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"Generalized ℓ_2 Synthesis"

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Abstract

A framework for optimal controller design with generalized l_2 objectives is presented. The allowable disturbances are constrained to be in a pre-specified set; the design objective is to ensure that the resulting output errors do not belong to another pre-specified set. The solution takes the form of an affine matrix inequality (AMI), which is both a necessary and sufficient condition for the posed problem to have a solution. In the simplest case, the resulting optimization reduces to standard $\mathcal{H}_\infty$ synthesis.

1 Introduction

In the standard $\mathcal{H}_\infty$ paradigm, the allowable disturbance class consists of arbitrary unit l_2 norm signals, while the design objective is to ensure that all output errors have l_2 norm less than one. In this paper, the allowable disturbance class and the design objectives are generalized to encompass a wide class of optimization problems. The underlying signal space is still taken to be l_2; as opposed to standard $\mathcal{H}_\infty$ synthesis, however, the allowable disturbance set and performance objective are general functions of the various inner products of the input and output variables. For example, denoting d as the exogenous disturbance and z as the output error, a specific choice is

$$\sum \langle d_k, d_k \rangle \leq 1, \ \ \sum \langle z_l, z_l \rangle < 1 \tag{1}$$

which leads to $\mathcal{H}_\infty$ optimization. Using the tools developed in this paper, other criteria such as

$$\langle d_k, d_k \rangle \leq 1 \ \forall k \tag{2}$$

$$\begin{bmatrix} \langle d_1, d_1 \rangle & \langle d_1, d_2 \rangle \\ \langle d_2, d_1 \rangle & \langle d_2, d_2 \rangle \end{bmatrix} - I \leq 0 \tag{3}$$

$$\textstyle\sum_l \sqrt{\langle z_l, z_l \rangle} < 1 \tag{4}$$

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and many others may be utilized. The motivation for considering these more general sets are twofold. The first is that many interesting problems may be cast in this framework; in Section 6, for example, the problem of controller synthesis for systems subject to a certain class of structured uncertainty is solved using the theory developed in this paper. The second is that these results extend the boundary for which convex synthesis methods yield globally optimal solutions.

The paper is organized as follows: after some mathematical preliminaries, the problem formulation is outlined. An analysis condition is then derived, which takes the form of an operator inequality. Using this condition, a method for constructing controllers which meet the performance objectives is presented, which takes the form of an affine matrix inequality (AMI). An example is then presented which makes use of the machinery developed in this paper.

2 Preliminaries

Most of the notation in this paper is standard. We restrict ourselves to discrete time systems, although most of the results in this paper extend to continuous time systems. The unit delay operator is denoted $\boldsymbol{\lambda}$. For $T \in \mathbb{Z}$, the truncation operator $\boldsymbol{P}_T$ is defined as

$$\boldsymbol{P}_T(d) = \begin{cases} d(t) & t \leq T \\ 0 & t > T \end{cases} \tag{5}$$

The Hilbert space of square summable sequences is denoted l_2; when the spatial structure is relevant, it is referred to as l_2^p. The inner product is denoted $\langle \cdot, \cdot \rangle$, while the norm is denoted $||\cdot||$. The induced l_2 norm of a bounded operator $\mathbf{A}$ over l_2 is denoted $||\mathbf{A}||$. The linear fractional transformation (LFT) between two operators $\mathbf{A}$ and $\mathbf{B}$ is denoted $\mathbf{A} \star \mathbf{B}$, and is defined as:

$$\mathbf{A} \star \mathbf{B} := \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{B}(\mathbf{I} - \mathbf{A}_{22}\mathbf{B})^{-1}\mathbf{A}_{21} \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

when the inverse of $(\mathbf{I} - \mathbf{A}_{22}\mathbf{B})$ is well defined. Given operator $\mathbf{A}$, $\mathbf{A} > 0 (\geq 0)$ denotes property $\langle d, \mathbf{A}d \rangle > 0 (\geq 0)\ \forall d \in l_2, ||d|| \neq 0$. The *adjoint* of $\mathbf{A}$ is denoted $\mathbf{A}^*$, and satisfies $\langle e, \mathbf{A}d \rangle = \langle \mathbf{A}^* e, d \rangle\ \forall d, e \in l_2$. The term *system* will be used to denote causal, finite dimensional, linear, time invariant operators over l_2. A system $\mathbf{G}$ is *stable* if it is bounded.
Given two subsets of l_2, S_1 and S_2, the *maximum distance* between S_1 and S_2 is defined as

$$\mathrm{d}\,(S_1, S_2) := \max\left(\sup_{s_1 \in S_1} \inf_{s_2 \in S_2} ||s_1 - s_2||, \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} ||s_2 - s_1|| \right) \tag{7}$$

The space of $m \times m$ symmetric matrices is denoted $\mathbb{R}_{\mathrm{S}}^{m \times m}$; $\mathbb{R}_{\mathrm{P}}^{m \times m}$ denotes the space of positive semi-definite symmetric matrices. Given two sets V and W, the *complement* of W in V is defined as $\{v \in V : v \notin W\}$ and is denoted $V^- W$; when V is clear from context, it is denoted $^- W$.

3 Problem Formulation

In order to state the problem, some definitions need to be introduced. For matrices $A \in \mathbb{R}^{pm \times pm}$ and $B \in \mathbb{R}^{m \times m}$, the *Trace Product* $C \in \mathbb{R}^{p \times p}$ of A and B is defined as

$$C =: A \textcircled{\scriptsize T} B \tag{8}$$

$$C_{[i,j]} := \textbf{trace}\left(A^{i,j} B\right) \tag{9}$$

$$A =: \begin{bmatrix} A^{1,1} & \dots & A^{1,p} \\ & \vdots & \\ A^{p,1} & \dots & A^{p,p} \end{bmatrix} \tag{10}$$

Thus C is a square matrix, each of whose elements is a linear combination of the elements of B. Given $d \in l_2^m$, define

$$\Lambda(d) := \sum_{t=-\infty}^{\infty} d(t)d(t)^* \in \mathbb{R}_{\mathbf{P}}^{m \times m} \tag{11}$$

Define the following sets for all $\epsilon \geq 0$:

$$\mathfrak{D}^\epsilon := \left\{\Lambda \in \mathbb{R}_{\mathbf{P}}^{m \times m} : D_k \textcircled{\scriptsize T} \Lambda - M_k \leq \epsilon I,\ 0 \leq k \leq C_{\mathrm{d}}\right\} \tag{12}$$

$$\mathfrak{E}^\epsilon := \left\{\Lambda \in \mathbb{R}_{\mathbf{P}}^{p \times p} : E_l \textcircled{\scriptsize T} \Lambda - P_l \leq \epsilon I,\ 0 \leq l \leq C_{\mathrm{e}}\right\} \tag{13}$$

$$\mathcal{D}^\epsilon := \{d \in l_2^m : \Lambda(d) \in \mathfrak{D}^\epsilon\} \tag{14}$$

$$\mathcal{E}^\epsilon := \{e \in l_2^p : \Lambda(e) \in \mathfrak{E}^\epsilon\} \tag{15}$$

where $M_k \in \mathbb{R}_{\mathbf{S}}^{m_k \times m_k}$, $P_l \in \mathbb{R}_{\mathbf{S}}^{p_l \times p_l}$, $m_k, p_l \in \mathbb{Z}^+$, and $D_k \in \mathbb{R}^{m_k m \times m_k m}$, $E_l \in \mathbb{R}^{p_l p \times p_l p}$. Denote $\mathfrak{D} := \mathfrak{D}^0$, $\mathfrak{E} := \mathfrak{E}^0$, $\mathcal{D} := \mathcal{D}^0$, $\mathcal{E} := \mathcal{E}^0$. It will be assumed that $0 < M_0, P_0 \in \mathbb{R}$ with $D_0 = I \in \mathbb{R}^{m \times m}$, $E_0 = I \in \mathbb{R}^{p \times p}$. This imposes constraints $\|d\|^2 \leq M_0$, $\|e\|^2 \leq P_0$, and ensures that sets $\mathcal{D}$ and $\mathcal{E}$ are bounded. It will also be assumed that $\mathfrak{D}$ and $\mathfrak{E}$ (and hence $\mathcal{D}$ and $\mathcal{E}$) are not empty sets.

Remarks: Since only the symmetric portions of $D_k \textcircled{\scriptsize T} \Lambda$ and $E_k \textcircled{\scriptsize T} \Lambda$ are required in constraints (12) and (13), it can be assumed that $D_k^{i,j} = D_k^{j,i}$, $E_l^{i,j} = E_l^{j,i}$. Furthermore, since $\Lambda(d)$ and $\Lambda(e)$ are symmetric, $D_k^{i,j}$ and $E_l^{i,j}$ can be assumed to be symmetric as well.

Consider the feedback interconnection of systems $\mathbf{G}$ and $\mathbf{K}$ in Figure 1. The closed loop map from d to z is $\mathbf{M} := \mathbf{G} \star \mathbf{K}$. $\mathbf{K}$ will be referred to as a *stabilizing controller* if the closed loop map of Figure 1 is internally stable [Zhou et al., 1995]; this corresponds to requiring that that the map from d, and signals injected anywhere in the loop, to z, y, and u be bounded and causal. The problem formulation is as follows:

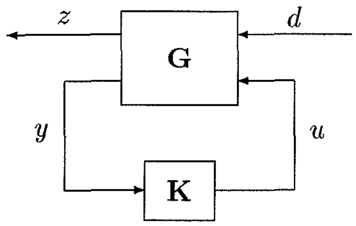

Figure 1: Synthesis Formulation

Generalized l_2 Synthesis

Given system $\mathbf{G}$ *and sets* $\mathcal{D}$ *and* $\mathcal{E}$*, find a stabilizing controller* $\mathbf{K}$ *such that*

$$\sup_{e\in\mathcal{E}}\ \sup_{d\in\mathcal{D}}\langle e, \mathbf{M}d\rangle < 1 \tag{16}$$

The term *Generalized* l_2 *Synthesis* stems from the fact that sets $\mathcal{D}$ and $\mathcal{E}$ which define the allowable disturbances and the cost criterion are not restricted to be balls in l_2. Standard $\mathcal{H}_\infty$ synthesis is a special case, with $\mathcal{D} = \left\{d \in l_2^m : \|d\|^2 \leq 1\right\}$ and $\mathcal{E} = \left\{e \in l_2^p : \|e\|^2 \leq 1\right\}$, which can readily be captured by $\mathfrak{D}$ and $\mathfrak{E}$.

Sets $\mathfrak{D}$ and $\mathfrak{E}$ are, in general, convex sets which can be used to constrain $\Lambda(d)$ and $\Lambda(e)$, respectively. In particular, each constraint in equations (12) and (13) is an AMI of variable dimension. As an illustrative example, consider the following sets in l_2^2

$$\begin{aligned} \mathcal{D}_1 &:= \left\{d \in l_2^2 : \Lambda(d) \leq I\right\} \\ \mathcal{D}_2 &:= \left\{d \in l_2^2 : \|d_1\|^2 \leq 1,\ \ \|d_2\|^2 \leq 1\right\} \end{aligned} \tag{17}$$

which can readily be captured in the format of equations (14) and (12). $\mathcal{D}_1 \subset \mathcal{D}_2$ since $\mathcal{D}_1$ possesses an additional constraint on $\langle d_1, d_2\rangle$. This is depicted in Figure 2. Thus for $\mathcal{D}_1$, signals which have unit energy content must be orthogonal to each other; adding constraint $\langle d_1, d_2\rangle = 0$ to $\mathcal{D}_2$ imposes an orthogonality constraint irrespective of the signal energy.

Matrix valued constraints, such as the one used to describe $\mathcal{D}_1$, appear naturally when dealing with certain types of uncertainty. For example, it is shown in [Paganini et al., 1994] that given $d_1, d_2 \in l_2^d$, there exists operator $\boldsymbol{\delta}$, $\|\boldsymbol{\delta}\| \leq 1$ such that $d_1 = \boldsymbol{\delta} I d_2$ if and only if $\Lambda(d_1) - \Lambda(d_2) \leq 0$. The framework in this paper and these types of constraints are utilized in [D'Andrea, 1996b] to extend the parameter varying synthesis results in [Packard, 1994] and [Apkarian and Gahinet, 1995] to systems subject to structured uncertainty.

Remarks:

- In general, any convex set in Λ may be approximated to any desired accuracy with the constraints of equation (12). This can, in fact, be achieved with only scalar valued

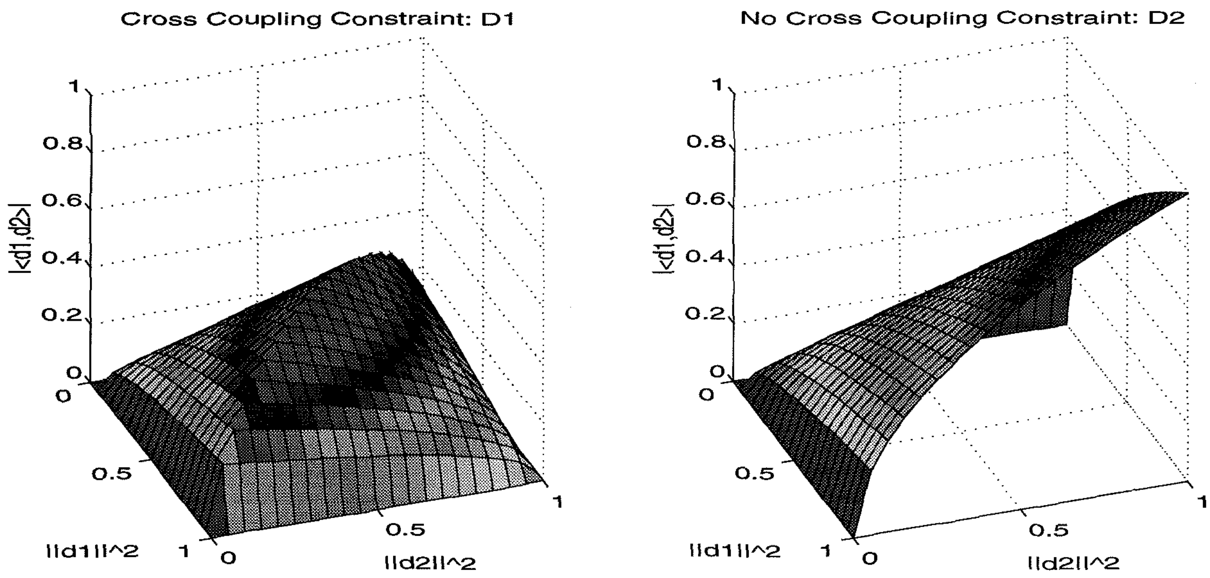

Figure 2: $\mathcal{D}_1$ and $\mathcal{D}_2$. Only the surface of each set is shown; all points below the surface are allowable.

constraints; the matrix valued constraints offer flexibility, and extend the class of convex sets which my be described without error ($\Lambda(d) \leq I$, for example).

- Note that 0 does not have to be an element of $\mathfrak{D}$ or of $\mathfrak{E}$, and consequently of $\mathcal{D}$ or $\mathcal{E}$; equivalently, matrices M_k and P_l are not restricted to be positive semi-definite. This allows us to consider very general convex sets, at the expense of complicating some of the proofs which follow.

The following Theorem states that sets $\mathcal{D}^\epsilon$ and $\mathcal{E}^\epsilon$ are in some sense continuous as a function of ϵ:

Theorem 1

$$\mathrm{d}\,(\mathcal{D}^\epsilon, \mathcal{D}) \stackrel{\epsilon \to 0}{\longrightarrow} 0, \quad \mathrm{d}\,(\mathcal{E}^\epsilon, \mathcal{E}) \stackrel{\epsilon \to 0}{\longrightarrow} 0 \tag{18}$$

The proof may be found in the Appendix. The following corollary follows immediately:

Corollary 1 *Given bounded* $\mathbf{M}$, *there exists* $\epsilon > 0$ *such that*

$$\sup_{e \in \mathcal{E}} \sup_{d \in \mathcal{D}} \langle e, \mathbf{M}d \rangle < 1 \;\; \Rightarrow \sup_{e \in \mathcal{E}^\epsilon} \sup_{d \in \mathcal{D}^\epsilon} \langle e, \mathbf{M}d \rangle < 1 \tag{19}$$

Thus for small enough ϵ, sets $\mathcal{D}$ and $\mathcal{E}$ are interchangeable with $\mathcal{D}^\epsilon$ and $\mathcal{E}^\epsilon$.

The allowable disturbances are directly specified by $\mathcal{D}$. The cost criterion, however, is only indirectly specified by $\mathcal{E}$. Define the following sets:

$$\mathcal{Z} := \left\{ z \in l_2^p : \sup_{e \in \mathcal{E}} \langle e, z \rangle < 1 \right\} \tag{20}$$

$$\mathfrak{Z} := \left\{ \Lambda \in \mathbb{R}_{\mathrm{P}}^{p \times p} : \Lambda = \Lambda(z),\ z \in \mathcal{Z} \right\} \tag{21}$$

Set $\mathfrak{Z}$ is not convex, but its complement is:

Lemma 1 *$^-\mathfrak{Z}$, the complement of $\mathfrak{Z}$ in $\mathbb{R}_{\mathrm{P}}^{p \times p}$, is closed and convex.*

Proof: Similar to the proof of Theorem 1, it can be shown that $\mathfrak{Z}$ is open, thus $^-\mathfrak{Z}$ is closed. To prove convexity, let $0 \leq \alpha \leq 1$ be given, and let $\Lambda_1, \Lambda_2 \in {}^-\mathfrak{Z}$. Define $\Lambda_0 := \alpha\Lambda_1 + (1-\alpha)\Lambda_2$. Let $\{z_1^k\}, \{z_2^k\} \in {}^-\mathcal{Z}$ be sequences such that $\Lambda(z_i^k) \overset{k \longrightarrow \infty}{\longrightarrow} \Lambda_i$. Note that for all $e_1, e_2 \in l_2$, and fixed k,

$$\begin{aligned} \langle \sqrt{\alpha} e_1 + \sqrt{1-\alpha}\boldsymbol{\lambda}^{\tau} e_2, \sqrt{\alpha} z_1^k + \sqrt{1-\alpha}\boldsymbol{\lambda}^{\tau} z_2^k \rangle &\overset{\tau \longrightarrow \infty}{\longrightarrow} \alpha\langle e_1, z_1^k\rangle + (1-\alpha)\langle e_2, z_2^k\rangle & (22) \\ \Lambda\left(\sqrt{\alpha} e_1 + \sqrt{1-\alpha}\boldsymbol{\lambda}^{\tau} e_2\right) &\overset{\tau \longrightarrow \infty}{\longrightarrow} \alpha\Lambda(e_1) + (1-\alpha)\Lambda(e_2) & (23) \end{aligned}$$

Thus by the continuity property of $\mathcal{E}$ established in Theorem 1 and the convexity of $\mathfrak{E}$, for fixed k,

$$\sup_{e \in \mathcal{E}} \langle e, \sqrt{\alpha} z_1^k + \sqrt{1-\alpha}\boldsymbol{\lambda}^{\tau} z_2^k \rangle - \left(\alpha \sup_{e \in \mathcal{E}} \langle e, z_1^k \rangle + (1-\alpha) \sup_{e \in \mathcal{E}} \langle e, z_2^k \rangle \right) \overset{\tau \longrightarrow \infty}{\longrightarrow} [0, \infty) \tag{24}$$

Thus there exists τ_k sufficiently large such that

$$\begin{aligned} z_k &:= (1 + 1/k)(\sqrt{\alpha} z_1^k + \sqrt{1-\alpha}\boldsymbol{\lambda}^{\tau_k} z_2^k) \in {}^-\mathcal{Z} & (25) \\ \overline{\sigma}\left(\Lambda(z_k) - \Lambda_0\right) &\leq \frac{1}{k}\overline{\sigma}\left(\Lambda_0\right) + 2\overline{\sigma}\left(\Lambda_1 - \Lambda(z_1^k)\right) + 2\overline{\sigma}\left(\Lambda_2 - \Lambda(z_2^k)\right) & (26) \end{aligned}$$

which implies that $\Lambda_0 \in {}^-\mathfrak{Z}$, as required. ■

The above result establishes a necessary condition for a given cost criterion to be compatible with the Generalized l_2 Synthesis formulation. For example, cost criterion $\|z\|^2 < 1$ is not incompatible since $\|z\|^2 \geq 1$ is a convex set in $\Lambda(z)$ (and can in fact be implemented by setting $\mathcal{E} = \{e \in l_2 : \|e\|^2 \leq 1\}$). Cost criterion $\|z_1\|^2 < 1$, $\|z_2\|^2 < 1$ is incompatible, since the set $\{\Lambda(z) : \|z_1\|^2 \geq 1\} \cup \{\Lambda(z) : \|z_2\|^2 \geq 1\}$ is not convex, as shown in Figure 3.

With this insight, a natural way to describe the optimization of equation (16) is in terms of a game, with the adversary's task to find d, with $\Lambda(d)$ in convex set $\mathfrak{D}$, such that $\Lambda(z) = \Lambda(Md)$ is in convex set $^-\mathfrak{Z}$ (modulo supremum arguments).

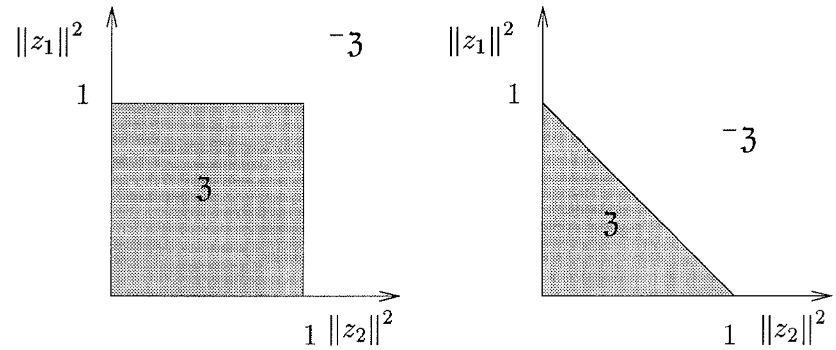

Figure 3: Cost criterion $\|z_1\|^2 < 1$, $\|z_2\|^2 < 1$ is incompatible, cost criterion $\|z\|^2 < 1$ is not incompatible.

4 Analysis Condition

The first step in providing a solution to the Generalized l_2 Synthesis problem is to obtain an analysis condition for the closed loop system, $\mathbf{M} = \mathbf{G} \star \mathbf{K}$. The main idea is the so called *S-procedure* [Yakubovich, 1971], the process of transforming a problem to one involving multipliers; motivated by the results in [Megretski and Treil, 1993, Paganini, 1995], it can be shown that the S-procedure applied to our problem formulation is lossless, or non-conservative. In order to state the analysis condition, the following notation needs to be introduced. Let $A \in \mathbb{R}^{pm \times pm}$ be given, with p and m fixed. The *Trace Transpose* $\grave{A} \in \mathbb{R}^{mp \times mp}$ of A is defined as

$$\grave{A} := \begin{bmatrix} \grave{A}^{1,1} & \cdots & \grave{A}^{1,m} \\ & \vdots & \\ \grave{A}^{m,1} & \cdots & \grave{A}^{m,m} \end{bmatrix} \quad (27)$$

$$\grave{A}^{k,l}_{[i,j]} := A^{i,j}_{[k,l]} \quad (28)$$

Theorem 2 *Given linear, time invariant, bounded operator* $\mathbf{M}$ *and sets* $\mathcal{D}$ *and* $\mathcal{E}$*, the following are equivalent:*

I. *The following supremum is satisfied:*

$$\sup_{e \in \mathcal{E}} \sup_{d \in \mathcal{D}} \langle e, \mathbf{M}d \rangle < 1 \quad (29)$$

II. *There exist* $0 < X_k \in \mathbb{R}_{\mathbf{P}}^{m_k \times m_k}, 0 \le k \le C_{\rm d}$ *and* $0 < Y_l \in \mathbb{R}_{\mathbf{P}}^{p_l \times p_l}, 0 \le l \le C_{\rm e}$ *such that*

$$\left\| Y^{-\frac{1}{2}} \mathbf{M} X^{-\frac{1}{2}} \right\| \quad < \quad 1 \tag{30}$$

$$X := \sum_{k=0}^{C_{\rm d}} \grave{D}_k \circledT X_k \quad > \quad 0 \tag{31}$$

$$Y := \sum_{l=0}^{C_{\rm e}} \grave{E}_l \circledT Y_l \quad > \quad 0 \tag{32}$$

$$T_x := \sum_{k=0}^{C_{\rm d}} \mathbf{trace}\,(M_k X_k) \quad < \quad 1 \tag{33}$$

$$T_y := \sum_{l=0}^{C_{\rm e}} \mathbf{trace}\,(P_l Y_l) \quad < \quad 1 \tag{34}$$

Before proving the above theorem, the following preliminary results need to be established.

Proposition 1 *Let* $A \in \mathbb{R}^{pm \times pm}$, $B \in \mathbb{R}^{m \times m}$, *and* $X \in \mathbb{R}^{p \times p}$ *be given. Then*

$$\mathbf{trace}\,((A \circledT B) X) = \mathbf{trace}\left((\grave{A} \circledT X) B\right) \tag{35}$$

Proof (Proposition):

$$\mathbf{trace}\,((A \circledT B) X) \quad = \quad \sum_{i,j=1}^{p} (A \circledT B)_{[i,j]} X_{[j,i]} = \sum_{i,j=1}^{p} X_{[j,i]} \sum_{k,l=1}^{m} A_{[k,l]}^{i,j} B_{[l,k]}, \tag{36}$$

$$\mathbf{trace}\left((\grave{A} \circledT X) B\right) \quad = \quad \sum_{k,l=1}^{m} (\grave{A} \circledT X)_{[k,l]} B_{[l,k]} = \sum_{k,l=1}^{m} B_{[l,k]} \sum_{i,j=1}^{p} \grave{A}_{[i,j]}^{k,l} X_{[j,i]} \tag{37}$$

■

The following lemma is a standard result in convex analysis:

Lemma 2 *[Rockafellar, 1970] Let* $\mathcal{K}_1$, $\mathcal{K}_2$ *be disjoint, convex sets in* $\mathbb{R}^d$, *where* $\mathcal{K}_1$ *is compact and* $\mathcal{K}_2$ *is closed. Then there exists vector* $x \in \mathbb{R}^d$ *and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\langle x, k_1 \rangle \le \alpha < \beta \le \langle x, k_2 \rangle \;\; \forall k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2 \tag{38}$$

Proof (Theorem):
II $\Rightarrow$ **I** : By Proposition 1 and equation (14), for all $d \in \mathcal{D}$,

$$\begin{aligned} \langle d, Xd \rangle &= \langle d, \left(\sum_{k=0}^{C_{\mathrm{d}}} \grave{D}_k \circledT X_k \right) d \rangle = \mathbf{trace} \left(\sum_{k=0}^{C_{\mathrm{d}}} \left(\grave{D}_k \circledT X_k \right) \Lambda(d) \right) && (39) \\ &= \sum_{k=0}^{C_{\mathrm{d}}} \mathbf{trace} \left(\left(D_k \circledT \Lambda(d) \right) X_k \right) && (40) \\ &\leq \sum_{k=0}^{C_{\mathrm{d}}} \mathbf{trace} \left(M_k X_k \right) < 1 && (41) \end{aligned}$$

Similarly, $\langle e, Ye \rangle < 1$ for all $e \in \mathcal{E}$. It thus follows that

$$\begin{aligned} 1 &> \left\| Y^{-\frac{1}{2}} \mathbf{M} X^{-\frac{1}{2}} \right\| && (42) \\ &= \sup_{\|\bar{e}\|^2 \leq 1} \sup_{\|\bar{d}\|^2 \leq 1} \langle Y^{-\frac{1}{2}} \bar{e}, \mathbf{M} X^{-\frac{1}{2}} \bar{d} \rangle = \sup_{\langle e, Ye \rangle \leq 1} \sup_{\langle d, Xd \rangle \leq 1} \langle e, \mathbf{M} d \rangle \\ &> \sup_{e \in \mathcal{E}} \sup_{d \in \mathcal{D}} \langle e, \mathbf{M} d \rangle && (43) \end{aligned}$$

I $\Rightarrow$ **II** : The proof of this claim is long and technical. The proof essentially consists of three parts. The first is to use use the separating plane argument of Lemma 2 to construct scaling matrices. In the second part, various arguments are employed to construct scale Y which satisfies the constraints of equations (32) and (34); most of the complications in this section arise from allowing matrices M_k and P_l to be arbitrary symmetric matrices, not necessarily positive semi-definite. In the third part, the problem data is manipulated so that the first two parts can be utilized to construct scale X which satisfies the constraints of equations (31) and (33), and yields the condition of equation (30). It will be assumed throughout the proof that $\mathbf{M} \neq \mathbf{0}$; if $\mathbf{M} = \mathbf{0}$, the proof is trivial.

Step 1: Separating Plane Argument

By the boundedness of $\mathcal{E}$ and Corollary 1, there exists $0 < \beta < \min(P_0^{-1}, 1/4)$ and $\epsilon > 0$ such that

$$\sup_{e \in \mathcal{E}^\epsilon} \sup_{d \in \mathcal{D}^\epsilon} \langle e, \mathbf{M} d \rangle + \beta + \beta^2 \|e\|^2 < 1 \tag{44}$$

Define the following matrix valued functions on l_2:

$$\begin{aligned} \sigma(d, e) &:= \langle e, \mathbf{M} d \rangle + \beta + \beta^2 \|e\|^2 - 1 && (45) \\ \Sigma_k^{\mathcal{D}}(d, e) &:= M_k \langle e, \mathbf{M} d \rangle - D_k \circledT \Lambda(d), \;\; 0 \leq k \leq C_{\mathrm{d}} && (46) \\ \Sigma_l^{\mathcal{E}}(e) &:= P_l - E_l \circledT \Lambda(e), \;\; 0 \leq l \leq C_{\mathrm{e}} && (47) \end{aligned}$$

the following constants,

$$B_e := max\left(\frac{1}{\beta^2}, \frac{P_0}{\beta^2}\right), \quad B_d := 2M_0 \sup_{||d||^2 \le M_0} \sup_{||e||^2 \le B_e} |\langle e, \mathbf{M}d\rangle|^2 \tag{48}$$

and corresponding bounded set:

$$\nabla := \left\{\Upsilon(d,e) = \left(\sigma(d,e), \Sigma_k^{\mathcal{D}}(d,e), \Sigma_l^{\mathcal{E}}(e)\right) : ||e||^2 \le B_e, \ \ ||d||^2 \le B_d\right\} \tag{49}$$

Define the following closed, convex set:

$$\mathbf{Z} := \left\{Z = \left(z, Z_k^{\mathcal{D}}, Z_l^{\mathcal{E}}\right) : z \in \mathbb{R}^+, Z_k^{\mathcal{D}} \in \mathbb{R}_{\mathbf{P}}^{m_k \times m_k}, Z_l^{\mathcal{E}} \in \mathbb{R}_{\mathbf{P}}^{p_l \times p_l}\right\} \tag{50}$$

Thus ∇ and $\mathbf{Z}$ live in the same finite dimensional real vector space. Equip this space with the following inner product:

$$\langle Z, \hat{Z}\rangle := z\hat{z} + \sum_{k=0}^{C_d} \mathbf{trace}\left(Z_k^{\mathcal{D}} \hat{Z}_k^{\mathcal{D}}\right) + \sum_{l=0}^{C_e} \mathbf{trace}\left(Z_l^{\mathcal{E}} \hat{Z}_l^{\mathcal{E}}\right) \tag{51}$$

Lemma 3 $\nabla \bigcap \mathbf{Z}$ *is the empty set.*

Proof (Lemma): Assume that there exists d, e such that equations (45,46,47) are all positive semi-definite. The constraints of equation (47) imply that $e \in \mathcal{E}$, and in particular, that $||e||^2 \le P_0$. Furthermore, by the upper bound on β, $\gamma := \langle e, \mathbf{M}d\rangle > 1/2$. Define $\bar{d} := \gamma^{-\frac{1}{2}} d$; constraint $k = 0$ in equation (46) implies that $||\bar{d}||^2 \le M_0$, ensuring that $\Upsilon(\bar{d}, e) \in \nabla$. By similar substitution, $\gamma M_k - \gamma D_k \circledT \Lambda(\bar{d}) \ge 0$, implying that $\bar{d} \in \mathcal{D}$, and $\langle e, \mathbf{M}\bar{d}\rangle = \sqrt{\gamma}$. If $\gamma \ge 1$, equation (44) is not satisfied. If $\gamma < 1$, $\sqrt{\gamma} > \gamma$, and equation (44) is not satisfied as well.
■

Corollary 2 *Let* $\overline{\nabla}$ *denote the closure of* ∇. $\overline{\nabla} \cap \mathbf{Z}$ *is the empty set.*

Proof (Corollary): Assume that $\overline{\nabla} \cap \mathbf{Z}$ is not the empty set. Thus for all $\epsilon > 0$, from the proof of Lemma 3,

$$\sup_{e \in \mathcal{E}^\epsilon} \sup_{d \in \mathcal{D}^\epsilon} \langle e, \mathbf{M}d\rangle + \beta + \beta^2 ||e||^2 + \epsilon \ge 1 \tag{52}$$

which contradicts equation (44). ■

Lemma 4 $\overline{\nabla}$ *is convex and compact.*

Proof (Lemma): The proof is essentially from [Paganini, 1995]. Let $\Upsilon_0 \in \mathrm{co}(\nabla)$, the convex hull of ∇. Thus

$$\Upsilon_0 = \sum_{k=0}^{N-1} \alpha_k \Upsilon\left(d_k, e_k\right), \alpha_k \ge 0, \sum_{k=0}^{N-1} \alpha_k = 1 \tag{53}$$

Define

$$f^\tau := \sum_{k=0}^{N-1} \sqrt{\alpha_k} \boldsymbol{\lambda}^{k\tau} d_k, \quad g^\tau := \sum_{k=0}^{N-1} \sqrt{\alpha_k} \boldsymbol{\lambda}^{k\tau} e_k \tag{54}$$

$$S_d := \sum_{k=0}^{N-1} \alpha_k \|d_k\|^2 \leq B_{\rm d}, \quad S_e := \sum_{k=0}^{N-1} \alpha_k \|e_k\|^2 \leq B_{\rm e} \tag{55}$$

$$d^\tau := \begin{cases} \frac{S_d}{\|f^\tau\|} f^\tau & \text{for } S_d > 0, \|f^\tau\| > 0 \\ 0 & \text{for } S_d \|f^\tau\| = 0 \end{cases}, \quad e^\tau := \begin{cases} \frac{S_e}{\|g^\tau\|} g^\tau & \text{for } S_e > 0, \|g^\tau\| > 0 \\ 0 & \text{for } S_e \|g^\tau\| = 0 \end{cases} \tag{56}$$

Thus $\Upsilon(d^\tau, e^\tau) \in \nabla \forall \tau$. Since $\mathbf{M}$ is linear, time invariant and $\Upsilon(\cdot)$ is quadratic in d and e, it follows that

$$\Upsilon(d^\tau, e^\tau) \overset{\tau\to\infty}{\longrightarrow} \sum_{k=0}^{N} \Upsilon(\sqrt{\alpha_k} d_k, \sqrt{\alpha_k} e_k) = \Upsilon_0 \tag{57}$$

The above argument demonstrates that $\Upsilon_0 \in \overline{\nabla}$. Thus $\text{co}(\nabla) \subset \overline{\nabla}$, and $\text{co}(\overline{\nabla}) \subset \overline{\text{co}(\nabla)} \subset \overline{\nabla}$, so $\overline{\nabla}$ is convex. ■

We are now in a position to invoke Lemma 2:

Proposition 2 *There exists $\hat{Z} \in \mathbf{Z}$, with $\hat{z} > 0$, $\hat{Z}_k^{\mathcal{D}} > 0$, $\hat{Z}_l^{\mathcal{E}} > 0$ such that for all $\Upsilon \in \overline{\nabla}$, $Z \in \mathbf{Z}$,*

$$\langle \hat{Z}, \Upsilon \rangle < 0 \leq \langle \hat{Z}, Z \rangle \tag{58}$$

Proof (Proposition): By Lemma 2, and by embedding $\overline{\nabla}$ and $\mathbf{Z}$ in $\mathbb{R}^d$, it follows that there exists α, β, and $\bar{Z}$ (not necessarily in $\mathbf{Z}$), such that for all $\Upsilon \in \overline{\nabla}$, $Z \in \mathbf{Z}$,

$$\langle \bar{Z}, \Upsilon \rangle \leq \alpha < \beta \leq \langle \bar{Z}, Z \rangle \tag{59}$$

Setting $Z = 0$ yields $\beta \leq 0$. Since $\mathbf{Z}$ is unbounded and $\overline{\nabla}$ is compact, however, $\beta = 0$, and each (matrix) element of $\bar{Z} \geq 0$. Since $\alpha < 0$ and $\overline{\nabla}$ is compact, a sufficiently small positive element can be added to each element of $\bar{Z}$, thus defining $\hat{Z}$ and yielding the required result. ■

Step 2: Constructing Y

Since $\hat{z}$ is positive, it can be assumed, without loss of generality, that $\hat{z} = 1$. Set $X_k := \hat{Z}_k^{\mathcal{D}}$, $Y_l := \hat{Z}_l^{\mathcal{E}}$. Denote pair (d, e) as *allowable* if $\|d\|^2 \leq B_{\rm d}$, $\|e\|^2 \leq B_{\rm e}$. Thus for all allowable

(d, e):

$$\begin{aligned}\langle e, \mathbf{M}d\rangle \leq \quad & 1 - \beta - \beta^2\|e\|^2 + X_0\|d\|^2 + Y_0\|e\|^2 + \langle d, \bar{X}d\rangle + \langle e, \bar{Y}e\rangle \\ & - \left(\bar{T}_x + X_0 M_0\right)\langle e, \mathbf{M}d\rangle - \left(\bar{T}_y + Y_0 P_0\right) \qquad (60)\\ \bar{X} \;:=\; & \sum_{k=1}^{C_\mathrm{d}} \grave{D}_k \circledT x_k, \quad \bar{Y} := \sum_{l=1}^{C_\mathrm{e}} \grave{E}_l \circledT y_l \qquad (61)\\ \bar{T}_x \;:=\; & \sum_{k=1}^{C_\mathrm{d}} \mathbf{trace}\,(M_k X_k), \quad \bar{T}_y := \sum_{l=1}^{C_\mathrm{e}} \mathbf{trace}\,(P_l Y_l) \qquad (62)\end{aligned}$$

Note that $\bar{T}_x$ and $\bar{T}_y$ do not necessarily have to be positive, since M_k and P_l are assumed to be arbitrary, symmetric matrices. Define

$$\hat{x} \;:=\; \sup\{X_0 :\ \text{equation (60) satisfied for some } Y_0 > 0\} \qquad (63)$$
$$\hat{y} \;:=\; \sup\{Y_0 :\ \text{equation (60) satisfied with } X_0 = \hat{x}\} \qquad (64)$$

Since Y_0 is uniformly bounded for all X_0 by considering $d = 0$, $e = 0$, the second supremum is finite. Since one can always find allowable d and e such that $\|d\|^2 - M_0\langle e, Md\rangle$ is negative, and Y_0 is bounded, the first supremum is finite as well. From the above construction, X_0 cannot be larger than $\hat{x}$ (with all other scales except Y_0 fixed), and Y_0 cannot be larger than $\hat{y}$ with all scales fixed and $X_0 = \hat{x}$. This leads to the following proposition:

Proposition 3 *Let $X_0 = \hat{x}$, $Y_0 = \hat{y}$, denote the left hand size of equation (60) by LHS, the right hand side by RHS. Then*

$$\sup_{\|e\|^2 \leq P_0} \ \sup_{\|d\|^2 \leq B_\mathrm{d}} \ LHS(d, e) - RHS(d, e) \;=\; 0 \qquad (65)$$
$$\sup_{\|e\|^2 \leq B_\mathrm{e}} \ \sup_{\|d\|^2 \leq B_\mathrm{d}/2} \ LHS(d, e) - RHS(d, e) \;=\; 0 \qquad (66)$$

Proof (Proposition): By the construction of $\hat{x}$ and $\hat{y}$, the above suprema must be less than or equal to 0. If the first supremum is negative, Y_0 can be increased and still satisfy equation (60), a contradiction. By the definition of B_d in equation (48),

$$\sup_{\|d\|^2 \leq B_\mathrm{d}} \ \sup_{\|e\|^2 \leq B_\mathrm{e}} \ M_0\langle e, \mathbf{M}d\rangle \leq \frac{B_\mathrm{d}}{\sqrt{2}} \qquad (67)$$

Thus for all $B_\mathrm{d}/2 \leq \|d\|^2 \leq B_\mathrm{d}$ and $\|e\|^2 \leq B_\mathrm{e}$, $\|d\|^2 - M_0\langle e, \mathbf{M}d\rangle \geq 0$. It thus follows that if the supremum in equation (66) is negative, X_0 can be increased and still satisfy equation (60), a contradiction as well. ∎

Lemma 5

$$T_y := \bar{T}_y + \hat{y}P_0 = 1 - \beta \qquad (68)$$

Proof (Lemma): Let $X_0 = \hat{x}$, $Y_0 = \hat{y}$. There exists an allowable sequence (d_k, e_k) such that

$$LHS(d_k, e_k) - RHS(d_k, e_k) \stackrel{k\to\infty}{\longrightarrow} 0 \tag{69}$$

By Proposition 3, $||d_k||^2$ can uniformly be taken to be strictly less than B_{d}, or $||e_k||^2$ can uniformly be taken to be strictly less than B_{e}. Similar to the proof of Lemma 4, however, by appropriately shifting signals in time these two cases can be combined to construct a sequence (d_k, e_k) satisfying equation (69) such that both $||d_k||^2$ and $||e_k||^2$ are strictly, uniformly bounded by B_{d} and B_{e}, respectively. It thus follows that there exists $\epsilon > 0$ such that $(1+\epsilon)(d_k, e_k)$ is allowable. Then

$$LHS\left((1+\epsilon)(d_k, e_k)\right) - RHS\left((1+\epsilon)(d_k, e_k)\right) \stackrel{k_i\to\infty}{\longrightarrow} \epsilon^2(1-\beta-T_y) \tag{70}$$

which implies that $1-\beta-T_y \le 0$. By setting $e=0$, $d=0$ in equation (60), however, $1-\beta-T_y \ge 0$, yielding the required result. ■

Define $X := \bar{X} + \hat{x}I,\ \ Y := \bar{Y} + \hat{y}I,\ \ T_x := \bar{T}_x + \hat{x}M_0$.

Lemma 6 $Y \ge \beta^2$, $X \ge 0$, $T_x \ge 0$.

Proof (Lemma): From equation (60), for negative values of $\langle e, \mathbf{M}d\rangle$

$$T_x \ge \frac{-\beta^2||e||^2 + \langle e, Ye\rangle + \langle d, Xd\rangle}{\langle e, \mathbf{M}d\rangle} - 1 \tag{71}$$

If $X \not\ge 0$, one can find a sequence (d_k, e_k) such that the numerator in equation (71) is bounded above by some negative value while the denominator is made arbitrarily small in magnitude. This would imply that T_x is unbounded. A similar argument holds for $Y \not\ge \beta^2$.
Finally, for all $d \in \mathcal{D}$, $\langle d, Xd\rangle \le T_x$, which by the positivity of X implies that $T_x \ge 0$. ■

Lemma 7

$$\sup_{||\bar{e}||^2\le 1}\ \sup_{d\in\mathcal{D}} \langle \bar{e}, Y^{-\frac{1}{2}}\mathbf{M}d\rangle < 1 \tag{72}$$

Proof (Lemma): Define $\bar{e} := Y^{\frac{1}{2}}e$. Thus for all $||\bar{e}||^2 = 1$

$$\langle \bar{e}, Y^{-\frac{1}{2}}\mathbf{M}d\rangle \le 1 - \epsilon + \langle d, Xd\rangle - T_x\langle \bar{e}, Y^{-\frac{1}{2}}\mathbf{M}d\rangle \tag{73}$$

where $\epsilon > 0$, and $||e||^2 \le \frac{1}{\beta^2}$ and is thus allowable. It thus follows that for all $d \in \mathcal{D}$, $||\bar{e}||^2 = 1$,

$$\langle \bar{e}, Y^{-\frac{1}{2}}\mathbf{M}d\rangle \le \frac{1 + T_x - \epsilon}{1 + T_x} < 1 \tag{74}$$

which yields the required result. ■

Step 3: Constructing X

It has been shown that

$$\sup_{e\in\mathcal{E}}\sup_{d\in\mathcal{D}}\langle e, \mathbf{M}d\rangle < 1 \implies \sup_{||e||^2\le 1}\sup_{d\in\mathcal{D}}\langle e, Y^{-\frac{1}{2}}\mathbf{M}d\rangle < 1 \tag{75}$$

where Y satisfies equation (32) and associated T_y satisfies equation (34). By defining

$$\bar{\mathcal{E}} := \left\{ e \in l_2^p : ||e||^2 \le 1 \right\} \tag{76}$$

the condition of Lemma 7 is equivalent to

$$\sup_{d\in\mathcal{D}}\sup_{\bar{e}\in\bar{\mathcal{E}}}\langle d, \mathbf{M}^*Y^{-\frac{1}{2}}\bar{e}\rangle < 1 \tag{77}$$

Thus by holding Y constant, replacing $\mathcal{E}$ with $\mathcal{D}$, $\mathcal{D}$ with $\bar{\mathcal{E}}$, e with d, d with $\bar{e}$, $\mathbf{M}$ with $\mathbf{M}^*Y^{-\frac{1}{2}}$, and repeating Step 1 and Step 2, scale X satisfying equation (31) may be constructed, with the associated T_x satisfying equation (33), and

$$\sup_{||\bar{d}||^2\le 1}\sup_{||\bar{e}||^2\le 1}\langle \bar{d}, X^{-\frac{1}{2}}\mathbf{M}^*Y^{-\frac{1}{2}}\bar{e}\rangle < 1 \tag{78}$$

which implies equation (30). ■

5 Synthesis Condition

In this section, the full solution to the Generalized l_2 Synthesis problem is presented. The solution takes the form of an AMI. The following result is from [Packard, 1994]:

Theorem 3 *Let* $\left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right]$ *be a minimal state space description for system* $\bar{\mathbf{G}}$. *There exists a stabilizing controller* $\mathbf{K}$ *for* $\bar{\mathbf{G}}$ *such that* $\left\|\bar{\mathbf{G}} \star \mathbf{K}\right\| < 1$ *if and only if there exist positive definite matrices* S *and* T *such that*

$$\bar{V}\left(\bar{R}^*\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}\bar{R} - \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}\right)\bar{V}^* < 0 \tag{79}$$

$$\bar{U}^*\left(\bar{R}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\bar{R}^* - \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\right)\bar{U} < 0 \tag{80}$$

$$\begin{bmatrix} S & I \\ I & T \end{bmatrix} \ge 0, \tag{81}$$

where

$$\bar{R} = \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_{11} \end{bmatrix}$$

$$\begin{aligned} \bar{V} = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 \end{bmatrix} &: \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \\ \bar{V}_1 & \bar{V}_2 \end{bmatrix} \text{invertible}, \quad \bar{C}_2\bar{V}_1^* + \bar{D}_{21}\bar{V}_2^* = 0 \\ \bar{U} = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix} &: \begin{bmatrix} \bar{B}_2 & \bar{U}_1 \\ \bar{D}_{12} & \bar{U}_2 \end{bmatrix} \text{invertible}, \quad \bar{U}_1^*\bar{B}_2 + \bar{U}_2^*\bar{D}_{12} = 0 \end{aligned} \tag{82}$$

Remarks: As discussed in [Packard, 1994], it can be assumed without loss of generality that $\begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$ is full row rank; thus there always exists a $\bar{V}$ satisfying equation (82). Similarly for $\bar{U}$.

We are now in a position to state and prove the main result of this paper.

Theorem 4 *Let* $\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$ *be a minimal state space representation for* **G**. *There exists a* **K** *which solves the Generalized* l_2 *Synthesis problem if and only if there exist scales* X *and* Y *satisfying the conditions of Theorem 2, and positive definite matrices* S, T, $\bar{X}$ *and* $\bar{Y}$, *such that*

$$V\left(R^*\begin{bmatrix} S & 0 \\ 0 & \bar{Y} \end{bmatrix} R - \begin{bmatrix} S & 0 \\ 0 & X \end{bmatrix}\right) V^* < 0 \tag{83}$$

$$U^*\left(R\begin{bmatrix} T & 0 \\ 0 & \bar{X} \end{bmatrix} R^* - \begin{bmatrix} T & 0 \\ 0 & Y \end{bmatrix}\right) U < 0 \tag{84}$$

$$\begin{bmatrix} S & I \\ I & T \end{bmatrix} \geq 0, \begin{bmatrix} X & I \\ I & \bar{X} \end{bmatrix} \geq 0, \begin{bmatrix} Y & I \\ I & \bar{Y} \end{bmatrix} \geq 0 \tag{85}$$

where

$$R = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$$

$$\begin{aligned} V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} &: \begin{bmatrix} C_2 & D_{21} \\ V_1 & V_2 \end{bmatrix} \text{invertible}, \quad C_2V_1^* + D_{21}V_2^* = 0 \\ U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} &: \begin{bmatrix} B_2 & U_1 \\ D_{12} & U_2 \end{bmatrix} \text{invertible}, \quad U_1^*B_2 + U_2^*D_{12} = 0 \end{aligned} \tag{86}$$

Proof: In the previous section, an analysis condition was derived which involved scales X and Y. If **M** is a bounded system and scales X and Y are fixed, equation (30) reduces to a standard $\mathcal{H}_\infty$ optimization problem, and results in the following corollary to Theorem 2:

Corollary 3 *The Generalized l_2 Synthesis problem is solvable if and only if there exist scales X and Y satisfying the conditions of Theorem 2 and a stabilizing controller* $\mathbf{K}$ *for* $\mathbf{G}$ *such that*

$$\left\| Y^{-\frac{1}{2}} (\mathbf{G} \star \mathbf{K}) X^{-\frac{1}{2}} \right\| < 1 \tag{87}$$

Define $\bar{\mathbf{G}} := \begin{bmatrix} Y^{-\frac{1}{2}} & \\ & I \end{bmatrix} \mathbf{G} \begin{bmatrix} X^{-\frac{1}{2}} & \\ & I \end{bmatrix}$. Then equation (87) is equivalent to $\|\bar{\mathbf{G}} \star \mathbf{K}\| < 1$. Furthermore,

$$\left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] := \left[\begin{array}{c|cc} A & B_1 X^{-\frac{1}{2}} & B_2 \\ \hline Y^{-\frac{1}{2}} C_1 & Y^{-\frac{1}{2}} D_{11} X^{-\frac{1}{2}} & Y^{-\frac{1}{2}} D_{12} \\ C_2 & D_{21} X^{-\frac{1}{2}} & D_{22} \end{array}\right] \tag{88}$$

is a minimal state space representation for $\bar{\mathbf{G}}$. Define $\tilde{X} := \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}$, $\tilde{Y} := \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix}$, and U, V, and R as in equation (86). Then $\bar{R} = \tilde{Y}^{-\frac{1}{2}} R \tilde{X}^{-\frac{1}{2}}$, and $\bar{V} := V \tilde{X}^{\frac{1}{2}}$, $\bar{U} := \tilde{Y}^{-\frac{1}{2}} U$ satisfy equations (82). Substituting into equations (79) and (80) yields

$$V \left(R^* \begin{bmatrix} S & 0 \\ 0 & Y^{-1} \end{bmatrix} R - \begin{bmatrix} S & 0 \\ 0 & X \end{bmatrix} \right) V^* < 0 \tag{89}$$

$$U^* \left(R \begin{bmatrix} T & 0 \\ 0 & X^{-1} \end{bmatrix} R^* - \begin{bmatrix} T & 0 \\ 0 & Y \end{bmatrix} \right) U < 0 \tag{90}$$

Finally, by Schur complement arguments [Zhou et al., 1995], if $\bar{X}$ satisfies matrix inequality $\begin{bmatrix} \bar{X} & I \\ I & X \end{bmatrix} \geq 0$, then $\bar{X} \geq X^{-1}$. Furthermore, $\bar{X} = X^{-1}$ satisfies the matrix inequality. This concludes the proof. ■

Remarks:

- A controller may be constructed as described in [Packard, 1994] using the state space description for $\bar{\mathbf{G}}$ (which includes scales X and Y) and scales S and T.
- The order of the resulting controller is less than or equal to the order of the plant. Thus the added complexity of sets $\mathcal{D}$ and $\mathcal{E}$ only manifests itself in the computation of the controller, not in the order of the controller itself. This is due to the fact that no dynamics are utilized when describing sets $\mathcal{D}$ and $\mathcal{E}$.
- The AMIs may be solved using standard convex optimization tools, such as The LMI Control Toolbox [Gahinet et al., 1994].

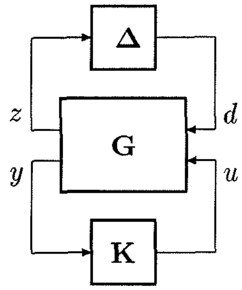

Figure 4: Synthesis for robust stability

6 Example

In this section, a class of synthesis problems where the plant is subject to structured uncertainty is solved. Consider the setup of Figure 4; variables z and d are partitioned into C_e and C_d components (not necessarily scalar valued). This partition induces a corresponding one for $\mathbf{\Delta}$:

$$d_k = \sum_{l=1}^{C_e} \mathbf{\Delta}_{kl} z_l, \quad 1 \leq k \leq C_d \tag{91}$$

For given C_e and C_d, the set $\mathbf{U\Delta}$ (where "U" symbolizes uncertainty) is defined as follows:

$$\mathbf{U\Delta} := \{\mathbf{\Delta} : \mathbf{\Delta} \text{ linear }, \|\mathbf{\Delta}_{kl}\| \leq 1\} \tag{92}$$

$\mathbf{K}$ is referred to as a *robustly stabilizing controller* for $\mathbf{G}$ and $\mathbf{U\Delta}$ if $\mathbf{K}$ internally stabilizes $\mathbf{G}$ and

$$\sup_{\mathbf{\Delta} \in \mathbf{U\Delta}} \left\| (\mathbf{I} - (\mathbf{G} \star \mathbf{K})\, \mathbf{\Delta})^{-1} \right\| < \infty \tag{93}$$

This condition establishes the stability and well-posedness of the closed loop system for all allowable uncertainty. If the condition above is satisfied, the closed loop system of Figure 4 is said to be *robustly stable.* Define

$$\mathcal{D} := \left\{ d \in l_2 : \|d_k\|^2 \leq 1, \;\; 1 \leq k \leq C_d \right\} \tag{94}$$

It is shown in [D'Andrea, 1995] that $\mathbf{K}$ is a robustly stabilizing controller if and only if $\mathbf{K}$ internally stabilizes $\mathbf{G}$ and

$$\sup_{d \in \mathcal{D}} \sum_{l=1}^{C_e} \|z_l\| < 1 \tag{95}$$

where $z = (\mathbf{G} \star \mathbf{K})\, d$. Note that $\left\{\Lambda(z) : \sum_{l=1}^{C_e} \|z_l\| \geq 1\right\}$ is a convex set. In fact, by defining

$$\mathcal{E} := \left\{e \in l_2 : \|e_l\|^2 \leq 1,\ \ 1 \leq l \leq C_e\right\} \tag{96}$$

the above reduces to a Generalized l_2 Synthesis problem; $\mathfrak{D}$ and $\mathfrak{E}$ can readily be constructed from the above $\mathcal{D}$ and $\mathcal{E}$.

The reader is referred to [D'Andrea, 1995] for an in depth treatment of this class of problems, and an exploration of the class of robust performance problems which are equivalent to the above robust stability condition.

7 Conclusions

The framework presented in this paper appears to be the natural extension to $\mathcal{H}_\infty$ optimization, in the sense that arbitrary convex sets are used to describe the allowable disturbances and the performance criterion. The resulting condition, an AMI, is both necessary and sufficient for the posed problem to have a solution. The example illustrated the strength of this approach. Since the solution is an AMI, other results which have AMI solutions can be combined with the Generalized l_2 Synthesis results to provide extremely powerful synthesis tools. For example, the gain scheduling results in [Packard, 1994] are extended in [D'Andrea, 1996b] to include the class of uncertainty presented in Section 6 of this paper.

For constraints of the form $\langle d_i, \boldsymbol{\Lambda}^T d_j\rangle = 0$ and the standard cost criterion $\|z\|^2 < 1$, a solution is provided in [D'Andrea, 1996a]. This allows one to impose correlation constraints on the disturbances, and can be shown to provide a solution to the mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ problem. A natural extension of this research is to combine it with the Generalized l_2 Synthesis results and allow dynamics in constraint sets $\mathcal{D}$ and $\mathcal{E}$.

Appendix

The following lemma states that $\mathfrak{D}^\epsilon$ is continuous in ϵ:

Lemma 8

$$\sup_{\Lambda_\epsilon \in \mathfrak{D}^\epsilon} \inf_{\Lambda \in \mathfrak{D}} \overline{\sigma}(\Lambda - \Lambda_\epsilon) \xrightarrow{\epsilon \to 0} 0 \tag{97}$$

Proof: Assume that equation (97) is not satisfied. There exists, therefore, a number $\delta > 0$ and a sequence $\left\{\Lambda^k\right\}$ with $\Lambda^k \in \mathfrak{D}^{\frac{1}{k}}$ such that

$$\inf_{\Lambda \in \mathfrak{D}} \overline{\sigma}\left(\Lambda - \Lambda^k\right) > \delta \ \ \forall k \tag{98}$$

Since $\mathfrak{D}^1$ is compact, there exists a subsequence of $\left\{\Lambda^k\right\}$ converging in $\mathfrak{D}^1$. Denote this limit by Λ_0. Since the trace product is a linear function of its argument (and is thus bounded), $\Lambda_0 \in \mathfrak{D}$. Substituting Λ_0 in equation (98) leads to a contradiction. ∎

Lemma 9 *$\exists C_0 > 0$ such that $\forall d \in l_2$ of compact support and $0 \le \tilde{\Lambda} \in \mathbb{R}_{\mathrm{P}}^{m\times m} \le C_1 I$ satisfying $\overline{\sigma}\left(\tilde{\Lambda} - \Lambda(d)\right) \le \delta^2 \le 1/2$, $\exists \tilde{d} \in l_2$ such that $\Lambda(\tilde{d}) = \tilde{\Lambda}$ and $\|d - \tilde{d}\|^2 \le C_0\delta$.*

Proof: Let $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} := T^*\tilde{\Lambda}T$, where T is unitary, $\overline{\sigma}(\Sigma_2) \le \delta$, $\underline{\sigma}(\Sigma_1) > \delta$. Define

$$\begin{aligned} V &:= T^*\Lambda(d)T \quad , \quad E := \Sigma - V \\ \bar{V} &:= \begin{bmatrix} (1-\delta)V_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \bar{E} := \Sigma - \bar{V} = \begin{bmatrix} (1-\delta)E_{11} + \delta\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \end{aligned} \tag{99}$$

where the partitions of V, E and $\bar{E}$ are consistent with Σ. Since T is unitary, $\overline{\sigma}(E_{11}) \le \delta^2$, and thus $0 \le \bar{E} \le \delta(C_1+1)$. Define $\bar{d} := T \begin{bmatrix} \sqrt{1-\delta}I & 0 \\ 0 & 0 \end{bmatrix} T^*d$. It follows that $\Lambda(\bar{d}) = T\bar{V}T^*$, and

$$\begin{aligned} \mathbf{trace}\left(\Lambda(d-\bar{d})\right) &= \left(1-\sqrt{1-\delta}\right)^2 \mathbf{trace}\left(\Sigma_1 - E_{11}\right) + \mathbf{trace}\left(\Sigma_2 - E_{22}\right) \\ &\le \left(1-\sqrt{1-\delta}\right)^2 \left(m\left(\overline{\sigma}(\Sigma_1) + \delta^2\right)\right) + m\left(\overline{\sigma}(\Sigma_2) + \delta^2\right) \\ &\le \delta m\left(C_1 + 3\right) \end{aligned} \tag{100}$$

Since $\bar{E} \ge 0$, there exists $Q \in \mathbb{R}_{\mathrm{P}}^{m\times m}$ such that $Q^2 = T\bar{E}T^*$. Define $\bar{\bar{d}} \in l_2^m$ as follows:

$$\begin{aligned} \begin{bmatrix} \bar{\bar{d}}(1) & \cdots & \bar{\bar{d}}(m) \end{bmatrix} &:= \begin{bmatrix} Q_1 & \cdots & Q_m \end{bmatrix} \\ \bar{\bar{d}}(t) &:= 0 \text{ otherwise} \end{aligned} \tag{101}$$

It thus follows that $\Lambda(\bar{\bar{d}}) = T\bar{E}T^*$. Finally, define $\tilde{d} := \bar{d} + \boldsymbol{\lambda}^\tau\bar{\bar{d}}$, where τ is any integer larger than the support of d. This yields $\Lambda(\tilde{d}) = \tilde{\Lambda}$. Furthermore,

$$\begin{aligned} \|d - \tilde{d}\|^2 &= \|d - \bar{d}\|^2 + \|\bar{\bar{d}}\|^2 \\ &\le \delta m\left(C_1 + 3\right) + \delta m\left(C_1 + 1\right) \end{aligned} \tag{102}$$

Defining $C_0 := m\left(2C_1 + 4\right)$ completes the proof. ■

We are now in a position to prove Theorem 1.

Proof of Theorem 1: Fix $0 < \delta < 1/2$. By Lemma 8, $\exists\epsilon > 0$ such that $\forall d_\epsilon \in \mathcal{D}^\epsilon$, $\exists\tilde{\Lambda} \in \mathfrak{D}$ such that $\overline{\sigma}\left(\tilde{\Lambda} - \Lambda(d_\epsilon)\right) \le \delta^2/2$. Furthermore, $\exists T \in \mathbb{Z}^+$ sufficiently large such that $\|d_\epsilon - \boldsymbol{P}_T d_\epsilon\|^2 \le \delta$ and $\overline{\sigma}\left(\tilde{\Lambda} - \Lambda(\boldsymbol{P}_T d_\epsilon)\right) \le \delta^2$. By Lemma 9, $\exists\tilde{d}$ such that $\Lambda(\tilde{d}) = \tilde{\Lambda}$, implying that $\tilde{d} \in \mathcal{D}$, and furthermore

$$\|d_\epsilon - \tilde{d}\|^2 \le \left(\|d_\epsilon - \boldsymbol{P}_T d_\epsilon\| + \|\boldsymbol{P}_T d_\epsilon - \tilde{d}\|\right)^2 \le 2C_0\delta \tag{103}$$

Since δ is arbitrary, this implies that $\mathrm{d}\left(\mathcal{D}^\epsilon, \mathcal{D}\right) \xrightarrow{\epsilon\to 0} 0$, as required. Similarly, $\mathrm{d}\left(\mathcal{E}^\epsilon, \mathcal{E}\right) \xrightarrow{\epsilon\to 0} 0$.
■

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