$\mathcal{L}_2$-Gain of Double Integrators with Saturation Nonlinearity

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Abstract

This paper uses quadratic surface Lyapunov functions to efficiently check if a double integrator in feedback with a saturation nonlinearity has $\mathcal{L}_2$-gain less than $\gamma > 0$. We show that for many of such systems, the $\mathcal{L}_2$-gain is non-conservative in the sense that this is approximately equal to the lower bound obtained by replacing the saturation with a constant gain of 1. These results allow the use of classical analysis tools like μ-analysis or IQCs to analyze systems with double integrators and saturations, including servo systems like some mechanical systems, satellites, hard-disks, CD players, etc.

1 Introduction

There are many control applications that can be modeled as a plant with a single integrator, a saturation nonlinearity, and a PI controller as shown in figure 1. One of the most simple one is the position control of a body with a PI controller and a power limit actuator. In this case, the force $F = m\ddot{x} + k\dot{x}$, where m and k represents the mass of the body and the coefficient of friction, respectively. Typically, if the position $x(t)$ is to track some reference command $u(t)$, a PI controller is used. In this case, $P(s) = (ms + k)^{-1}$.

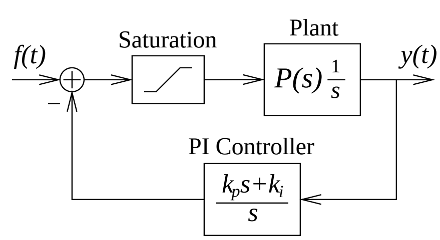

Figure 1: PI position control system with power limited actuator

Not only systems satisfying the Newton's law $F = ma$ can be modeled as in figure 1. Many servo systems, including mechanical systems, are often modeled this way. A double integrator system may be used as a simple model for satellite control, modeling the relation between the angular position and velocity and the reaction jets. Other examples are the control of a hard-disk drive head, the laser beam of a CD, etc.

Analysis of saturation systems with double integrators has been done for many years. As explained in [6], in order to perform robustness analysis the system is typically transformed into one shown in figure 2, where the saturation is treated as an uncertainty. The problem with this approach is that it gives us a nominal plant that is marginally unstable, preventing us to apply some classical analysis tools such as the Popov criterion, μ-analysis, and Integral Quadratic Constraints (IQCs).

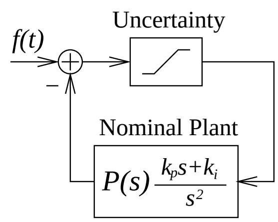

Figure 2: Nominal system and uncertainty

An alternative is to encapsulate the unstable operator in an artificial feedback loop which defines a bounded operator. Robustness analysis can then be performed on the transformed system which consists of bounded operators. Assuming $P(s)$ is stable, this leaves us with the double integrator and the saturation to worry about. A possible loop transformation is shown in figure 3. In order to analyze the system, we mush first check if Δ is a bounded operator. In this case, Δ is a double integrator in feedback interconnection with a saturation nonlinearity, where the output consists of signals from both the first and second integrator. The question whether the system $\ddot{x} = sat(-k_1 x - k_2 \dot{x} + u)$ has finite $\mathcal{L}_2$-gain from u to x, $\dot{x}$, or $\ddot{x}$, has been posted as an open problem [2]. It has been shown, meanwhile, that the $\mathcal{L}_2$-gain from u to x is infinite [7], and the $\mathcal{L}_2$-gain from u to $\dot{x}$ is also infinite [6]. This means the loop transformation in figure 3 does not result in a finite $\mathcal{L}_2$-gain operator Δ.

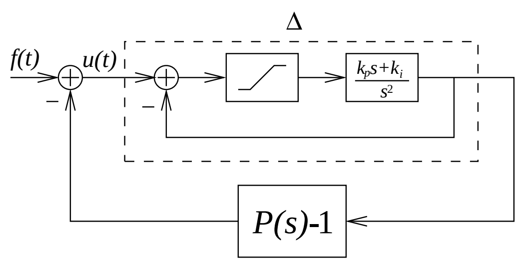

Figure 3: Loop transformation with an *unstable* operator Δ

In this paper, we propose the loop decomposition shown in figure 4, where k_1, k_2, and $G(s)$ are functions of k_p, k_i, and $P(s)$, and $G(s)$ is stable (see appendix A for details). The loops of both systems in figures 1 and 4 are identical and analysis properties can be inferred from one to another and vice versa. The low-pass filter is used to exclude high frequency content from the feedback loop, as expected from real applications. In [3], it is shown that for $k_1 = k_2 = 1$ and $\alpha = 0$, the $\mathcal{L}_2$-gain of Δ is finite, but no upper bound of this gain is given. The goal of this paper is, for given $k_1 > 0$, $k_2 \geq 0$, and $\alpha > 0$, to give sufficient conditions to (1) check if the $\mathcal{L}_2$-gain of Δ is finite and (2) find upper bounds on the $\mathcal{L}_2$-gain of Δ. We show that our method is *not conservative* for many values of k_1, k_2, and α since we are able to find upper bounds on the $\mathcal{L}_2$-gain of Δ that are approximately equal to lower bounds obtained when the saturation is replaced by a unity constant gain. The method is based on constructing quadratic Lyapunov functions on the switching surface associated with the saturation system. The construction of such Lyapunov functions is done by solving a set of LMIs.

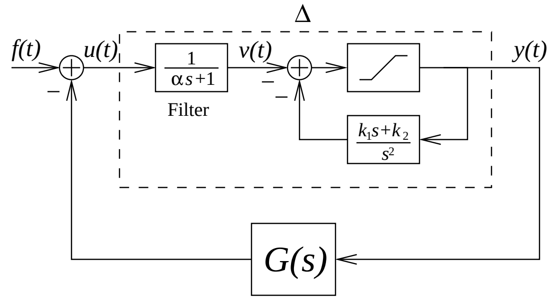

Figure 4: Loop transformation with *stable* operators

This paper is organized as follows. The following section contains the main result of the paper. There, conditions in the form of LMIs are given to check if γ is an upper bound of the $\mathcal{L}_2$-gain of Δ in figure 4. This section also contains several illustrative examples. Section 3 proves the main result and section 4 gives conclusions. Finally, computational details can be found in appendix.

2 Main Results

2.1 Preliminaries

Let $\mathcal{L}_2$ denote the space of all functions $f : [0, \infty) \to \mathbb{R}$ which are square summable, i.e.,

$$\|f\|^2 = \int_0^\infty f^2(t)dt < \infty$$

The extended space $\mathcal{L}_{2e}$ consists of all functions $f(t)$ which satisfy $P_T f(t) \in \mathcal{L}_2$, for all $T \geq 0$, where P_T is a truncation operator defined as $(P_T f)(t) = f(t)$ if $t \leq T$ and $(P_T f)(t) = 0$ otherwise.

We say that the $\mathcal{L}_2$-gain from input u to output y of some system is less than $\gamma \geq 0$ if

$$\int_0^T y^2(t)dt \leq \gamma \int_0^T u^2(t)dt \tag{1}$$

for all $T \geq 0$, and all $u \in \mathcal{L}_{2e}$. The $\mathcal{L}_2$-gain γ^* of the system from u to y is the infimum over all γ such that (1) is satisfied.

Consider the operator Δ in figure 4. For given k_1, k_2, α, we are interested in finding an upper bound of the $\mathcal{L}_2$-gain of Δ. The following proposition gives an easy way to find a lower bound of the $\mathcal{L}_2$-gain of Δ. The proof, based on the fact that the saturation behaves linearly for small inputs, can be found in appendix B.

Proposition 2.1 *Consider the system Δ in figure 4. The $\mathcal{L}_2$-gain γ_L of the same system but with the saturation replaced by a constant gain of* 1 *is a lower bound of the $\mathcal{L}_2$-gain of Δ, i.e.,* $0 \leq \gamma_L \leq \gamma^*$.

Note that when the saturation is replaced by a constant gain of 1, the system becomes linear. Thus, γ_L is simply the square of the $\mathcal{H}_\infty$ of the linear system

$$\frac{Y(s)}{U(s)} = \frac{s^2}{(\alpha s + 1)(s^2 + k_1 s + k_2)}$$

From this expression we immediately see that it is necessary $k_1 > 0$, $k_2 \geq 0$, and $\alpha > 0$, or otherwise $\gamma_L = \infty$. When $k_2 = 0$ the original system is reduced to a single integrator which was studied in [5, 9]. Hence, this case will just be briefly discussed in section 2.4. The proof of the following proposition can also be found in appendix B.

Proposition 2.2 *Consider the system Δ in figure 4. If there exists an $\alpha = \alpha_1 > 0$ such that the $\mathcal{L}_2$-gain of Δ is finite then the $\mathcal{L}_2$-gain is finite for any $\alpha > 0$.*

A state-space representation of system Δ in figure 4 is

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & k_2 x_2 \\ \dot{x}_2 & = & y \\ \dot{v} & = & -\frac{1}{\alpha} v + \frac{1}{\alpha} u \\ y & = & sat(-x_1 - k_1 x_2 - v) \end{array} \right. \tag{2}$$

Let $x = [x_1\ x_2\ v]'$ and $C = [1\ k_1\ 1]$. In the state-space, the system can be seen as piecewise linear system, with 3 cells and two switching surfaces (see figure 5). The switching surfaces are

$$S = \{x \in \mathbb{R}^n :\ Cx = 1\}$$

and $\underline{S} = -S$. When $Cx \geq 1$, $\dot{x}_2 = -1$, when $Cx \leq -1$, $\dot{x}_2 = 1$, and, finally, when $-1 \leq Cx \leq 1$, $\dot{x}_2 = -Cx$.

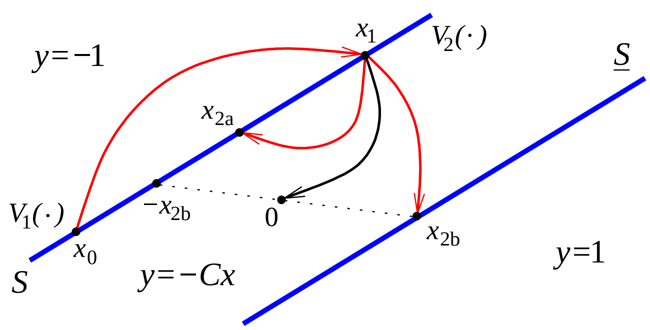

Figure 5: Possible trajectories in the state-space

2.2 Double Integrator

Assume that $k_2 > 0$. The following matrices will be needed in the main result. For some $T > 0$, let

$$W_a(T) = \begin{pmatrix} \frac{k_1}{k_2} - \frac{T}{2} \\ 0 \\ \frac{k_1}{k_2} + \frac{T}{2} \\ 0 \end{pmatrix}, \quad W_b(T) = \begin{pmatrix} -\frac{1}{k_2 T} & \frac{1}{k_2 T} \\ 1 & 0 \\ -\frac{1}{k_2 T} & \frac{1}{k_2 T} \\ 0 & 1 \end{pmatrix}$$

and

$$W_j(T) = \frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}} \begin{pmatrix} 1 \\ -e^{-\frac{T}{\alpha}} \end{pmatrix} \begin{pmatrix} 1 & -e^{-\frac{T}{\alpha}} \end{pmatrix}$$

Define also

$$A = \begin{pmatrix} 0 & k_2 & 0 \\ -1 & -k_1 & -1 \\ 0 & 0 & -\frac{1}{\alpha} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\alpha} \end{pmatrix}$$

and

$$H = \begin{pmatrix} A & BB'/\gamma \\ -C'C & -A' \end{pmatrix}, \quad e^{HT} = \begin{pmatrix} e_{11}(T) & e_{12}(T) \\ e_{21}(T) & e_{22}(T) \end{pmatrix} \tag{3}$$

where each $e_{ij}(T)$ are square matrices of dimension 3 by 3 and

$$W_t(T) = \begin{bmatrix} e_{22}e_{12}^{-1} & \frac{1}{2}\left(e_{21} - e_{22}e_{12}^{-1}e_{11} - (e_{12}^{-1})'\right) \\ \frac{1}{2}\left(e_{21} - e_{22}e_{12}^{-1}e_{11} - (e_{12}^{-1})'\right)' & e_{12}^{-1}e_{11} \end{bmatrix}$$

where the notation $e_{ij} = e_{ij}(T)$ was used for simplification. Finally, define

$$W_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -k_1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -k_1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We are now ready for the main result of the paper. In this result, we drop the argument (T) for simplification.

Theorem 2.1 *Consider the system Δ in figure 4. Given $k_1, k_2, \alpha > 0$, let $\gamma \geq \gamma_L$. Let also $p > 0$ be a 2 by 2 diagonal matrix and $g \in \mathbb{R}^2$. Define*

$$P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \quad G = \begin{pmatrix} g \\ -g \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} -g \\ -g \end{pmatrix}$$

If

$$R_1(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_j - W_b'PW_b & -W_b'(PW_a + G) \\ -(W_a'P + G')W_b & -T - W_a'PW_a - 2W_a'G \end{pmatrix} > 0 \tag{4}$$

$$R_{2a}(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_2'W_tW_2 - P & W_2'W_tW_1 - G \\ W_1'W_tW_2 - G' & W_1'W_tW_1 \end{pmatrix} > 0 \tag{5}$$

$$R_{2b}(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_2'W_tW_2 - P & W_2'W_tW_3 - \bar{G} \\ W_3'W_tW_2 - \bar{G}' & W_3'W_tW_3 \end{pmatrix} > 0 \tag{6}$$

for all $T > 0$ then the $\mathcal{L}_2$-gain of Δ is less or equal than γ.

The last theorem gives us a set of infinite dimensional LMIs that, when satisfied, guarantee that Δ not only has finite $\mathcal{L}_2$-gain, but also that this is upper bounded by γ. This allows us to write an IQC of the form

$$\int_0^T y^2(t)dt \leq \gamma \int_0^T u^2(t)dt \tag{7}$$

which, in turn, allows us to perform robustness and performance analysis on the system in figure 4 or, equivalently, on the original system in figure 1.

The method of proof is as follows. First, inequality (7) is satisfied if for every $u \in \mathcal{L}_{2e}$ there exists a Lyapunov function $V(\cdot)$ such that the solution $x(t)$ from the initial state $x(0) = 0$ satisfies

$$\int_{T_i}^{T_f} \left[\gamma u^2(t) - y^2(t)\right] dt \geq V(x(T_f)) - V(x(T_i)) \tag{8}$$

for all $0 \leq T_i \leq T_f$. To see this, let $T_i = 0$. Then, $V(x(0)) = 0$ and $V(x(T_f)) \geq 0$, since V is a Lyapunov function.

Figure 5 shows possible trajectories of (2) starting at S. Depending on the control input u, a trajectory may enter the region where $y = -1$. Since $u \in \mathcal{L}_{2e}$, a switch must eventually occur at some point $x_1 \in S$. The control u may also be such that the trajectory enters the linear region where $y = -Cx$. In this case, there are three possibilities: the trajectory does not switch again and goes to zero as $t \to \infty$, it returns to S, or it intersects $\underline{S}$. Since the system is symmetric around the origin, for analysis purposes, any other trajectories can be reduced the ones just described.

Second, define two Lyapunov functions V_1 and V_2 on the switching surface S. Condition (8) is satisfied if

$$\int_{0}^{T_1} \left[\gamma u_1^2(t) - y^2(t)\right] dt \geq V_2(x_1) - V_1(x_0) \tag{9}$$

$$\int_{0}^{T_{2a}} \left[\gamma u_{2a}^2(t) - y^2(t)\right] dt \geq V_1(x_{2a}) - V_2(x_1) \tag{10}$$

$$\int_{0}^{T_{2b}} \left[\gamma u_{2b}^2(t) - y^2(t)\right] dt \geq V_1(-x_{2b}) - V_2(x_1) \tag{11}$$

for all $x_0, x_1, x_{2a}, -x_{2b} \in S$, and $T_1, T_{2a}, T_{2b} > 0$, and where $u_1(t) \in \mathcal{L}_2$ is such that a trajectory starting at x_0 satisfies $x_1 = x(T_1)$ and $y = -1$, $t \in [0, T_1]$, and $u_i(t) \in \mathcal{L}_2$, $i = 2a, 2b$ is such that a trajectory starting at x_1 satisfies $x_i = x(T_i)$ and $y = -Cx$, $t \in [0, T_i]$.

Finally, under certain assumptions, the inputs u_i, $i = 1, 2a, 2b$, that minimize the integrals on the left side of the above inequalities can be explicitly found. If the Lyapunov functions are chosen to be quadratic functions, the result are conditions (4)-(6). The details of the proof can be found in section 3.

2.3 Examples

In order to solve an infinite dimensional set of LMIs, there are some extra steps we need to take to make this solution computationally attractive. Obviously, it is not possible to solve the three quadratic inequalities for all $T > 0$. The idea is to find a finite sequence of times $\{T_i\}$ defined on some bounded set $\mathcal{T} = (0, T_+]$ such that it is sufficient (4)-(6) are satisfied in $\mathcal{T}$ to prove the desired result. It is then necessary to guarantee they are also satisfied in $T \in (T_+, \infty)$, and $T \in (T_i, T_{i+1})$ for all $T_i, T_{i+1} \in \mathcal{T}$. The latest can be guaranteed by estimating bounds on the derivative of each condition (4)-(6) between $T_i, T_{i+1} \in \mathcal{T}$. Conditions to guarantee that (4)-(6) are also satisfied in $T \in (T_+, \infty)$, for some $0 < T_+ < \infty$, are given in propositions C.1 and C.2.

The following examples were processed in `matlab` code. The latest version of this software is available at [4]. Before presenting the examples, we briefly explain the `matlab` function we developed. The user supplies $k_1 > 0$, $k_2 \geq 0$ (the case when $k_2 = 0$ results in the single integrator which will be dealt in the next section), and $\alpha > 0$. If all three conditions (4)-(6) are satisfied for all $T \in \mathcal{T}$, the function returns a graphic showing the minimum eigenvalues of each $R_i(T)$, which, obviously, must be positive for all $T \in \mathcal{T}$.

Example 2.1 Let $k_1 = 0.5$, $k_2 = 2$, and $\alpha = 2$. In this example, we find the smallest upper bound γ of the $\mathcal{L}_2$-gain of Δ in figure 4 using theorem 2.1. A lower bound can be found by computing the linear gain, i.e., the $\mathcal{L}_2$-gain of Δ when the saturation nonlinearity is replaced by a constant gain of 1. Here, this is $\gamma_L = 0.8892297$. Using the software described above, we found an upper bound of the $\mathcal{L}_2$-gain of Δ of $\gamma = 0.8892299$. Note that the difference between the upper and lower bound is smaller than 2×10^{-7}, i.e., the precision is less than $2.15 \times 10^{-5}\%$.

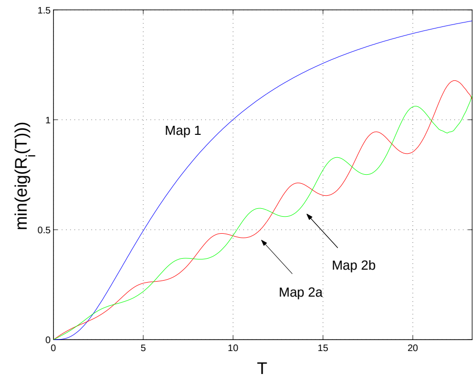

Figure 6: Minimum eigenvalues of $R_i(T)$, $i = 1, 2a, 2b$

Figure 6 shows the minimum eigenvalues of $R_i(T)$, $i = 1, 2a, 2b$. For visualization purposes, the minimum eigenvalues of $R_{2a}(T)$ and $R_{2b}(T)$ were scaled by 2×10^6. ■

Example 2.2 Let $k_1 = k_2 = 1$. In this example we find the smallest upper bound γ of the $\mathcal{L}_2$-gain of Δ for different values of $\alpha > 0$. The left side of figure 7 shows the lower bound γ_L and the upper bound γ on the $\mathcal{L}_2$-gain of Δ. The right side of figure 7 plots $\gamma - \gamma_L$. Logarithmic scales were used for better visualization.

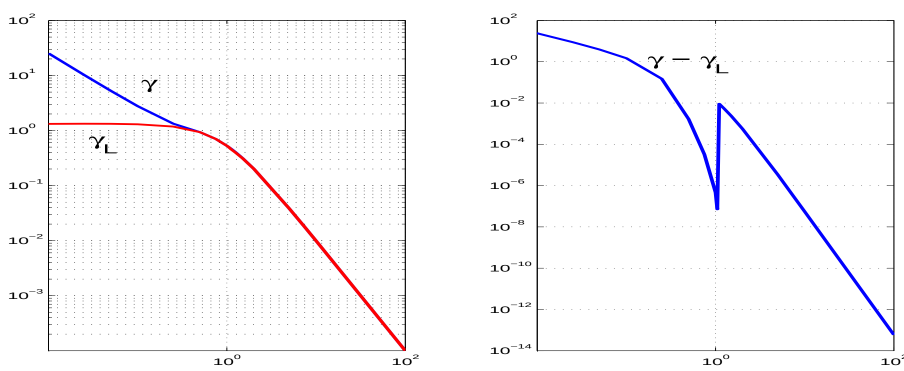

Figure 7: γ and γ_L as a function of α (left) and $\gamma - \gamma_L$ (right)

From this figure we can see that the difference between the upper and lower bound goes to zero as α goes to infinity. In fact, for $\alpha > 0.5$ the difference between γ and γ_L is less than

0.76%. For $\alpha > 5$ this difference is already smaller than 0.009% and less than 6×10^{-8}% for $\alpha > 100$.

If $\gamma \geq \gamma_L$ is chosen small enough, the Hamiltonian matrix H in (3) has pure imaginary eigenvalues. For $\alpha \geq 0.5$, it turns out that for *all* $\gamma > \gamma_L$ such that H has no pure imaginary eigenvalues, it was *always* possible to find p, g such that conditions (4)-(6) are satisfied. In other words, numerically we found that for $\alpha \geq 0.5$ conditions (4)-(6) are satisfied if and only if H has no pure imaginary eigenvalues. Thus, for $\alpha \geq 0.5$, figure 7 also shows the smallest γ such that H does not have pure imaginary eigenvalues. For $\alpha < 0.5$, however, we encountered several numerical problems and γ tended to be higher than the smallest γ such that H has no pure imaginary eigenvalues.

Several questions can now be raised: is the gap between γ and γ_L increasing as α approaches zero due to numerical errors, conservatism of the method, or the fact that the $\mathcal{L}_2$-gain of the system is just larger than γ_L, and this gap increases as α approaches zero? Or is true that $\gamma = \gamma_L$ or $\gamma \approx \gamma_L$ for all $\alpha > 0$? Answers to such questions are currently under investigation.

For sure, this example shows that our method is *not conservative*, except maybe for small values of α, since the upper and lower bounds of the $\mathcal{L}_2$-gain of Δ are almost identical. ■

2.4 Single Integrator

When $k_2 = 0$, the system reduces to a single integrator. This class of system has been studied before in [5, 9]. For completeness, in this section we briefly explain how some matrices and vectors in conditions (4)-(6) can be changed in order to allow to check for the $\mathcal{L}_2$-gain of systems with a single integrator. For some $T > 0$, let $W_a(T) = (\, k_1 T \quad 0\,)'$ and $W_b(T) = (\,1 \quad 1\,)'$. Define also

$$A = \begin{pmatrix} -k_1 & -1 \\ 0 & -\frac{1}{\alpha} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{1}{\alpha} \end{pmatrix}$$

and, finally,

$$W_1 = \begin{pmatrix} \frac{1}{k_1} \\ 0 \\ \frac{1}{k_1} \\ 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -\frac{1}{k_1} & 0 \\ 1 & 0 \\ 0 & -\frac{1}{k_1} \\ 0 & 1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} -\frac{1}{k_1} \\ 0 \\ \frac{1}{k_1} \\ 0 \end{pmatrix}$$

With all the other variables defined as in section 2.2, we have a result similar to theorem 2.1, but with $k_2 = 0$.

Theorem 2.2 *Consider the system Δ in figure 4. Given $k_1 > 0$, $k_2 = 0$, and $\alpha > 0$, let $\gamma \geq \gamma_L$. Let also $p > 0$ and g be real numbers. Define*

$$P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \quad G = \begin{pmatrix} g \\ -g \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} -g \\ -g \end{pmatrix}$$

If

$$\begin{pmatrix} W_b'(W_j - P)W_b & W_b'((W_j - P)W_a - G) \\ (W_a'(W_j - P) - G')W_b & -T + W_a'(W_j - P)W_a - 2W_a'G \end{pmatrix} > 0$$

$$R_{2a}(T) > 0$$

$$R_{2b}(T) > 0$$

for all $T > 0$, where $R_{2a}(T)$ and $R_{2b}(T)$ are defined as in (5) and (6), respectively, then the $\mathcal{L}_2$-gain of Δ is less or equal than γ.

The proof of this result is similar to the proof of theorem 2.1 and is therefore omitted here.

3 Proof of Theorem 2.1

In this section, we show that if conditions (4)-(6) are satisfied then so are conditions (9)-(11). But, before we do, consider conditions (9) and (10). If $x_1 = x_0 = x_{2a} \in S$ and $T_1 = T_{2a} = 0$ then it results that the left side of both conditions is equal to zero, i.e.,

$$\begin{aligned} 0 &\geq V_2(x_0) - V_1(x_0) \\ 0 &\geq V_1(x_0) - V_2(x_0) \end{aligned}$$

which means that $V_1(\cdot) = V_2(\cdot)$, i.e., the Lyapunov functions must be identical. So, from now on, we consider $V(\cdot) = V_1(\cdot) = V_2(\cdot)$.

A notion that will be usefully throughout the rest of the proof is the notion of *impact map*. An impact map is simply a map from one switching surface to the next switching surface. There are three impact maps of interest associated with a saturation system (see figure 5). The fist impact map (impact map 1) takes points $x_0 \in S$ and maps them back to $x_1 \in S$ such that the trajectory stays in the region where $y = -1$. The second impact map (impact map $2a$) takes points from $x_1 \in S$ and also maps them back to $x_{2a} \in S$, but this time the trajectory stays in the region where $y = -Cx$. Finally, the third impact map (impact map $2b$) takes points from from $x_1 \in S$ and maps them to $x_{2b}\underline{S}$ such that the trajectory stays in the region where $y = -Cx$.

Each of these impact maps is associated with each condition (9)-(11). We will start with impact map 1 and condition (9).

3.1 Impact Map 1

The first map we consider is the map that leaves S and returns to S and the trajectory remains in the region where where $Cx \geq 1$. Here, $y = -1$ and therefore $\dot{x}_2 = -1$. Let $x_0, x_1 \in S$ and $T > 0$. For simplicity, write $x(0) = x_0 = [x_{10}\ x_{20}\ v_0]'$ and $x(T) = x_T = [x_{1T}\ x_{2T}\ v_T]'$. Note that, in this region, only the last state v is controllable. The first two states x_1 and x_2 do not depend on the input. Integrating, we get $x_2(t) = -t + x_{20}$ and, at $t = T$,

$$x_{2T} = -T + x_{20}$$

This means that $\dot{x}_1(t) = -k_2 t + k_2 x_{20}$. Integrating, and evaluating at $t = T$ we get

$$x_{1T} = -k_2 \frac{T^2}{2} + k_2 x_{20} T + x_{10}$$

Since $x_0, x_1 \in S$, it is also true that

$$\begin{cases} x_{10} + k_1 x_{20} + v_0 = 1 \\ x_{1T} + k_1 x_{2T} + v_T = 1 \end{cases}$$

This gives us four equations with six variables. Let the free variables be v_0 and v_T and define

$$\begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} x_{2T} \\ v_T \\ x_{20} \\ v_0 \end{pmatrix} = W_a(T) + W_b(T) \begin{pmatrix} v_T \\ v_0 \end{pmatrix}$$

Next, we solve the following minimization problem

$$J^* = \min_{u \in \mathcal{L}_2} \int_0^T \Big(\gamma u^2(t) - y(t)\Big)\, dt$$

subject to $\dot{v} = -\frac{1}{\alpha}v + \frac{1}{\alpha}u$, $v(0) = v_0$, $v(T) = v_T$, and u is such that $Cx(t) \geq 1$, $t \in [0, T]$. In order to find an explicit solution for u, we simplify the problem and ignore the fact that $Cx(t) \geq 1$, $t \in [0, T]$. The problem then becomes a standard $\mathcal{H}_2$ optimization problem where the solution can be found in appendix D. In this case,

$$J^* = \begin{pmatrix} v_T \\ v_0 \end{pmatrix}' W_j(T) \begin{pmatrix} v_T \\ v_0 \end{pmatrix} - T$$

Define a quadratic surface Lyapunov function $V(\cdot)$ in S as

$$V(\Delta_i) = \Delta_i' p \Delta_i + 2\Delta_i' g$$

where $p = p' > 0$. Hence,

$$\begin{aligned} V(\Delta_1) - V(\Delta_0) &= \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix}' P \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix} + 2 \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix}' G \\ &= \begin{pmatrix} v_T \\ v_0 \end{pmatrix}' W_b' P W_b \begin{pmatrix} v_T \\ v_0 \end{pmatrix} + 2 \begin{pmatrix} v_T \\ v_0 \end{pmatrix}' W_b'(PW_a + G) + W_a' P W_a + 2W_a' G \end{aligned}$$

where $W_a = W_a(T)$ and $W_b = W_b(T)$ were used for simplification. Finally,

$$J^* > V(\Delta_1) - V(\Delta_0)$$

is equivalent to (4).

The reason why $p > 0$ is chosen a diagonal matrix versus a symmetric one comes from the following proposition.

Proposition 3.1 *Let*

$$p = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix} > 0$$

If $p_3 \neq 0$ then (4) is never satisfied for large enough values of $T > 0$.

The proof can be found in appendix C.1.

3.2 Impact Map $2a$

The next map we consider is the map that leaves S and returns to S and the trajectory remains in the region where where $-1 \leq Cx \leq 1$. This means $y = -Cx$, or $\dot{x}_2 = -Cx$. In this region, the system is linear given by $\dot{x} = Ax + Bu$. Let $x_1 = [x_{10}\ x_{20}\ v_0]', x_{2a} = [x_{1T}\ x_{2T}\ v_T]' \in S$ and $T > 0$. Here, all the states are controllable and finding the optimal cost J^* follows just like in appendix D, yielding

$$J^* = \begin{pmatrix} x_{2a} \\ x_1 \end{pmatrix}' W_t(T) \begin{pmatrix} x_{2a} \\ x_1 \end{pmatrix}$$

Since $x_1, x_{2a} \in S$

$$\begin{pmatrix} x_{2a} \\ x_1 \end{pmatrix} = W_1 + W_2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}$$

where $\Delta_1 = [x_{20}\ v_0]'$ and $\Delta_{2a} = [x_{2T}\ v_T]'$. Hence

$$J^* = \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' W_2' W_t W_2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix} + 2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' W_2' W_t W_1 + W_1' W_t W_1$$

On the other hand,

$$V(\Delta_{2a}) - V(\Delta_1) = \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' P \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix} + 2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' G$$

Finally,

$$J^* > V(\Delta_{2a}) - V(\Delta_1)$$

is equivalent to (5).

3.3 Impact Map $2b$

The last map we consider is the map from S to $\underline{S}$ and where the trajectory remains in the same region as the previous map. The proof for this map is similar to the one from impact map $2a$. The difference is that

$$\begin{pmatrix} x_{2b} \\ x_1 \end{pmatrix} = W_1 + W_3 \begin{pmatrix} \Delta_{2b} \\ \Delta_1 \end{pmatrix}$$

since $x_1 \in S$ and $x_{2b} \in \underline{S}$. This means $-x_{2b} \in S$ which results in

$$J^* > V(-\Delta_{2b}) - V(\Delta_1)$$

which is equivalent to (6).

4 Conclusions

This paper gives conditions in the form of LMIs that, when satisfied, guarantee a system with a double integrator in feedback with a saturation nonlinearity has finite $\mathcal{L}_2$-gain. Moreover, for a large class of such systems, we showed that the linear $\mathcal{L}_2$-gain of the system, i.e., the $\mathcal{L}_2$-gain of the same system but with the saturation nonlinearity replaced by a constant gain of 1, is approximately equal to the $\mathcal{L}_2$-gain of the original system. These results allow the use of classical analysis tools like μ-analysis or IQCs to analyze systems with double integrators and saturations, including servo systems like some mechanical systems, satellites, hard-disks, CD players, etc.

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Appendix

A Loop Transformation to Find Stable Operators

In this section, we show how to chose k_1, k_2, and $G(s)$ as functions of k_p, k_i, and $P(s)$ so that $G(s)$ is a proper stable system and the systems in figures 1 and 4 are equivalent, in the sense that both loops are identical. In other words, analysis properties can be inferred from one to another and vice versa. First, let $P(s)$ be written as

$$P(s) \stackrel{\text{def}}{=} \frac{n(s)}{d(s)} = \frac{\xi_m s^m + \cdots + \xi_1 s + \xi_0}{s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}$$

where $m \leq n$, $\xi_0 \neq 0$ or otherwise the system would have only one integrator, and also $\beta_0 \neq 0$ or otherwise the system would have three integrators and therefore be unstable (Sussmann and Yang [10] showed that a chain of n integrators, $n \geq 3$, cannot be stabilized by bounded linear feedback).

Proposition A.1 *Let*

$$k_1 = \frac{\xi_0}{\beta_0}k_p + \frac{1}{\beta_0}\left(\xi_1 - \xi_0\frac{\beta_1}{\beta_0}\right)k_i, \qquad k_2 = \frac{\xi_0}{\beta_0}k_i$$

and the proper system

$$G(s) = \frac{\bar{n}(s)}{d(s)}\,(\alpha s + 1)$$

where the degree of $\bar{n}(s)$ is strictly less than the degree of $d(s)$. Then, the systems in figures 1 and 4 are have identical loops and analysis properties can be inferred from one to another and vice versa. Moreover, $G(s)$ is stable if and only if $P(s)$ is stable.

The proof, omitted here, is based on replacing the above equalities in the system in figure 4 and showing that this loop is indeed identical to the one in figure 1.

B Proof of Propositions 2.1 and 2.2

Proof of proposition 2.1: Consider the system Δ_L obtained from system Δ in figure 4 with the saturation replaced by a constant gain of 1, and let γ_L be the respective $\mathcal{L}_2$-gain. For simplicity, and without loss of generality, assume there exist a control input $u_L^* \in \mathcal{L}_2$ such that $\|y_L\|^2 = \gamma_L\|u_L^*\|^2$ (a similar argument can be applied if such $u_L^* \in \mathcal{L}_2$ does not exist by considering a sequence of $u_i \in \mathcal{L}_2$ resulting in γ_i arbitrarily close to γ_L). Since Δ_L is linear, u_L^* can be scaled such that $|y_L(t)| \leq 1$. Hence, by applying such input u_L^* to Δ, we obtain $\|y\|^2 = \gamma_L\|u_L^*\|^2$ since the saturation never leaves the linear region. This means that γ_L is a lower bound of the $\mathcal{L}_2$-gain of Δ. ■

Proof of proposition 2.2: Let $\gamma_1 < \infty$ be an upper bound of the $\mathcal{L}_2$-gain of Δ when $\alpha = \alpha_1 > 0$. Let now $\alpha > 0$ and consider the following subsystem

$$\frac{U(s)}{\bar{U}(s)} = \frac{\alpha_1 s + 1}{\alpha s + 1}$$

The $\mathcal{L}_2$-gain of this subsystem is $\bar{\gamma} = \max(1, \alpha_1/\alpha)$. Then, the $\mathcal{L}_2$-gain γ of Δ when $\alpha > 0$ ($\mathcal{L}_2$-gain from $\bar{u}$ to y) is upper bounded by $\bar{\gamma}\gamma_1$, i.e., $\gamma \leq \bar{\gamma}\gamma_1 < \infty$. ■

C Computational Details

In order to be able to solve for the parameters of the Lyapunov function, we need to first to solve several computational issues associated with conditions (4)-(6) in theorem 2.1. In particular, we need to guarantee the conditions are satisfied for large enough values of T. We start with impact map 1.

C.1 Impact Map 1

We start by proving proposition 3.1.

Proof of proposition 3.1: Assume $p_3 \neq 0$. The first step is to write $R_1(T)$ explicitly. After some manipulation, we find that

$$R_1(T) = \begin{pmatrix} \frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}} - p_2 + \frac{2}{k_2 T}p_3 & -\frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}}e^{-\frac{T}{\alpha}} - \frac{2}{k_2 T}p_3 & -\frac{p_1}{k_2} - (\frac{k_1}{k_2} - \frac{T}{2})p_3 - g_2 \\ -\frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}}e^{-\frac{T}{\alpha}} - \frac{2}{k_2 T}p_3 & \frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}}e^{-\frac{2T}{\alpha}} + p_2 + \frac{2}{k_2 T}p_3 & \frac{1}{k_2}p_1 + (\frac{k_1}{k_2} - \frac{T}{2})p_3 + g_2 \\ -\frac{p_1}{k_2} - (\frac{k_1}{k_2} - \frac{T}{2})p_3 - g_2 & \frac{1}{k_2}p_1 + (\frac{k_1}{k_2} - \frac{T}{2})p_3 + g_2 & T\left(2p_1\frac{k_1}{k_2} + 2g_1 - 1\right) \end{pmatrix}$$

When T is large, $R_1(T)$ tends to

$$R_1(T) \rightarrow \begin{pmatrix} 2\gamma\alpha - p_2 & -\frac{2}{k_2 T}p_3 & \frac{T}{2}p_3 \\ -\frac{2}{k_2 T}p_3 & p_2 & \frac{T}{2}p_3 \\ \frac{T}{2}p_3 & \frac{T}{2}p_3 & T\left(2p_1\frac{k_1}{k_2} + 2g_1 - 1\right) \end{pmatrix}$$

This matrix, however, is not positive definite for large enough T. To see this, consider the sub-matrix

$$\begin{pmatrix} p_2 & \frac{T}{2}p_3 \\ \frac{T}{2}p_3 & T\left(2p_1\frac{k_1}{k_2} + 2g_1 - 1\right) \end{pmatrix}$$

Assuming $2p_1\frac{k_1}{k_2} + 2g_1 > 1$, it is necessary that

$$p_2\left(2p_1\frac{k_1}{k_2} + 2g_1 - 1\right)T > \frac{p_3^2}{4}T^2$$

which is not true for large enough T. ■

Proposition 3.1 establishes that $p > 0$ must be a diagonal matrix, i.e., $p = \mathrm{diag}(p_1, p_2)$, where $p_1, p_2 > 0$. Hence, $R_1(T)$ reduces to

$$R_1(T) = \begin{pmatrix} \frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}} - p_2 & -\frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}}e^{-\frac{T}{\alpha}} & -\frac{1}{k_2}p_1 - g_2 \\ -\frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}}e^{-\frac{T}{\alpha}} & \frac{2\gamma\alpha}{1-e^{-\frac{2T}{\alpha}}}e^{-\frac{2T}{\alpha}} + p_2 & \frac{1}{k_2}p_1 + g_2 \\ -\frac{1}{k_2}p_1 - g_2 & \frac{1}{k_2}p_1 + g_2 & T\left(2p_1\frac{k_1}{k_2} + 2g_1 - 1\right) \end{pmatrix}$$

From the main diagonal of $R_1(T)$, we see immediately that it is necessary that

$$2p_1\frac{k_1}{k_2} + 2g_1 - 1 > 0 \tag{12}$$

and

$$0 < p_2 < 2\gamma\alpha \tag{13}$$

The next proposition guarantees that if the previous inequalities are satisfied, for large enough $T > 0$ condition $R_1(T) > 0$ is always satisfied. The proof, omitted here, is based on showing that for large enough T all the eigenvalues of $R_1(T)$ are positive. Note that $R_1(T)$ is a 3 by 3 matrix. Thus, the characteristic polynomial is of the form $\lambda^3 + \phi_2\lambda^2 + \phi_1\lambda + \phi_0 = 0$. The roots of this third order polynomial are all positive if and only if $\phi_2 < 0$, $\phi_1 > 0$, $\phi_0 < 0$, and $\phi_0 > \phi_1\phi_2$.

Proposition C.1 *If both (12) and (13) are satisfied then there exists a $0 \leq T_{1+} < \infty$ such that $R_1(T) > 0$ for all $T \geq T_{1+}$.*

Note that T_{1+} in the last proposition can be found explicitly, although this is not done here.

C.2 Impact Maps $2a$ and $2b$

The goal of this section is to give a similar result to proposition C.1 for impact maps $2a$ and $2b$. Lets start by decomposing the Hamiltonian matrix H in $H = V\Sigma U$ where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

and D is such that all its eigenvalues are in the left-half plane. Define matrices M and m such that $W_2 = \mathrm{diag}(M, M)$ and $W_1 = \begin{pmatrix} m' & m' \end{pmatrix}'$. Define also

$$L_1 = \begin{pmatrix} M'V_{22}V_{12}^{-1}M & 0 \\ 0 & M'U_{22}^{-1}U_{21}M \end{pmatrix}, \quad L_{2a} = \begin{pmatrix} M'V_{22}V_{12}^{-1}m \\ M'U_{22}^{-1}U_{21}m \end{pmatrix}, \quad L_{2b} = \begin{pmatrix} -M'V_{22}V_{12}^{-1}m \\ M'U_{22}^{-1}U_{21}m \end{pmatrix}$$

and $L_3 = m'(V_{22}V_{12}^{-1} + U_{22}^{-1}U_{21})m$. Denote $G_a = G$ and $G_b = \bar{G}$. Then, after some manipulation of $R_{2a}(T)$ and $R_{2a}(T)$, we get

$$R_{2i\infty} = \lim_{T\to\infty} R_{2i}(T) = \begin{pmatrix} L_1 - P & L_{2i} - G_i \\ L_{2i}' - G_i' & L_3 \end{pmatrix}$$

for $i = a, b$. Then, the following proposition follows.

Proposition C.2 *If $R_{2i\infty} > 0$, $i = a, b$, then there exist $0 \le T_{i+} < \infty$ such that $R_{2i}(T_{i+}) > 0$ for all $T \ge T_{i+}$, $i = a, b$.*

D Review of $\mathcal{H}_2$ Optimization

In this appendix, we are interested in minimizing over $u \in \mathcal{L}_2$ the functional

$$J(u(\cdot)) = \int_0^T \left(\gamma u^2(t) - y^2(t)\right) dt \tag{14}$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \quad x(0) = x_0, \quad x(T) = x_T \end{aligned}$$

and $T > 0$ (for $T = 0$, $J(u(\cdot)) = 0$). This is a typical linear quadratic optimal control problem which can be solved using the well known Pontryagin's maximum principle. For completeness, we include here the derivation of the solution. Let

$$L(u, x, \psi) = \int_0^T \left(\gamma u^2 - x'C'Cx + 2\psi'(\dot{x} - Ax - Bu)\right) dt$$

where ψ is the adjoint vector. It is a well known result that if u^* is the solution to (14) then u^* minimizes L. Thus,

$$\begin{aligned} \frac{\partial L}{\partial x} &= -2\dot{\psi} - 2C'Cx - 2A'\psi = 0 \\ \frac{\partial L}{\partial u} &= 2\gamma u - 2B'\psi = 0 \end{aligned}$$

From the last equality,

$$u^*(t) = \frac{1}{\gamma}B'\psi(t)$$

yielding the linear time-invariant system known as the Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = H \begin{pmatrix} x \\ \psi \end{pmatrix}$$

where H is defined as in section 2.2 and $x_0 = x(0)$, $x_T = x(T)$ are given. Then, the optimal cost $J(u^*) = J^*(x_0, x_T, T)$ is given by

$$J^*(x_0, x_T, T) = \psi_T' x_T - \psi_0' x_0$$

where $\psi_T = \psi(T)$ and $\psi_0 = \psi(0)$.

The solution to the Hamiltonian system is simply the solution of a linear time-invariant autonomous system

$$\begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} = e^{Ht} \begin{bmatrix} x_0 \\ \psi_0 \end{bmatrix}$$

In order to find $J^*(x_0, x_T, T)$ we still need to write ψ_0 and ψ_T as functions of T, x_0, and x_T. At $t = T$, the solution to the Hamiltonian system is

$$\begin{pmatrix} x_T \\ \psi_T \end{pmatrix} = e^{HT} \begin{pmatrix} x_0 \\ \psi_0 \end{pmatrix}$$

Then,

$$\psi_0 = e_{12}^{-1} \left(x_T - e_{11} x_0 \right)$$

and

$$\psi_T = \left(e_{21} - e_{12}^{-1} e_{11} \right) x_0 + e_{12}^{-1} x_T$$

where $e_{ij} = e_{ij}(T)$ are defined as in section 2.2. Finally, the optimal cost $J(u^*) = J^*(x_0, x_T, T)$ can be written as

$$J^*(x_0, x_T, T) = \begin{pmatrix} x_T \\ x_0 \end{pmatrix}' W_t(T) \begin{pmatrix} x_T \\ x_0 \end{pmatrix} \tag{15}$$

where $W_t(T)$ is the symmetric matrix defined as in section 2.2.

Optimal control references are numerous. For more detailed and general solutions see, for example, [1, 8].

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