CODING SUBSET SHIFT BY SUBGROUP CONJUGACY

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ABSTRACT

We present a Borel reduction from a subset shift equivalence relation of a countable group to a subgroup conjugacy relation of a free product. The technique gives a much shorter proof of an earlier result of Thomas and Velickovic.

For Borel equivalence relations E and F on standard Borel spaces X and Y, respectively, we say that E is *Borel reducible to* F, denoted by $E \leqslant_B F$, if there is a Borel function $f : X \to Y$ such that

$$x \, E \, y \iff f(x) \, F \, f(y) \quad \text{for } x, y \in X.$$

Intuitively, if E is Borel reducible to F, then the equivalence relation E (or some classification problem giving rise to E) can be thought to be no more complicated than F.

Countable Borel equivalence relations are Borel equivalence relations all of whose classes are countable. It is well known that there is a *universal* countable Borel equivalence relation, that is, there is a relation E_∞ such that for any countable Borel equivalence relation E, $E \leqslant_B E_\infty$ holds [**1**]. One realization of the universal countable equivalence relation E_∞ is the orbit equivalence relation of the shift action of F_n on the space of its subsets, where $n > 1$ and F_n is the free group with n generators.

In [**2**], Thomas and Velickovic proved, among other things, two other realizations. One is the isomorphism relation of finitely generated groups, which explains why it is essentially impossible to have a satisfactory classification theory for finitely generated groups. The other is the orbit equivalence relation of the conjugacy action of F_2 on the space of its subgroups, the proof of which is a continuation of the method they used to prove the first result. Their method of proof is to eventually code the subset shift equivalence relation of F_2 by the subgroup conjugacy relation of F_2. In the following we give a more general coding technique, which is also much shorter than Thomas and Velickovic's proof.

For an arbitrary group G, let $E_s(G)$ be the orbit equivalence relation of the shift action of G on the space of its subsets. Notice that the space of subsets of G can be identified with the space 2^G with the product topology, and the shift action of G on 2^G is Borel. Let $E_c(G)$ denote the orbit equivalence relation of the conjugacy action of G on the space of its subgroups. We denote the space of subgroups of G by $\mathrm{Sg}(G)$. Then $\mathrm{Sg}(G)$ is a Borel subset of 2^G, and hence can be given a Polish topology. We shall prove the following theorem.

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THEOREM 1. *Let G be a countable group, and let H be a nontrivial cyclic group. Then $E_s(G) \leqslant_B E_c(G * H)$.*

The above theorem gives immediately the following sufficient condition for $E_c(G)$ to be universal countable.

THEOREM 2. *If $G = K * H$ where K has a nonabelian free subgroup and H is nontrivial cyclic, then $E_c(G)$ is a universal countable Borel equivalence relation. In particular, $E_c(F_n)$ is universal countable for $n \geqslant 3$.*

To see that this implies the universality of subgroup conjugacy of F_2, note as in [**2**] that F_3 can be embedded as a malnormal subgroup of F_2. (Recall that a subgroup H of a group G is *malnormal* if $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H$. In this case, if A, B are subgroups of H, then A and B are conjugate in H if and only if they are conjugate in G. Thus $E_c(H) \leqslant_B E_c(G)$.)

Now we turn to a proof of Theorem 1. Let G be a nontrivial countable group, and let H be a nontrivial cyclic group. If G is finite, then there are only finitely many equivalence classes for $E_s(G)$; therefore to establish $E_s(G) \leqslant_B E_c(G * H)$, it suffices to show that $E_c(G * H)$ has infinitely many equivalence classes.

LEMMA 1. *There are infinitely many equivalence classes for $E_c(G * H)$.*

Proof. It suffices to find infinitely many pairwise nonconjugate subgroups of $G * H$. Let $g \in G$, $h \in H$ be nonidentity elements, and for each $n > 0$, let K_n be the subgroup of $G * H$ generated by $(gh)^n$. Since the elements $(gh)^n$ for $n > 0$ are in cyclically reduced form, it follows that if $n \neq m$, then $(gh)^n$ is not conjugate to either $(gh)^m$ or $(gh)^{-m}$. Consequently, if $0 < n < m$, then K_n and K_m are nonconjugate.

From now on, assume that G is infinite. We shall first consider the shift action of G on the invariant set $\Omega = \{A \subseteq G \mid A \text{ is infinite}\}$. Ω can be viewed as an invariant Borel (in fact, G_δ) subset of 2^G. Moreover, there are only countably many $E_s(G)$ equivalence classes in the complement of Ω. The following is our key lemma.

LEMMA 2. $E_s(G) \restriction \Omega \leqslant_B E_c(G * H)$.

Proof. Let $h \in H$ be a generator of H. We define a map $K : \Omega \to \mathrm{Sg}(G * H)$ by letting

$$K(A) = \langle xhx^{-1} \ : \ x \in A \rangle,$$

the subgroup of $G * H$ generated by the set $\{xhx^{-1} \mid x \in A\}$. K is obviously a Borel map. It remains only to verify that it is a reduction.

Suppose that $A, B \in \Omega$, and that there exists $g \in G$ such that $A = gB$. Then

$$\begin{aligned} K(A) &= \langle xhx^{-1} \ : \ x \in A \rangle \\ &= \langle (gy)h(gy)^{-1} \ : \ y \in B \rangle \\ &= \langle g(yhy^{-1})g^{-1} \ : \ y \in B \rangle \\ &= g\,K(B)\,g^{-1}. \end{aligned}$$

So $A\,E_s(G)\,B$ implies $K(A)\,E_c(G * H)\,K(B)$.

Suppose now that $A, B \in \Omega$, and that $\gamma \in G * H$ is such that $K(A) = \gamma K(B) \gamma^{-1}$. For each $x \in A$, let $w_x \in K(B)$ be the element such that

$$xhx^{-1} = \gamma w_x \gamma^{-1}. \tag{1}$$

We fix a map $x \mapsto w_x$ from A into $K(B)$. This map must be an injection.

Let us make some observations and conventions concerning the terms appearing in equation (1). First note that the sequence x, h, x^{-1} is a reduced sequence in $G * H$. On the right-hand side, we assume that:

(a) γ and γ^{-1} are initially expressed by reduced sequences in $G * H$;

(b) $w_x \in K(B)$ is initially expressed by a sequence $y_1, y_2, \ldots, y_k$, where for each $1 \leqslant i \leqslant k$, we have $y_i = z_i h z_i^{-1}$ or $z_i h^{-1} z_i^{-1}$, for some $z_i \in B$.

Note that the reduced sequence in $G * H$ corresponding to w_x actually has the form

$$u_1,\ h^{i_1},\ u_2,\ h^{i_2},\ \ldots,\ u_m,\ h^{i_m},\ u_{m+1},$$

where for $j = 1, \ldots, k+1$, we have $i_j \in \mathbb{Z} \setminus \{0\}$, $u_j \in G$ and the product $u_1 \cdots u_j \in B$. Moreover, the reduced sequence for w_x for any $x \in A$ is never trivial.

Now equation (1) implies that starting from the assumed initial expressions, there is a step-by-step cancellation procedure which can be performed on the right-hand side of (1) and which eventually leads to the reduced expression on the left-hand side. Since in any such procedure the length of the sequence on the right-hand side is monotonically decreasing, we conclude that some occurrence of h on the right-hand side is preserved throughout the cancellation procedure, to give rise to the occurrence of h on the left-hand side. This occurrence of h is called the *preserved occurrence* of h in the cancellation procedure. We let $\Delta \subseteq \Omega$ be the set of elements $x \in A$ for which the preserved occurrence of h in some cancellation procedure is in the original expression for w_x.

We claim that $A \setminus \Delta$ is finite. In fact, if $x \in A \setminus \Delta$, then in some cancellation procedure, the preserved occurrence of h is from γ or γ^{-1}. We shall show that there is at most one $x \in A \setminus \Delta$ such that this happens for γ, and similar arguments work for γ^{-1}. Therefore $A \setminus \Delta$ has at most two elements. Suppose that $x_1, x_2 \in A \setminus \Delta$ and that x_1, x_2 both have the property mentioned above. Then in each cancellation procedure involved, the preserved occurrence of h must be the first one in γ, because γ is assumed to be expressed by a reduced sequence. Then we can write $\gamma = khu$ for $k \in G$ and $u \in G * H$. Comparing both sides of the equations

$$x_1 h x_1^{-1} = khuw_{x_1}\gamma^{-1},$$
$$x_2 h x_2^{-1} = khuw_{x_2}\gamma^{-1},$$

we must have $x_1 = k = x_2$.

The preceding claim implies that Δ is nonempty (in fact, it is infinite). Now fix some element $x_0 \in \Delta$. By our assumption, we can write

$$\begin{aligned} x_0 h x_0^{-1} &= \gamma w_{x_0} \gamma^{-1} \\ &= \gamma u_{x_0} z_{x_0} h z_{x_0}^{-1} v_{x_0} \gamma^{-1}, \end{aligned}$$

where $z_{x_0} \in B$, $u_{x_0}, v_{x_0} \in K(B)$, and the displayed occurrence of h on the last line is the preserved occurrence of h in some cancellation procedure. This implies that

$$x_0 = \gamma u_{x_0} z_{x_0},$$
$$x_0^{-1} = z_{x_0}^{-1} v_{x_0} \gamma^{-1}.$$

Therefore $v_{x_0} = u_{x_0}^{-1}$. Let $g = x_0 z_{x_0}^{-1} \in G$. Then we have $\gamma = g u_{x_0}^{-1}$ and $x_0 = g z_{x_0}$.

We claim that $A = gB$. First we check that $A \subseteq gB$. Let $x \in A$. If $x \in \Delta$, then

$$\begin{aligned} xhx^{-1} &= \gamma w_x \gamma^{-1} \\ &= \gamma u_x z_x h z_x^{-1} v_x \gamma^{-1} \end{aligned}$$

for some $z_x \in B$, $u_x, v_x \in K(B)$, and the displayed occurrence of h in the last expression is the preserved occurrence in some cancellation procedure. But then $\gamma u_x = g u_{x_0}^{-1} u_x \in G$. This implies that $u_{x_0}^{-1} u_x = 1$, since it is easy to see that $K(B) \cap G = \{1\}$. Thus $u_{x_0} = u_x$, and then $x = \gamma u_x z_x = g z_x$. Suppose that $x \in A \setminus \Delta$, and that the preserved occurrence of h in any cancellation procedure is in γ. Then we have

$$\begin{aligned} xhx^{-1} &= \gamma w_x \gamma^{-1} \\ &= g u_{x_0}^{-1} w_x u_{x_0} g^{-1}. \end{aligned}$$

Recall, in this case, that the preserved occurrence of h must be its first occurrence in γ. Hence the sequence representing $u_{x_0}^{-1}$ must begin with zhz^{-1} for some $z \in B$, and

$$xhx^{-1} = g(zhz^{-1})\cdots,$$

where the displayed occurrence of h is the preserved occurrence. Comparing both sides of the equation, we have that $x = gz$. The argument for the γ^{-1} case is similar.

Finally, we verify that $gB \subseteq A$. To see this, let $z \in B$, and notice that

$$(gz)h(gz)^{-1} = \gamma u_{x_0} z h z^{-1} u_{x_0}^{-1} \gamma^{-1}.$$

Since $u_{x_0} z h z^{-1} u_{x_0}^{-1} \in K(B)$ and $K(A) = \gamma K(B) \gamma^{-1}$, we have that $(gz)h(gz)^{-1} \in K(A)$. But then we must have $gz \in A$.

Therefore, by the preceding two paragraphs, we have $A = gB$. This completes the proof of Lemma 2.

We are now ready to prove Theorem 1. In the following proof, we keep the same notation as used in the earlier part of this paper.

Proof of Theorem 1. Let G be a countable group, and let H be a nontrivial cyclic group. If G is finite, then Lemma 1 establishes that $E_s(G) \leqslant_B E_c(G * H)$. If G is infinite, then Ω is uncountable. Since $E_s(G)$ is a countable Borel equivalence relation, it follows that there are uncountably many $E_s(G)$ equivalence classes in Ω. Let Γ be an arbitrary invariant Borel subset of Ω which contains countably infinitely many $E_s(G)$ equivalence classes. Let Λ be the smallest $E_c(G * H)$ invariant subset of $\mathrm{Sg}(G*H)$ which contains the image of Γ under the Borel reduction $K : \Omega \to \mathrm{Sg}(G*H)$. Then, by a standard argument, Λ is Borel and also contains countably infinitely many equivalence classes. Now we denote by Γ' the union of Γ with the complement of Ω, thus there are countably infinitely many $E_s(G)$ equivalence classes in Γ'. It is then easy to define a Borel reduction K' to witness $E_s(G) \restriction \Gamma' \leqslant_B E_c(G * H) \restriction \Lambda$. Then K', together with $K \restriction \Omega \setminus \Gamma$, gives a Borel reduction to witness $E_s(G) \leqslant_B E_c(G * H)$.

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References

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