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# Optimality of Nonlinear Design Techniques: A Converse HJB Approach

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**Abstract**

The issue of optimality in nonlinear controller design is confronted by using the converse HJB approach [1] to classify dynamics under which certain design schemes are optimal. In particular, the techniques of Jacobian linearization, pseudo-Jacobian linearization, and feedback linearization are analyzed. Finally, the conditions for optimality are applied to the 2-D nonlinear oscillator, where simple, nontrivial examples are produced in which the various design techniques are optimal.

## 1 Introduction

Determination of the optimal feedback law for nonlinear optimal control problems require solutions of Hamilton-Jacobi-Bellman (HJB) partial differential equations. Difficulties in solving the HJB equation for high dimensional systems have precluded their use except in specific areas, and has motivated the study of alternative control techniques. Many of such alternative techniques attempt to "approximately" solve the HJB by using some simplified scheme (Jacobian linearization). Others merely trade an attempt at optimality for some other quality, such as guaranteed global stability (feedback linearization). Finally, other techniques attempt to combine guaranteed stability with at least some notion of optimality (control Lyapunov functions and inverse optimality).

In this paper we ask the following question: For what class of systems is a nonlinear control design technique optimal in the HJB sense? To answer this, we use the notion of Converse HJB optimality, which is the idea that if the solution to the HJB equation is known (i.e. the value function $V$), then the HJB equation can be used to characterize all systems for which this $V$ solves the optimal control problem. In this way, it is possible derive equations which classify what properties a system must have in order for a certain design technique to be optimal (merely by chance). Furthermore, we present methods to derive systems for which various techniques will be optimal, and use this to generate examples. Specifically, we analyze the techniques of Jacobian linearization, pseudo-Jacobian linearization, and feedback linearization.

This paper is organized as follows. Section 2 briefly introduces the optimal control problem, which leads to Hamilton-Jacobi-Bellman partial differential equations. In contrast to the optimal control problem, Section 3 formulates the Converse optimal control problem. In Section 4, the ideas of converse optimality are used to provide conditions under which the techniques of Jacobian linearization, pseudo-Jacobian linearization, and feedback lineariztion can produce the optimal controller. Section 5 specializes this analysis to the 2-D nonlinear oscillator and Section 6 uses the results of Section 5 to produce examples which demonstrate when each of the techniques is optimal. Conclusions are presented in section 7.

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## 2 Problem Statement

The fundamental problem of optimal control design involves minimizing a performance objective subject to certain underlying dynamics. Our formulation of the optimal control problem will be as follows: Consider the nonlinear system,

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0 \tag{1}$$

with state $x \in \mathbf{R}^n$, and control $u \in \mathbf{R}^m$. With these dynamics, we associate a performance objective,

$$J(u) = \int_0^\infty (q(x) + u^T u)dt \tag{2}$$

with $q \in \mathbf{C}$, $q(0) = 0$, and $q(x) \geq 0$, $\forall x$.

**Definition 1** ***Optimal Control Problem (OCP):*** *Find a state-feedback control law $u = \Phi(x)$ such that the performance objective (2) is minimized, subject to the nonlinear system dynamics (1).*

A standard technique for solving the optimal control problem is to use a dynamic programming approach [2]: Let $V$ denote the *value function*, defined as follows:

$$V(x) = \min_{u(\cdot)} \int_0^\infty (q(x) + u^T u)dt \tag{3}$$

then under appropriate technical conditions ($V \in \mathbf{C}^1$), the OCP can be reduced to solving the Hamilton-Jacobi-Bellman (HJB) pde:

$$V_x f - \frac{1}{4} V_x g g^T V_x^T + q = 0 \tag{4}$$

where $V_x = \frac{\partial V}{\partial x}$. If the HJB can be solved, then the optimal control $u^*$ is given by:

$$u^* = -\frac{1}{2} g^T \frac{\partial V^T}{\partial x} \tag{5}$$

The HJB equation (4) is difficult to solve analytically, in particular there is no efficient algorithm available when the problem dimension is high. Thus reducing the optimal control problem to the HJB cannot be viewed as a general, practical method.

In this paper, our approach will not rely on being able to solve the HJB, rather we will focus our attention on the so-called converse HJB approach.

## 3 Converse Optimal Control Problem

The converse optimal control problem contrasts with the optimal control problem in that it proceeds backwards, i.e. starting from the solution of the HJB and working toward the dynamics that produced that solution.

**Definition 2** ***Converse HJB (CoHJB) Problem:*** *Given a performance objective $J$ and a value function $V$, find the class of nonlinear systems for which this $V$ is the solution of the Optimal Control Problem (HJB) (4).*

To solve the CoHJB problem is to solve the HJB (4) for the dynamics, $f$ and $g$, assuming that the value function $V$ is known. In this sense, the CoHJB can be viewed as solving an algebraic equation, in contrast to the Optimal Control Problem which requires the solution of a p.d.e. In this sense, the difficulties in solving the Optimal Control Problem, or equivalently, the HJB are avoided. Of course, the CoHJB is useless for design, but is excellent for generating benchmark examples on which various control schemes can be tested [1]. Furthermore, we will see that it can help to distinguish the class of systems for which various control techniques will in fact result in the optimal controller. For a complete exposition of the CoHJB approach see [1] or [4].

Before delving into these aspects, we will first derive conditions under which the converse optimal control problem is well-posed.

## 3.1 Admissibility

Recall that the Converse HJB problem proceeds from a performance objective and the value function. Equivalently, we may specify the pair $(V, q)$ where $V$ is the value function, and $q(x)$ is the function that corresponds to the cost $J$ (cf. eqn. (2)). In some cases, there might not be any sensible dynamics that correspond to this choice of $V$ and $q$. This motivates the following definition.

**Definition 3** *The pair $(V, q)$ is said to be* ***admissible*** *for the CoHJB problem if there exists a continuous $g$ and $f$, with $f(0) = 0$ such that $f, g, V$ and $q$ satisfy the HJB (4). Furthermore, we define the set $\mathcal{A}$ to be the set of all admissible pairs, $(V, q)$.*

A more explicit characterization of the admissible set $\mathcal{A}$ is given by the following theorem:

**Theorem 1** *$(V, q) \in \mathcal{A}$ if and only if $q$ can be factored into the product:*

$$q = V_x h \tag{6}$$

*for $h \in \mathbf{C}$, $h(0) = 0$.*

**Proof:**
$(\Rightarrow)$ Assume $(V, q) \in \mathcal{A}$, then there exist a $g$ and $f$, $f(0) = 0$ such that the HJB is satisfied:

$$V_x f - \frac{1}{4} V_x g g^T V_x^T + q = 0 \tag{7}$$

Hence,

$$q = V_x \left(\frac{1}{4} g g^T V_x^T - f\right) \tag{8}$$

Since $\frac{1}{4} g g^T V_x^T - f \in \mathbf{C}$ and

$$\frac{1}{4} g(0) g(0)^T V_x(0)^T - f(0) = 0$$

$(f(0) = 0$ and $V_x(0) = 0)$ then taking $h = \frac{1}{4} g g^T V_x^T - f$ satisfies the theorem.

$(\Leftarrow)$ Assume $q = V_x h$, with $h \in \mathbf{C}$, $h(0) = 0$. For any $g \in \mathbf{C}$ , let,

$$f = \frac{1}{4} g g^T V_x^T + h \tag{9}$$

hence $f \in \mathbf{C}$, $f(0) = 0$ and $f, g, V$ and $q$ satisfy the HJB. Therefore $(V, q) \in \mathcal{A}$. $\diamond$

# 4 Converse HJB Optimality of Control Schemes

In this section, we derive conditions under which various control schemes solve the OCP (optimal control problem). In all cases we shall begin with the implicit assumption that we are only considering pairs $(V, q) \in \mathcal{A}$.

## 4.1 Notation

First we establish notation concerning the linearization of the nonlinear system (1) that will be used later. This notation is not only needed to describe a controller designed by Jacobian linearization, but since any controller that is globally optimal must also be locally optimal, it is useful for pinning down parameters that otherwise might appear to be free in various techniques.

We will use the following conventions for differentiating between the linear portion of a system, and the purely nonlinear portion: Let

$$\dot{x} = f(x) + g(x)u = Ax + \hat{f}(x) + (B + \hat{g}(x))u \tag{10}$$

where $A$ and $B$ represent the linear portion of the nonlinear system and are given by:

$$A = \frac{\partial f(0)}{\partial x} \qquad B = g(0) \tag{11}$$

Both $\hat{f}$ and $\hat{g}$ denote the nonlinear portions.

We will also designate the portions of $q(x)$ and $V$ that correspond to the linearized version of the optimal control problem: Expand $q$ as:

$$q = x^T Qx + \hat{q}(x) \tag{12}$$

with $\hat{q}(x)$ of order $O(x^3)$. Then $A$, $B$ and $Q$ correspond to a linear quadratic regulator problem whose solution is given by the Riccati equation:

$$A^T P + PA - PBB^T P + Q = 0 \tag{13}$$

Since locally (i.e. in a neighborhood of the origin) $V$ must agree with the solution of the linearized problem, it will be of the form:

$$V = x^T Px + \hat{V}(x) \tag{14}$$

where $\hat{V}(x)$ is of order $O(x^3)$.

## 4.2 Design Techniques

In this section we introduce our three benchmark design techniques: Jacobian linearization, pseudo-Jacobian linearization, and feedback linearization. In each case, we shall require that the controller produced by each of these methods be optimal when applied to any linear system. This is equivalent to requiring that they be locally optimal (i.e. in a neighborhood of the origin), since the nonlinear system is locally approximated by its linearization.

**Jacobian linearization (JL)**

Our technique for Jacobian Linearization is standard. First consider the linearized system, and design a linear controller which is optimal for the linearized system.

More specifically, given the nonlinear system:

$$\dot{x} = f(x) + g(x)u \qquad f(0) = 0 \tag{15}$$

the Jacobian linearized controller is found by linearizing the system around the origin ($x = 0$) resulting in the linearized system:

$$\dot{x} = Ax + Bu \tag{16}$$

where $A$ and $B$ are given in (11). Solving the corresponding Riccati equation (13) provides the appropriate control action:

$$u_{jl} = -B^T Px \tag{17}$$

**Pseudo-Jacobian linearization (PJL)**

In addition to standard Jacobian linearization, we propose an alternative scheme which we will refer to as *pseudo-Jacobian linearization.* The idea is to use the solution of the Riccati equation $P$ which corresponds to the linearized system, but substitute $g(x)$ for $B$ in the JL controller, controller. i.e.:

$$u_{pjl} = -g^T Px \tag{18}$$

**Feedback Linearization (FL)**

For feedback linearization, we consider dynamics that possess a special form:

$$\left.\begin{array}{rcl} \dot{x}_1 & = & A_1 x \\ \dot{x}_2 & = & f_2(x) + g_2(x)u \end{array}\right\} = f(x) + g(x)u \tag{19}$$

with $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$ and $g_2(x)$ assumed to be invertible.

In order that the FL controller be optimal for the linearized system, we require that it take the following form:

$$u_{fl} = -g_2^{-1}(x)(f_2(x) - [0\ I](A - BB^T P)x) \tag{20}$$

with $[0\ I]$ being $n_2 \times n$ (i.e. $[0\ I]$ picks off the bottom $n_2$ rows of $A - BB^T P$). This results in the correct closed-loop dynamics:

$$\dot{x} = (A - BB^T P)x. \tag{21}$$

## 4.3 Optimality

We will show how by assuming that the value function $V$ is known, and requiring that the optimal control be equal to the control under consideration, equations can be derived from manipulation of the HJB which must be satisfied. This approach will be analyzed in detail for Jacobian linearization, with the approach for pseudo-Jacobian linearization and feedback linearization being similar, and left to Appendix A.

- **Jacobian Linearization (JL)**

Recall that the JL controller is given by:

$$u_{jl} = -B^T P x \tag{22}$$

where $B = g(0)$ and $P$ solves the Riccati equation (13). Now, assuming that JL is globally optimal, i.e. $u_{jl} = u^*$, gives from equation (5):

$$-\frac{1}{2} g^T V_x^T = -B^T P x. \tag{23}$$

Using this result in the HJB (4) gives:

$$V_x f - x^T P B B^T P x + q = 0 \tag{24}$$

or in its more expanded form,

$$(2x^T P + \hat{V}_x)(Ax + \hat{f}) - x^T P B B^T P x + x^T Q x + \hat{q} = 0. \tag{25}$$

Finally using the fact the $P$ is the solution to the Riccati equation (13), leaves

$$2x^T P \hat{f} + \hat{V}_x A x + \hat{V}_x \hat{f} + \hat{q} = 0 \tag{26}$$

or

$$2x^T P \hat{f} + \hat{V}_x f + \hat{q} = 0 \tag{27}$$

which characterizes optimality for the JL design technique.

Hence we have derived the following necessary condition for JL to be optimal:

**Theorem 2** *If Jacobian linearization is optimal, then the following relationship holds:*

$$2x^T P \hat{f} + \hat{V}_x f + \hat{q} = 0.$$

Taking this analysis one step further, if $g$ is invertible then (23) can be solved for $V_x$ in which case we obtain:

$$V_x = 2x^T PBg^{-1}. \tag{28}$$

Continuing by substituting (28) into the HJB (4) yields:

$$2x^T PBg^{-1} f - x^T PBB^T Px + q = 0 \tag{29}$$

Using the Riccati equation (13) to substitute for $x^T Qx$ in the above gives:

$$2x^T P(Bg^{-1} f - Ax) + \hat{q} = 0 \tag{30}$$

This leads to the following statement:

**Theorem 3** *If Jacobian Linearization is optimal and $g$ is invertible, then the following relationship must hold:*

$$2x^T P(Bg^{-1} f - Ax) + \hat{q} = 0$$

If $g$ is not invertible (but assume $g^T g$ is invertible), then we may still write:

$$V_x = 2x^T PB(g^T g)^{-1} g^T + g_\perp^T \tag{31}$$

where:

$$g_\perp^T g = 0 \tag{32}$$

Substitution into the HJB yields,

$$(2x^T PB(g^T g)^{-1} g^T + g_\perp^T) f - x^T PBB^T Px + q = 0. \tag{33}$$

Finally, using the fact that $P$ satisfies the Riccati equation (13) gives the following condition for optimality:

$$2x^T P((B(g^T g)^{-1} g^T + g_\perp^T) f - Ax) + \hat{q} = 0 \tag{34}$$

**Theorem 4** *If Jacobian linearization is optimal, then there exists a $g_\perp$, such that $g_\perp^T g = 0$ and the following is satisfied:*

$$2x^T P((B(g^T g)^{-1} g^T + g_\perp^T) f - Ax) + \hat{q} = 0.$$

## 4.4 Summary of equations for optimality

The following presents a summary of the equations characterizing the necessary conditions which systems must satisfy for the following design techniques to be optimal. Derivations of the equations for pseudo-Jacobian linearization and feedback linearization can be found in Appendix A.

- Jacobian Linearization: $u(x) = -B^T Px$
  - $2x^T P\hat{f} + \hat{V}_x f + \hat{q} = 0$
  - $g$ invertible: $2x^T P(Bg^{-1} f - Ax) + \hat{q} = 0$
  - $g$ not invertible, ($g_\perp^T g = 0$) $2x^T P((B(g^T g)^{-1} g^T + g_\perp^T) f - Ax) + \hat{q} = 0$
- Pseudo-Jacobian Linearization: $u(x) = -g(x)Px$
  - $2x^T P\hat{f} + \hat{V}_x f - x^T P(gg^T - BB^T)Px + \hat{q} = 0$
  - $g$ invertible: $2x^T P\hat{f} - x^T P(gg^T - BB^T)Px + \hat{q} = 0$
  - $g$ not invertible, ($g_\perp^T g = 0$): $2x^T P\hat{f} + g_\perp^T f - x^T P(gg^T - BB^T)Px + \hat{q} = 0$
- Feedback Linearization: $u(x) = -g_2^{-1}(x)(f_2(x) - [0\ I](A - BB^T P)x)$
  - $V_x f - (\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1} (\hat{f}_2 + [0\ I]BB^T Px) + q = 0$
  - $2(\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1} f_2 + g_{2_\perp}^T A_1 x - (\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1} (\hat{f}_2 + [0\ I]BB^T Px) + q = 0$

## 5 2-D Nonlinear Oscillator

The true usefulness of the previous results is that they can be used to generate example systems for which one of the design techniques will be optimal. This allows for a nice comparison of the various techniques, on somewhat biased examples where one of the techniques is optimal. It presents the opportunity to understand what the optimal technique is doing correct, and why the other techniques fail to achieve this optimality.

We will use the 2-D nonlinear oscillator as our benchmark example, and explicitly produce systems where each of the above three techniques is optimal.

Consider the following dynamics which we refer to as the 2-D nonlinear oscillator.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u, \end{cases} \tag{35}$$

For this system, the HJB can be written as,

$$V_1 x_2 + V_2 f - \frac{1}{4}(V_2 g)^2 + q(x) = 0 \tag{36}$$

where

$$V = \min_u \int_0^\infty (q(x) + u^2)dt \tag{37}$$

and $V_1 = \frac{\partial V}{\partial x_1}$, $V_2 = \frac{\partial V}{\partial x_2}$.

**Admissible $V$ and $q$ for the 2-D oscillator**
Since in the 2-D oscillator, the dynamics have been constrained a-priori by assuming $\dot{x}_1 = x_2$, Theorem 1 does not strictly apply in its present form. Instead, we are able to use the structure of the dynamics to obtain more specific conditions under which the pair $(V, q)$ will be admissible. First note that the HJB (36) can be solved explicitly for $f$:

$$f = \frac{1}{4}V_2 g^2 - \frac{(V_1 x_2 + q(x))}{V_2} \tag{38}$$

This leads to the following characterization of admissibility:

**Theorem 5** *$(V, q) \in \mathcal{A}$ for the 2-D nonlinear oscillator iff*

$$\frac{(V_1 x_2 + q(x))}{V_2} < \infty, \quad \forall x \tag{39}$$

*and*

$$\lim_{x \to 0} \frac{(V_1 x_2 + q(x))}{V_2} = 0 \tag{40}$$

**Remarks:** From equation (38), it is clear that for continuous $g$, and $V \in C^1$, we have:

$$f(0) = -\lim_{x \to 0} \frac{(V_1 x_2 + q(x))}{V_2} = 0 \tag{41}$$

from which follows equation (40)

Furthermore, to insure $f$ is continuous, we impose (39):

$$\frac{(V_1 x_2 + q(x))}{V_2} < \infty, \quad \forall x. \tag{42}$$

See Appendix B for a derivation of admissible *quadratic* $q$ and $V$.

## 5.1 Generating Optimal Systems

Below, we briefly outline a procedure for creating systems for which JL will be optimal. A summary of useful equations for all three methods is also provided.

- **Jacobian Linearization**

Consider the linearization of the 2-D nonlinear oscillator (35) at the origin:

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} -a_1- \\ -a_2- \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\partial f(0)}{\partial x_1} & \frac{\partial f(0)}{\partial x_2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ g(0) \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} -p_1- \\ -p_2- \end{bmatrix}$$

be the solution of the Riccati equation:

$$A^T P + PA - PBB^T P + Q = 0$$

where $q = x^T Qx + O(x^3)$. Recall that $V$ must be of the form,

$$V = x^T Px + O(x^3).$$

since locally the system looks like its linearization, and hence, $V$ must locally look like the value function for the linearized system (i.e. $x^T Px$).

To determine how to generate systems for which Jacobian linearization is optimal, we proceed as in the general case by first designing a locally optimal (i.e. for the nonlinear system linearized at the origin) controller using Jacobian linearization:

$$u_{jl} = -B^T Px = -g(0)p_2 x$$

For this controller to be *globally* optimal, it must satisfy $u_{jl} = u^*$ or

$$-g(0)p_2 x = -\frac{1}{2} g V_2$$

which gives $g = \frac{2g(0)p_2 x}{V_2}$ and from (38) yields:

$$f = \frac{(g(0)p_2 x)^2}{V_2} - \frac{V_1 x_2}{V_2} - \frac{q(x)}{V_2}$$

So, given admissible $V$ and $q$, as well as choosing g(0), we use the quadratic portion of $V$ to determine $P$ and finally the above equations to determine $f$ and $g$ for which Jacobian Linearization will be optimal.

In a similar manner, equations may be derived for other techniques. We summarize the results below. For a derivation of these equations, see Appendix B.

- Jacobian Linearization: $u^* = -g(0)p_2 x$

$$g = \frac{2g(0)p_2 x}{V_2} \tag{43}$$

$$f = \frac{(g(0)p_2 x)^2}{V_2} - \frac{V_1 x_2 + q(x)}{V_2} \tag{44}$$

- Pseudo-Jacobian Linearization: $u^* = -g(x)p_2x$

$$\begin{aligned} g &= \text{arbitrary} && (45) \\ f &= \frac{1}{2}p_2xg^2 - \frac{V_1x_2 + q(x)}{2p_2x} && (46) \end{aligned}$$

- Feedback Linearization: $u^* = -(1/g)(f + kx)$

$$\begin{aligned} g^2 &= \frac{4([g(0)^2p_2 - a_2]x)}{V_2} - \frac{4}{V_2^2}(V_1x_2 + q(x)) && (47) \\ f &= [g(0)^2p_2 - a_2]x - \frac{2}{V_2}(V_1x_2 + q(x)) && (48) \end{aligned}$$

Note that each of the above equations places even further restrictions on admissible $V$ and $q$. For instance, in JL, we would like that $V_2$ divides $p_2x$. In pseudo JL ($u^* = -g(x)p_2x$), any admissible $V$ of the form $V = x^TPx + O(x_1^3)$ will work (i.e. $V_2 = 2p_2x$). FL seems to require more restrictive conditions than JL and PJL. In addition to choosing $V$ and $q$ so that the resulting $f$ and $g$ are continuous, we need $g > 0$ for all $x$ so that the system will be feedback linearizable. This can be a tricky process. Finally, it is interesting to note that for $V$ and $q$ quadratic, $f$ and $g$ can only come from a linear system for JL or FL to be optimal.

# 6 Examples

Three examples are presented in this section, each being optimal for one of our three benchmark techniques.

## 6.1 Example 1: JL optimal

This first example was created by following the procedure outlined in the previous section. By choosing an admissible pair $(V, q)$ to be:

$$\begin{aligned} V &= 2x_1^2 + cos^2(x_1) - 2(1 + x_2)e^{-x_2} + 1 \\ q &= x_2^2 \end{aligned}$$

and requiring that $g(0) = 1$, equations (43) and (44) were used to generate $f$ and $g$:

$$\begin{aligned} f &= -2e^{x_2}x_1 + e^{x_2}cos(x_1)sin(x_1) \\ g &= e^{x_2} \end{aligned}$$

This leads to the following dynamics for the 2-D nonlinear oscillator:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2e^{x_2}x_1 + e^{x_2}cos(x_1)sin(x_1) + e^{x_2}u \end{cases} \qquad (49)$$

with performance objective:

$$J = \int_0^\infty x_2^2 + u^2.$$

By design, Jacobian linearization produces the optimal controller which is:

$$u_{jl} = -x_2.$$

A simulation of the system from the initial condition $[0, 10]$ and using $u_{jl}$ is given in Figure 1. The upper left corner shows state trajectories of $x_1$ and $x_2$ as a function of time. The upper right corner is a phase portrait with $x_1$ on the x-axis and $x_2$ on the y-axis. This lower left corner plots the control action $u$ as a

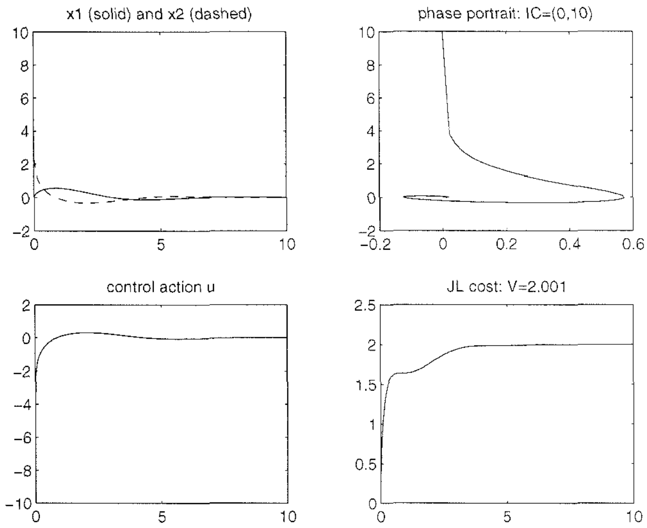


Figure 1: Ex. 1: Jacobian Linearization, initial condition $[0, 10]$

function of time, and the lower right corner is an integration of the cost $J$ as as a function of time. From the bottom right plot in Figure 1, we see that the optimal cost is 2, which is achieved by the JL controller.

The pseudo-Jacobian linearized controller is:

$$u_{pjl} = -e^{x_2}x_2$$

which differs from the JL controller only in the addition of the $e^{x_2}$ term. Unfortunately, this term has drastic effects on the amount of control action used and results in trajectories and control action as shown in Figure 2. The state trajectories are very similar to those for the JL controller, with the major difference coming in the initial amount of control action expended. This large initial burst of control is clearly unnecessary and accounts for the increase in cost by a factor of 25 over JL.

The Feedback linearized controller is given by a somewhat more complicated expression:

$$u_{fl} = -(e^{-x_2})(-2e^{x_2}x_1 + e^{x_2}cos(x_1)sin(x_1) + x_1 + x_2).$$

Simulation results for $u_{fl}$ are shown in Figure 3. The trajectories and control action differ dramatically from the optimal. It is clear that the optimal trajectories, as shown in Figure 1, do not correspond to a linear system, yet FL forces the closed loop to be linear. This results in a cost that exceeds 500.

An interesting note is that the explanation that FL does not perform well because it cancels "beneficial" drift nonlinearities does not apply. For example, the control

$$u = -(e^{-x_2})(-2e^{x_2}x_1 + e^{x_2}cos(x_1)sin(x_1)) - x_1 - x_2$$

cancels the same nonlinearities as $u_{fl}$, but simulation results indicate that this controller performs much closer to the optimal. The cost for the above initial conditions was within 1% of optimal.

## 6.2 Example 2: PJL optimal

In this example, we analyze the following system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{2}(x_2e^{2x_1} - 2x_1 - x_2) + e^{x_1}u\end{aligned}$$

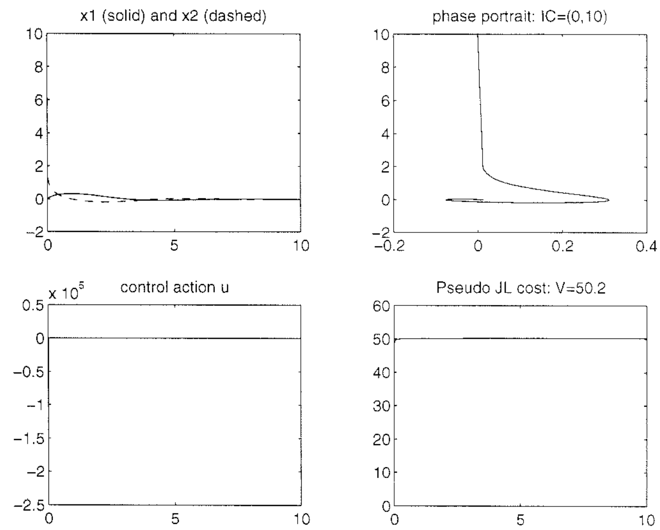


Figure 2: Ex. 1: Pseudo-Jacobian Linearization, initial condition $[0, 10]$

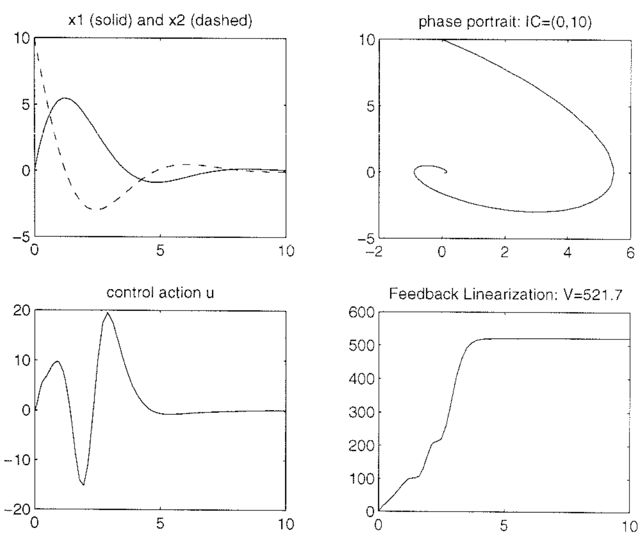


Figure 3: Ex. 1: Feedback Linearization, initial condition $[0, 10]$

subject to the cost:

$$J = \int_0^\infty x_2^2 + u^2.$$

This example was generated using $V = x_1^2 + x_2^2$ and $q = x_2^2$ so that pseudo-Jacobian linearization would be optimal:

$$u_{pjl} = -e^{x_1} x_2$$

Simulation results of the optimal from the initial condition $[-5, 0]$ are shown in Figure 4. The optimal cost from this initial condition turns out to be 25.

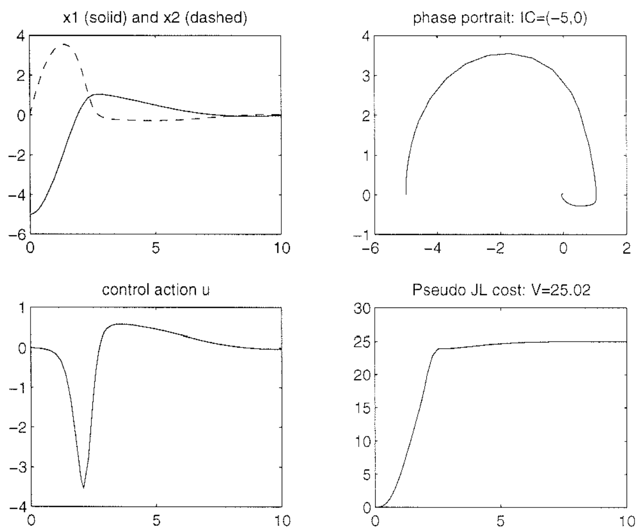


Figure 4: Ex. 2: Pseudo-Jacobian Linearization, initial condition $[-5, 0]$

Simulation results for the JL controller:

$$u_{jl} = -x_2$$

are given in Figure 5. This controller actually does not stabilize the system, as can be seen in the figure. It is interesting to note that it fails for exactly the opposite reason that PJL performed poorly in the previous example. In the previous example, PJL used too much control effort when it was unnecessary. In this example, it is JL that uses too little control effort, and fails to stabilize the system.

The FL controller is given by:

$$u_{fl} = -\frac{1}{2} x_2 (e^{2x_1} + 1) e^{-x_1} \tag{50}$$

with simulation results in Figure 6. Again it results in an exceedingly high cost ($J = 4381$) because it fails to take advantage of nonlinearities in the system and forces the system to perform globally as a linear system. The advantage is that stability is guaranteed, but at the expense of large unnecessary control actions.

## 6.3 Example 3: FL optimal

In the final example, we create a system where feedback linearization is optimal. For this system we choose the admissible pair $(V, q)$ as:

$$\begin{aligned} V &= e^{x_1^2 + x_2^2} - 1 \\ q &= x_2^2 \end{aligned}$$

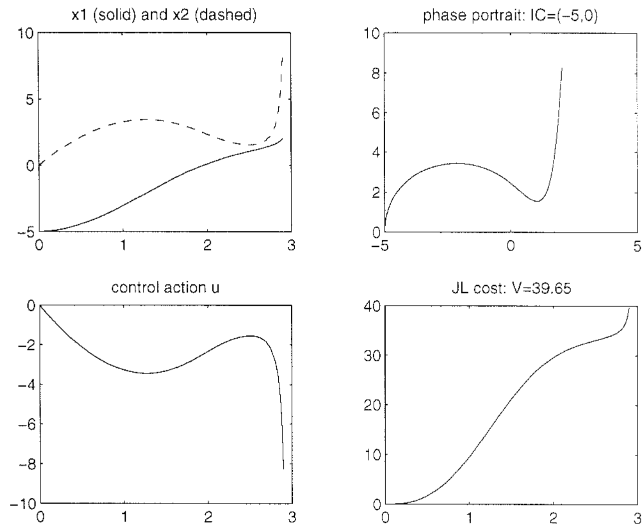


Figure 5: Ex. 2: Jacobian Linearization, initial condition [−5, 0]

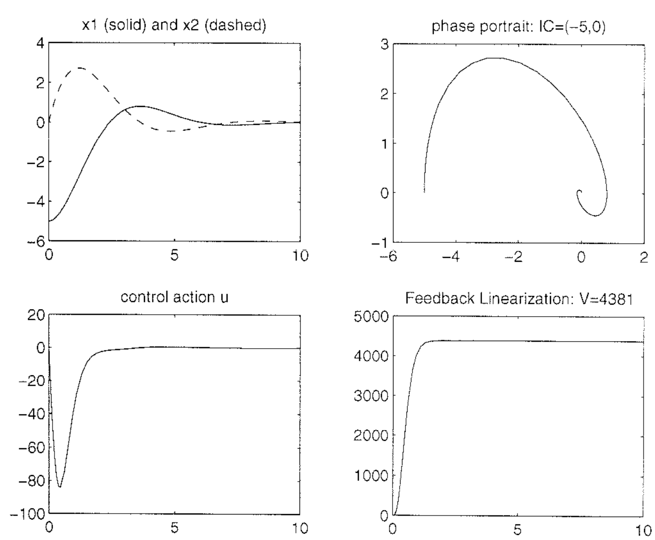


Figure 6: Ex. 2: Feedback Linearization, initial condition [−5, 0]

and iterate using equations (48) and (47) to arrive at:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2 - e^{-(x_1^2+x_2^2)}x_2 + \sqrt{2e^{-(x_1^2+x_2^2)} - e^{-2(x_1^2+x_2^2)}}u\end{aligned}$$

The optimal/feedback linearized controller is:

$$u_{fl} = -\sqrt{2e^{-(x_1^2+x_2^2)} - e^{-2(x_1^2+x_2^2)}}e^{x_1^2+x_2^2}x_2 \tag{51}$$

and simulation results of this system from the initial condition [1, 1] are presented in Figure 7.

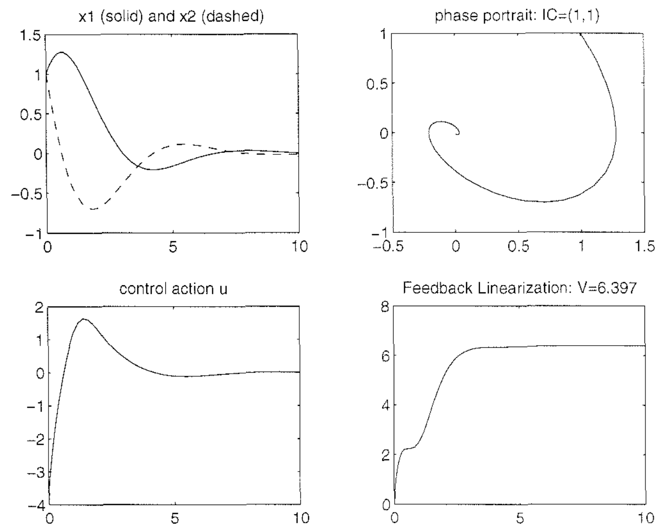


Figure 7: Ex. 3: Feedback Linearization, initial condition [1, 1]

In this example, both JL:

$$u_{jl} = -x_2$$

and PJL

$$u_{pjl} = -\sqrt{2e^{-(x_1^2+x_2^2)} - e^{-2(x_1^2+x_2^2)}}x_2$$

result in instability, despite using quite different control actions. The results from JL and PJL are plotted in Figures 8 and 9 respectively. In this case, merely the guarantee of stability provided by FL is a large advantage over the JL and PJL designs.

# 7 Conclusions

In this paper we demonstrated that by approaching the issue of optimality and the HJB pde from the converse point of view, interesting examples and necessary conditions for design techniques to be optimal can be derived. Specifically, we examined three simple nonlinear techniques: Jacobian linearization, pseudo-Jacobian linearization and feedback linearization. For each of these techniques, necessary conditions for optimality were presented, but perhaps more useful, equations were provided which aid in producing nontrivial nonlinear examples for which each technique will result in the optimal controller. Three such examples were generated, each being optimal for only one of the techniques, and suboptimal to unstable for the others.

Specifically, we observed the following properties of the three techniques under consideration. When Jacobian linearization or pseudo-Jacobian linearization was found to be stabilizing, they tended to outperform

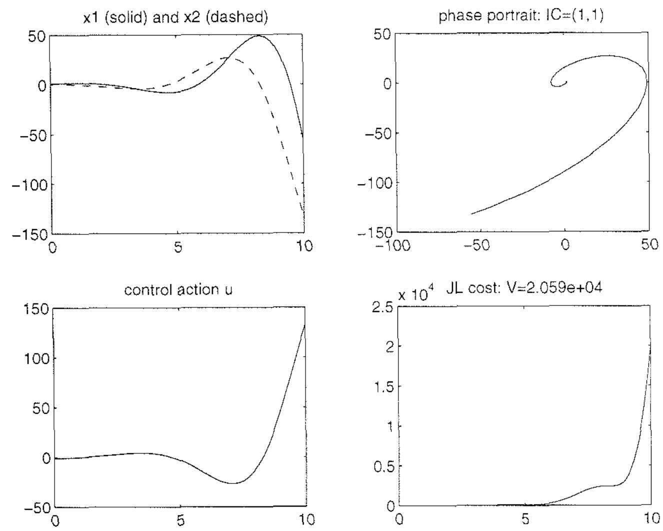


Figure 8: Ex. 3: Jacobian Linearization, initial condition [1, 1]

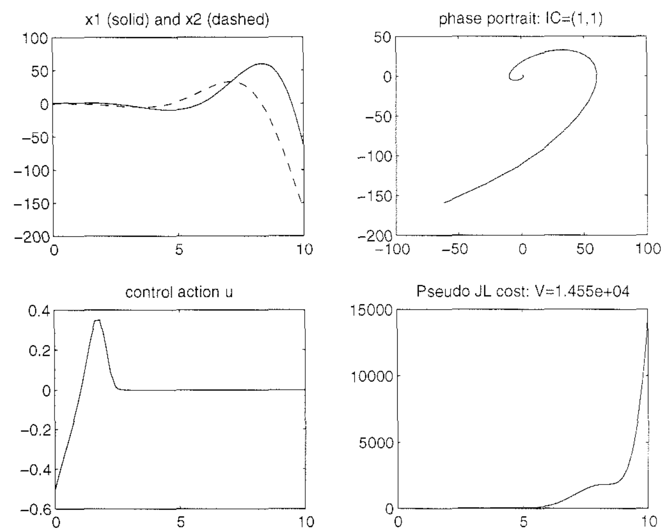


Figure 9: Ex. 3: Pseudo-Jacobian Linearization, initial condition [1, 1]

feedback linearization. Furthermore, it was much more difficult to concoct systems for which feedback linearization was optimal, tending to indicate that such systems are more scarce than systems for which JL or PJL is optimal. PJL clearly seemed to require the least restrictive conditions for which it could be optimal. On the other hand, these findings were to be expected considering that feedback linearization guarantees global stability, and it is reasonable to expect that a tradeoff must be made for such a requirement.

Our hope is that by providing methods to generate examples where various techniques perform poorly and well, not only can we establish benchmarks for testing techniques, but hopefully provide a glimpse into the workings and failing of each technique. If we can gain insight into the nonlinear control problem by such an approach, then perhaps new techniques may emerge which address these failings, and ultimately provide better solutions.

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# A Optimality of Design Techniques

## A.1 Pseudo Jacobian Linearization (PJL)

Assume that the pseudo-Jacobian linearized controller $u_{pjl} = -g^T(x)Px$ is optimal, then

$$-\frac{1}{2}g^T V_x^T = -g^T Px. \tag{52}$$

Using this in the HJB (4) gives:

$$V_x f - xPgg^T Px + q = 0 \tag{53}$$

Using the fact the $P$ satisfies the Riccati equation (13) and substituting for $Q$ leaves:

$$2x^T P\hat{f} + \hat{V}_x f - x^T P(gg^T - BB^T)Px + \hat{q} = 0 \tag{54}$$

If $g$ is invertible, then from (52):

$$V_x = 2x^T P \tag{55}$$

Using this in the HJB (4) gives:

$$2x^T Pf - x^T Pgg^T Px + q = 0 \tag{56}$$

Using the fact that $P$ satisfies the Riccati equation yields

$$2x^T P\hat{f} - x^T P(gg^T - BB^T)Px + \hat{q} = 0 \tag{57}$$

On the other hand, if $g$ is not invertible, then from (52):

$$V_x = 2x^T P + g_\perp^T \tag{58}$$

where $g_\perp^T g = 0$. Using this in the HJB (4) gives:

$$2x^T Pf + g_\perp^T f - x^T Pgg^T Px + q = 0 \tag{59}$$

and the fact that $P$ satisfies the Riccati equation leaves

$$2x^T P\hat{f} + g_\perp^T f - x^T P(gg^T - BB^T)Px + \hat{q} = 0. \tag{60}$$

## A.2 Feedback Linearization (FL)

We consider systems of the form:

$$\left.\begin{array}{rcl} \dot{x}_1 &=& A_1 x \\ \dot{x}_2 &=& f_2(x) + g_2(x)u \end{array}\right\} = f(x) + g(x)u \tag{61}$$

where $g_2(x)$ is assumed to be invertible.

The feedback linearized controller is formed as follows:

$$u_{fl} = -g_2^{-1}(x)(f_2(x) - [0\ I](A - BB^T P)x) \tag{62}$$

For this controller to be optimal, we must have:

$$-\frac{1}{2}g^T V_x^T = -g_2^{-1}(x)(f_2(x) - [0\ I](A - BB^T P)x) = -g_2^{-1}(x)(\hat{f}_2(x) + [0\ I]BB^T Px) \tag{63}$$

Substituting this into the HJB gives:

$$V_x f - (\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1} (\hat{f}_2 + [0\ I]BB^T Px) + q = 0 \tag{64}$$

A similar characterization can be given as follows: we may write $V_x$ as:

$$V_x = [g_{2\perp}^T\ ,\ 2(\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1}] \tag{65}$$

then the HJB equation can be written as follows:

$$2(\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1} f_2 + g_{2\perp}^T A_1 x - (\hat{f}_2 + [0\ I]BB^T Px)^T (g_2 g_2^T)^{-1} (\hat{f}_2 + [0\ I]BB^T Px) + q = 0 \tag{66}$$

# B Optimality for the 2-D Nonlinear Oscillator

Recall that the 2-D nonlinear oscillator is given by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + g(x)u \end{aligned}$$

**Pseudo-Jacobian linearization**
If $u_{pjl}$ is indeed the optimal controller, then:

$$-gp_2 x = -\frac{1}{2} g V_2 \tag{67}$$

which implies,

$$V_2 = 2p_2 x \tag{68}$$

(The notation $p_2$ etc. was established in Section 5.1.) Hence for any admissible system with $V_2$ of the form $V_2 = 2p_2 x$, this control scheme will be optimal and have $f$ given by:

$$f = \frac{1}{2} p_2 x g^2 - \frac{V_1 x_2 + q(x)}{2p_2 x} \tag{69}$$

Another way to characterize value functions, $V$, that have $V_2 = 2p_2 x$, is that $V$ will be of the form $V = x^T Px + O(x_1^3)$.

**Feedback linearization**
A FL controller for the 2-D oscillator will take the form:

$$u_{fl} = -\frac{1}{g}(f + k_1 x_1 + k_2 x_2) \tag{70}$$

For this controller to be optimal, $u_{fl} = u^*$ or equivalently

$$-\frac{1}{g}(f + k_1 x_1 + k_2 x_2) = -\frac{1}{2} g V_2 \tag{71}$$

which gives,

$$g^2 = \frac{2}{V_2}(f + k_1 x_1 + k_2 x_2) \tag{72}$$

Substituting (72) in the HJB (36) yields,

$$V_1 x_2 + V_2 f - \frac{1}{4} V_2 (2(f + k_1 x_1 + k_2 x_2)) + q(x) = 0 \tag{73}$$

and according to (73), $f$ is given by,

$$f = k_1 x_1 + k_2 x_2 - \frac{2}{V_2}(V_1 x_2 + q(x)) \tag{74}$$

To find $g$ we use (74) to substitute for $f$ in (72), which yields,

$$g^2 = \frac{4}{V_2}(k_1 x_1 + k_2 x_2) - \frac{4}{V_2^2}(V_1 x_2 + q(x)) \tag{75}$$

Additionally, we require $g^2 > 0$ for the system to be globally feedback linearizable.

Similar to JL, the FL controller $u_{fl}$ is designed so that the closed loop system is optimal in a neighborhood of the origin, with respect to the linearized system. Since the closed loop system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \tag{76}$$

and the closed loop corresponding to the LQR solution of the linearized system is,

$$\dot{x} = A - BB^T P \tag{77}$$

where $A$,$B$ and $P$ are as previously defined in (11) and (13). Hence we have:

$$A - BB^T P = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \tag{78}$$

or

$$k = [k_1 \;\; k_2] = g(0)^2 p_2 - a_2 = [g(0)^2 p_{21} - \frac{\partial f(0)}{\partial x_1} \;\; g(0)^2 p_{22} - \frac{\partial f(0)}{\partial x_2}] \tag{79}$$

In summary, for FL of 2-D oscillator to be optimal, the following conditions are to be satisfied:

$$f = [g(0)^2 p_2 - a_2]x - \frac{2}{V_2}(V_1 x_2 + q(x)) \tag{80}$$

$$g^2 = \frac{4([g(0)^2 p_2 - a_2]x)}{V_2} - \frac{4}{V_2^2}(V_1 x_2 + q(x)) > 0 \tag{81}$$

$$\tag{82}$$

Unfortunately, there does not appear to be a general, systematic procedure for choosing $V$ and $q$ such that equations (81 and 80) are satisfied. Hence, determining systems "backward" for which FL is optimal is in general a trial and error process.

## B.1 Admissible Quadratic $q$ and $V$

As a special case of interest, the set of admissible quadratic $V$ and $q$ will be characterized.
Assume the following form for $V$ and $q$:

$$V = p_1 x_1^2 + 2p_{12} x_1 x_2 + p_2 x_2^2 \tag{83}$$

$$q = q_1 x_1^2 + 2q_{12} x_1 x_2 + q_2 x_2^2 \tag{84}$$

hence,

$$V_1 = 2p_1 x_1 + 2p_{12} x_2 \tag{85}$$

$$V_2 = 2p_{12} x_1 + 2p_2 x_2 \tag{86}$$

According to the Theorem 5, $V$ and $q$ are admissible if the following condition is satisfied.

$$\frac{(2p_1x_1 + 2p_{12}x_2)x_2 + q_1x_1^2 + 2q_{12}x_1x_2 + q_2x_2^2}{2p_{12}x_1 + 2p_2x_2} < \infty \tag{87}$$

In order to satisfy (39), the denominator has to be a factor of the numerator, i.e. there exist $a$ and $b$ such that:

$$(2p_{12}x_1 + 2p_2x_2)(ax_1 + bx_2) = (2p_1x_1 + 2p_{12}x_2)x_2 + q_1x_1^2 + 2q_{12}x_1x_2 + q_2x_2^2 \tag{88}$$

This leads to the following condition,

$$p_{12}^2q_2 = 2p_{12}^3 - p_2^2q_1 - 2p_1p_{12}p_2 + 2p_{12}p_2q_{12}. \tag{89}$$

Equation (89) characterizes all admissible quadratic $V$ and $q$ for the 2-D nonlinear oscillator.