Proceedings of the 1999 IEEE
International Conference on Robotics & Automation
Detroit, Michigan • May 1999

# Propulsion and Control of Deformable Bodies in an Ideal Fluid

Richard Mason and Joel Burdick
Dept. of Mechanical Engineering, California Institute of Technology
Mail Code 104-44, Pasadena, CA 91125

**Abstract:** *Motivated by considerations of shape changing propulsion of underwater robotic vehicles, this paper analyses the mechanics of deformable bodies operating in an ideal fluid. The application of methods from geometric mechanics results in a compact and insightful formulation of the problem. We develop an explicit formula for the fluid mechanical connection, in terms of the fluid potential function, for this class of systems. The connection can be used to analyze many issues in motion planning and control. The theory is illustrated by application to an amoeba-like device.*

## 1 Introduction

This paper considers the problem of self-propulsion of deformable bodies in an idealized inviscid and irrotational fluid. We are primarily motivated by an interest in robotic underwater vehicles that propel and steer themselves by changes in shape [1, 2]. Aquatic animals propel themselves using a variety of fluid dynamic effects [3]. The analysis presented in this paper is most suited to studying systems whose propulsive movements are analogous to the movements of amoeba. That is, our "robot amoeba" would swim through a fluid by using deformations of its surface. We want to know how a swimmer of this kind should move in order to propel itself, and how effective this kind of swimming would be.

Within the fluid mechanics community, there is a significant prior literature related to this topic, which has largely been motivated by the study of deformable bubbles in a fluid [4, 5, 6, 7]. The study of the propulsive movements of aquatic organisms by biologists and fluid mechanicians also has a long history. For example, see [3, 8] and references therein. There has been considerable prior effort to design and control propeller driven underwater vehicles. However, only recently have serious efforts emerged to study, build, and control underwater vehicles that move and steer by changes of shape, and not by propellers [1, 2]. A key issue that has not been addressed in all of these prior works is a rigorous foundation for the design of trajectory generation and feedback control laws that can select patterns of body deformations that produce accurate motion. This paper addresses this problem for a limited class of swimmers.

This paper has the following goals and contributions. First, we take a fresh approach to the fluid mechanics of this problem, employing recently developed ideas of symmetry and Lagrangian reduction to develop a new and compact formulation of the relevant equations of motion. In particular, we show that the relationship between shape deformation and body velocity can be described by a *connection*, for which we give an exact formula in terms of the fluid potential function. The connection allows one to answer questions relevant to control and motion planning. The spirit of this work is close in nature to that of Shapere and Wilczek [9], who studied some analogous problems for the case of Stokes flow. Next we develop a novel expression for the net body displacement that results from one cycle of shape deformation. A direct result of this analysis is a formula that measures the "effectiveness" of a given propulsor. Finally, we show how to use our formulation to develop trajectory planning schemes for such systems.

This work fits nicely into a series of recent efforts to understand robot locomotion using methods of geometric mechanics and geometric nonlinear control theory. In particular, prior efforts [10, 11, 12] have shown that principal fiber bundles and their associated connections, as well as methods of Lagrangian reduction, are powerful ideas for unifying the study of locomotory systems. This paper gives further evidence that these ideas are useful for fluid locomotion as well.

As an illustrative example we consider the motion of a planar robot "amoeba" with three modes of deformation. Chen et al. [13] have recently constructed an actual mechanical amoeba which moves very much like the example system studied here. Our analysis indicates that macroscopic amoebae of this type will be relatively slow swimmers. However, we are interested in this system more as a test of the theory than as a practical application. More importantly, the results presented in Section 4 are quite general, and do not depend upon the amoeba-like shape of the example.

0-7803-5180-0-5/99 $10.00 

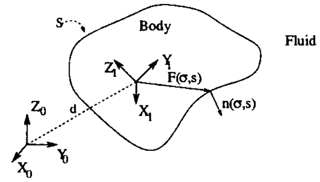

Figure 1: Schematic of deformable swimming body

## 2 Background

Real amoebae are microscopically small. They operate at a very low Reynolds number and the relevant fluid equations are those of creeping flow [9]. However, if we were to build a macroscopic "robot amoeba" and expect it to swim through water, the Reynolds number of its ambient flow would be much higher and inertial forces will dominate instead of viscous ones. We thus make the reasonable idealization that the amoeba is a connected deformable body swimming through an inviscid and incompressible fluid. We also assume that the fluid is irrotational everywhere, and that the amoeba cannot generate vorticity in the fluid. Unless the amoeba grows sharp fins, this too is a reasonable assumption.

Any "amoeba" robot which we might actually construct would have a finite number of actuators. Indeed, we would like to use as few actuators as possible. Therefore, rather than allow the boundary of the amoeba to be infinitely variable, we assume that its shape can be described by a finite number, $n_s$, of *shape* variables, $s$. The space of all possible shapes, denoted by $\mathcal{S}$, is a finite-dimensional manifold.

We fix a frame, $\mathcal{F}_B$, to the body of the swimmer and let $\mathcal{F}_W$ denote a fixed reference frame. (See Figure 1.) The location of the $\mathcal{F}_B$ is given by $g(t) \in \mathrm{SE}(d)$, $d = 2, 3$. In coordinates, elements of SE($d$) can be represented by homogeneous matrices, $g$, whose form is given in Eq. (1). The matrix $R \in \mathrm{SO}(d)$ describes the orientation of $\mathcal{F}_B$ with respect to $\mathcal{F}_W$, while $\vec{p} \in \mathbb{R}^d$ is the position of $\mathcal{F}_B$'s origin. The velocity of the moving reference frame, as seen by an observer in $\mathcal{F}_B$, is $g^{-1}\dot{g}$:

$$g = \begin{pmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{pmatrix} \qquad g^{-1}\dot{g} = \begin{pmatrix} \hat{\omega} & \vec{\xi} \\ \vec{0}^T & 0 \end{pmatrix} \tag{1}$$

where $\hat{\omega}$ is a $d \times d$ skew symmetric matrix and $\vec{\xi} \in \mathbb{R}^d$. The quantity $g^{-1}\dot{g}$ is an element of the Lie algebra of SE($d$), $\mathfrak{se}(d)$. We shall denote by "$\vee$" the identification of $\mathfrak{se}(d)$ with $\mathbb{R}^{\frac{d(d+1)}{2}}$: $(g^{-1}\dot{g})^\vee = [\xi^T \ \omega^T]^T$, where $\xi$ and $\omega$ are the linear and angular body velocities.

The swimmer's smooth surface, $\Sigma$, is parameterized by a coordinate chart that is a function of $(d-1)$ parameters $\sigma_1, \cdots, \sigma_{(d-1)}$, or by an atlas of $(d-1)$-dimensional charts. The surface parameters themselves are functions of the shape variables, $s_1, \cdots, s_{n_s}$.

Given the assumptions described above, the fluid motion around the swimmer is described by potential flow, and its domain, $\mathcal{D}$, is assumed to be unbounded. In the most general case, the ambient fluid undergoes non-uniform motion. We will assume that the ambient flow is quiescent. The Kirchoff principle for potential flow around a rigid body [14] can be extended to show that the general fluid potential, $\phi$, for the fluid surrounding a deformable body will take the form:

$$\phi = \sum_{i=1}^{3}(\xi_i\phi_i^g + \omega_i\phi_{i+3}^g) + \sum_{j=1}^{n_s}\phi_j^s\dot{s}_j. \tag{2}$$

The terms $\{\phi_i^g\}$ are the standard Kirchoff potentials for a rigid body—in this case, for the deformable body at a fixed shape, $s$. The term $\phi^s$ terms represent the contribution to the total potential due to body deformations. Let $F(\vec{\sigma}, s)$ denote the location of a surface point with respect to a body fixed frame. The normal vector to the surface at that point is denoted $n(\vec{\sigma}, s)$. Then the instantaneous velocity, in the body frame, of the surface point parameterized by $\vec{\sigma}$ is $\vec{\xi} + \vec{\omega} \times F(\vec{\sigma}, s) + \frac{\partial}{\partial s_i}F(\vec{\sigma}, s)\dot{s}_i$. At the body surface, the fluid velocity and surface velocity in the normal direction must match, leading to the following boundary conditions.

$$\begin{aligned} \nabla\phi_i^g \cdot \vec{n}(\vec{\sigma}, s) &= n_i(\vec{\sigma}, s) & i = 1,2,3 \\ \nabla\phi_i^g \cdot \vec{n}(\vec{\sigma}, s) &= (F(\vec{\sigma}, s) \times \vec{n}(\vec{\sigma}, s))_i & i = 4,5,6 \\ -\nabla\phi_i^s \cdot n(\vec{\sigma}, s) &= \frac{\partial}{\partial s_i}F(\vec{\sigma}, s) \cdot n(\vec{\sigma}, s) \ \forall s_i & \end{aligned} \tag{3}$$

These form separate Neumann problems for the Laplace equation ($\nabla^2\phi = 0$) for each term $\phi_i^g$ or $\phi_i^s$. A unique solution (up to a constant) exists for each term [15].

The total kinetic energy of the constant density fluid and deformable body system is:

$$\begin{aligned} T_{total} &= T_{body} + T_{fluid} = \tfrac{1}{2}\dot{q}^T\Lambda(q)\dot{q} + \frac{\rho_0}{2}\int_{\mathcal{D}} \|\nabla\phi\|^2 \, dV \\ &= \tfrac{1}{2}(\dot{g}^T \ \dot{s}^T)\begin{pmatrix} \Lambda^{gg}(q) & \Lambda^{gs}(q) \\ (\Lambda^{gs})^T(q) & \Lambda^{ss}(q) \end{pmatrix}\begin{pmatrix} \dot{g} \\ \dot{s} \end{pmatrix} - \frac{\rho_0}{2}\int_{\Sigma} \phi(\nabla\phi \cdot n) \, d\Sigma \end{aligned} \tag{4}$$

where $\rho_0$ is the fluid density, and $\Lambda(q)$ is the kinetic energy metric (or mass matrix) of the deformable body (in the absence of surrounding fluid). Because the potentials take the form of Equation (2) the total kinetic energy can be put in the form

$$T_{total} = \tfrac{1}{2}\dot{q}^T M(q)\dot{q}.$$

For the moment we will ignore any additional potential forces acting on the deformable body, and hence the system's Lagrangian is equivalent to $T_{total}$. One could next derive the governing mechanics from the Euler-Lagrange equations. Instead we take a more abstract approach.

# 3 Ideas from Geometric Mechanics

New insight can be obtained by applying methods of geometric mechanics to the system described in the previous section. In particular we wish to find symmetries that lead to reduction. This section provides a brief summary of relevant ideas. More extensive background can be found in [16, 17]. In the interest of brevity, certain technicalities are omitted.

## 3.1 Principal Fiber Bundles

Let $Q$ denote the configuration space of the deformable swimmer, which consists of its position, $g \in \mathrm{SE}(d)$, and its shape, $s \in \mathcal{S}$. Hence, the configuration space is $\mathrm{SE}(d) \times M$, and a configuration $q \in Q$ can be given local coordinates $q = (g, s)$. This configuration space has a surprisingly rich structure due to the fact that $\mathrm{SE}(d)$ is a Lie group. Recall that every Lie group, $G$, has an associated Lie algebra, denoted $\mathfrak{g}$. In our context, the Lie algebra of $\mathrm{SE}(d)$, denoted by $\mathfrak{se}(d)$, consists of the velocities of $\mathcal{F}_B$ relative to $\mathcal{F}_W$, as seen by a body fixed observer. Elements of $\mathfrak{se}(d)$ can be represented by matrices of the form $g^{-1}\dot{g}$, as seen in Eq. (1).

If the swimmer's initial body fixed frame position is $h \in G$, and it is displaced by an amount $g$, then its final position is $gh$. This *left translation* can be thought of as a map $L_g : G \to G$ given by $L_g(h) = gh$. The left translation induces a *left action* of $G$ on $Q$. A left action is a smooth mapping $\Phi : G \times Q \to Q$ such that: (1) $\Phi_e(q) = q$ for all $q \in Q$; and (2) $\Phi_g(\Phi_h(q)) = \Phi_{L_g h}(q)$ for all $g, h \in G$ and $q \in Q$. Omitting some technicalities, the configuration space $Q$ endowed with such an action is a *principal fiber bundle.* $Q$ is called the *total space,* $\mathcal{S}$ the *base space* (or *shape space*), and $G$ the *structure group.* The *canonical projection* $\pi : Q \to M = Q/G$ is a differentiable projection onto the second coordinate factor: $\pi(q) = \pi(g, s) = s$. The sets $\pi^{-1}(s) \subset Q$ for $r \in \mathcal{S}$ are the *fibers*, and $Q$ is the union over $\mathcal{S}$ of its fibers. The usefulness of this structure will become more apparent in Section 3.3.

## 3.2 Symmetries

In Lagrangian mechanics, symmetries result in conservation laws. By a symmetry, we mean an invariance of the Lagrangian with respect to some operation.

**Definition 1** *The* lifted action *is the map* $T\Phi_g : T_qQ \to T_{\Phi(q)}Q : (q, v) \mapsto (\Phi_g(q), T_q\Phi_g(v))$ *for all* $g \in G$ *and* $q \in Q$. *I.e.,* $T_q\Phi_g$ *is the Jacobian of the group action. For left translation on* $G$, $T\Phi_g$ *has the coordinate form:*

$$T_q\Phi_h\dot{q} = \begin{pmatrix} T_gL_h\dot{g} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} h\dot{g} \\ \dot{s} \end{pmatrix}. \tag{5}$$

Based on the group action and lifted action, we can introduce the following notions of symmetry, or *invariance.*

**Definition 2** *A Lagrangian function,* $L\colon TQ \to \mathbb{R}$, *is said to be* $G$-invariant *if it is invariant with respect to the lifted action, i.e., if*

$$L(q, v_q) \quad = \quad L(\Phi_h(q), T_q\Phi_h v_q)$$

*for all* $h \in G$ *and all* $v_q \in T_qQ$.

## 3.3 Connections

To analyze and control propulsion of a deformable body, one would like to systematically derive an expression that answers the question: "If I wiggle the body's surface, how does the body move?" The relationship between shape and position changes can be formalized in terms of a *connection*, an intrinsic mathematical structure that is associated with a principal fiber bundle. We begin with some necessary technical definitions.

**Definition 3** *If* $\xi \in \mathfrak{g}$, *the vector field on* $Q$ *denoted by* $\xi_Q$ *and given by*

$$\xi_Q(q) = \frac{d}{dt}\Phi_{\exp(t\xi)}(q)\big|_{t=0} \tag{6}$$

*is called the* infinitesimal generator *of the action corresponding to* $\xi$. *The vertical space,* $V_qQ \subset T_qQ$, *is defined as*

$$V_qQ = ker(T_q\pi) = \{v |\ v = \xi_Q(q) \ \ \forall\ \xi \in \mathfrak{g}\}.$$

**Definition 4** *The* connection one-form, $\Gamma(q)\colon T_qQ \to \mathfrak{g}$, *is a Lie-algebra valued one-form having the following properties:*

*(i)* $\Gamma(q)$ *is linear in its action on* $T_qQ$.

*(ii)* $\Gamma(q)\xi_Q = \xi$ *for* $\xi \in \mathfrak{g}$.

*(iii)* $\Gamma(q)\dot{q}$ *is equivariant, i.e. it transforms as* $\Gamma(\Phi_h(q))T_q\Phi_h(q)\dot{q} = \mathrm{Ad}_h\,\Gamma(q)\dot{q}$.[1]

The *horizontal space* is the kernel of the connection one-form, $H_qQ = \{z \mid \Gamma(q)z = 0\}$, and is complementary to $V_qQ$. It can be shown that the connection one-form can be expressed in local coordinates $q = (g, s)$ as follows:

$$\Gamma(q)\dot{q} = Ad_g(A(s)\dot{s} + g^{-1}\dot{g}). \tag{7}$$

where $A\colon T\mathcal{S} \to \mathfrak{g}$ is termed *the "local" form of the connection.* Hence, any $\dot{q} = (\dot{g}, \dot{s})$ which lies in $H_qQ$ must satisfy a constraint of the following form:

$$g^{-1}\dot{g} = -A(s)\dot{s}. \tag{8}$$

[1]The adjoint action $\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}$ is defined as $\mathrm{Ad}_h\,\xi = T_{h^{-1}}L_h(T_eR_{h^{-1}}\xi)$ for $\xi \in \mathfrak{g}$.

Note that *the local connection plays a central role in the ensuing analysis.*

For Lagrangian systems with symmetries, such as the one studied here, the conservation laws that arise from symmetries constrain the overall system motion. This constraint can be expressed as a connection, called the *mechanical connection* [16]. It can be shown that the *mechanical connection* is given by the expression

$$\Gamma(q)v_q = I^{-1}(q)\mathrm{J}(v_q). \tag{9}$$

where the *momentum map*, $\mathrm{J} : TQ \to \mathfrak{g}^*$, satisfies:

$$\langle \mathrm{J}(v_q); \xi \rangle = \langle\langle v_q, \xi_Q \rangle\rangle,$$

for all $\xi \in \mathfrak{g}$ and $v_q \in T_qQ$. The expression $\langle\langle \cdot, \cdot \rangle\rangle$ denotes inner produce with respect to the kinetic energy metric. The *locked inertia tensor* is the map, $I(q) : \mathfrak{g} \to \mathfrak{g}^*$ which satisfies:

$$\langle I(q)\xi; \eta \rangle = \langle\langle \xi_Q, \eta_Q \rangle\rangle \text{ for all } \xi, \eta \in \mathfrak{g}$$

## 4 The Dynamics Revisited

We now revisit the mechanics of the deformable swimmer in light of the ideas presented in the last section. We first show that a Lagrangian that is invariant with respect to a group action $\Phi_h$ induces a *reduced Lagrangian.* This is a general result that is independent of the particular fluid mechanical model that is studied in this paper.

**Proposition 4.1** *[18] If $L(q, \dot{q})$ is a G-invariant Lagrangian, i.e. it satisfies Definition 2, then the* reduced Lagrangian, *$l: TQ/G \to I\!R$, can be expressed as:*

$$l(s, \dot{s}, \xi) = \begin{pmatrix} \xi^T & \dot{s}^T \end{pmatrix} \begin{pmatrix} I(s) & IA(s) \\ A^T I(s) & m(s) \end{pmatrix} \begin{pmatrix} \xi \\ \dot{s} \end{pmatrix} - V(s). \tag{10}$$

*where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$, $\dot{s} \in T_r S$, $I(s)$ is the locked inertia matrix, and $A(s)$ is the local form of the mechanical connection.*

To apply Prop. 4.1 to our particular fluid mechanical problem, we must show that the fluid mechanical Lagrangian is invariant with respect to a group action. As seen in Section 2, the Lagrangian is a function of $\phi$ and $\nabla\phi$. Hence, invariance is based on the invariance of $\phi$ and $\nabla\phi$. Note that the potential, $\phi$, is defined with respect to $\mathcal{F}_B$. Straightforward calculations based on this fact and an analysis of the boundary conditions can be used to prove the following fact.

**Proposition 4.2** *In the case of quiescent ambient flow, the potential, $\phi$, defined by Equation (2) and subject to boundary conditions (3) is* SE$(d)$*-invariant. Similarly, $\nabla\phi$ is* SE$(d)$*-invariant.*

This proposition can not in general be extended to the case of a non-quiescent ambient flow. The following is a direct consequence of Prop. 4.2.

**Proposition 4.3** *The fluid's kinetic energy, considered as the Lagrangian $T_{fluid}$: $TQ \to I\!R$ and given by*

$$T_{fluid} = \frac{\rho_0}{2}\int_{\mathcal{D}} ||\nabla\phi||^2\, dV = -\frac{\rho_0}{2}\int_{\Sigma} \phi(\nabla\phi \cdot n)\, dS$$

*is invariant with respect to a left* SE$(d)$*-action.*

As a corollary to Prop. 4.1 and Prop. 4.3, we can state that the kinetic energy of a deformable swimmer in an inviscid irrotational fluid takes the following special form.

**Proposition 4.4** *The kinetic energy of the deformable swimmer, Equation (4), assumes the reduced form:*

$$T_{total}(s, \dot{s}, \xi) = \frac{1}{2}(\xi^T, \dot{s}^T) \begin{pmatrix} I_d & I_d A_d \\ A_d^T I_d & m_d \end{pmatrix} \begin{pmatrix} \xi \\ \dot{s} \end{pmatrix} \tag{11}$$

*where $I_d$ is the $6 \times 6$ "locked added inertia" matrix, with entries:*

$$(I_d)_{ij} = \Lambda^{gg}_{ij}(s) - \frac{\rho_0}{2}\int_{\Sigma} (\phi^g_i(\nabla\phi^g_j \cdot n) + \phi^g_j(\nabla\phi^g_i \cdot n))\, dS \tag{12}$$

*where $\Lambda^{gg}(s)$ is the locked inertia matrix of the deformable body system (considered in the absence of the fluid) and the second term is the classical added fluid mass/inertia. Meanwhile $I_d A_d$ is a $(6 \times n_r)$ matrix with entries:*

$$(I_d A_d)_{ij} = \Lambda^{gs}_{ij}(s) - \frac{\rho_0}{2}\int_{\Sigma} (\phi^g_i(\nabla\phi^s_j \cdot n) + \phi^s_j(\nabla\phi^g_i \cdot n))\, dS \tag{13}$$

*Hence, the extremely useful local form of the* fluid mechanical connection, *$A_d$, can be computed as:*

$$A_d(s) = I_d^{-1}(s)(I_d A_d)(s) \tag{14}$$

When the symmetry principles are taken into account, the governing equations of motion that one derives from the Euler-Lagrange mechanical equations *reduce* to this form:

$$\begin{aligned} g^{-1}\dot{g} &= -A_d(s)\dot{s} + I_d^{-1}(s)\mu && (15) \\ \dot{\mu} &= ad^*_\xi \mu && (16) \\ M(s)\ddot{s} &= T(s)\tau - B(s, \dot{s}) - C(s) && (17) \end{aligned}$$

where $\mu$ is the system's momentum, in body coordinates. The variable $\tau$ represents the "shape forces." The first

Figure 2: The variables $s_1(t), s_2(t), s_3(t)$ correspond to 3 deformation modes. The $1^{st}$ and $2^{nd}$ modes together yield motion in the $x$-direction. The $1^{st}$ and $1^{st}$ modes together yield motion in the $y$-direction.

equation is the connection, modified to include the possibility that the swimmer starts with non-zero momentum. The second equation describes the evolution of the momentum, as seen in body coordinates. In spatial coordinates, this momentum is constant since it is a conserved quantity. We will hereafter assume that the swimmer starts with zero momentum, thereby eliminating the second equation and simplifying the first to the form of Eq. (8). The third equation is known as the "shape" dynamics and is only a function of the shape varibles. *For the purposes of control analysis and trajectory generation, we need only focus on the connection.*

## 5 Planar Amoeba Example

As an example, we consider the propulsive movements of a roughly circular device whose boundary shape is modulated by a "small" amount. Bearing in mind that a practical robot amoeba should have as few actuators as possible, we restrict the possible deformations of its boundary to a set parameterized by three variables $s_1(t), s_2(t), s_3(t)$ as follows. Fix a frame in the body of the amoeba, and let the shape of the amoeba be described in polar coordinates in the body frame by the equation (see Figure 2):

$$F(\sigma, s) = r_0[1 + \epsilon(s_1 \cos(2\sigma) + s_2 \cos(3\sigma) + s_3 \sin(3\sigma))]$$

(Note that the centroid $C$ of the deformed amoeba is not, in general, located at the origin of the body frame; we will return to this point later.) The perfectly irrotational fluid surrounding the amoeba has density $\rho$. The potential $\phi$ is determined by the surface boundary conditions, by the requirement that $u = \nabla\phi$ go to zero as $r$ approaches infinity, and by the requirement that there be no circulation around the amoeba. The surface boundary condition in polar coordinates is:

$$(\nabla\phi \cdot \vec{n})|_\Sigma = \begin{bmatrix}\dot{x}\cos\sigma + \dot{y}\sin\sigma & -\dot{x}\sin\sigma + \dot{y}\cos\sigma\end{bmatrix}^T \cdot \vec{n} + \epsilon r_0 \begin{bmatrix}\dot{s}_1\cos(2\sigma) + \dot{s}_2\cos(3\sigma) + \dot{s}_3\sin(3\sigma) & 0\end{bmatrix}^T \cdot \vec{n}$$

Solving Laplace's equation by separation of variables:

$$\phi = \phi_1\dot{x} + \phi_2\dot{y} + \phi_3\omega + \phi_1^s\dot{s}_1 + \phi_2^s\dot{s}_2 + \phi_3^s\dot{s}_3 + \mathcal{O}(\epsilon^3)$$

where, using the notation $c(\sigma) = \cos(\sigma)$, $s(\sigma) = \sin(\sigma)$,

$$\begin{aligned}\phi_1 = & -\frac{r_0^2}{r}c(\sigma) + \epsilon\left(\frac{r_0^2}{r}s_1c(\sigma) + \frac{r_0^3}{r^2}s_2c(2\sigma) - \frac{r_0^4}{r^3}s_1c(3\sigma) - \frac{r_0^5}{r^4}s_2c(4\sigma)\right. \\ & \left. + \frac{r_0^3}{r^2}s_3s(2\sigma) - \frac{r_0^5}{r^4}s_3s(4\sigma)\right) + \epsilon^2\left(-\frac{r_0^2}{2r}(3s_1^2 + 5s_2^2 + 5s_3^2)c(\sigma)\right. \\ & -2\frac{r_0^3}{r^2}s_1s_2c(2\sigma) + \frac{r_0^4}{4r^3}s_1^2c(3\sigma) + \frac{r_0^5}{r^4}s_1s_2c(4\sigma) - \frac{r_0^6}{4r^5}(5s_1^2 - 3s_2^2 + 3s_3^2)c(5\sigma) \\ & -3\frac{r_0^7}{r^6}s_1s_2c(6\sigma) - \frac{7r_0^8}{4r^7}(s_2^2 - s_3^2)c(7\sigma) - \frac{2r_0^3}{r^2}s_1s_3s(2\sigma) + \frac{r_0^5}{r^4}s_1s_3s(4\sigma) \\ & \left. + \frac{3r_0^6}{2r^5}s_2s_3s(5\sigma) - \frac{3r_0^7}{r^6}s_1s_3s(6\sigma) - \frac{7r_0^8}{2r^7}s_2s_3s(7\sigma)\right)\end{aligned}$$

$$\begin{aligned}\phi_2 = & -\frac{r_0^2}{r}s(\sigma) + \epsilon\left(\frac{r_0^3}{r^2}s_3c(2\sigma) + \frac{r_0^5}{r^4}s_3c(4\sigma) - \frac{r_0^2}{r}s_1s(\sigma) - \frac{r_0^3}{r^2}s_2s(2\sigma) - \frac{r_0^5}{r^3}s_1s(3\sigma)\right. \\ & \left. -\frac{r_0^5}{r^4}s_2s(4\sigma)\right) + \epsilon^2\left(\frac{2r_0^3}{r^2}s_1s_3c(2\sigma) + \frac{r_0^5}{r^4}s_1s_3c(4\sigma) + \frac{3r_0^6}{2r^5}s_2s_3c(5\sigma) + \frac{3r_0^7}{r^6}s_1s_3c(6\sigma)\right. \\ & + \frac{7r_0^8}{2r^7}s_2s_3c(7\sigma) - \frac{r_0^2}{2r}(3s_1^2 + 5s_2^2 + 5s_3^2)s(\sigma) - \frac{2r_0^3}{r^2}s_1s_2s(2\sigma) - \frac{r_0^4}{4r^3}s_1^2s(3\sigma) \\ & \left. -\frac{r_0^5}{r^4}s_1s_2s(4\sigma) - \frac{r_0^6}{4r^5}(5s_1^2 + 3s_2^2 - 3s_3^2)s(5\sigma) - \frac{3r_0^7}{r^6}s_1s_2s(6\sigma) - \frac{7r_0^8}{4r^7}(s_2^2 - s_3^2)s(7\sigma)\right)\end{aligned}$$

$$\begin{aligned}\phi_3 = & \ \epsilon\left(\frac{r_0^5}{r^3}s_3c(3\sigma) - \frac{r_0^4}{r^2}s_1s(2\sigma) - \frac{r_0^5}{r^3}s_2s(3\sigma)\right) \\ & + \epsilon^2\left(\frac{3r_0^3}{r}s_1s_3c(\sigma) + \frac{3r_0^7}{r^5}s_1s_3c(5\sigma) + \frac{7r_0^8}{2r^6}s_2s_3c(6\sigma) - \frac{3r_0^3}{r}s_1s_2s(\sigma)\right. \\ & \left. -\frac{5r_0^6}{4r^4}s_1^2s(4\sigma) - \frac{3r_0^7}{r^5}s_1s_2s(5\sigma) - \frac{7r_0^8}{4r^6}(s_2^2 - s_3^2)s(6\sigma)\right)\end{aligned}$$

$$\phi_1^s = \epsilon\left(-\frac{r_0^4}{2r^2}c(2\sigma)\right) + \epsilon^2\left(\frac{r_0^2}{2}s_1\ln(r) - \frac{5r_0^6}{8r^4}s_1c(4\sigma) - \frac{3r_0^7}{5r^5}s_2c(5\sigma) - \frac{3r_0^7}{5r^5}s_3s(5\sigma)\right)$$

$$\begin{aligned}\phi_2^s = & \ \epsilon\left(-\frac{r_0^5}{3r^3}c(3\sigma)\right) + \epsilon^2\left(\frac{r_0^2}{2}s_2\ln(r) - \frac{r_0^3}{r}s_1c(\sigma) - \frac{3r_0^7}{5r^5}s_1c(5\sigma)\right. \\ & \left. -\frac{7r_0^8}{12r^6}s_2c(6\sigma) - \frac{7r_0^8}{12r^6}s_3s(6\sigma)\right)\end{aligned}$$

$$\begin{aligned}\phi_3^s = & \ \epsilon\left(-\frac{r_0^5}{3r^3}s(3\sigma)\right) + \epsilon^2\left(\frac{r_0^2}{2}s_3\ln(r) + \frac{7r_0^8}{12r^6}s_3c(6\sigma) - \frac{r_0^3}{r}s_1s(\sigma)\right. \\ & \left. -\frac{3r_0^7}{5r^5}s_1s(5\sigma) - \frac{7r_0^8}{12r^6}s_2s(6\sigma)\right)\end{aligned}$$

From $\phi$ we can readily find the rightmost terms of equations (12) and (13); it remains to find $\Lambda^{gg}$ and $\Lambda^{gs}$.

To compute $\Lambda^{gg}$ and $\Lambda^{gs}$, we must make some assumptions regarding the amoeba's internal structure. For this example we assume that the amoeba is homogeneous (which implies that the center of mass is located at the centroid) and has the kinetic energy of an instantaneously rigid body of mass $M$ with the same center-of-mass velocity and angular velocity. The velocity of the centroid is given by

$$\dot{C} = \begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} + \epsilon^2 r_0 \begin{bmatrix}s_1\dot{s}_2 + \dot{s}_1 s_2 - s_1 s_3\omega \\ s_1\dot{s}_3 + \dot{s}_1 s_3 + s_1 s_2\omega\end{bmatrix} + \mathcal{O}(\epsilon^3)$$

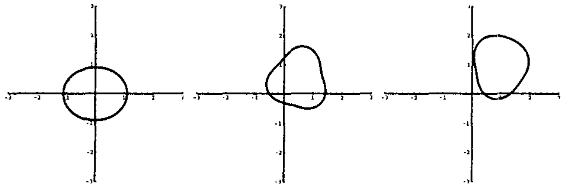

Figure 3: Computer animation of a planar amoeba whose density equals that of the surrounding fluid. There are roughly 15.3 oscillations between snapshots.

so the matrices $\Lambda^{gg}$ and $\Lambda^{gs}$ are given to order $\epsilon^2$ by

$$\Lambda^{gg} \approx M \begin{bmatrix} 1 & 0 & -\epsilon^2 r_0 s_1 s_3 \\ 0 & 1 & \epsilon^2 r_0 s_1 s_2 \\ -\epsilon^2 r_0 s_1 s_3 & \epsilon^2 r_0 s_1 s_2 & r_0^2(\frac{1}{2} + \frac{5}{4}\epsilon^2(\sum_{i=1}^3 s_i^2)) \end{bmatrix}$$

$$\Lambda^{gs} = M \begin{bmatrix} \epsilon^2 r_0 s_2 & \epsilon^2 r_0 s_1 & 0 \\ \epsilon^2 r_0 s_3 & 0 & \epsilon^2 r_0 s_1 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3)$$

Using formulas (12) and (13), to find $I_d$ and $I_d A_d$, we end with a local connection form $A_d = (I_d)^{-1}(I_d A_d)$.

$$A_d = \epsilon^2 \begin{bmatrix} r_0(1-\mu)s_2 & r_0\, s_1 & 0 \\ r_0(1-\mu)s_3 & 0 & r_0\, s_1 \\ 0 & \frac{-2\pi r_0{}^2 \rho s_3}{M} & \frac{2\pi r_0{}^2 \rho s_2}{M} \end{bmatrix} + \mathcal{O}(\epsilon^3) \tag{18}$$

where $\mu = (2\pi r_0^2 \rho)/(M + \pi r_0^2 \rho)$. After using the connection to derive the motion of the frame, we may easily derive the velocity of the centroid:

$$\dot{C} = \epsilon^2 r_0 \left( \mu \begin{bmatrix} \dot{s}_1 s_2 \\ \dot{s}_1 s_3 \end{bmatrix} + \frac{2\rho\pi r_0^3}{M} s_1 \begin{bmatrix} \dot{s}_2 s_3^2 - \dot{s}_3 s_2 s_3 \\ \dot{s}_3 s_2^2 - \dot{s}_2 s_2 s_3 \end{bmatrix} \right) + \mathcal{O}(\epsilon^3)$$

As seen above, the abstract approach espoused in this paper leads to a surprisingly succint description of the essential governing equations.

## 6 Displacement by Periodic Motion

Recall that the infinitesimal relationship between shape changes and body velocity is described by the local form of the connection:

$$\dot{g} = -gA(s)\dot{s} = -gA_i(s)\dot{s}^i, \tag{19}$$

where the index $i$ implies summation. We would like to find a solution for this equation that will aid in designing or evaluating motions that arise from shape variations. Because SE($d$) is a Lie group, the solution to Eq. (19) will generally have the form

$$g(t) = g(0)e^{z(t)}$$

where $z \in \mathfrak{se}(d)$. An expansion for the Lie algebra valued function $z(t)$ has been given by Magnus [19].

$$z = \overline{A} + \frac{1}{2}\overline{[\overline{A}, A]} + \frac{1}{3}\overline{[\overline{[\overline{A}, A]}, A]} + \frac{1}{12}\overline{[\overline{A}, \overline{[\overline{A}, A]}]} + \cdots$$

$$\overline{A}(t) \equiv \int_0^t A(\tau)\dot{s}(\tau)d\tau$$

where $[\cdot,\cdot]$ is the Jacobi-Lie bracket on $\mathfrak{g}$.

To obtain useful results, examine the group displacement resulting from a periodic path $\alpha\colon [0,T] \to M$ such that $\alpha(0) = \alpha(T)$. Taylor expand $A_i$ about $\alpha(0)$ and then judiciously regroup, simplify, apply integration by parts, and use the fact that the path is cyclic [20].

$$\begin{aligned} z(\alpha) &= -\frac{1}{2}F_{ij}(0)\int_\alpha ds^i\, ds^j \qquad (20) \\ &+ \frac{1}{3}(F_{ij,k} - [A_i, F_{jk}])(0)\int_\alpha ds^i\, ds^j\, ds^k + \cdots \end{aligned}$$

where

$$F_{ij} \equiv A_{j,i} - A_{i,j} - [A_i, A_j] \tag{21}$$

is termed the *curvature* of the connection, the notation ",j" indicates differentiation with respect to $s_j$, and the following short-hand notation is used

$$\int_\alpha ds^i ds^j ds^k \equiv \int_0^T \int_0^{t_k} \int_0^{t_j} \dot{s}^i(t_i)dt_i\; \dot{s}^j(t_j)dt_j\; \dot{s}^k(t_k)dt_k.$$

Summation over indices is implied. The connection, $A$, and its curvature, $F$, are evaluated at $\alpha(0)$ so that the coefficients of the integrals are constants. For a complete discussion of this series, see [20].

Thus, our geometric analysis shows that for small boundary deformations, the displacement of the body over one period of shape deformation is proportional to the curvature of the connection. *The curvature is an excellent measure of the effectiveness of the swimmer.* This result further bolsters the central role of the fluid mechanical connection in the analysis of deformable swimmers.

For proportionally small deformations, the displacement experienced during one deformation cycle is:

$$g_{disp} = e^{z(\alpha)} \simeq \exp\left(-\tfrac{1}{2}F_{ij}\int_\alpha ds^i\, ds^j\right) \tag{22}$$

The term $-\frac{1}{2}\sum_{i,j} F_{ij}\int_\alpha ds^i\, ds^j$ is Lie-algebra valued, and therefore it will take the form of $g^{-1}\dot{g}$ in Eq. (1). The exponential of such a matrix is:

$$\begin{bmatrix} e^{\hat{\omega}} & (I - e^{\hat{\omega}})(\hat{\omega}\xi) + (\vec{\omega}\cdot\xi)\vec{\omega} \\ \vec{0}^T & 1 \end{bmatrix} \tag{23}$$

**Example** (*continued*): Consider the case of our idealized planar amoeba. Since the curvature $F$ of the connection is a skew symmetric quantity, there are only three independent non-zero curvature terms, $F_{12}$, $F_{13}$, and $F_{23}$. Let us assume that the first deformation mode is forced periodically by input $s_1 = \cos(\Omega t)$, while the second mode is periodically forced by input $s_2 = \sin(\Omega t)$. For the chosen input forcing functions, the integral terms associated with the $F_{13}$ and $F_{23}$ terms are zero. From Eq.s (18) and (21) we see that

$$\hat{A}_1 = \epsilon^2 r_0 \frac{M - \pi r_0^2 \rho}{M + \pi r_0^2 \rho} \begin{bmatrix} 0 & 0 & s_2 \\ 0 & 0 & s_3 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3) \qquad (24)$$

$$\hat{A}_2 = \epsilon^2 r_0 \begin{bmatrix} 0 & \frac{2\pi r_0 \rho}{M} s_3 & s_1 \\ -\frac{2\pi r_0 \rho}{M} s_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3) \qquad (25)$$

$$F_{12} = \frac{2\pi r_0^3 \rho}{M + \pi r_0^2 \rho} \epsilon^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3) \qquad (26)$$

Therefore, the displacement over one period of forcing is

$$\begin{aligned} g_{\text{disp}} &\simeq \exp\left( -\tfrac{1}{2} F_{12}(0) \int_0^{\frac{2\pi}{\Omega}} \left( \int_0^t \dot{s}_1(\tau) d\tau \right) \dot{s}_2(t) dt \right. \\ &\qquad \left. -\tfrac{1}{2} F_{21}(0) \int_0^{\frac{2\pi}{\Omega}} \left( \int_0^t \dot{s}_2(\tau) d\tau \right) \dot{s}_1(t) dt \right) \\ &= \exp(-\pi F_{12}(0)) \end{aligned} \qquad (27)$$

If we discard the high-order terms in $F_{12}$ and exponentiate only the curvature proportional to $\epsilon^2$, we find

$$g_{\text{disp}} \approx \begin{bmatrix} 1 & 0 & \frac{-2\epsilon^2 \pi^2 r_0^3 \rho}{M + \pi r_0^2 \rho} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (28)$$

Thus, to $\mathcal{O}(\epsilon^2)$, each oscillation results in a displacement of $(-2\pi\epsilon^2 r_0 \frac{\pi r_0^2 \rho}{M + \pi r_0^2 \rho})$ along the $x$-axis. The simplicity with which this result can be obtained is a direct result of the geometric approach outlined in Sections 3, 4.

## 7 Rectilinear Motion Planning

The series expansion outlined in Section 6 can be used as a basis for developing motion planning algorithms, as it directly relates control inputs to net displacement. Interested readers are referred to Ref. [20] for more details. Here we take a simplified approach to our example.

Consider the two-degree-of-freedom problem where we do not wish the amoeba to rotate, but wish the centroid to follow a path in the plane. Suppose that we choose sinusoidal inputs with time-varying amplitude, as follows: $s_1(t) = \cos(\Omega t)$, $s_2(t) = -a(t)\sin(\Omega t)$, $s_3(t) = -b(t)\sin(\Omega t)$. Then

$$\dot{C} = \epsilon^2 r_0 \frac{2\rho\pi r_0^2}{\rho\pi r_0^2 + M} \begin{bmatrix} a(t)\Omega\sin^2(\Omega t) \\ b(t)\Omega\sin^2(\Omega t) \end{bmatrix} + \mathcal{O}(\epsilon^3)$$

Thus, moving the centroid of the amoeba along a given curve in the plane is remarkably easy. (By contrast, moving the body frame origin along a given curve, using this form of input, would be more complicated, since the velocity of the body frame origin depends on the derivatives $\dot{a}(t)$ and $\dot{b}(t)$ to leading order.) At any point along the curve, we make the velocity of the centroid tangent to the curve by appropriate choice of $a$ and $b$ ($b/a$ is the slope of the curve). As long as the curve is smooth, $a$ and $b$ are smooth functions of time.

This method steers the centroid along a path in the plane, but not necessarily at a desired speed at any given instant. In particular, the velocity of the centroid will periodically vanish (when $\sin^2(\Omega t)$ vanishes). Figure 3 shows snapshosts of a computer simulation of this model as it tracks a straight line with unit slope.

We must also note that the average velocity of the centroid is disappointingly low, on the order of $\epsilon^2 r_0 \Omega$ (which is the norm of the connection's curvature!). Each shape change cycle moves the amoeba a distance on the order of $\epsilon^2 r_0$. Thus if $\epsilon = 0.1$, then 100 oscillations are required to move the amoeba one body radius. Mechanically feasible amoeba will be relatively slow swimmers.

## 8 Optimal Control Analysis

We now demonstrate that the sinusoidal inputs used in the last two sections are "optimal" inputs, according to one natural measure of performance. First we restrict ourselves to motion along the $x$-axis. Therefore we have one base variable $C_x$ and two shape variables, $s_1$ and $s_2$ (we can neglect $s_3$). Assume that our control inputs are $u_1 = \dot{s}_1$ and $u_2 = \dot{s}_2$, so $\dot{C}_x = \epsilon^2 r_0 \mu u_1 s_2$. Suppose we choose a minimum-control-effort performance index:

$$J(0) = \frac{1}{2}\int_0^T u^T u \, dt = \frac{1}{2}\int_0^T (u_1^2 + u_2^2)\, dt \qquad (29)$$

Assume that $C_x(0) = 0$ and for simplicity $s_2(0) = 0$. At time $T$ we require a final state $s_1(T) = s_1(0)$, $s_2(T) = s_2(0)$, and $C_x(0) = d$. The Hamiltonian and costate equations for this optimal control problem are

$$H = \frac{1}{2}(u_1^2 + u_2^2) + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 \epsilon^2 r_0 \mu u_1 s_2$$

$$-(\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3) = (0, \lambda_3 \epsilon^2 r_0 \mu u_1, 0)$$

From this we see that $\lambda_1$ and $\lambda_3$ are constants, while

$$\lambda_2(t) = \lambda_2(T) + \lambda_3 \epsilon^2 r_0 \mu \int_t^T u_1(t')\, dt'$$

The stationarity condition is

$$\begin{bmatrix} u_1 + \lambda_1 + \lambda_3 \epsilon^2 r_0 \mu s_2 \\ u_2 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A solution to these equations is

$$\begin{aligned} s_1 &= a\cos(N\pi t/T) \\ s_2 &= a\sin(N\pi t/T) \\ C_x &= -\epsilon^2 r_0 \mu a^2 \frac{N\pi}{T}[\frac{t}{2} - \frac{\sin((2N\pi/T)t)}{(4N\pi/T)}] \end{aligned}$$

where $a = s_1(0)$ and $N = -\frac{2d}{\epsilon^2 a^2 r_0 \mu \pi}$.

To summarize, optimal inputs for $s_1(t)$ and $s_2(t)$ are sinusoidal functions of time, 90 degrees out of phase. "Optimal" inputs are those which cause the amoeba to swim a given distance along the $x$-axis in a given time, while minimizing control effort as defined in (29). By symmetry, sinusoids 90 degrees out of phase in $s_1(t)$ and $s_3(t)$ will cause optimal motion along the $y$-axis. Further, by making $s_2(t)$ and $s_3(t)$ each oscillate 90 degrees out of phase with $s_1(t)$, we can drive the amoeba in any direction in the plane, as seen in Section 7.

## 9 Conclusion

We have shown how the tools of geometric mechanics can be used to analyze a general class of smooth deformable swimming robots. We have shown how to derive the connection form which relates the shape of the robot to its propulsion through the fluid, and how the curvature of the connection relates to the efficiency of the propulsion. We have examined an example "amoeba" robot with a particular set of actuators and analyzed its performance. Robot "amoebae" like the example will be easily steered in any direction, but will be slow swimmers. Rapidly swimming macroscopic robots must shed vorticity, and this type of movement is analyzed in Ref. [2].

## References


[1] D.S. Barrett. *Propulsive Efficiency of a Flexible Hull Undersea Vehicle.* PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 1996.

[2] S. Kelly, R. Mason, C. Anhalt, R. Murray, and J. Burdick. Modeling and experimental investigation of carangiform locomotion for control. In *Proc. American Control Conference*, 1998.

[3] S. Childress. *Mechanics of Swimiing and Flying.* Cambridge University Press, Cambridge, 1981.

[4] P.G. Saffman. The self-propulsion of a deformable body in a perfect fluid. *J. Fluid Mechanics*, 28:285–289, 1967.

[5] T. Brooke Benjamin and Albert T. Ellis. Self-propulsion of asymmetrically vibrating bubbles. *J. Fluid Mech.*, 212:65–80, 1990.

[6] A. Galper and T. Miloh. Generalized kirchhoff equations for a deformable body moving in a weakly non-uniform flow field. *Proc. Roy. Soc. London A*, 446:169–193, 1994.

[7] T. Miloh. Hydrodynamical self-propulsion of deformable bodies and oscillating bubbles. In T. Miloh, editor, *Mathematical Approaches in Hydrodynamics*, pages 21–36. SIAM, Philadelphia, 1991.

[8] T. Wu, C.J. Brokaw, and C. Brennan, editors. *Swimming and Flying in Nature.* Plenum Press, New York, 1975.

[9] Alfred Shapere and Frank Wilczek. Geometry of self-propulsion at low Reynolds number. *Journal of Fluid Mechanics*, 198:557–585, January 1989.

[10] J.P. Ostrowski and J.W. Burdick. The mechanics of undulatory robotic locomotion. *Int. J. of Robotics Research*, 17(7):683–701, July 1998.

[11] S. Kelly and R. Murray. Geometric phases and robotic locomotion. *J. Robotic Systems*, 12(6):417–431, June 1995.

[12] P. S. Krishnaprasad and Dimitris P. Tsakiris. *G*-snakes: Nonholonomic kinematic chains on Lie groups. In *Proc. for the* $33^{rd}$ *IEEE Conference on Decision and Control*, Lake Buena Vista, FL, Dec 1994.

[13] I-Ming Chen, Hsi-Shang Li, and Arnaud Cathala. Amoebot – metamorphic underwater vehicle. In *Fifth International Conference on Control, Automation, Robotics, and Vision*, Singapore, 1998.

[14] V. Voinov, O. Voinov, and A. Petrov. Hydrodynamic interaction between bodies in a perfect incompressible fluid and their motion in non-uniform streams. *Prikl. Math. mech.*, 37:680–689, 1973.

[15] A.W. Naylor and G.R. Sell. *Linear Operator Theory in Engineering and Science.* Springer–Verlag, 1982.

[16] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry.* Springer-Verlag, New York, 1994.

[17] J.E. Marsden and J. Scheurle. The reduced Euler-Lagrange equations. *Fields Institute Comm.*, 1:139–164, 1993.

[18] J.P. Ostrowski. *The Mechanics and Control of Undulatory Robotic Locomotion.* PhD thesis, California Institute of Technology, Pasadena, CA, Sept. 1995.

[19] Wilhelm Magnus. On the exponential solutions of differential equations for a linear operator. *Communications on Pure and Applied Mathematics*, VII:649–673, 1954.

[20] J.E. Radford and J.W. Burdick. Local motion planning for nonholonomic control systems evolving on principal bundles. In *Proc. Mathematical Theory of Networks and Systems*, July 1998.