CDS

TECHNICAL MEMORANDUM NO. CIT-CDS 97-007
May, 1997

"A Stability Criterion for Systems with Nuetrally Stable Modes and Deadzone Nonlinearities"

Ulf Jönsson*

Control and Dynamical Systems
California Institute of Technology
Pasadena, California 91125

A Stability Criterion for Systems with Neutrally Stable Modes and Deadzone Nonlinearities

Ulf Jönsson*
Control and Dynamical Systems
California Institute of Technology
Mail Stop 116 81
Passadena, CA 911 25, USA
Email: ulfj@cds.caltech.edu

May 14, 1997

Abstract

Stability analysis is considered for feedback interconnections of deadzone nonlinearities with linear systems that has a neutrally stable mode. Such systems do not have a unique equilibrium point and the standard techniques from passivity and Lyapunov theory cannot be applied. A stability criterion that generalizes the Popov criterion for this class of systems is derived in this report and several examples will prove its applicability.

1 Introduction

Stability of linear time invariant systems in feedback interconnection with various nonlinearities have been studied extensively in the litterature, see for example, [1, 4, 6, 8, 9]. The case when the linear system is neutrally stable and when the nonlinearity is in the sector $[0, k]$ for some $k > 0$ is particularly hard. Such systems are often called critical in the absolute stability litterature. This term refers to the fact that such systems often are on the dividing line between unstable and (locally) exponentially stable systems. Neutral stability of a linear system means that there are simple modes on imaginary axis. This is a common situation in control applications where the integrator in a PI regulator corresponds to such a neutraly stable mode.

We will focus our attention on systems with a deadzone nonlinearity. We consider the simplest possible case with a single-input single-output linear time

*The author is also with the Department of Automatic Control at Lund Institute of Technology.

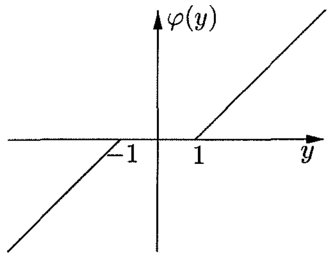

Figure 1: The deadzone nonlinearity

invariant system in a negative feedback interconnection with a deadzone nonlinearity:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= -\varphi(y). \end{aligned} \tag{1}$$

Here $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, and $C \in \mathbf{R}^{1\times n}$, where we assume that A has a simple eigenvalue at the origin and that all the other eigenvalues are strictly in the left half plane. It is further assumed that the mode corresponding to the eigenvalue at the origin is controllable and observable. This means that the transfer function for the linear part of the system has the form

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s}G_1(s),$$

where G_1 is stable, i.e., it has all poles strictly in the left half plane. The deadzone nonlinearity is defined by

$$\varphi(x) = \begin{cases} x - 1, & x > 1, \\ 0, & |x| \le 1, \\ x + 1, & x < -1. \end{cases}$$

See also Figure 1. It is important to note that the origin is not a unique equilibrium point for the system in (1). It is easy to see that the set of fixed points is $\{x : Ax = 0,\ Cx \in [-1, 1]\}$.

We will derive a stability result for the system in (1) based on a technique of separating the system response corresponding to the neutrally stable modes from the response of the stable modes. The technique has been used before by Yakubovich to derive Popov criteria for neutraly stable systems, see for example [8]. We obtain stronger results by using an additional multiplier in the criterion. The extra multiplier belongs to the causal subset of the multipliers in Zames and Falbs stability result for slope restricted nonlinearities, [10]

Our result can be adapted to more general situations with multi-input multi-output systems in feedback interconnection with perturbations consisting of, for example, deadzone nonlinearities and other nonlinearities in a diagonal structure.

Notation

The following notation will be used in the report:

- $\sigma_{\max}(M), \sigma_{\min}(M), \lambda_{\max}(M)$, and $\lambda_{\min}(M)$ denotes the maximum and minimum singular values and eigenvalues, respectively, of a matrix M.
- $\mathbf{L}_2[0,\infty)$ denotes the space of square integrable functions $f : [0,\infty) \to \mathbf{R}^n$, with norm defined by

$$\|f\|^2 = \int_0^\infty |f|^2 \, dt.$$

  The dimension n will be clear from the context.
- We use the standard notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \stackrel{\text{def}}{=} C(sI - A)^{-1}B + D$$

2 Main Result

The next theorem gives a stability criterion for the system in (1). Examples in Section 3 show that the multiplier H in the criterion is useful and we thus have an important extension of the corresponding Popov criterion.

Theorem 1. *Assume that*

(*i*) $\lim_{s\to 0} sG(s) > 0$,

(*ii*) *there exists* $\varepsilon > 0$ *such that*

$$Re\ (1 + j\omega\lambda + H(j\omega))(G(j\omega) + 1) \geq \varepsilon, \quad \forall \omega \in (0,\infty)$$

for some $\lambda \geq 0$ *and for some strictly proper and stable transfer function* $H(s)$ *with corresponding weighting function* h *satisfying* $\int_{-\infty}^{\infty} |h(t)| dt < 1$.

Then the solution to (1) is stable in the sense that

(*a*) *there exists* $c > 0$ *such that* $|x(t)| \leq c|x_0|$, *for all* $(x_0, t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+$,

(*b*) $x(t) \to \{x : Ax = 0, Cx \in [-1,1]\}$ *as* $t \to \infty$.

Remark 1. Note that condition (*i*) of the theorem statement is necessary in general. For example, consider the system in (1) when

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Hence, we have

$$G(s) = \frac{s-1}{s(s+1)},$$

which satisfies (ii) when $\lambda = 0$ and $H \equiv 0$. However, $\lim_{s\to 0} sG(s) = -1$, and it can verified that the system is unstable.

Remark 2. The theorem also holds if, for example, the deadzone nonlinearity saturates at high gains. The proof is essentially the same except that the last part must be slightly modified. The convergence to the fixed point set will for the case with saturated deadzone nonlinearities typically be slower than for the unsaturated case considered in the report.

Remark 3. The proof relies on the use of Lemma 1 in the appendix. This lemma is formulated more generally than necessary for the particular case studied in the report. It is useful for various extensions to multivariable cases where we have a critical system with many nonlinearities.

Proof of Main Theorem

We will for the proof of Theorem 1 transform the system in (1) such that $u \mapsto -u$ and $y \mapsto -y$, i.e.,

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= -Cx, \\ u &= \varphi(y). \end{aligned} \tag{2}$$

The first step of the proof is to derive two integral quadratic constraints (IQCs) for the deadzone φ. The first is the standard Popov IQC: Let $\lambda \geq 0$, then

$$\int_0^t \lambda u(\tau) dy(\tau) = \lambda \int_{y(0)}^{y(t)} \varphi(\sigma) d\sigma \geq -\lambda \int_0^{y(0)} \varphi(\sigma) d\sigma \geq -\gamma |y(0)|^2,$$

where $\gamma = \lambda/2$. For the second IQC, let H be as stated in the theorem. Then

$$\begin{aligned} \varphi(y)[y - \varphi(y) + h * (y - \varphi(y))] &\geq \varphi(y)[y - \varphi(y) - \sup_{y\in\mathbf{R}} |y - \varphi(y)| \cdot \|h\|_1] \\ &\geq \varphi(y)[y - \varphi(y) - 1] = 0, \end{aligned} \tag{3}$$

where $\|h\|_1 = \int_{-\infty}^{\infty} |h(t)| dt$. Let us introduce a state space realization $H(s) = C_H(sI - A_H)^{-1}B_H$, where A_H is Hurwitz. Then the integral (which is nonegative by the inequality (3))

$$2\int_0^t u(\tau)[y(\tau) - u(\tau) + (h * (y - u))(\tau)] d\tau$$

can be computed as (note that we have $y = -Cx$ in (2))

$$\int_0^t v^T M v \, dt,$$

where

$$v = \begin{bmatrix} C_H \\ 0 \end{bmatrix} z + \begin{bmatrix} -Cx - u \\ u \end{bmatrix}, \tag{4}$$

$$\dot{z} = A_H z + B_H(-Cx - u), \quad z(0) = 0, \tag{5}$$

and

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{6}$$

Next we note that condition (ii) in the theorem statement can be written as

$$\Phi(j\omega)^* M \Phi(j\omega) \leq -\varepsilon I, \quad \forall \omega \in (0, \infty), \tag{7}$$

where M is defined in (6) and where

$$\Phi(s) = \begin{bmatrix} (1 + H(s))(-G(s) - 1) - \lambda s G(s) \\ 1 \end{bmatrix}.$$

It is straightforward to verify that Φ has the state space realization

$$\Phi = \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right] \stackrel{\text{def}}{=} \left[\begin{array}{cc|c} A_H & -B_H C & -B_H \\ 0 & A & B \\ \hline C_H & -C - \lambda CA & -1 - \lambda CB \\ 0 & 0 & 1 \end{array}\right],$$

where it is easy to verify that $C_\Phi^T M C_\Phi = 0$ and that (A_Φ, B_Φ) is stabilizable.

We also notice that condition (i) in the theorem statement gives

$$\lim_{s \to 0} s(1 + \lambda s + H(s))(-G(s) - 1) = -(1 + H(0)) \lim_{s \to 0} sG(s) = k < 0, \tag{8}$$

where we use that $|H(0)| < 1$.

From (7) and (8) it follows that the conditions of Lemma 1 in the appendix are satisfied. For our special case the lemma implies that:

a. there exists a nonsingular matrix T such that

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right] \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}^{-1} = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & 0 & k \\ \hline C_1 & C_2 & D \end{array}\right],$$

where $D = D_\Phi$, and $C_2^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$,

b. there exists a scalar $P_2 = -1/k > 0$,

c. there exists a matrix $P_1 = P_1^T > 0$ such that the LMI (15) in the appendix holds.

Let us now define the new states obtained by using T as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T \begin{bmatrix} z \\ x \end{bmatrix},$$

where x is the state in (2) and where z is the state in (5). Furthermore, let

$$P = T^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} T, \quad V_1(t) = \begin{bmatrix} z \\ x \end{bmatrix}^T P \begin{bmatrix} z \\ x \end{bmatrix} = x_1^T P_1 x_1 + x_2^T P_2 x_2,$$

and

$$V_2(t) = 2\lambda \int_0^t u(\tau) dy(\tau) + \int_0^t v^T M v \, dt,$$

where $u = \varphi(y)$, and where v and M are defined in (4) and (6). From the discussion above we know that $V_2(t) \geq -\gamma |y(0)|^2$, where $\gamma > 0$. Finally, let $V(t) = V_1(t) + V_2(t)$. Differentiation and the use of (15) in the appendix gives

$$\dot{V} = \begin{bmatrix} x_1 \\ u \end{bmatrix}^T \begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D \end{bmatrix}^T \begin{bmatrix} 0 & P_1 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix} \leq -\varepsilon(|x_1|^2 + |u|^2) \tag{9}$$

for some positive $\varepsilon > 0$. The following conclusions can be drawn

1. $V(t)$ is monotonically decreasing function, i.e., $V(t) \leq V(0)$ for all $t \geq 0$. This implies that $|x(t)|^2 + |z(t)|^2 \leq \frac{1}{\lambda_{\min}(P)} \left(\lambda_{\max}(P) + \gamma \sigma_{\max}^2(C)\right) |x_0|^2$ for all $t \geq 0$.

2. Integration of (9) gives

$$\int_0^t (|x_1|^2 + |u|^2) dt \leq \frac{1}{\varepsilon}(V(0) - V(t)) \leq \frac{1}{\varepsilon}(\sigma_{\max}(P) + \gamma \sigma_{\max}^2(C))|x_0|^2$$

for all $t \geq 0$. Hence, $x_1, u \in \mathbf{L}_2[0, \infty)$. Furthermore, $\dot{x}_1 = A_1 x_1 + B_1 u$, which implies that also $\dot{x}_1 \in \mathbf{L}_2[0, \infty)$. This means that $x_1(t) \to 0$ as $t \to \infty$, which in particular means that all modes in the system (2) that corresponds to nonzero eigenvalues converges to zero.

We are now at the last step of the proof. We can find a nonsingular transformation matrix S such that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Sx, \quad z_1 \in \mathbf{R}^{n-1}, \quad z_2 \in \mathbf{R}$$

and such that the system in (2) is transformed into

$$\begin{aligned} \dot{z}_1 &= \widehat{A}_1 z_1 + \widehat{B}_2 \varphi(\widehat{C}_1 z_1 + z_2) \\ \dot{z}_2 &= -b\varphi(\widehat{C}_1 z_1 + z_2) \end{aligned}$$

where $\widehat{A}_1$ is Hurwitz and $b > 0$. The fixed points of (2) and its transformed version above are related as $\{x : Ax = 0,\ Cx \in [-1,1]\} = \{S^{-1}z : z_1 = 0,\ z_2 \in [-1,1]\}$, where $z^T = \begin{bmatrix} z_1^T & z_2 \end{bmatrix}$. It follows from the second conclusion above that $z_1 \to 0$ as $t \to \infty$. Hence, the proof follows if we can show that $z_2 \to [-1,1]$ as $t \to \infty$.

Let $0 < \varepsilon < 1$ and let $T_{1\varepsilon}$ be such that $|z_1(t)| \leq \varepsilon/(2\sigma_{\max}(\widehat{C}_1))$ for all $t \geq T_{1\varepsilon}$. We note that the set $\{z_2 : |z_2| \leq 1 + \varepsilon\}$ is invariant when $t \geq T_{1\varepsilon}$. In fact, if $|z_2| = 1 + \varepsilon$, then

$$\frac{d}{dt} z_2^2 = -2bz_2\varphi(\widehat{C}_1 z_1 + z_2) < 0,$$

since $|\widehat{C}_1 z_1| \leq \varepsilon/2$ when $t \geq T_{1\varepsilon}$. We also know from the first conclusion above that $|z_2(T_{1\varepsilon})| \leq (\sigma_{\max}(S)\kappa)^{1/2}|x_0|$, where $\kappa = \left(\lambda_{\max}(P) + \gamma\sigma_{\max}^2(C)\right)/\lambda_{\min}(P)$.

Assuming that this is larger than $1+\varepsilon$, then we have the following differential inequality

$$\frac{d}{dt}|z_2| \leq -b|z_2| + b(1 + \varepsilon/2), \text{ when } t \geq T_{1\varepsilon} \text{ and } |z_2(t)| > 1 + \varepsilon.$$

Hence,

$$|z_2(t)| \leq e^{-b(t-T_{1\varepsilon})}|z_2(T_{1\varepsilon})| + \int_{T_{1\varepsilon}}^{t} e^{-b(t-\tau)} b(1 + \varepsilon/2) d\tau, \quad t \geq T_{1\varepsilon}.$$

This implies that there exists $T_{2\varepsilon}$ such that $|z_2(T_{2\varepsilon})| \leq 1 + \varepsilon$ for all $t \geq T_{2e}$. In conclusion, we have $|z_2(t)| \leq 1+\varepsilon$, for all $t \geq T_{1\varepsilon} + T_{2\varepsilon}$. The proof follows since ε can be taken arbitrarily small.

3 Examples

We will in this section give several examples that illustrates the use of Theorem 1. In the first example we only need the Popov part in condition (*ii*) of the theorem.

Example 1. Let us consider the case when

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0.9 \end{bmatrix}, \tag{10}$$

which means that

$$G(s) = \frac{0.9}{s(s+1)^2}.$$

We have $\lim_{s \to 0} sG(s) = 0.9$, i.e., condition (*i*) of Theorem 1 is satisfied. If we take $H = 0$ and $\lambda = 1$ then condition (*ii*) is satisfied, see Figure 2. In Figure 3 we show a simulation of the system for the case when the initial condition is $x_0 = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}$.

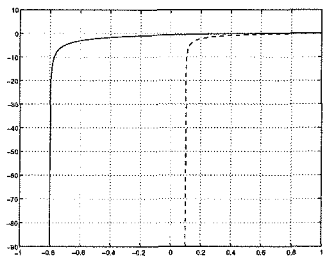

Figure 2: The solid line shows the Nyquist curve of $G(s) + 1$, which is not strictly in the right half plane. The dashed line shows the Nyqist curve of $(1 + s)(G(s) + 1)$, which is strictly in the right half plane. Hence, condition (ii) of Theorem 1 holds.

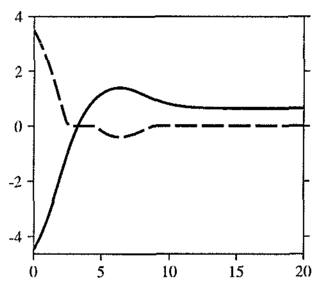

Figure 3: Simulation of the system in (1) when the system matrices are as in (10) and when the initial condition is as stated above. The solid line corresponds to y and the dashed line corresponds to u.

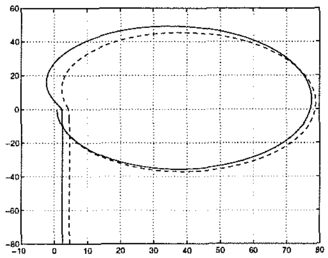

Figure 4: The solid line shows the Nyquist curve of $G(s) + 1$, which is not strictly in the right half plane. The dashed line shows the Nyqist curve of $(1 + H(s))(G(s) + 1)$, which is strictly in the right half plane. Hence, condition (ii) of Theorem 1 holds. The two Nyquist curves overlap at low frequencies.

In the next example we will need to use the multiplier H.

Example 2. For this example we let

$$A = \begin{bmatrix} -1.2 & -0.48 & -0.064 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 80 & 4.8 & 0.096 & 0.0006 \end{bmatrix}, \tag{11}$$

which means that

$$G(s) = 80\frac{(s + 0.02)^3}{s(s + 0.4)^3}.$$

We see that $\lim_{s \to 0} sG(s) = 0.01$, which means that condition (i) of Theorem 1 is satisfied. If we take $\lambda = 0$ and $H(s) =$ then condition (ii) is satisfied, see Figure 4. A simulation of the system is shown for the case when the initial condition is $x_0 = \begin{bmatrix} 0 & 1 & -5 & 0 \end{bmatrix}^T$, see Figure 5.

In our last example we use both the Popov part and the multiplier H.

Example 3. In this example we use

$$A = \begin{bmatrix} -10 & -35 & -50 & -24 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{12}$$

and

$$C = \begin{bmatrix} 0 & 350 & 21 & 0.385 & 0.0021 \end{bmatrix}, \tag{13}$$

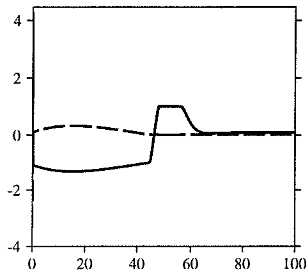

Figure 5: Simulation of the system in (1) when the system matrices are as in (11) and when the initial condition is as stated above. The solid line corresponds to y and the dashed line corresponds to u.

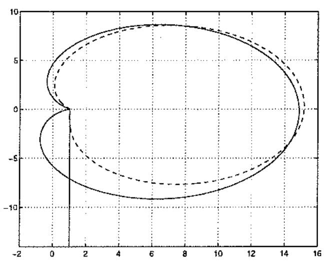

Figure 6: The solid line shows the Nyquist curve of $G(s) + 1$, which is not strictly in the right half plane. The dashed line shows the Nyqist curve of $(1 + \lambda s + H(s))(G(s) + 1)$, which is strictly in the right half plane. Hence, condition (ii) of Theorem 1 holds.

with corresponding transfer function

$$G(s) = 350\frac{(s+0.01)(s+0.02)(s+0.03)}{s(s+1)(s+2)(s+3)(s+4)}.$$

We have $\lim_{s\to 0} sG(s) = 8.75\cdot 10^{-5}$, thus condition (i) of Theorem 1 is satisfied. If we take $\lambda = 0.1$ and $H(s) = 0.29/(s+0.3)$ then condition (ii) is satisfied, see Figure 6. A simulation of the system is shown for the case when the initial condition is $x_0 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, see Figure 7.

4 Conclusions

We have derived a stability criterion for feedback interconnections of deadzone nonlinearities with a linear system that has a pole at the origin. Several numer-

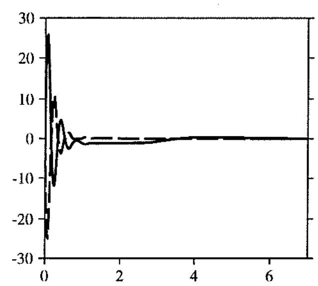

Figure 7: Simulation of the system in (1) when the system matrices are as in (12),(13), and when the initial condition is as stated above. The solid line corresponds to y and the dashed line corresponds to u.

ical examples prove that our stability criterion gives a substantial improvement of the Popov criterion.

An improvement of Theorem 1 would be to allow λ to be any real number and $H(s)$ to be noncausal. Such results are possible for the case when A is Hurwitz, [2]. New results by Megretski indicate that it is possible to obtain the desired improvement, [3]. Megretski's proof technique is different.

Appendix: A KYP Lemma

The following version of the Kalman-Yakubovich-Popov Lemma is a key tool for the proof of Theorem 1.

Lemma 1 (KYP). *Let $M = M^T$ and let*

$$\Phi = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Assume that

- *(a) (A, B) is stabilizable and that all eigenvalues of A are strictly in the left halfplane except for n_s simple eigenvalues at the origin,*
- *(b) if $x \in \mathcal{N}(A)$ then $x \in \mathcal{N}(C^T MC)$, and if $x \notin \mathcal{N}(A)$ then $x^T C^T MCx \geq 0$. $\mathcal{N}(M)$ denotes the nullspace of a matrix M.*

Then the following conditions are equivalent

- *(i) There exists $\varepsilon > 0$ such that*

$$\Phi(j\omega)^* M \Phi(j\omega) \leq -\varepsilon I, \quad \forall \omega \in (0, \infty) \tag{14}$$

(ii) There exist matrices $P_1 = P_1^T \in \boldsymbol{R}^{(n-n_s)\times(n-n_s)}$ *and* $P_2 = P_2^T \in \boldsymbol{R}^{n_s \times n_s}$ *and a state space transformation such that*

(a) in the new coordinates Φ *has the representation*

$$\Phi = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & 0 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

(b) $P_1 > 0$ *and it satisfies the LMI*

$$\begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D \end{bmatrix}^T \begin{bmatrix} 0 & P_1 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D \end{bmatrix} < 0 \tag{15}$$

(c) $P_2 B_2 = -C_2^T M D$.

Furthermore,

- $\lim_{s\to 0} s\Phi^T(s) M \Phi(s) \leq 0$ *if and only if* $P_2 \geq 0$,
- $\lim_{s\to 0} s\Phi^T(s) M \Phi(s) \leq 0$ *and has rank* n_s *if and only if* $P_2 > 0$.

In particular, if $\lim_{s\to 0} s\Phi^T(s) M \Phi(s) < 0$, *then* $P_2 > 0$.

We need the following lemma in the proof of Lemma 1.

Lemma 2. *Assume that* $A, B \in \boldsymbol{R}^{n_1 \times n_2}$, *and that* A *has full row rank, i.e.* $\text{rank}(A) = n_1$. *Then the following statements are equivalent*

1. $A^T B = B^T A$.
2. $\exists P = P^T \in \boldsymbol{R}^{n_1 \times n_1}$, *such that* $B = PA$.

Furthermore,

- $A^T B + B^T A \geq 0$ *if and only if* $P \geq 0$,
- $A^T B + B^T A \geq 0$ *and has rank* n_1 *if and only if* $P > 0$.

In particular, if $A^T B + B^T A > 0$, *then* $P > 0$.

Proof. ($2 \Rightarrow 1$) This implication is obvious. ($1 \Rightarrow 2$) This is proven by construction. Let $P = A_R^T B^T$, where A_R is the right inverse $A_R = A^T(AA^T)^{-1}$. Using $B^T A = A^T B$ and $AA_R A = A$, we get

$$A^T(PA - B) = A^T A_R^T A^T B - A^T B = 0$$

which implies that $PA = B$, since A^T has full column rank. Next we notice that

$$A^T(P - P^T)A = A^T A_R^T A^T B - B^T A A_R A = A^T B - B^T A = 0$$

which implies that $P = P^T$, since A has full row rank.

For the last part we note that the sufficiency is trivial. For the necessity we note that

$$A^T(P + P^T)A = A^T A_R^T A^T B + B^T A A_R A = A^T B + B^T A.$$

Hence, if

- $A^T B + B^T A \geq 0$, then $P \geq 0$ since A has full row rank,
- $A^T B + B^T A > 0$, then A and B must be square invertible matrices and it follows that $P > 0$,
- $A^T B + B^T A \geq 0$ has rank n_1, then P must have full rank and it follows that $P > 0$.

□

Proof of Lemma 1: ($(i) \Rightarrow (ii)$) We notice that there must exist a state space transformation as stated where B_2 has full row rank by the stabilizability asumption (a) and where $\begin{bmatrix} C_1 & C_2 \end{bmatrix}^T MC_2 = 0$ and $C_1^T MC_1 \geq 0$ by condition (b). Hence,

$$\Phi(j\omega)^* M\Phi(j\omega) = \Phi_1(j\omega)^* M\Phi_1(j\omega) + \frac{1}{j\omega}(D^T MC_2B_2 - B_2^T C_2^T MD)$$

where

$$\Phi_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D \end{array}\right].$$

For (14) to hold it is necessary that

$$\lim_{\omega \to 0} j\omega\Phi(j\omega)^* M\Phi(j\omega) = D^T MC_2B_2 - B_2^T C_2^T MD = 0$$

With $A = B_2$ and $B = -C_2^T MD$ it follows from Lemma 2 that this is equivalent with the existence of $P_2 \in \mathbf{R}^{n_s \times n_s}$ satisfying $P_2 = P_2^T$, and $P_2B_2 = -C_2^T MD$. Furthermore,

$$\lim_{s \to 0} s\Phi^T(s)M\Phi(s) = D^T MC_2B_2 + B_2^T C_2^T MD$$

Thus, the positivity conditions on P_2 follows from the assumptions on this residue and the last part of Lemma 2. The existence of a matrix $P_1 \in \mathbf{R}^{(n-n_s)\times(n-n_s)}$ satisfying the stated conditions follows from the standard version of the KYP Lemma, see for example [7] or [5]. Note that P_1 must be positive definite since A_1 is Hurwitz and since $C_1^T MC_1 \geq 0$.

The other direction of the proof is now straightforward.

References

[1] C.A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties.* Academic Press, New York, 1975.

[2] U. Jönsson. Stability analysis with Popov multipliers and integral quadratic constraints. *Systems and Control Letters*, 1997. To Appear.

[3] A. Megretski. Personal communication.

[4] K.S. Narendra and J.H. Taylor. *Frequency Domain Criteria for Absolute Stability.* Academic Press, New York, 1973.

[5] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters*, 28(1):7–10, 1996.

[6] J.C. Willems. *The Analysis of Feedback Systems.* MIT Press, Cambridge, Massachusetts, 1971.

[7] J.C. Willems. The least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 16(6):621–634, 1971.

[8] V.A. Yakubovich. Frequency conditions for the absolute stability of control systems with several nonlinear or linear nonstationary blocks. *Avtomatika i Telemekhanika*, 6:5–30, June 1967.

[9] G Zames. On the input-output stability of nonlinear time-varying feedback systems—part II: Conditions involving circles in the frequency plane and sector nonlinearities. *IEEE Transactions on Automatic Control*, 11(3):465–476, July 1966.

[10] G. Zames and P.L. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal of Control*, 6(1):89–108, 1968.