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"Necessary and Sufficient Conditions for Robust Gain Scheduling"

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Necessary and Sufficient Conditions for Robust Gain Scheduling

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Abstract

Recent results in the design of controllers for parameter dependent systems are extended to systems with plant uncertainty. The solution takes the form of an affine matrix inequality (AMI), which is both a necessary and sufficient condition for the posed problem to have a solution. The results in this paper may be used for the design of gain scheduled controllers for a class of uncertain systems.

1 Introduction

It has generally been accepted by the control community that accounting for a system's uncertainty is crucial in the design process [Doyle and Stein, 1981, Packard and Doyle, 1993]. This realization motivated much of the research activity in $\mathcal{H}_\infty$ theory in the last decade, culminating with the state space formulas in [Doyle et al., 1989], and more recently, the affine matrix inequality (AMI) approaches of [Scherer, 1992, Gahinet and Apkarian, 1994]. In [Packard, 1994, Apkarian and Gahinet, 1995], the AMI approach was used to extend $\mathcal{H}_\infty$ theory to the design of parameter varying controllers for parameter varying systems; these results allow one to design gain scheduled controllers which achieve guaranteed performance and stability objectives. One of the drawbacks of the theory, however, is that plant uncertainty cannot directly be incorporated in the design process.

In this paper, the results in [Packard, 1994] are extended to allow for structured uncertainty in the given system. In this formulation, the controller to be designed has access to the time varying parameters, but does not have access to the plant uncertainty. This is achieved by combining the parameter varying framework with the Generalized l_2 Synthesis framework in [D'Andrea, 1996a], and recent results in controller design for uncertain systems in [D'Andrea, 1995]. The solution takes the form of an AMI, which is both a necessary and sufficient condition for the posed problem to have a solution.

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The paper is organized as follows: after some mathematical preliminaries, the robust gain scheduling problem formulation is outlined. The Generalized l_2 Synthesis results in [D'Andrea, 1996a] are then reviewed. The robust gain scheduling problem is then cast in the Generalized l_2 Synthesis framework, and an analysis condition is derived. Using this condition, a method for constructing controllers which meet the performance objectives is presented, which takes the form of an AMI. An example is then presented which makes use of the machinery developed in this paper.

2 Preliminaries

Most of the notation in this paper is standard. We restrict ourselves to discrete time systems, although most of the results in this paper extend to continuous time systems. The unit delay operator is denoted $\boldsymbol{\lambda}$. The Hilbert space of square summable sequences is denoted l_2; when the spatial structure is relevant, it is referred to as l_2^p. The inner product is denoted $\langle \cdot, \cdot \rangle$, while the norm is denoted $||\cdot||$. The induced l_2 norm of a bounded operator $\mathbf{A}$ over l_2 is denoted $||\mathbf{A}||$. The linear fractional transformation (LFT) between two operators $\mathbf{A}$ and $\mathbf{B}$ is denoted $\mathbf{A} \star \mathbf{B}$, and is defined as:

$$\mathbf{A} \star \mathbf{B} := \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{B}(\mathbf{I} - \mathbf{A}_{22}\mathbf{B})^{-1}\mathbf{A}_{21} \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

when the inverse of $(\mathbf{I} - \mathbf{A}_{22}\mathbf{B})$ is well defined. Given operator $\mathbf{A}$, $\mathbf{A} > 0 (\geq 0)$ denotes property $\langle d, \mathbf{A}d \rangle > 0 (\geq 0)\ \forall d \in l_2, ||d|| \neq 0$. The *adjoint* of $\mathbf{A}$ is denoted $\mathbf{A}^*$, and satisfies $\langle e, \mathbf{A}d \rangle = \langle \mathbf{A}^* e, d \rangle\ \forall d, e \in l_2$. The term *system* will be used to denote causal, finite dimensional, linear, time invariant operators over l_2. A system $\mathbf{G}$ is *stable* if it is bounded.
Given two subsets of l_2, S_1 and S_2, the *maximum distance* between S_1 and S_2 is defined as

$$\mathrm{d}\left(S_1, S_2\right) := \max\left(\sup_{s_1 \in S_1} \inf_{s_2 \in S_2} ||s_1 - s_2||, \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} ||s_2 - s_1|| \right) \tag{2}$$

The space of $m \times m$ symmetric matrices is denoted $\mathbb{R}_{\mathrm{S}}^{m\times m}$; $\mathbb{R}_{\mathrm{P}}^{m \times m}$ denotes the space of positive semi-definite symmetric matrices. For all $\epsilon \geq 0$, $r \in \mathbb{R}$ is said to be $\mathrm{O}(\epsilon)$ if there exists C such that $|r| \leq C\epsilon$.

3 Robust Gain Scheduling problem formulation

Consider the block diagram of Figure 1. $\mathbf{G}_0$ is the given system, $\boldsymbol{\Delta}^{\mathbf{u}}$ is the plant uncertainty, and $\boldsymbol{\Delta}^{\mathbf{p}}$ is a structured operator (to be defined) which parametrizes the plant and which the controller $\mathbf{K}$ (to be designed) has access to.

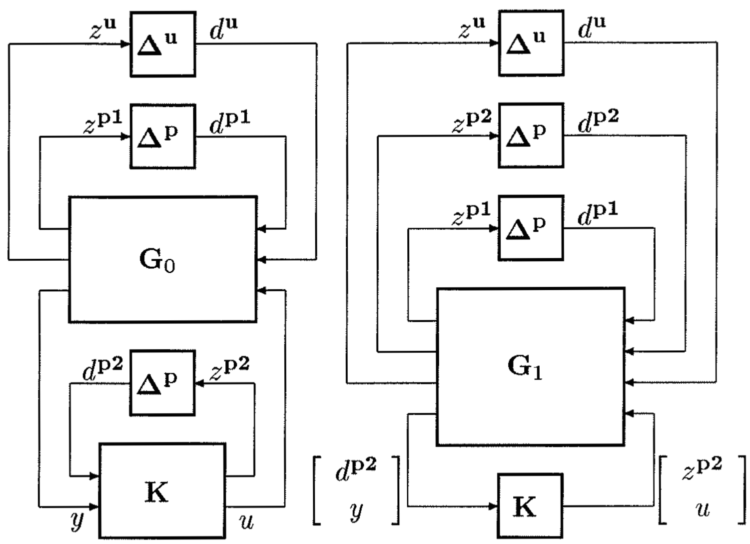


Figure 1: Gain Scheduling Design

Partition variables $z^{\mathbf{u}}$ and $d^{\mathbf{u}}$ into $C_{\mathbf{e}}^{\mathbf{u}}$ and $C_{\mathbf{d}}^{\mathbf{u}}$ components (not necessarily scalar valued). This partition induces a corresponding one for $\mathbf{\Delta}^{\mathbf{u}}$:

$$d_k^{\mathbf{u}} = \sum_{l=1}^{C_{\mathbf{e}}^{\mathbf{u}}} \mathbf{\Delta}_{kl}^{\mathbf{u}} z_l^{\mathbf{u}}, \quad 1 \leq k \leq C_{\mathbf{d}}^{\mathbf{u}} \tag{3}$$

The set $\mathbf{U\Delta}$ (where "U" symbolizes uncertainty) is defined as follows:

$$\mathbf{U\Delta} := \{\mathbf{\Delta}^{\mathbf{u}} : \mathbf{\Delta}^{\mathbf{u}} \text{ linear }, \|\mathbf{\Delta}_{kl}^{\mathbf{u}}\| \leq 1\} \tag{4}$$

The plant uncertainty is then assumed to be in set $\mathbf{U\Delta}$. The plant parameter $\mathbf{\Delta}^{\mathbf{p}}$ is assumed to be in set $\boldsymbol{P\Delta}$:

$$\boldsymbol{P\Delta} := \{\mathbf{diag}[\boldsymbol{\delta}_1 I, \cdots, \boldsymbol{\delta}_{C^{\mathbf{p}}} I] : \boldsymbol{\delta}_i \text{ linear }, \|\boldsymbol{\delta}_i\| \leq 1\} \tag{5}$$

where the identities above are of arbitrary, but fixed dimension. It is required to find system $\mathbf{K}$ such that the closed loop system is robustly stable. More precisely, construct system $\mathbf{G}_1$ from $\mathbf{G}_0$ such that the two closed loop system in Figure 1 are identical, ie.,

$$\mathbf{\Delta}^{\mathbf{u}} \star (\mathbf{\Delta}^{\mathbf{p}} \star \mathbf{G}_0) \star (\mathbf{\Delta}^{\mathbf{p}} \star \mathbf{K}) = \mathbf{\Delta}^{\mathbf{u}} \star (\mathbf{\Delta}^{\mathbf{p}} \star (\mathbf{\Delta}^{\mathbf{p}} \star (\mathbf{G}_1 \star \mathbf{K}))) \tag{6}$$

Define the following uncertainty set:

$$\boldsymbol{PP\Delta} := \{\mathbf{diag}[\boldsymbol{\delta}_1 I, \boldsymbol{\delta}_1 I, \cdots, \boldsymbol{\delta}_{C^{\mathbf{p}}} I, \boldsymbol{\delta}_{C^{\mathbf{p}}} I] : \boldsymbol{\delta}_i \text{ linear }, \|\boldsymbol{\delta}_i\| \leq 1\} \tag{7}$$

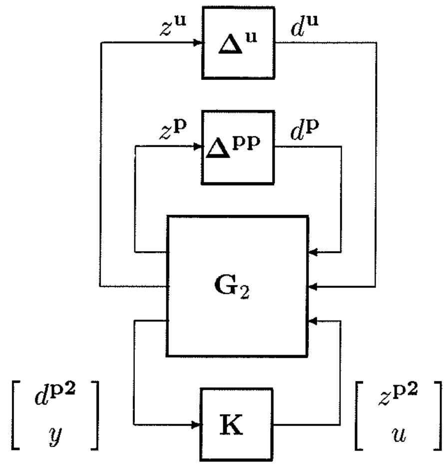


Figure 2: Equivalent System

Thus the multiplicity of each operator in $\boldsymbol{P\Delta}$ has been doubled in size. By rearranging $\mathbf{G}_1$, a new system $\mathbf{G}_2$ can be constructed such that the closed loop systems of Figure 1 and the one in Figure 2 are identical, ie.,

$$\mathbf{\Delta}^{\mathrm{u}} \star (\mathbf{\Delta}^{\mathrm{p}} \star (\mathbf{\Delta}^{\mathrm{p}} \star (\mathbf{G}_1 \star \mathbf{K}))) = \mathbf{\Delta}^{\mathrm{u}} \star (\mathbf{\Delta}^{\mathrm{pp}} \star (\mathbf{G}_2 \star \mathbf{K})) \tag{8}$$

The problem formulation is as follows:

Robust Gain Scheduling

Find system $\mathbf{K}$ *which internally stabilizes* $\mathbf{G}_2$ *and satisfies*

$$\sup_{\mathbf{\Delta}^{\mathrm{u}} \in \mathbf{U\Delta}} \ \sup_{\mathbf{\Delta}^{\mathrm{pp}} \in \boldsymbol{PP\Delta}} \left\| \left(\mathbf{I} - (\mathbf{G}_2 \star K)\, \mathbf{\Delta} \right)^{-1} \right\| < \infty \tag{9}$$

where $\mathbf{\Delta} := \mathbf{diag}[\mathbf{\Delta}^{\mathrm{pp}}, \mathbf{\Delta}^{\mathrm{u}}]$.

Remarks:

- In practice, the operators in set $\boldsymbol{P\Delta}$ will be time varying bounded real parameters, not arbitrary bounded operators on l_2. Thus the above condition may be conservative.
- Note that $\mathbf{G}_1$, and hence $\mathbf{G}_2$, is a highly structured system matrix; system $\mathbf{K}$ has full access to $\mathbf{\Delta}^{\mathrm{p}}$. This is the key fact utilized in [Packard, 1994] to solve the gain scheduling problem when $\mathbf{\Delta}^{\mathrm{u}}$ is unstructured, or one full block.
- Note that the multiplicity of each $\boldsymbol{\delta}_i$ in $\mathbf{\Delta}^{\mathrm{p}}$ which $\mathbf{K}$ has access to is assumed to be the same as that which affects system $\mathbf{G}_0$; it is conceivable that allowing more copies of each $\boldsymbol{\delta}_i$ might lead to better performance controllers. It has been shown in [Packard, 1994],

however, that one can always do as well with a duplicate copy of $\mathbf{\Delta^P}$ for the controller $\mathbf{K}$. This is analogous to standard $\mathcal{H}_\infty$ optimization (and is in fact intimately related), where the order of the controller can always be assumed to be equal to that of the plant.

- The above is a robust stability problem. As in standard μ theory [Packard and Doyle, 1993] however, many robust performance problems may be converted to robust stability problems; this is explored in Section 6.1.

4 Review of Generalized l_2 Synthesis

In this section, the problem formulation in [D'Andrea, 1996a] is outlined, and some of the main results reviewed.

In order to state the problem, some definitions need to be introduced. For matrices $A \in \mathbb{R}^{pm \times pm}$ and $B \in \mathbb{R}^{m \times m}$, the *Trace Product* $C \in \mathbb{R}^{p \times p}$ of A and B is defined as

$$\begin{aligned} C &=: A \textcircled{\scriptsize T} B && (10) \\ C_{[i,j]} &:= \mathbf{trace}\left(A^{i,j} B\right) && (11) \\ A &=: \begin{bmatrix} A^{1,1} & \dots & A^{1,p} \\ & \vdots & \\ A^{p,1} & \dots & A^{p,p} \end{bmatrix} && (12) \end{aligned}$$

Thus C is a square matrix, each of whose elements is a linear combination of the elements of B. Given $d \in l_2^m$, define

$$\Lambda(d) := \sum_{t=-\infty}^{\infty} d(t)d(t)^* \in \mathbb{R}_{\mathrm{P}}^{m \times m} \quad (13)$$

Define the following sets for all $\epsilon \geq 0$:

$$\begin{aligned} \mathcal{D}^\epsilon &:= \{d \in l_2^m : D_k \textcircled{\scriptsize T} \Lambda(d) - M_k \leq \epsilon I,\ \ 0 \leq k \leq C_{\mathrm{d}}\} && (14) \\ \mathcal{E}^\epsilon &:= \{e \in l_2^p : E_l \textcircled{\scriptsize T} \Lambda(e) - P_l \leq \epsilon I,\ \ 0 \leq l \leq C_{\mathrm{e}}\} && (15) \\ & && (16) \end{aligned}$$

where $M_k \in \mathbb{R}_{\mathrm{S}}^{m_k \times m_k}$, $P_l \in \mathbb{R}_{\mathrm{S}}^{p_l \times p_l}$, $m_k, p_l \in Z^+$, and $D_k \in \mathbb{R}^{m_k m \times m_k m}$, $E_l \in \mathbb{R}^{p_l m \times p_l p}$. Denote $\mathcal{D} := \mathcal{D}^0$, $\mathcal{E} := \mathcal{E}^0$. It will be assumed that $0 < M_0, P_0 \in \mathbb{R}$ with $D_0 = I \in \mathbb{R}^{m \times m}$, $E_0 = I \in \mathbb{R}^{p \times p}$. This imposes constraints $\|d\|^2 \leq M_0$, $\|e\|^2 \leq P_0$, and ensures that sets $\mathcal{D}$ and $\mathcal{E}$ are bounded. It will also be assumed that $\mathcal{D}$ and $\mathcal{E}$ are not empty sets.

Consider the feedback interconnection of systems $\mathbf{G}$ and $\mathbf{K}$ in Figure 3. The closed loop map from d to z is $\mathbf{M} := \mathbf{G} \star \mathbf{K}$. $\mathbf{K}$ will be referred to as a *stabilizing controller* if the closed loop map of Figure 3 is internally stable [Zhou et al., 1995]; this corresponds to requiring that that the map from d, and signals injected anywhere in the loop, to z, y, and u be bounded and causal. The problem formulation is as follows:

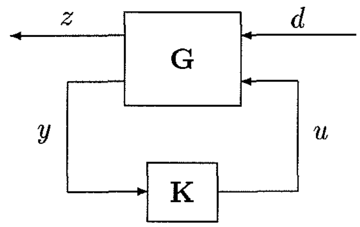


Figure 3: Synthesis Formulation

Generalized l_2 Synthesis [D'Andrea, 1996a]

Given system $\mathbf{G}$ *and sets* $\mathcal{D}$ *and* $\mathcal{E}$*, find a stabilizing controller* $\mathbf{K}$ *such that*

$$\sup_{e \in \mathcal{E}} \sup_{d \in \mathcal{D}} \langle e, \mathbf{M}d \rangle < 1 \tag{17}$$

The term *Generalized* l_2 *Synthesis* stems from the fact that sets $\mathcal{D}$ and $\mathcal{E}$ which define the allowable disturbances and the cost criterion are not restricted to be balls in l_2. Standard $\mathcal{H}_\infty$ synthesis is a special case, with $\mathcal{D} = \left\{ d \in l_2^m : \|d\|^2 \leq 1 \right\}$ and $\mathcal{E} = \left\{ e \in l_2^p : \|e\|^2 \leq 1 \right\}$. In [D'Andrea, 1996a], the issues of what types of constraints may be imposed via sets $\mathcal{D}$ and $\mathcal{E}$, and the types of optimization problems that ensue are explored.

The following Theorem states that sets $\mathcal{D}^\epsilon$ and $\mathcal{E}^\epsilon$ are in some sense continuous as a function of ϵ:

Theorem 1 *[D'Andrea, 1996a]*

$$\mathrm{d}\left(\mathcal{D}^\epsilon, \mathcal{D}\right) \overset{\epsilon \to 0}{\longrightarrow} 0, \quad \mathrm{d}\left(\mathcal{E}^\epsilon, \mathcal{E}\right) \overset{\epsilon \to 0}{\longrightarrow} 0 \tag{18}$$

Let $A \in \mathbb{R}^{pm \times pm}$ be given, with p and m fixed. The *Trace Transpose* $\grave{A} \in \mathbb{R}^{mp \times mp}$ of A is defined as

$$\grave{A} := \begin{bmatrix} \grave{A}^{1,1} & \dots & \grave{A}^{1,m} \\ & \vdots & \\ \grave{A}^{m,1} & \dots & \grave{A}^{m,m} \end{bmatrix} \tag{19}$$

$$\grave{A}^{k,l}_{[i,j]} := A^{i,j}_{[k,l]} \tag{20}$$

The following theorem provides an alternate condition for a controller $\mathbf{K}$ to solve the Generalized l_2 Synthesis problem:

Theorem 2 *[D'Andrea, 1996a] Given linear, time invariant, bounded operator* $\mathbf{M}$ *and sets* $\mathcal{D}$ *and* $\mathcal{E}$*, the following are equivalent:*

I. *The following supremum is satisfied:*

$$\sup_{e \in \mathcal{E}} \sup_{d \in \mathcal{D}} \langle e, \mathbf{M}d \rangle < 1 \tag{21}$$

II. *There exist* $0 < X_k \in \mathbb{R}_{\mathrm{P}}^{m_k \times m_k}, 0 \leq k \leq C_{\mathrm{d}}$ *and* $0 < Y_l \in \mathbb{R}_{\mathrm{P}}^{p_l \times p_l}, 0 \leq l \leq C_{\mathrm{e}}$ *such that*

$$\left\| Y^{-\frac{1}{2}} \mathbf{M} X^{-\frac{1}{2}} \right\| \;<\; 1 \tag{22}$$

$$X := \sum_{k=0}^{C_{\mathrm{d}}} \grave{D}_k \textcircled{\scriptsize T} X_k \;>\; 0 \tag{23}$$

$$Y := \sum_{l=0}^{C_{\mathrm{e}}} \grave{E}_l \textcircled{\scriptsize T} Y_l \;>\; 0 \tag{24}$$

$$T_x := \sum_{k=0}^{C_{\mathrm{d}}} \mathbf{trace}\,(M_k X_k) \;<\; 1 \tag{25}$$

$$T_y := \sum_{l=0}^{C_{\mathrm{e}}} \mathbf{trace}\,(P_l Y_l) \;<\; 1 \tag{26}$$

It is shown in [D'Andrea, 1996a] how searching for a $\mathbf{K}$ such that $\mathbf{M} = \mathbf{G} \star \mathbf{K}$ satisfies the above conditions may be converted to an AMI. The above theorem is generalized below to encompass a larger class of disturbances d. This will, in general, result in a condition which cannot directly be converted to an AMI. By considering systems $\mathbf{G}$ which have a special structure, however, AMIs which solve the corresponding synthesis problem may be derived.

Given linear, time invariant bounded operator $\mathbf{M}$, define the following set in l_2:

$$\bar{\mathcal{D}} := \left\{ d \in l_2^m : D_k \textcircled{\scriptsize T} \Lambda(d) + \bar{D}_k \textcircled{\scriptsize T} \Lambda(\mathbf{M}d) - M_k \leq 0, \;\; 0 \leq k \leq C_{\mathrm{d}} \right\} \tag{27}$$

$\bar{\mathcal{D}}$ differs from $\mathcal{D}$ in that an extra term involving $\Lambda(\mathbf{M}d)$ is included in the constraints. As in the definition of $\mathcal{D}$, it is assumed that $0 < M_0 \in R$, with $D_0 = I \in \mathbb{R}^{m \times m}$, and $\bar{D}_0 = 0 \in \mathbb{R}^{m \times m}$, implying that $\|d\|^2 \leq M_0$. $\bar{\mathcal{D}}^\epsilon$ can be defined analogously to $\mathcal{D}^\epsilon$.

The following corollary is an extension of Theorem 2:

Corollary 1 *Given linear, time invariant, bounded operator* $\mathbf{M}$ *and sets* $\bar{\mathcal{D}}$ *and* $\mathcal{E}$, *the following are equivalent:*

I. *There exists* $\epsilon > 0$ *such that the following supremum is satisfied:*

$$\sup_{e \in \mathcal{E}^\epsilon} \sup_{d \in \bar{\mathcal{D}}^\epsilon} \langle e, \mathbf{M}d \rangle < 1 \tag{28}$$

II. *There exist* $0 < X_k \in \mathbb{R}_{\mathbf{P}}^{m_k \times m_k}, 0 \le k \le C_{\mathbf{d}}$ *and* $0 < Y_l \in \mathbb{R}_{\mathbf{P}}^{p_l \times p_l}, 0 \le l \le C_{\mathbf{e}}$ *such that*

$$\left\| Y^{-\frac{1}{2}} \mathbf{M} X^{-\frac{1}{2}} \right\| \quad < \quad 1 \tag{29}$$

$$X := \sum_{k=0}^{C_{\mathbf{d}}} \grave{D}_k \textcircled{T} X_k + \mathbf{M}^* \left(\grave{\bar{D}}_k \textcircled{T} X_k \right) \mathbf{M} \quad > \quad 0 \tag{30}$$

$$Y := \sum_{l=0}^{C_{\mathbf{e}}} \grave{E}_l \textcircled{T} Y_l \quad > \quad 0 \tag{31}$$

$$T_x := \sum_{k=0}^{C_{\mathbf{d}}} \mathbf{trace}\,(M_k X_k) \quad < \quad 1 \tag{32}$$

$$T_y := \sum_{l=0}^{C_{\mathbf{e}}} \mathbf{trace}\,(P_l Y_l) \quad < \quad 1 \tag{33}$$

Remarks:

- The proof of the above claim is essentially equivalent to that of Theorem 2 in [D'Andrea, 1996a] when $D_k \textcircled{T} \Lambda(d)$ is replaced by $D_k \textcircled{T} \Lambda(d) + \bar{D}_k \textcircled{T} \Lambda(\mathbf{M}d)$ and $\grave{D}_k \textcircled{T} X_k$ is replaced by $\mathbf{M}^* \left(\grave{\bar{D}}_k \textcircled{T} X_k \right) \mathbf{M}$. Note, however, that X is now a positive definite operator, not a constant matrix.
- Statement **I** in Corollary 1 is stronger than statement **I** in Theorem 2. This simplifies the proof that statement **I** implies statement **II** since the continuity property presented in Theorem 1 is not required; this continuity property has not been proved for $\bar{\mathcal{D}}$. Conversely, one can only infer from the proof of Theorem 2 in [D'Andrea, 1996a] that statement **II** implies statement **I** in Corollary 1 when $\epsilon = 0$; since the inequalities in statement **II** are strict, however, the result follows for some $\epsilon > 0$.

5 Converting to Generalized l_2 Synthesis setup

The Robust Gain Scheduling problem will now be converted to the modified Generalized l_2 Synthesis setup of Corollary 1. This will result in a scaled $\mathcal{H}_\infty$ condition, which will later be converted to an AMI.

Define the following sets:

$$\bar{\mathcal{D}}^\epsilon \quad := \quad \left\{ d \in l_2 : d = (d^{\mathbf{p}}, d^{\mathbf{u}}),\ \Lambda\left(d_k^{\mathbf{p}}\right) - \Lambda\left((\mathbf{M}d)_k^{\mathbf{p}}\right) \le \epsilon I,\ \|d_k^{\mathbf{u}}\|^2 \le 1 + \epsilon \right\} \tag{34}$$

$$\mathcal{E}^\epsilon \quad := \quad \left\{ e \in l_2 : e = (e^{\mathbf{p}}, e^{\mathbf{u}}), \|e^{\mathbf{p}}\|^2 \le \epsilon,\ \|e_l^{\mathbf{u}}\|^2 \le 1 + \epsilon \right\} \tag{35}$$

which can readily be put in the form of equations (27) and (15).

Theorem 3 *System* $\mathbf{K}$ *solves the Robust Gain Scheduling problem if and only if* $\mathbf{K}$ *internally stabilizes* $\mathbf{G}_2$ *and there exists an* $\epsilon > 0$ *such that the following two conditions are satisfied:*

$$\sup_{d \in \bar{\mathcal{D}}^\epsilon} \sup_{e \in \mathcal{E}^\epsilon} \langle e, \mathbf{M}d \rangle < 1 \tag{36}$$

$$\bar{\mathcal{D}}^\epsilon \text{ is bounded.} \tag{37}$$

where $\mathbf{M} := \mathbf{G}_2 \star \mathbf{K}$.

Proof: The following preliminary results are required:

Lemma 1 *[D'Andrea, 1995] Given* $x_j, y \in l_2$, $\sum \|x_j\| \geq \|y\|$ *if and only if there exist linear operators* $\mathbf{\Delta}_j$, $\|\mathbf{\Delta}_j\| \leq 1$, *such that* $y = \sum \mathbf{\Delta}_j x_j$.

Lemma 2 *[Paganini et al., 1994] Given* $x, y \in l_2$, *there exists a linear operator* $\boldsymbol{\delta}$, $\|\boldsymbol{\delta}\| \leq 1$, *such that* $y = \boldsymbol{\delta} I x$ *if and only if* $\Lambda(x) - \Lambda(y) \geq 0$.

Sufficiency: Assume that $\mathbf{K}$ does not solve the Robust Gain Scheduling problem. It suffices to show that either of equations (36) or (37) are not satisfied. Fix $\epsilon > 0$. By equation (9), there exists $n = (n^{\mathrm{p}}, n^{\mathrm{u}}) \in l_2$, $\bar{z} = (\bar{z}^{\mathrm{p}}, \bar{z}^{\mathrm{u}}) \in l_2$, such that $\bar{z} = n + M\mathbf{\Delta}\bar{z}$, $\|n\| \leq \epsilon$, and either of the following two conditions are satisfied:

$$\sum_{l=1}^{C_{\mathrm{e}}^{\mathrm{u}}} \|\bar{z}_l^{\mathrm{u}}\| = 1 \quad , \quad \|\bar{z}^{\mathrm{p}}\| \leq \frac{1}{\sqrt{\epsilon}} \tag{38}$$

$$\sum_{l=1}^{C_{\mathrm{e}}^{\mathrm{u}}} \|\bar{z}_l^{\mathrm{u}}\| \leq 1 \quad , \quad \|\bar{z}^{\mathrm{p}}\| = \frac{1}{\sqrt{\epsilon}} \tag{39}$$

Let $d = (d^{\mathrm{p}}, d^{\mathrm{u}}) := \mathbf{\Delta}\bar{z}$, $z = (z^{\mathrm{p}}, z^{\mathrm{u}}) := \bar{z} - n = \mathbf{M}d$. If equation (38) is satisfied, then by Lemmas 1 and 2 and the norm bound on n the following conditions must be satisfied:

$$\|d_k^{\mathrm{u}}\| \leq 1, \;\; \Lambda\left(d_k^{\mathrm{p}}\right) - \Lambda\left(z_k^{\mathrm{p}}\right) \;\; \leq \;\; \mathrm{O}\left(\sqrt{\epsilon}\right) I \tag{40}$$

$$\sum_{l=1}^{C_{\mathrm{e}}^{\mathrm{u}}} \|z_l^{\mathrm{u}}\| \;\; \geq \;\; 1 - \mathrm{O}\left(\epsilon\right) \tag{41}$$

From equation (41), one can find $e \in \mathcal{E}$ such that $\langle e, \mathbf{M}d \rangle \geq 1 - \mathrm{O}(\epsilon)$.
If equation (39) is satisfied,

$$\|d_k^{\mathrm{u}}\| \leq 1, \;\; \Lambda\left(d_k^{\mathrm{p}}\right) - \Lambda\left(z_k^{\mathrm{p}}\right) \leq \mathrm{O}\left(\sqrt{\epsilon}\right) I, \;\; \|z^{\mathrm{p}}\| \geq \frac{1}{\sqrt{\epsilon}} - \mathrm{O}\left(\epsilon\right) \tag{42}$$

Since $\mathbf{M}$ is bounded, $\|d^{\mathrm{p}}\| \geq 1/\mathrm{O}\left(\sqrt{\epsilon}\right)$. Since ϵ is arbitrary, at least one of equations (36) and (37) are not satisfied.

Necessity: Assume that for all $\epsilon > 0$, equation (36) is not satisfied. Define $z = \mathbf{M}d$. Then

$$\|d_k^{\mathbf{u}}\|^2 \leq 1+\epsilon, \;\; \Lambda\left(d_k^{\mathbf{p}}\right) - \Lambda\left(z_k^{\mathbf{p}}\right) \leq \epsilon I, \;\; \sum_{l=1}^{C_{\mathbf{e}}^{\mathbf{u}}} \|z_l^{\mathbf{u}}\| \geq 1 - \mathrm{O}\left(\epsilon\right) \tag{43}$$

It follows that there exists $\bar{z}, n \in l_2$ such that $\bar{z} = z + n$ with

$$\Lambda\left(d_k^{\mathbf{p}}\right) - \Lambda\left(\bar{z}_k^{\mathbf{p}}\right) \leq 0, \;\; \sum_{l=1}^{C_{\mathbf{e}}^{\mathbf{u}}} \|\bar{z}_l^{\mathbf{u}}\| \geq 1+\epsilon, \;\; \|n\| \leq \mathrm{O}\left(\sqrt{\epsilon}\right) \tag{44}$$

By Lemmas 1 and 2, there exists $\mathbf{\Delta}$ such that $d = \mathbf{\Delta}\bar{z}$, yielding

$$\bar{z} = \left(\mathbf{I} - \mathbf{M\Delta}\right)^{-1} n, \;\; \|\bar{z}\| \geq 1, \;\; \|n\| \leq \mathrm{O}\left(\sqrt{\epsilon}\right) \tag{45}$$

Since ϵ is arbitrary, equation (9) is not satisfied.
Assume that for all $\epsilon > 0$, equation (37) is not satisfied. By appropriate scaling, the following equations are satisfied:

$$\|d_k^{\mathbf{u}}\| \leq \epsilon, \;\; \Lambda\left(d_k^{\mathbf{p}}\right) - \Lambda\left(z_k^{\mathbf{p}}\right) \leq \epsilon I, \;\; \|d^{\mathbf{p}}\| \geq 1 \tag{46}$$

It follows that there exists $\bar{z}, n \in l_2$ such that $\bar{z} = z + n$ with

$$\Lambda\left(d_k^{\mathbf{p}}\right) - \Lambda\left(\bar{z}_k^{\mathbf{p}}\right) \leq 0, \;\; \sum_{l=1}^{C_{\mathbf{e}}^{\mathbf{u}}} \|\bar{z}_l^{\mathbf{u}}\| \geq \epsilon, \;\; \|n\| \leq \mathrm{O}\left(\sqrt{\epsilon}\right) \tag{47}$$

By Lemmas 1 and 2, there exists $\mathbf{\Delta}$ such that $d = \mathbf{\Delta}\bar{z}$, yielding

$$\bar{z} = \left(\mathbf{I} - \mathbf{M\Delta}\right)^{-1} n, \;\; \|\bar{z}\| \geq \|d^{\mathbf{p}}\| \geq 1, \;\; \|n\| \leq \mathrm{O}\left(\sqrt{\epsilon}\right) \tag{48}$$

Since ϵ is arbitrary, equation (9) is not satisfied. ■

We are now in a position to invoke Corollary 1:

Theorem 4 *System* $\mathbf{K}$ *solves the Robust Gain Scheduling problem if and only if* $\mathbf{K}$ *internally stabilizes* $\mathbf{G}_2$ *and there exists scales*

$$X_{\mathbf{u}} = \mathbf{diag}[x_1 I, \cdots, x_{C_{\mathbf{d}}^{\mathbf{u}}} I], \;\; x_k > 0, \;\; \sum_{k=1}^{C_{\mathbf{d}}^{\mathbf{u}}} x_k < 1 \tag{49}$$

$$Y_{\mathbf{u}} = \mathbf{diag}[y_1 I, \cdots, y_{C_{\mathbf{e}}^{\mathbf{u}}} I], \;\; y_l > 0, \;\; \sum_{l=1}^{C_{\mathbf{e}}^{\mathbf{u}}} y_l < 1 \tag{50}$$

$$X_{\mathbf{p}} = \mathbf{diag}[X_1, \cdots, X_{C^{\mathbf{p}}}], \;\; X_k > 0 \tag{51}$$

such that

$$\left\| \begin{bmatrix} X_{\mathbf{p}}^{\frac{1}{2}} & 0 \\ 0 & Y_{\mathbf{u}}^{-\frac{1}{2}} \end{bmatrix} \mathbf{M} \begin{bmatrix} X_{\mathbf{p}}^{-\frac{1}{2}} & 0 \\ 0 & X_{\mathbf{u}}^{-\frac{1}{2}} \end{bmatrix} \right\| < 1 \tag{52}$$

Proof: First, note that equation (52) is equivalent to

$$\mathbf{M}^* \begin{bmatrix} X_{\mathbf{p}} & 0 \\ 0 & Y_{\mathbf{u}}^{-1} \end{bmatrix} \mathbf{M} - \begin{bmatrix} X_{\mathbf{p}} & 0 \\ 0 & X_{\mathbf{u}} \end{bmatrix} < 0 \tag{53}$$

Sufficiency: Assume that the above scales exist, and that equation (53) is satisfied. It follows that there exists $Y_{\mathbf{p}} > 0$, sufficiently large, and $\delta > 0$, sufficiently small, such that

$$\mathbf{M}^* Y^{-1} \mathbf{M} - X \;<\; 0 \tag{54}$$

$$Y := \begin{bmatrix} Y_{\mathbf{p}} & 0 \\ 0 & Y_{\mathbf{u}} \end{bmatrix} \;>\; 0 \tag{55}$$

$$X := \begin{bmatrix} X_{\mathbf{p}} & 0 \\ 0 & X_{\mathbf{u}} \end{bmatrix} - \mathbf{M}^* \begin{bmatrix} X_{\mathbf{p}} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{M} \;>\; 0 \tag{56}$$

$$(1+\delta) \sum_{k=1}^{C_{\mathbf{d}}^{\mathbf{u}}} x_k + \delta \sum_{k=1}^{C_{\mathbf{d}}^{\mathbf{u}}} \mathbf{trace}\left(X_k^{\mathbf{p}}\right) \;<\; 1 \tag{57}$$

$$(1+\delta) \sum_{l=1}^{C_{\mathbf{e}}^{\mathbf{u}}} y_l + \delta \mathbf{trace}\left(Y^{\mathbf{p}}\right) \;<\; 1 \tag{58}$$

Since $\delta > 0$, it can readily be verified that there exists $\epsilon > 0$ such that $\langle e, Ye \rangle \leq 1 \; \forall e \in \mathcal{E}^\epsilon$ and that $\langle d, Xd \rangle \leq 1 \; \forall d \in \bar{\mathcal{D}}^\epsilon$. Since $X > 0$, this implies that $\bar{\mathcal{D}}^\epsilon$ is bounded, verifying condition (37). Invoking the same arguments used in the sufficiency proof of Theorem 2 in [D'Andrea, 1996a], it follows that equation (36) is satisfied as well.

Necessity: All the conditions of Corollary 1 are satisfied, with the exception of additional constraints $\|e\|^2 \leq P_0$, $\|d\|^2 \leq M_0$ for some $P_0, M_0 > 0$, which are not explicitly included in sets $\mathcal{E}$ and $\bar{\mathcal{D}}$; this is necessary, since the proof of Theorem 2 in [D'Andrea, 1996a] explicitly depends on the presence of these constraints. Since $\mathcal{E}^\epsilon$ is bounded, by construction, and it is given that $\bar{\mathcal{D}}^\epsilon$ is bounded, there exist constants $P_0, M_0 > 0$, sufficiently large, such that constraints $\|e\|^2 \leq P_0$, $\|d\|^2 \leq M_0$ do not affect sets $\mathcal{E}^\epsilon$ and $\bar{\mathcal{D}}^\epsilon$. In addition, there exists $\delta > 0$ such that constraint $\|e^{\mathbf{p}}\|^2 \leq 0$ in $\mathcal{E}$ may be replaced by $\|e^{\mathbf{p}}\|^2 \leq \delta$, since $\mathbf{M}$ and $\bar{\mathcal{D}}$ are bounded. The following scales can thus be constructed, as per Corollary 1, such that

$\mathbf{M}^*\tilde{Y}^{-1}\mathbf{M} - \tilde{X} < 0$:

$$\begin{aligned}
\tilde{X} &= -\mathbf{M}^* \begin{bmatrix} \tilde{X}_{\mathbf{p}} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{M} + \begin{bmatrix} \tilde{X}_{\mathbf{p}} & 0 \\ 0 & \tilde{X}_{\mathbf{u}} \end{bmatrix} + \tilde{X}_0 I > 0 && (59)\\
\tilde{Y} &= \begin{bmatrix} \tilde{Y}_{\mathbf{p}} & 0 \\ 0 & \tilde{Y}_{\mathbf{u}} \end{bmatrix} + \tilde{Y}_0 I > 0 && (60)\\
\tilde{X}_{\mathbf{u}} &= \mathbf{diag}[\tilde{x}_1 I, \cdots, \tilde{x}_{C_{\mathrm{d}}^{\mathrm{u}}} I], \;\; \tilde{x}_k > 0 && (61)\\
\tilde{Y}_{\mathbf{u}} &= \mathbf{diag}[\tilde{y}_1 I, \cdots, \tilde{y}_{C_{\mathrm{e}}^{\mathrm{u}}} I], \;\; \tilde{y}_l > 0 && (62)\\
\tilde{X}_{\mathbf{p}} &= \mathbf{diag}[\tilde{X}_1, \cdots, \tilde{X}_{C^{\mathrm{P}}}], \;\; \tilde{X}_k > 0 && (63)\\
\tilde{Y}_{\mathbf{p}} &> 0 && (64)\\
\sum_{k=1}^{C_{\mathrm{d}}^{\mathrm{u}}} \tilde{x}_k + M_0 \tilde{X}_0 &< 1, \;\; \sum_{l=1}^{C_{\mathrm{e}}^{\mathrm{u}}} \tilde{y}_l + \tilde{Y}_0 P_0 + \delta \mathbf{trace}\left(\tilde{Y}_{\mathbf{p}}\right) < 1 && (65)
\end{aligned}$$

Define the following scales:

$$X_{\mathbf{p}} := \tilde{X}_{\mathbf{p}} + \tilde{X}_0 I, \;\; X_{\mathbf{u}} := \tilde{X}_{\mathbf{u}} + \tilde{X}_0 I, \;\; Y_{\mathbf{u}} := \tilde{Y}_{\mathbf{u}} + \tilde{Y}_0 I \tag{66}$$

Since $P_0 \geq C_{\mathrm{e}}^{\mathrm{u}}$ and $M_0 \geq C_{\mathrm{d}}^{\mathrm{u}}$ for constraints $\|e\|^2 \leq P_0$ and $\|d\|^2 \leq M_0$ to be inactive, these scalings satisfy the first three conditions of Theorem 4. This leads to the following matrix inequality:

$$\mathbf{M}^* \begin{bmatrix} X_{\mathbf{p}} + \tilde{Y}_{\mathbf{p}}^{-1} - \tilde{X}_0 I & 0 \\ 0 & Y_{\mathbf{u}}^{-1} \end{bmatrix} \mathbf{M} - \begin{bmatrix} X_{\mathbf{p}} & 0 \\ 0 & X_{\mathbf{u}} \end{bmatrix} < 0 \tag{67}$$

Since $\delta > 0$, $\tilde{Y}_{\mathbf{p}}$ is bounded. Furthermore, $\tilde{X}_0 < M_0^{-1}$. Thus M_0 can be chosen sufficiently large such that $\tilde{Y}_{\mathbf{p}}^{-1} - \tilde{X}_0 I \geq 0$. This yields equation (53), as required. ■

6 Synthesis Condition

The condition of Theorem 4 and the particular structure of $\mathbf{G}_2$ could be used to convert the Robust Gain Scheduling problem to an AMI. Much of this development, however, would duplicate the main results in [Packard, 1994]. The condition of Theorem 4 will instead be converted to a form for which the results in [Packard, 1994] may be applied, and an AMI constructed from the resulting conditions.

For fixed scales $X_{\mathbf{u}}$ and $Y_{\mathbf{u}}$, the conditions of Theorem 4 are equivalent to the Gain Scheduling problem of [Packard, 1994] in Figure 4; it is required to find a nominally internally stabilizing controller $\mathbf{K}$ such that

$$\sup_{\mathbf{\Delta}^{\mathrm{P}} \in \boldsymbol{P\Delta}} \; \sup_{\|\bar{d}^{\mathrm{u}}\| \leq 1} \left\| \left(\left(\mathbf{\Delta}^{\mathrm{P}} \star \bar{\mathbf{G}}_0 \right) \star \left(\mathbf{\Delta}^{\mathrm{P}} \star \mathbf{K} \right) \right) \bar{d}^{\mathrm{u}} \right\| < 1 \tag{68}$$

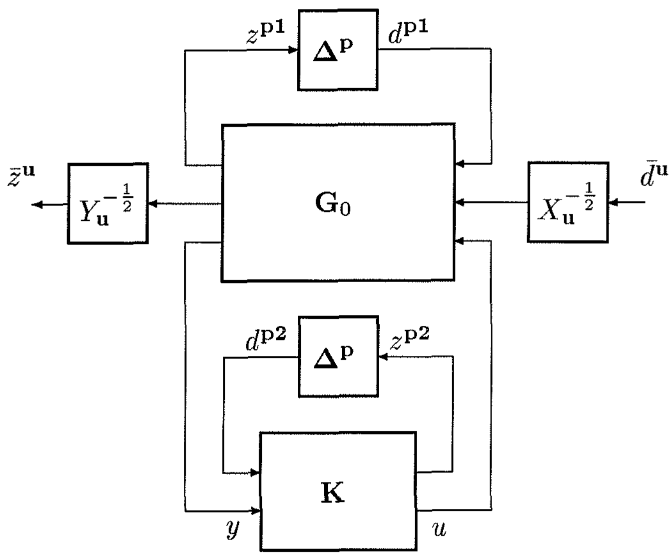


Figure 4: Equivalent Gain Scheduling problem

where

$$\bar{\mathbf{G}}_0 := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Y_{\mathrm{u}}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{G}_0 \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & X_{\mathrm{u}}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{69}$$

In [Packard, 1994], scalings which commute with $\mathbf{\Delta}^{\mathrm{p}}$ are introduced to convert the above problem to a scaled $\mathcal{H}_\infty$ condition; these scales are in fact the X_{p} of the previous section, modulo the transformation from $\mathbf{G}_0$ to $\mathbf{G}_2$. The following result is from [Packard, 1994]:

Theorem 5 *Let* $\left[\begin{array}{c|ccc} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} & \bar{B}_{21} \\ \hline \bar{A}_{21} & \bar{A}_{22} & \bar{B}_{12} & \bar{B}_{22} \\ \bar{C}_{11} & \bar{C}_{12} & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{D}_{21} & \bar{D}_{22} \end{array}\right]$ *be a minimal state space description for system* $\bar{\mathbf{G}}_0$. *There exists a nominally internally stabilizing controller* $\mathbf{K}$ *such that equation (68) is satisfied if there exist structured positive definite matrices* X *and* Y *satisfying an AMI (with structure and AMI outlined in [Packard, 1994]) such that*

$$\bar{V}\left(\bar{R}^* \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \bar{R} - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}\right) \bar{V}^* < 0 \tag{70}$$

$$\bar{U}^*\left(\bar{R} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \bar{R}^* - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}\right) \bar{U} < 0 \tag{71}$$

where

$$\bar{R} = \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_{11} \end{bmatrix}$$

$$\bar{V} = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 \end{bmatrix} \; : \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \\ \bar{V}_1 & \bar{V}_2 \end{bmatrix} \text{invertible}, \quad \bar{C}_2\bar{V}_1^* + \bar{D}_{21}\bar{V}_2^* = 0 \tag{72}$$

$$\bar{U} = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix} \; : \begin{bmatrix} \bar{B}_2 & \bar{U}_1 \\ \bar{D}_{12} & \bar{U}_2 \end{bmatrix} \text{invertible}, \quad \bar{U}_1^*\bar{B}_2 + \bar{U}_2^*\bar{D}_{12} = 0$$

Remarks:

- As discussed in [Packard, 1994], it can be assumed without loss of generality that $\begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$ is full row rank; thus there always exists a $\bar{V}$ satisfying equation (72). Similarly for $\bar{U}$.
- As stated in the theorem, the above condition is only sufficient for the existence of a $\mathbf{K}$ such that equation (68) is satisfied. As argued in the previous section, however, the above condition is also necessary (again assuming that the $\mathbf{\Delta^P}$ are arbitrary linear operators).

We are now in a position to state and prove the main result of this paper.

Theorem 6 *Let* $\left[\begin{array}{c|ccc} A_{11} & A_{12} & B_{11} & B_{21} \\ \hline A_{21} & A_{22} & B_{12} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{array}\right]$ *be a minimal state space description for system* $\mathbf{G}_0$. *There exists a* $\mathbf{K}$ *which solves the Robust Gain Scheduling problem if and only if there exist scales* $X_\mathbf{u}$ *and* $Y_\mathbf{u}$ *satisfying the conditions of Theorem 4, positive definite matrices* $\bar{X}_\mathbf{u}$ *and* $\bar{Y}_\mathbf{u}$*, and structured positive definite matrices* X *and* Y *satisfying an AMI (with structure and AMI outlined in [Packard, 1994]), such that*

$$V\left(R^*\begin{bmatrix} Y & 0 \\ 0 & \bar{Y}_\mathbf{u} \end{bmatrix}R - \begin{bmatrix} Y & 0 \\ 0 & X_\mathbf{u} \end{bmatrix}\right)V^* < 0 \tag{73}$$

$$U^*\left(R\begin{bmatrix} X & 0 \\ 0 & \bar{X}_\mathbf{u} \end{bmatrix}R^* - \begin{bmatrix} X & 0 \\ 0 & Y_\mathbf{u} \end{bmatrix}\right)U < 0 \tag{74}$$

$$\begin{bmatrix} X_\mathbf{u} & I \\ I & \bar{X}_\mathbf{u} \end{bmatrix} \geq 0, \; \begin{bmatrix} Y_\mathbf{u} & I \\ I & \bar{Y}_\mathbf{u} \end{bmatrix} \geq 0 \tag{75}$$

where

$$R = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$$

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \; : \begin{bmatrix} C_2 & D_{21} \\ V_1 & V_2 \end{bmatrix} \text{invertible}, \quad C_2V_1^* + D_{21}V_2^* = 0 \tag{76}$$

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \; : \begin{bmatrix} B_2 & U_1 \\ D_{12} & U_2 \end{bmatrix} \text{invertible}, \quad U_1^*B_2 + U_2^*D_{12} = 0$$

Proof: By the definition of $\bar{\mathbf{G}}_0$, defining

$$\left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] := \left[\begin{array}{c|cc} A & B_1 X_{\mathbf{u}}^{-\frac{1}{2}} & B_2 \\ \hline Y_{\mathbf{u}}^{-\frac{1}{2}} C_1 & Y_{\mathbf{u}}^{-\frac{1}{2}} D_{11} X_{\mathbf{u}}^{-\frac{1}{2}} & Y_{\mathbf{u}}^{-\frac{1}{2}} D_{12} \\ C_2 & D_{21} X_{\mathbf{u}}^{-\frac{1}{2}} & D_{22} \end{array}\right] \tag{77}$$

yields a minimal state space representation for $\bar{\mathbf{G}}_0$. Define $\tilde{X} := \begin{bmatrix} I & 0 \\ 0 & X_{\mathbf{u}} \end{bmatrix}$, $\tilde{Y} := \begin{bmatrix} I & 0 \\ 0 & Y_{\mathbf{u}} \end{bmatrix}$, and U, V, and R as in equation (76). Then $\bar{R} = \tilde{Y}^{-\frac{1}{2}} R \tilde{X}^{-\frac{1}{2}}$, and $\bar{V} := V \tilde{X}^{\frac{1}{2}}$, $\bar{U} := \tilde{Y}^{-\frac{1}{2}} U$ satisfy equations (72). Substituting into equations (70) and (71) yields

$$V \left(R^* \begin{bmatrix} Y & 0 \\ 0 & Y_{\mathbf{u}}^{-1} \end{bmatrix} R - \begin{bmatrix} Y & 0 \\ 0 & X_{\mathbf{u}} \end{bmatrix} \right) V^* < 0 \tag{78}$$

$$U^* \left(R \begin{bmatrix} X & 0 \\ 0 & X_{\mathbf{u}}^{-1} \end{bmatrix} R^* - \begin{bmatrix} X & 0 \\ 0 & Y_{\mathbf{u}} \end{bmatrix} \right) U < 0 \tag{79}$$

Finally, by Schur complement arguments [Zhou et al., 1995], if $\bar{X}_{\mathbf{u}}$ satisfies matrix inequality $\begin{bmatrix} \bar{X}_{\mathbf{u}} & I \\ I & X_{\mathbf{u}} \end{bmatrix} \geq 0$, then $\bar{X}_{\mathbf{u}} \geq X_{\mathbf{u}}^{-1}$. Furthermore, $\bar{X}_{\mathbf{u}} = X_{\mathbf{u}}^{-1}$ satisfies the matrix inequality. This concludes the proof. ■

Remarks:

- A controller may be constructed as described in [Packard, 1994] using the state space description for $\bar{\mathbf{G}}_0$ (which includes scales $X_{\mathbf{u}}$ and $Y_{\mathbf{u}}$) and scales X and Y.
- The affine matrix inequalities may be solved using standard convex optimization tools, such as The LMI Control Toolbox [Gahinet et al., 1994].

6.1 Example

Consider the setup of Figure 5. Given system $\mathbf{P}$, it is required to design system $\mathbf{K}$ such that disturbance d_1 and measurement noise d_2 have a small effect on plant output z. The plant is subject to multiplicative, unstructured uncertainty $\bar{\mathbf{\Delta}}^{\mathbf{u}}$, weighted by $\mathbf{W}_t$. In addition, plant $\mathbf{P} \star \mathbf{\Delta}^{\mathbf{P}}$ is a function of time varying parameters $\mathbf{\Delta}^{\mathbf{P}}$ in set $\boldsymbol{P\Delta}$. The system to be designed, $\mathbf{K}$, has access to these parameters as well. The exact problem formulation is to find a system $\mathbf{K}$ such that the closed loop system is robustly stable and

$$\sup_{\|d_1\| \leq 1, \|d_2\| \leq 1} \sup_{\|\bar{\mathbf{\Delta}}^{\mathbf{u}}\| \leq 1} \sup_{\mathbf{\Delta}^{\mathbf{P}} \in \boldsymbol{P\Delta}} \|z\| < 1 \tag{80}$$

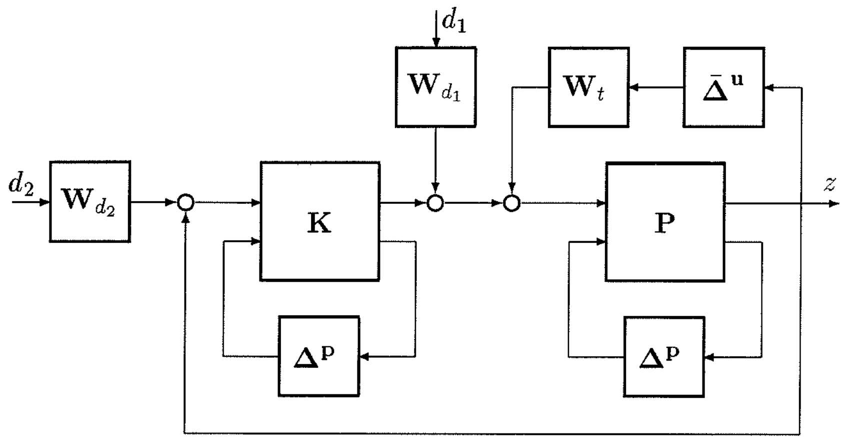
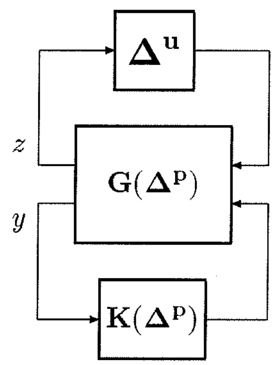


Figure 5: Robust Gain Scheduled Disturbance Rejection



$$\mathbf{\Delta}^{\mathbf{u}} = \begin{bmatrix} \bar{\mathbf{\Delta}}^{\mathbf{u}} \\ \mathbf{\Delta}^{\mathbf{u}}_{d_1} \\ \mathbf{\Delta}^{\mathbf{u}}_{d_2} \end{bmatrix}$$

Figure 6: Equivalent Problem

Note that if either of d_1 or d_2 are vector valued signals, they can further be partitioned and bounded in norm separately. The above can be converted to the problem setup of Figure 1 using techniques similar to those in [D'Andrea, 1995]. The idea is to first apply the tools in [D'Andrea, 1995], with plant $\mathbf{P} \star \mathbf{\Delta}^{\mathbf{p}}$ and controller $\mathbf{K} \star \mathbf{\Delta}^{\mathbf{p}}$, to convert the above robust performance problem to the robust stability problem of Figure 6. The $\mathbf{\Delta}^{\mathbf{p}}$s can then be unwrapped from $\mathbf{G}(\mathbf{\Delta}^{\mathbf{p}})$ and $\mathbf{K}(\mathbf{\Delta}^{\mathbf{p}})$ to yield the block diagram in Figure 1.

7 Conclusions

The results in this paper can be used to synthesize parameter dependent controllers for parameter dependent systems subject to a class of structured uncertainty. The resulting AMI condition is both necessary and sufficient for the posed problem to have a solution when the parameters and plant uncertainty are assumed to be arbitrary norm bounded operators on

l_2. In practice, the conditions may be conservative since the plant parameters will be time varying gains, and the plant uncertainty may have additional structure, such as being linear time invariant or parametric.

The results presented in this paper only consider a restricted class of uncertainty and performance objectives. In general, any type of constraints which fit into the Generalized l_2 Synthesis framework may be imposed. For example, the mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ results in [D'Andrea, 1996b] may be incorporated in the above development, allowing robust gain scheduling subject to white noise disturbances.

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