# Stability of Zeno Equilibria in Lagrangian Hybrid Systems

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***Abstract*— This paper presents both necessary and sufficient conditions for the stability of Zeno equilibria in Lagrangian hybrid systems, i.e., hybrid systems modeling mechanical systems undergoing impacts. These conditions for stability are motivated by the sufficient conditions for Zeno behavior in Lagrangian hybrid systems obtained in [11]—we show that the same conditions that imply the existence of Zeno behavior near Zeno equilibria imply the stability of the Zeno equilibria. This paper, therefore, not only presents conditions for the stability of Zeno equilibria, but directly relates the stability of Zeno equilibria to the existence of Zeno behavior.**

## I. Introduction

Zeno behavior occurs in a hybrid system when an infinite number of discrete transitions occur in a finite amount of time. Despite the simplicity of the definition of Zeno behavior, understanding this behavior on a fundamental level presents difficult and intriguing problems in hybrid systems. Can simple conditions for the existence of Zeno behavior be obtained? How does the existence of Zeno behavior relate to the convergence properties, or stability, of hybrid systems? In order to obtain an intuitive understanding of this phenomena, and help to answer some of the fundamental questions that arise when studying Zeno behavior, it is useful to study it in the context of hybrid systems that model real world systems.

In this paper, we study hybrid systems modeling mechanical systems undergoing impacts: *Lagrangian hybrid systems*. In particular, we consider a configuration space, a Lagrangian modeling a mechanical systems, and a *unilateral constraint function* that gives the set of admissible configurations for this system. From this data, we obtain a Lagrangian hybrid system. Moreover, hybrid systems of this form commonly display Zeno behavior (when an infinite number of collisions occur in a finite amount of time), and therefore provide the ideal class of systems in which to gain an intuitive understanding of Zeno behavior.

In [11], sufficient conditions for the existence of Zeno behavior in Lagrangian hybrid systems were presented. These conditions were obtained by considering *Zeno equilibria*—subsets of the continuous domains of a hybrid system that are fixed points of the discrete dynamics but not the continuous dynamics. It was shown that one need only check the sign of the second derivative of the unilateral constraint function evaluated at a Zeno equilibria to verify the existence of Zeno behavior. These conditions, and the framework in which they were presented, naturally raises the question: can similar conditions for the stability of Zeno equilibria in Lagrangian hybrid systems be obtained?

The main result of this paper are both necessary and sufficient conditions for the stability of Zeno equilibria in Lagrangian hybrid systems. Moreover, the sufficient conditions that we obtain are *exactly* that same as the conditions for the existence of Zeno behavior presented in [11]. That is, given a Zeno equilibrium point of a Lagrangian hybrid system, if the second derivative of the unilateral constraint function evaluated as this point is negative, then this point is stable and the hybrid system is Zeno. This result is appealing not only because it presents conditions for the stability of Zeno equilibria, but relates the stability of such equilibria to Zeno behavior and vice versa. That is, this paper allows for a deeper insight into the relationship between stability of Zeno equilibria and Zeno behavior in hybrid systems modeling mechanical systems undergoing impacts.

Due to the subtle and complex nature of Zeno behavior, it has been studied in many forms and from many different perspectives. Most of the conditions for Zeno behavior are necessary and tend to be very conservative; see [20] for general hybrid systems, and [7], [19] for linear complementarity systems. Until recently, sufficient conditions for Zeno behavior were more rare [2]. Necessary and sufficient conditions for Zeno behavior in a significantly different class of controlled hybrid systems were found in [9]. We also note that this paper studies Zeno behavior in Lagrangian hybrid systems, which were studied in [1], [3] and [4] as motivated by [6].

## II. Lagrangian Hybrid Systems

In this section, we introduce the notion of a hybrid Lagrangian and the associated Lagrangian hybrid system. Hybrid Lagrangians of this form have been studied in the context of Zeno behavior and reduction; see [1], [3], [4] and [10]. First, we review the notion of a simple hybrid system.

***Definition 1:*** A *simple hybrid system* is a tuple:

$$\mathscr{H} = (D, G, R, f),$$

where

- $D$ is a smooth manifold called the *domain*,
- $G$ is an embedded submanifold of $D$ called the *guard*,
- $R$ is a smooth map $R : G \to D$ called the *reset map*,
- $f$ is a vector field on the manifold $D$.

This paper focuses on *simple* hybrid systems, having a single domain, guard and reset map. A general hybrid system (see [13]), which is not discussed here, consists of a

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collection of domains, guards, reset maps and vector fields as indexed by an oriented graph.

**Hybrid executions.** An *execution* of a simple hybrid system $\mathscr{H}$ is a tuple $\chi^{\mathscr{H}} = (\Lambda, \mathcal{I}, \mathcal{C})$, where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is an indexing set.
- $\mathcal{I} = \{I_i\}_{i\in\Lambda}$ is a *hybrid interval* where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, $N$ finite. Here, $\tau_i, \tau_{i+1}, \tau_N \in I\!R$ and $\tau_i \leq \tau_{i+1}$.
- $\mathcal{C} = \{c_i\}_{i\in\Lambda}$ is a collection of integral curves of $f$, i.e., $\dot{c}_i(t) = f(c_i(t))$ for $t \in I_i$, $i \in \Lambda$,

And the following conditions hold for every $i, i+1 \in \Lambda$:

(i) $c_i(\tau_{i+1}) \in G$,
(ii) $R(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1})$,
(iii) $\tau_{i+1} = \min\{t \in I_i : c_i(t) \in G\}$.

The *initial condition* for the execution is $c_0(\tau_0)$.

**Lagrangians.** Let $Q$ be the $n$-dimensional *configuration space* for a mechanical system (assumed to be a smooth manifold) and $TQ$ the tangent bundle of $Q$. In this paper, we will consider Lagrangians, $L : TQ \to I\!R$, describing mechanical, or robotic, systems, which are Lagrangians of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{1}$$

where $M(q)$ is the (positive definite) inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy and $V(q)$ is the potential energy. In this case, the Euler-Lagrange equations yield the (unconstrained) equations of motion for the system:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = 0, \tag{2}$$

where $C(q, \dot{q})$ is the *Coriolis matrix* (cf. [15]) and $N(q) = \frac{\partial V}{\partial q}(q)$. Setting $x = (q, \dot{q})$, the Lagrangian vector field, $f_L$, associated to $L$ takes the familiar form:

$$\dot{x} = f_L(x) = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q)) \end{pmatrix}. \tag{3}$$

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

***Definition 2:*** A *simple hybrid Lagrangian* is defined to be a tuple

$$\mathbf{L} = (Q, L, h),$$

where

- $Q$ is the configuration space,
- $L : TQ \to I\!R$ is a hyperregular Lagrangian,
- $h : Q \to I\!R$ provides a unilateral constraint on the configuration space; we assume that $h^{-1}(0)$ is a smooth manifold.

**Simple Lagrangian hybrid systems.** For a Lagrangian (1), there is an associated dynamical system (3). Similarly, given a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ the *simple Lagrangian hybrid system* (**SLHC**) associated to $\mathbf{L}$ is the simple hybrid system:

$$\mathscr{H}_{\mathbf{L}} = (D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, f_{\mathbf{L}}).$$

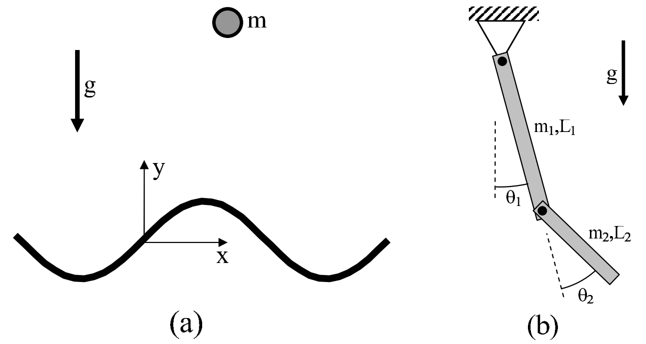


Fig. 1. (a) The bouncing ball on a sinusoidal surface (b) The double pendulum

First, we define

$$\begin{aligned} D_{\mathbf{L}} &= \{(q, \dot{q}) \in TQ : h(q) \geq 0\}, \\ G_{\mathbf{L}} &= \{(q, \dot{q}) \in TQ : h(q) = 0 \text{ and } dh(q)\dot{q} \leq 0\}, \end{aligned}$$

where

$$dh(q) = \left(\frac{\partial h}{\partial q}(q)\right)^T = \begin{pmatrix} \frac{\partial h}{\partial q_1}(q) & \cdots & \frac{\partial h}{\partial q_n}(q) \end{pmatrix}.$$

In this paper, we adopt the reset map ([6]):

$$R_{\mathbf{L}}(q, \dot{q}) = (q, P_{\mathbf{L}}(q, \dot{q})),$$

which based on the *impact equation*

$$\begin{aligned} &P_{\mathbf{L}}(q, \dot{q}) = \\ &\quad \dot{q} - (1+e)\tfrac{dh(q)\dot{q}}{dh(q)M(q)^{-1}dh(q)^T} M(q)^{-1}dh(q)^T, \end{aligned} \tag{4}$$

where $0 \leq e \leq 1$ is the *coefficient of restitution*, which is a measure of the energy dissipated through impact. This reset map corresponds to rigid-body collision law under the assumption of *frictionless impact*. Examples of more complicated collision laws that account for friction can be found in [6], [8].

Finally, $f_{\mathbf{L}} = f_L$ is the Lagrangian vector field associated to $L$ in (3).

***Example 1 (Ball):*** The first running example of this paper is a planar model of a ball bouncing on a sinusoidal surface (cf. Fig. 1(a)). The ball is modelled as a point mass $m$. In this case

$$\mathbf{B} = (Q_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}}),$$

where $Q_{\mathbf{B}} = I\!R^2$, and the configuration is the position of the ball $q = (x, y)$,

$$L_{\mathbf{B}}(x, \dot{x}) = \frac{1}{2}m\|\dot{q}\|^2 - mgy.$$

Finally, we make the problem interesting by considering the sinusoidal constraint function

$$h_{\mathbf{B}}(q) = y - \sin(x) \geq 0.$$

So, for this example, there are trivial dynamics and a nontrivial constraint function.

***Example 2 (Double Pendulum):*** Our second running example is a constrained double pendulum with a mechanical stop (cf. Fig. 1(b)). The double pendulum consists of two rigid links of masses $m_1, m_2$, lengths $L_1, L_2$, and uniform mass distribution, which are attached by passive joints, while a mechanical stop dictates the range of motion of the second link.

The example serves as a simplified model of a leg with a passive knee and a mechanical stop, which is widely investigated in the robotics literature in the context of passive dynamics of bipedal walkers (cf. [18], [14]). In this case

$$\mathbf{P} = (Q_{\mathbf{P}}, L_{\mathbf{P}}, h_{\mathbf{P}}),$$

where $Q_{\mathbf{P}} = \mathbb{S}^1 \times \mathbb{S}^1$, $q = (\theta_1, \theta_2)$, and

$$\begin{aligned} L_{\mathbf{P}}(q,\dot{q}) \;\; &= \;\; \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + (\tfrac{1}{2}m_1 L_1 + m_2 L_1) g\cos\theta_1 \\ &\quad + \tfrac{1}{2} m_2 L_2 g \cos(\theta_1 + \theta_2), \end{aligned}$$

with the elements of the $2\times 2$ inertia matrix $M(q)$ given by

$$\begin{aligned} &M_{11} = m_1 L_1^2/3 + m_2(L_1^2 + L_2^2/3 + L_1 L_2 \cos\theta_2) \\ &M_{12} = M_{21} = m_2(3L_1 L_2 \cos\theta_2 + 2L_2^2)/6 \\ &M_{22} = m_2 L_2^2/3. \end{aligned}$$

Finally, the constraint that represents the mechanical stop is given by $h_{\mathbf{P}}(q) = \theta_2 \geq 0$. So, for this example, there are nontrivial dynamics and a trivial constraint function.

## III. Zeno Behavior and Zeno Equilibria

This section discusses Zeno behavior and the corresponding notion of Zeno equilibria. More importantly, we state the sufficient conditions for Zeno behavior that will motivate the main result of this paper in that our sufficient conditions for the stability of Zeno equilibria utilize exactly the same conditions; that is, in Lagrangian hybrid systems, the existence of Zeno behavior and the stability of Zeno equilibria can be detected with the same simple and easily verifiable conditions.

**Zeno behavior.** An execution $\chi^{\mathscr{H}}$ is *Zeno* if $\Lambda = \mathbb{N}$ and

$$\lim_{i\to\infty} \tau_i = \tau_\infty < \infty.$$

Here $\tau_\infty$ is called the *Zeno time*. If $\chi^{\mathscr{H}_{\mathbf{L}}}$ is a Zeno execution of a Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$, then its *Zeno point* is defined to be

$$x_\infty = (q_\infty, \dot{q}_\infty) = \lim_{i\to\infty} c_i(\tau_i) = \lim_{i\to\infty} (q_i(\tau_i), \dot{q}_i(\tau_i)).$$

These limit points are intricately related to a type of equilibrium point that are unique to hybrid systems: Zeno equilibria.

***Definition 3:*** A Zeno equilibrium point of a simple hybrid system $\mathscr{H}$ is a point $x^* \in G$ such that

- $R(x^*) = x^*$,
- $f(x^*) \neq 0$.

**Zeno equilibria.** If $\mathscr{H}_{\mathbf{L}}$ is a Lagrangian hybrid system, then due to the special form of these systems we find that the point $(q^*, \dot{q}^*)$ is a Zeno equilibria iff $\dot{q}^* = P_{\mathbf{L}}(q, \dot{q}^*)$, with $P_{\mathbf{L}}$ given in (4). In particular, the special form of $P_{\mathbf{L}}$ implies that this hold iff $dh(q^*)\dot{q}^* = 0$. Therefore the set of all Zeno equilibria for a Lagrangian hybrid system is given by the hypersurfaces in $G_{\mathbf{L}}$:

$$Z = \{(q,\dot{q}) \in G_{\mathbf{L}} : dh(q)\dot{q} = 0\}.$$

Note that if $\dim(Q) > 1$, the Zeno equilibria in Lagrangian hybrid systems are always non-isolated (see [10])—this motivates the study of such equilibria.

**Sufficient conditions for Zeno behavior.** Let $\ddot{h}(q,\dot{q})$ be the acceleration of $h(t)$ along trajectories of the unconstrained dynamics (2), given by:

$$\begin{aligned} &\ddot{h}(q,\dot{q}) = \\ &\dot{q}^T H(q)\dot{q} + dh(q) M(q)^{-1}(-C(q,\dot{q})\dot{q} - N(q)), \end{aligned} \tag{5}$$

where $H(q)$ is the Hessian of $h$ at $q$. The following theorem, which was proven in [11], provides sufficient conditions for existence of Zeno executions in the vicinity of a Zeno equilibrium point.

***Theorem 1 ([11]):*** *Let $\mathscr{H}_{\mathbf{L}}$ be a simple Lagrangian hybrid system and Let $(q^*, \dot{q}^*)$ be a Zeno equilibrium point of $\mathscr{H}_{\mathbf{L}}$. Then if $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, there exists a neighborhood $W \subset D_{\mathbf{L}}$ of $(q^*, \dot{q}^*)$ such that for every $(q_0, \dot{q}_0) \in W$, there is a unique Zeno execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ of $\mathscr{H}_{\mathbf{L}}$ with $c_0(\tau_0) = (q_0, \dot{q}_0)$.*

## IV. Stability of Zeno Equilibria

In this section, we present and prove the main result of this paper: sufficient conditions for the stability of Zeno equilibria. In particular, we introduce a type of stability that Zeno equilibria in **SLHS** can display: bounded-time local stability (**BTLS**). We show that the same conditions on the coefficient of restitution and the second derivative of the unilateral constraint function implies this type of stability. Conversely, if these conditions are not satisfied, the Zeno equilibrium point is *not* **BTLS**.

***Definition 4:*** Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$. Then $x^*$ is defined as *bounded-time locally stable* if for each open neighborhood $U \subseteq TQ$ of $x^*$ and $\epsilon_t > 0$, there exists another open neighborhood $W$ of $x^*$, such that for every initial conditions $c_0(\tau_0) \in W \cap D_{\mathbf{L}}$, the corresponding execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ is Zeno, and satisfies $c_i(t) \in U$ for all $t \in I_i$ and $i \in \Lambda$, while its Zeno time satisfies $\tau_\infty - \tau_0 < \epsilon_t$.

### A. Statement of Main Result

We now present the main result of the paper: conditions for **BTLS** of Zeno equilibria of **SLHS**.

***Theorem 2:*** *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$. Then the following two conditions hold:*

(i) *If $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, then $x^*$ is* **BTLS**.
(ii) *If $\ddot{h}(q^*, \dot{q}^*) > 0$, then $x^*$ is not* **BTLS**.

For part (i), we not only prove the existence of the neighborhood $W$ for given $U$, but also provide an explicit relation between $W$ and $U$. For the sake of concreteness and simplicity, we use a *local coordinate chart* for small neighborhoods of $x^*$. Therefore, we can identify both $q$ and $\dot{q}$ with elements of $\mathbb{R}^n$, and use the induced Euclidean norm $\|\cdot\|$ to define neighborhoods of $x^* = (q^*, \dot{q}^*)$ as

$$N(\epsilon_q, \epsilon_v) = \{(q,\dot{q}) \in D_{\mathbf{L}} : \; \|q - q^*\| < \epsilon_q, \; \|\dot{q} - \dot{q}^*\| < \epsilon_v\}$$

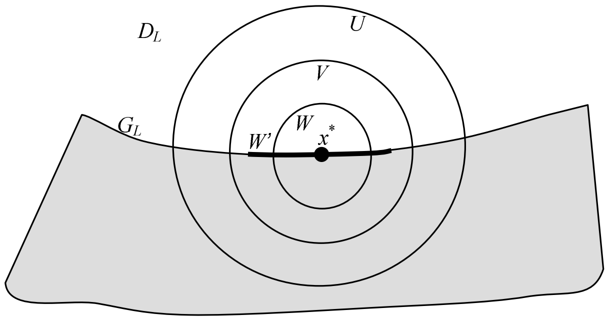


Fig. 2. Illustration of the neighborhoods $U, V, W$ and $W'$ of $x^*$.

Using this notation, for a given $U$ there exist $\epsilon_q$ and $\epsilon_v$ such that $U \subseteq N(\epsilon_q, \epsilon_v)$. Assuming that $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, our goal is to construct a neighborhood $W = N(\delta_q, \delta_v)$ that satisfies the requirements given in Definition 4.

### B. Proof of Main Result

The rest of this section proves Theorem 2 in stages through a series of lemmas. Before presenting these lemmas, we will first give a general outline of the proof. In particular, the proof of part (i) of Theorem 2 is divided into three steps:

1) We define an intermediate neighborhood $V \subset U$, such that any execution that stays within $V$ at all times is guaranteed to be Zeno.
2) We define another neighborhood $W' \subset G_{\mathbf{L}} \cap V$, which lies *on the guard* $G_{\mathbf{L}}$, such that any execution whose first discrete event $c_0(\tau_1)$ lies within $W'$ is guaranteed to stay within $V$.
3) We construct the neighborhood $W$, such that any execution with initial conditions within $W$ is guaranteed to pass through a point of $W'$ at time $\tau_1$, and thus it is a Zeno execution that stays within U, as required. An illustration of these neighborhoods appear in Fig. 2.

We now formally proceed through these steps in order to establish the main result of the paper.

**Step 1.** We begin by defining the intermediate neighborhood $V = N(\epsilon'_q, \epsilon'_v)$, where $\epsilon'_q < \epsilon_q$ and $\epsilon'_v < \epsilon_v$ are chosen so that for

$$\begin{aligned} a_{min} &= -\max_{(q,\dot{q}) \in V} \ddot{h}(q, \dot{q}), \\ a_{max} &= -\min_{(q,\dot{q}) \in V} \ddot{h}(q, \dot{q}), \end{aligned}$$

The following conditions hold:

$$a_{max} > a_{min} > 0 \qquad \text{and} \qquad e\frac{a_{max}}{a_{min}} < 1. \tag{6}$$

Note that the fact that $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, along with the continuity of $\ddot{h}(q, \dot{q})$, imply that such $\epsilon'_q, \epsilon'_v$ exist. This definition of $V$ implies that when $(q(t), \dot{q}(t)) \in V$, $h(q(t))$ satisfies the second-order differential inclusion

$$\ddot{h}(q(t), \dot{q}(t)) \in [-a_{max}, -a_{min}]. \tag{7}$$

For simplicity of notation, for an execution $\chi^{\mathscr{H}_{\mathbf{L}}}$, let

$$\begin{aligned} v_i^- &= dh(q_{i-1}(\tau_i))\dot{q}_{i-1}(\tau_i), \\ v_i^+ &= dh(q_i(\tau_i))\dot{q}_i(\tau_i). \end{aligned}$$

Note that (4) implies that $v_i^+ = -ev_i^-$. Also, let $T_i = \tau_i - \tau_{i-1}$, which is the time difference between consecutive collisions. The following lemma states that any execution which is bounded within $V$ is guaranteed to be Zeno.

***Lemma 1:*** *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangia hybrid system such that $\ddot{h}(q^*, \dot{q}^*) < 0$ and $e < 1$, and let $V = N(\epsilon'_q, \epsilon'_v)$ be a neighborhood of $x^*$ that satisfies (6). Then for any execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ such that $c_i(t) \in V$ for all $t \in I_i$ and $i \in \Lambda$, the discrete-time series of $v_i^+$ and $T_i$ satisfy:*

$$e\sqrt{\frac{a_{min}}{a_{max}}} \le \frac{v_{i+1}^+}{v_i^+} \le e\sqrt{\frac{a_{max}}{a_{min}}}, \tag{8}$$

$$\frac{T_{i+1}}{T_i} \le e\frac{a_{max}}{a_{min}}. \tag{9}$$

*Therefore, $\chi^{\mathscr{H}_{\mathbf{L}}}$ is Zeno.*

**Setup for Lemma 1.** In order to prove Lemma 1, we will utilize methods from optimal control. (The idea of using results from optimal control to analyze stability of differential inclusions also appears in the work of Liberzon and Margaliot [12].) We, therefore, briefly review the basic form of Pontryagin's maximum principle based on its presentation in [5], though we adopt a slightly different notation.

Consider a control system

$$\dot{x} = f(x, u), \tag{10}$$

where $x \in I\!R^n$ and $u \in \Omega \subseteq I\!R^m$, where $\Omega$ is a convex set of admissible controls. A solution to (10) on a time interval $[t_0, t_f]$ is a pair $(x(t), u(t))$ satisfying (10) and $u(t) \in \Omega$ for all $t \in [t_0, t_f]$; the initial and final conditions of $x(t)$ are denoted $x_0 = x(t_0)$ and $x_f = x(t_f)$. The design goal is to find a solution to (10) that minimizes a given cost function $P(x_f, t_f)$ [1]; note that the end condition $x_f$ and the end time $t_f$, may be either specified or "free".

Using calculus of variations techniques, the solution of this problem is given as follows. First, define the *Hamiltonian*, given by $H(x, u, \lambda, t) = \lambda(t)^T f(x, u)$, where $\lambda \in I\!R^n$ is called *the co-state vector*. The co-state dynamic equations are then given by $\dot{\lambda} = -\frac{\partial H}{\partial x}$, and the optimal control satisfies $u^*(t) = \mathrm{argmin} H$. The end condition is given by $[\frac{\partial P}{\partial x_f} - \lambda(t_f)]^T \delta x_f = 0$, where if a particular state variable $x_i$ is specified, then its variation $\delta x_i(t_f)$ vanishes, and if it is not specified, then it gives an end condition for the corresponding co-state variable $\lambda_i(t_f)$. In case where the terminal time $t_f$ is not specified, an additional condition on $H(t_f)$ is given by $\frac{\partial P}{\partial t_f} + H(t_f) = 0$.

*Proof:* [*of Lemma 1*] We begin by proving (8). Let $\chi^{\mathscr{H}_{\mathbf{L}}}$ be an execution such that $c_i(t) = (q_i(t), \dot{q}_i(t)) \in V$ for all $t \in I_i = [\tau_i, \tau_{i+1}]$ and $i \in \Lambda$. Moreover,

$$\ddot{h}(q_i(t), \dot{q}_i(t)) \in [-a_{max}, -a_{min}] \tag{11}$$

[1]Many textbooks also consider an integral cost function of the form $J = \int_{t_0}^{t_f} g(x, u, t)dt$. This cost function can be incorporated into the formulation here by using an additional state variable $z$, whose dynamics is given by $\dot{z} = g(x, u, t)$. The cost function is then simply given by $P = z(t_f)$.

for all $t \in I_i$. The main idea of this proof is that, choosing a state vector $x = (x_1, x_2) = (h(q), \dot{h}(q, \dot{q}))$, (11) can be stated as a control system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{12}$$

where $u \in [-a_{max}, -a_{min}]$.

To prove (8), consider the cost function: $P(x_f, t_f) = (x_f)_2$ for the control system (12). The Hamiltonian is given by $H = \lambda_1 x_2 + \lambda_2 u$. The co-state dynamic equations are then $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = \lambda_1$, indicating that $\lambda_1(t)$ is constant and $\lambda_1(t)$ is a linear function. The end condition gives $\lambda_2(t) = 1$. The maximum principle then implies that the optimal input $u^*(t)$ is either $a_{min}$ or $a_{max}$, and depends solely on the sign of $\lambda_2(t)$, which is a linear function that has at most one zero-crossing point. Therefore, $u^*$ is a piecewise-constant function with at most one switching point, and we can set $u^*(t) = -u_1$ for $t \in [t_0, t_s]$ and $u^*(t) = -u_2$ for $t \in [t_s, t_f]$, where $t_s$ is the switching time, and $u_1, u_2 \in \{a_{max}, a_{min}\}$.

Now, for the execution $\chi^{\mathscr{H}_\mathbf{L}}$ and taking $t_0 = \tau_i$ and $t_f = \tau_{i+1}$, direct integration of $\ddot{h}(q_i(t), \dot{q}_i(t)) = u(t)$ gives

$$\begin{aligned} &\dot{h}(q_i(\tau_{i+1}), \dot{q}_i(\tau_{i+1})) = \\ &\qquad -\sqrt{(v_i^+ - u_1 t_s)^2 + 2u_2(v_i^+ t_s - u_1 t_s^2/2)}, \end{aligned}$$

whose critical value is attained at $t_s^* = v_i^+/u_1$, i.e. it satisfies $\dot{h}(q_i(t_s), \dot{q}_i(t_s)) = 0$. It then follows that the minimum and maximum values of $\dot{h}(q_i(\tau_{i+1}), \dot{q}_i(\tau_{i+1}))$ are given by

$$-\sqrt{\tfrac{a_{max}}{a_{min}}} v_i^+ < \dot{h}(q_i(\tau_{i+1}), \dot{q}_i(\tau_{i+1})) = v_{i+1}^- < -\sqrt{\tfrac{a_{min}}{a_{max}}} v_i^+.$$

Using the fact that $v_i^+ = -e v_i^-$, one obtains (8).

To prove (9), consider the differential inclusion (11) for two consecutive time intervals $I_{i-1} = [\tau_{i-1}, \tau_i]$ and $I_i = [\tau_i, \tau_{i+1}]$. That is, we consider two control systems as defined in (12). The initial conditions and final conditions for the first control system are given by:

$$\begin{aligned} x_1(\tau_{i-1}) &= h(q_{i-1}(\tau_{i-1})) = 0. \\ x_2(\tau_{i-1}) &= \dot{h}(q_{i-1}(\tau_{i-1}), \dot{q}_{i-1}(\tau_{i-1})) = v_{i-1}^+ \\ x_1(\tau_i) &= h(q_i(\tau_i)) = 0. \\ x_2(\tau_i) &= \dot{h}(q_{i-1}(\tau_i), \dot{q}_{i-1}(\tau_i)) = v_i^- \end{aligned}$$

where $v_i^-$ and $\tau_i$ are not specified. The initial and final conditions for the second control system are:

$$\begin{aligned} x_1(\tau_i) &= h(q_i(\tau_i)) = 0. \\ x_2(\tau_i) &= \dot{h}(q_i(\tau_i), \dot{q}_i(\tau_i)) = v_i^+ = -e v_i^- \\ x_1(\tau_{i+1}) &= h(q_i(\tau_{i+1})) = 0. \\ x_2(\tau_{i+1}) &= \dot{h}(q_i(\tau_{i+1}), \dot{q}_i(\tau_{i+1})) = v_{i+1}^- \end{aligned}$$

where $\tau_{i+1}$ and $v_{i+1}^-$ are not specified.

The goal is to find a solution to the two control systems in which $T_{i+1}/T_i$ is maximized where, again, $T_i = \tau_i - \tau_{i-1}$. It is easy to see that for a given $v_i^+$, $T_{i+1}$ is maximized by simply taking $u(t) = -a_{min}$ for $t \in I_i$, and its maximum value is given by $T_{i+1}^* = 2v_i^+/a_{min}$. The problem then reduces to maximizing the ratio $v_i^+/T_i$ for a solution to the second control system. Note that the definition of the Hamiltonian $H$ and the derivation of the co-state dynamic equation for $\lambda(t)$ are also identical to those derived in the proof of (8). Setting $t_0 = \tau_{i-1} = 0$, the cost function to be minimized in this problem is given by $P(x_f, t_f) = (x_f)_2/t_f$ where here $t_f = \tau_i$. As before, the maximum principle implies that the optimal input $u^*(t)$ is either $a_{min}$ or $a_{max}$, and depends solely on the sign of $\lambda_2(t)$. Using the end condition for $\lambda_2$ gives $\lambda_2(t_f) = 1/t_f$, which implies that $\lambda_1(t) = c_1$ and $\lambda_2(t) = 1/t_f + c_1(t_f - t)$. The additional condition on $H(t_f)$ gives $x_2(t_f)(c_1 - 1/t_f^2) + u(t_f)/t_f = 0$. Since $x_2(t_f)$ and $u(t_f)$ are both negative, we conclude that $c_1 - 1/t_f^2 < 0$. This implies that $\lambda_2(t)$ does not cross zero, and is positive for $t \in [0, t_f]$. Therefore, minimization of the cost function is obtained by taking $u(t) = -a_{max}$ for $t \in [0, t_f]$, and the maximum value for $T_{i+1}/T_i$ is consequently $e\frac{a_{max}}{a_{min}}$. ■

**Step 2.** As the next step towards computing the neighborhood W, we compute the neighborhood $W' \subset G_\mathbf{L} \cap V$, of initial conditions *on the guard* $G_\mathbf{L}$ (i.e. corresponding to a collision), such that any execution with initial conditions in $W'$ stays within $V$.

In order to construct $W'$ for given neighborhoods $U$ and $V$, we first define the following scalars:

$$\begin{aligned} e' &= e\frac{a_{max}}{a_{min}} \\ e'' &= e\sqrt{\frac{a_{max}}{a_{min}}} \\ \beta &= \|\dot{q}^*\| + \epsilon_v' \\ \eta &= \max_{(q,\dot{q})\in V} \frac{\|M^{-1}(q) dh(q)^T\|}{dh(q) M(q) dh(q)^T} \\ \zeta &= \max_{(q,\dot{q})\in V} \left\| M^{-1}(q) \left( C(q,\dot{q})\dot{q} + N(q) \right) \right\|. \end{aligned} \tag{13}$$

The following lemma completes the definition of $W'$.

***Lemma 2:*** *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_\mathbf{L}$ such that $\ddot{h}(q^*, \ddot{q}^*) < 0$ and $e < 1$, and let $V = N(\epsilon_q', \epsilon_v')$ be a neighborhood of $x^*$ that satisfies (6). For a given $\epsilon_t' > 0$, let $W'$ be the neighborhood defined as follows:*

$$\begin{aligned} W' \;=\; & \{(q,\dot{q}) \in TQ : \; h(q) = 0, \; \|q - q^*\| < \delta_q', \\ & \|\dot{q} - \dot{q}^*\| < \delta_v' \text{ and } dh(q)\dot{q} < -v_{1max} < 0\}. \end{aligned} \tag{14}$$

*such that $\delta_q', \delta_v'$ and $v_{1max}$ satisfy the conditions:*

$$\delta_q' < \epsilon_q', \; \delta_v' < \epsilon_v' \text{ and } v_{1max} < \min\{c_1, c_2, c_3\} \tag{15}$$

*where*

$$\begin{aligned} c_1 &= \tfrac{a_{min}(1-e')}{2e} \epsilon_t' \\ c_2 &= \tfrac{a_{min}(1-e')}{2e\beta} (\epsilon_q' - \delta_q') \\ c_3 &= \left( \epsilon_v' - \delta_v' \right) / \left( \tfrac{(1+e)\eta}{1-e''} + \tfrac{2e\zeta}{a_{min}(1-e')} \right). \end{aligned}$$

*Then each execution $\chi^{\mathscr{H}_\mathbf{L}}$ such that $c_0(\tau_1) \in W'$ is Zeno and satisfies $c_i(t) \in V$ for all $t \in I_i$ and $i \geq 1$. Moreover,* *the corresponding Zeno time satisfies*

$$\tau_\infty - \tau_1 < \epsilon_t'. \qquad (16)$$

*Proof:* It is easy to see that (14) implies that $W' \subset V$. Treating $\tau_1$ as the initial time, the initial conditions of $\chi^{\mathscr{H}_\mathbf{L}}$ are thus lying inside $V$. In the following, we first assume that $\chi^{\mathscr{H}_\mathbf{L}}$ stays within $V$, and thus the differential inclusion (11) for $h(q_i(t))$ holds for all $i \in \Lambda$, then we show that conditions (15) imply that $(q_i(t), \dot{q}_i(t))$ actually stays within $V$ for all $t \in I_i$ and $i \in \Lambda$. First, we assume that the execution $\chi^{\mathscr{H}_\mathbf{L}}$ stays within $V$, and show that its Zeno time satisfies (16). By assumption, during the time interval $t \in [\tau_1, \tau_2]$, $h(q_1(t))$ satisfies the differential inclusion (11) (with $i = 1$) with initial conditions $h(\tau_1) = 0$, $\dot{h}(\tau_1) = v_1^+$ and an end condition $h(\tau_2) = 0$. It is easily shown that $T_2 = \tau_2 - \tau_1$ is bounded by

$$T_2 \leq \frac{2v_1^+}{a_{min}} \leq \frac{2ev_{1max}}{a_{min}}.$$

Using Lemma 1, the sequence $T_i$ satisfies (9), and is thus bounded by a geometric series with the factor $e'$. Since conditions (6) imply that $e' < 1$, the total execution time is bounded by $\tau_\infty - \tau_1 = \sum T_i \leq \frac{T_2}{1-e'}$. Combining the two inequalities above, one gets

$$\tau_\infty - \tau_1 \leq \frac{2ev_{1max}}{a_{min}(1-e')}. \qquad (17)$$

Setting $v_{1max} < c_1$ in (15) then verifies that the bound (16) is satisfied.

Next, let $q(t) = q_i(t)$ for $t \in [\tau_i, \tau_{i+1}]$; this is well defined since $q_i(\tau_{i+1}) = q_{i+1}(\tau_{i+1})$ for all $i \in \Lambda$, i.e., $q(t)$ does not change through the collisions. Since $(q_i(t), \dot{q}_i(t))$ are assumed to remain within $V$ for all $t \in I_i$ and $i \in \Lambda$, i.e., during the duration of the execution $t \in [\tau_1, \tau_\infty]$, the change in $q(t)$ is bounded by $\|q_\infty - q_1(\tau_1)\| \leq \beta(t - \tau_1)$ where $\beta$, defined in (13), is the maximum norm of $\dot{q}$ in $V$. Using the bound (17) and the triangle inequality $\|q(t) - q^*\| \leq \|q_1(\tau_1) - q^*\| + \|q(t) - q_1(\tau_1)\|$, the condition $v_{1max} < c_2$ in (15) then verifies that $\|q(t) - q^*\| \leq \epsilon_q'$ for all $t \in [\tau_1, \tau_\infty]$, and the $q$-component of $c_i(t)$ is guaranteed to stay within $V$.

Finally, the change in the velocity $\dot{q}$ during the execution is decomposed into its dicrete and continuous parts, as follows. Let us denote $\Delta_i^{(1)} = \dot{q}_i(\tau_i) - \dot{q}_{i-1}(\tau_i)$ and $\Delta_i^{(2)} = \dot{q}_i(\tau_{i+1}) - \dot{q}_i(\tau_i)$. The total change in $\dot{q}$ is thus bounded by $\|\dot{q}_\infty - \dot{q}_1(\tau_1)\| \leq \Delta^{(1)} + \Delta^{(2)}$, where $\Delta^{(j)} = \sum_{i=1}^{\infty} \|\Delta_i^{(j)}\|$ for $j = 1, 2$. Assume that $(q_i(t), \dot{q}_i(t)) \in V$ for all $t \in I_i$ and $i \in \Lambda$. The velocity change due to a single collision at time $\tau_i$ is given in (4), and is thus bounded by $\|\Delta_i^{(1)}\| \leq (1 + e)\eta \left|v_i^-\right|$, where $\eta$ is defined in (13). Using Eq. (8) in Lemma 1 along with the relation $v_i^+ = -ev_i^-$, the sequence $\left|v_i^-\right|$ is bounded by $\left|v_i^-\right| \leq (e'')^{i-1}\left|v_1^-\right| < v_{1max}$. Thus, $\Delta^{(1)}$ is bounded by the sum of a geometric series as $\Delta^{(1)} < \frac{(1+e)\eta}{1-e''} v_{1max}$.

The continuous part is bounded by $\Delta^{(2)} \leq \zeta(\tau_\infty - \tau_1)$, where $\zeta$, defined in (13), is the maximum norm of $\ddot{q}$ in $V$, and $\tau_\infty - \tau_1$ is bounded according to (17). Using the bounds obtained on $\Delta^{(1)}, \Delta^{(2)}$ and the triangle inequality $\|\dot{q}_\infty - \dot{q}^*\| \leq \Delta^{(1)} + \Delta^{(2)} + \delta_v'$, the condition $v_{1max} < c_3$ in (15) then verifies that $\|\dot{q}_\infty - \dot{q}^*\| \leq \epsilon_v'$. By our construction, it is clear that this inequality also holds when replacing $\dot{q}_\infty$ with $\dot{q}_i(t)$ for all $t \in I_i$ and $i \in \Lambda$, and thus the $\dot{q}$-component of $c_i(t)$ is guaranteed to stay within $V$. ■

**Step 3.** At this final stage, for a given $\epsilon_t'' > 0$, we define the neighborhood $W$ as

$$W = N(\delta_q, \delta_v),$$

where $\delta_q < \delta_q'$ and $\delta_q < \delta_q'$ satisfy:

$$\begin{aligned} &\text{(i)}\ \frac{dh(q)\dot{q} + \sqrt{(dh(q)\dot{q})^2 - a_{min}h(q)}}{a_{min}} < \min\{\frac{\delta_q' - \delta_q}{\beta},\ \frac{\delta_v' - \delta_v}{\zeta},\ \epsilon_t''\} \\ &\text{(ii)}\ \left(2h(q) + \frac{(dh(q)\dot{q})^2}{a_{min}}\right) a_{max} < (v_{1max})^2 \end{aligned} \qquad (18)$$

for all $(q, \dot{q}) \in N(\delta_q, \delta_v) \cap D_\mathbf{L}$.

Note that since $h(q^*) = 0$ and $dh(q^*)\dot{q}^* = 0$, continuity of $h(q)$ and $dh(q)$ imply that such $\delta_q, \delta_v$ exist. The following lemma states that if the initial condition are within $W$, then at the first collision time $\tau_1$, $(q, \dot{q})$ are within $W'$.

***Lemma 3:*** *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_\mathbf{L}$ such that $\ddot{h}(q^*, \ddot{q}^*) < 0$ and $e < 1$, and let $V$, $W'$ and $W$ be the neighborhoods of $x^*$ defined in (6), (14) and (18) respectively. Then each execution $\chi^{\mathscr{H}_\mathbf{L}}$ such that $c_0(\tau_0) \in W \cap D_\mathbf{L}$ satisfies $c_0(t) \in V$ for $t \in I_0$, and $c_0(\tau_1) \in W'$ and $\tau_1 - \tau_0 < \epsilon_t''$.*

*Proof:* From the definition of $W$ and $V$, it is clear that the initial condition satisfies $c_0(\tau_0) \in V$. We first assume that $c_0(t)$ stays within $V$ for $t \in I_0$, therefore $h(q_0(t))$ satisfies the differential inclusion (11) (for $i = 0$). Then we prove that under the conditions on $W$ in (18), a finite $\tau_1$ does exist, and $c_0(t)$ actually remains within $V$ for $t \in [\tau_0, \tau_1]$. Since $c_0(\tau_0) \in V$, the differential inclusion implies $\ddot{h}(q_0(t), \dot{q}_0(t)) < 0$ Therefore, there exists some finite $\tau_1$ such that $h(q_0(\tau_1)) = 0$.

Assume that $h(q_0(t))$ satisfies the differential inclusion (11) for $t \in [\tau_0, \tau_1]$, with initial conditions $h(q_0(\tau_0)) = h_0 \geq 0$ and $\dot{h}(q_0(\tau_0), \dot{q}_0(\tau_0))) = v_0^+$, and the end condition $h(q_0(\tau_1)) = 0$. It is easy to show (even without using Pontryagin's maximum principle) that the "free" end conditions for $\tau_1$ and $\dot{h}(q_0(\tau_1), \dot{q}_0(\tau_1))$ are bounded by

$$\tau_1 - \tau_0 < \frac{dh(q_0(\tau_0))\dot{q}_0(\tau_0) + \sqrt{(dh(q_0(\tau_0))\dot{q}_0(\tau_0))^2 - a_{min}h(q_0(\tau_0))}}{a_{min}} \qquad (19)$$

$$\left|\dot{h}(q_0(\tau_1), \dot{q}_0(\tau_1))\right| < \sqrt{\left(2h_0 + \frac{(v_0^+)^2}{a_{min}}\right) a_{max}}. \qquad (20)$$

Assuming that $(q_0(t), \dot{q}_0(t)) \in V$, the total change in $q_0(t)$ and $\dot{q}_0(t)$ for $t \in [\tau_0, \tau_1]$ are bounded by

$$\|q_0(t) - q^*\| < \beta(t - \tau_0), \qquad \|\dot{q}_0(t) - \dot{q}^*\| < \zeta(t - \tau_0).$$

Using the bound on $\tau_1 - \tau_0$ in (19) and the triangle inequality, item (i) in (18) then implies that both $(q_0(t), \dot{q}_0(t))$ actually stays within $V$ for $t \in [\tau_0, \tau_1]$. Moreover, the bound in (20), along with item (ii) in (18) imply that $(q_0(\tau_1), \dot{q}_0(\tau_1)) \in W'$. Finally, condition (i) in (18) also implies that $\tau_1 - \tau_0 < \epsilon_t''$. ■

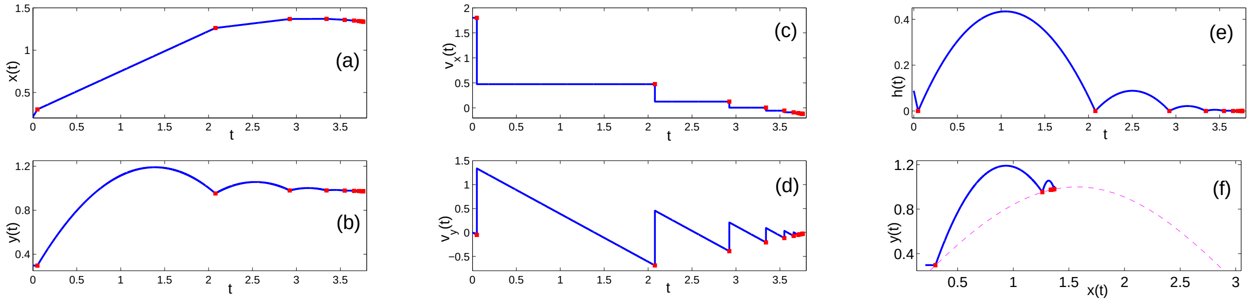


Fig. 3. Simulation results for the ball example with initial velocities $v_x(0) = 1.8$ and $v_y(0) = 0$.

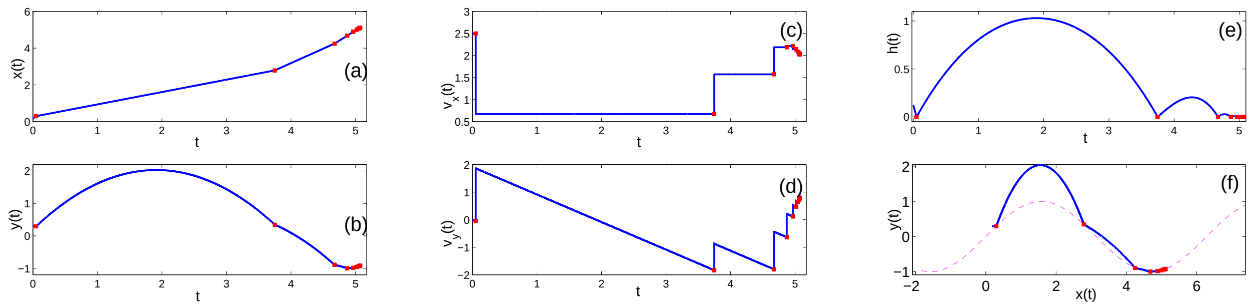


Fig. 4. Simulation results for the ball example with initial velocities $v_x(0) = 2.5$ and $v_y(0) = 0$.

We now combine the results above to complete the proof of Theorem 2.

*Proof:* [*of Theorem 2*] First, we prove part (i). Assume that $e < 1$ and $\ddot{h}(q^*, \ddot{q}^*) < 0$. For the given neighborhood $U$ and $\epsilon_t > 0$, pick $\epsilon_q, \epsilon_t$ such that $U \subseteq N(\epsilon_q, \epsilon_t)$. Next, choose $\epsilon_q' < \epsilon_q$ and $\epsilon_v' < \epsilon_v$ such that the neighborhood $V = N(\epsilon_q', \epsilon_v')$ satisfies (6). Next, choose $\epsilon_t' < \epsilon_t$ and $\delta_q', \delta_v'$ and $v_{1max}$ such that (15) is satisfied. The neighborhood $W'$ is then defined in (14). Finally, choose $\epsilon_t'' < \epsilon_t - \epsilon_t'$, and $\delta_q < \delta_q'$, $\delta_v < \delta_v'$ that satisfy (18), and define $W = N(\delta_q, \delta_v)$. Consider an execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ with initial conditions $c_0(\tau_0) \in W \cap D_{\mathbf{L}}$. Lemma 3 implies that $c_0(t) \in V$ for $t \in I_0$, and $c_0(\tau_1) \in W'$ and $\tau_1 - \tau_0 < \epsilon_t''$. Lemma 2 then implies that $c_i(t) \in V$ for all $t \in I_i$ and $i \geq 1$, that $\chi^{\mathscr{H}_{\mathbf{L}}}$ is Zeno, and that $\tau_\infty - \tau_1 < \epsilon_t'$. Therefore, $\chi^{\mathscr{H}_{\mathbf{L}}}$ is Zeno, stays within $V \subset U$, and its Zeno time satisfies $\tau_\infty - \tau_0 < \epsilon_t$.

We now prove part (ii) of the theorem in case where $\ddot{h}(q^*, \dot{q}^*) > 0$. First, choose $0 < a_0 < \ddot{h}(q^*, \dot{q}^*)$. Next, choose an open neighborhood $\bar{U}$ of $x^*$ such that $\ddot{h}(q, \dot{q}) > a_0$ for any $(q, \dot{q}) \in \bar{U}$, and define

$$h_{max} = \max\{h(q)|(q, \dot{q}) \in \bar{U}\}.$$

Choose any initial condition $(q_0, \dot{q}_0) \in \bar{U} \cap D_{\mathbf{L}}$ such that $dh(q_0)\dot{q}_0 > 0$, and assume that the corresponding execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ satisfies $c_0(t) \in \bar{U}$ for all $t \in I_0$. Then by construction, $h(q_0(t))$ satisfies $\ddot{h}(q_0(t), \dot{q}_0(t)) > a_0$ for all $t \in I_0$, and its initial conditions are $h(q_0(\tau_0)) \geq 0$ and $\dot{h}(q_0(\tau_0), \dot{q}_0(\tau_0)) > 0$. It is easily seen that there exists a time $t' \geq \tau_0 + \sqrt{2a_0 h_{max}}$ such that $h(q_0(t')) > h_{max}$, and thus $(q_i(t'), \dot{q}_i(t') \notin \bar{U}$. Therefore, the execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ cannot be bounded within $\bar{U}$ by setting the initial conditions arbitrarily close to $x^*$, in contradiction with the assumption and with the definition of stability. ■

## V. Simulation Results

In this section, we present numerical simulations of the first example considered at the beginning of this paper.

***Example 3 (Ball):*** Continuing with Example 1, by direct computation the condition for stability of a Zeno equilibrium point $(q, \dot{q})$ in this system as given in Theorem 2 is:

$$\ddot{h}(q, \ddot{q}) = v_x^2 \sin(x) - g < 0$$

where we denote $\dot{q} = (v_x, v_y)$. This indicates that Zeno equilibrium points that satisfy $\sin(x) < 0$ (i.e. near the minima) are more likely to attract Zeno executions. Moreover, setting the horizontal velocity $v_x$ sufficiently small increases the chances of exhibiting Zeno convergence even at points such that $\sin(x) > 0$ (i.e. near the maxima). For the sake of simplicity, we take $m = 1$, $g = 1$ and $e = 0.5$.

We simulate this system under two different sets of initial conditions, where in both cases the initial conditions at $t = 0$ are chosen such that at $t_1 = 0.05$, a first collision occurs at $x(t_1) = 0.3$, $y(t_1) = \sin(0.3)$. In the first case, the initial velocities are chosen as $v_x(0) = 1.8$ and $v_y(0) = 0$. The execution was simulated until a collision time $\tau_k$ at which the collision velocity $dh(q(\tau_k))\dot{q}(\tau_k)$ is less than $10^{-10}$. Figures 3(a)-(f) show the simulation results of this running example. Figures 3(a),(b),(c),(d),(e) show the time plots of $x(t), y(t), v_x(t), v_y(t)$ and $h(q(t))$, respectively. The points of collision events are marked with squares ("■"). Figure 3(f) plots $x(t)$ vs. $y(t)$, with the constraint surface $y = \sin(x)$ appearing as a dashed curve. This simulation results in a Zeno execution that converges at a Zeno time $t_\infty = 3.761$ to the Zeno equilibrium point $q^* = (1.337, 0.973)$ and $\dot{q}^* = (-0.121, -0.028)$. This Zeno point is close to a maximum point of the surface; note that the horizontal velocity $v_x$ is significantly decreased from its initial value, so that $\ddot{h}(q^*, \ddot{q}^*) = -0.986 < 0$ and the stability condition is satisfied. Note, too, that the motion of $h(q(t))$ in the vicinity of the Zeno point is remarkably similar to that of a simple bouncing ball (cf. Figure 3(e)).

In the second case, the initial velocities are chosen as $v_x(0) = 2.5$ and $v_y(0) = 0$. Figures 4(a)-(f) show the simulation results under these initial conditions. This simulation results in a Zeno execution that converges at a Zeno time $t_\infty = 5.0731$ to the Zeno equilibrium point $q^* = (5.114, -0.920)$ and $\dot{q}^* = (2.023, 0.791)$. One can see that the trajectory is initially "repelled" from the maximum point due to the large horizontal velocity, and attracted towards the next minimum point, while the horizontal velocity is *increased*, such that $\ddot{h}(q^*, \ddot{q}^*) = -4.766$ satisfies the stability condition in Theorem 2.

***Example 4 (Double Pendulum):*** In the second running example (Example 2) consisting of a double pendulum with a mechanical stop, the condition for stability of Zeno equilibria given in Theorem 2 is

$$\ddot{h}(q, \dot{q}) = \frac{g \sin \theta_1}{\tilde{L}} < 0, \text{ where } \tilde{L} = \frac{(4m_1+3m_2)L_1L_2}{3(m_1(L_1+2L_2)m_2L_2)}.$$

This indicates that only points at which $\sin \theta_1 < 0$ (i.e. the link $L_1$ is inclined to the left) can be stable Zeno equilibria. Simulation results of this system, which are not shown here due to space limitations, are quite similar to those of the ball example. The reader is referred to [16] for simulation results of the *completed double-pendulum system* (i.e. executions are also carried *beyond* the Zeno points).

## VI. Conclusion

In this paper we analyzed the stability of Zeno equilibria of simple Lagrangian hybrid systems, deriving sufficient conditions for stability and for instability of such equilibria. The stability conditions presented are analogous to determining the local stability of equilibrium points of a nonlinear continuous system by computing the eigenvalues of the linearization. This paper provides *almost necessary and sufficient conditions* for stability of Zeno equilibria, where the exceptional intermediate case of $\ddot{h}(q^*, \dot{q}^*) = 0$ is analogous to the case where the linearization of a continuous system has eigenvalues on the imaginary axis, and stability cannot be determined via linearization. This analogy motivates future investigation of techniques for *global* stability analysis of Zeno equilibria, where a promising direction is the use of Lyapunov-like functions as was already done in the analysis of isolated Zeno equilibrium points [10].

The fact that Zeno behavior is fundamentally a modeling phenomena indicates that the conditions used to detect Zeno behavior can be used to "complete" the hybrid system model. That is, carry an execution past the Zeno point by switching to a holonomically constrained dynamical system. This has been studied to a limited degree in [4], but the result presented in this paper can be used to complete hybrid systems in a formal manner. This is the subject of the companion paper [16].

Finally, the paper analyzes stability only for *simple* Lagrangian hybrid systems, i.e. systems with a single domain and a single guard. The extension to mechanical systems with multiple unilateral constraints is still a challenging open problem, although preliminary results for stability of a specific two-constraint mechanical system were obtained in [17]. This extension, along with the completion process described above, will enable the analysis of complex mechanical and robotic systems with intermittent contacts, such as bipedal walkers with knees (e.g. [18] and [14]), under a unified framework of Lagrangian hybrid systems.

## References


[1] A. D. Ames, "A categorical theory of hybrid systems," Ph.D. dissertation, University of California, Berkeley, 2006.

[2] A. D. Ames, A. Abate, and S. Sastry, "Sufficient conditions for the existence of Zeno behavior," ser. 44th IEEE Conference on Decision and Control and European Control Conference ECC, 2005.

[3] A. D. Ames and S. Sastry, "Routhian reduction of hybrid lagrangians and lagrangian hybrid systems," in *American Control Conference*, 2006.

[4] A. D. Ames, H. Zheng, R. D. Gregg, and S. Sastry, "Is there life after Zeno? Taking executions past the breaking (Zeno) point," in *25th American Control Conference*, Minneapolis, MN, 2006.

[5] M. Athans and P. L. Falb, *Optimal Control: An Introduction to the Theory and Its Applications*. McGraw-Hill, 1966.

[6] B. Brogliato, *Nonsmooth Mechanics*. Springer-Verlag, 1999.

[7] M. K. Camlibel and J. M. Schumacher, "On the Zeno behavior of linear complementarity systems," in *40th IEEE Conference on Decision and Control*, 2001.

[8] A. Chatterjee and A. Ruina, "A new algebraic rigid body collision law based on impulse space considerations," *Journal of Applied Mechanics*, vol. 65, no. 4, pp. 939–951, 1998.

[9] M. Heymann, F. Lin, G. Meyer, and S. Resmerita, "Analysis of Zeno behaviors in a class of hybrid systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 376–384, 2005.

[10] A. Lamperski and A. D. Ames, "Lyapunov-like conditions for the existence of Zeno behavior in hybrid and Lagrangian hybrid systems," in *IEEE Conference on Decision and Control*, 2007.

[11] ——, "Sufficient conditions for Zeno behavior in lagrangian hybrid systems," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science. Springer-Verlag, 2008.

[12] D. Liberzon and M. Margaliot, "Lie-algebraic stability conditions for nonlinear switched systems and differential inclusions," *Systems and Control Letters*, vol. 55, no. 1, pp. 8–16, 2006.

[13] J. Lygeros, K. H. Johansson, S. Simic, J. Zhang, and S. Sastry, "Dynamical properties of hybrid automata," *IEEE Transactions on Automatic Control*, vol. 48, pp. 2– 17, 2003.

[14] T. McGeer, "Passive walking with knees," in *IEEE International Conference on Robotics and Automation*, 1990.

[15] R. M. Murray, Z. Li, and S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1993.

[16] Y. Or and A. D. Ames, "A formal approach to completing Lagrangian hybrid system models," 2008, submitted to CDC08.

[17] Y. Or and E. Rimon, "On the hybrid dynamics of planar mechanisms supported by frictional contacts. II: Stability of two-contact rigid body postures," in *IEEE International Conference on Robotics and Automation*, 2008, to appear.

[18] J. Pratt and G. A. Pratt, "Exploiting natural dynamics in the control of a planar bipedal walking robot," in *Proceedings of the 36th Annual Allerton Conferenceon Communications, Control and Computing*, Monticello, IL, 1998.

[19] J. Shen and J.-S. Pang, "Linear complementarity systems: Zeno states," *SIAM Journal on Control and Optimization*, vol. 44, no. 3, pp. 1040–1066, 2005.

[20] J. Zhang, K. H. Johansson, J. Lygeros, and S. Sastry, "Zeno hybrid systems," *Int. J. Robust and Nonlinear Control*, vol. 11, no. 2, pp. 435–451, 2001.