroup of* T.

PROOF: We intend to use $F \equiv \pi^{-1}(\mu_0)$ as the model space for the fibres of $\pi : P \to \mathcal{O}$. To verify that F (and all the pre-images $\pi^{-1}(\mu)$) are submanifolds of P, we prove that π is a surjective submersion first so that the pre-image theorem applies. Choose $\mu \in \mathcal{O}$. Since we assume P is non-empty, there is $g \in G$, $w \in W$ and $x \in P$ such that $\mathbf{J}(x) = g \cdot w$. By the equivariance of $\mathbf{J}$, the entire orbit $G \cdot w$ is in the image of J. Since $\pi_{\mathcal{O}}$ maps orbits onto $\mathcal{O}$, there is $\nu \in G \cdot w$ with $\pi_{\mathcal{O}}(\nu) = \mu$. Then as $G \cdot w \subset \mathrm{Im}\,\mathbf{J}$, there exists $x' \in P$ such that $\mathbf{J}(x') = \nu$. This implies $(\pi_{\mathcal{O}} \circ \mathbf{J})(x') = \mu$. But μ was arbitrary, so $\pi \equiv \pi_{\mathcal{O}} \circ \mathbf{J}$ is surjective. At any $x \in P$, $\mathrm{T}_{\mathbf{J}(x)}(G \cdot \mathbf{J}(x)) \subset \mathrm{Im}\ \mathrm{T}_x\mathbf{J}$ (by the equivariance of $\mathbf{J}$). Since $\pi_{\mathcal{O}}$ maps $G \cdot \mathbf{J}(x)$ diffeomorphically onto $\mathcal{O}$, $\mathrm{T}_x\pi = \mathrm{T}_{\mathbf{J}(x)}\pi_{\mathcal{O}} \circ \mathrm{T}_x\mathbf{J}$ must be surjective, and so π is a submersion.

Let U be an open G_{μ_0}-invariant neighborhood of μ_0 in $\mathcal{O}$ with the property that

$$g \cdot \mu_0 \in U \Rightarrow g^{-1} \cdot \mu_0 \in U \ . \tag{12}$$

Continuity of the group action guarantees that arbitrarily small such neighborhoods exist. Put $U' \equiv \phi_{\mu_0}^{-1}(U)$. If we choose U and accordingly $U' \subset G/G_{\mu_0}$ sufficiently small, then there exists an injective immersion $s : U' \to G$ satisfying

$$s(gG_{\mu_0})G_{\mu_0} = gG_{\mu_0} \ . \tag{13}$$

In other words s is a local section for the fibre bundle $G \to G/G_{\mu_0}$. The claim is that $\psi : U \times F \to \pi^{-1}(U)$ defined by

$$\psi(\mu, x) \equiv s(\phi_{\mu_0}^{-1}(\mu)) \cdot x$$

is a local bundle chart (recall that $\phi_{\mu_0} : G/G_{\mu_0} \to G \cdot \mu_0$ is the G-equivariant diffeomorphism defined by $\phi_{\mu_0}(gG_{\mu_0}) \equiv g \cdot \mu_0$). Indeed we observe that $\psi(g \cdot \mu_0, x) = s(gG_{\mu_0}) \cdot x$, so

$$\pi(\psi(g \cdot \mu_0, x)) = \pi(s(gG_{\mu_0}) \cdot x) = s(gG_{\mu_0}) \cdot \pi(x) = s(gG_{\mu_0}) \cdot \mu_0 = g \cdot \mu_0 \ ,$$

by (13). Therefore $\pi(\psi(\mu, x)) = \mu$ for all $\mu \in U$. Furthermore ψ is a diffeomorphism for it has smooth inverse

$$\begin{aligned} \psi^{-1}(y) &= (\pi(y), g^{-1} \cdot y) \\ \text{where } g &\equiv s \circ \phi_{\mu_0}^{-1} \circ \pi(y) \ , \end{aligned}$$

which is well-defined on account of (12).

So far we only have a chart for an open neighborhood of $\pi^{-1}(\mu_0)$. For each $\mu \in \mathcal{O}$ we now define $U_\mu \equiv g(U)$ where $g \in \phi_{\mu_0}^{-1}(\mu)$; U_μ is well-defined since U is G_{μ_0}-invariant. The family $\{U_\mu \mid \mu \in \mathcal{O}\}$ is an open cover for $\mathcal{O}$. Since $\mathcal{O}$ is compact there is a finite subcover $\{U_{\mu_1}, \dots, U_{\mu_k}\}$. The family $\{\pi^{-1}(U_{\mu_1}), \dots, \pi^{-1}(U_{\mu_k})\}$ is an open cover for P. For each k we arbitrarily choose some $g_k \in G$ such that $\mu_k = g_k \cdot \mu_0$. The group G acts naturally on $\mathcal{O} \times P$ according to $g \cdot (\mu, x) \equiv (g \cdot \mu, g \cdot x)$. The map $\psi_k : U_{\mu_k} \times F \to \pi^{-1}(U_{\mu_k})$ defined by

$$\psi_k(\mu, x) \equiv g_k \cdot \psi(g_k^{-1} \cdot (\mu, x))$$

is a local fibre bundle chart. Indeed we have

$$\pi(\psi_k(g \cdot \mu_0, x)) = g_k \cdot \pi(\psi(g_k^{-1} g \cdot \mu_0, g_k^{-1} g \cdot \mu_0)) = g_k \cdot (g_k^{-1} g \cdot \mu_0) = g \cdot \mu_0 \ .$$

Therefore $\pi(\psi_k(\mu, x)) = \mu$ for all $\mu \in U_{\mu_k}$. Furthermore ψ_k is a diffeomorphism for it has smooth inverse

$$\psi_k^{-1}(y) = g_k \cdot \psi^{-1}(g_k^{-1} \cdot y) \ .$$

The transition functions $\psi_j^{-1} \circ \psi_k : U_{\mu_k} \times F \to U_{\mu_j} \times F$ have the form of local vector bundle isomorphisms, as required; they are given by

$$\psi_j^{-1} \circ \psi_k(\mu, x) = (g_j^{-1} g_k \cdot \mu, x) \ .$$

This completes the proof that $\pi : P \to \mathcal{O}$ is a fibre bundle. Since $g_j^{-1} g_k \in G_{\mu_0}$, the group of the bundle is a subgroup of $T = G_{\mu_0}$. (This group need not be T itself since G_{μ_0} need not act faithfully on F.)
□

The fibres of π are symplectic submanifolds of P. The proof is provided by:

4.5 **Theorem** (Guillemin and Sternberg (1983)) *Let Z be a submanifold of $\mathfrak{g}^*$ such that $\mathbf{J} \pitchfork Z$ and*

$$\mathrm{T}_z Z \oplus \mathrm{T}_z(G \cdot z) = \mathfrak{g}^*$$

at all $z \in Z$. Then $\mathbf{J}^{-1}(Z)$ is a symplectic submanifold of P.

4.6 **Corollary** *The fibres of $\pi : P \to \mathcal{O}$ are symplectic. In particular, $\pi : P \to \mathcal{O}$ has a natural Ehresmann connection $\alpha : \mathrm{T}P \to \mathrm{Ker}\ \mathrm{T}\pi$ defined fibre-wise by projecting $\mathrm{T}_x P$ onto* $\mathrm{Ker}\ \mathrm{T}_x\pi$ *along its ω-orthogonal complement* $(\mathrm{Ker}\ \mathrm{T}_x\pi)^\omega$.

Proof: By definition every fibre of $\pi : P \to \mathcal{O}$ is of the form $\mathbf{J}^{-1}(Z)$ for some Weyl chamber Z in $\mathfrak{g}^*$ (see 3.15*c*). $\mathbf{J} \pitchfork Z$ follows from momentum map equivariance. This chamber intersects all co-adjoint orbits transversally, so the preceding theorem applies.
□

4.7 **Definition** Let $(P_1, \omega_1, G_1, \mathbf{J}^1, H_1)$ and $(P_2, \omega_2, G_2, \mathbf{J}^2, H_2)$ be two Hamiltonian G-systems. We say that the *G-spaces* $(P_1, \omega_1, G_1, \mathbf{J}^1)$ and $(P_2, \omega_2, G_2, \mathbf{J}^2)$ are *equivalent* if there exists a Lie group isomorphism $\varphi : G_1 \to G_2$ and a symplectic diffeomorphism $\phi : P_1 \to P_2$ such that

a. $\varphi(g) \cdot \phi(x) = \phi(g \cdot x)$, and

b. $J^2_{\mathrm{T}\phi\cdot\xi} \circ \phi = J^1_\xi$

for all $g \in G$, $x \in P_1$ and $\xi \in \mathfrak{g}_1$. If furthermore $H_2 \circ \phi = H_1$, then we say that the corresponding *G-systems* are equivalent.

In this language we can summarize the above results by:

4.8 **Theorem** *Let G be a compact connected Lie group and let $(P, \omega, G, \mathbf{J}, H)$ be a Hamiltonian G-system for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. Apply 3.15 to obtain the map $\pi_{\mathcal{O}} : \mathfrak{g}^*_{\mathrm{reg}} \to \mathcal{O}$. Then the map $\pi \equiv \pi_{\mathcal{O}} \circ \mathbf{J}$ is a well-defined G-equivariant surjective submersion. In fact $\pi : P \to \mathcal{O}$ admits the structure of a fibre bundle (in the sense of 4.4). Every fibre $\pi^{-1}(\mu)$ is invariant with respect to the flow of X_H, and is a symplectic submanifold of P. The subgroup of G mapping points of a fibre $\pi^{-1}(\mu)$ to points of the same fibre is the maximal torus G_μ (see (11)). For each $\mu \in \mathcal{O}$ write $\omega_\mu \equiv \omega|\pi^{-1}(\mu)$ and $H_\mu \equiv H|\pi^{-1}(\mu)$. A G_μ-equivariant momentum map $\mathbf{J}^\mu : \pi^{-1}(\mu) \to \mathfrak{g}^*_\mu$ for the action of G_μ on $\pi^{-1}(\mu)$ is defined by $\mathbf{J}^\mu(x) \equiv \mathbf{J}(x)|\mathfrak{g}_\mu$. The systems $\mathcal{S}_\mu \equiv (\pi^{-1}(\mu), \omega_\mu, G_\mu, \mathbf{J}^\mu, H_\mu)$, $\mu \in \mathcal{O}$, are then mutually equivalent Hamiltonian G_μ-systems.*

4.9 **Remark** Stated simply, the theorem states that every Hamiltonian G-system with regular momenta is just a fibering by mutually equivalent, dynamically invariant subsystems, in which the group actions are by maximal tori of G. By *dynamically invariant* we mean invariant with respect to the flow of the G-invariant Hamiltonian. We can always construct a representative fibre in the fibration as follows: Let W be a Weyl chamber in $\mathfrak{g}^*$ and let $T \subset G$ be the corresponding maximal torus (i.e. $T \equiv G_w$ where w is any point in W). Put $F \equiv \mathbf{J}^{-1}(W)$ and let $i_F : F \to P$ denote the inclusion. The submanifold F is symplectic and is T-invariant. A equivariant momentum map $\mathbf{J}^F : F \to \mathfrak{t}^*$ for the action of T on F is given by $\mathbf{J}^F(x) \equiv \mathbf{J}(x)|\mathfrak{t}$. We call $(F, i_F^*\omega, T, \mathbf{J}^F)$ the *Hamiltonian T-space associated with* $(P, \omega, G, \mathbf{J})$ and $(F, i_F^*\omega, T, \mathbf{J}^F, i_F^* H)$ the *Hamiltonian T-system associated with* $(P, \omega, G, \mathbf{J}, H)$.

We emphasize that although the regular orbit $\mathcal{O}$ is completely arbitrary, we obtain precisely the same fibres for π if we make a different choice. Recalling the example of the spherical pendulum discussed in the Introduction, this amounts to the observation that while we used the unit momentum sphere to parameterize the family of planar pendulum subsystems, we could just have well chosen any other sphere in momentum space (centered at the origin) to do so.

The horizontal spaces

Since the fibres of $\pi : P \to \mathcal{O}$ are symplectic, there is a natural Ehresmann connection (see e.g. Kobayashi and Nomizu (1963)) $\alpha : \mathrm{T}P \to \ker \mathrm{T}\pi$ defined fibre-wise by projecting $\mathrm{T}_x P$ onto $\ker \mathrm{T}_x\pi$ along $\mathrm{Hor}_x \equiv (\ker \mathrm{T}_x\pi)^\omega$ ($(\cdot)^\omega$ denotes ω-orthogonal complement). In this section we show that each horizontal space Hor_x can be naturally identified with a subspace of $\mathfrak{g}$, or alternatively with a subspace of $\mathfrak{g}^*$, and give an associated formula for the restriction of ω to Hor_x. These results are useful in deducing theorems about a G-space with regular momenta from theorems about its associated T-space.

To begin, let us recall the elementary fact that connects the momentum map, the group orbits in P and the symplectic structure of P, at a point $x \in P$:

4.10 **Lemma** Ker $\mathrm{T}_x\mathbf{J} = (\mathrm{T}_x(G\cdot x))^\omega$

($(\cdot)^\omega$ denotes ω-orthogonal complement.)
Proof:

$$\begin{aligned} v \in \mathrm{Ker}\ \mathrm{T}_x\mathbf{J} \ &\Leftrightarrow\ \mathrm{T}_x\mathbf{J}\cdot v = 0 \\ &\Leftrightarrow\ \langle \mathrm{T}_x\mathbf{J}\cdot v, \xi\rangle = 0\ \ \forall \xi \in \mathrm{T}_\mu\mathfrak{g} \cong \mathfrak{g} \\ &\Leftrightarrow\ \langle d_x J_\xi, v\rangle = 0\ \ \forall \xi \in \mathfrak{g} \\ &\Leftrightarrow\ \omega(\xi_P(x), v) = 0\ \ \forall \xi \in \mathfrak{g} \\ &\Leftrightarrow\ \omega(u, v) = 0\ \ \forall u \in \mathrm{T}_x(G\cdot x) \\ &\Leftrightarrow\ v \in (\mathrm{T}_x(G\cdot x))^\omega\ \ . \end{aligned}$$

□

4.11 **Corollary** $\mathrm{Hor}_x \subset \mathrm{T}_x(G\cdot x)$

Proof: Since $\pi \equiv \pi_{\mathcal{O}} \circ \mathbf{J}$, Ker $\mathrm{T}_x\pi \supset$ Ker $\mathrm{T}_x\mathbf{J}$. Applying 4.10 gives Ker $\mathrm{T}_x\pi \supset (\mathrm{T}_x(G\cdot x))^\omega$. Taking ω-orthogonal complements: (Ker $\mathrm{T}_x\pi)^\omega \subset \mathrm{T}_x(G\cdot x)$.
□

4.12 **Proposition** *Let $x \in P$ and write $\mathfrak{t} \equiv \mathfrak{g}_{\mathbf{J}(x)}$, so that $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{g}$ (by 3.2 and (10)). Then in the notation of Sect. 3:*

a. $\xi \mapsto \xi_P(x)$ *maps* $\mathfrak{t}^\perp \subset \mathfrak{g}$ *isomorphically onto* Hor_x.

b. $\mathrm{T}_x\mathbf{J}$ *maps* Hor_x *isomorphically onto* $\mathrm{T}_{\mathbf{J}(x)}(G\cdot\mathbf{J}(x)) \cong \underline{\mathfrak{t}}^\perp \subset \mathfrak{g}^*$.

c. *For any* $\xi, \eta \in \mathfrak{g}$, $\omega(\xi_P(x), \eta_P(x)) = \langle \mathbf{J}(x), [\xi, \eta]\rangle$

4.13 **Remark** If we equip the co-adjoint orbit $G\cdot\mathbf{J}(x)$ with its natural symplectic structure (Kirillov, 1976), then 4.12*c* implies that the isomorphism $\mathrm{T}_x\mathbf{J} : \mathrm{Hor}_x \to \mathrm{T}_{\mathbf{J}(x)}(G\cdot\mathbf{J}(x))$ is symplectic. We also remark that one might think of 4.12*a* as local freeness of the action of G on P 'transverse' to fibres of π.

PROOF: First note that the image of Hor_x under $\mathrm{T}_x\mathbf{J}$ is contained in $\mathrm{T}_{\mathbf{J}(x)}(G \cdot \mathbf{J}(x))$, because $\mathrm{Hor}_x \subset \mathrm{T}_x(G \cdot x)$ and $\mathbf{J}$ is equivariant. Since $\dim \mathrm{Hor}_x = \dim \mathcal{O} = \dim \mathfrak{t}^\perp = \dim \underline{\mathfrak{t}}^\perp$, it suffices in proving 4.12*a* to show that

$$\mathfrak{g}_x \cap \mathfrak{t}^\perp = \{0\} \ , \tag{14}$$

and in the case of 4.12*b* to show that

$$\ker \mathrm{T}_x\mathbf{J} \cap \mathrm{Hor}_x = \{0\} \ . \tag{15}$$

By the equivariance of $\mathbf{J}$, $\mathfrak{g}_x \subset \mathfrak{g}_{\mathbf{J}(x)} = \mathfrak{t}$. Equation (14) follows since $\mathfrak{t} \cap \mathfrak{t}^\perp = \{0\}$ by 3.4. Equation (15) is true because $\ker \mathrm{T}_x\mathbf{J} \subset \ker \mathrm{T}_x\pi$ and $\ker \mathrm{T}_x\pi \cap \mathrm{Hor}_x = \{0\}$. To prove 4.12*c*, recall that as $\mathbf{J}$ is equivariant, $\{J_\eta, J_\xi\} = J_{[\xi,\eta]}$ (see e.g. Abraham and Marsden (1978, Theorem 4.2.8) but note that we are adopting a different sign convention: $\{f, h\} \equiv X_f \lrcorner X_h \lrcorner \omega$). We have

$$\begin{aligned}
\omega(\xi_P(x), \eta_P(x)) &= \omega(X_{J_\xi}(x), X_{J_\eta}(x)) \\
&= \{J_\eta, J_\xi\}(x) \\
&= J_{[\xi,\eta]}(x) \\
&= \langle \mathbf{J}(x), [\xi, \eta] \rangle \ .
\end{aligned}$$

□

5 The canonical T-action

The action

Recall (Theorem 4.8) that on every fibre $\pi^{-1}(\mu)$ we have the maximal torus G_μ acting, and that up to equivalence these G_μ-spaces ($\mu \in \mathcal{O}$) are all the same. Using the Abelianness of the G_μ, $\mu \in \mathcal{O}$, we will be able to 'glue' these actions together into a single Abelian action of $T = G_{\mu_0}$ acting *on the whole of P*. With respect to this new action, the fibres of π will be invariant.

For any $\mu \in \mathcal{O}$ define the group isomorphism $\kappa_\mu : T \to G_\mu$ by

$$\kappa_\mu(t) \equiv gtg^{-1} \ ,$$

where g is any element of G such that $\mu = g \cdot \mu_0$. For any $t' \in T$, $(gt')t(gt')^{-1} = g(t'tt'^{-1})g^{-1} = gtg^{-1}$, since T is Abelian. This verifies that κ_μ does not depend on the choice of g. The new action $(t, x) \mapsto \Psi_t(x) : T \times P \to P$ is defined by

$$\begin{aligned}
& \Psi_t(x) \equiv \Phi_h(x) \ , \\
\text{where} \quad & h \equiv \kappa_\mu(t) \\
\text{and} \quad & \mu \equiv \pi(x) \ .
\end{aligned}$$

We will also write

$$t \cdot_T x \equiv \Psi_t(x) \ , \quad t \in T \ .$$

In this notation we have the identity

$$t \cdot_T (g \cdot x_0) = (gt) \cdot x_0 \ , \tag{16}$$

for any $x_0 \in \pi^{-1}(\mu_0)$. This identity also serves to define the T-action. It should be clear that up to conjugacy this action is independent of the particular choice of $\mu_0 \in \mathcal{O}$. We denote the new infinitessimal generators by

$$\xi_P^T(x) \equiv \frac{d}{d\tau} \exp(\tau\xi) \cdot_T x \Big|_{\tau=0} \ .$$

One verifies that the infinitessimal generators of the T and G actions are related for any $\xi \in \mathfrak{t} \subset \mathfrak{g}$ by

$$\xi_P^T(x) = (\mathrm{Ad}_g \xi)_P(x) \ , \tag{17}$$

where $g \in G$ is chosen to satisfy $g^{-1} \cdot x \in \pi^{-1}(\mu_0)$.

The momentum map

Recall (see 3.15) that we have a map $\pi_W : \mathfrak{g}^*_{\mathrm{reg}} \to W$, where W is the Weyl chamber through μ_0. Assuming $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, $\pi_W \circ \mathbf{J} : P \to W \subset \mathfrak{g}^*$ is well-defined. Recall also (see 3.15a) that the natural projection $i : \mathfrak{g}^* \to \mathfrak{t}^*$ defined by $i(\mu) \equiv \mu | \mathfrak{t}$ maps W injectively onto an open set $\mathfrak{t}^*_0$.

5.1 **Proposition** *The T-action defined above is Hamiltonian. Indeed, under the assumptions of Theorem 4.8, a T-equivariant momentum map $\mathbf{j} : P \to \mathfrak{t}^*$ for this action is given by $\mathbf{j} \equiv i \circ \pi_W \circ \mathbf{J}$.*

Proof: First note that $\mathbf{j}$ is G-invariant because $\pi_W : \mathfrak{g}^*_{\mathrm{reg}} \to W$ maps co-adjoint orbits to points. Indeed $\mathbf{j}(g \cdot x) = \pi_W(\mathbf{J}(g \cdot x))|\mathfrak{t} = \pi_W(g \cdot (\mathbf{J}(x))|\mathfrak{t} = \pi_W(\mathbf{J}(x))|\mathfrak{t} = \mathbf{j}(x)$. The T-equivariance of $\mathbf{j}$ then follows because the co-adjoint action of T on $\mathfrak{t}$ is trivial (T is Abelian), and because $\mathbf{j}$ maps T-orbits to points. Let $x \in P$ be given. Define $\mu \equiv \pi(x)$ and let $g \in G$ be chosen such that $\mu = g \cdot \mu_0$, so that for all $x' \in \pi^{-1}(\mu)$, $g^{-1} \cdot \mathbf{J}(x') \in W$. For any $\xi \in \mathfrak{t}$ we have

$$\begin{aligned} \langle \mathbf{j}(x'), \xi \rangle &= \langle \pi_W(\mathbf{J}(x')), \xi \rangle \\ &= \langle \pi_W(g^{-1} \cdot \mathbf{J}(x')), \xi \rangle \\ &= \langle g^{-1} \cdot \mathbf{J}(x'), \xi \rangle \\ &= \langle \mathbf{J}(x'), \mathrm{Ad}_g \xi \rangle \ . \end{aligned}$$

This implies

$$j_\xi(x') = J_{\mathrm{Ad}_g \xi}(x') \ \forall x' \in \pi^{-1}(\mu) \ ,$$

giving

$$d_x j_\xi | \mathrm{Ker}\ \mathrm{T}_x \pi = d_x J_{\mathrm{Ad}_g \xi} | \mathrm{Ker}\ \mathrm{T}_x \pi \ . \tag{18}$$

According to 4.11, vectors in Hor_x are tangent to an orbit. Since $\mathbf{j}$ is G-invariant, so is j_ξ. We may conclude that

$$d_x j_\xi | \mathrm{Hor}_x = 0 \ . \tag{19}$$

We claim that a similar result holds for $J_{\mathrm{Ad}_g \xi}$ at x. Suppose then that $v \in \mathrm{Hor}_x$. Then $\langle d_x J_{\mathrm{Ad}_g \xi}, v \rangle = \langle \mathrm{T}_x \mathbf{J} \cdot v, \mathrm{Ad}_g \xi \rangle$. Now v is tangent to a G-orbit. It follows by the equivariance of $\mathbf{J}$ that $\mathrm{T}_x\mathbf{J}\cdot v$ is tangent to the co-adjoint orbit through $\mathbf{J}(x)$. Therefore $\mathrm{T}_x\mathbf{J}\cdot v = \mathrm{ad}^*_\eta \mathbf{J}(x)$ for some $\eta \in \mathfrak{g}^*$. We then have

$$\begin{aligned}\langle d_x J_{\mathrm{Ad}_g\xi}, v\rangle &= \langle \mathbf{J}(x), [\eta, \mathrm{Ad}_g\xi]\rangle \\ &= \langle \mathrm{Ad}^*_g\mathbf{J}(x), [\mathrm{Ad}_{g^{-1}}\eta, \xi]\rangle \\ &= -\langle g^{-1}\cdot \mathbf{J}(x), \mathrm{ad}_\xi \mathrm{Ad}_{g^{-1}}\eta\rangle \\ &= -\langle \mathrm{ad}^*_\xi(g^{-1}\cdot\mathbf{J}(x)), \mathrm{Ad}_{g^{-1}}\eta\rangle \ .\end{aligned}$$

We argue that the right hand side is zero as follows: Since $\mathfrak{t}$ is the maximal Abelian subalgebra associated with the Weyl chamber W and $g^{-1}\cdot\mathbf{J}(x) \in W$, it follows that $\mathfrak{g}_{g^{-1}\cdot\mathbf{J}(x)} = \mathfrak{t}$. In particular $\xi \in \mathfrak{g}_{g^{-1}\cdot\mathbf{J}(x)}$. Therefore $\mathrm{ad}^*_\xi(g^{-1}\cdot\mathbf{J}(x)) = 0$ making the right hand side of the above zero as claimed. Since $v \in \mathrm{Hor}_x$ was arbitrary, we conclude that

$$d_x J_{\mathrm{Ad}_g\xi}|\mathrm{Hor}_x = 0 \ . \tag{20}$$

Since Ker $\mathrm{T}_x\pi$ and Hor_x are complementary spaces we conclude from (18), (19) and (20) that

$$d_x j_\xi = d_x J_{\mathrm{Ad}_g\xi} \ .$$

Therefore $X_{j_\xi}(x) = X_{J_{\mathrm{Ad}_g\xi}}(x) \equiv (\mathrm{Ad}_g\xi)_P(x)$. Appealing to (17):

$$X_{j_\xi}(x) = \xi^T_P(x) \ ,$$

which proves that $\mathbf{j}$ delivers the infinitessimal generators for the T-action.
□

6 On the classification of G-spaces with regular momenta

Reduction to the Abelian case

Recall that by 4.8 (see also 4.9) we may associate with any Hamiltonian G-space $(P, \omega, G, \mathbf{J})$ for which $\mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$, a lower dimensional Hamiltonian T-space $(F, \omega_F, T, \mathbf{J}_F)$, where T is the maximal torus of G (assuming G is non-Abelian). This T-space is uniquely defined up to equivalence. We will prove the following.

6.1 **Theorem** *Let G be a compact connected Lie group. Then two Hamiltonian G-spaces with regular momenta are equivalent $\Leftrightarrow$ their associated T-spaces are equivalent (see 4.7 for the definition of the equivalence of Hamiltonian G-spaces).*

The $\Rightarrow$ part is obvious. To prove the $\Leftarrow$ part we need 4.12a. As we have remarked, this amounts to local freeness of the action 'transverse' to the fibres of the symplectic fibration (the lower dimensional T-spaces).
Proof of $\Leftarrow$: Let $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$ be Hamiltonian G-spaces with $\mathbf{J}^i(P_i) \subset \mathfrak{g}^*_{\mathrm{reg}}$. Applying 4.8, we obtain the symplectic fibrations $\pi_1 : P_1 \to \mathcal{O}$ and $\pi_2 : P_2 \to \mathcal{O}$, where $\mathcal{O} \equiv G\cdot\mu_0$, for some (arbitrary) point $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$. Fix representative fibres $F_1 \equiv \pi_1(\mu_0)$ and $F_2 \equiv \pi_2(\mu_0)$ and denote the corresponding inclusions by $i_1 : F_1 \to P_1$ and $i_2 : F_2 \to P_2$. Then the maximal torus $T \equiv G_{\mu_0}$ acts on $(F_1, i_1^*\omega_1)$ and $(F_2, i_2^*\omega_2)$, with corresponding momentum maps $\mathbf{J}^{F_i} : F_i \to \mathfrak{t}^*_0$ defined by $\mathbf{J}^{F_i}(x) \equiv \mathbf{J}^i(x)|\mathfrak{t}$. Our hypothesis is that the Hamiltonian

T-spaces $(F_1, i_1^*\omega_1, T, \mathbf{J}^{F_1})$ and $(F_2, i_2^*\omega_2, T, \mathbf{J}^{F_2})$ are equivalent. Thus we have a T-equivariant symplectic diffeomorphism $\varphi : F_1 \to F_2$ such that $\mathbf{J}^{F_2} \circ \varphi = \mathbf{J}^{F_1}$. We now describe how to extend φ to a G-equivariant symplectic diffeomorphism $\phi : P_1 \to P_2$.

Let $x \in P_1$. Pick $g \in G$ to satisfy $\pi_1(x) = g \cdot \mu_0 \in \mathcal{O}$ so that, by the G-equivariance of π_1, $g^{-1} \cdot x \in F_1$. Then we define

$$\phi(x) \equiv g \cdot \varphi(g^{-1} \cdot x) \quad (g^{-1} \cdot x \in F_1) \ .$$

To show that ϕ is well-defined (i.e. independent of the choice of g) we must show that we can replace g by gt for any $t \in T = G_{\mu_0}$ in the definition with no effect. This follows easily from the T-equivariance of φ.

To show that ϕ is G-equivariant, let $x \in P_1$ with $\pi_1(x) = g \cdot \mu_0$ as before, and suppose $h \in G$. Then $\pi_1(h \cdot x) = hg \cdot \mu_0$, so that

$$\begin{aligned} \phi(h \cdot x) &= hg \cdot \varphi(g^{-1}h^{-1} \cdot (h \cdot x)) \\ &= h \cdot [g \cdot \varphi(g^{-1} \cdot x)] \\ &= h \cdot \varphi(x). \end{aligned}$$

This proves the G-equivariance.

By construction we have $\pi_2 \circ \phi = \pi_1$. It is not difficult to show using the G-equivariance of ϕ, π_1 and π_2 that $\mathbf{J}^{F_2} \circ \varphi = \mathbf{J}^{F_1}$ implies $\mathbf{J}^2 \circ \phi = \mathbf{J}^1$.

An inverse for ϕ is given easily enough. Let $x \in P_2$ and pick $g \in G$ to satisfy $\pi_2(x) = g \cdot \mu_0$. Then define $\psi : P_2 \to P_1$ by $\psi(x) \equiv g \cdot \varphi^{-1}(g^{-1} \cdot x)$. One shows that ψ is well-defined in the same way as above and verifies immediately that $\phi \circ \psi = \mathrm{id}_{P_2}$ and $\psi \circ \phi = \mathrm{id}_{P_1}$.

For any $g \in G$, the map $x \mapsto g \cdot x$ sending a fibre $\pi_i^{-1}(\mu)$ to the fibre $\pi_i^{-1}(g \cdot \mu)$ ($i = 1$ or 2) is a symplectic diffeomorphism. Since $\varphi : \pi_1^{-1}(\mu_0) \to \pi_2^{-1}(\mu_0)$ is a symplectic diffeomorphism, it follows that the restriction of ϕ to any fibre of $\pi_1 : P_1 \to \mathcal{O}$ is a symplectic diffeomorphism onto a fibre of $\pi_2 : P_2 \to \mathcal{O}$. The tangent space $\mathrm{T}_x P_1$ at each $x \in P_1$ splits into complementary subspaces $\ker \mathrm{T}_x \pi_1$ and $\mathrm{Hor}_x^1 \equiv (\ker \mathrm{T}_x \pi_1)^\omega$. The preceding argument implies that $\mathrm{T}\phi : \mathrm{T}P_1 \to \mathrm{T}P_2$ maps $\ker \mathrm{T}_x \pi_1$ symplectically and isomorphically onto $\ker \mathrm{T}_{\phi(x)} \pi_2$. By 4.12a and the G-equivariance of ϕ, it follows that $\mathrm{T}\phi$ maps Hor_x^1 isomorphically onto $\mathrm{Hor}_{\phi(x)}^2 \equiv (\ker \mathrm{T}_{\phi(x)} \pi_2)^\omega$. To see this note that $\mathfrak{g}_{\mathbf{J}^1(x)} = \mathfrak{g}_{\mathbf{J}^2(\phi(x))}$ since $\mathbf{J}^2 \circ \phi = \mathbf{J}^1$. By 4.12c and the G-equivariance of ϕ we have

$$\begin{aligned} \omega_2(\mathrm{T}\phi \cdot \xi_{P_1}(x), \mathrm{T}\phi \cdot \eta_{P_1}(x)) &= \omega_2(\xi_{P_2}(\phi(x)), \eta_{P_2}(\phi(x))) \\ &= \langle \mathbf{J}_2(\phi(x)), [\xi, \eta] \rangle \\ &= \langle \mathbf{J}_1(x), [\xi, \eta] \rangle \\ &= \omega_1(\xi_{P_1}(x), \eta_{P_1}(x)). \end{aligned}$$

The isomorphism $\mathrm{T}\phi : \mathrm{Hor}_x^1 \to \mathrm{Hor}_{\phi(x)}^2$ is therefore symplectic. Since $\ker \mathrm{T}_{\phi(x)} \pi_i$ and $\mathrm{Hor}_{\phi(x)}^i$ are complementary subspaces of $\mathrm{T}_{\phi(x)} P_i$ ($i = 1, 2$), it follows that $\mathrm{T}\phi : \mathrm{T}_x P_1 \to \mathrm{T}_{\phi(x)} P_2$ is a symplectic isomorphism and that $\phi : P_1 \to P_2$ is therefore a symplectic diffeomorphism.
□

A generalization of Delzant's Theorem

To illustrate the application of 6.1, we will deduce a generalization of Delzant's theorem on the classification of integrable T-spaces. Here is Delzant's result:

6.2 **Theorem** (Delzant (1988)) *Let T be an n-torus and let $(P_1, \omega_1, T, \mathbf{J}^1)$ and $(P_2, \omega_2, T, \mathbf{J}^2)$ be two Hamiltonian T-spaces. Then if:*

a. *P_1 and P_2 are* compact

b. *T acts faithfully on P_1 and P_2*

c. $\dim P_1 = \dim P_2 = 2 \dim T$

d. $\mathbf{J}(P_1) = \mathbf{J}(P_2)$

then the two spaces are equivalent.

6.3 **Remark** Recall that a group action $(g, x) \mapsto \Phi(x) : G \times X \to X$ is *faithful* if $\Phi_g = \Phi_h$ implies $g = h$. Conditions 6.2b and 6.2c are often referred to as *complete integrability.* Condition 6.2c is the special Abelian case of the more general requirement $\dim P_i = \dim G + \text{rank } G$.

Here is a non-Abelian generalization of Delzant's theorem:

6.4 **Corollary** *Let G be a compact connected Lie group and let $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$ be two Hamiltonian G-spaces with regular momenta. Let W be any Weyl chamber in $\mathfrak{g}^*$ and let $T \subset G$ be the associated maximal torus (i.e. $T \equiv G_{\mu_0}$ where $\mu_0 \in W$ is arbitrary). Then if:*

a. *P_1 and P_2 are compact*

b. *T acts faithfully on $(\mathbf{J}^1)^{-1}(W)$ and $(\mathbf{J}^2)^{-1}(W)$*

c. $\dim P_1 = \dim P_2 = \dim G + \text{rank } G$

d. $\mathbf{J}^1(P_1) = \mathbf{J}^2(P_2)$

then the G-spaces are equivalent.

6.5 **Remark** Condition 6.4b is just the requirement that in the lower dimensional T-spaces associated with $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$, the action of T be faithful.

Proof: With no loss of generality, take W to be the Weyl chamber in $\mathfrak{g}^*$ containing μ_0, and write $F_1 \equiv (\mathbf{J}^1)^{-1}(W) = \pi_1^{-1}(\mu_0)$ and $F_2 \equiv (\mathbf{J}^2)^{-1}(W) = \pi_2^{-1}(\mu_0)$, where $\pi_1 : P_1 \to \mathcal{O}$ and $\pi_2 : P_2 \to \mathcal{O}$ are the symplectic fibrations onto the co-adjoint orbit $\mathcal{O} \equiv G \cdot \mu_0$. The maximal torus $T \equiv G_{\mu_0}$ acts on F_1 and F_2, which are Hamiltonian T-spaces. One can argue that F_1 and F_2 are compact using 6.4a and 4.4. The momentum maps $\mathbf{J}^{F_1} : F_1 \to \mathfrak{t}_0^*$ and $\mathbf{J}^{F_2} : F_2 \to \mathfrak{t}_0^*$ are given by $\mathbf{J}^{F_i}(x) \equiv \mathbf{J}(x)|\mathfrak{g}$. By d, $\mathbf{J}^{F_1}(F_1) = \mathbf{J}^{F_2}(F_2)$. We have $\dim F_i = \dim P_i - \dim \mathcal{O} = \dim P_i - \dim G + \text{rank } G$. Applying 6.4$c$, we obtain $\dim F_i = 2\text{rank } G = 2 \dim T$. By 6.4$b$ the action of T on F_1 and F_2 is faithful. Whence the conditions of Delzant's theorem are met and the T-spaces associated with the G-spaces $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$ are equivalent. By 6.1 the G-spaces themselves are equivalent.
□

7 Symplectic reduction and integrability

Symplectic reduction in G-spaces with regular momenta

A group action $(x, g) \mapsto \Phi_g(x) : X \times G \to G$ is *free* if $G_x = \{g \in G \mid \Phi_g(x) = x\}$ is trivial for all $X \in x$. In the case that G acts freely, the reduction in dimension of a Hamiltonian system with symmetry is provided by *Marsden-Weinstein reduction*:

7.1 **Theorem** (Marsden and Weinstein (1974), Meyer (1973))
Let $(P, \omega, G, \Phi, \mathbf{J})$ be a Hamiltonian G-space and assume G acts freely and properly. Then for any $\mu \in \mathbf{J}(P)$, $\mathbf{J}^{-1}(\mu)$ is a submanifold of P and $P_\mu \equiv \mathbf{J}^{-1}(\mu)/G_\mu$ admits the structure of a smooth manifold, with respect to which the natural projection $\gamma_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is a surjective submersion. There is a unique symplectic form ω_μ on P_μ with the property that $\gamma_\mu^* \omega_\mu = i_\mu^* \omega$, where $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ is the inclusion.

7.2 **Remark** The symplectic manifolds P_μ are called the *reduced spaces*. Of course the action is automatically proper when G is compact.

The following result connects the reduced spaces of a G-space with regular momenta with the reduced spaces of its associated Hamiltonian T-space.

7.3 **Lemma** *Let G be a compact and connected Lie group. Let $(P, \omega, G, \mathbf{J})$ be a G-space with regular momenta with G acting freely, and choose any $\mu \in \mathbf{J}(P) \subset \mathfrak{g}^*_{\mathrm{reg}}$. Let W denote the Weyl chamber through μ and construct the Hamiltonian T-space $(F, \omega_F, T, \mathbf{J}^F)$ associated with $(P, \omega, G, \mathbf{J})$ as described in 4.9. Let p denote the image of μ under the natural projection $\mathfrak{g}^* \to \mathfrak{t}^*$. Then the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ is precisely the reduced space $(\mathbf{J}^F)^{-1}(p)/T_p = (\mathbf{J}^F)^{-1}(p)/T$.*

In other words, symplectic reduction in a G-space with regular momenta gives the same reduced spaces as symplectic reduction in its associated Hamiltonian T-space.

Proof: Since $\mu \in W$, $G_\mu = T$. It is clear that $\mathbf{J}^{-1}(\mu) \subset F$ since $F = \mathbf{J}^{-1}(W)$ and $\mu \in W$. Recall (see Sect. 3) that W is a connected component of $\underline{\mathfrak{t}} \cap \mathfrak{g}^*_{\mathrm{reg}}$, so that for all $x \in F$, $\mathbf{J}(x) \in \underline{\mathfrak{t}} \equiv \mathrm{Ann}\, \mathfrak{t}^\perp$. In particular, $\mathbf{J}(x) - \mu \in \mathrm{Ann}\, \mathfrak{t}^\perp$ for all $x \in F$. Denoting by $i : \mathfrak{g}^* \to \mathfrak{t}^*$ the natural projection, and putting $p \equiv i(\mu)$, we therefore obtain

$$\begin{aligned} x \in \mathbf{J}^{-1}(\mu) \subset F &\iff \langle \mathbf{J}(x) - \mu, \xi \rangle = 0 \quad \forall \xi \in \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp \\ &\iff \langle \mathbf{J}(x) - \mu, \xi \rangle = 0 \quad \forall \xi \in \mathfrak{t} \\ &\iff \mathbf{J}(x)|\mathfrak{t} = \mu|\mathfrak{t} \\ &\iff \mathbf{J}^F(x) = i(\mu) \\ &\iff x \in (\mathbf{J}^F)^{-1}(p) \ . \end{aligned}$$

In other words, $\mathbf{J}^{-1}(\mu) = (\mathbf{J}^F)^{-1}(p)$, so that $\mathbf{J}^{-1}(\mu)/G_\mu = (\mathbf{J}^F)^{-1}(p)/T$.
□

Integrability

If H is a G-invariant Hamiltonian, then there exists for each μ in Theorem 7.1, a Hamiltonian $H_\mu : P_\mu \to \mathbb{R}$ such that the Hamiltonian vector fields $X_H|\mathbf{J}^{-1}(\mu)$ and X_{H_μ} are γ_μ-related.

It can then be shown that if the integral curves of the vector field on P_μ are known, then the integral curves in $\mathbf{J}^{-1}(\mu) \subset P$ can be reconstructed by simply solving *linear* ordinary differential equations with time dependent coefficients (see Marsden (1992) for details and further references). In the context of symmetry one natural definition of 'integrability' is then that the reduced spaces are zero-dimensional. In a G-space with regular momenta (G acting freely), all the reduced spaces have the dimension $\dim P - \dim G - \operatorname{rank} G$. Whence:

7.4 **Definition** Suppose that $(P, \omega, G, \mathbf{J})$ is a G-space with regular momenta and that G acts freely. We call this space *integrable* if $\dim P = \dim G + \operatorname{rank} G$. Furthermore if $H : P \to \mathbb{R}$ is G-invariant we refer to $(P, \omega, G, \mathbf{J}, H)$ as an *integrable* G-system.

By virtue of 7.3, integrability of a G-space with regular momenta is immediately equivalent to integrability of the associated Hamiltonian T-space, i.e. in each fibre of $\pi : P \to \mathcal{O}$ (which is also an invariant manifold for the flow generated by any G-invariant Hamiltonian). Although it is possible (locally) to transform the 'non-commutative' integrability of P into 'commutative' (Arnold-Liouville) type integrability of P (Mishchenko and Fomenko, 1978), this process is unnatural in the sense that it rather arbitrarily glues together the lower dimensional tori arising from the (commutative) integrability in each fibre of π (these tori are the orbits in P of the canonical T-action described in Sect. 5), to form Liouville tori (tori of half the dimension of P). Marle (1983) has shown us that a more natural procedure is to abandon the classical 'action-angle' normal form in favor of non-Abelian generalizations. This will be described in Sect. 8 in the context of G-spaces with regular momenta.

Dual pairs

Recall that in Sect. 5 we constructed a natural T-action on P and its associated momentum map $\mathbf{j} : P \to \mathfrak{t}_0^*$, for any G-space with regular momenta (G compact and connected). In the special case that G acts freely and $(P, \omega, G, \mathbf{J})$ is integrable this map has a special interpretation. In the language of Weinstein (1983), the maps $\mathfrak{g}^* \xleftarrow{\mathbf{J}} P \xrightarrow{\mathbf{j}} \mathfrak{t}_0^*$ form a *dual pair*. We leave the verification of this fact to the interested reader. The existence of this dual pair is not unrelated to the previous discussion of integrability; for the connection between dual pairs and the Mishchenko-Fomenko theory, refer to Adams and Ratiu (1988). In the next section we will see what this dual pair looks like when we put the G-space into 'normal form'.

8 The non-Abelian generalization of action-angle coordinates

When one performs the classical construction of action-angle coordinates for an integrable Hamiltonian system (see e.g. Duistermaat (1980)) one is showing that the phase space (or an open subset) is symplectomorphic to an open subset of $\mathbb{T}^n \times \mathbb{R}^n$, equipped with the canonical symplectic structure $\sum_{j=1}^n dq_j \wedge dp_j$ $q_1, \ldots, q_n, p_1, \ldots, p_n$ denote the standard coordinates on $\mathbb{T}^n \times \mathbb{R}^n$. Furthermore one shows that the Hamiltonian function depends only on the $\mathbb{R}^n$ part: $H(q,p) = H(p)$. If we replace $\mathbb{T}^n$ by a general compact connected Lie group G, then a natural generalization of $\mathbb{T}^n \times \mathbb{R}^n$ turns out to be $G \times \mathfrak{t}_0^*$, where $\mathfrak{t}_0^*$ is any one of the open subsets of $\mathfrak{t}^*$ corresponding to a Weyl chamber in $\mathfrak{g}^*$ (see Sect. 3) and $\mathfrak{t} \subset \mathfrak{g}$ is a maximal Abelian subalgebra. The natural symplectic structure for this space comes from realizing it as the Hamiltonian T-space associated with T^*G, after making T^*G a G-space with regular momenta by removing the 'irregular' points. We outline this first. We include a formula for the Poisson bracket on $G\times\mathfrak{t}_0^*$, which is relevant in generalizing the method of Lie-transforms in perturbation theory (Perry, 1996). Secondly we show that (at least locally) any integrable G-space with regular momenta is equivalent to an open subset $G\times U$ of $G\times\mathfrak{t}_0^*$ and we reduce the construction of the equivalence to the construction of an equivalence between the associated lower dimensional T-space, and an open subset of $T\times\mathfrak{t}^*\cong\mathbb{T}^n\times\mathbb{R}^n$. We may interpret the latter as the construction of conventional action-angle coordinates. Since the construction will be G-equivariant, a G-invariant Hamiltonian becomes a function depending only on the $\mathfrak{t}_0^*$ part: $H(g,p)=H(p)$, $(g,p)\in G\times\mathfrak{t}_0^*$. Furthermore, if the action-angle coordinates can be constructed globally in the T-space then the G-space equivalence also applies globally.

The non-Abelian model space (normal form)

Let G be a compact connected Lie group and put $P=\mathrm{T}^*G$; P has the natural symplectic structure provided by $\omega=-d\Theta$, where Θ is the canonical one-form on P defined by

$$\langle\Theta,\zeta\rangle=\langle\tau_{\mathrm{T}^*G}(\zeta),\mathrm{T}\tau_G^*\cdot\zeta\rangle\ .$$

The maps $\tau_{\mathrm{T}^*G}:\mathrm{T}(\mathrm{T}^*G)\to\mathrm{T}^*G$ and $\tau_G^*:\mathrm{T}^*G\to G$ denote the canonical projections. Let G act on itself according to:

$$g\cdot h\equiv R_{g^{-1}}(h)\equiv hg^{-1}\ \ ,\ h\in G\ .$$

Let G act on P by the cotangent lifted action:

$$g\cdot x\equiv\mathrm{T}_*R_{g^{-1}}\cdot x\ \ ,\ x\in\mathrm{T}^*G\ \ ,$$

where the (covariant) cotangent lift $\mathrm{T}_*\phi:\mathrm{T}^*G\to\mathrm{T}^*G$ of a diffeomorphism $\phi:G\to G$ is defined by

$$\langle\mathrm{T}_*\phi\cdot p,u\rangle=\langle p,\mathrm{T}\phi^{-1}\cdot u\rangle\ \ ,\ \forall u\in\mathrm{T}_gG;g\equiv\tau_G^*(p)\ .$$

An equivariant momentum map $\mathbf{J}:P\to\mathfrak{g}^*$ is provided by

$$\mathbf{J}(x)=\mathrm{T}_*L_g\cdot x\ \ ,$$

where $g\equiv\tau_G^*(x)$ and $L_g(h)\equiv gh$. The momentum map is surjective. We replace P by $\mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$, so that $(P,\omega,G,\mathbf{J})$ is a G-space with regular momenta. Apply 3.15 and 4.8. Then, identifying $\mathrm{T}^*_{\mathrm{id}}G$ with $\mathfrak{g}^*$, we have $\pi(\mathrm{T}_*L_{g^{-1}}\cdot\mu)=\pi_{\mathcal{O}}(\mu)$, so that

$$\pi^{-1}(\mu_0)=\{\mathrm{T}_*L_g\cdot\mu\mid g\in G\text{ and }\mu\in W\}$$

is a representative fibre in the fibering of P by mutually equivalent T-spaces. We next construct a diffeomorphism $\phi:G\times\mathfrak{t}_0^*\to\pi^{-1}(\mu_0)$ and see what symplectic structure we pull back from $\pi^{-1}(\mu_0)$, which by 4.8 is a symplectic submanifold of P. Let $i:\underline{\mathfrak{t}}\to\mathfrak{t}^*$ denote the natural isomorphism (see 3.15a). Define the diffeomorphism $\phi:G\times\mathfrak{t}_0^*\to\pi^{-1}(\mu_0)$ by

$$\phi(g,p)\equiv\mathrm{T}_*L_g\cdot i^{-1}(p)\ .$$

We write $N \equiv G \times \mathfrak{t}_0^*$ ('N' stands for 'normal form'). For all $(\xi, \tau) \in \mathfrak{g} \times \mathfrak{t}^*$ define the vector field $(\xi, \tau)_N$ on $G \times \mathfrak{t}_0^*$ by

$$(\xi, \tau)_N(g, p) \equiv \frac{d}{dt}(g \exp(t\xi), p + t\tau)\Big|_{t=0} \ . \tag{21}$$

Note that $(\xi, \tau) \mapsto (\xi, \tau)_N(g, p) : \mathfrak{g} \times \mathfrak{t}^* \to \mathrm{T}_{(g,p)}N$ is an isomorphism at every $(g, p) \in N$. One verifies that[7]

$$[(\xi_1, \tau_1)_N, (\xi_2, \tau_2)_N] = ([\xi_1, \xi_2], 0)_N \ . \tag{22}$$

One computes

$$\begin{aligned}
\langle \phi^*\Theta, (\xi, \tau)_N(g, p)\rangle &= \langle \mathrm{T}_* L_g \cdot i^{-1}(p), \frac{d}{dt}\tau_G^*\Big(\phi(g \exp(t\xi), p + t\tau)\Big)\Big|_{t=0}\rangle \\
&= \langle i^{-1}(p), \mathrm{T}L_{g^{-1}} \cdot \frac{d}{dt}\tau_G^*(\mathrm{T}_* L_{g\exp(t\xi)} \cdot i^{-1}(p + t\tau))\Big|_{t=0}\rangle \\
&= \langle i^{-1}(p), \frac{d}{dt} g^{-1} g \exp(t\xi)\Big|_{t=0}\rangle \\
&= \langle i^{-1}(p), \xi\rangle = \langle p, \sigma(\xi)\rangle
\end{aligned} \tag{23}$$

where $\sigma : \mathfrak{g} \to \mathfrak{t}$ is the projection onto $\mathfrak{t}$ along $[\mathfrak{t}, \mathfrak{g}]$. From this we compute

$$((\xi_1, \tau_1)_N \lrcorner\, d((\xi_2, \tau_2)_N \lrcorner\, \phi^*\Theta))(g, p) = \frac{d}{dt}\langle p + t\tau_1, \sigma(\xi_2)\rangle\Big|_{t=0} = \langle \tau_1, \sigma(\xi_2)\rangle \ . \tag{24}$$

Applying the well-known formula

$$v \lrcorner\, u \lrcorner\, d\beta = u \lrcorner\, d(v \lrcorner\, \beta) - v \lrcorner\, d(u \lrcorner\, \beta) - [u, v] \lrcorner\, \beta \ ,$$

with $\beta = \phi^*\Theta$, we obtain from (22),(23) and (24)

$$\phi^*\omega((\xi_1, \tau_1)_N(g, p), (\xi_2, \tau_2)_N(g, p)) = \langle \tau_2, \sigma(\xi_1)\rangle - \langle \tau_1, \sigma(\xi_2)\rangle + \langle p, \sigma[\xi_1, \xi_2]\rangle \ .$$

Notice that we have realized $G \times \mathfrak{t}_0^*$ as a T-space, but that G also acts naturally on $G \times \mathfrak{t}_0^*$, according to $g' \cdot (g, p) \equiv (g'g, p)$. In fact this action is Hamiltonian; an associated momentum map $\mathbf{J}^N : G \times \mathfrak{t}_0^* \to \mathfrak{g}^*$ is provided by $\mathbf{J}^N(g, p) \equiv g \cdot i^{-1}(p)$. To show this note that $\phi^*\omega = -d\phi^*\Theta$ and that the Lie derivative of $\phi^*\Theta$ with respect to any infinitessimal generator of the G-action vanishes. It follows from Cartan's 'magic formula' that we may take $J_\xi^N = \xi_N \lrcorner\, \phi^*\Theta$, where ξ_N is the infinitessimal generator corresponding to $\xi \in \mathfrak{g}$. Equation (23) gives the claimed result.

We have proven:

8.1 **Proposition** *Let G be a compact connected Lie group, W a Weyl chamber in $\mathfrak{g}^*$ and T the associated maximal torus of G (i.e. $T \equiv G_w$ where w is any point in W). Denote by $\mathfrak{t}_0^*$ the (open and connected) image of W under the natural projection $\mathfrak{g}^* \to \mathfrak{t}^*$, and let $i : \underline{\mathfrak{t}} \to \mathfrak{t}^*$ denote the natural isomorphism (see 3.7c). Write $N \equiv G \times \mathfrak{t}_0^*$ and for each $(\xi, \tau) \in \mathfrak{g} \times \mathfrak{t}^*$ let $(\xi, \tau)_N$ be the vector field on N defined in (21) above. This gives an identification of each*

[7] We define the bracket on the Lie algebra of a Lie group using the *left* invariant vector fields.

$$\begin{tikzcd} & G \times \mathfrak{t}_0^* \arrow[dl, "\mathbf{J}^N"'] \arrow[dr, "\mathbf{j}^N"] & \\ \mathfrak{g}^*_{\mathrm{reg}} \arrow[rr] & & \mathfrak{g}^*_{\mathrm{reg}}/G \simeq \mathfrak{t}_0^* \end{tikzcd}$$

Figure 1: The dual pair for the normal form of an integrable G-space with regular momenta

tangent space $\mathrm{T}_{(g,p)}N$ *with* $\mathfrak{g} \times \mathfrak{t}^*$. *Then* N *admits a natural symplectic structure defined by* $\omega_N = -d\Theta_N$, *where*

$$\langle \Theta_N, (\xi, \tau)_N(g,p)\rangle \equiv \langle p, \sigma\xi\rangle$$

and $\sigma : \mathfrak{g} \to \mathfrak{t}$ *is the projection along* $\mathfrak{t}^\perp \equiv [\mathfrak{t}, \mathfrak{g}]$. *In fact we have*

$$\omega_N\Big((\xi_1,\tau_1)_N(g,p), (\xi_2,\tau_2)_N(g,p)\Big) = \omega^p\Big((\xi_1,\tau_1),(\xi_2,\tau_2)\Big) \ ,$$

where ω^p *is the skew-symmetric bi-linear form on* $\mathfrak{g} \times \mathfrak{t}^*$ *defined by*

$$\omega^p\Big((\xi_1,\tau_1),(\xi_2,\tau_2)\Big) \equiv \langle \tau_2, \sigma\xi_1\rangle - \langle \tau_1, \sigma\xi_2\rangle + \langle p, \sigma[\xi_1,\xi_2]\rangle \ .$$

The action of G *on* N *defined by* $g' \cdot (g,p) \equiv (g'g, p)$ *admits an equivariant momentum map* $\mathbf{J}^N : N \to \mathfrak{g}^*$ *defined by*

$$\mathbf{J}^N(g,p) \equiv g \cdot i^{-1}(p) = \mathrm{Ad}^*_{g^{-1}}\, i^{-1}(p) \ .$$

The proposition states that $G \times \mathfrak{t}_0^*$ is a Hamiltonian G-space. One immediately verifies that the space is in fact an *integrable* G-space with regular momenta, in the sense of 7.4. Applying the construction of Sect. 5 we obtain a map $\mathbf{j}^N : G \times \mathfrak{t}_0^* \to \mathfrak{t}_0^*$, which turns out to be just the projection onto the second component: $\mathbf{j}^N(g,p) = p$. The maps $\mathfrak{g}^*_{\mathrm{reg}} \xleftarrow{\mathbf{J}^N} N \xrightarrow{\mathbf{j}^N} \mathfrak{t}_0^*$ form a (full) dual pair (see Sect. 7 and Fig. 1). We have 'duality' because the T-orbits are the fibres of $\mathbf{J}^N$ (the momentum map of the G-action) while the G-orbits are the fibres of $\mathbf{j}^N$ (the momentum map of the T-action).

The Poisson bracket on $G \times \mathfrak{t}_0^*$

For any smooth function $f : G \times \mathfrak{t}_0^* \to \mathbb{R}$ define the vector-valued functions $\frac{\partial f}{\partial g} : G \times \mathfrak{t}_0^* \to \mathfrak{g}^*$ and $\frac{\partial f}{\partial p} : G \times \mathfrak{t}_0^* \to \mathfrak{t}$ by

$$\langle \frac{\partial f}{\partial g}(g,p), \xi\rangle \equiv \langle df, (0,\xi)_N(g,p)\rangle = \left.\tfrac{d}{dt} f(g\exp(t\xi), p)\right|_{t=0} \quad \forall \xi \in \mathfrak{g} \ , \tag{25}$$

$$\langle \tau, \frac{\partial f}{\partial p}(g,p)\rangle \equiv \langle df, (0,\tau)_N(g,p)\rangle = \left.\tfrac{d}{dt} f(g, p + t\tau)\right|_{t=0} \quad \forall \tau \in \mathfrak{t}^* \ . \tag{26}$$

Let X_f^N denote the Hamiltonian vector field on $N \equiv G \times \mathfrak{t}_0^*$ corresponding to Hamiltonian $f : N \to \mathbb{R}$, i.e. $X_f^N \lrcorner\, \omega_N = df$.

8.2 **Lemma** *$X_f^N(g,p) = (\xi,\tau)_N(g,p)$, where*

$$\begin{aligned}\xi &\equiv \frac{\partial f}{\partial p}(g,p) + \Lambda_p \frac{\partial f}{\partial g}(g,p)\\ \tau &\equiv -i\sigma^* \frac{\partial f}{\partial g}(g,p)\\ \Lambda_p &\equiv \lambda_p \circ (\mathrm{id} - \sigma^*)\end{aligned}$$

and where $\sigma^* : \mathfrak{g}^* \to \underline{\mathfrak{t}}$ is the projection along $\underline{\mathfrak{t}}^\perp$, i.e. $\langle \sigma^*\mu, \xi\rangle = \langle \mu, \sigma\xi\rangle$, $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$.

Recall that $\lambda_p : \underline{\mathfrak{t}}^\perp \to \mathfrak{t}^\perp$ is the isomorphism defined in 3.17. It is defined for all $p \in \mathfrak{t}_0^*$ but 'blows up' as p approaches the boundary of $\mathfrak{t}_0^*$ (the chamber 'walls'). The linear map Λ_p maps $\mathfrak{g}^*$ onto $\mathfrak{t}^\perp \subset \mathfrak{g}$. The proof of 8.2 is a routine calculation we have relegated to Appendix A.

The Poisson bracket on $N \equiv G \times \mathfrak{t}_0^*$ is defined by $\{f,h\}_N \equiv X_f \lrcorner X_h \lrcorner \omega_N$.

8.3 **Corollary** (The bracket on $G \times \mathfrak{t}_0^*$) *Dropping the '$(g,p)$' argument from '$\frac{\partial f}{\partial g}(g,p)$', etc:*

$$\{f,h\}_N(g,p) = \langle \frac{\partial f}{\partial g}, \frac{\partial h}{\partial p}\rangle - \langle \frac{\partial h}{\partial g}, \frac{\partial f}{\partial p}\rangle - \langle p, [\Lambda_p \frac{\partial f}{\partial g}, \Lambda_p \frac{\partial h}{\partial g}]\rangle,$$

where $\Lambda_p : \mathfrak{g}^* \to \mathfrak{g}$ is the map defined above.

The proof of the corollary follows easily from 8.2, and is left to the reader.

The Abelian case

Here we review the case of a Hamiltonian toral action. In particular, we clarify what we mean by constructing action-angle coordinates in the context of T-spaces, T being an n-torus.

Let T be an n-torus and suppose that $G = T$ so that $N = T \times \mathfrak{t}^*$. The symplectic and Poisson structures described in the preceding paragraphs are just the canonical ones (note that we can identify $T \times \mathfrak{t}^*$ with $\mathbb{T}^n \times \mathbb{R}^n$). Of course the momentum map for this space is just the natural projection $T \times \mathfrak{t}^* \to \mathfrak{t}^*$. Recall that that two G-spaces $(P_1, \omega_1, G, \mathbf{J}^1)$ and $(P_2, \omega_2, G, \mathbf{J}^2)$ are *equivalent* if there exists a G-equivariant symplectic diffeomorphism $\phi : P_1 \to P_2$ such that $\mathbf{J}^2 \circ \phi = \mathbf{J}^1$. We say that the G-systems $(P_1, \omega_1, G, \mathbf{J}^1, H_1)$ and $(P_2, \omega_2, G, \mathbf{J}^2, H_2)$ are equivalent if $H_2 \circ \phi = H_1$ is also true. Note that equivalent spaces/systems necessarily have identical momentum map images: $\mathbf{J}^1(P_1) = \mathbf{J}^2(P_2)$.

8.4 **Definition** We say that a T-space $(F, \omega_F, T, \mathbf{J}^F)$ *admits global action-angle coordinates* if $\mathbf{J}^F(F) \subset \mathfrak{t}^*$ is open and the space is equivalent to $T \times \mathbf{J}^F(F)$.

Let $(F, \omega_F, T, \mathbf{J}^F)$ be a Hamiltonian T-space and assume that T acts freely. Put $U = \mathbf{J}^F(F)$. Recall (see 7.4) that this space is *integrable* if $\dim F = 2 \dim P$. We now address the problem of constructing global action-angle coordinates for such a space. Duistermaat (1980) has studied this problem in the case of a Lagrangian fibration whose fibres are compact and connected. Now the momentum map $\mathbf{J}^F : F \to U$ is a surjective submersion, and the *connected components* of a fibre $(\mathbf{J}^F)^{-1}(p)$ turn out to be Lagrangian tori. But unfortunately we cannot assume a priori that $(\mathbf{J}^F)^{-1}(p)$ is connected or compact, or that $\mathbf{J}^F : F \to U$ is fibrating (i.e. a (locally trivial) fibre bundle). This kind of problem also occurs in trying to prove 'convexity theorems' for non-compact spaces (see e.g. Hilgert, Neeb and Plank (1994)[8]). On the other hand the situation is simpler than that considered by Duistermaat (1980) in the sense that we already have global *actions*; these are just appropriate components of the momentum map.

8.5 **Proposition** *Let $(F, \omega_F, T, \mathbf{J}^F)$ be a Hamiltonian T-space in which T acts freely, and assume that the space is integrable. Then the space admits global action-angle coordinates $\Leftrightarrow$ the following conditions hold:*

a. *The fibres of $\mathbf{J}^F : F \to \mathbf{J}^F(F)$ are connected*

b. *$\mathbf{J}^F : F \to \mathbf{J}^F(F)$ admits a Lagrangian section[9] $s : \mathbf{J}^F(F) \to F$*

Proof of $\Rightarrow$: 8.5a is obvious. 8.5b follows since the map $p \mapsto (\theta, p) : \mathfrak{t}^* \to T \times \mathfrak{t}^*$ is a Lagrangian section for any $\theta \in T$.

□

Proof of $\Leftarrow$: Let $U = \mathbf{J}^F(F)$. Since T acts freely $\mathbf{J}^F : F \to U$ is a surjective submersion. By the pre-image theorem its fibres are submanifolds of F of dimension $\dim F - \dim T$. Since the space is integrable this dimension is $\dim T$. Using the momentum map equivariance we then conclude that the connected component of a fibre is a T-orbit. It follows from 8.5a that each fibre is a *single* T-orbit. In other words $U \cong F/T$ and $\mathbf{J}^F : F \to U \cong F/T$ is just the natural projection. Therefore the map $\phi : T \times U \to F$ defined by $\phi(\theta, p) \equiv \theta \cdot s$ is a diffeomorphism (remember T acts freely). Appealing to 4.12c and the fact that $s : U \to F$ is Lagrangian, one easily verifies that ϕ is in fact a symplectomorphism. The details are left to the reader (but see also the proof of 8.8 below).

□

If in addition to condition 8.5a one already knows that $\mathbf{J}^F : F \to \mathbf{J}^F(F)$ is *fibrating*, then topological obstructions to the construction of a Lagrangian section can be characterized by algebraic invariants of the fibration (Duistermaat, 1980). Note that if the symplectic form ω_F is exact then an arbitrary section can always be modified to obtain a Lagrangian section. The relevant details are recalled in Appendix B. Although 8.5 gives necessary and sufficient conditions for global action-angle coordinates, the following stronger sufficient conditions may be more readily verified in examples.

8.6 **Proposition** *Let $(F, \omega_F, T, \mathbf{J}^F)$ be a T-space in which T acts freely. Suppose that the space is integrable and that F is connected. Then if furthermore,*

a. *$\mathbf{J}^F : F \to \mathbf{J}^F(F)$ is proper*

b. *$\mathbf{J}^F(F) \subset \mathfrak{t}^*$ is convex*

then the T-space admits global action-angle coordinates.

[8] Note that these results do not apply immediately here since we cannot assume that $\mathbf{J}^F$ is proper as a map *into* $\mathfrak{t}^*$. Indeed examples of spaces satisfying the conditions of 8.5 can be constructed such that $\mathbf{J}^F(F)$ is non-convex.

[9] By *section* we mean an injective immersion $s : \mathbf{J}^F(F) \to F$ that satisfies $\mathbf{J}^F \circ s = \mathrm{id}$. By *Lagrangian* we mean that $s^*\omega_F = 0$ and that $\dim s(\mathbf{J}^F(F)) = \dim P/2$.

We have stated 8.6 mainly for the sake of completeness. The result is not used elsewhere in this paper. A proof of 8.6 is sketched in Appendix A.

8.7 **Remarks** a. Assuming that all other hypotheses are true, one can always arrange for conditions 8.6a and 8.6b to be satisfied by replacing F in 8.6 by an appropriate T-invariant open neighborhood V of an orbit.

b. Suppose that $(F, \omega_F, T, \mathbf{J}^F)$ is the T-space associated with a G-space $(P, \omega, G, \mathbf{J})$ with regular momenta. Then F is connected if P is connected, T acts freely if G acts freely, and the T-space is integrable if the G-space is integrable (in the sense of 7.4). Furthermore $\mathbf{J}^F(F) \cong \mathbf{j}(P)$, where $\mathbf{j} : P \to \mathfrak{t}^*$ is the momentum map associated with the natural T-action on P constructed in Sect. 5.

Putting an integrable non-Abelian G-space into normal form

Here is the main result of this section.

8.8 **Theorem** *Let G be a compact and connected Lie group. Let $(P, \omega, G, \mathbf{J}, H)$ be a G-system with regular momenta, in which G acts freely and which is integrable in the sense of 7.4. Let $\pi : P \to \mathcal{O}$ be the fibering of P by mutually equivalent (integrable) Hamiltonian T-spaces described in 4.8 and let $(F, i_F^*\omega, T, \mathbf{J}^F)$ be a representative fibre, as described in 4.9. Let $\mathfrak{t}_0^* = i(W)$ (where $W = \pi(F)$ is the associated Weyl chamber in $\mathfrak{g}^*$ and $i : \underline{\mathfrak{t}} \to \mathfrak{t}^*$ is the natural isomorphism; see 3.7c). Assume that this T-space admits global action-angle coordinates in the sense of 8.4, and let $s : U \to F$ be an associated Lagrangian section of $\mathbf{J}^F : F \to U$ (see 8.5), where $U \equiv \mathbf{J}^F(F) \subset \mathfrak{t}_0^*$. Define $\hat{s} : U \to P$ by $\hat{s} = i_F \circ s$ ($i_F : F \to P$ denotes inclusion). Then the map $\phi : G \times U \to P$ defined by*

$$\phi(g, p) \equiv g \cdot \hat{s}(p)$$

is a G-equivariant symplectomorphism for which $\mathbf{J} \circ \phi = \mathbf{J}^N$, where $N \equiv G \times U \subset G \times \mathfrak{t}_0^*$ is equipped with its natural G-space structure (see 8.1). The pulled-back Hamiltonian $H \circ \phi$ is of the form $(H \circ \phi)(g, p) = h(p)$ for some $h : U \to \mathbb{R}$.

In other words if the T-space associated with an integrable Hamiltonian G-system $(P, \omega, G, \mathbf{J}, H)$ with regular momenta admits global action-angle coordinates, then the G-system is equivalent to the 'normal form' $(G \times U, \omega_N, G, \mathbf{J}^N, h)$, with Hamiltonian of the form $h(g, p) = h(p)$.

8.9 **Remarks** a. The map $\hat{s} : U \to P$ is an isotropic section for $\mathbf{j} : P \to U = \mathbf{j}(p)$. By *isotropic* we mean that $s^*\omega = 0$. Indeed if the fibres of $\mathbf{j} : P \to U$ are connected, then one may take $\hat{s} : U \to P$ to be *any* isotropic section of $\mathbf{j}$ and the claims of the theorem again follow. This fact will be obvious from the proof of 8.8 (given below). In particular, if one can find such a section it is not necessary to first construct the G-space's associated T-space. In Example 9.5 we will construct the isotropic section directly.

b. We can always arrange for the global action-angle coordinates to apply by replacing P by an open subset $G(V)$ — see Remark 8.7a. In other words, the normal form always applies locally in the neighborhood of a G-orbit.

Proof: Let $\mathbf{j} : P \to \mathfrak{t}_0^*$ be the momentum map associated with the natural action of T on P described in Sect. 5. Note that T acts freely because G acts freely and that $\mathbf{j}(P) = U$. Since G acts freely $\mathbf{J} : P \to \mathfrak{g}^*$ is a submersion (by 4.2). It follows that $\mathbf{j} : P \to U$ is a surjective submersion. Using the equivariance of $\mathbf{J}$, one can show that for any $p \in \mathfrak{t}_0^*$, $\mathbf{j}^{-1}(p) = G(\mathbf{J}^{-1}(i^{-1}(p))) = G((\mathbf{J}^F)^{-1}(p))$. By 8.5, $(\mathbf{J}^F)^{-1}(p)$ is connected. Since G is connected, it follows that every fibre $\mathbf{j}^{-1}(p)$ is connected. Each of these fibres is a submanifold of dimension $\dim G$ and a union of G-orbits ($\mathbf{j}$ is G-equivariant). By the connectedness each fibre is therefore a *single* G-orbit. In other words $U \cong P/G$ and $\mathbf{j} : P \to U \cong P/G$ is just the natural projection. The map $\hat{s} : U \to P$ is a section for $\mathbf{j} : P \to U$. It follows that the map $\phi : G \times U \to P$ defined in the statement of the theorem is a diffeomorphism (remember G acts freely). Since $s : U \to F$ is a Lagrangian section, $\hat{s} : U \to P$ is an isotropic section, i.e. $\hat{s}^*\omega = 0$. To verify that ϕ is symplectic, it suffices to show that $\phi^*\omega$ and ω_N agree on pairs of tangent vectors of the following three forms:

a. $(\xi_1, 0)_N(g,p)$ and $(\xi_2, 0)_N(g,p)$

b. $(\xi_1, 0)_N(g,p)$ and $(0, \tau_2)_N(g,p)$

c. $(0, \tau_1)_N(g,p)$ and $(0, \tau_2)_N(g,p)$

where $\xi_1, \xi_2 \in \mathfrak{g}$, $\tau_1, \tau_2 \in \mathfrak{t}^*$ and $(g,p) \in G \times U$ are arbitrary. First we compute

$$\begin{aligned} \mathrm{T}\phi \cdot (\xi, 0)_N(g,p) &= \frac{d}{dt}\phi(g\exp(t\xi), p)\Big|_{t=0} \\ &= (g\cdot\xi)_P(g\cdot\hat{s}(p)) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \mathrm{T}\phi \cdot (0, \tau)_N(g,p) &= \frac{d}{dt}\phi(g, p + t\tau)\Big|_{t=0} \\ &= \frac{d}{dt} g\cdot\hat{s}(p+t\tau)\Big|_{t=0} \ . \end{aligned} \tag{28}$$

Case *a*: Using (27):

$$\begin{aligned} \phi^*\omega((\xi_1,0)_N(g,p),(\xi_2,0)_N(g,p)) &= \omega((g\cdot\xi_1)_P(g\cdot\hat{s}(p)),(g\cdot\xi_2)_P(g\cdot\hat{s}(p))) \\ &= \langle \mathbf{J}(g\cdot\hat{s}(p)), [g\cdot\xi_1, g\cdot\xi_2]\rangle \qquad \text{by 4.12}c \\ &= \langle g\cdot\mathbf{J}(\hat{s}(p)), g\cdot[\xi_1,\xi_2]\rangle \\ &= \langle \mathbf{J}(\hat{s}(p)), [\xi_1,\xi_2]\rangle \ . \end{aligned} \tag{29}$$

Now $\hat{s}(p) \in F$, so $\mathbf{J}(\hat{s}(p)) \in W$. In particular we can write $\mathbf{J}(\hat{s}(p)) = (\pi_W \circ \mathbf{J})(\hat{s}(p))$, so that (29) becomes

$$\begin{aligned} \phi^*\omega((\xi_1,0)_N(g,p),(\xi_2,0)_N(g,p)) &= \langle(\pi_W\circ\mathbf{J})(\hat{s}(p)),[\xi_1,\xi_2]\rangle \\ &= \langle(i\circ\pi_W\circ\mathbf{J})(\hat{s}(p)),\sigma[\xi_1,\xi_2]\rangle \\ &= \langle(\mathbf{j}\circ\hat{s})(p),\sigma[\xi_1,\xi_2]\rangle \\ &= \langle p,\sigma[\xi_1,\xi_2]\rangle \\ &= \omega_N((\xi_1,0)_N(g,p),(\xi_2,0)_N(g,p)) \ . \end{aligned}$$

CASE b: Using (27) and (28):

$$\begin{aligned} \phi^*\omega((\xi_1,0)_N(g,p),(0,\tau_2)_N(g,p)) &= \omega((g\cdot\xi_1)_P(g\cdot\hat{s}(p)),\frac{d}{dt}g\cdot\hat{s}(p+t\tau_2)\Big|_{t=0}) \\ &= \langle dJ_{g\cdot\xi_1},\frac{d}{dt}g\cdot\hat{s}(p+t\tau_2)\Big|_{t=0}\rangle \\ &= \frac{d}{dt}J_{g\cdot\xi_1}(g\cdot\hat{s}(p+t\tau_2))\Big|_{t=0} \\ &= \frac{d}{dt}\langle\mathbf{J}(g\cdot\hat{s}(p+t\tau_2)),g\cdot\xi_1\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle g\cdot\mathbf{J}(\hat{s}(p+t\tau_2)),g\cdot\xi_1\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle\mathbf{J}(\hat{s}(p+t\tau_2)),\xi_1\rangle\Big|_{t=0} \ . \end{aligned} \tag{30}$$

Now $\hat{s}(p+t\tau_2)\in F$, so $\mathbf{J}(\hat{s}(p+t\tau_2))\in W$. In particular we can write $\mathbf{J}(\hat{s}(p+t\tau_2)) = \pi_W\circ\mathbf{J}(\hat{s}(p+t\tau_2))$, so that (30) becomes

$$\begin{aligned} \phi^*\omega((\xi_1,0)_N(g,p),(0,\tau_2)_N(g,p)) &= \frac{d}{dt}\langle(\pi_W\circ\mathbf{J})(\hat{s}(p+t\tau_2)),\xi_1\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle(i\circ\pi_W\circ\mathbf{J})(\hat{s}(p+t\tau_2)),\sigma(\xi_1)\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle(\mathbf{j}\circ\hat{s})(p+t\tau_2),\sigma(\xi_1)\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle p+t\tau_2,\sigma(\xi_1)\rangle\Big|_{t=0} \\ &= \langle\tau_2,\sigma(\xi_1)\rangle \\ &= \omega_N((\xi_1,0)_N(g,p),(0,\tau_2)_N(g,p)) \ . \end{aligned}$$

CASE c: Write $g\cdot x=\Phi_g(x)$, $x\in P$. Using (28),

$$\phi^*\omega((0,\tau_1)_N(g,p),(0,\tau_2)_N(g,p)) = \hat{s}^*\Phi_g^*\omega(\frac{d}{dt}(p+t\tau_1)\Big|_{t=0},\frac{d}{dt}(p+t\tau_2)\Big|_{t=0}) \ .$$

But the action of G is symplectic, so $\Phi_g^*\omega=\omega$. Therefore we have $\hat{s}^*\Phi_g^*\omega=\hat{s}^*\omega=0$. So we have

$$\begin{aligned} \phi^*\omega((0,\tau_1)_N(g,p),(0,\tau_2)_N(g,p)) &= 0 \\ &= \omega_N((0,\tau_1)_N(g,p),(0,\tau_2)_N(g,p)) \ . \end{aligned}$$

The G-equivariance of ϕ is obvious. Also, we have

$$\begin{aligned} \mathbf{J}(\phi(g,p)) &= \mathbf{J}(g\cdot\hat{s}(p)) \\ &= g\cdot\mathbf{J}(\hat{s}(p)) \\ &= g\cdot i^{-1}(p) \ , \end{aligned}$$

where $i:\underline{\mathbf{t}}\to\mathbf{t}^*$ is the natural isomorphism. This proves that $\mathbf{J}\circ\phi=\mathbf{J}^N$.
□

9 Examples

Normal forms

We now present two examples of normal forms for integrable G-spaces with regular momenta. We seek to give both a demonstration of the general theory as well as an indication of how one might describe these spaces in a form more convenient in applications (e.g. to perturbation theory). Only the first example is described in detail. Later in this section we shall show that the first example gives the normal form for a spherical pendulum free of external forces, while the second gives the normal form for the Euler-Poinsot rigid body.

9.1 **Example** (The normal form for $G = SO(3)$) Consider the normal form $N \equiv G \times \mathfrak{t}_0^*$, for an integrable G-space with regular momenta in the case that $G = SO(3)$. Make the identifications $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathbb{R}^3$ described in 3.12. Take $\mu_0 \equiv e_3$. Let W be the Weyl chamber containing μ_0 and let $\mathcal{O}$ be the regular orbit through μ_0. The co-adjoint action is given by $g \cdot \mu = g\mu$, $\mu \in \mathbb{R}^3$, so that $\mathcal{O}$ is the unit sphere S^2 and $W = \{(0,0,t) \mid t \in (0,\infty)\} \cong (0,\infty)$ (see 3.7b). We have $\mathfrak{t} \cong \underline{\mathfrak{t}} \cong \text{span}\,\{e_3\} \cong \mathbb{R}$ and $\mathfrak{t}^\perp \cong \underline{\mathfrak{t}}^\perp \cong \text{span}\,\{e_1, e_2\} \cong \mathbb{R}^2$, where $\mathfrak{t} \equiv \mathfrak{g}_{\mu_0}$. The projections $\sigma : \mathbb{R}^3 \to \mathbb{R}$ and $\sigma^* : \mathbb{R}^3 \to \mathbb{R}$ are given by $\sigma(\xi_1, \xi_2, \xi_3) = \sigma^*(\xi_1, \xi_2, \xi_3) = \xi_3$. The isomorphism $i : \mathbb{R} \to \mathbb{R}$ is just the identity, so that $\mathfrak{t}_0^* \equiv i(W) = (0, \infty)$.

Let $\xi \mapsto \hat{\xi} : \mathbb{R}^3 \to \mathfrak{so}(3)$ denote the isomorphism defined by

$$\hat{\xi} u = \xi \times u \ , \quad \forall u \in \mathbb{R}^3$$

($\mathfrak{so}(3)$ denotes the real skew-symmetric 3×3 matrices). Then the vector fields on $N \cong SO(3) \times (0, \infty)$ defined in (21) can be written as

$$(\xi, \tau)_N(g, p) = \frac{d}{dt}(g e^{t\hat{\xi}}, p + t\tau)\Big|_{t=0} \ , \quad (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \ .$$

The symplectic structure on $SO(3) \times (0, \infty)$ given in 8.1 is $\omega_N = -d\Theta_N$, where

$$\langle \Theta_N, (\xi, \tau)_N(g, p) \rangle = p\xi_3 \ .$$

To obtain a more explicit formula for the Poisson bracket on N we need to compute the isomorphism $\lambda_p : \mathbb{R}^2 \to \mathbb{R}^2$ (see 3.17). First note that the infinitessimal co-adjoint action is given by $-\operatorname{ad}_\xi^* \mu = \xi \times \mu$. One then easily computes

$$\lambda_p(\mu_1, \mu_2) = \frac{1}{p}(\mu_2, -\mu_1) \ , \quad p \in (0, \infty) \ .$$

The map $\Lambda_p : \mathbb{R}^3 \to \mathbb{R}^3$ (see 8.2) is therefore given by

$$\Lambda_p(\mu_1, \mu_2, \mu_3) = \frac{1}{p}(\mu_2, -\mu_1, 0) \ , \quad p \in (0, \infty) \ .$$

Given our identifications we can write the derivative maps $\frac{\partial f}{\partial p} : N \to \mathbb{R}^3$ and $\frac{\partial f}{\partial g} : N \to \mathbb{R}^3$ as

$$\begin{aligned} \frac{\partial f}{\partial p}(g,p) &= \frac{d}{dt} f(g, p+t)\Big|_{t=0} \\ \frac{\partial f}{\partial g}(g,p) &= (\frac{\partial f}{\partial g_1}(g,p), \dots, \frac{\partial f}{\partial g_3}(g,p)) \end{aligned}$$

where

$$\frac{\partial f}{\partial g_j}(g,p) \equiv \frac{d}{dt} f(g e^{t\hat{e}_j}, p)\Big|_{t=0} \ .$$

The Poisson bracket on $SO(3) \times (0,\infty)$ (see 8.3) is now easily calculated as

$$\{f,h\}_N = \frac{\partial f}{\partial g_3}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial g_3}\frac{\partial f}{\partial p} + \frac{1}{p}\left(\frac{\partial f}{\partial g_1}\frac{\partial h}{\partial g_2} - \frac{\partial h}{\partial g_1}\frac{\partial f}{\partial g_2}\right) \ .$$

Notice that the bracket 'blows up' as $p \to 0$, i.e. as p approaches the boundary of $\mathfrak{t}_0^* = (0,\infty)$.
□

9.2 **Example** (The normal form for $G = SO(3) \times S^1$) Consider the normal form for an integrable G-space with regular momenta $N \equiv G \times \mathfrak{t}_0^*$ in the case that $G = SO(3) \times S^1$. We make the (by now obvious) identifications $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathbb{R}^3 \times \mathbb{R}^1$. Following the same procedure as in the previous example, we find that $\mathfrak{t}_0^* \cong (0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$, i.e. $\mathfrak{t}_0^*$ is the (open) right hand half of the plane $\mathbb{R}^2$. Thus $N \cong SO(3) \times S^1 \times (0,\infty) \times \mathbb{R}$. We write a point $(g,p) \in N = G \times \mathfrak{t}_0^*$ as $(g,p) = ((B,\theta),(p_1,p_2))$. Defining

$$\begin{aligned}
\frac{\partial f}{\partial p_j}(g,p) &\equiv \frac{d}{dt} f(g, p + t e_j)\Big|_{t=0} \quad j = 1,2 \\
\frac{\partial f}{\partial g_j}(g,p) &\equiv \frac{d}{dt} f((B e^{t\hat{e}_j}, \theta), p)\Big|_{t=0} j = 1,2,3 \\
\frac{\partial f}{\partial g_4}(g,p) &\equiv \frac{d}{dt} f((B, \theta + t), p)\Big|_{t=0}
\end{aligned}$$

we compute

$$\begin{aligned}
\{f,h\}_N = & \frac{\partial f}{\partial g_3}\frac{\partial h}{\partial p_1} - \frac{\partial h}{\partial g_3}\frac{\partial f}{\partial p_1} \\
& + \frac{\partial f}{\partial g_4}\frac{\partial h}{\partial p_2} - \frac{\partial h}{\partial g_4}\frac{\partial f}{\partial p_2} \\
& + \frac{1}{p_1}\left(\frac{\partial f}{\partial g_1}\frac{\partial h}{\partial g_2} - \frac{\partial h}{\partial g_1}\frac{\partial f}{\partial g_2}\right) \ .
\end{aligned}$$

Note that the bracket 'blows-up' as $p_1 \to 0$, i.e. as $p = (p_1, p_2)$ approaches the boundary of $\mathfrak{t}_0^* = (0,\infty) \times \mathbb{R}$.
□

The spherical pendulum

We now revisit the problem of the spherical pendulum described in the Introduction in the light of the general theory.

9.3 **Example** (The spherical pendulum without gravity) Consider a spherical pendulum of mass M and length L. It is well-known that in the absence of external forces, this system is described by the Hamiltonian G-system $(P', \omega, G, \mathbf{J}, H)$, where

$$P' \equiv \mathrm{T}S^2 \cong \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \cdot y = 0, \|x\| = 1\}$$

$$\begin{aligned} \omega &\equiv -d\Theta & \langle \Theta, \frac{d}{dt}(x_t, y_t)\Big|_{t=0} \rangle &\equiv y_0 \cdot \frac{d}{dt} x_t \Big|_{t=0} \\ G &\equiv SO(3) & g \cdot (x, y) &\equiv (gx, gy) \\ \mathbf{J}(x, y) &\equiv x \times y & H(x, y) &= \frac{1}{2ML^2} \|y\|^2 \ . \end{aligned}$$

Note that we are making the same identification $\mathfrak{g}^* \cong \mathbb{R}^3$ we made in the previous example (see 3.12), and that $\|\cdot\|$ denotes Euclidean distance. Since $\mathfrak{g}^*_{\rm reg} = \mathbb{R}^3 \backslash \{0\}$, we make $(P', \omega, G, \mathbf{J})$ a G-space with regular momenta by replacing P' by $P \equiv \mathbf{J}^{-1}(\mathfrak{g}^*_{\rm reg}) = \mathrm{T}^+ S^2 \subset P'$, where $\mathrm{T}^+ S^2$ denotes the tangent bundle of S^2 with the zero section removed:

$$\mathrm{T}^+ S^2 \equiv \{(x, y) \in \mathrm{T} S^2 \mid y \neq 0\} \ .$$

Notice that G acts freely on $\mathrm{T}^+ S^2$ and that the space is integrable in the sense of 7.4. Let μ_0, $\mathcal{O}$ and W be chosen as in 9.1. Applying Proposition 3.15, we have the maps $\pi_{\mathcal{O}} : \mathbb{R}^3 \backslash \{0\} \to S^2$ and $\pi_W : \mathbb{R}^3 \backslash \{0\} \to (0, \infty)$ defined by $\pi_{\mathcal{O}}(\mu) \equiv \mu / \|\mu\|$ and $\pi_W(\mu) \equiv \|\mu\|$. The symplectic fibration $\pi : \mathrm{T}^+ S^2 \to S^2$ of Theorem 4.8 is given by $\pi(x, y) = \pi_{\mathcal{O}}(\mathbf{J}(x, y)) = y / \|y\|$. Before constructing a representative fibre let us construct the fibre-preserving action by a maximal torus of $G = SO(3)$. We have $T \equiv G_{\mu_0} = G_{e_3} = \{e^{\theta \hat{e}_3} | \theta \in [0, 2\pi)\}$ (i.e. T is the rotations about the e_3-axis), which we identify with $S^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$. In the notation of Sect. 5, we have for each $\mu \in S^2$ the group isomorphism $\kappa_\mu : S^1 \to G_\mu \subset SO(3)$ given by $\kappa_\mu(\theta) = g e^{\theta \hat{e}_3} g^{-1}$, where $g \in SO(3)$ is to satisfy $\mu = g e_3$. Indeed it is easily verified that $\kappa_\mu(\theta) = e^{\theta \hat{\mu}}$. It follows that the T-action on $\mathrm{T}^+ S^2$ is given by

$$\theta \cdot_T (x, y) \equiv (e^{\theta \hat{\mu}} x, e^{\theta \hat{\mu}} y) \ , \quad \text{where} \quad \mu \equiv \frac{x \times y}{\|y\|} \ .$$

According to 5.1, an associated momentum map $j : \mathrm{T}^+ S^2 \to \mathbb{R}$ is given by

$$j(x, y) \equiv \pi_W(\mathbf{J}(x, y)) = \|x \times y\| = \|y\| \ .$$

A representative fibre $F \equiv \pi^{-1}(e_3)$ in the fibering of $\mathrm{T}^+ S^2$ by mutually equivalent Hamiltonian T-spaces, is diffeomorphic to $S^1 \times (0, \infty)$. Indeed the reader may verify that $\phi_1 : S^1 \times (0, \infty) \to F$ defined by

$$\phi_1(\theta, I) \equiv (R_\theta e_1, I R_\theta e_2) \ , \quad \text{where} \quad R_\theta \equiv e^{\theta \hat{e}_3}$$

is a diffeomorphism. To compute the symplectic structure pulled back by ϕ_1, let us begin by denoting by $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial I}$ the standard vector fields on $S^1 \times (0, \infty)$. One then computes $\frac{\partial}{\partial \theta} \lrcorner \, \phi_1^* \Theta = I$ and $\frac{\partial}{\partial I} \lrcorner \, \phi_1^* \Theta = 0$. For example we have

$$\begin{aligned} \langle \phi_1^* \Theta, \frac{\partial}{\partial \theta}(\theta, I) \rangle &= \langle \Theta, \frac{d}{dt}(R_{\theta+t} e_1, I R_{\theta+t} e_2)\Big|_{t=0} \rangle \\ &= I R_\theta e_2 \cdot \frac{d}{dt} R_{\theta+t} e_1 \Big|_{t=0} \\ &= I R_\theta e_2 \cdot R_\theta (e_3 \times e_1) \\ &= I e_2 \cdot (e_3 \times e_1) = I \ . \end{aligned}$$

Since it follows that $\phi_1^* \Theta = I d\theta$, the symplectic structure on $S^1 \times (0, \infty)$ is $\phi_1^* \omega = -d \phi_1^* \Theta = d\theta \wedge dI$. The maximal torus $T = G_{e_3}$ of $SO(3)$ acts on $F = \pi^{-1}(e_3)$ as a subgroup of

$SO(3)$. The action of T on $S^1 \times (0, \infty)$ that makes the diffeomorphism $\phi_1 : S^1 \times (0, \infty) \to F$ equivariant is given by $\theta' \cdot (\theta, I) = (\theta + \theta', I)$ (we are identifying T with $S^1 \equiv \mathbb{R}^n / 2\pi\mathbb{Z}^n$). An associated equivariant momentum map for this action is just the coordinate function I. The Hamiltonian pulled back by ϕ_1 is given by

$$(\phi_1^* H)(\theta, I) = H(R_\theta e_1, I R_\theta e_2) = \frac{1}{2ML^2} I^2 \ .$$

We immediately recognize $(S^1 \times (0, \infty), d\theta \wedge dI, S^1, I, \frac{1}{2ML^2} I^2)$ as the S^1-system corresponding to a *planar* pendulum of mass M and length L (without gravity), with points corresponding to non-positive angular momentum removed. We have just constructed this system as the T-system corresponding to the G-system $(P, \omega, G, \mathbf{J}, H)$ describing the spherical pendulum (without gravity). We can therefore think of the spherical pendulum with 3D rotational symmetry as an S^2-family of subsystems equivalent to the corresponding planar pendulum with 2D rotational symmetry.

The above T-space obviously admits global action-angle coordinates in the sense of 8.4; an associated Lagrangian section $s : (0, \infty) \to F$ is given by $s(I) \equiv (e_1, I e_2)$. By Theorem 8.8 the G-space of the spherical pendulum $(P, \omega, G, \mathbf{J}) = (\mathrm{T}^+ S^2, \omega, SO(3), \mathbf{J})$ is equivalent to the normal form $SO(3) \times (0, \infty)$ described in detail in Example 9.1. The equivalence is given by the G-equivariant symplectomorphism $\phi_2 : SO(3) \times (0, \infty) \to \mathrm{T}^+ S^2$ defined by $\phi_2(g, p) \equiv g \cdot s(p) = (R_\theta e_1, p R_\theta e_2)$. The pulled-back Hamiltonian on $SO(3) \times (0, \infty)$ is given by $\phi_2^* H(g, p) = \frac{1}{2ML^2} p^2$.
□

The Euler-Poinsot rigid body

We next turn to the description of a rigid body as an $SO(3)$-space.

9.4 **Example** (The Euler-Poinsot rigid body and its associated T-system.) Consider a rigid body fixed at but free to rotate about a point O that is motionless in an appropriate inertial frame of reference. Let I_1, I_2, I_3 denote the body's principal moments of inertia about O. In the absence of external moments, it is well-known that this system is described by the following Hamiltonian G-system:

$$\begin{aligned} P' \equiv \mathrm{T}SO(3) &\cong \{ R\hat{m} \mid R \in SO(3), m \in \mathbb{R}^3 \} \\ &= \{ \hat{n} R \mid R \in SO(3), n \in \mathbb{R}^3 \} \end{aligned}$$

$$\begin{array}{ll} \omega \equiv -d\Theta & \langle \Theta, \frac{d}{dt} R_t \hat{m}_t \Big|_{t=0} \rangle \equiv \langle\langle R_0 \hat{m}_0, \frac{d}{dt} R_t \Big|_{t=0} \rangle\rangle \\ G \equiv SO(3) & g \cdot (R\hat{m}) \equiv (gR)\hat{m} \\ \mathbf{J}(\hat{n}R) \equiv n & H(R\hat{m}) = \frac{1}{2I_1} m_1^2 + \frac{1}{2I_2} m_2^2 + \frac{1}{2I_3} m_3^2 \end{array}$$

where $\mu \mapsto \hat{\mu} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the isomorphism defined in 9.1, and the Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on $SO(3)$ is defined by

$$\langle\langle R\hat{a}, R\hat{b} \rangle\rangle \equiv a \cdot b = \langle\langle \hat{a} R, \hat{b} R \rangle\rangle \ ,$$

where $a \cdot b \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$. To make the G-system a G-system with regular momenta we replace $P = \mathrm{T}SO(3)$ with $P \equiv \mathbf{J}^{-1}(\mathfrak{g}^*_{\mathrm{reg}}) = \mathbf{J}^{-1}(\mathbb{R}^3 \backslash \{0\}) = T^+SO(3) \subset P'$, where

$$\mathrm{T}^+SO(3) \equiv \{R\hat{m} \in \mathrm{T}SO(3) \mid m \neq 0\} \ ,$$

i.e. $\mathrm{T}^+SO(3)$ is just $\mathrm{T}SO(3)$ with its zero section removed. Let μ_0, $\mathcal{O}$ and W be as in 9.1. The symplectic fibration $\pi : T^+SO(3) \to S^2$ is given by $\pi(\hat{n}R) \equiv n/\|n\|$. The fibres are diffeomorphic to $SO(3) \times (0, \infty)$. Consider for example $F \equiv \pi^{-1}(e_3)$; the map $\phi_3 : SO(3) \times (0, \infty) \to F \subset \mathrm{T}SO(3)$ defined by $\phi_3(g, p) \equiv (pe_3)\hat{\ } g^{-1}$ is a diffeomorphism. But we have already seen in 9.1 that $SO(3) \times (0, \infty)$ has a natural symplectic structure ω_N, and in 9.3 that $SO(3) \times (0, \infty)$ is symplectomorphic to T^+S^2, the phase space of the spherical pendulum (with points corresponding to zero angular momentum removed). Curiously, it turns out that with respect to this structure ϕ_3 is *anti*-symplectic, i.e. $\phi_3^* i_F^* \omega = -\omega_N$, where $i_F : F \to \mathrm{T}^+SO(3)$ denotes inclusion. (The proof is left to the reader. It is easiest to first show that $\phi_3^* i_F^* \Theta = -\Theta_N$.) It follows that F (and therefore every fibre of $\pi : \mathrm{T}^+SO(3) \to S^2$) is anti-symplectomorphic to T^+S^2. The anti-symplectomorphism is $\phi_4 : \mathrm{T}^+S^2 \to F$ where $\phi_4 \equiv \phi_3 \circ {\phi_2}^{-1}$ (see 9.2 for the definition of ϕ_2). The dynamics in the fibre F is conjugate via ϕ_4 to dynamics on T^+S^2 if we take as the Hamiltonian on T^+S^2 the function $-\phi_4^* H$. We leave it to the reader to verify that

$$\phi_4^* H(x, y) = \frac{1}{2I_1} b_1^2 + \frac{1}{2I_2} b_2^2 + \frac{1}{2I_3} b_3^2 \ , \quad \text{where } b \equiv x \times y \ . \tag{31}$$

Finally, let us not forget that F is a T-space, where $T \equiv G_{\mu_0} \cong S^1$. The action of S^1 on T^+S^2 that makes ϕ_4 equivariant turns out to be

$$\theta \cdot (x, y) \equiv (e^{-\theta \hat{\mu}} x, e^{-\theta \hat{\mu}} y) \ , \quad \text{where } \mu \equiv b/\|b\| = x \times y / \|y\| \ .$$

Notice incidentally that modulo the minus signs, this is the same S^1 action on T^+S^2 we encountered in 9.3. One verifies that $\phi_4^* H$ given by (31) is indeed S^1-invariant. If we equip T^+S^2 with the *negative* of the standard symplectic structure (as defined in 9.3), then $\phi_4 : \mathrm{T}^+S^2 \to F$ is *symplectic* (rather than anti-symplectic). With respect to this structure the momentum map $\mathbf{J}^F : \mathrm{T}^+S^2 \to \mathbb{R}$ corresponding to the above S^1-action is given by $\mathbf{J}^F(x, y) \equiv \|y\|$.

To summarize, the Hamiltonian T-system associated with $(P, \omega, G, \mathbf{J}, H)$ describing the rigid body is $(\mathrm{T}^+S^2, \omega_-, S^1, \mathbf{J}^F, \phi_4^* H)$, where ω_- denotes the negative of the standard symplectic structure on $\mathrm{T}^+S^2 \subset \mathrm{T}S^2$, and $\phi_4^* H$ is the Hamiltonian given in (31). In this sense the rigid body is an S^2-family of subsystems, each equivalent to the spherical pendulum; the usual Hamiltonian is substituted with the peculiar one given in (31). Note that in the special case of all moments of inertia equal to I, this Hamiltonian is the standard one for a spherical pendulum (without gravity) of moment of inertia I about the hinged point. It might be interesting to give a physical interpretation of this Hamiltonian in the general case.
□

The previous example gave a description of a rigid body as an $SO(3)$-space. Note that the Hamiltonian enjoyed a fixed $SO(3)$-symmetry, independent of the moments of inertia of the body. This symmetry is perhaps the most natural but it does not lead to an *integrable* Hamiltonian G-space. If we *fix* the Hamiltonian H, i.e. fix particular moments of inertia, then we can enlarge the symmetry group to $SO(3)\times\mathbb{R}$ by generating an $\mathbb{R}$-action using the flow of the corresponding fixed Hamiltonian vector field. Of course H is invariant with respect to this new action by energy conservation. Unfortunately the group $SO(3)\times\mathbb{R}$ is not compact. While it is possible to replace the $SO(3)\times\mathbb{R}$-action by an $SO(3)\times S^1$-action on an open subset of the phase space, such that H is $SO(3)\times S^1$-invariant, this 'compactification' procedure cannot be performed globally in general. In the special case that two moments of inertia are equal, this procedure can be applied to an open dense subset of phase space. Furthermore the corresponding S^1-action then has the interpretation of the symmetry of the body's inertial ellipsoid about one axis. One says that the $SO(3)$-action corresponds to a *spatial* symmetry, while the S^1-action corresponds to a *body* symmetry. For simplicity we discuss only the axisymmetric case here but make the remark that one obtains (locally) the same G-space normal form in the general case.

9.5 **Example** (The axisymmetric Euler-Poinsot rigid body in normal form) Consider the Euler-Poinsot rigid body as described in the previous example. Assume that $I_1=I_2\equiv I$. Define P', ω and H as before and define $G\equiv SO(3)\times S^1$. Let an element $g\equiv(B,\theta)$ of G act on P' according to

$$\begin{aligned}&(B,\theta)\cdot(R\hat{m})\equiv BR\hat{m}R_\theta=(BRR_\theta)(R_\theta m)^\wedge\\ \text{i.e.}\quad &(B,\theta)\cdot(\hat{n}R)=B\hat{n}RR_\theta=(Bn)^\wedge(BRR_\theta)\ .\end{aligned}$$

Recall that $R_\theta=e^{\theta\hat{e}_3}$. The Hamiltonian H is invariant with respect to this action. Furthermore the action is Hamiltonian, an equivariant momentum map $\mathbf{J}:P'\to\mathbb{R}^3\times\mathbb{R}^1$ being given by

$$\begin{aligned}\mathbf{J}(R\hat{m})&\equiv(n,m_3)\\ \text{where}\quad n&\equiv Rm,\end{aligned}$$

and where we write $m=(m_1,m_2,m_3)$. So that G acts freely and the image of the momentum map is contained in $\mathfrak{g}^*_{\mathrm{reg}}$, we replace P' by $P\subset P'$ defined by

$$P\equiv\{R\hat{m}\in TSO(3)\mid\quad m_1\neq0\ \text{ or }\ m_2\neq0\},$$

i.e. we remove points in phase space corresponding to body angular momentum lying on the e_3-axis. Note that $\mathfrak{g}^*_{\mathrm{reg}}=(\mathbb{R}^3\backslash\{0\})\times\mathbb{R}$, so that $\mathbf{J}(P)\subset\mathfrak{g}^*_{\mathrm{reg}}$ is immediately verified. The only difference between the G-system $(P,\omega,G,\mathbf{J},H)$ and the system described in the previous example is that we are assuming $I_1=I_2$, that we have enlarged the symmetry group, and that we have removed more points from the phase space (although $P\subset P'$ is still open and dense).

Fix $\mu_0\equiv(e_3,0)\in\mathbb{R}^3\times\mathbb{R}=\mathfrak{g}^*$. The co-adjoint orbit through μ_0 is $\mathcal{O}\equiv S^2\times\{0\}\cong S^2$ and the chamber through μ_0 can be identified with $W\cong\mathfrak{t}^*_0\equiv(0,\infty)\times\mathbb{R}$. The symplectic fibration $\pi:P\to S^2$ of 4.8 is given by $\pi(\hat{n}R)=n/\|n\|$. Notice that this is the same as in 9.4. The momentum map $\mathbf{j}:P\to(0,\infty)\times\mathbb{R}$ associated with the canonical T-action described in Sect. 5 is given by $\mathbf{j}(R\hat{m})=(\|m\|,m_3)$.

The space $(P,\omega,G,\mathbf{J})$ is integrable (in the sense of 7.4). We now describe the construction of the integrable G-space normal form for $(P,\omega,G,\mathbf{J})$. First we note that

$$U\equiv\mathbf{j}(P)=\{(p_1,p_2)\in(0,\infty)\times\mathbb{R}\mid\ |p_2|<p_1\}$$

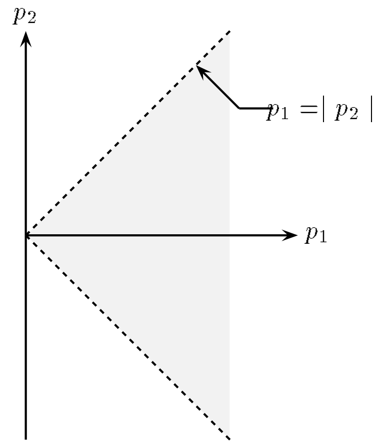

Figure 2: The image U of $\mathbf{j} : P \to \mathfrak{t}^*$ (shaded) for the axisymmetric Euler-Poinsot rigid body

(see Fig. 2), and that the fibres of $\mathbf{j}$ are connected. It is not too difficult to see that a section $\hat{s} : U \to P$ for $\mathbf{j} : P \to U$ is given by

$$\begin{aligned} \hat{s}(p_1, p_2) &\equiv R\hat{m} \\ \text{where} \quad R &= \mathrm{id}_{3\times 3} \\ \text{and} \quad m &\equiv (0, \sqrt{p_1^2 - p_2^2}, p_2) \ . \end{aligned}$$

It is trivial to check that $\hat{s}^*\Theta = 0$, where Θ is the 1-form defined in 9.4. It follows immediately that $\hat{s}$ is isotropic. According to 8.9a we may apply Theorem 8.8 as follows: define $\phi : G \times U \to P$ by

$$\begin{aligned} \phi((B, \theta), (p_1, p_2)) &\equiv (B, \theta) \cdot \hat{s}(p_1, p_2) \\ &= (BR_\theta)(R_{-\theta} m)^\wedge \\ \text{where} \quad m &\equiv (0, \sqrt{p_1^2 - p_2^2}, p_2) \ . \end{aligned}$$

Then ϕ is a G-equivariant symplectic diffeomorphism from the normal form $(G \times U, \omega_N, G, \mathbf{J}^N)$ (an open subset of the G-space described concretely in 9.2), and the G-space $(P, \omega, G, \mathbf{J})$, describing an axisymmetric ($I_1 = I_2$) Euler-Poinsot rigid body. Furthermore we have $\mathbf{J} \circ \phi = \mathbf{J}^N$ and the Hamiltonian $\phi^* H$ pulled back to $G \times U$ is given by $\phi^* H(g, p) = h(p)$, where

$$h(p_1, p_2) \equiv \frac{1}{2I} p_1^2 + (\frac{1}{I_3} - \frac{1}{I}) p_2^2 \ .$$

$\square$

For the construction of classical action-angle coordinates for the axisymmetric rigid body, see e.g. Tantalo (1994) or Hanßmann (1995).

A Proof details

Proof of 8.2: Define $\xi \in \mathfrak{g}$ and $\tau \in \mathfrak{t}^*$ as in the statement of the lemma. It suffices to verify that $(\xi, \tau)_N \lrcorner\, \omega_N = df$, where ω_N is as defined in 8.1. We have for all $x \in \mathfrak{g}$, $y \in \mathfrak{t}^*$, $g \in G$

and $p \in \mathfrak{t}_0^*$

$$\begin{aligned}
&\langle (\xi,\tau)_N \lrcorner\, \omega_N, (x,y)_N(g,p)\rangle = \langle y, \sigma\xi\rangle - \langle \tau, \sigma x\rangle + \langle p, \sigma[\xi,x]\rangle \\
&= \langle i\sigma^* \frac{\partial f}{\partial g}(g,p), \sigma x\rangle + \langle y, \frac{\partial f}{\partial p}(g,p)\rangle + \langle i^{-1}(p), [\Lambda_p \frac{\partial f}{\partial g}(g,p), x] + [\frac{\partial f}{\partial p}(g,p), x]\rangle \\
&= \langle \frac{\partial f}{\partial g}(g,p), \sigma x\rangle + \langle y, \frac{\partial f}{\partial p}(g,p)\rangle + \langle i^{-1}(p), \Lambda_p \frac{\partial f}{\partial g}(g,p), x]\rangle \ . \qquad (32)
\end{aligned}$$

We have used the fact $\langle i^{-1}(p), [\frac{\partial f}{\partial p}(g,p), x]\rangle = 0$, which is true because $i^{-1}(p) \in \mathrm{Ann}\ [\mathfrak{t}, \mathfrak{g}]$ and $\frac{\partial f}{\partial p}(g,p) \in \mathfrak{t}$. Applying 3.17,

$$\begin{aligned}
\langle i^{-1}(p), [\Lambda_p \frac{\partial f}{\partial g}(g,p), x]\rangle &= \langle \lambda_p^{-1} \Lambda_p \frac{\partial f}{\partial g}(g,p), x\rangle \\
&= \langle (\mathrm{id} - \sigma^*) \frac{\partial f}{\partial g}(g,p), x\rangle \\
&= \langle \frac{\partial f}{\partial g}(g,p), (\mathrm{id} - \sigma)x\rangle \ ,
\end{aligned}$$

so that (32) becomes

$$\begin{aligned}
\langle (\xi,\tau)_N \lrcorner\, \omega_N, (x,y)_N(g,p)\rangle &= \langle \frac{\partial f}{\partial g}(g,p), x\rangle + \langle y, \frac{\partial f}{\partial p}(g,p)\rangle \\
&= \langle df, (x,y)_N(g,p)\rangle \ .
\end{aligned}$$

□

PROOF OF 8.6: Put $U \equiv \mathbf{J}^F(F)$. Since T acts freely, the map $\mathbf{J}^F : F \to U$ is a surjective submersion. Since this map is also proper, it can be shown that $\mathbf{J}^F : F \to U$ is fibrating, i.e. it is a locally trivial fibre bundle (see e.g. Bates and Śniatycki (1992)). We show that the fibres of $\mathbf{J}^F$ must be connected. To do this construct a covering space $\tilde{\mathbf{J}}^F : \tilde{U} \to U$ as follows: For each $p \in U$ let Q_p denote the set of connected components of $(\mathbf{J}^F)^{-1}(p)$, and for each $x \in F$ let $[x]$ denote the element of $Q_{\mathbf{J}^F(x)}$ that contains x. Define $\tilde{U} \equiv \cup_{p \in U} Q_p$ and $\tilde{\mathbf{J}}^F : \tilde{U} \to U$ by $\tilde{\mathbf{J}}^F([x]) \equiv \mathbf{J}^F(x)$. Using the local triviality of $\mathbf{J}^F : F \to U$, it is not difficult to show that $\tilde{U}$ admits the structure of a (smooth) connected manifold with respect to which $\tilde{\mathbf{J}}^F : \tilde{U} \to U$ is a (smooth) covering map. The multiplicity (number of sheets) of this cover is the number of connected components in each fibre $(\mathbf{J}^F)^{-1}(p)$. But since $U \subset \mathfrak{t}_0^*$ is convex, it is simply connected; from covering space theory it follows that $\tilde{\mathbf{J}}^F : \tilde{U} \to U$ is a diffeomorphism. The multiplicity of the cover is therefore one, proving that the fibres of $\mathbf{J}^F$ are connected.

The next step is to construct a section for $\mathbf{J}^F : F \to U$. Equip the bundle $\mathbf{J}^F : F \to U$ with an Ehresmann connection (see e.g. Kobayashi and Nomizu (1963)). This is always possible by giving F the structure of a (smooth) Riemannian manifold. That one can do so in the C^∞ category is a standard partition of unity argument. In the real-analytic category this follows from the Whitney-Morrey-Grauert Embedding Theorem (Grauert, 1952). Let x_0 be any point of F, and define a smooth section $s : U \to F$ as follows: For any point $p \in U$, let $\gamma : [0,1] \to U$ denote the 'straight line' path joining $p_0 \equiv \mathbf{J}^F(x_0)$ to p, i.e. $\gamma(t) \equiv p_0 + t(p - p_0)$. This path is well-defined since we assume U is convex. Use the connection to lift γ to a path $\tilde{\gamma} : [0,1] \to F$ with $\tilde{\gamma}(0) = x_0$, and define $s(p) \equiv \tilde{\gamma}(1)$.

We now use the section $s : U \to F$ to show that the fibre bundle $\mathbf{J}^F : F \to U$ is globally trivial. By the integrability assumption, each fibre of the bundle is a submanifold of dimension $\dim T$, and a union of T orbits ($\mathbf{J}^F$ is T-equivariant). By the connectedness proven above, each fibre is therefore a *single* T-orbit. In other words $U \cong F/T$ and $\mathbf{J}^F : F \to U \cong F/T$ is just the natural projection. Since s is a smooth section, the map $\phi : T \times U \to F$ defined by $\phi(\tau, p) \equiv \tau \cdot s(p)$ is a diffeomorphism (remember T acts freely).

Since $\mathbf{J}^F : F \to U$ is globally trivial, F (smoothly) deformation retracts onto some fibre T'. The fibres are just T-orbits, which are Lagrangian since T is Abelian (applying the formula in 4.12*c*). Because of the deformation retraction, the inclusion $i : T' \to F$ induces an isomorphism in de Rham cohomology. Since T' is Lagrangian $i^*\omega_F = 0$, implying that the cohomology class of ω_F vanishes, i.e. ω_F is exact. Applying B.1 from Appendix B, we can modify $s : U \to F$ to obtain a Lagrangian section $\tilde{s} : U \to F$. Proposition 8.6 now follows from 8.5.

□

B On constructing Lagrangian sections

Here we recall how to modify an arbitrary section of an integrable T-space's momentum map to obtain a Lagrangian section.

B.1 **Proposition** *Let $(F, \omega, T, \mathbf{J}^F)$ be an integrable T-space in which T acts freely. Put $U \equiv \mathbf{J}^F(F)$. Assume that the symplectic form is exact: $\omega = -d\Theta$. Then if $\mathbf{J}^F : F \to U$ admits a section $s : U \to F$, then it admits a Lagrangian section $\tilde{s} : U \to F$.*

The following proof comes from Sect. 44 of Guillemin and Sternberg (1984b). All we have done here is translate this proof into the language of integrable T-spaces.

Proof: By integrability and momentum equivariance each fibre of $\mathbf{J}^F : F \to U$ is a submanifold whose connected components are T-orbits. Let α be a one-form on U. Then with each $p \in U \subset \mathfrak{t}^*$, α associates a Lie algebra element $\hat{\alpha}(p) \in \mathfrak{t}$ defined by

$$\langle \alpha, \nu \rangle \equiv \langle \nu, \hat{\alpha}(p) \rangle$$

for all $\nu \in \mathrm{T}_p\mathfrak{g}^* \cong \mathfrak{g}^*$. We may associate with the one-form α a vector field X^α on F, tangent to the fibres of $\mathbf{J}^F$ (i.e. tangent to the T-orbits), defined by

$$X^\alpha(x) \equiv (\hat{\alpha}(\mathbf{J}^F(x)))_F(x) \ ,$$

where $(\hat{\alpha}(\mathbf{J}^F(x)))_F$ is the infinitessimal generator of $\hat{\alpha}(\mathbf{J}^F(x)) \in \mathfrak{t}$. Since X^α is tangent to the fibres of $\mathbf{J}^F : F \to U$, whose connected components are compact, we have a well-defined flow associated with X^α, which we denote by $(x, t) \mapsto \alpha_t(x) : F \times (-\infty, \infty) \to F$. The 'time-one map' α_1 is a diffeomorphism of F preserving the fibres of $\mathbf{J}^F : F \to U$. In general α_1 is not symplectic. The idea is to define $\tilde{s} \equiv \alpha_1 \circ s$ and see if α can be chosen such that $\tilde{s}^*\omega = 0$. We have

$$
\begin{aligned}
\tilde{s}^*\omega &= s^*\alpha_1^*\omega \\
&= s^*\omega + s^*\int_0^1 (\frac{d}{dt}\alpha_t^*\omega)dt \\
&= s^*\omega + s^*\int_0^1 (\alpha_t^*(X^\alpha \lrcorner\, d\omega) + \alpha_t^* d(X^\alpha \lrcorner\, \omega))dt \\
&= s^*\omega + \int_0^1 s^*\alpha_t^* d(X^\alpha \lrcorner\, \omega)dt \\
&= s^*\omega + \int_0^1 d(s^*\alpha_t^*(X^\alpha \lrcorner\, \omega))dt \quad . \qquad (33)
\end{aligned}
$$

For any $u \in \mathrm{T}_p U \cong \mathfrak{t}^*$ we have

$$
\begin{aligned}
\langle s^*\alpha_t^*(X^\alpha \lrcorner\, \omega), u\rangle &= \langle X^\alpha \lrcorner\, \omega, \mathrm{T}(\alpha_t \circ s)\cdot u\rangle \\
&= \langle dJ^F_{\hat{\alpha}(p)}, \mathrm{T}(\alpha_t \circ s)\cdot u\rangle \\
&= \frac{d}{d\tau} J^F_{\hat{\alpha}(p)}((\alpha_t \circ s)(p+\tau u))\Big|_{\tau=0} \\
&= \frac{d}{d\tau}\langle(\mathbf{J}^F \circ \alpha_t \circ s)(p+\tau u), \hat{\alpha}(p)\rangle\Big|_{\tau=0} \\
&= \langle \alpha, \frac{d}{d\tau}(p+\tau u)\Big|_{\tau=0}\rangle \\
&= \langle \alpha, u\rangle \quad .
\end{aligned}
$$

Therefore $s^*\alpha_t^*(X^\alpha \lrcorner\, \omega) = \alpha$. Using this in (33) gives

$$\tilde{s}^*\omega = s^*\omega + d\alpha \quad . \qquad (34)$$

Choosing $\alpha = s^*\Theta$ gives $\tilde{s}^*\omega = 0$ as required.
□

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